Finite volume advection on spherical meshes in global Cartesian coordinates

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A while ago Hilary suggested a test of zonal solid body rotation on a lat-lon mesh using a Courant number (C) of 1 and a first-order FTBS (forward-in-time, backward-in-space) scheme that should yield a perfect result. I've been unable to achieve exactly $C = 1$ because there is a mismatch between face fluxes and cell volumes. This document shows how I calculate face fluxes and explains why there is a mismatch with cell volumes.

Flux calculations

In OpenFOAM, all calculations are performed in global Cartesian coordinates and all meshes are three-dimensional. Here we consider a lat-lon spherical mesh with a single layer of cells.

The flux at a face, ϕ, is the integral of the wind normal to the face:

$$\phi = \int_f \boldsymbol{u} \cdot \hat{\boldsymbol{n}} \, df \tag{1}$$

where f is the face area and $\boldsymbol{u}$ is the wind. For zonal solid body rotation $\boldsymbol{u} = u_0 \cos(\theta)\hat{\boldsymbol{x}}$ where θ is a latitude and $\hat{\boldsymbol{x}}$ is a zonal unit vector. On a lat-lon grid, fluxes are non-zero through faces that are North–South aligned. These faces have the shape on an annular sector bounded by an inner radius r_1, outer radius r_2 and latitudes θ_1 and θ_2. Integrating the wind over such a face gives

$$\phi = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} u_0 \cos\theta r \, dr \, d\theta \tag{2}$$

$$= u_0 \left(\sin\theta_2 - \sin\theta_1\right) \frac{r_2^2 - r_1^2}{2} \tag{3}$$

Note that $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} = 1$ for any North–South aligned face.

Mismatch with cell volume

Starting with the definition of the multidimensional Courant number

$$C = \frac{\Delta t}{2V} \sum_f \phi_f \tag{4}$$

On a lat-lon grid there are equal fluxes through the two North–South aligned cells:

$$C = \frac{\Delta t}{2V} 2u_0 \left(\sin\theta_2 - \sin\theta_1\right) \frac{r_2^2 - r_1^2}{2} \tag{5}$$

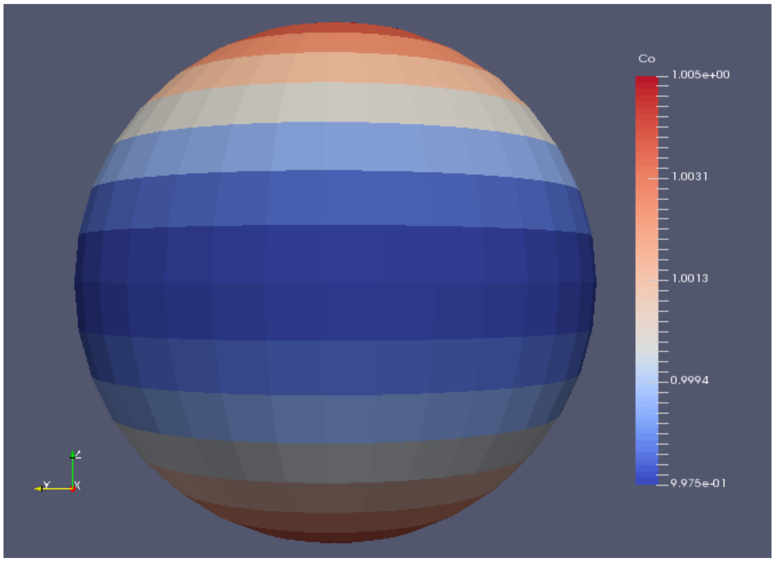

Figure 1: Courant numbers on a coarse lat-lon mesh. Courant numbers converge on one when the mesh is refined.

and assuming $C = 1$ then

$$\frac{V}{\Delta t} = u_0 \left(\sin\theta_2 - \sin\theta_1\right) \frac{r_2^2 - r_1^2}{2} \tag{6}$$

However, when we calculate $V/\Delta t$ and compare it to the face fluxes we find that they are not equal: that is, the Courant number is not exactly one. The actual Courant number varies with latitude as seen in figure 1.