Royal Commision Yanbu Colleges & Institutes

MATH 002 Engineering & Sciences Mathematics 2

Lecture Notes

Module III

In this Module we will learn the following topics

1. Using Fundamental Identities
2. Verifying Trigonometric Identities
3. Solving Trigonometric Equations
4. Sum and Difference Formulas
5. Multiple-Angle and Product-to-Sum Formulas

Text Book: Algebra & Trigonometry 11th Edition. (2020). Ron Larson
CENGAGE Learning. ISBN-13: 978-0-357-45208-0

MODULE-III ANALYTIC TRIGONOMETRY (To be covered in 12 Teaching Hours)						
Article #	**CLOs**	**Page #**	**Class work Problems**	**Week #**	**Practice Problems**	**Summative Assessment**
7.1	**2.2**	**508**	**Example-02, Example-05, CW03, 31, 37, 39, 41**	**8**	**32, 40, 42, 44, 45**	**Quiz # 3 Tentatively in Week # 10/11** **Assignment # 3 will be in week # 10/11 from Module-III**
7.2	**2.2**	**515**	**Example-01, Example-02, Q15, 19, 29**	**8**	**17, 18, 20, 32, 35, 38**	
7.3	**2.2**	**522**	**Q07, 09, 13, 14, 21**	**9**	**10, 15, 16, 26**	
7.4	**2.2**	**535**	**Example-01, Example-02, Example-05, Q37, 39, 63**	**9**	**26, 38, 40, 42, 60**	
7.5	**2.2**	**541**	**Q13, 14, 15, 47, 49, 52, 55**	**10**	**16, 17, 18, 50, 51, 56, 57**	

Objectives:

- **Apply trigonometric identities to simplify the trigonometric expressions.**

7.1 Fundamental Trigonometric Identities

Objectives:

- **Recognize the fundamental trigonometric identities.**
- **Use the fundamental identifies and algebra rules to simplify trigonometric expressions.**

(a) Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

(b) Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

(c) Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Note:

Pythagorean identities are sometime used in radical form such as

$\sin x = \pm\sqrt{1 - \cos^2 x}$ or $\cos x = \pm\sqrt{1 - \sin^2 x}$

$\tan x = \pm\sqrt{\sec^2 x - 1}$ or $\sec x = \pm\sqrt{1 + \tan^2 x}$

$\cot x = \pm\sqrt{\csc^2 x - 1}$ or $\csc x = \pm\sqrt{1 + \cot^2 x}$

where the sign depends upon the choice of x.

(d) Cofunction Identities

$$\boldsymbol{sin\left(\frac{\pi}{2}-x\right)=cosx} \qquad \boldsymbol{cos\left(\frac{\pi}{2}-x\right)=sinx}$$

$$\boldsymbol{tan\left(\frac{\pi}{2}-x\right)=cotx} \qquad \boldsymbol{cot\left(\frac{\pi}{2}-x\right)=tanx}$$

$$\boldsymbol{sec\left(\frac{\pi}{2}-x\right)=cscx} \qquad \boldsymbol{csc\left(\frac{\pi}{2}-x\right)=secx}$$

(e) Even/Odd Identities

$$\boldsymbol{sin(-x)=-sinx} \qquad \boldsymbol{csc(-x)=-cscx}$$

$$\boldsymbol{cos(-x)=cosx} \qquad \boldsymbol{sec(-x)=secx}$$

$$\boldsymbol{tan(-x)=-tanx} \qquad \boldsymbol{cot(-x)=-cotx}$$

CW01:Example #2 Page-509

Factor the expression and use the fundamental identities to simplify the expression.

$\boldsymbol{sin\ x\ cos^2x-sin\ x}$

Solution:

CW02: Example # 05 Page- 510

Use the fundamental identities to simplify the expression $\boldsymbol{sin\vartheta+cot\vartheta cos\vartheta}$.

Solution:

CW03:

Use the fundamental identities to simplify the expression $\boldsymbol{sin(-\vartheta) + tan\vartheta cos\vartheta}$.

Solution:

CW04: Question # 31 Exercise: 7.1

Use the fundamental identities to simplify the expression $\boldsymbol{tan\vartheta\ csc\vartheta}$.

Solution:

CW05: Question # 37 Exercise: 7.1

Use the fundamental identities to simplify the expression $\boldsymbol{\frac{1-sin^2x}{csc^2x-1}}$.

Solution:

CW06: Question # 39 Exercise: 7.1

Perform the addition or subtraction and use the fundamental identities to simplify the expression $\frac{1}{1+cosx}+\frac{1}{1-cosx}$.

Solution:

CW07: Question # 41 Exercise: 7.1

Perform the addition or subtraction and use the fundamental identities to simplify the expression $\frac{cosx}{1+sinx}-\frac{cosx}{1-sinx}$.

Solution:

PRACTICE QUESTIONS FOR THE STUDENTS TO SOLVE

Question # 32 Exercise: 7.1

Use the fundamental identities to simplify the expression $tan(-x)\ cosx$.

Solution:

Question # 45 Exercise: 7.1

Use the fundamental identities to simplify the expression $\frac{sin^2y}{1-cosy}$.

Solution:

Question # 40 Exercise: 7.1

Perform the addition or subtraction and use the fundamental identities to simplify the expression $\frac{1}{secx+1}-\frac{1}{secx-1}$.

Solution:

Question # 42 Exercise: 7.1

Perform the addition or subtraction and use the fundamental identities to simplify the expression $\frac{sinx}{1+cosx}+\frac{sinx}{1-cosx}$.

Solution:

Question # 44 Exercise: 7.1

Perform the addition or subtraction and use the fundamental identities to simplify the expression $\frac{cosx}{1+sinx}+\frac{1+sinx}{cosx}$.

Solution:

7.2 Verifying Trigonometric Identities

Objectives:

- **Use the fundamental identities and algebra rules to verify trigonometric identifies.**

Guidelines for verifying trigonometric identities:

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note, which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. When the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Always try something. Even making an attempt that leads to a dead end can provide insight.

CW01: Example 01 Page- 516

Verify the trigonometric identity $\frac{sec^2\vartheta-1}{sec^2\vartheta} = sin^2\vartheta$

Solution:

CW02: Example 02 Page- 516

Verify the trigonometric identity $2sec^2\alpha = \frac{1}{1-sin\alpha} + \frac{1}{1+sin\alpha}$**.**

Solution:

CW:03 Question 15 Exercise: 7.2

Verify the trigonometric identity $\frac{1}{tanx} + \frac{1}{cotx} = tanx + cotx$**.**

Solution:

CW:04 Question 19 Exercise: 7.2

Verify the trigonometric identity $\frac{1}{cosx+1}+\frac{1}{cosx-1}=-2\ cscx\ cotx.$

Solution:

CW:05 Question 29 Exercise: 7.2

Verify the trigonometric identity $secx-cosx=sinx\ tanx.$

Solution:

PRACTICE QUESTIONS FOR THE STUDENTS TO SOLVE

Question 17 Exercise: 7.2

Verify the trigonometric identity $\frac{1+sin\vartheta}{cos\vartheta} + \frac{cos\vartheta}{1+sin\vartheta} = 2sec\,\vartheta.$

Solution:

Question 18 Exercise: 7.2

Verify the trigonometric identity $\frac{cos\vartheta\ cot\vartheta}{1-sin\vartheta} - 1 = csc\,\vartheta$ **.**

Solution:

Question 20 Exercise: 7.2

Verify the trigonometric identity $\boldsymbol{cosx - \frac{cosx}{1-tanx} = \frac{sinx\ cosx}{sinx-cosx}}$ **.**

Solution:

Question 32 Exercise: 7.2

Verify the trigonometric identity $\boldsymbol{\frac{csc(-x)}{sec(-x)} = -cotx.}$

Solution:

Question 35 Exercise: 7.2
Verify the trigonometric identity $(\mathbf{1}+\boldsymbol{siny})[(\mathbf{1}+\boldsymbol{sin}(-\boldsymbol{y})]=\boldsymbol{cos}^{\mathbf{2}}\boldsymbol{y}$**.**
Solution:

Question 38 Exercise: 7.2
Verify the trigonometric identity $\frac{\boldsymbol{cosx}-\boldsymbol{cosy}}{\boldsymbol{sinx}+\boldsymbol{siny}}+\frac{\boldsymbol{sinx}-\boldsymbol{siny}}{\boldsymbol{cosx}+\boldsymbol{cosy}}=\mathbf{0}$**.**
Solution:

7.3 Solving Trigonometric Equation

Objectives:

- **Solve Trigonometric Equations using algebraic techniques.**

To solve a trigonometric equation, use standard algebraic techniques (when possible) such as collecting like terms, extracting square roots, and factoring. Your preliminary goal in solving a trigonometric equation is to isolate the trigonometric function on one side of the equation. For example, to solve the equation $\boldsymbol{2sinx = 1}$, divide each side by $\mathbf{2}$ to obtain $\boldsymbol{sinx} = \frac{1}{2}$.

To solve for $\boldsymbol{x}$, note in the graph of $\boldsymbol{y = sinx}$ below that the equation $\boldsymbol{sinx} = \frac{1}{2}$ has solutions $\boldsymbol{x = \frac{\pi}{6}}$ and $\boldsymbol{x = \frac{5\pi}{6}}$ in the interval $[\mathbf{0}\,, \boldsymbol{2\pi})$. Moreover, because $\boldsymbol{sinx}$ has a period of $\boldsymbol{2\pi}$, there are infinitely many other solutions, which can be written as $\boldsymbol{x = \frac{\pi}{6} + 2n\pi}$ and $\boldsymbol{x = \frac{5\pi}{6} + 2n\pi}$, where $\boldsymbol{n}$ is an integer.

CW:01 Question 07 Exercise: 7.3

Verify that each x-value is a solution of the equation $\boldsymbol{tanx - \sqrt{3} = 0}$.

(a) $\boldsymbol{x = \frac{\pi}{3}}$ **(b)** $\boldsymbol{x = \frac{4\pi}{3}}$

Solution:

CW:02 Question 09 Exercise: 7.3

Verify that each x-value is a solution of the equation $3tan^2(2x) - 1 = 0$.

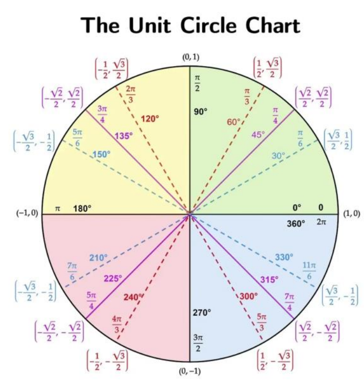
(a) $x = \frac{\pi}{12}$ **(b)** $x = \frac{5\pi}{12}$

Solution:

CW:03 Question 11 Exercise: 7.3

Solve the trigonometric equation $\sqrt{3}cscx - 2 = 0$.

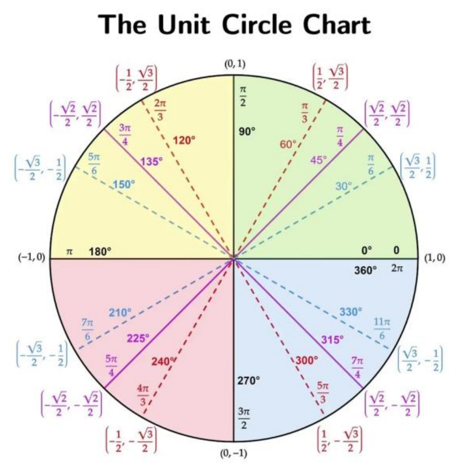
Solution:



CW:04 Question 14 Exercise: 7.3

Solve the trigonometric equation $\boldsymbol{tanx + \sqrt{3} = 0}$.

Solution:



CW:05 Question 21 Exercise: 7.3

Solve the trigonometric equation $\boldsymbol{sinx\,(sinx + 1) = 0}$.

Solution:

PRACTICE QUESTIONS FOR THE STUDENTS TO SOLVE

Question 10 Exercise: 7.3

Verify that each x-value is a solution of the equation $2\cos^2(4x) - 1 = 0$.

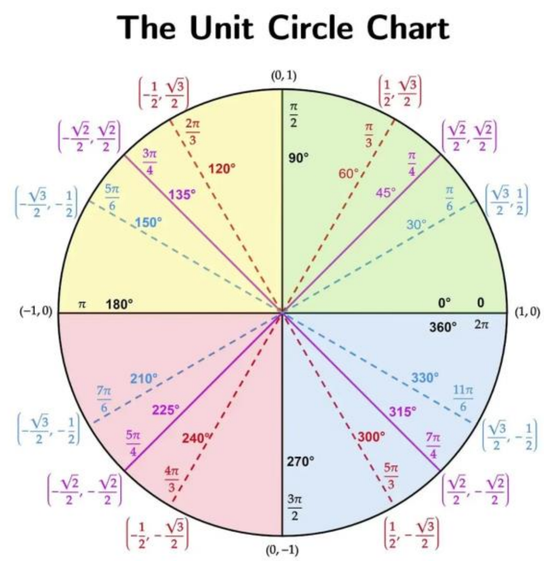
(a) $x = \frac{\pi}{16}$ **(b)** $x = \frac{3\pi}{16}$

Solution:

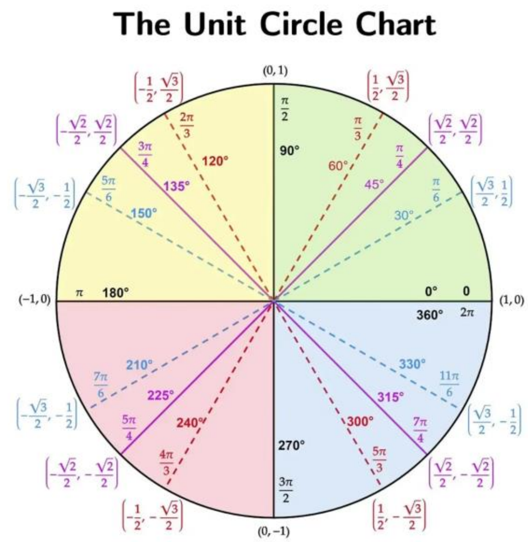
Question 15 Exercise: 7.3

Solve the trigonometric equation $\cos x + 1 = -\cos x$.

Solution:



Question 16 Exercise: 7.3
Solve the trigonometric equation $3sinx + 1 = sinx$.
Solution:



Question 26 Exercise: 7.3
Solve the trigonometric equation $secx\ cscx = 2cscx$.
Solution:

7.4 Sum and Difference Formulas

Objectives:

- **Use sum and difference formulas to evaluate trigonometric functions and verify identities.**

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

CW01: Example 01 Page- 534

Find the exact value of $\sin \frac{\pi}{12}$. (using sum and difference formula)

Solution:

CW02: Example 02 Page- 535

Find the exact value of $\cos 75^{\circ}$. (using sum and difference formula)

Solution:

CW03: Example 05 Page- 536

Verify the cofunction identity $\boldsymbol{cos\left(\frac{\pi}{2}-x\right)=sinx}$.

Solution:

CW:04 Question 37 Exercise: 7.4

Find the exact value of $\boldsymbol{sin\frac{\pi}{12}\;\; cos\frac{\pi}{4}+cos\frac{\pi}{12}\;\; sin\frac{\pi}{4}}$. (using sum and difference formula)

Solution:

CW:05 Question 39 Exercise: 7.4

Find the exact value of $\boldsymbol{cos130^{o}\; cos10^{o}+sin130^{o}\; sin10^{o}}$. (using sum and difference formula)

Solution:

CW:06 Question 63 Exercise: 7.4

Verify the identity $\boldsymbol{tan\left(\vartheta-\frac{\pi}{4}\right)=\frac{tan\vartheta-1}{tan\vartheta+1}}$.

Solution:

PRACTICE QUESTIONS FOR THE STUDENTS TO SOLVE

Question 26 Exercise: 7.4

Find the exact values of the sine, cosine, and tangent of the angle $\boldsymbol{\vartheta = 15^{o}}$. (using sum and difference formulas)

Solution:

Question 38 Exercise: 7.4

Find the exact value of $\boldsymbol{cos\frac{\pi}{16}\ \ cos\frac{3\pi}{16} - sin\frac{\pi}{16}\ \ sin\frac{3\pi}{16}}$. (using sum and difference formula)

Solution:

Question 40 Exercise: 7.4

Find the exact value of $\boldsymbol{sin100^{o}\ cos40^{o} - cos100^{o}\ sin40^{o}}$**.** (using sum and difference formula)

Solution:

Question 42 Exercise: 7.4

Find the exact value of $\boldsymbol{\frac{tan25^{o}+tan110^{o}}{1-tan25^{o}\ tan110^{o}}}$**.** (using sum and difference formulas)

Solution:

Question 60 Exercise: 7.4

Verify the identity $\boldsymbol{sin\left(\frac{\pi}{2}+x\right) = cosx}$**.**

Solution:

7.5 Multiple-Angles and Product-to-Sum Formulas

Objectives:

- Use double-angle, product-to-sum and sum-to-product formulas to rewrite and evaluate trigonometric expressions.

In this section, you will study four other categories of trigonometric identities.

1. The first category involves functions of multiple angles such as $\boldsymbol{sinkx}$ and $\boldsymbol{coskx}$.
2. The second category involves squares of trigonometric functions such as $\boldsymbol{sin^2x}$.
3. The third category involves functions of half-angles such as $\boldsymbol{sin\frac{x}{2}}$.
4. The fourth category involves products of trigonometric functions such as $\boldsymbol{sinxcosy}$.

Double-Angle-Formulas:

$$\boldsymbol{sin2u = 2sinu\,cosu}$$

$$\boldsymbol{\begin{aligned} cos2u &= cos^2u - sin^2u \\ &= 2cos^2u - 1 \\ &= 1 - 2sin^2u \end{aligned}}$$

$$\boldsymbol{tan2u = \frac{2tanu}{1-tan^2u}}$$

CW01: Question 13 Exercise: 7.5
Use a double-angle formula to rewrite the expression $\boldsymbol{6\,sinx\,cosx}$.
Solution:

CW02: Question 14 Exercise: 7.5
Use a double-angle formula to rewrite the expression $\boldsymbol{sinx\,cosx}$.
Solution:

CW03: Question 15 Exercise: 7.5

Use a double-angle formula to rewrite the expression $6cos^2x - 3$.

Solution:

Product-to-Sum Formulas:

$$sinu\ sinv = \frac{1}{2}\left[cos(u - v) - cos(u + v)\right]$$

$$cosu\ cosv = \frac{1}{2}\left[cos(u - v) + cos(u + v)\right]$$

$$sinu\ cosv = \frac{1}{2}\left[sin(u + v) + sin(u - v)\right]$$

$$cosu\ sinv = \frac{1}{2}\left[sin(u + v) - sin(u - v)\right]$$

CW04: Question 47 Exercise: 7.5

Use the product-to-sum formula to rewrite the product $sin5\vartheta\ sin3\vartheta$ as sum or difference.

Solution:

CW05: Question 49 Exercise: 7.5

Use the product-to-sum formula to rewrite the product $cos2\vartheta\, cos4\vartheta$ as sum or difference.

Solution:

Sum-to-Product Formulas:

$$sinu + sinv = 2sin\left(\frac{u+v}{2}\right)cos\left(\frac{u-v}{2}\right)$$

$$sinu - sinv = 2cos\left(\frac{u+v}{2}\right)sin\left(\frac{u-v}{2}\right)$$

$$cosu + cosv = 2cos\left(\frac{u+v}{2}\right)cos\left(\frac{u-v}{2}\right)$$

$$cosu - cosv = -2sin\left(\frac{u+v}{2}\right)sin\left(\frac{u-v}{2}\right)$$

CW06: Question 52 Exercise: 7.5

Use the sum-to-product formula to rewrite the sum $sin3\vartheta + sin\vartheta$ as product.

Solution:

CW07: Question 55 Exercise: 7.5

Use the sum-to-product formula to find the exact value of $sin75^{o} + sin15^{o}$.

Solution:

PRACTICE QUESTIONS FOR THE STUDENTS TO SOLVE

Question 16 Exercise: 7.5

Use a double-angle formula to rewrite the expression $\boldsymbol{cos^2 x - \frac{1}{2}}$**.**

Solution:

Question 17 Exercise: 7.5

Use a double-angle formula to rewrite the expression $\boldsymbol{4 - 8sin^2 x}$**.**

Solution:

Question 18 Exercise: 7.5

Use a double-angle formula to rewrite the expression $\boldsymbol{10sin^2 x - 5}$**.**

Solution:

Question 50 Exercise: 7.5

Use the product-to sum formula to rewrite the product $sin(x+y)\ cos(x-y)$ as a sum or difference.

Solution:

Question 51 Exercise: 7.5

Use the sum-to-product formula to rewrite the difference $sin5\vartheta - \ sin3\vartheta$ as a product.

Solution:

Question 56 Exercise: 7.5

Use the sum-to-product formula to find the exact value of $cos120^{o} + \ cos60^{o}$.
Solution:

Question 57 Exercise: 7.5

Use the sum-to-product formula to find the exact value of $cos\frac{3\pi}{4} - cos\frac{\pi}{4}$.
Solution: