Informed search

CE417: Introduction to Artificial Intelligence
Sharif University of Technology
Spring 2016

Soleymani

“Artificial Intelligence: A Modern Approach”, Chapter 3

Outline

- Best-first search
- Greedy best-first search
- A^* search
- Finding heuristics

Informed search

- When exhaustive search is impractical, heuristic methods are used to speed up the process of finding a satisfactory (or optimal) solution.

Best-first search

- Idea: use an evaluation function $f(n)$ for each node and expand the most desirable unexpanded node
  - More general than "$g(n)$ = cost so far to reach n"
  - Evaluation function provides an upper bound on the desirability (lower bound on the cost) that can be obtained through expanding a node

- Implementation: priority queue with decreasing order of desirability (search strategy is determined based on evaluation function)

- Special cases:
  - Greedy best-first search
  - A* search
  - Uniform-cost search

Heuristic Function

- Incorporating problem-specific knowledge in search
  - Information more than problem definition
  - In order to come to an optimal solution as rapidly as possible

- Heuristic function can be used as a component of $f(n)$

- $h(n)$: estimated cost of cheapest path from n to a goal
  - Depends only on n (not path from root to n)
  - If n is a goal state then $h(n)$=0
  - $h(n) \geq 0$

- Examples of heuristic functions include using a rule-of-thumb, an educated guess, or an intuitive judgment

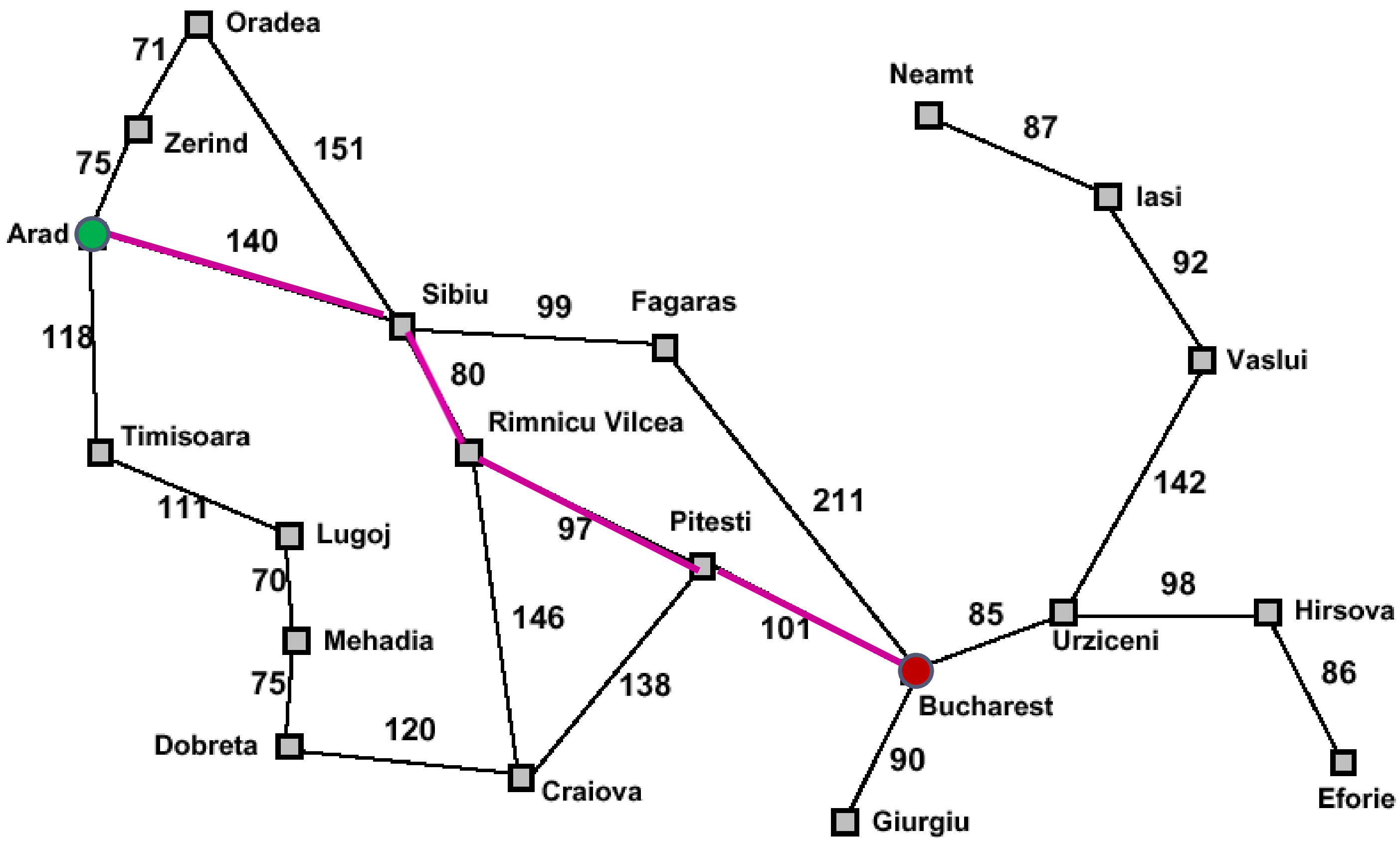
Greedy best-first search

- Evaluation function $f(n) = h(n)$
  - e.g., $h_{SLD}(n)$ = straight-line distance from *n* to Bucharest

- Greedy best-first search expands the node that appears to be closest to goal

Greedy

Romania with step costs in km



Straight–line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
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Lugoj	244
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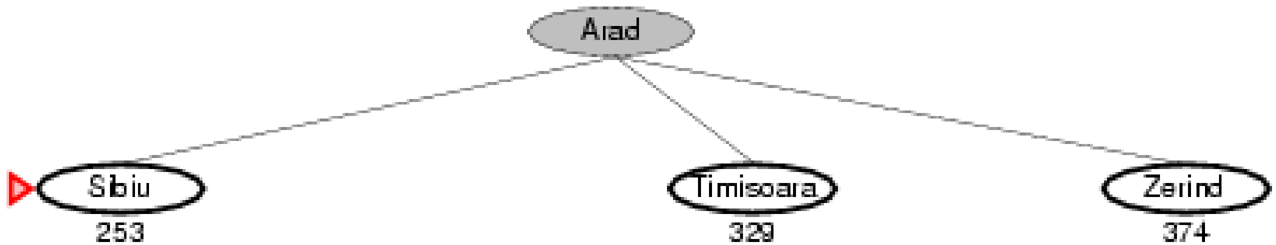
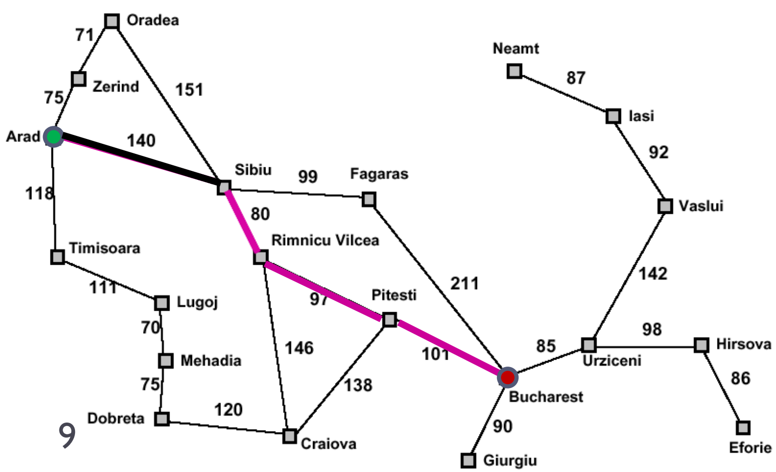
Greedy best-first search example

Greedy best-first search example





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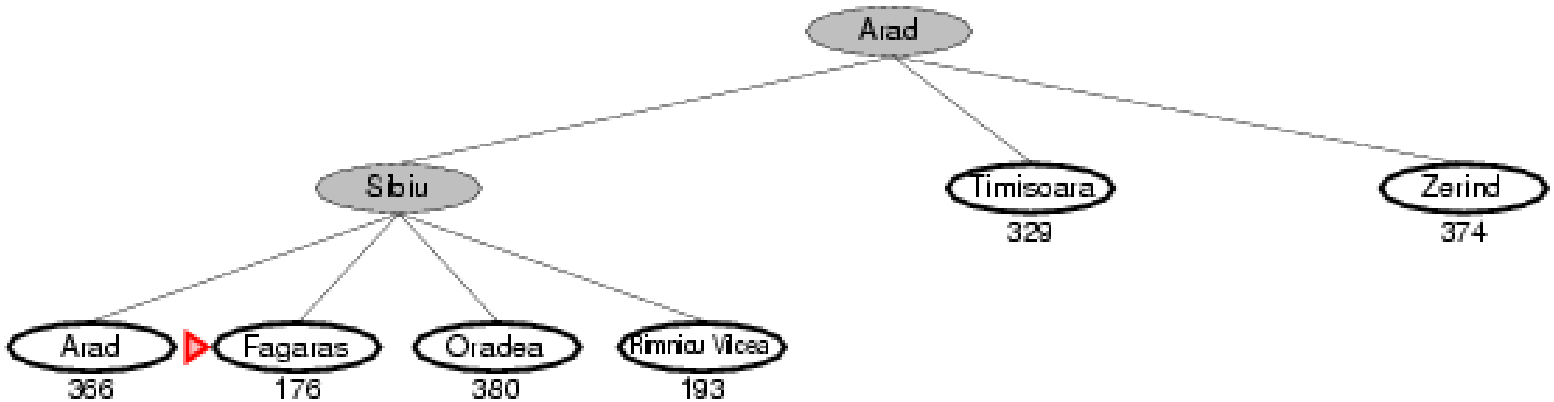

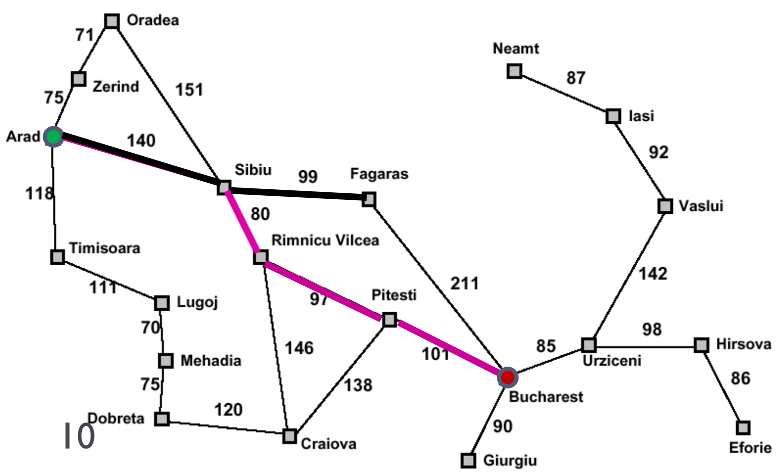

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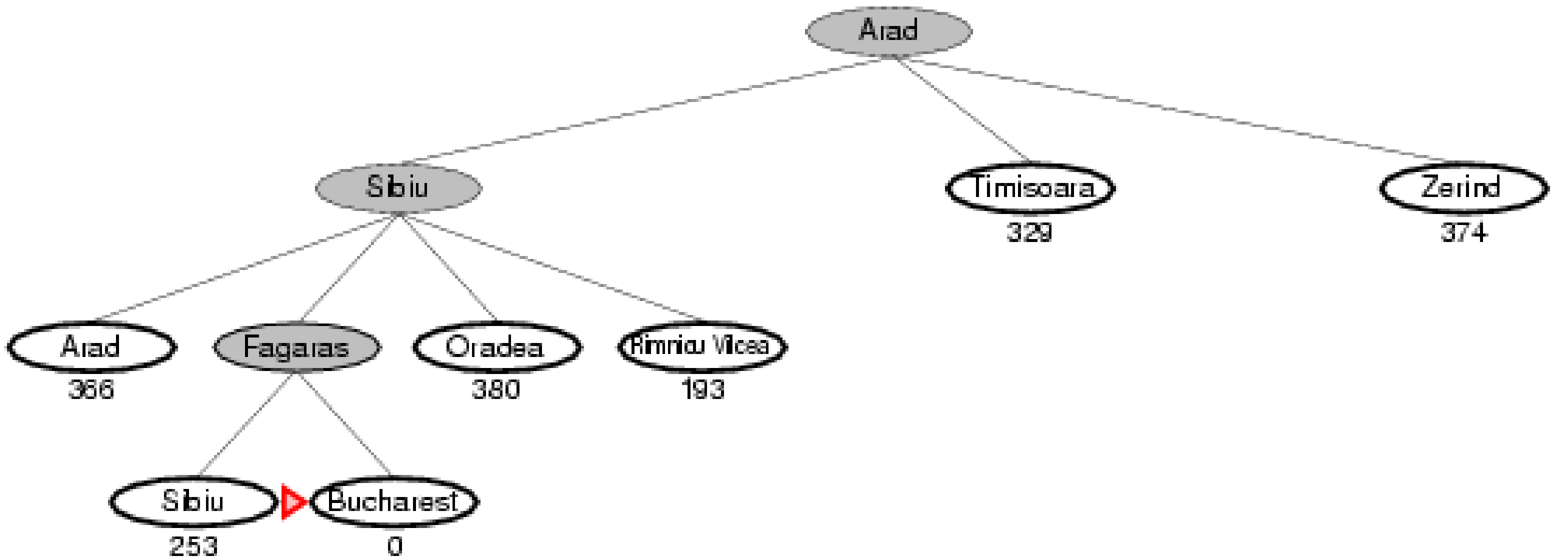

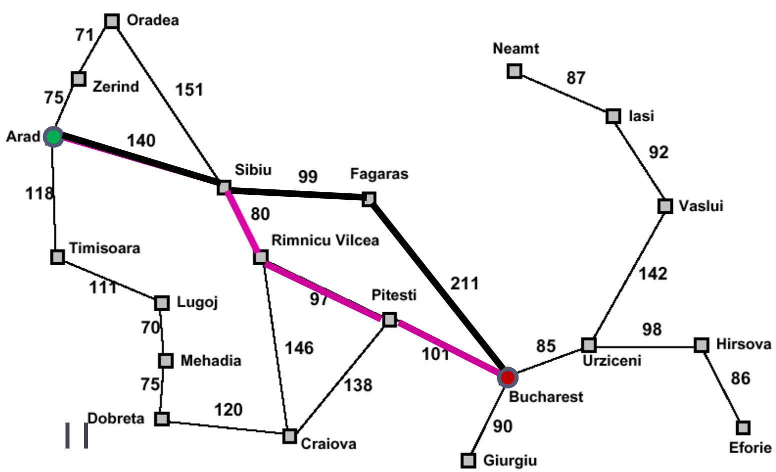

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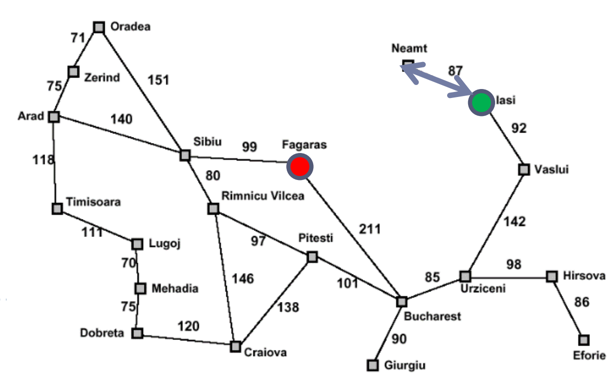
Properties of greedy best-first search

- Complete? No
  - Similar to DFS, only graph search version is complete in finite spaces
  - Infinite loops, e.g., (Iasi to Fagaras) Iasi → Neamt → Iasi → Neamt

- Time
  - $O(b^m)$, but a good heuristic can give dramatic improvement

- Space
  - $O(b^m)$: keeps all nodes in memory

- Optimal? No



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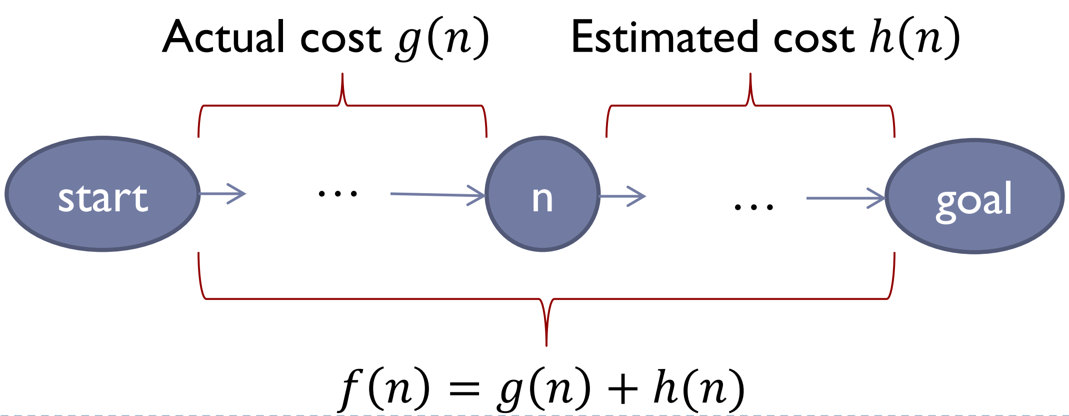
A* search

- Idea: minimizing the total estimated solution cost
- Evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
  - $h(n)$ = estimated cost of the cheapest path from n to goal
  - So, $f(n)$ = estimated total cost of path through n to goal

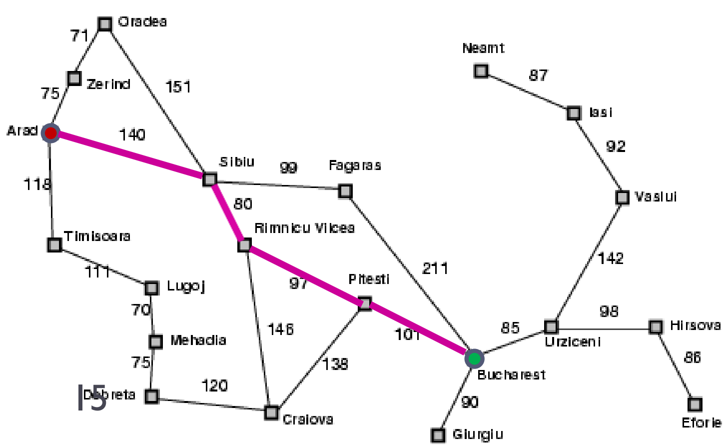


A* search

- Combines advantages of uniform-cost and greedy searches

- A* can be complete and optimal when $h(n)$ has some properties

A* search: example



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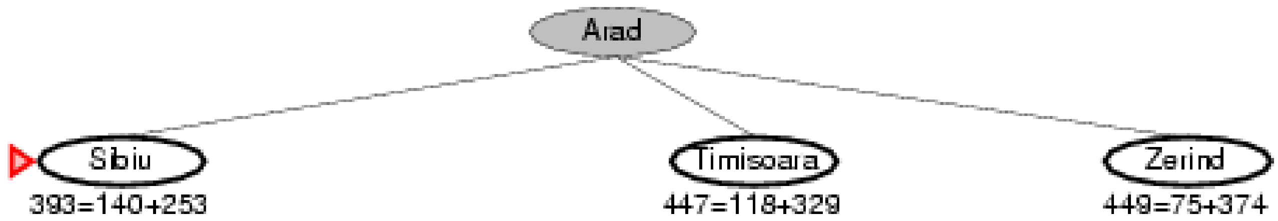
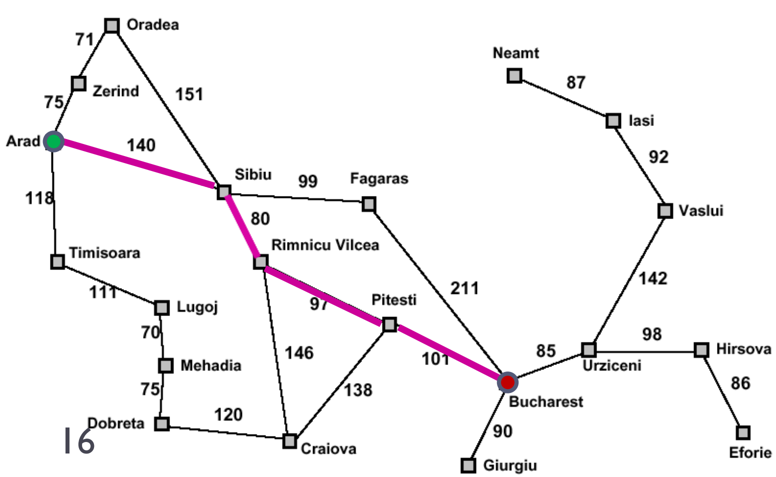
A* search: example





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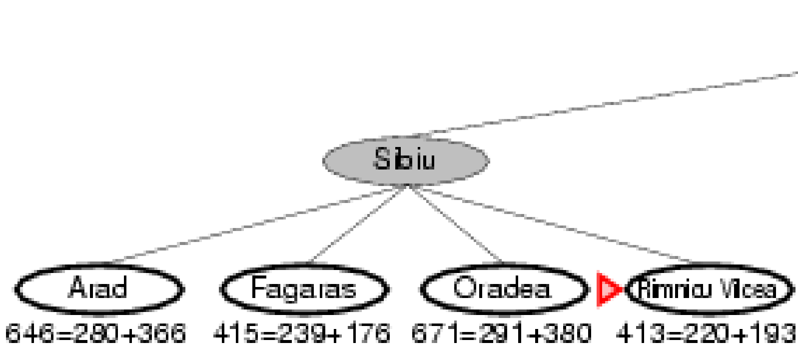

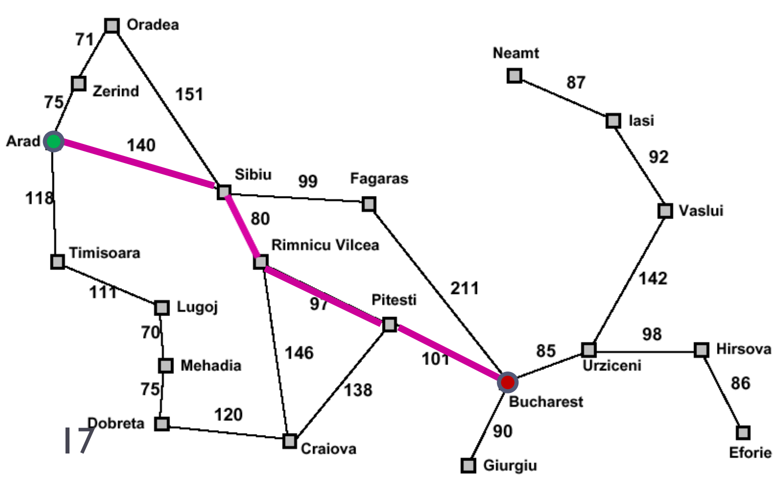

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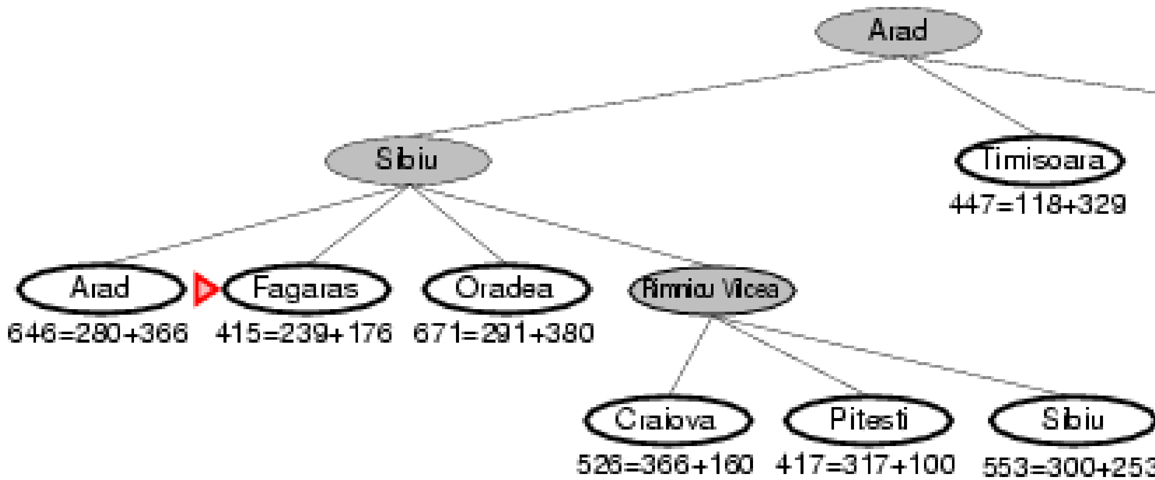

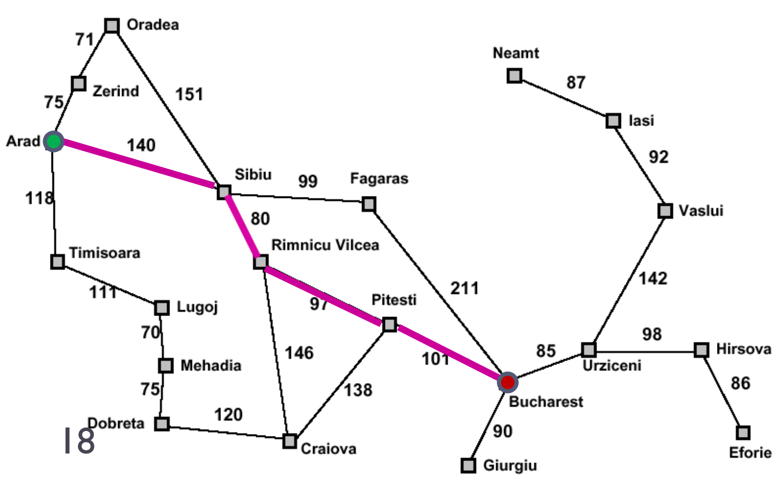

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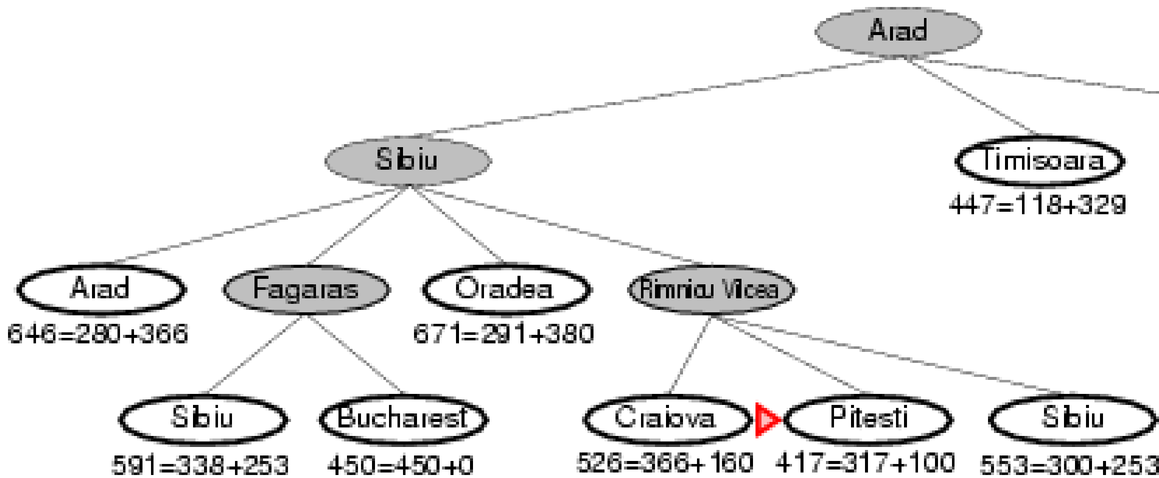

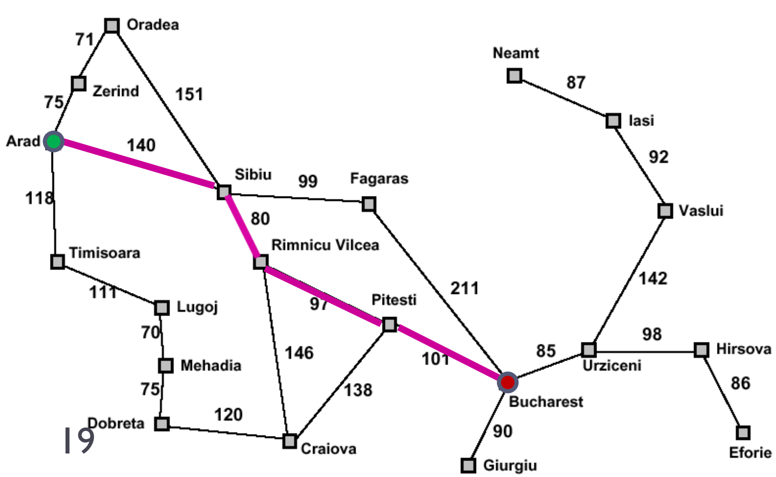

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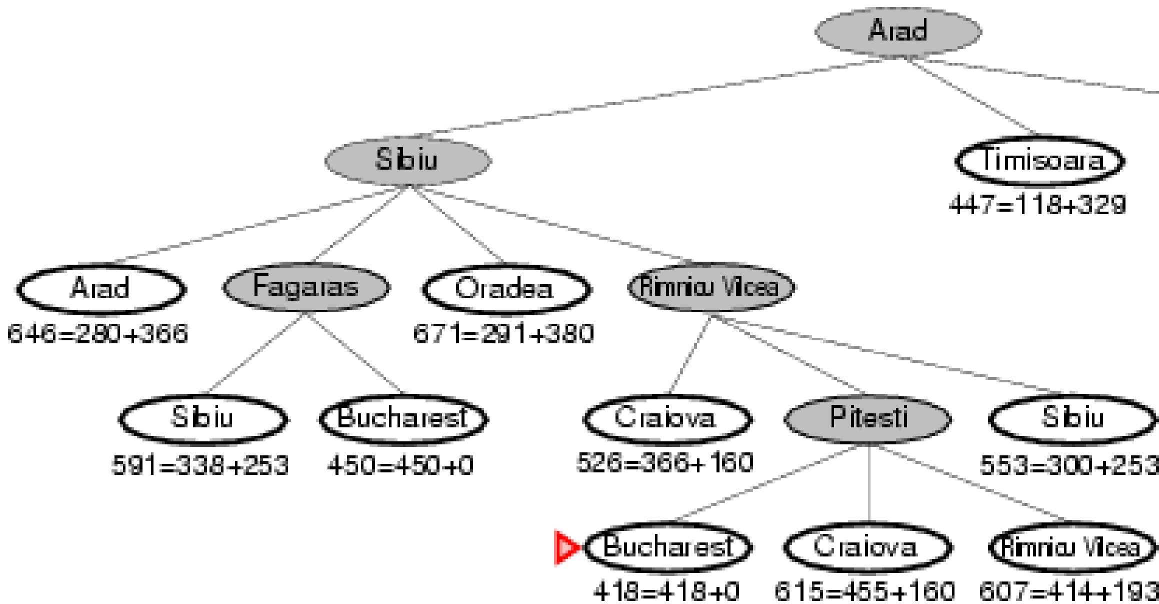

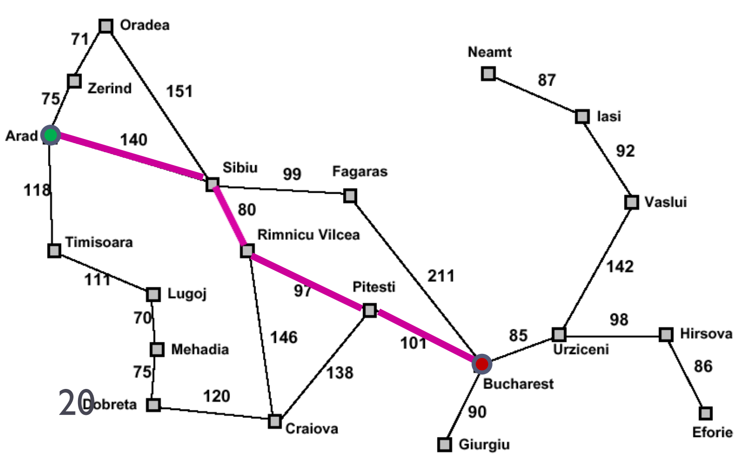

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Conditions for optimality of A*

- Admissibility: $h(n)$ be a lower bound on the cost to reach goal
  - Condition for optimality of TREE−SEARCH version of A*
- Consistency (monotonicity): $h(n) \leq c(n, a, n') + h(n')$
  - Condition for optimality of GRAPH−SEARCH version of A*

Admissible heuristics

- Admissible heuristic $h(n)$ never overestimates the cost to reach the goal (optimistic)
  - $h(n)$ is a lower bound on path cost from n to goal

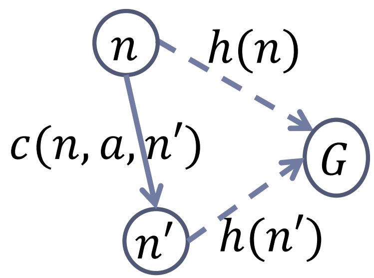
$$\forall n, h(n) \leq h^*(n)$$

  where $h^*(n)$ is the real cost to reach the goal state from n

    - Example: $h_{SLD}(n) \leq$ the actual road distance

Consistent heuristics

- Triangle inequality



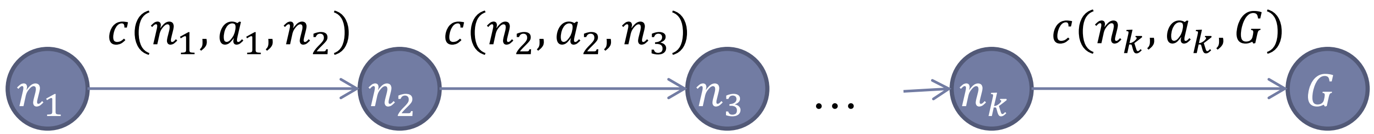
for every node n and every successor n' generated by any action a

$$h(n) \leq c(n,a,n') + h(n')$$

$c(n,a,n')$: cost of generating n' by applying action to n

Consistency vs. admissibility

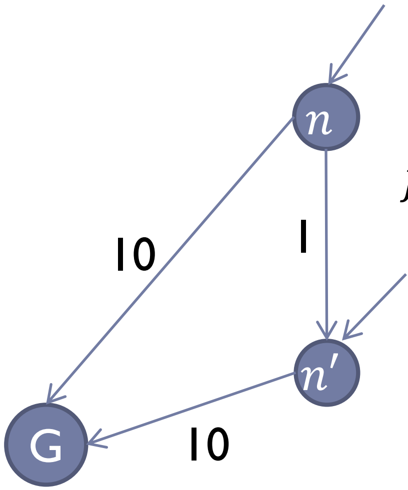
- Consistency ⇒ Admissblity
  - All consistent heuristic functions are admissible
  - Nonetheless, most admissible heuristics are also consistent



$$
\begin{aligned}
h(n_1) &\le c(n_1, a_1, n_2) + h(n_2) \\
&\le c(n_1, a_1, n_2) + c(n_2, a_2, n_3) + h(n_3) \\
&\dots \\
&\le \textstyle\sum_{i=1}^{k} c(n_i, a_i, n_{i+1}) + 0
\end{aligned}
$$

$\Rightarrow h(n_1) \le$ cost of (every) path from n_1 to goal
$\le$ cost of optimal path from n_1 to goal

Admissible but not consistent: Example



$g(n) = 5$
$h(n) = 9$
$f(n) = 14$

$g(n') = 6$
$h(n') = 6$
$f(n') = 12$

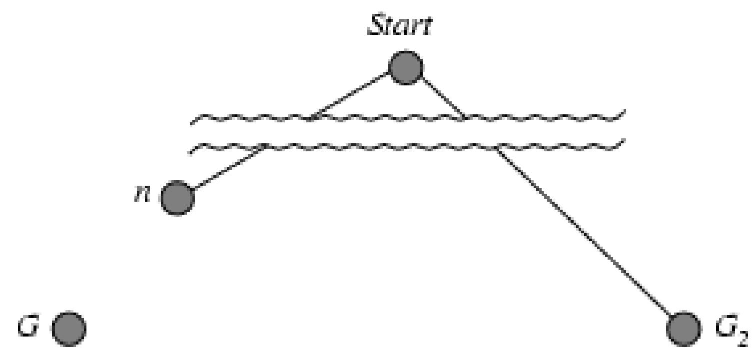
$c(n, a, n') = 1$
$h(n) = 9$
$h(n') = 6$
$\Longrightarrow h(n) \nleq h(n') + c(n, a, n')$

- f (for admissible heuristic) may decrease along a path
- Is there any way to make h consistent?

$$\bar{h}(n') = \max(h(n'), \bar{h}(n) - c(n, a, n'))$$

Optimality of A* (admissible heuristics)

- **Theorem**: If $h(n)$ is admissible, A* using `TREE-SEARCH` is optimal
- Assumptions: G_2 is a suboptimal goal in the frontier, n is an unexpanded node in the frontier and it is on a shortest path to an optimal goal G.



I. $h(G_2) = 0 \Rightarrow f(G_2) = g(G_2)$

II. $h(G) = 0 \Rightarrow f(G) = g(G)$

III. G_2 is suboptimal $\Rightarrow g(G_2) > g(G)$

IV. I, II, III $\Rightarrow f(G_2) > f(G)$

V. h is admissible $\Rightarrow h(n) \le h^*(n)$

$$\Rightarrow g(n) + h(n) \le g(n) + h^*(n)$$

$$\Rightarrow f(n) \le f(G) \overset{IV}{\Rightarrow} f(n) < f(G_2)$$

A* will never select G_2 for expansion

Optimality of A* (consistent heuristics)

Theorem: If $h(n)$ is consistent, A* using `GRAPH-SEARCH` is optimal

Lemma1: if $h(n)$ is consistent then $f(n)$ values are non-decreasing along any path

Proof: Let n' be a successor of n

I. $f(n') = g(n') + h(n')$
II. $g(n') = g(n) + c(n, a, n')$
III. $I, II \Rightarrow f(n') = g(n) + c(n, a, n') + h(n')$
IV. $h(n)$ is consistent $\Rightarrow h(n) \leq c(n, a, n') + h(n')$
V. $III, IV \Rightarrow f(n') \geq g(n) + h(n) = f(n)$

Optimality of A* (consistent heuristics)

⇒The sequence of nodes expanded by A* is in non-decreasing order of $f(n)$

Since $h = 0$ for goal nodes (f is the true cost for goal nodes), the first selected goal node for expansion provides an optimal solution. Indeed, we cannot reach a goal node with lower value of f because of the non-decreasing order of f.

We can also show that:

If A* selects a node n for expansion, the optimal solution to that node has been found.

Proof by contradiction: Another frontier node n' must exist on the optimal path from initial node to n (using graph separation property). Moreover, based on Lemma 1, $f(n') \leq f(n)$ and thus n' would have been selected first.

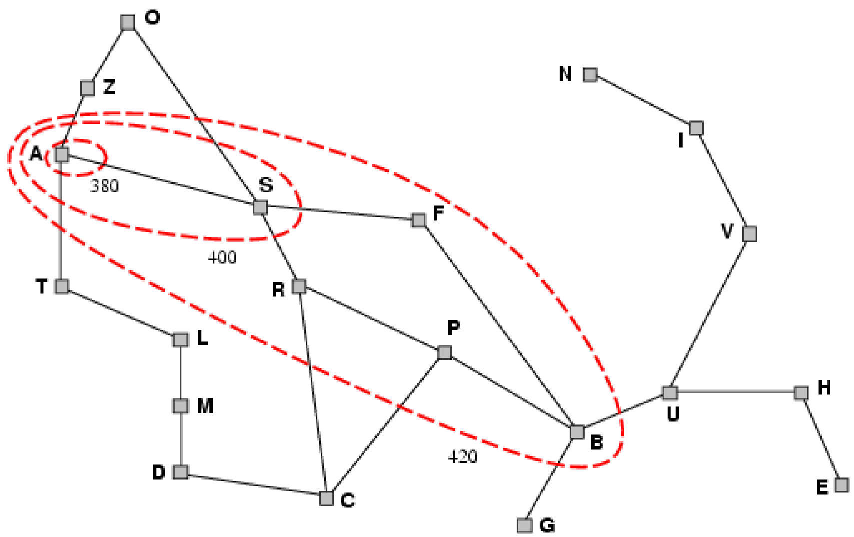
Admissible vs. consistent (tree vs. graph search)

- Consistent heuristic: When selecting a node for expansion, the path with the lowest cost to that node has been found

- When an admissible heuristic is not consistent, a node will need repeated expansion, every time a new best (so-far) cost is achieved for it.

Contours in the state space

- A^* (using `GRAPH-SEARCH`) expands nodes in order of increasing f value
- Gradually adds "f-contours" of nodes
  - Contour i has all nodes with $f = f_i$ where $f_i < f_{i+1}$

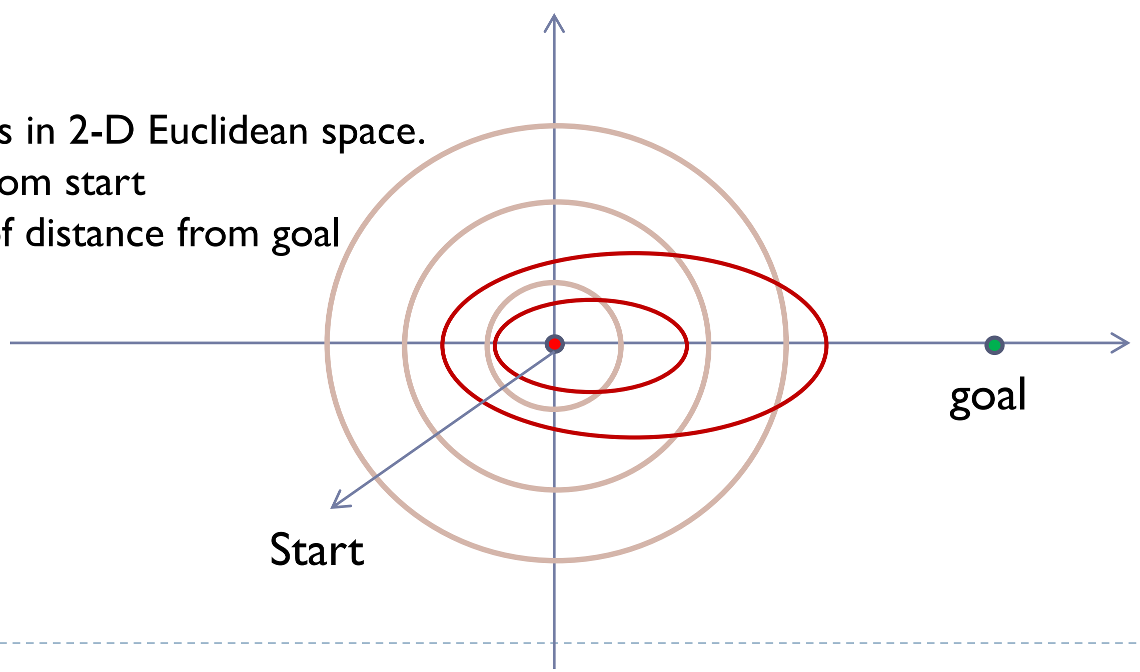


A* expands all nodes with f(n) < C*
A* expands some nodes with f(n) = C* (nodes on the goal contour)
A* expands no nodes with f(n) > C* $\Longrightarrow$ pruning

A^* search vs. uniform cost search

- Uniform-cost search (A^* using $h(n) = 0$) causes circular bands around initial state
- A^* causes irregular bands
  - More accurate heuristics stretched toward the goal (more narrowly focused around the optimal path)

States are points in 2-D Euclidean space.
g(n)=distance from start
h(n)=estimate of distance from goal

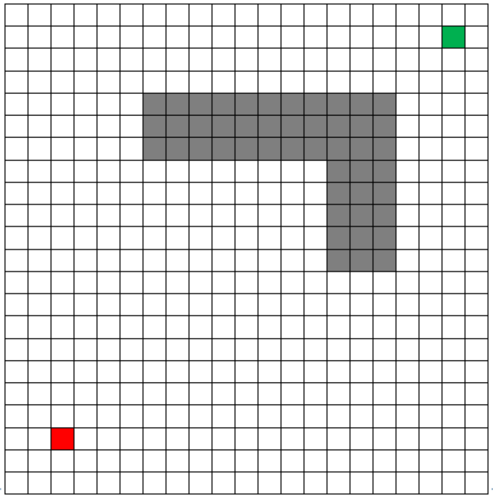


Properties of A*

- Complete?
  - Yes if nodes with $f \leq f(G) = C^*$ are finite
    - Step cost$\geq \varepsilon > 0$ and b is finite
- Time?
  - Exponential
    - But, with a smaller branching factor
      - b^{h^*-h} or when equal step costs $b^{d\times\frac{h^*-h}{h^*}}$
    - Polynomial when $|h(x) - h^*(x)| = O(log\ h^*(x))$
  - However, A* is optimally efficient for any given consistent heuristic
    - No optimal algorithm of this type is guaranteed to expand fewer nodes than A* (except to node with $f = C^*$)
- Space?
  - Keeps all leaf and/or explored nodes in memory
- Optimal?
  - Yes (expanding node in non-decreasing order of f)

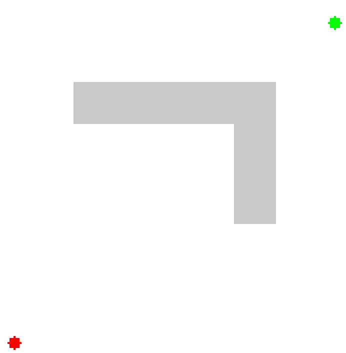
Robot navigation example

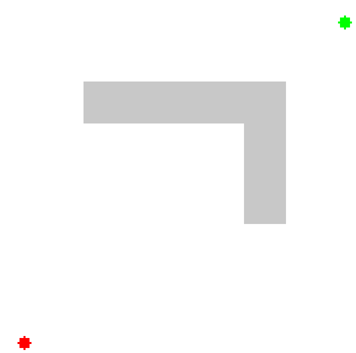
- Initial state? Red cell
- States? Cells on rectangular grid (except to obstacle)
- Actions? Move to one of 8 neighbors (if it is not obstacle)
- Goal test? Green cell
- Path cost? Action cost is the Euclidean length of movement



A* vs. UCS: Robot navigation example

- Heuristic: Euclidean distance to goal
- Expanded nodes: filled circles in red & green
  - Color indicating g value (red: lower, green: higher)
- Frontier: empty nodes with blue boundary
- Nodes falling inside the obstacle are discarded



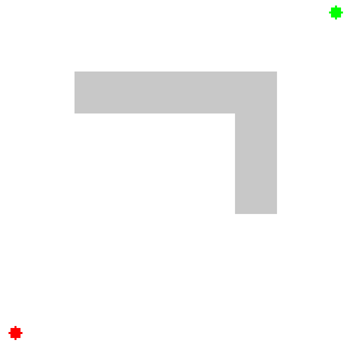


Adopted from: http://en.wikipedia.org/wiki/Talk%3AA*_search_algorithm

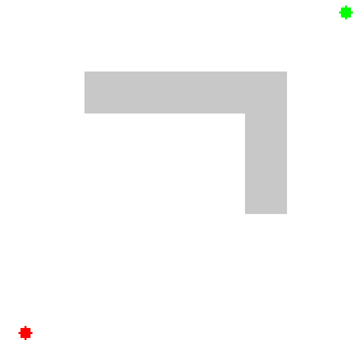
Robot navigation: Admissible heuristic

- Is Manhattan $d_M(x, y) = |x_1 - y_1| + |x_2 - y_2|$ distance an admissible heuristic for previous example?

A*: inadmissible heuristic



$h = 5 * h\_SLD$



$h = h\_SLD$

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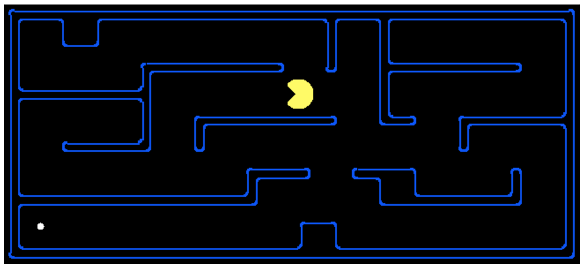
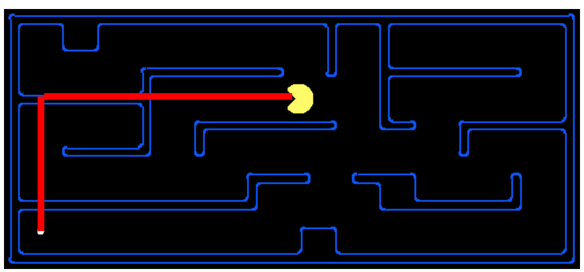
A*, Greedy, UCS: Pacman

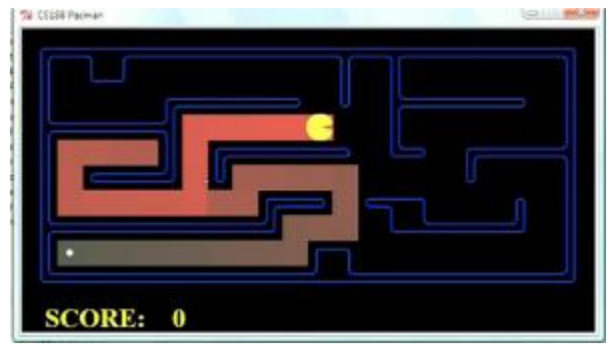


Heuristic: Manhattan distance



Color: expanded in which iteration (red: lower)

UCS



Greedy

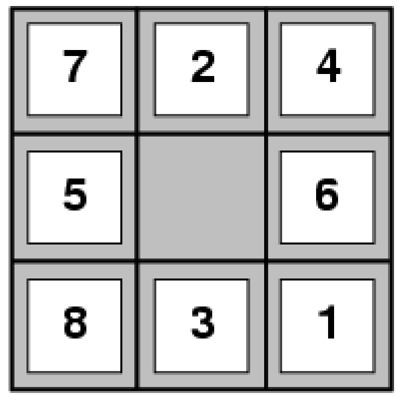
A*

Adapted from Dan Klein’s slides

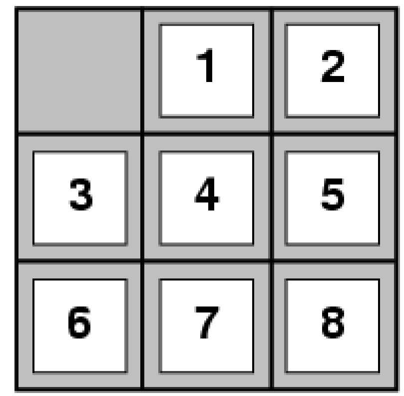
A* difficulties

- Space is the main problem of A*
- Overcoming space problem while retaining completeness and optimality
  - IDA*, RBFS, MA*, SMA*
- A* time complexity
  - Variants of A* trying to find suboptimal solutions quickly
  - More accurate but not strictly admissible heuristics

8-puzzle problem: state space



Start State

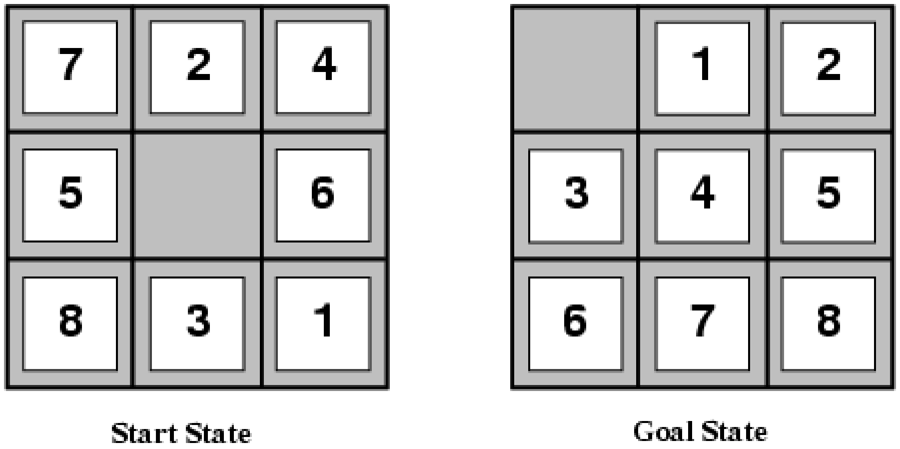


Goal State

- $b \approx 3$, average solution cost for random **8-puzzle** ≈ 22
- **Tree search**: $\mathrm{b} \approx 3, \mathrm{d} \approx 22 \Longrightarrow 3^{22} \approx 3.1 \times 10^{10}$states
- **Graph search**: $9!/2 \approx 181{,}440$ states for **8-puzzle**
  - 10^{13} for 15-puzzle

Admissible heuristics: 8-puzzle

- $h_1(n)$ = number of misplaced tiles
- $h_2(n)$ = sum of Manhattan distance of tiles from their target position
  - i.e., no. of squares from desired location of each tile



- $h_1(S) =$ 8
- $h_2(S) =$ 3+1+2+2+2+3+3+2 = 18

Effect of heuristic on accuracy

- N: number of generated nodes by A*
- d: solution depth

- Effective branching factor b^*: branching factor of a uniform tree of depth d containing $N + 1$ nodes.

$$N + 1 = 1 + b^* + (b^*)^2 + \cdots + (b^*)^d$$

- Well-defined heuristic: b^* is close to one

Comparison on 8-puzzle

Search Cost (N)

d	IDS	A*(h_1)	A*(h_2)
6	680	20	18
12	3644035	227	73
24	--	39135	1641

Effective branching factor (b^*)

d	IDS	A*(h_1)	A*(h_2)
6	2.87	1.34	1.30
12	2.78	1.42	1.24
24	--	1.48	1.26

Heuristic quality

If $\forall n, h_2(n) \geq h_1(n)$ (both admissible)

then h_2 dominates h_1 and it is better for search

- Surely expanded nodes: $f(n) < C^* \Rightarrow h(n) < C^* - g(n)$
  - If $h_2(n) \geq h_1(n)$ then every node expanded for h_2 will also be surely expanded with h_1 (h_1 may also causes some more node expansion)

More accurate heuristic

- Max of admissible heuristics is admissible (while it is a more accurate estimate)

$$h(n) = \max(h_1(n), h_2(n))$$

- How about using the actual cost as a heuristic?
  - $h(n) = h^*(n)$ for all n
    - Will go straight to the goal ?!
  - Trade of between accuracy and computation time

Generating heuristics

- Relaxed problems
  - Inventing admissible heuristics automatically
- Sub-problems (pattern databases)
- Learning heuristics from experience

Relaxed problem

- Relaxed problem: Problem with fewer restrictions on the actions

- Optimal solution to the relaxed problem may be computed easily (without search)

- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
  - The optimal solution is the shortest path in the super-graph of the state-space.

Relaxed problem: 8-puzzle

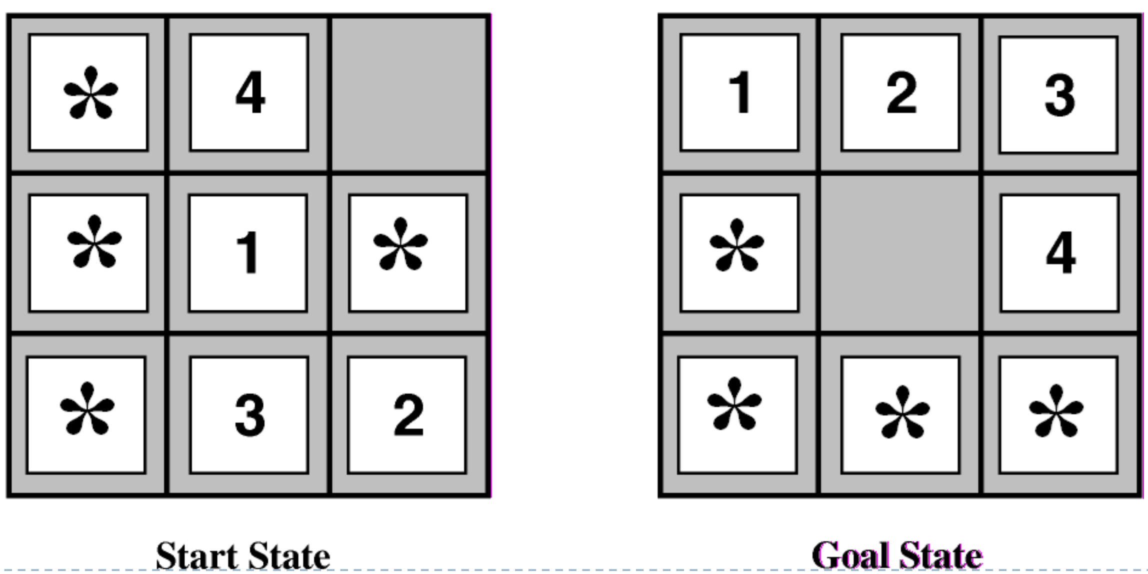
- 8-Puzzle: move a tile from square A to B if <u>A is adjacent (left, right, above, below) to B and B is blank</u>
  - Relaxed problems
    1) can move from A to B if A is adjacent to B (ignore whether or not position is blank)
    2) can move from A to B if B is blank (ignore adjacency)
    3) can move from A to B (ignore both conditions)

- Admissible heuristics for original problem ($h_1(n)$ and $h_2(n)$) are optimal path costs for relaxed problems
  - First case: <u>a tile can move to any adjacent square</u> $\Rightarrow$ $h_2(n)$
  - Third case: <u>a tile can move anywhere</u> $\Rightarrow$ $h_1(n)$

Sub-problem heuristic

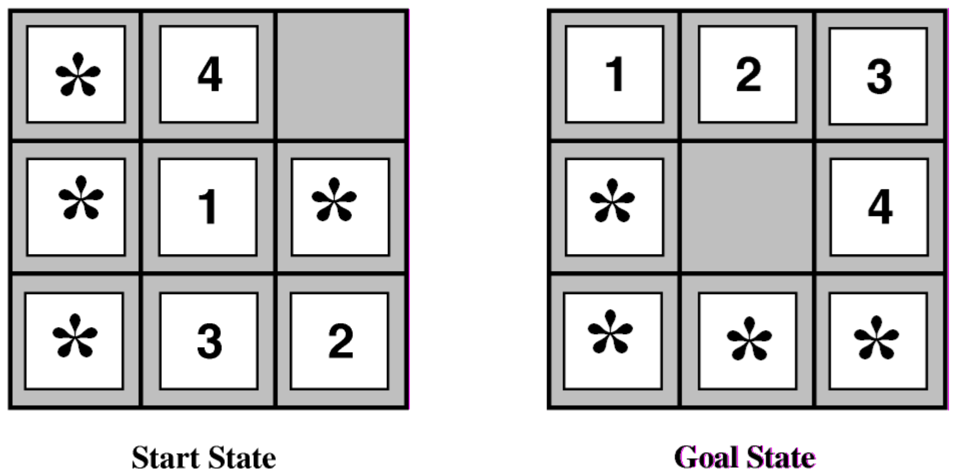
- The cost to solve a sub-problem
  - Store exact solution costs for every possible sub-problem

- Admissible?
  - The cost of the optimal solution to this problem is a lower bound on the cost of the complete problem



Pattern databases heuristics

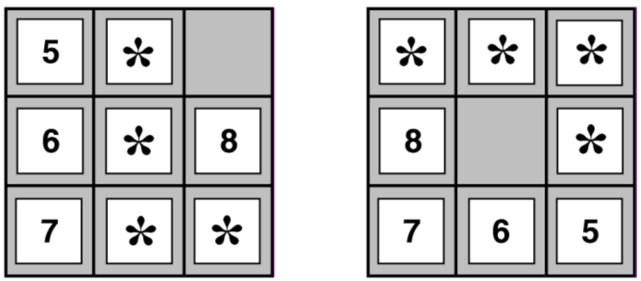
- Storing the exact solution cost for every possible sub-problem instance



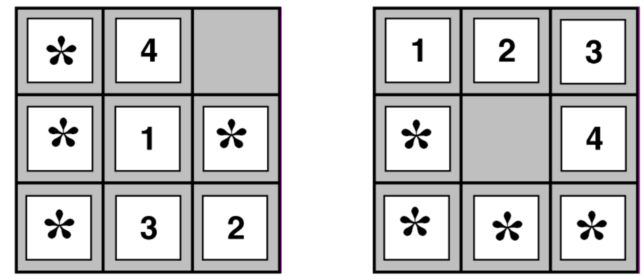
- Combination (taking maximum) of heuristics resulted by different sub-problems
  - 15-Puzzle: 10^3 times reduction in no. of generated nodes vs. h_2

Disjoint pattern databases

- Adding these pattern-database heuristics yields an admissible heuristic?!



Start State Goal State



Start State Goal State

- Dividing up the problem such that each move affects only one sub-problem (disjoint sub-problems) and then adding heuristics
  - 15-puzzle: 10^4 times reduction in no. of generated nodes vs. h_2
  - 24-Puzzle: 10^6 times reduction in no. of generated nodes vs. h_2
  - Can Rubik's cube be divided up to disjoint sub-problems?

Learning heuristics from experience

- Machine Learning Techniques
  - Learn $h(n)$ from samples of optimally solved problems (predicting solution cost for other states)

- Features of state (instead of raw state description)
  - 8-puzzle
    - number of misplaced tiles
    - number of adjacent pairs of tiles that are not adjacent in the goal state
  - Linear Combination of features