Charney–Phillips cell centre reconstruction

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`volVectorField x = fvc::reconstruct(surfaceScalarField xf)` is implemented as

$$x = \left(\sum_f \frac{1}{|\mathbf{S}_f|} \mathbf{S}_f \otimes \mathbf{S}_f \right)^{-1} \cdot \left(\frac{1}{|\mathbf{S}_f|} \mathbf{S}_f x_f \right) \tag{1}$$

where x is defined at cell centres and x_f is defined at face centres. The surface area vector $\mathbf{S}_f$ has magnitude equal to the area of face f and is directed outward normal to f. Cell centre potential temperature θ_C is reconstructed from surrounding values of θ_f

$$\theta_C = \hat{\mathbf{g}} \cdot \left\{ \left(\sum_f \frac{1}{|\mathbf{S}_f|} \mathbf{S}_f \otimes \mathbf{S}_f \right)^{-1} \cdot \left(\sum_f \frac{1}{|\mathbf{S}_f|} \mathbf{S}_f \left(\theta_f \hat{\mathbf{g}} \cdot \hat{\mathbf{n}} |\mathbf{S}_f| \right) \right) \right\} \tag{2}$$

where $\hat{\mathbf{g}}$ is the unit vector of gravitational acceleration and the surface normal unit vector $\hat{\mathbf{n}} = \mathbf{S}_f / |\mathbf{S}_f|$. Simplifying we get

$$\theta_C = \hat{\mathbf{g}} \cdot \left\{ \left(\sum_f \frac{1}{|\mathbf{S}_f|} \mathbf{S}_f \otimes \mathbf{S}_f \right)^{-1} \cdot \left(\sum_f \frac{1}{|\mathbf{S}_f|} \mathbf{S}_f \left(\theta_f \hat{\mathbf{g}} \cdot \mathbf{S}_f \right) \right) \right\} \tag{3}$$

Notice that $\theta_f \hat{\mathbf{g}} \cdot \mathbf{S}_f / |\mathbf{S}_f| = 0$ on vertical faces and $|\theta_f \hat{\mathbf{g}} \cdot \mathbf{S}_f| / |\mathbf{S}_f| = \theta_f$ on horizontal faces.

Now consider a two-dimensional cell in the x–z plane having dimensions 1×1 such that $|\mathbf{S}_f| = 1 \, \forall f$. The top, right, bottom and left surface area vectors are $\mathbf{S}_T = \hat{\boldsymbol{k}}$, $\mathbf{S}_R = \hat{\boldsymbol{\imath}}$, $\mathbf{S}_B = -\hat{\boldsymbol{k}}$ and $\mathbf{S}_L = -\hat{\boldsymbol{\imath}}$ respectively.

$$\begin{aligned}
\theta_C &= \hat{\mathbf{g}} \cdot \left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \times 1 \end{bmatrix} + \begin{bmatrix} 1 \times 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \times -1 \end{bmatrix} + \begin{bmatrix} -1 \times -1 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \cdot \left(\theta_B + \theta_T \right) \hat{\boldsymbol{k}} \right\} && (4) \\
&= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot \left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \theta_B + \theta_T \end{bmatrix} \right\} && (5) \\
&= \begin{bmatrix} 0 \\ \frac{\theta_B + \theta_T}{2} \end{bmatrix} && (6)
\end{aligned}$$