Von Neumann stability analysis for `cubicUpwindCPCFit` advection scheme

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Start with the flux form equation, discretised in space, continuous in time:

$$\frac{\partial \phi_j}{\partial t} = -u\frac{\phi_R - \phi_L}{\Delta x} \tag{1}$$
$$\phi_L = \alpha_u \phi_{j-1} + \alpha_d \phi_j \tag{2}$$
$$\phi_R = \beta_u \phi_j + \beta_d \phi_{j+1} \tag{3}$$

Von Neumann stability analysis with perfect time discretisation

$$\phi_j^n = A^n e^{ijk\Delta x} \tag{4}$$
$$t = n\Delta t \tag{5}$$
$$\frac{\partial \phi_j}{\partial t} = \frac{\partial}{\partial t}\left(A^{t/\Delta t}\right) e^{ijk\Delta x} \tag{6}$$
$$= \frac{\ln A}{\Delta t} A^n e^{ikj\Delta x} \tag{7}$$
$$\frac{\ln A}{\Delta t} = -\frac{u}{\Delta x}\left(\beta_u + \beta_d e^{ik\Delta x} - \alpha_u e^{-ik\Delta x} - \alpha_d\right) \tag{8}$$
$$\ln A = -c\left(\beta_u - \alpha_d + \beta_d e^{ik\Delta x} - \alpha_u e^{-ik\Delta x}\right) \tag{9}$$
$$= -c\left(\beta_u - \alpha_d + \beta_d \cos k\Delta x + i\beta_d \sin k\Delta x - \alpha_u \cos k\Delta x - i\alpha_u \sin k\Delta x\right) \tag{10}$$

let $\Re = \beta_u - \alpha_d + \beta_d \cos k\Delta x - \alpha_u \cos k\Delta x$ and $\Im = \beta_d \sin k\Delta x - \alpha_u \sin k\Delta x$, then

$$\ln A = -c\left(\Re + i\Im\right) \tag{11}$$
$$A = e^{-c\Re} e^{-ic\Im} \tag{12}$$
$$|A| = e^{-c\Re} = \exp\left(-c\left(\beta_u - \alpha_d + (\beta_d - \alpha_u)\cos k\Delta x\right)\right) \tag{13}$$
$$\arg(A) = -c\Im = -c\left(\beta_d + \alpha_u\right)\sin k\Delta x \tag{14}$$

For stability we need $|A| \leq 1$ and $\arg(A) < 0$ for $c > 0$, so

$$\beta_u - \alpha_d + (\beta_d - \alpha_u)\cos k\Delta x \geq 0 \quad \forall k\Delta x \quad \text{and} \tag{15}$$
$$\beta_d + \alpha_u > 0 \tag{16}$$

TODO: *not sure what to make of these inequalities, Hilary got a little further but I didn't follow all of it. But no matter, let's continue. . .*

Imposing the additional constraints that $\alpha_u = \beta_u$ and $\alpha_d = \beta_d$:

$$|A| = \exp\left(-c\left(\alpha_u - \alpha_d\right)\left(1 - \cos k\Delta x\right)\right) \tag{17}$$

and given $1 - \cos k\Delta x \geq 0$ for well-resolved waves

$$\alpha_u - \alpha_d \geq 0 \tag{18}$$
$$\alpha_u \geq \alpha_d \tag{19}$$

and from eqn (16)

$$\alpha_d + \alpha_u > 0 \tag{20}$$
$$\alpha_u > -\alpha_d \quad \text{hence} \tag{21}$$
$$\alpha_u > |\alpha_d| \tag{22}$$

Additionally, we do not want more damping than an upwind scheme (where $\alpha_u = \beta_u = 1$, $\alpha_d = \beta_d = 0$), having an amplification factor, A_{up}:

$$|A_{\text{up}}| = \exp\left(-c\left(1 - \cos k\Delta x\right)\right) \tag{23}$$

So we need $|A| \geq |A_{\text{up}}|$:

$$-c\left(\alpha_u - \alpha_d\right)\left(1 - \cos k\Delta x\right) \geq -c\left(1 - \cos k\Delta x\right) \tag{24}$$
$$\alpha_u - \alpha_d \leq 1 \tag{25}$$
$$\alpha_u \leq 1 + \alpha_d \tag{26}$$

which provides an upper bound on α_u. Combining with eqn (22) we can bound α_u on both sides:

$$|\alpha_d| < \alpha_u \leq 1 + \alpha_d \tag{27}$$

Now, assume that $\alpha_u + \alpha_d = 1$ (or $\alpha_d = 1 - \alpha_u$), then

$$1 - \alpha_u < \alpha_u \leq 1 + (1 - \alpha_u) \tag{28}$$
$$0.5 < \alpha_u \leq 1 \tag{29}$$

we use only the lower bound TODO: *why?* to obtain

$$|\alpha_d| < \alpha_u \leq 1 + \alpha_d \quad \text{and} \quad 0.5 < \alpha_u \tag{30}$$