Thinking about 2D flow over a hill using Bernoulli's principle

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Earlier in my PhD I studied gravity waves that were induced by flow over a hill in the two-dimensional test case by Schär et al. (2002) using a numerical model of the Euler equations (Shaw and Weller, 2016). Bernoulli's principle states that, for incompressible steady-state flow in a pipe, energy is conserved along a streamline (Vallis, 2006)

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\underbrace{\frac{1}{2}\boldsymbol{u}^2}_{\text{kinetic}} + \underbrace{gz}_{\text{potential}} + \underbrace{\frac{p}{\rho_0}}_{\text{internal}}\right) = 0 \tag{1}$$

with a flow $\boldsymbol{u}$, gravitational acceleration g, height z, pressure p, and fixed density ρ_0. We might think of atmospheric flow as fluid flowing through a pipe (figure 1). In two dimensions, the width of the pipe is the distance between the upper and lower boundaries. In this conceptual model, we can think of a hill as a constriction in the pipe.

Given the incompressible, steady state flow, the flux across the inlet boundary, outlet boundary, or any vertical cross section through the interior, must be equal. Hence, the fluid speed must be larger above the hill (or, conceptually, in the region of the constriction in the pipe). The fluid must also gain gravitational potential energy as it ascends over the hill. Bernoulli's principle then tells us that, in order to conserve energy, the pressure must decrease in the region above the hill. Since pressure is proportional to temperature in an incompressible fluid, then temperature also decreases above the hill.

A few thoughts arise following on from this discussion:

- Tracing streamlines that start at different heights we find that streamlines near the lower boundary will have greater changes in potential energy than streamlines near the upper boundary. If this is true, then pressure and temperature will decrease the most following streamlines at the ground of the hill, and neither pressure nor temperature will vary along the upper boundary.

- If we can get a neutrally stable flow test case working, we should look at our numerical model pressure and temperature fields and see if their perturbations match Bernoulli's principle.

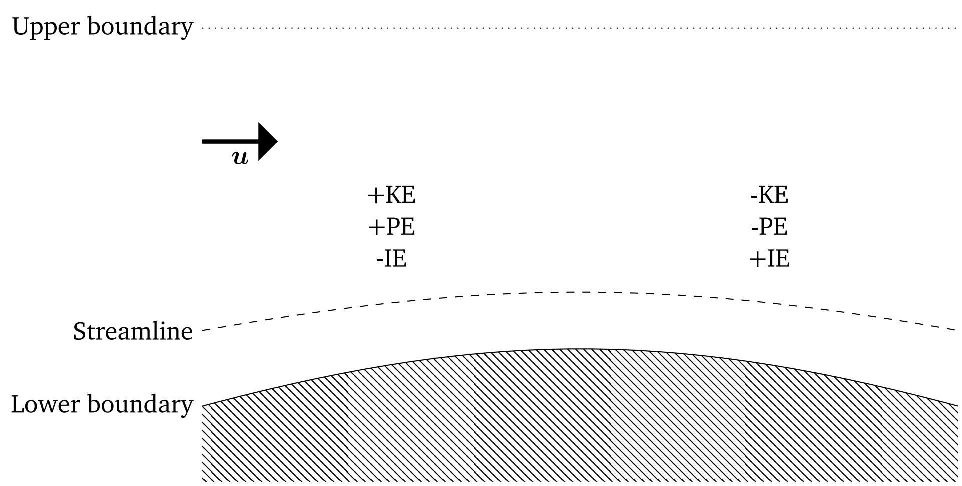

Figure 1: Fluid flowing from left to right over a hill. Following a streamline, kinetic energy (KE), potential energy (PE) increase and internal energy (IE) decreases as the pipe narrows on the windward side of the hill. The reverse is true as the pipe widens on the lee side of the hill.

- Can we explain the fluid behaviour using the Euler equations?
- How does the fluid behave if we consider that the upper boundary is a free surface? Then Bernoulli's principle of energy conservation through a pipe no longer applies.

Some other ideas for further consideration:

- How does a stratified fluid behave? How can we explain the perturbations to potential temperature and vertical velocity that we've seen in two-dimensional numerical simulations?
- Think about flow regimes in relation to the Froude number. How is this derived, what is the threshold value and why?

References

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