

# A high-order correction to the one-dimensional cubicFit transport scheme

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A transport scheme is ‘super-convergent’ when its order of convergence is higher on uniform meshes than on non-uniform meshes. For example, the transport scheme by Skamarock and Gassmann (2011) is super-convergent because it is first-order on non-uniform meshes and third-order on uniform meshes. Without a high-order correction, the one-dimensional cubicFit transport scheme is not super-convergent because it is second-order convergent on both uniform and non-uniform meshes.

Here I describe a correction to cubicFit that results in fourth-order convergence on uniform meshes. The correction technique that I use is inspired by the Taylor series expansion used by Skamarock and Gassmann (2011). The corrected cubic scheme retains second-order convergence on non-uniform meshes with improved absolute accuracy compared to the uncorrected scheme.

The one-dimensional linear transport of a dependent variable  $\phi$  is given by

$$\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x} \quad (1)$$

where  $u$  is a constant, positive velocity. The term on the right-hand side of equation (1) is called the flux divergence. The finite volume method offers one way to discretise the flux divergence by considering flux across faces of a cell,

$$-u \frac{\partial \phi}{\partial x} \approx -u \frac{\phi_R - \phi_L}{\Delta x} \quad (2)$$

where  $\phi_L$  and  $\phi_R$  are approximate values of  $\phi$  at the left and right faces respectively, and  $\Delta x$  is the distance between the faces. The cubicFit scheme is used to approximate face values  $\phi_L$  and  $\phi_R$  from surrounding cell centre values. In one dimension, the cubicFit scheme exactly interpolates the value of a dependent variable  $\phi$  at face  $f$  using the neighbouring downwind and three upwind cell centre values. This arrangement is shown in figure 1.

The one-dimensional cubic interpolation is

$$\phi = a_1 + a_2x + a_3x^2 + a_4x^3. \quad (3)$$

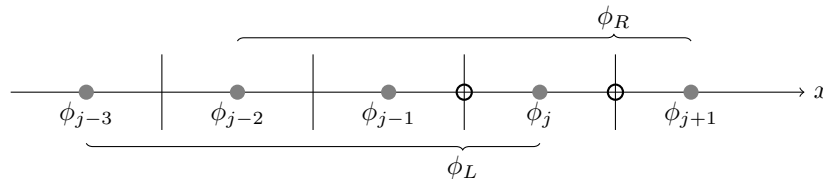


Figure 1: The one-dimensional cubicFit transport scheme interpolates face values  $\phi_L$  and  $\phi_R$  using four-point, upwind-biased stencils of cell-centre values.

Assuming a uniform mesh with  $\Delta x = 1$  and choosing the position of  $\phi_{i+1/2}$  to be  $x = 0$ , equation (3) is evaluated at the cell centres  $\phi_{i-2}, \dots, \phi_{i+1}$  to form the matrix equation

$$\mathbf{B}\mathbf{a} = \phi \quad (4)$$

$$\begin{bmatrix} 1 & -5/2 & 25/4 & -125/8 \\ 1 & -3/2 & 9/4 & -27/8 \\ 1 & -1/2 & 1/4 & -1/8 \\ 1 & 1/2 & 1/4 & 1/8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix}. \quad (5)$$

The unknown coefficients  $\mathbf{a}$  are found by inverting  $\mathbf{B}$  such that  $\mathbf{a} = \mathbf{B}^{-1}\phi$ . The inverse matrix is

$$\mathbf{B}^{-1} = \frac{1}{48} \begin{bmatrix} 3 & -15 & 45 & 15 \\ 2 & -6 & -42 & 46 \\ -12 & 60 & -84 & 36 \\ -8 & 24 & -24 & 8 \end{bmatrix} \quad (6)$$

Since  $x = 0$  was chosen to be the position  $i + 1/2$  then

$$\phi_{i+1/2} = a_1 = \frac{1}{16} \begin{bmatrix} 1 \\ -5 \\ 15 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix}. \quad (7)$$

The finite difference method offers another way to discretise the flux divergence of cell  $i$  with a cubic approximation using cell centre values  $\phi_{i-2}, \dots, \phi_{i+1}$ . A matrix equation is constructed using equation (3) evaluated at every cell centre. For convenience, assume that  $\Delta x = 1$  and that  $x = 0$  at the cell centre position of  $\phi_i$ , hence the matrix equation becomes

$$\mathbf{B}\mathbf{a} = \phi \quad (8)$$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (9)$$

The unknown coefficients  $\mathbf{a}$  are found by calculating the inverse matrix,

$$\mathbf{B}^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 0 & 6 & 0 \\ 1 & -6 & 3 & 2 \\ 0 & 3 & -6 & 3 \\ -1 & 3 & -3 & 1 \end{bmatrix} \quad (10)$$

To calculate the flux divergence I calculate the derivative  $\partial\phi/\partial x = a_2 + 2a_3x + 3a_4x^2$ . Evaluating the flux divergence at  $\phi_i$  where  $x = 0$  then  $\partial\phi_i/\partial x = a_2$ . Hence I find that the finite difference weighted sum is

$$-u \frac{\partial\phi_i}{\partial x} = -ua_2 = -u \cdot \frac{1}{6} \begin{bmatrix} 1 \\ -6 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (11)$$

The cubic finite difference approximation given in equation (11) is conservative on uniform meshes. This

can be demonstrated by decomposing the weights vector,

$$-u \cdot \frac{1}{6} \begin{bmatrix} 1 \\ -6 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} = -u \cdot \frac{1}{6} \left( \begin{bmatrix} 0 \\ -1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 5 \\ 2 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (12)$$

$$= -u \left( \frac{1}{6} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \end{bmatrix} \right). \quad (13)$$

Notice that the flux divergence has been rewritten as the difference between right and left fluxes (equation 2).

### Calculating the high-order correction

The high-order correction to the cubicFit scheme is calculated as the difference between the cubic finite difference approximation (equation 13) and the uncorrected cubicFit approximation (equation 7),

$$\text{correction}(\phi_{i+1/2}) = \left( \frac{1}{6} \begin{bmatrix} 0 \\ -1 \\ 5 \\ 2 \end{bmatrix} - \frac{1}{16} \begin{bmatrix} 1 \\ -5 \\ 15 \\ 5 \end{bmatrix} \right) \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (14)$$

$$= \frac{1}{48} \begin{bmatrix} -3 \\ 7 \\ -5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (15)$$

which can be decomposed into a linear combination of second derivatives where  $\partial_x^2 \phi_i = \phi_{i-1} - 2\phi_i + \phi_{i+1}$ ,

$$= \frac{1}{48} \left( -3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (16)$$

$$= \frac{1}{48} (-3\partial_x^2 \phi_{i-1} + \partial_x^2 \phi_i). \quad (17)$$

Applying this correction using the three-point approximation of the second derivative results in third-order convergence on uniform meshes and second-order convergence on non-uniform meshes.

Alternatively, the second derivative can be calculated from equation (3) such that  $\partial_x^2 \phi = 2a_3 + 6a_4x$  where  $a_3$  and  $a_4$  can be calculated using equation (6). This approach results in fourth-order convergence on uniform meshes and second-order convergence on non-uniform meshes. Curiously, this approach yields the correction

$$\text{correction}(\phi_{i+1/2}) = \frac{1}{48} \left( -3 \cdot \frac{4}{5} \begin{bmatrix} -27 \\ 141 \\ -210 \\ 87 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} -33 \\ 159 \\ -219 \\ 93 \end{bmatrix} \right) \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (18)$$

$$= \frac{1}{48} \cdot \begin{bmatrix} 0.8 \\ -4.4 \\ 6.4 \\ -2.8 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \quad (19)$$

whose coefficients are similar in reverse order to those in equation 15.

## References

Skamarock, W. C., and A. Gassmann, 2011: Conservative transport schemes for spherical geodesic grids: High-order flux operators for ODE-based time integration. *Mon. Wea. Rev.*, **139**, 2962–2975, doi:10.1175/MWR-D-10-05056.1.