cubicFit and highOrderFit matrix equations

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Here we make a comparison between two transport schemes: cubicFit (Shaw et al., 2017) and highOrderFit, which is based on the high-order formulation by Devendran et al. (2017). Both schemes form a matrix equation that is solved to find coefficients used to calculate the flux. We define a one-dimensional, four-point, upwind-biased stencil (figure 1) with equispaced cell centres. For cubicFit we approximate a field ϕ using a cubic polynomial

$$\phi = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{1}$$

that interpolates the four stencil points. A matrix equation is formed in order to calculate the unknown coefficients $a_1 \dots a_4$,

$$\begin{bmatrix} 1 & x_{uuu} & x_{uuu}^2 & x_{uuu}^3 \\ 1 & x_{uu} & x_{uu}^2 & x_{uu}^3 \\ 1 & x_u & x_u^2 & x_u^3 \\ 1 & x_d & x_d^2 & x_d^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix}. \tag{2}$$

If the equispaced cell centres are positioned at $x_{uuu} = -2.5$, $x_{uu} = -1.5$, $x_u = -0.5$, $x_d = 0.5$ then

$$\begin{bmatrix} 1 & -2.5 & 6.25 & -15.625 \\ 1 & -1.5 & 2.25 & -3.375 \\ 1 & -0.5 & 0.25 & -0.125 \\ 1 & 0.5 & 0.25 & 0.125 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix}. \tag{3}$$

For highOrderFit, we solve the matrix equation

$$\begin{bmatrix} \mathfrak{m}_{uuu}^0/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^1/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^2/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^3/\mathfrak{m}_{uuu}^0 \\ \mathfrak{m}_{uu}^0/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^1/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^2/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^3/\mathfrak{m}_{uu}^0 \\ \mathfrak{m}_u^0/\mathfrak{m}_u^0 & \mathfrak{m}_u^1/\mathfrak{m}_u^0 & \mathfrak{m}_u^2/\mathfrak{m}_u^0 & \mathfrak{m}_u^3/\mathfrak{m}_u^0 \\ \mathfrak{m}_d^0/\mathfrak{m}_d^0 & \mathfrak{m}_d^1/\mathfrak{m}_d^0 & \mathfrak{m}_d^2/\mathfrak{m}_d^0 & \mathfrak{m}_d^3/\mathfrak{m}_d^0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} \tag{4}$$

where $\mathfrak{m}_V^p = \int_V x^p dV$ is the pth moment of volume V, and the zeroth moment $\mathfrak{m}_V^0$ is equal to the volume. If the equispaced cells each have $\mathfrak{m}^0 = 1$ with the cell centres positioned as before, then

$$\begin{bmatrix} 1 & -2.5 & 6.\dot{3} & -16.25 \\ 1 & -1.5 & 2.\dot{3} & -3.75 \\ 1 & -0.5 & 0.\dot{3} & -0.25 \\ 1 & 0.5 & 0.\dot{3} & 0.25 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix}. \tag{5}$$

Notice how the matrix in equation 5 is similar to, but not equal to, the matrix in equation 3.

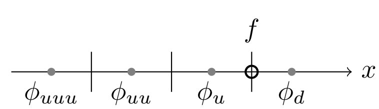

Figure 1: One-dimensional four-point upwind-biased stencil used to approximate the flux at face f.

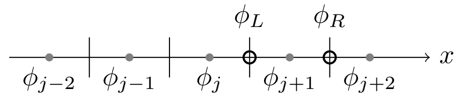

Figure 2: One-dimensional fluxes through a cell ϕ_{j+1} using four-point upwind-biased stencils to approximate fluxes ϕ_L and ϕ_R.

Taylor series analysis

Let's construct a Taylor series approximation centred at ϕ_L for the stencil points $\phi_{j-2} \dots \phi_{j+1}$ (figure 2)

$$\phi_{j-2} = \phi_L - \frac{5\Delta x}{2}\phi'_L + \frac{25\Delta x^2}{8}\phi''_L - \frac{125\Delta x^3}{48}\phi'''_L + \frac{625\Delta x^4}{384}\phi''''_L + \mathcal{O}(\Delta x^5) \tag{6}$$

$$\phi_{j-1} = \phi_L - \frac{3\Delta x}{2}\phi'_L + \frac{9\Delta x^2}{8}\phi''_L - \frac{27\Delta x^3}{48}\phi'''_L + \frac{81\Delta x^4}{384}\phi''''_L + \mathcal{O}(\Delta x^5) \tag{7}$$

$$\phi_j = \phi_L - \frac{\Delta x}{2}\phi'_L + \frac{\Delta x^2}{8}\phi''_L - \frac{\Delta x^3}{48}\phi'''_L + \frac{\Delta x^4}{384}\phi''''_L + \mathcal{O}(\Delta x^5) \tag{8}$$

$$\phi_{j+1} = \phi_L + \frac{\Delta x}{2}\phi'_L + \frac{\Delta x^2}{8}\phi''_L + \frac{\Delta x^3}{48}\phi'''_L + \frac{\Delta x^4}{384}\phi''''_L + \mathcal{O}(\Delta x^5) \tag{9}$$

combining (8) and (9)

$$\phi_j + \phi_{j+1} = 2\phi_L + \frac{\Delta x^2}{4}\phi''_L + \frac{\Delta x^4}{192}\phi''''_L + \mathcal{O}(\Delta x^5) \tag{10}$$

and combining (6) and (7)

$$5\phi_{j-1} - 3\phi_{j-2} = 2\phi_L - \frac{15\Delta x^2}{4}\phi''_L + 5\Delta x^3\phi'''_L - 256/64\Delta x^4\phi''''_L + \mathcal{O}(\Delta x^5) \tag{11}$$

then combine (10) and (11)

$$15\phi_j + 15\phi_{j+1} + 5\phi_{j-1} - 3\phi_{j-2} = 32\phi_L + \mathcal{O}(\Delta x^3). \tag{12}$$

Construct the same Taylor series approximation centred at ϕ_R for the stencil points $\phi_{j-1} \dots \phi_{j+2}$, and substitute into the transport equation

$$\frac{\partial \phi}{\partial t} = -u\frac{\phi_R - \phi_L}{\Delta x} \tag{13}$$

$$= -u\frac{3\phi_{j-2} - 8\phi_{j-1} - 10\phi_j + 15\phi_{j+2} + \mathcal{O}(\Delta x^3)}{32\Delta x} \tag{14}$$

Hence, this discretisation is second-order accurate. So how, then, can higher-order terms be cancelled to achieve higher than second-order accuracy?

References

Devendran, D., D. Graves, H. Johansen, and T. Ligocki, 2017: A fourth-order Cartesian grid embedded boundary method for Poisson's equation. *Comm. App. Math. Comp. Sci.*, **12 (1)**, 51–79, doi:10.2140/camcos.2017.12.51.

Shaw, J., H. Weller, J. Methven, and T. Davies, 2017: Multidimensional method-of-lines transport for atmospheric flows over steep terrain using arbitrary meshes. *J. Comp. Phys.*, **344**, 86–107, doi:10.1016/j.jcp.2017.04.061.