# Von Neumann stability analysis for `cubicUpwindCPCFit` advection scheme on a uniform mesh

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We will perform two stability analyses: the first assuming perfect timestepping and the second using forward Euler timestepping. The spatial discretisation is the same in both analyses so we will consider it first.

Start with the flux form equation, discretised in space, continuous in time:

$$\frac{\partial \phi_j}{\partial t} = -u\frac{\phi_R - \phi_L}{\Delta x} \tag{1}$$
$$\phi_L = 0.06\phi_{j-3} - 0.31\phi_{j-2} + 0.94\phi_{j-1} + 0.31\phi_j \tag{2}$$
$$\phi_R = 0.06\phi_{j-2} - 0.31\phi_{j-1} + 0.94\phi_j + 0.31\phi_{j+1} \tag{3}$$
$$\frac{\partial \phi_j}{\partial t} = -\frac{u}{\Delta x}\left(-0.06\phi_{j-3} + 0.37\phi_{j-2} - 1.25\phi_{j-1} + 0.63\phi_j + 0.31\phi_{j+1}\right) \tag{4}$$

## Perfect timestepping

$$\phi_j^n = A^n e^{ijk\Delta x} \tag{5}$$
$$t = n\Delta t \tag{6}$$
$$\frac{\partial \phi_j}{\partial t} = \frac{\partial}{\partial t}\left(A^{t/\Delta t}\right)e^{ijk\Delta x} \tag{7}$$
$$= \frac{\ln A}{\Delta t}A^n e^{ikj\Delta x} \tag{8}$$
$$\frac{\ln A}{\Delta t} = -\frac{u}{\Delta x}\left(-0.06\phi_{j-3} + 0.37\phi_{j-2} - 1.25\phi_{j-1} + 0.63\phi_j + 0.31\phi_{j+1}\right) \tag{9}$$
$$\ln A = -c\left(-0.06e^{-3ik\Delta x} + 0.37e^{-2ik\Delta x} - 1.25e^{-ik\Delta x} + 0.63 + 0.31e^{ik\Delta x}\right) \tag{10}$$

Let

$$\Re = -0.06\cos 3k\Delta x + 0.37\cos 2k\Delta x + (-1.25 + 0.31)\cos k\Delta x + 0.63 \quad \text{and} \tag{11}$$
$$\Im = 0.06\sin 3k\Delta x - 0.37\sin 2k\Delta x + (1.25 + 0.31)\sin k\Delta x \tag{12}$$

then

$$\ln A = -c\left(\Re + i\Im\right) \tag{13}$$
$$A = e^{-c\Re}e^{-ic\Im} \tag{14}$$
$$|A| = e^{-c\Re} = \exp\left(-c\left(-0.06\cos 3k\Delta x + 0.37\cos 2k\Delta x - 0.94\cos k\Delta x + 0.63\right)\right) \tag{15}$$

Realising that $\cos(\cdot) \in [-1, +1]$ then

$$\exp\left(-c\left(0.63 + 0.06 + 0.37 + 0.94\right)\right) \leq |A| \leq \exp\left(-c\left(0.63 - 0.06 - 0.37 - 0.94\right)\right) \tag{16}$$
$$\exp\left(-2c\right) \leq |A| \leq \left(-0.74c\right) \tag{17}$$

Hence the scheme is unconditionally stable.

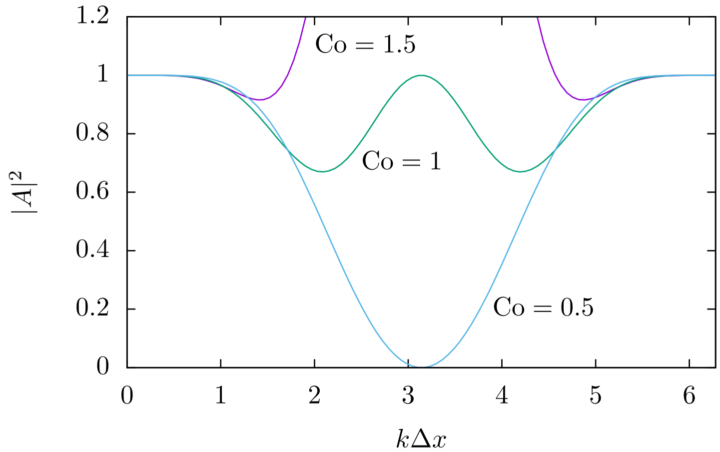


Figure 1: Amplification factor as a function of $k\Delta x$ with forward Euler timestepping with a selection of Courant numbers, Co.

## Forward Euler

We now discretise time using the forward Euler method. Continuing from equation (4):

$$\phi_j^{(n+1)} = \phi_j^{(n)} - c\left(-0.06\phi_{j-3} + 0.37\phi_{j-2} - 1.25\phi_{j-1} + 0.63\phi_j + 0.31\phi_{j+1}\right) \tag{18}$$

$$A = 1 - c\left(-0.06e^{-3ik\Delta x} + 0.37e^{-2ik\Delta x} - 1.25e^{-ik\Delta x} + 0.63 + 0.31e^{ik\Delta x}\right) \tag{19}$$

Let

$$\Re = 1 - c\left(\underbrace{-0.06\cos 3k\Delta x + 0.37\cos 2k\Delta x - 0.94\cos k\Delta x + 0.63}_{=\alpha}\right) \quad \text{and} \tag{20}$$

$$\Im = -c\left(\underbrace{0.06\sin 3k\Delta x - 0.37\sin 2k\Delta x + 0.56\sin k\Delta x}_{=\beta}\right) \tag{21}$$

then

$$|A|^2 = \Re^2 + \Im^2 \tag{22}$$

$$= (1-\alpha c)^2 + \beta^2 c^2 \tag{23}$$

$$= 1 - 2\alpha c + (\alpha^2 + \beta^2)c^2 \tag{24}$$

We know that we want $|A|^2 \leq 1$ and figure 1 shows that this condition is met for all wavenumbers using $c \leq 1$.