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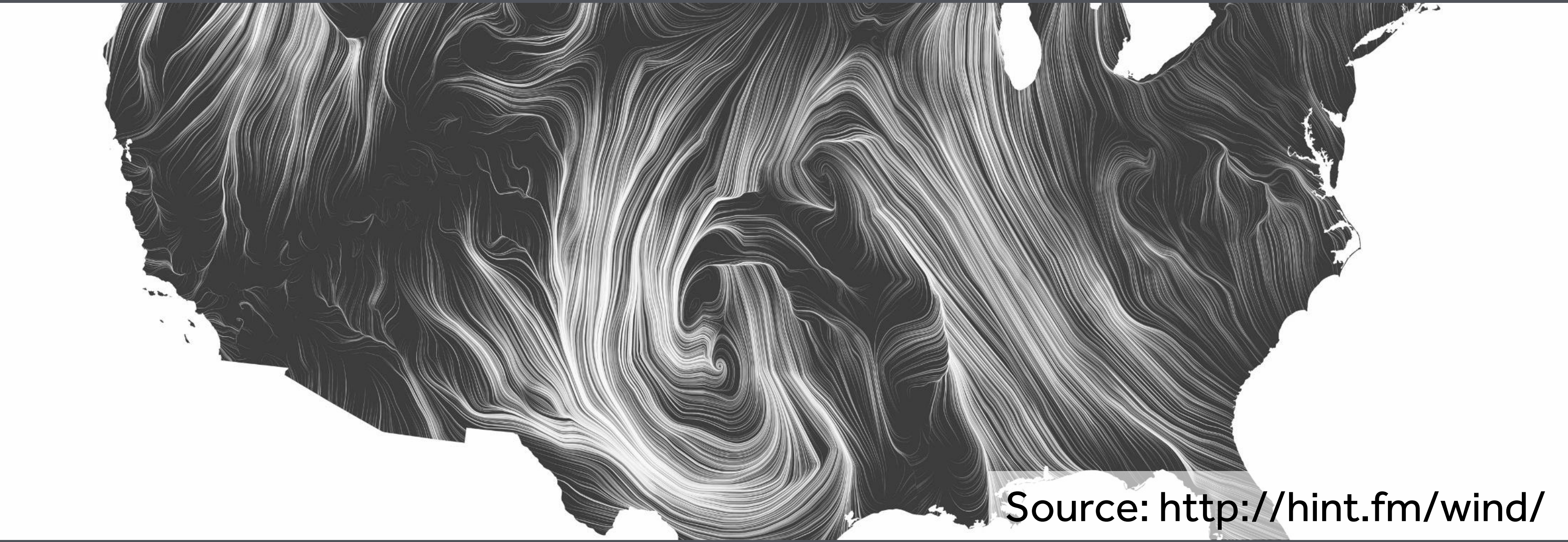
DISCRETE VECTOR CALCULUS ON ARAKAWA C GRIDS



James Shaw

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LIMITLESS **POTENTIAL** | LIMITLESS **OPPORTUNITIES** | LIMITLESS **IMPACT**

DISCLAIMER

OVERVIEW

1. Divergence and curl operators
2. Primal and dual meshes
3. Computing div & curl in bulk
4. Next steps

Nicolaides, R., 1992: Direct discretization of planar div-curl problems. *SIAM J. Numerical Analysis*, **29**, 32–56

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DIVERGENCE

$$\nabla \cdot \mathbf{F}(p) = \lim_{V \to \{p\}} \frac{1}{|V|} \int_{\Gamma} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}\Gamma$$

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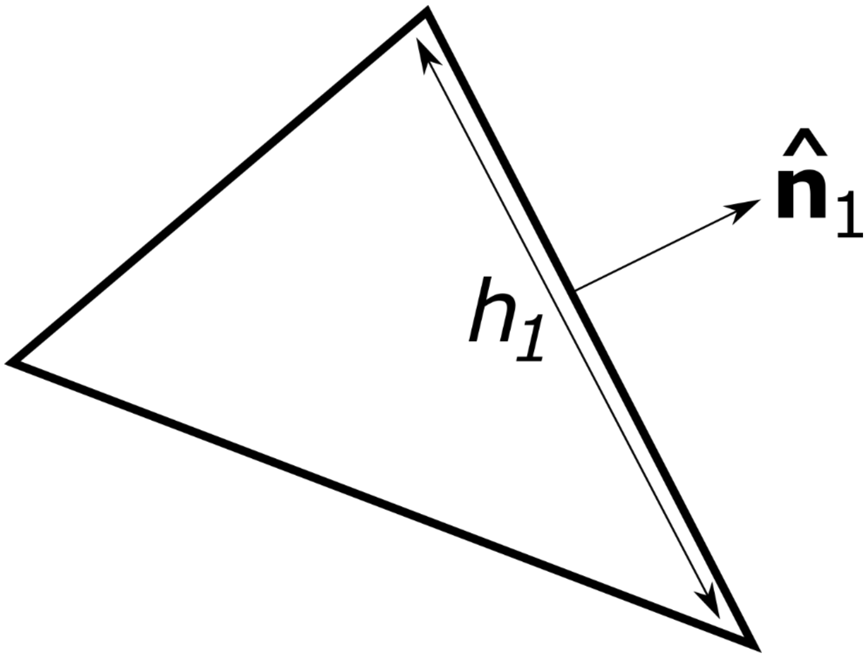

DISCRETE DIVERGENCE

CURL

$$(\nabla \times \mathbf{F}(p)) \cdot \hat{\mathbf{n}} = \lim_{A \to \{p\}} \frac{1}{|A|} \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r}$$

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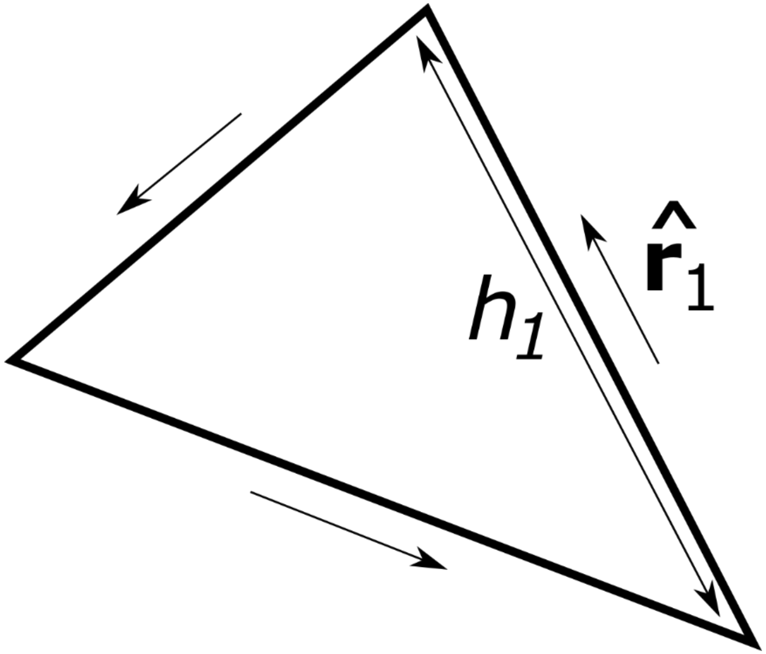
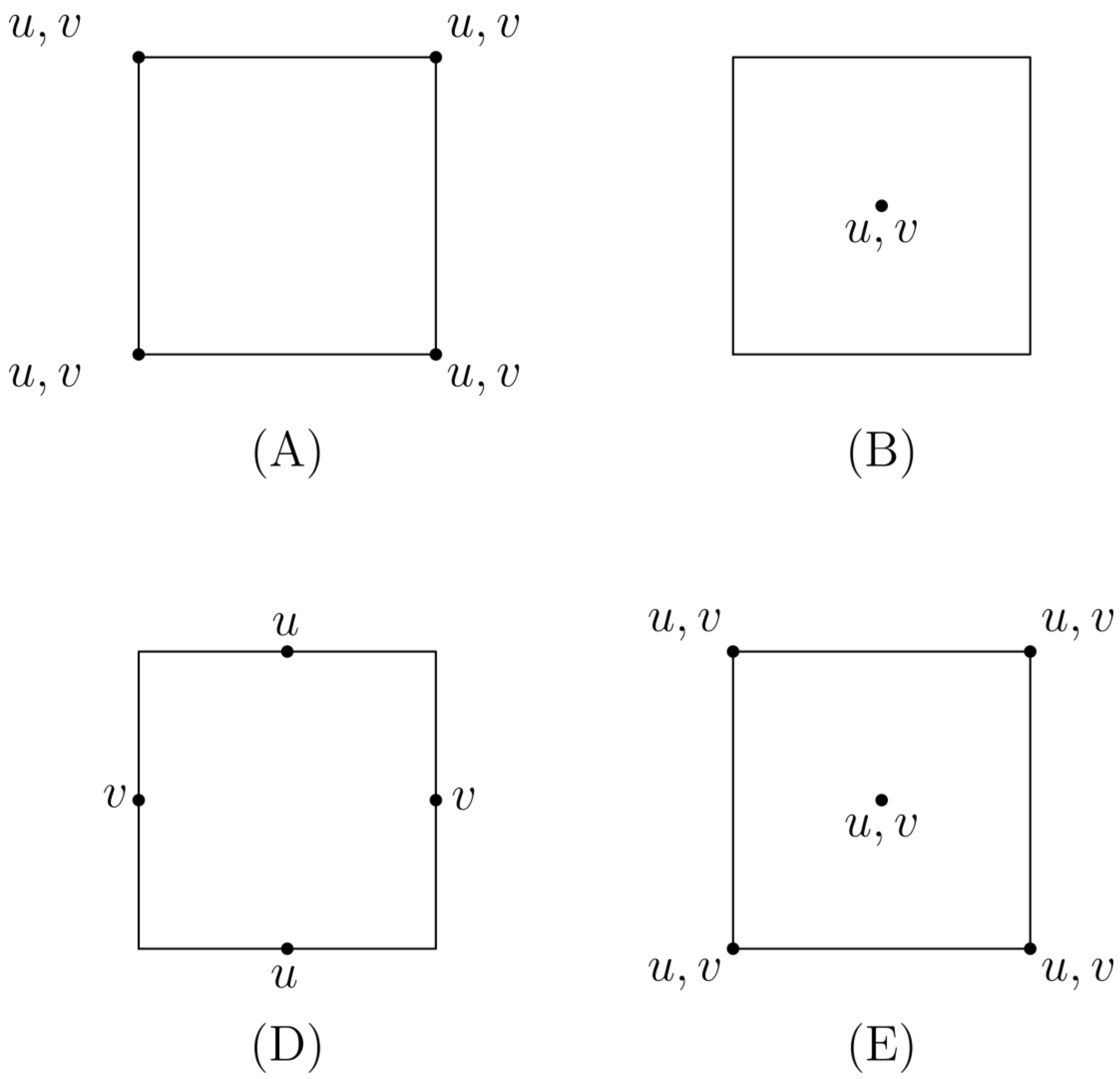
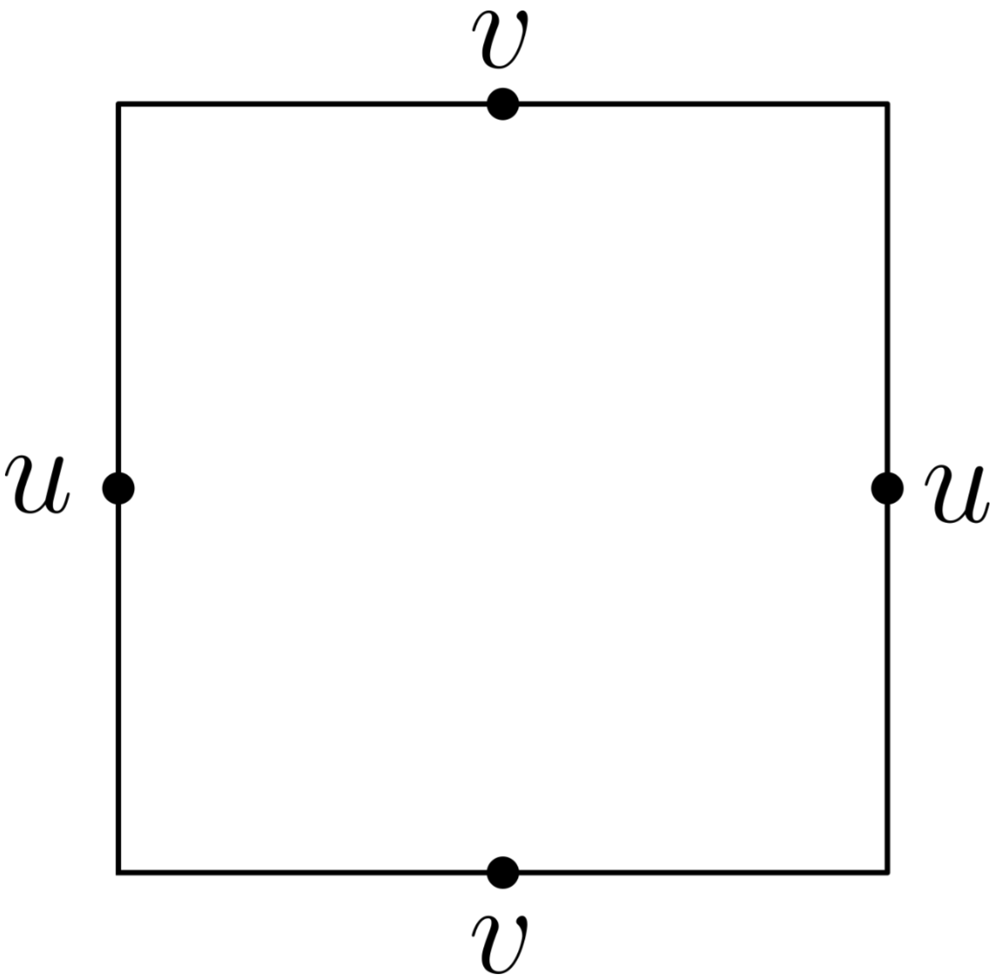


DISCRETE CURL



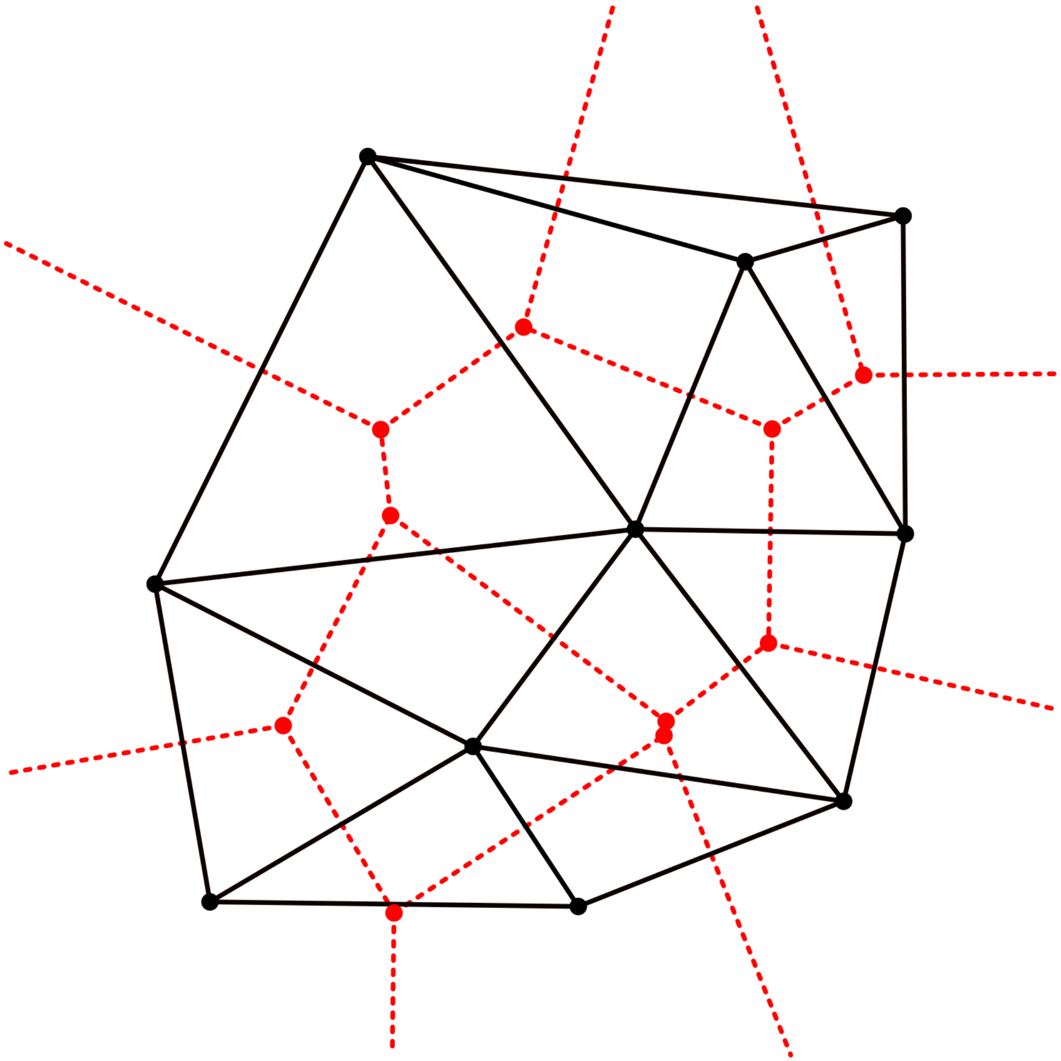
ARAKAWA GRIDS

Source: JuliusSimplus, CC BY-SA 3.0



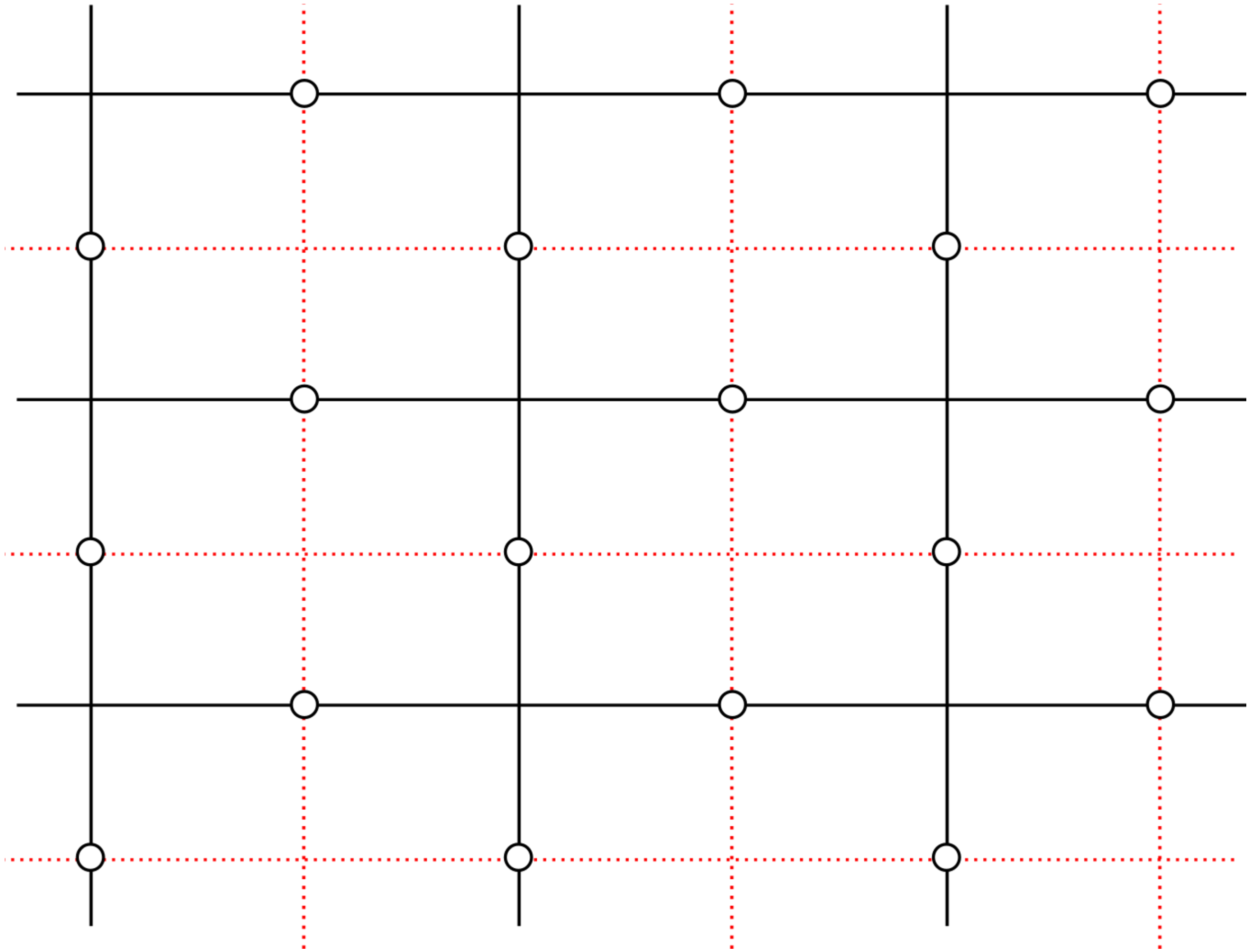
ARAKAWA C GRID

Source: JuliusSimplus, CC BY-SA 3.0

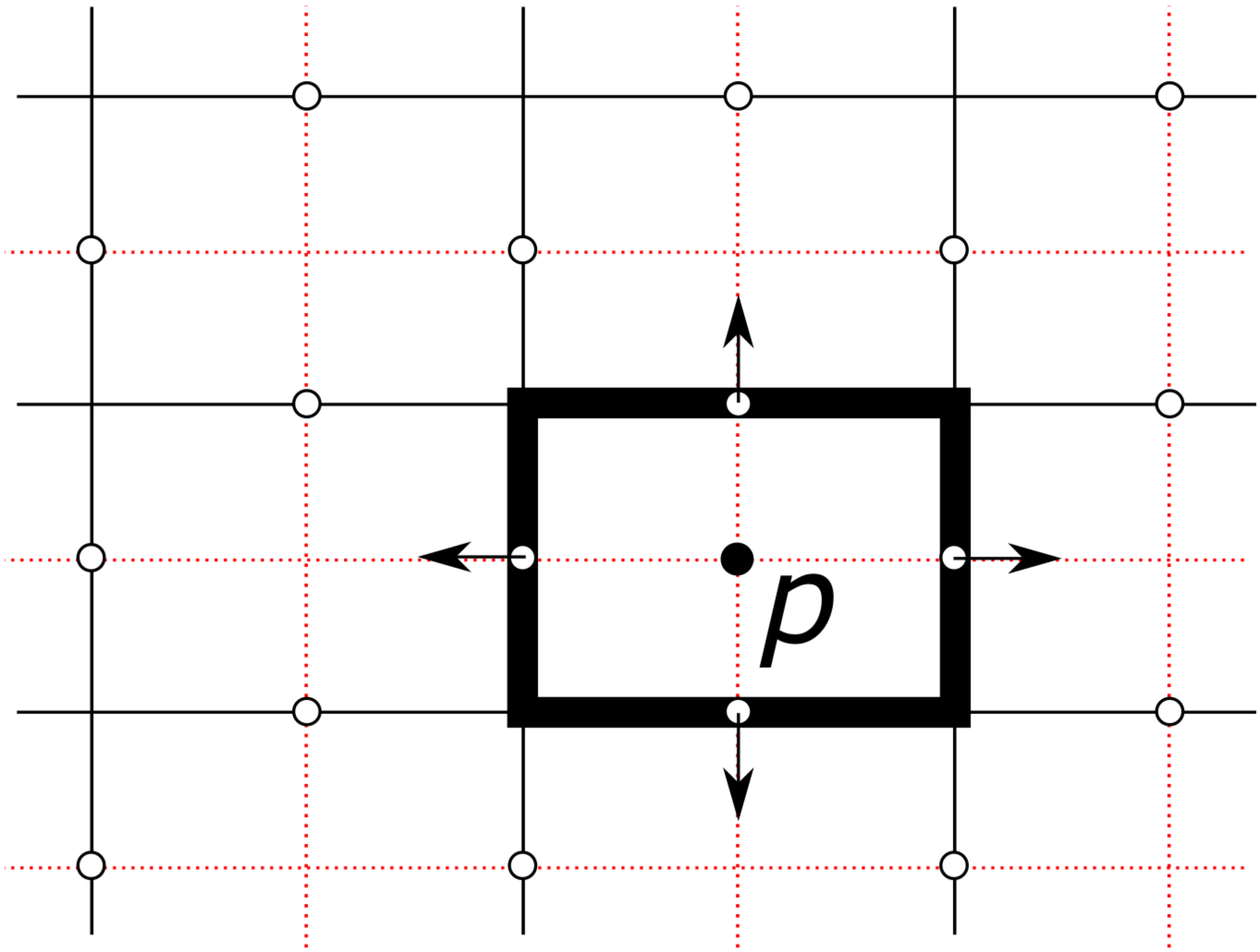


Source: Hferee, CC BY-SA 3.0

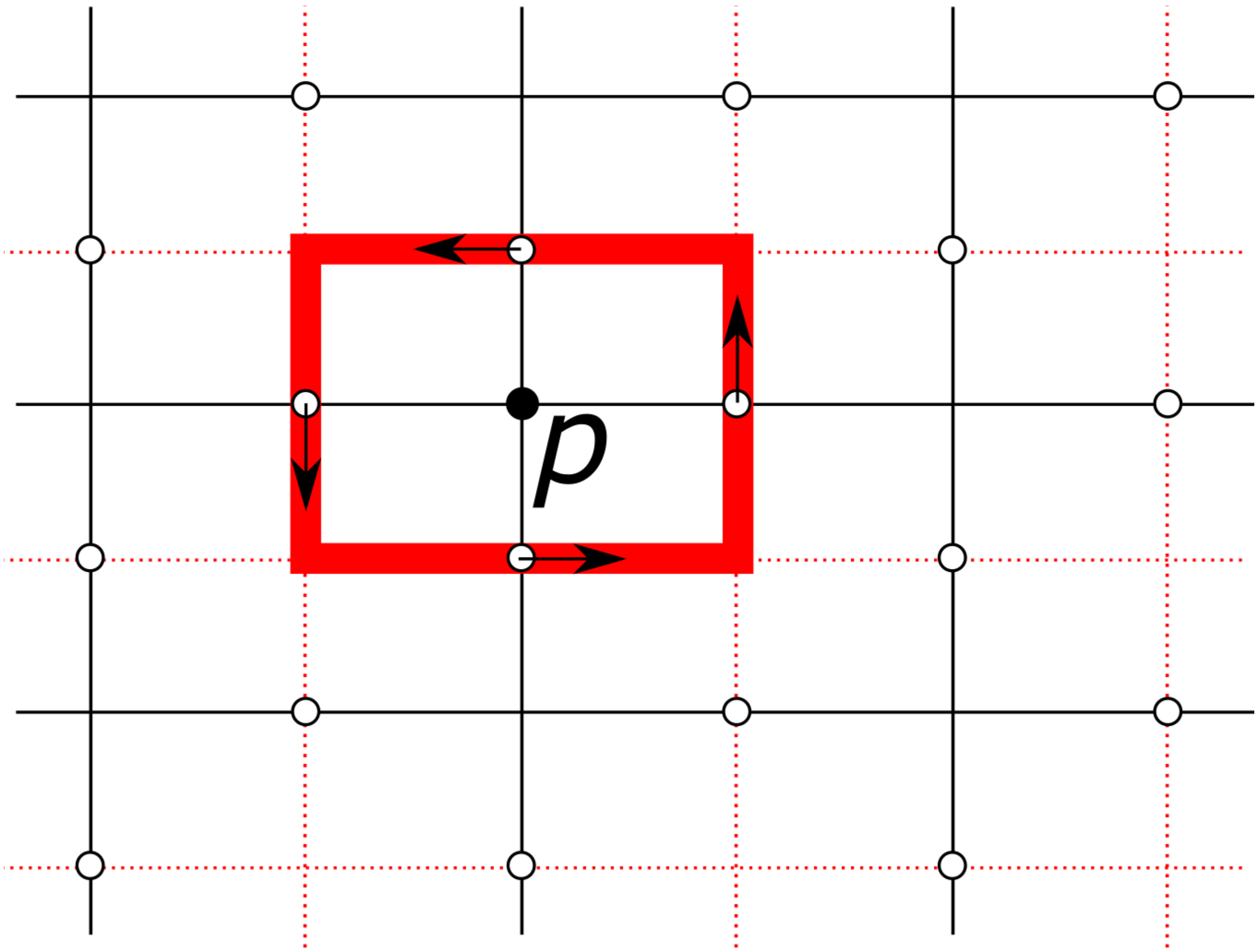
PRIMAL AND DUAL MESHES



PRIMAL AND DUAL MESHES



DISCRETE DIVERGENCE ON THE C GRID



DISCRETE CURL ON THE C GRID

COMPUTING DIV & CURL IN BULK

COMPUTING DIV & CURL IN BULK

1. Number primal vertices
2. Determine primal edge orientation
3. Determine dual edge orientation
4. Compute dual edge – primal cell incidence matrix (for divergence), or primal edge – dual cell incidence matrix (for curl)
5. Multiply by edge length vector and velocity

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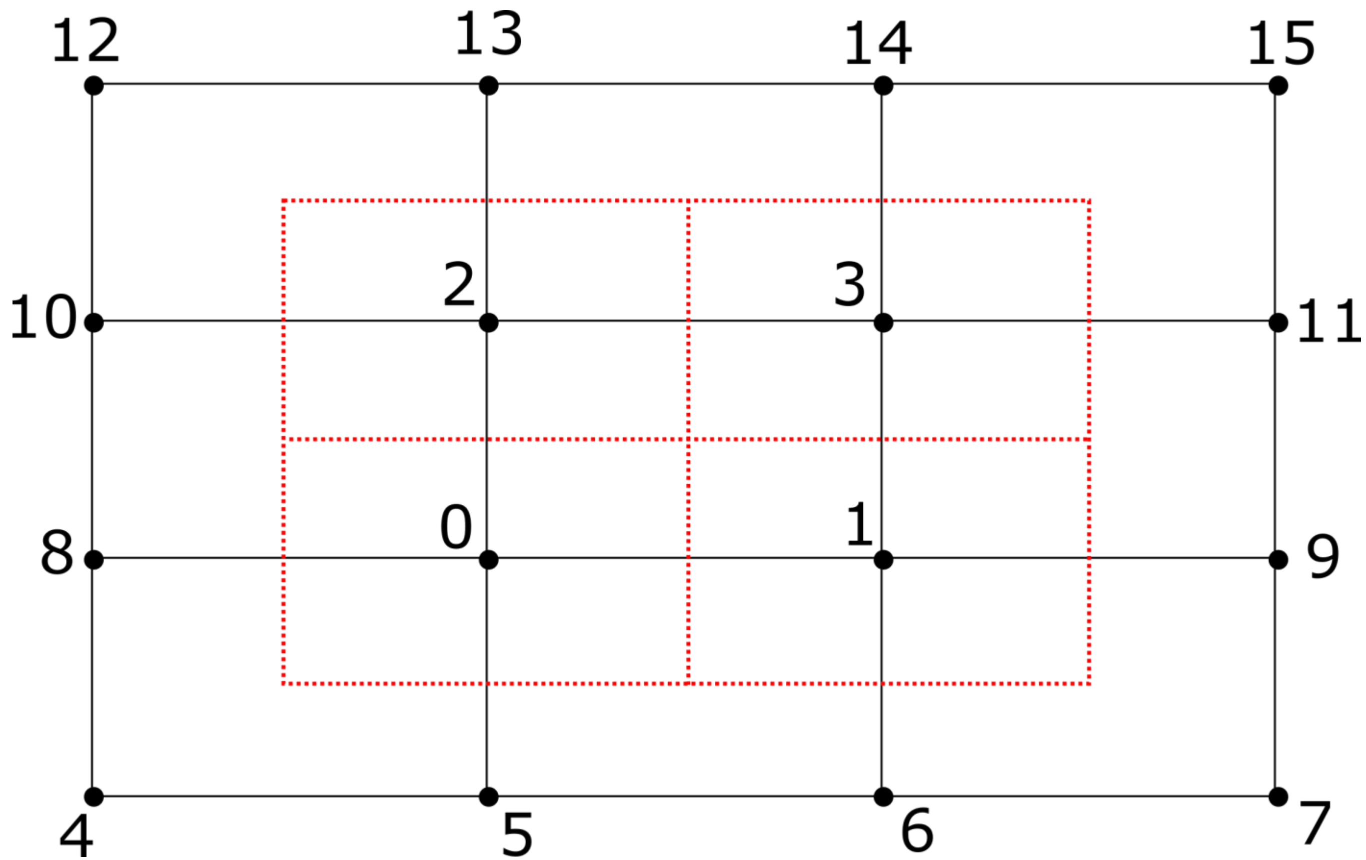
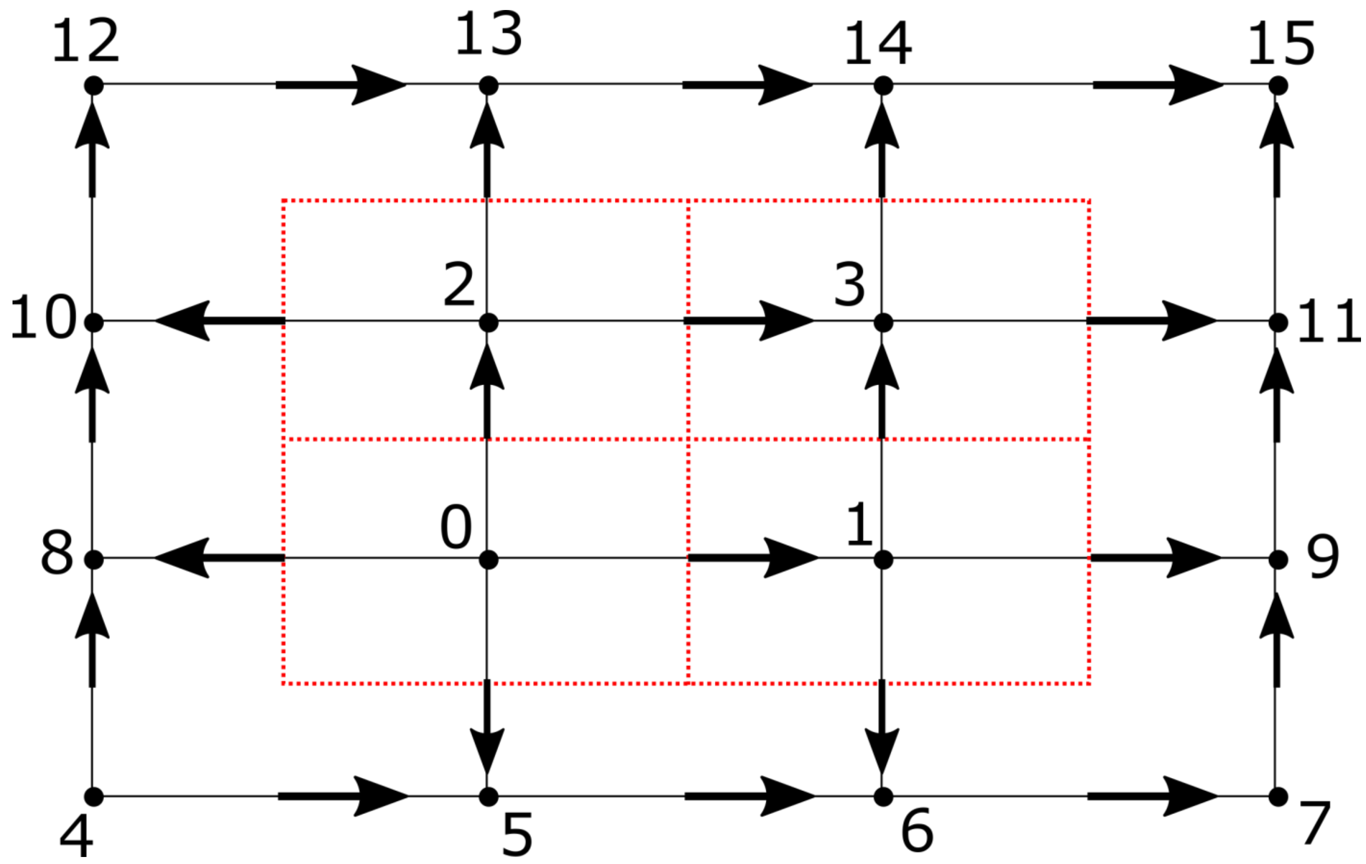
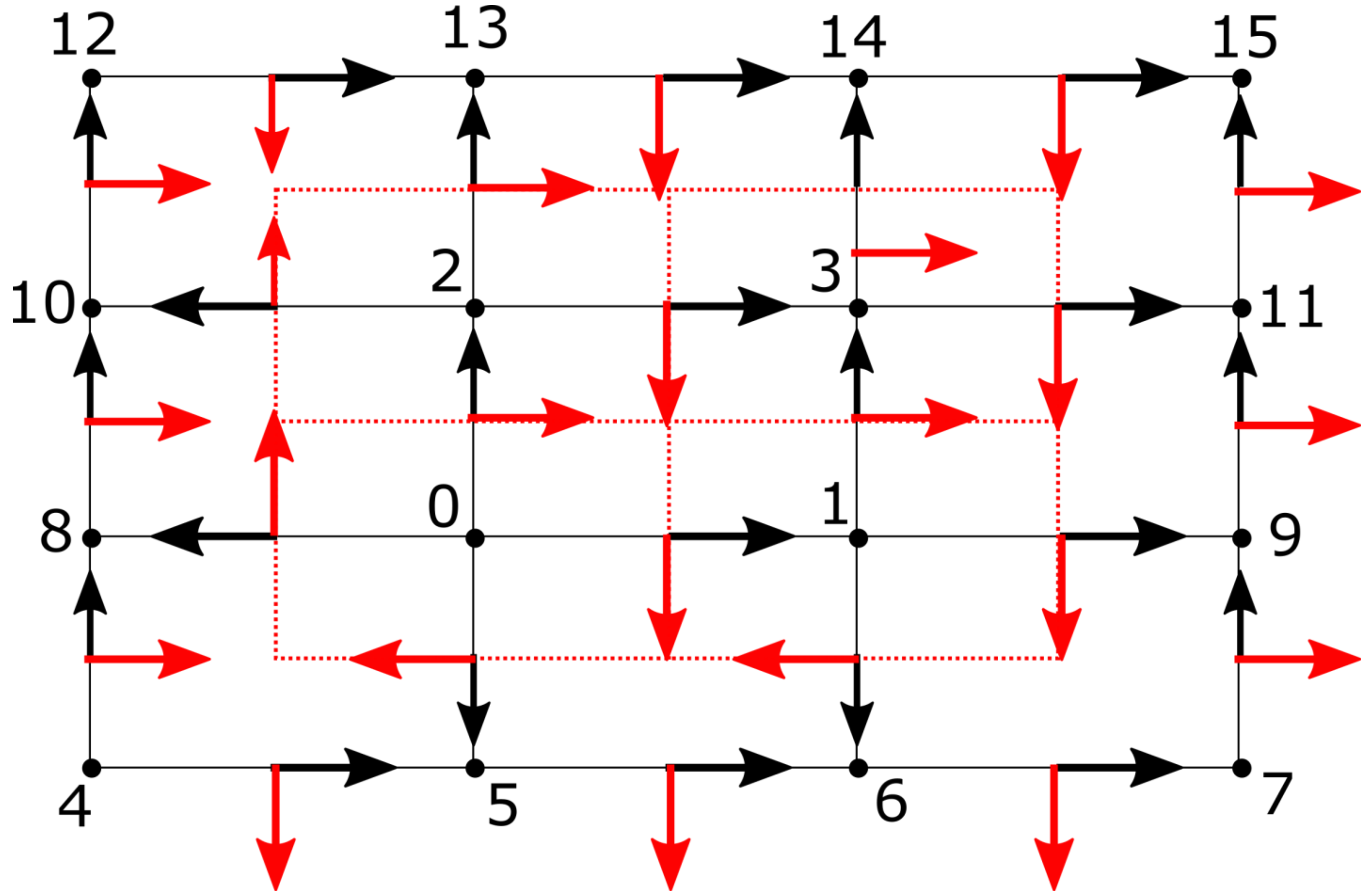


NUMBER PRIMAL VERTICES



PRIMAL EDGE ORIENTATION

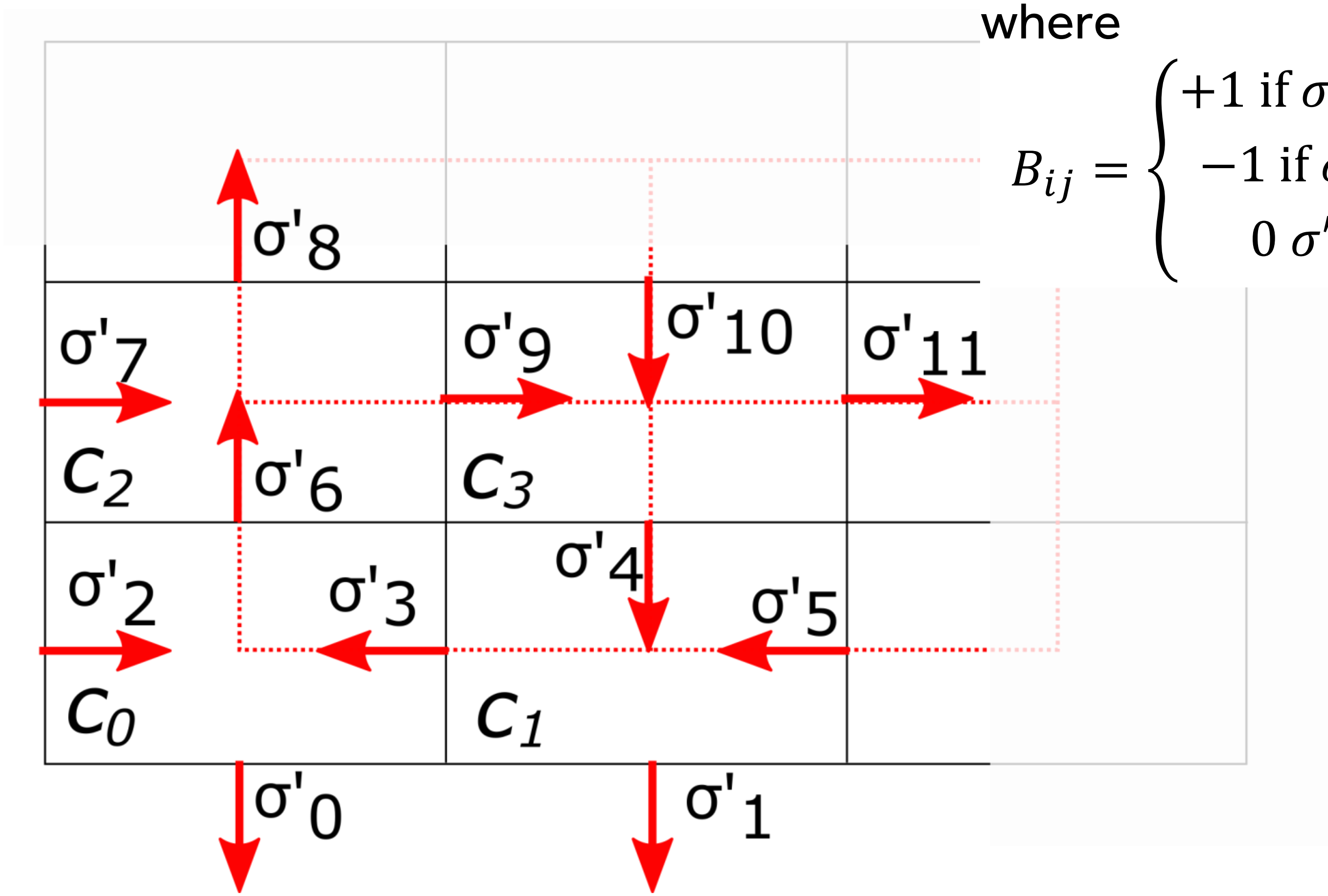


DUAL EDGE ORIENTATION

$$B = \begin{pmatrix} \sigma'_0, c_0 & \cdots & \sigma'_0, c_3 \\ \vdots & \ddots & \vdots \\ \sigma'_{11}, c_0 & \cdots & \sigma'_{11}, c_3 \end{pmatrix}$$

where

$$B_{ij} = \begin{cases} +1 \text{ if } \sigma'_i \text{is directed out of } c_j \\ -1 \text{ if } \sigma'_i \text{ is directed into } c_j \\ 0 \ \sigma'_i \text{ does not meet } c_j \end{cases}$$



EDGE –CELL INCIDENCE MATRIX

NEXT STEPS

1. Helmholtz decomposition
2. De Rham complex and Hodge operators
3. Transform between covariant and contravariant velocity components

HELMHOLTZ DECOMPOSITION

- Can find streamfunction, Ψ, and velocity potential, χ, by inverting a product of incidence matrices

$$\nabla \cdot \mathbf{u} = \nabla^2 \chi, \nabla \times \chi = 0$$

$$\nabla \times \mathbf{u} = \nabla^2 \Psi, \nabla \cdot \Psi = 0$$

LIMITLESS **POTENTIAL** | LIMITLESS **OPPORTUNITIES** | LIMITLESS **IMPACT**

DE RHAM COMPLEX & HODGE OPERATORS

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathcal{V} \arrow[r, "\mathsf{GRAD}"] \arrow[d, dotted, "\mathsf{H}_\alpha^{\mathcal{V}\tilde{\mathcal{C}}}"'] & \mathcal{E} \arrow[r, "\mathsf{CURL}"] \arrow[d, dotted, "\mathsf{H}_\alpha^{\mathcal{E}\tilde{\mathcal{F}}}"'] & \mathcal{F} \arrow[r, "\mathsf{DIV}"] \arrow[d, dotted, "\mathsf{H}_\alpha^{\mathcal{F}\tilde{\mathcal{E}}}"'] & \mathcal{C} \arrow[d, dotted, "\mathsf{H}_\alpha^{\mathcal{C}\tilde{\mathcal{V}}}"'] \\
\tilde{\mathcal{C}} & \tilde{\mathcal{F}} \arrow[l, "\widetilde{\mathsf{DIV}}"] & \tilde{\mathcal{E}} \arrow[l, "\widetilde{\mathsf{CURL}}"] & \tilde{\mathcal{V}} \arrow[l, "\widetilde{\mathsf{GRAD}}"]
\end{tikzcd}
$$

Source: J. Bonelle & A. Ern, Compatible Discrete Operators schemes for Stokes equations, 2015

SUMMARY

1. Divergence and curl operators
2. Arakawa grids
3. Primal and dual meshes
4. Calculating div & curl using incidence matrices
5. Helmholtz decomposition, de Rham complex and co/contra-variant velocities