Want to find an analytic solution to wobblyTracerAdvection where we prescribe a velocity field as the streamfunction of the coordinate transform, $z^\star$.

$$
\begin{aligned}
z^\star &= H\frac{z-h}{H-h} && (1)\\
u &= u_0\frac{\partial z^\star}{\partial z} && (2)\\
h &= h_0\cos^2(\alpha x)\cos^2(\beta x) && (3)\\
\alpha &= \frac{\pi}{\lambda} \quad , \quad \beta = \frac{\pi}{2a} && (4)\\
\mathrm{d}t &= \frac{\mathrm{d}x}{u} && (5)\\
\int \mathrm{d}t &= \int \frac{H-h}{u_0 H}\mathrm{d}x && (6)\\
t &= \frac{x}{u_0} - \frac{1}{u_0 H}\int h(x)\mathrm{d}x && (7)\\
&= \frac{x}{u_0} - \frac{h_0}{u_0 H}\int \cos^2\alpha x\cos^2\beta x\mathrm{d}x && (8)\\
&= \frac{x}{u_0} - \frac{h_0}{u_0 H}\int \left[\frac{1}{2}+\frac{1}{2}\cos 2\alpha x\right]\left[\frac{1}{2}+\frac{1}{2}\cos 2\beta x\right]\mathrm{d}x \quad \left(\text{using } \cos^2 X = \frac{1}{2}+\frac{1}{2}\cos 2X\right) && (9)\\
&= \frac{x}{u_0} - \frac{h_0}{u_0 H}\int \frac{1}{4}+\frac{1}{4}\cos 2\alpha x cos 2\beta x + \frac{1}{4}\cos 2\alpha x + \frac{1}{4}\cos 2\beta x\,\mathrm{d}x && (10)\\
&= \frac{x}{u_0} - \frac{h_0}{4u_0 H}\left[x + \int \cos 2\alpha x\cos 2\beta x\,\mathrm{d}x + \int \cos 2\alpha x\,\mathrm{d}x + \int \cos 2\beta x\,\mathrm{d}x\right] + C && (11)\\
&= \frac{x}{u_0} - \frac{h_0}{4u_0 H}\left[x + \int \cos 2\alpha x\cos 2\beta x\,\mathrm{d}x + \frac{\sin 2\alpha x}{2\alpha} + \frac{\sin 2\beta x}{2\beta}\right] + C && (12)\\
&= \frac{x}{u_0} - \frac{h_0}{4u_0 H}\left[x + \frac{1}{2}\int \cos 2\left(\alpha+\beta\right)x + \cos 2\left(\alpha-\beta\right)x\,\mathrm{d}x + \frac{\sin 2\alpha x}{2\alpha} + \frac{\sin 2\beta x}{2\beta}\right] + C && (13)\\
&\left(\text{using } \cos X\cos Y = \frac{1}{2}\left[\cos(X-Y)+\cos(X+Y)\right]\right)\\
t &= \frac{x}{u_0} - \frac{h_0}{16u_0 H}\left[4x + \frac{\sin 2(\alpha+\beta)x}{\alpha+\beta} + \frac{\sin 2(\alpha-\beta)x}{\alpha-\beta} + 2\left(\frac{\sin 2\alpha x}{\alpha}+\frac{\sin 2\beta x}{\beta}\right)\right] + C && (14)
\end{aligned}
$$

Now do the same for w

$$w = -u_0 \frac{\partial z^\star}{\partial x} \tag{15}$$

$$= u_0 H \frac{\partial h}{\partial x} \frac{H - z}{(H-h)^2} \tag{16}$$

$$\frac{\partial h}{\partial x} = -h_0 \left[\beta \cos^2 \alpha x \sin 2\beta x + \alpha \cos^2 \beta x \sin 2\alpha x\right] \tag{17}$$

$$\mathrm{d}t = \frac{\mathrm{d}z}{w} \tag{18}$$

$$\int \mathrm{d}t = \frac{(H-h)^2}{u_0 H} \left(\frac{\partial h}{\partial x}\right)^{-1} \int \frac{1}{H-z}\,\mathrm{d}z \tag{19}$$

$$t = \frac{(H-h)^2}{u_0 H h_0} \left(\beta \cos^2 \alpha x \sin 2\beta x + \alpha \cos^2 \beta x \sin 2\alpha x\right)^{-1} \ln(H-z) + c(x) \tag{20}$$

$$z = H - \exp\left[\frac{u_0 H h_0 (t - c(x))}{(H-h)^2} \left(\beta \cos^2 \alpha x \sin 2\beta x + \alpha \cos^2 \beta x \sin 2\alpha x\right)\right] \tag{21}$$