Comparison of one-dimensional cubicFit and highOrderFit

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Here we make a comparison between two transport schemes: cubicFit (Shaw et al., 2017) and highOrderFit, which is based on the high-order formulation by Devendran et al. (2017). Both schemes form a matrix equation that is solved to find coefficients used to calculate the flux. We define a one-dimensional, four-point, upwind-biased stencil (figure 1) with equispaced cell centres. For cubicFit we approximate a field ϕ using a cubic polynomial

$$\phi = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{1}$$

that interpolates the four stencil points. A matrix equation is formed in order to calculate the unknown coefficients $a_1 \dots a_4$,

$$\begin{bmatrix} 1 & x_{uuu} & x_{uuu}^2 & x_{uuu}^3 \\ 1 & x_{uu} & x_{uu}^2 & x_{uu}^3 \\ 1 & x_u & x_u^2 & x_u^3 \\ 1 & x_d & x_d^2 & x_d^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} . \tag{2}$$

If the equispaced cell centres are positioned at $x_{uuu} = -2.5$, $x_{uu} = -1.5$, $x_u = -0.5$, $x_d = 0.5$ then

$$\begin{bmatrix} 1 & -2.5 & 6.25 & -15.625 \\ 1 & -1.5 & 2.25 & -3.375 \\ 1 & -0.5 & 0.25 & -0.125 \\ 1 & 0.5 & 0.25 & 0.125 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} . \tag{3}$$

For highOrderFit, we solve the matrix equation

$$\begin{bmatrix} \mathfrak{m}_{uuu}^0/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^1/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^2/\mathfrak{m}_{uuu}^0 & \mathfrak{m}_{uuu}^3/\mathfrak{m}_{uuu}^0 \\ \mathfrak{m}_{uu}^0/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^1/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^2/\mathfrak{m}_{uu}^0 & \mathfrak{m}_{uu}^3/\mathfrak{m}_{uu}^0 \\ \mathfrak{m}_{u}^0/\mathfrak{m}_{u}^0 & \mathfrak{m}_{u}^1/\mathfrak{m}_{u}^0 & \mathfrak{m}_{u}^2/\mathfrak{m}_{u}^0 & \mathfrak{m}_{u}^3/\mathfrak{m}_{u}^0 \\ \mathfrak{m}_{d}^0/\mathfrak{m}_{d}^0 & \mathfrak{m}_{d}^1/\mathfrak{m}_{d}^0 & \mathfrak{m}_{d}^2/\mathfrak{m}_{d}^0 & \mathfrak{m}_{d}^3/\mathfrak{m}_{d}^0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} \tag{4}$$

where $\mathfrak{m}_V^p = \int_V x^p dV$ is the pth moment of volume V, and the zeroth moment $\mathfrak{m}_V^0$ is equal to the volume. If the equispaced cells each have $\mathfrak{m}^0 = 1$ with the cell centres positioned as before, then

$$\begin{bmatrix} 1 & -2.5 & 6.\dot{3} & -16.25 \\ 1 & -1.5 & 2.\dot{3} & -3.75 \\ 1 & -0.5 & 0.\dot{3} & -0.25 \\ 1 & 0.5 & 0.\dot{3} & 0.25 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{uuu} \\ \phi_{uu} \\ \phi_u \\ \phi_d \end{bmatrix} . \tag{5}$$

Notice how the matrix in equation (5) is similar to, but not equal to, the matrix in equation (3).

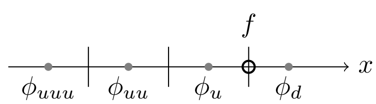

Figure 1: One-dimensional four-point upwind-biased stencil used to approximate the flux at face f.

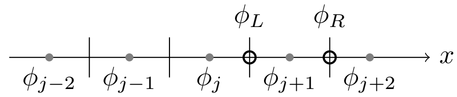

Figure 2: One-dimensional fluxes through a cell ϕ_{j+1} using four-point upwind-biased stencils to approximate fluxes ϕ_L and ϕ_R.

Taylor series analysis

To find a polynomial that interpolates the 4-point upwind-biased stencil we construct Taylor series approximations centred at ϕ_L for the stencil points $\phi_{j-2} \ldots \phi_{j+1}$

$$\begin{bmatrix} 1 & -\frac{5}{2} & \frac{1}{2!}\left(-\frac{5}{2}\right)^2 & \frac{1}{3!}\left(-\frac{5}{2}\right)^3 \\ 1 & -\frac{3}{2} & \frac{1}{2!}\left(-\frac{3}{2}\right)^2 & \frac{1}{3!}\left(-\frac{3}{2}\right)^3 \\ 1 & -\frac{1}{2} & \frac{1}{2!}\left(-\frac{1}{2}\right)^2 & \frac{1}{3!}\left(-\frac{1}{2}\right)^3 \\ 1 & \frac{1}{2} & \frac{1}{2!}\left(\frac{1}{2}\right)^2 & \frac{1}{3!}\left(\frac{1}{2}\right)^3 \end{bmatrix} \begin{bmatrix} \phi_L \\ \phi_L' \\ \phi_L'' \\ \phi_L''' \end{bmatrix} = \begin{bmatrix} \phi_{j-2} \\ \phi_{j-1} \\ \phi_j \\ \phi_{j+1} \end{bmatrix}. \tag{6}$$

Solving this system results in the fourth-order accurate approximation,

$$\phi_L = \frac{1}{16}\phi_{j-2} - \frac{5}{16}\phi_{j-1} + \frac{15}{16}\phi_j + \frac{5}{16}\phi_{j+1} + \mathcal{O}(\Delta x^5) \tag{7}$$

The approximate flux divergence (figure 2) is then

$$\phi_R - \phi_L = -\frac{1}{16}\phi_{j-2} + \frac{6}{16}\phi_{j-1} - \frac{20}{16}\phi_j + \frac{20}{16}\phi_{j+1} + \frac{5}{16}\phi_{j+2} \tag{8}$$

but this not a fourth-order accurate flux divergence approximation as we will see next. A fourth-order accurate approximation to the flux divergence is derived from a Taylor series approximation of $\partial\phi/\partial x$ centred at ϕ_{j+1} (figure 2),

$$\begin{bmatrix} 1 & -3 & \frac{1}{2!}(-3)^2 & \frac{1}{3!}(-3)^3 & \frac{1}{4!}(-3)^4 \\ 1 & -2 & \frac{1}{2!}(-2)^2 & \frac{1}{3!}(-2)^3 & \frac{1}{4!}(-2)^4 \\ 1 & -1 & \frac{1}{2!}(-1)^2 & \frac{1}{3!}(-1)^3 & \frac{1}{4!}(-1)^4 \\ 1 & 0 & \frac{1}{2!}0^2 & \frac{1}{3!}0^3 & \frac{1}{4!}0^4 \\ 1 & 1 & \frac{1}{2!}1^2 & \frac{1}{3!}1^3 & \frac{1}{4!}1^4 \end{bmatrix} \begin{bmatrix} \phi_{j+1} \\ \phi_{j+1}' \\ \phi_{j+1}'' \\ \phi_{j+1}''' \\ \phi_{j+1}'''' \end{bmatrix} = \begin{bmatrix} \phi_{j-2} \\ \phi_{j-1} \\ \phi_j \\ \phi_{j+1} \\ \phi_{j+2} \end{bmatrix} \tag{9}$$

which yields

$$\left.\frac{\partial\phi}{\partial x}\right|_{j+1} = -\frac{1}{12}\phi_{j-2} + \frac{6}{12}\phi_{j-1} - \frac{18}{12}\phi_j + \frac{10}{12}\phi_{j+1} + \frac{3}{12}\phi_{j+2} + \mathcal{O}(\Delta x^4). \tag{10}$$

To find the fourth-order face flux we invert the matrix in equation (5) and find that

$$\phi_L = \frac{1}{12}\phi_{j-2} - \frac{5}{12}\phi_{j-1} + \frac{13}{12}\phi_j + \frac{3}{12}\phi_{j+1}. \tag{11}$$

Using equation (9) and equation (11) we can verify that $\partial\phi/\partial x|_{j+1} = \phi_R - \phi_L + \mathcal{O}(\Delta x^4)$ as desired (figure 3).

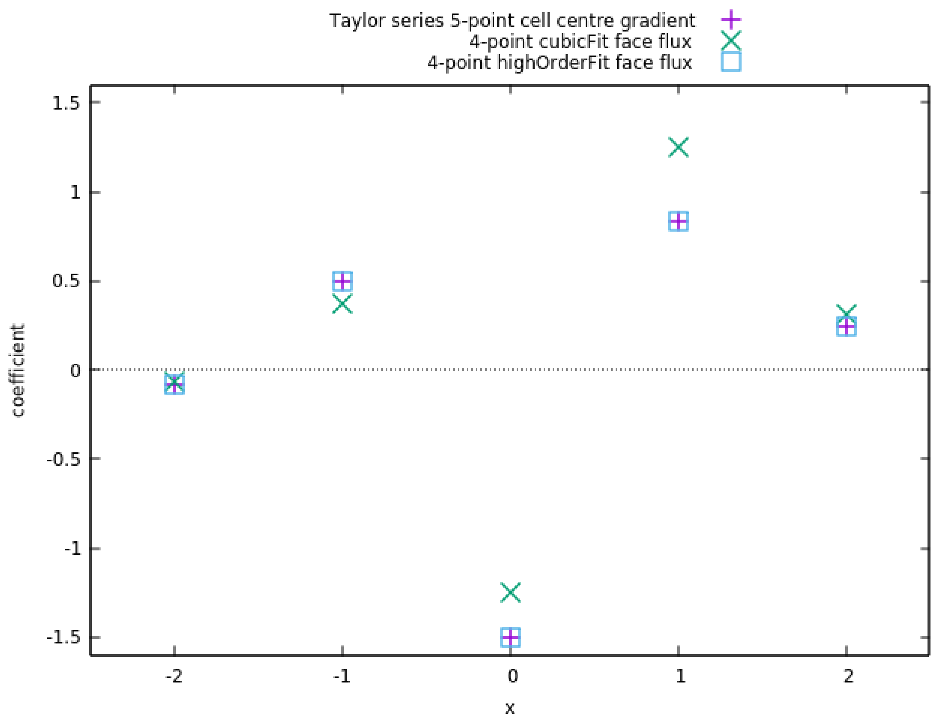

Figure 3: Flux divergence cofficients using a 5-point Taylor series approximation (equation 10), cubicFit approximation (equation 8) and highOrderFit approximation (equation 11).

References

Devendran, D., D. Graves, H. Johansen, and T. Ligocki, 2017: A fourth-order Cartesian grid embedded boundary method for Poisson's equation. *Comm. App. Math. Comp. Sci.*, **12 (1)**, 51–79, doi:10.2140/camcos.2017.12.51.

Shaw, J., H. Weller, J. Methven, and T. Davies, 2017: Multidimensional method-of-lines transport for atmospheric flows over steep terrain using arbitrary meshes. *J. Comp. Phys.*, **344**, 86–107, doi:10.1016/j.jcp.2017.04.061.