A high-order correction to the one-dimensional cubicFit transport scheme

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A transport scheme is 'super-convergent' when its order of convergence is higher on uniform meshes than on non-uniform meshes. For example, the transport scheme by Skamarock and Gassmann (2011) is super-convergent because it is first-order on non-uniform meshes and third-order on uniform meshes. Without a high-order correction, the one-dimensional cubicFit transport scheme is not super-convergent because it is second-order convergent on both uniform and non-uniform meshes.

Here I describe a correction to cubicFit that results in fourth-order convergence on uniform meshes. The correction technique that I use is inspired by the Taylor series expansion used by Skamarock and Gassmann (2011). The corrected cubic scheme retains second-order convergence on non-uniform meshes with improved absolute accuracy compared to the uncorrected scheme.

The one-dimensional linear transport of a dependent variable ϕ is given by

$$\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x} \tag{1}$$

where u is a constant, positive velocity. The term on the right-hand side of equation (1) is called the flux divergence. The finite volume method offers one way to discretise the flux divergence by considering flux across faces of a cell,

$$-u \frac{\partial \phi}{\Delta x} \approx -u \frac{\phi_R - \phi_L}{\Delta x} \tag{2}$$

where ϕ_L and ϕ_R are approximate values of ϕ at the left and right faces respectively, and Δx is the distance between the faces. The cubicFit scheme is used to approximate face values ϕ_L and ϕ_R from surrounding cell centre values. In one dimension, the cubicFit scheme exactly interpolates the value of a dependent variable ϕ at face f using the neighbouring downwind and three upwind cell centre values. This arrangement is shown in figure 1.

The one-dimensional cubic interpolation is

$$\phi = a_1 + a_2 x + a_3 x^2 + a_4 x^3. \tag{3}$$

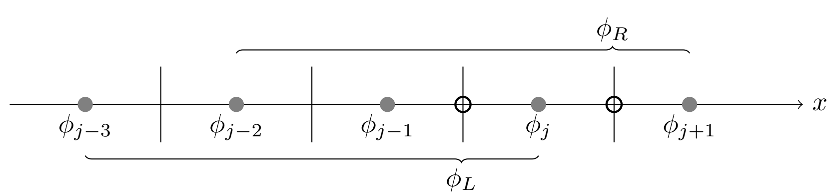

Figure 1: The one-dimensional cubicFit transport scheme interpolates face values ϕ_L and ϕ_R using four-point, upwind-biased stencils of cell-centre values.

Assuming a uniform mesh with $\Delta x = 1$ and choosing the position of $\phi_{i+1/2}$ to be $x = 0$, equation (3) is evaluated at the cell centres $\phi_{i-2}, \ldots, \phi_{i+1}$ to form the matrix equation

$$\mathbf{B}\mathbf{a} = \boldsymbol{\phi} \tag{4}$$

$$\begin{bmatrix} 1 & -5/2 & 25/4 & -125/8 \\ 1 & -3/2 & 9/4 & -27/8 \\ 1 & -1/2 & 1/4 & -1/8 \\ 1 & 1/2 & 1/4 & 1/8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix}. \tag{5}$$

The unknown coefficients $\mathbf{a}$ are found by inverting $\mathbf{B}$ such that $\mathbf{a} = \mathbf{B}^{-1}\boldsymbol{\phi}$. The inverse matrix is

$$\mathbf{B}^{-1} = \frac{1}{48} \begin{bmatrix} 3 & -15 & 45 & 15 \\ 2 & -6 & -42 & 46 \\ -12 & 60 & -84 & 36 \\ -8 & 24 & -24 & 8 \end{bmatrix} \tag{6}$$

Since $x = 0$ was chosen to be the position $i + 1/2$ then

$$\phi_{i+1/2} = a_1 = \frac{1}{16} \begin{bmatrix} 1 \\ -5 \\ 15 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi i + 1 \end{bmatrix}. \tag{7}$$

The finite difference method offers another way to discretise the flux divergence of cell i with a cubic approximation using cell centre values $\phi_{i-2}, \ldots, \phi_{i+1}$. A matrix equation is constructed using equation (3) evaluated at every cell centre. For convenience, assume that $\Delta x = 1$ and that $x = 0$ at the cell centre position of ϕ_i, hence the matrix equation becomes

$$\mathbf{B}\mathbf{a} = \boldsymbol{\phi} \tag{8}$$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \tag{9}$$

The unknown coefficients $\mathbf{a}$ are found by calculating the inverse matrix,

$$\mathbf{B}^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 0 & 6 & 0 \\ 1 & -6 & 3 & 2 \\ 0 & 3 & -6 & 3 \\ -1 & 3 & -3 & 1 \end{bmatrix} \tag{10}$$

To calculate the flux divergence I calculate the derivative $\partial\phi/\partial x = a_2 + 2a_3 x + 3a_4 x^2$. Evaluating the flux divergence at ϕ_i where $x = 0$ then $\partial\phi_i/\partial x = a_2$. Hence I find that the finite difference weighted sum is

$$-u\frac{\partial\phi_i}{\partial x} = -ua_2 = -u \cdot \frac{1}{6} \begin{bmatrix} 1 \\ -6 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{i-2} \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \end{bmatrix} \tag{11}$$

The cubic finite difference approximation given in equation (11) is conservative on uniform meshes. This can be demonstrated by decomposing the weights vector,

$$-u \cdot \frac{1}{6}\begin{bmatrix}1\\-6\\3\\2\end{bmatrix} \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} = -u \cdot \frac{1}{6}\left(\begin{bmatrix}0\\-1\\5\\2\end{bmatrix} - \begin{bmatrix}-1\\5\\2\\0\end{bmatrix}\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{12}$$

$$= -u\left(\frac{1}{6}\begin{bmatrix}-1\\5\\2\end{bmatrix} \cdot \begin{bmatrix}\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} - \frac{1}{6}\begin{bmatrix}-1\\5\\2\end{bmatrix} \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\end{bmatrix}\right). \tag{13}$$

Notice that the flux divergence has been rewritten as the difference between right and left fluxes (equation 2).

Calculating the high-order correction

The high-order correction to the cubicFit scheme is calculated as the difference between the cubic finite difference approximation (equation 13) and the uncorrected cubicFit approximation (equation 7),

$$\text{correction}(\phi_{i+1/2}) = \left(\frac{1}{6}\begin{bmatrix}0\\-1\\5\\2\end{bmatrix} - \frac{1}{16}\begin{bmatrix}1\\-5\\15\\5\end{bmatrix}\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{14}$$

$$= \frac{1}{48}\begin{bmatrix}-3\\7\\-5\\1\end{bmatrix} \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{15}$$

which can be decomposed into a linear combination of second derivatives where $\partial_x^2\phi_i = \phi_{i-1} - 2\phi_i + \phi_{i+1}$,

$$= \frac{1}{48}\left(-3\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\-2\\1\end{bmatrix}\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{16}$$

$$= \frac{1}{48}\left(-3\partial_x^2\phi_{i-1} + \partial_x^2\phi_i\right). \tag{17}$$

Applying this correction using the three-point approximation of the second derivative results in third-order convergence on uniform meshes and second-order convergence on non-uniform meshes.

Alternatively, the second derivative can be calculated from equation (3) such that $\partial_x^2\phi = 2a_3 + 6a_4x$ where a_3 and a_4 can be calculated using equation (6). This approach results in fourth-order convergence on uniform meshes and second-order convergence on non-uniform meshes. Curiously, this approach yields the correction

$$\text{correction}(\phi_{i+1/2}) = \frac{1}{48}\left(-3 \cdot \begin{bmatrix}0\\1\\-2\\1\end{bmatrix} + \begin{bmatrix}-1\\4\\-5\\2\end{bmatrix}\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{18}$$

$$= \frac{1}{48} \cdot \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} = \frac{1}{48}\left(\begin{bmatrix}-1\\2\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\2\\-1\end{bmatrix}\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix} \tag{19}$$

$$= -\frac{1}{48}\left(\partial_x^2\phi_{i-1} + \partial_x^2\phi_i\right) \cdot \begin{bmatrix}\phi_{i-2}\\\phi_{i-1}\\\phi_i\\\phi_{i+1}\end{bmatrix}. \tag{20}$$

References

Skamarock, W. C., and A. Gassmann, 2011: Conservative transport schemes for spherical geodesic grids: High-order flux operators for ODE-based time integration. *Mon. Wea. Rev.*, **139**, 2962–2975, doi:10.1175/MWR-D-10-05056.1.