Von Neumann stability analysis for `cubicUpwindCPCFit` advection scheme

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The stabilisation procedure for the `cubicUpwindCPCFit` advection scheme comprises several steps. One of the steps selects a suitable subset of polynomial terms using a constraint

$$-c_p + 0.01 \leq \min(c_u, c_d)\ \forall\, p \tag{1}$$

where c_i is the coefficient associated with cell i, c_u is the upwind coeficient, c_d is the downwind coefficient, and p is the set of peripheral points. Collectively, the upwind and downwind points are called the central points, and the upwind and downwind coefficients are called the central coefficients. Peripheral points are those points that are not central points. This constraint ensures that peripheral cells do not have very negative values. The magic constant 0.01 was found to improve accuracy for a few particular stencils near the lower boundary of BTF meshes.

This constraint is present in the current codebase[1] and supercedes a previous, stronger constraint[2]:

$$|c_p| \leq \min(c_u, c_d) \tag{2}$$

This constraint ensures that peripheral coefficients are smaller than any central coefficient.

This document presents two analyses that begin to justify these constraints. Section 1 analyses the matrix inverse for a one-dimensional quadratic equation. Section 2 performs a von Neumann analysis of a one-dimensional linear advection scheme with three points. Finally, we note the similarity between the two results.

1 Matrix inversion of a quadratic equation

An interpolating quadratic equation, $\phi = a_1 + a_2 x + a_3 x^2$, is found for three data, $\phi_{uu}, \phi_u, \phi_d$, that are known at points x_{uu}, x_u, x_d. This forms a matrix equation

$$\underbrace{\begin{bmatrix} 1 & x_{uu} & x_{uu}^2 \\ 1 & x_u & x_u^2 \\ 1 & x_d & x_d^2 \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\boldsymbol{a}} = \underbrace{\begin{bmatrix} \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix}}_{\boldsymbol{\phi}} \tag{3}$$

which can be inverted to solve for $\boldsymbol{a}$, that is $\boldsymbol{a} = \boldsymbol{B^{-1}\phi}$. In fact, we only need to calculate a_1:

$$a_1 = \begin{bmatrix} \underbrace{\dfrac{x_u x_d^2 - x_u^2 x_d}{\det \boldsymbol{B}}}_{c_{uu}} & \underbrace{\dfrac{x_{uu}^2 x_d - x_{uu} x_d^2}{\det \boldsymbol{B}}}_{c_u} & \underbrace{\dfrac{x_{uu} x_u^2 - x_{uu}^2 x_u}{\det \boldsymbol{B}}}_{c_d} \end{bmatrix} \begin{bmatrix} \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} \tag{4}$$

[1] `https://github.com/hertzsprung/AtmosFOAM/commit/ffaf71`
[2] `https://github.com/hertzsprung/AtmosFOAM/commit/208899`

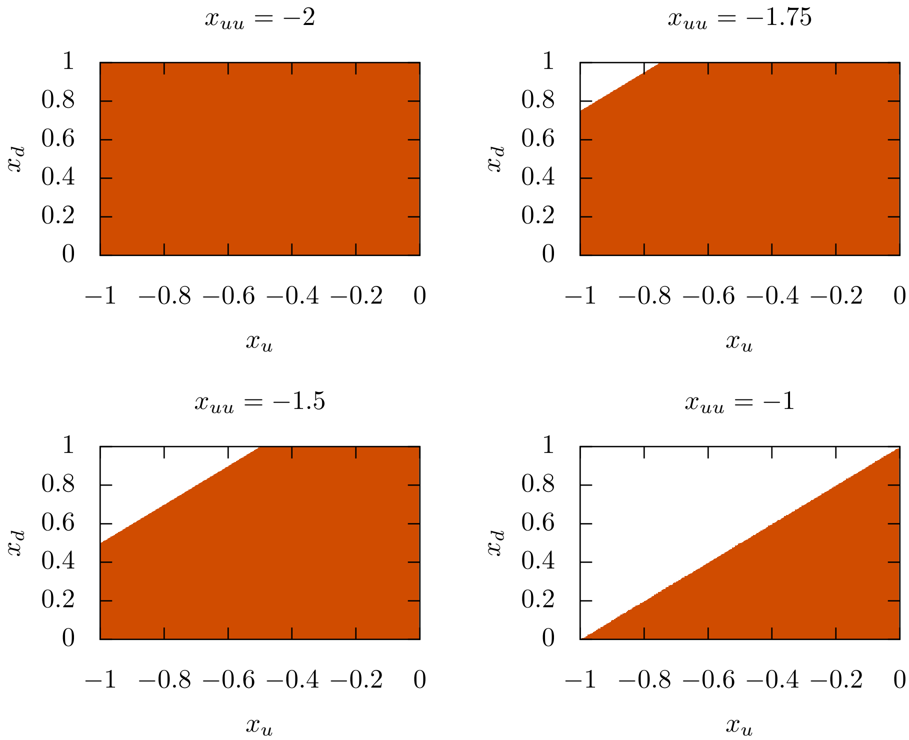

Figure 1: Regions in which $c_{uu} < c_u$ and $c_{uu} < c_d$

We choose x_{uu}, x_u, x_d such that $x_{uu} < x_u < 0$ and $x_d > 0$ and we wish to find the constraint involving x_{uu}, x_u and x_d such that $|c_{uu}| \leq |c_u|$ and $c_{uu} \leq |c_d|$. To do so, we choose four different values for x_{uu} between -2 and -1, and allow $-1 < x_u < 0$ and $0 < x_d < 1$. Figure 1 highlights in orange those regions in which the constraint is satisfied. By inspection, we find that the constraint is

$$x_u - x_d \geq x_{uu} \tag{5}$$

2 Von Neumann analysis of a three-point, 1D linear advection scheme

Start with the flux form equation, discretised in space, continuous in time:

$$\frac{\partial \phi_j}{\partial t} = -u \frac{\phi_R - \phi_L}{\Delta x} \tag{6}$$

$$\phi_L = \alpha_{uu}\phi_{j-2} + \alpha_u \phi_{j-1} + \alpha_d \phi_j \tag{7}$$

$$\phi_R = \alpha_{uu}\phi_{j-1} + \alpha_u \phi_j + \alpha_d \phi_{j+1} \tag{8}$$

We perform a Von Neumann stability analysis with perfect time discretisation

$$\phi_j^n = A^n e^{ijk\Delta x} \tag{9}$$

$$t = n\Delta t \tag{10}$$

$$\frac{\partial \phi_j}{\partial t} = \frac{\partial}{\partial t}\left(A^{t/\Delta t}\right) e^{ijk\Delta x} \tag{11}$$

$$= \frac{\ln A}{\Delta t} A^n e^{ikj\Delta x} \tag{12}$$

$$\frac{\ln A}{\Delta t} = -\frac{u}{\Delta x}\left(\alpha_{uu}\left(e^{-ik\Delta x} - e^{-2ik\Delta x}\right) + \alpha_u\left(1 - e^{-ik\Delta x}\right) + \alpha_d\left(e^{ik\Delta x} - 1\right)\right) \tag{13}$$

$$A = \exp\left(-C\left[\alpha_{uu}\left(e^{-ik\Delta x} - e^{-2ik\Delta x}\right) + \alpha_u\left(1 - e^{-ik\Delta x}\right) + \alpha_d\left(e^{ik\Delta x} - 1\right)\right]\right) \tag{14}$$

where $C = u\Delta t/\Delta x$ is the Courant number. Stability requires $|A| \leq 1$. Let $z = \alpha_{uu}\left(e^{-ik\Delta x} - e^{-2ik\Delta x}\right) + \alpha_u\left(1 - e^{-ik\Delta x}\right) + \alpha_d\left(e^{ik\Delta x} - 1\right)$ and $\Re$ and $\Im$ are the real and imaginary parts of z respectively, then

$$|e^{-Cz}| \leq 1 \tag{15}$$

$$|e^{-C(\Re + i\Im)}| \leq 1 \tag{16}$$

$$|e^{-C\Re} e^{-i\Im}| \leq 1 \tag{17}$$

$$|e^{-C\Re}| \leq 1 \tag{18}$$

So $\Re \geq 0$, then

$$\alpha_{uu}\left(\cos k\Delta x - \cos 2k\Delta x\right) + \alpha_u\left(1 - \cos k\Delta x\right) + \alpha_d\left(\cos k\Delta x - 1\right) \geq 0 \tag{19}$$

$$\alpha_u - \alpha_d + \left(\alpha_{uu} - \alpha_u + \alpha_d\right)\cos k\Delta x - \alpha_{uu}\cos 2k\Delta x \geq 0 \tag{20}$$

Now we want to find inequalities that constrain α_u, α_{uu} and α_d. If $\cos k\Delta x = -1$ and $\cos 2k\Delta x = 1$ then $\alpha_u \geq \alpha_{uu} + \alpha_d$. So we have the constraint

$$\alpha_u - \alpha_d \geq \alpha_{uu} \tag{21}$$

This constraint has the same form as equation (5).

Von Neumann analysis of a two-point, 1D linear advection scheme

Start with the flux form equation, discretised in space, continuous in time:

$$\frac{\partial \phi_j}{\partial t} = -u\frac{\phi_R - \phi_L}{\Delta x} \tag{22}$$

$$\phi_L = \alpha_u \phi_{j-1} + \alpha_d \phi_j \tag{23}$$

$$\phi_R = \beta_u \phi_j + \beta_d \phi_{j+1} \tag{24}$$

Von Neumann stability analysis with perfect time discretisation

$$\phi_j^n = A^n e^{ijk\Delta x} \tag{25}$$

$$t = n\Delta t \tag{26}$$

$$\frac{\partial \phi_j}{\partial t} = \frac{\partial}{\partial t}\left(A^{t/\Delta t}\right) e^{ijk\Delta x} \tag{27}$$

$$= \frac{\ln A}{\Delta t} A^n e^{ikj\Delta x} \tag{28}$$

$$\frac{\ln A}{\Delta t} = -\frac{u}{\Delta x}\left(\beta_u + \beta_d e^{ik\Delta x} - \alpha_u e^{-ik\Delta x} - \alpha_d\right) \tag{29}$$

$$\ln A = -c\left(\beta_u - \alpha_d + \beta_d e^{ik\Delta x} - \alpha_u e^{-ik\Delta x}\right) \tag{30}$$

$$= -c\left(\beta_u - \alpha_d + \beta_d \cos k\Delta x + i\beta_d \sin k\Delta x - \alpha_u \cos k\Delta x + i\alpha_u \sin k\Delta x\right) \tag{31}$$

let $\Re = \beta_u - \alpha_d + \beta_d \cos k\Delta x - \alpha_u \cos k\Delta x$ and $\Im = \beta_d \sin k\Delta x + \alpha_u \sin k\Delta x$, then

$$\ln A = -c\left(\Re + i\Im\right) \tag{32}$$
$$A = e^{-c\Re}e^{-ic\Im} \tag{33}$$
$$|A| = e^{-c\Re} = \exp\left(-c\left(\beta_u - \alpha_d + \left(\beta_d - \alpha_u\right)\cos k\Delta x\right)\right) \tag{34}$$
$$\arg(A) = -c\Im = -c\left(\beta_d + \alpha_u\right)\sin k\Delta x \tag{35}$$

For stability we need $|A| \leq 1$ and $\arg(A) < 0$ for $c > 0$, so

$$\beta_u - \alpha_d + \left(\beta_d - \alpha_u\right)\cos k\Delta x \geq 0 \quad \forall k\Delta x \quad \text{and} \tag{36}$$
$$\beta_d + \alpha_u > 0 \tag{37}$$

TODO: *not sure what to make of these inequalities, Hilary got a little further but I didn't follow all of it. But no matter, let's continue...*

Imposing the additional constraints that $\alpha_u = \beta_u$ and $\alpha_d = \beta_d$:

$$|A| = \exp\left(-c\left(\alpha_u - \alpha_d\right)\left(1 - \cos k\Delta x\right)\right) \tag{38}$$

and given $1 - \cos k\Delta x \geq 0$ for well-resolved waves

$$\alpha_u - \alpha_d \geq 0 \tag{39}$$
$$\alpha_u \geq \alpha_d \tag{40}$$

and from eqn (37)

$$\alpha_d + \alpha_u > 0 \tag{41}$$
$$\alpha_u > -\alpha_d \quad \text{hence} \tag{42}$$
$$\alpha_u > |\alpha_d| \tag{43}$$

Additionally, we do not want more damping than an upwind scheme (where $\alpha_u = \beta_u = 1$, $\alpha_d = \beta_d = 0$), having an amplification factor, A_{up}:

$$|A_{\text{up}}| = \exp\left(-c\left(1 - \cos k\Delta x\right)\right) \tag{44}$$

So we need $|A| \geq |A_{\text{up}}|$:

$$-c\left(\alpha_u - \alpha_d\right)\left(1 - \cos k\Delta x\right) \geq -c\left(1 - \cos k\Delta x\right) \tag{45}$$
$$\alpha_u - \alpha_d \leq 1 \tag{46}$$
$$\alpha_u \leq 1 + \alpha_d \tag{47}$$

which provides an upper bound on α_u. Combining with eqn (43) we can bound α_u on both sides:

$$|\alpha_d| < \alpha_u \leq 1 + \alpha_d \tag{48}$$

Now, assume that $\alpha_u + \alpha_d = 1$ (or $\alpha_d = 1 - \alpha_u$), then

$$1 - \alpha_u < \alpha_u \leq 1 + \left(1 - \alpha_u\right) \tag{49}$$
$$0.5 < \alpha_u \leq 1 \tag{50}$$

we use only the lower bound TODO: *why?* to obtain

$$|\alpha_d| < \alpha_u \leq 1 + \alpha_d \quad \text{and} \quad 0.5 < \alpha_u \tag{51}$$