A Probability and Statistics Refresher

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Review of Basic Probability Results

- The HTTP log of a Web server was analyzed and file sizes were collected:

File	File Size (KB)
1	2.3
2	3.7
3	10.4
4	2.2
5	7.3
6	102.0
7	2.9
8	4.0
9	30.0
10	1.2
11	3.4
12	20.0
13	3.5
14	9.0
15	2.8

Sample vs. Population

- Population: all possible file sizes (very large or infinite)
- Sample: a finite number of file sizes.
- Q: How we present measured data?

Probability 101

- Random Variable (r.v.): a variable that takes one of a specified set of values with a given probability.
  - X: size of file retrieved from a Web server
- Cumulative Distribution Function (CDF):

$$F_X(x) = P[X \leq x]$$

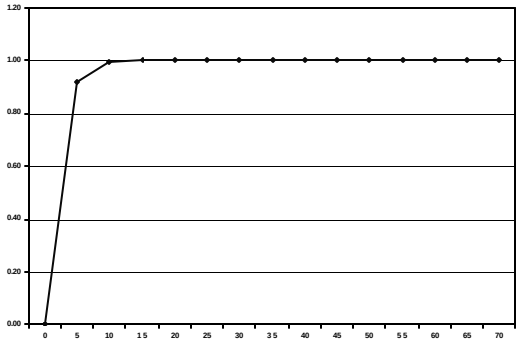
Probability 101

- Properties of a CDF:

$0 \le F_X(x) \le 1$

$\lim_{x \to \infty} F_X(x) = 1$

$b > a \quad \Rightarrow F_X(b) \ge F_X(a)$



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Probability 101

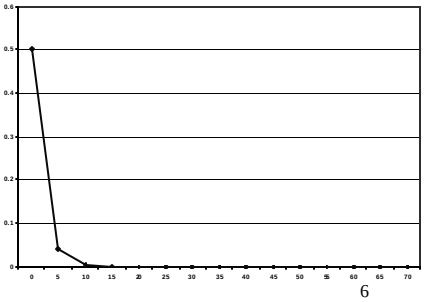
- Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Properties of the pdf:

$\int_{x=a}^{x=b} f_X(x)dx = P[a \le X \le b]$

$\int_{x=0}^{x=\infty} f_X(x)dx = 1$



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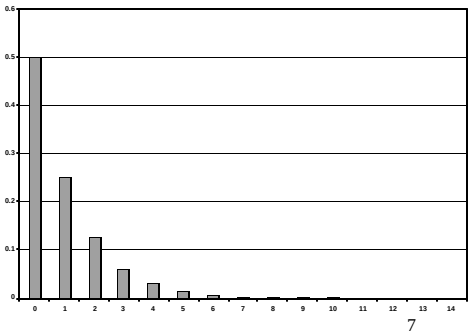
Probability Mass Function

- For discrete r.v.s:
  - Values: $x_1, \ldots, x_n$
  - Probabilities: $p_1, \ldots, p_n$
- Property:

$$\sum_{i=1}^{n} p_i = 1$$



Probability 101

- Mean or Expected Value

$$E[X] = m = \sum_{k=1}^{n} x_k \times p_k = \int_{x=0}^{\infty} x f_X(x) dx$$

- Variance

$$s^2 = Var(X) = E[(x-m)^2] = \sum_{k=1}^{n} (x_k - m)^2 \times p_k$$

$$= \int_{x=0}^{\infty} (x-m)^2 f_X(x) dx$$

- Standard Deviation

$$s$$

Probability 101

- Coefficient of Variation:

$$C_X = \frac{s}{m}$$

- a -percentile: value of x at which the CDF takes the value a . E.g., if the 90-percentile of the file size is 2Kbytes, then

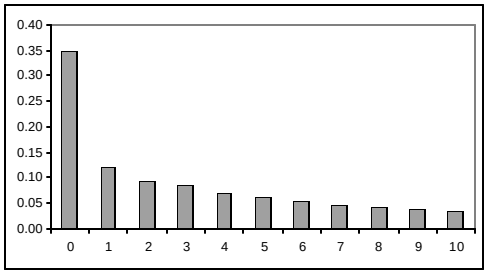
$$P[X \leq 2KB] = 0.9$$

- Median: 50-percentile.

Discrete Probability Distributions

Discrete Probability Distribution

- Distribution: set of all possible values and their probabilities.



Number of I/Os per Transaction	Probability
0	0.350
1	0.120
2	0.095
3	0.085
4	0.070
5	0.060
6	0.054
7	0.048
8	0.043
9	0.040
10	0.035
	1.000

Moments of a Discrete Random Variable

- Expected Value:

$$m = E[X] = \sum_{\forall i} X_i \times P[X_i]$$

- k-th moment:

$$m = E[X^k] = \sum_{\forall i} X_i^k \times P[X_i]$$

Number of I/Os per Transaction	Probability	For First Moment (average)	For Second Moment
0	0.350	0.000	0.000
1	0.120	0.120	0.120
2	0.095	0.190	0.380
3	0.085	0.255	0.765
4	0.070	0.280	1.120
5	0.060	0.300	1.500
6	0.054	0.324	1.944
7	0.048	0.336	2.352
8	0.043	0.344	2.752
9	0.040	0.360	3.240
10	0.035	0.350	3.500
	1.000	**2.859**	**17.673**

mean

second moment

Central Moments of a Discrete Random Variable

- k-th central moment:

$$E[(X-\overline{X})^k] = \sum_{\forall i} (X_i - \overline{X})^k \times P[X_i]$$

- The variance is the second central moment:

$$s^2 = E[(X-\overline{X})^2] = E[X^2 + (\overline{X})^2 - 2X\overline{X}]$$
$$= E[X^2] + (\overline{X})^2 - 2(\overline{X})^2 =$$
$$= E[X^2] - (\overline{X})^2$$

Central Moments of a Discrete Random Variable

Number of I/Os per Transaction	Probability	For First Moment (average)	For Second Moment	For Second Central Moment
0	0.350	0.000	0.000	2.8609
1	0.120	0.120	0.120	0.4147
2	0.095	0.190	0.380	0.0701
3	0.085	0.255	0.765	0.0017
4	0.070	0.280	1.120	0.0911
5	0.060	0.300	1.500	0.2750
6	0.054	0.324	1.944	0.5328
7	0.048	0.336	2.352	0.8231
8	0.043	0.344	2.752	1.1365
9	0.040	0.360	3.240	1.5085
10	0.035	0.350	3.500	1.7848
	1.000	**2.859**	**17.673**	**9.4991**

average *variance*

Properties of the Mean

- The mean of the sum is the sum of the means.

$$E[X+Y]=E[X]+E[Y]$$

- If X and Y are independent random variables, then the mean of the product is the product of the means.

$$E[XY]=E[X]E[Y]$$

Discrete Random Variables

- Binomial
- Hypergeometric
- Negative Binomial
- Geometric
- Poisson

The Binomial Distribution

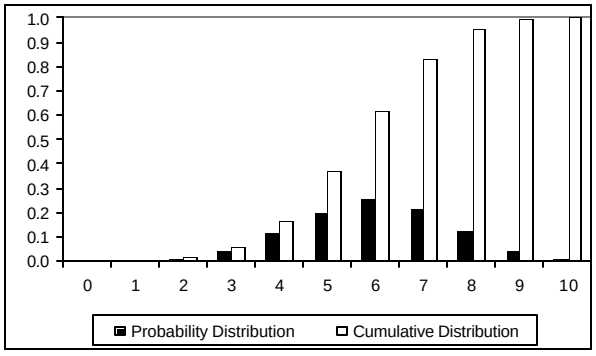
- Distribution: based on carrying out independent experiments with two possible outcomes:
  - Success with probability p and
  - Failure with probability $(1-p)$.
- A binomial r.v. counts the number of successes in n trials.

$$P[X=k]=\binom{n}{k}p^k(1-p)^{n-k}=\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

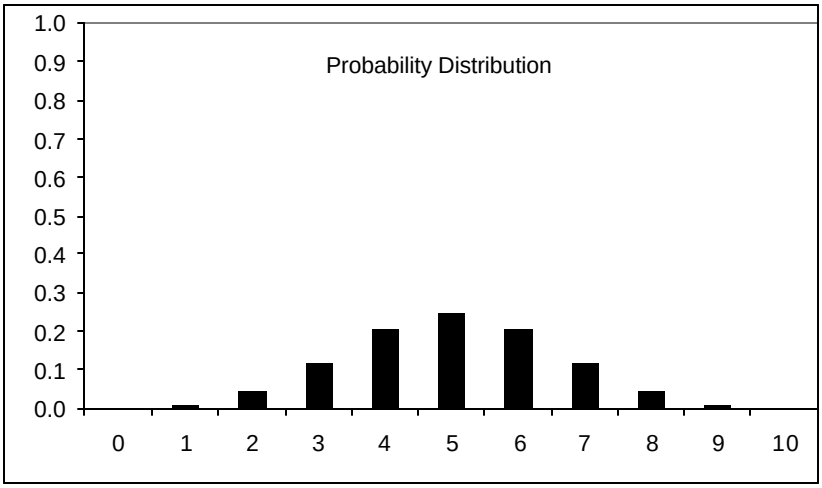
The Binomial Distribution

Success Probability	**0.6**	**(p)**
Number of Attempts	**10**	**(n)**

Number of Attempts (k)	Probability k successful attempts in n	Cumulative
0	0.000105	0.000105
1	0.001573	0.001678
2	0.010617	0.012295
3	0.042467	0.054762
4	0.111477	0.166239
5	0.200658	0.366897
6	0.250823	0.617719
7	0.214991	0.832710
8	0.120932	0.953643
9	0.040311	0.993953
10	0.006047	1.000000



Shape of the Binomial Distribution



$p = 0.5$ symmetric for any n.

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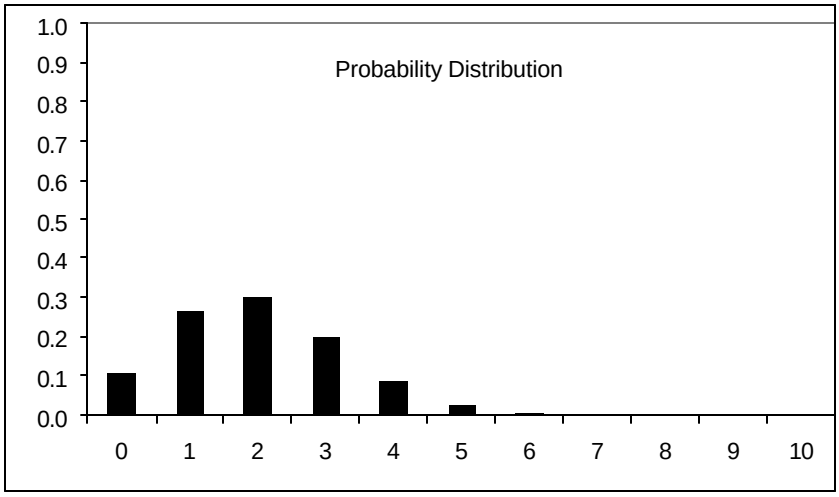
Shape of the Binomial Distribution



$p = 0.2$ right skewed

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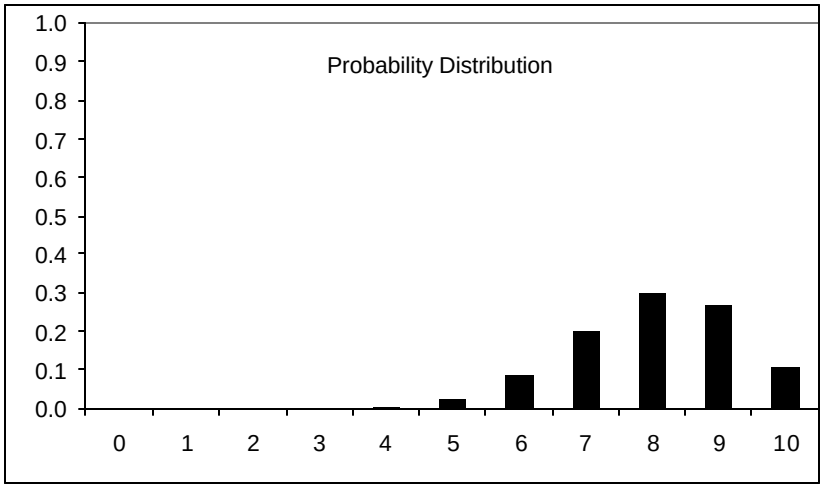
Shape of the Binomial Distribution



$p = 0.8$ left skewed

Moments of the Binomial Distribution

- Average: $n\,p$
- Variance: $np(1-p)$
- Standard Deviation: $\sqrt{np(1-p)}$
- Coefficient of Variation:

$$\frac{\sqrt{np(1-p)}}{np} = \sqrt{\frac{1-p}{np}}$$

Hypergeometric Distribution

- Binomial was based on experiments with equal success probability.
- Hypergeometric: not all experiments have the same success probability.
- Given a sample size of n out of a population of size N with A known successes in the population, the probability of k successes is

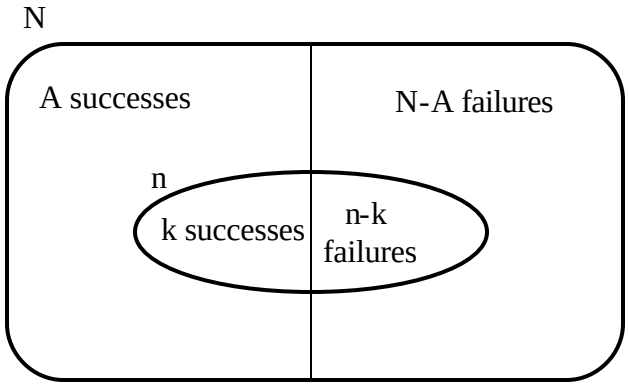
choose k successes out of A successes in the population

choose (n-k) failures from N-A failures in the population

$$P[X = k] = \frac{\binom{A}{k}\binom{N-A}{n-k}}{\binom{N}{n}}$$

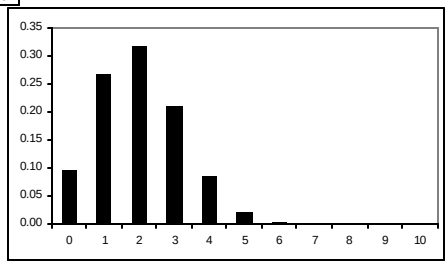
total # of possible samples

Hypergeometric Distribution



Hypergeometric Distribution

No. successes in sample k	sample size n	no. successes in population A	population size N	
0	20	10	100	0.09511627
1	20	10	100	0.26793316
2	20	10	100	0.31817063
3	20	10	100	0.20920809
4	20	10	100	0.08410730
5	20	10	100	0.02153147
6	20	10	100	0.00354136
7	20	10	100	0.00036793
8	20	10	100	0.00002300
9	20	10	100	0.00000078
10	20	10	100	0.00000001



Moments of the Hypergeometric

- Average: $\frac{nA}{N}$
- Standard Deviation: $\sqrt{\frac{nA(N-A)}{N^2}}\sqrt{\frac{N-n}{N-1}}$
- If the sample size is less than 5% of the population, the binomial is a good approximation for the hypergeometric.

Negative Binomial Distribution

- Probability of success is equal to p and is the same on all trials.
- Random variable X counts the number of trials until the k-th success is observed.

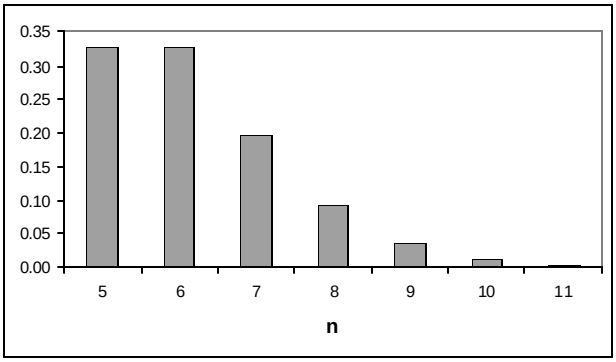
$$P[X = n] = \binom{n-1}{k-1}(1-p)^{n-k}\, p^{k}$$

S	F	F	S	. . .	F	S
1	2	3	4		n-1	n

Negative Binomial Distribution

Success probability	0.8

k	n	Prob[X=n]
1	1	0.800000
1	2	0.160000
1	3	0.032000
1	4	0.006400
5	5	0.327680
5	6	0.327680
5	7	0.196608
5	8	0.091750
5	9	0.036700
5	10	0.013212
5	11	0.004404



Moments of the Negative Binomial Distribution

- Average: $\frac{k}{p}$
- Standard Deviation: $\sqrt{\frac{k(1-p)}{p^2}}$
- Coefficient of Variation: $\sqrt{\frac{1-p}{k}}$

Geometric Distribution

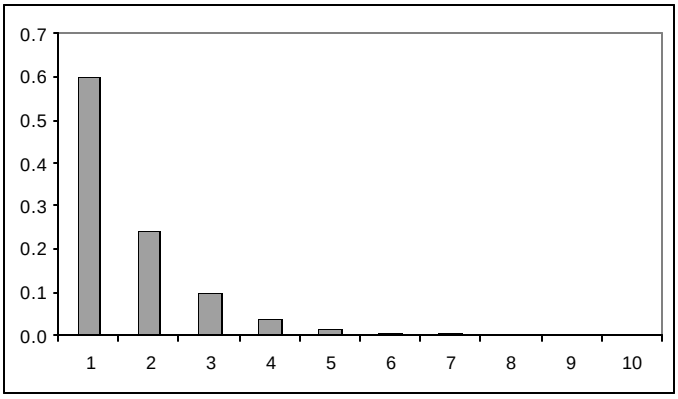
- Special case of the negative binomial with k=1.
- Probability that the first success occurs after n trials is

$$p[X = n] = p(1-p)^{n-1} \qquad n = 1,2,...$$

Geometric Distribution

Success probability	0.6

n	P[X=n]
1	0.6000
2	0.2400
3	0.0960
4	0.0384
5	0.0154
6	0.0061
7	0.0025
8	0.0010
9	0.0004
10	0.0002



Moments of the Geometric Distribution

- Average: $\frac{1}{p}$
- Standard Deviation: $\sqrt{\frac{1-p}{p^2}}$
- Coefficient of Variation: $\sqrt{1-p} \leq 1$

Poisson Distribution

- Used to model the number of arrivals over a given interval, e.g.,
  - Number of requests to a server
  - Number of failures of a component
  - Number of queries to the database.
- A Poisson distribution usually arises when arrivals come from a large number of independent sources.

Poisson Distribution

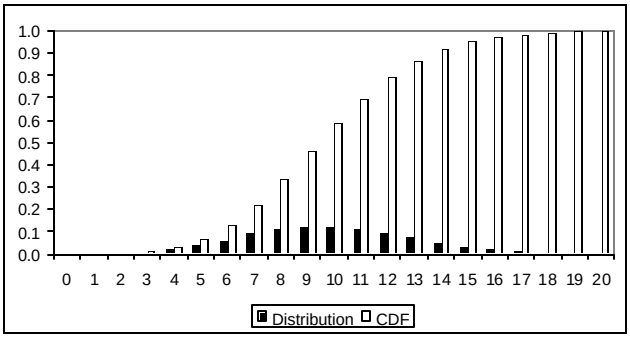
- Distribution: $P[X = k] = \frac{l^k e^{-l}}{k!} \qquad k = 0,1,...,\infty$
- Counting arrivals in an interval of duration *t*:

$$P[k \text{ arrivals in } [0, t)] = \frac{(lt)^k e^{-lt}}{k!} \qquad k = 0,1,...,\infty$$

- Average=Variance=λ

Poisson Distribution

Lambda	10	
K	Poisson Distribution	CDF
0	0.00005	0.0000
1	0.00045	0.0005
2	0.00227	0.0028
3	0.00757	0.0103
4	0.01892	0.0293
5	0.03783	0.0671
6	0.06306	0.1301
7	0.09008	0.2202
8	0.11260	0.3328
9	0.12511	0.4579
10	0.12511	0.5830
11	0.11374	0.6968
12	0.09478	0.7916
13	0.07291	0.8645
14	0.05208	0.9165
15	0.03472	0.9513
16	0.02170	0.9730
17	0.01276	0.9857
18	0.00709	0.9928
19	0.00373	0.9965
20	0.00187	0.9984



Continuous Random Variables

Relevant Functions

- Probability density function (pdf) of r.v. X: $f_X(x)$

$$P[a \le X \le b] = \int_a^b f_X(x)dx$$

- Cumulative distribution function (CDF):

$$F_X(x) = P[X \le x]$$

- Tail of the distribution (reliability function):

$$R_X(x) = P[X > x] = 1 - F_X(x)$$

Moments

- k-th moment: $E[X^k] = \int_{-\infty}^{+\infty} x^k f_X(x)dx$
- Expected value (mean): first moment

$$m = E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx$$

- k-th central moment:

$$E[(X-m)^k] = \int_{-\infty}^{+\infty} (x-m)^k f_X(x)dx$$

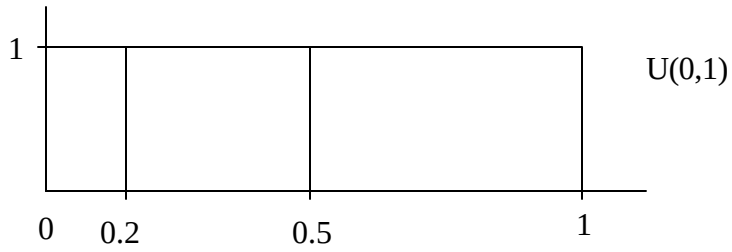
- Variance: second central moment

$$s^2 = E[(X-m)^2] = \int_{-\infty}^{+\infty} (x-m)^2 f_X(x)dx$$

The Uniform Distribution

- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$
- Mean: $m = \frac{a+b}{2}$
- Variance: $s^2 = \frac{(b-a)^2}{12}$

The Uniform Distribution



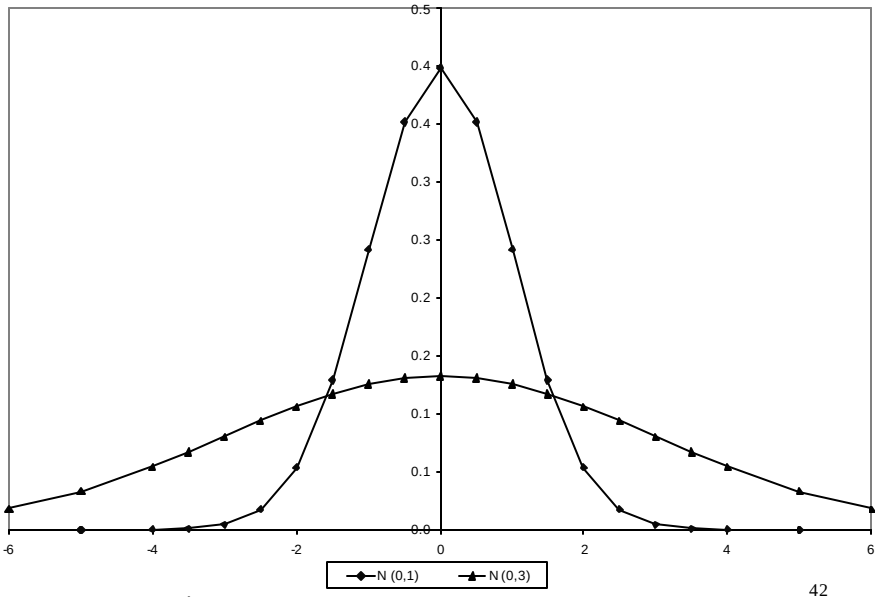
P[0.2<X<0.5]=(0.5-0.2)x1.0=0.3

The Normal Distribution $N(m, s)$

- Many natural phenomena follow a normal distribution.
- The normal distribution can be used to approximate the binomial and the Poisson distributions.
- Two parameters: mean and standard deviation.

$$f_X(x) = \frac{1}{\sqrt{2p}\, s} e^{-(1/2)[(x-m)/s]^2}$$

The Normal Distribution $N(m, s)$



The Standard Normal Distribution

- To use tables for computing values related to the normal distribution, we need to standardize a normal r.v. as

$$Z = \frac{X - m}{s}$$

standard normal score

- Given X, compute a Z value z.
- Find the area value in a Table (Prob [0<Z<z]).

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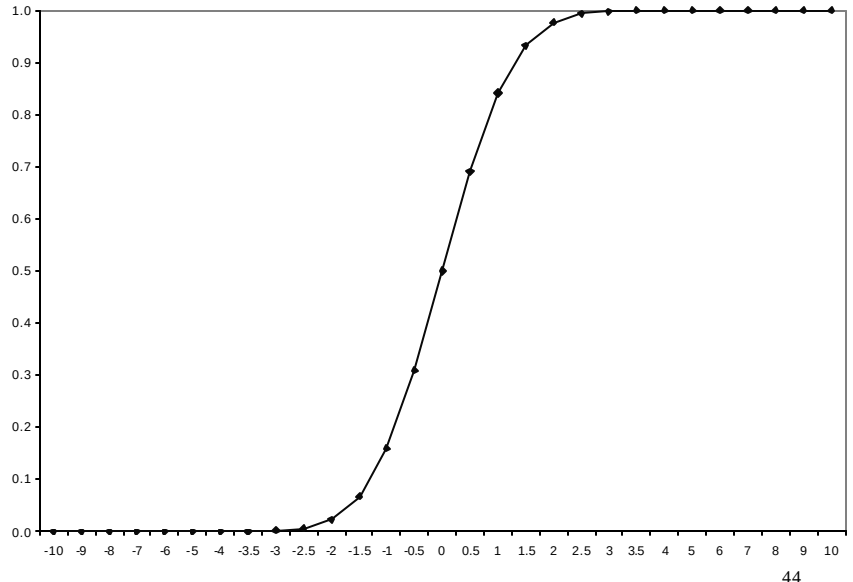
Normal CDF



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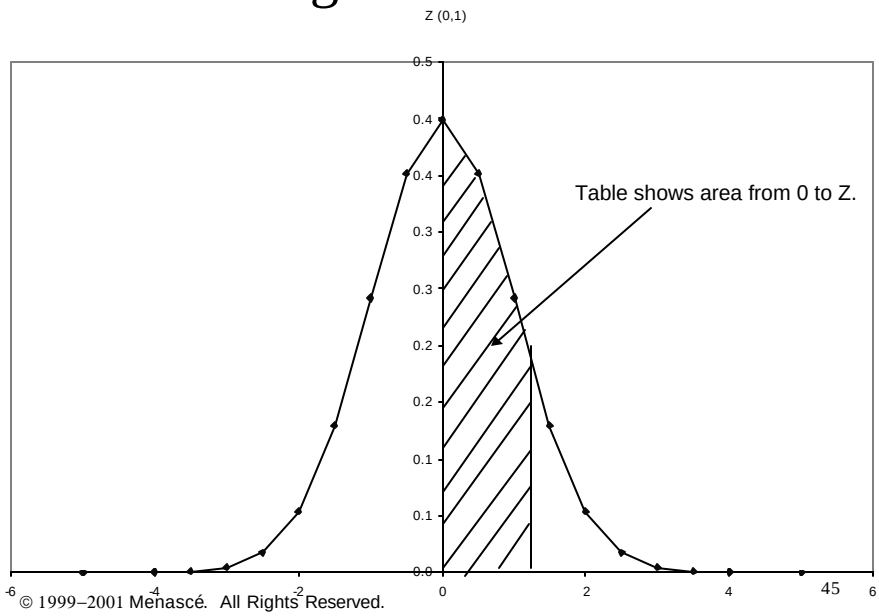
Using Normal Tables



The Normal as an Approximation to the Binomial Distribution

- The normal can approximate the binomial if the variance of the binomial

$$np(1-p) \geq 10$$

- Binomial: $m = np$

$$s = \sqrt{np(1-p)}$$

- Transformation: $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$

The Normal as an Approximation to the Binomial Distribution

- Consider a binomial r.v. X with average 50 and variance 25. What is $P[50 \le X \le 60]$?

- Transformation: $Z = \frac{X - 50}{\sqrt{25}} = \frac{60 - 50}{5} = 2.0$
- Using the table, the area between 50 and 60 for Z=2.0 is 0.4772. So,

$$P[50 \le X \le 60] = 0.4772$$

The Normal as an Approximation to the Poisson Distribution

- The normal can approximate the Poisson if the $\lambda > 5$.

- Poisson: $m = l$
  $s = \sqrt{l}$

- Transformation: $Z = \frac{X - l}{\sqrt{l}}$

The Exponential Distribution

- Widely used in queuing systems to model the inter-arrival time between requests to a system.
- If the inter-arrival times are exponentially distributed then the number of arrivals in an interval *t* has a Poisson distribution and vice-versa.

$$f_X(x) = l e^{-l.x} \qquad F_X(x) = 1 - e^{-l.x} \qquad x \geq 0$$

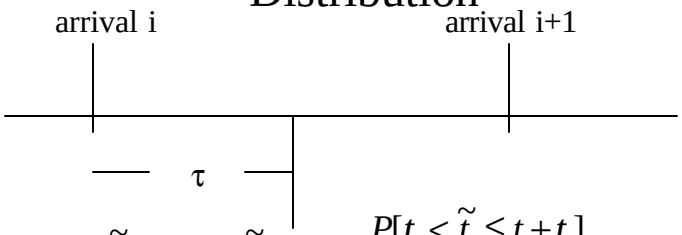
The Exponential Distribution

- Mean and Standard Deviation:

$$m = s = 1/l$$

- The coefficient of variation is 1. The exponential is the only continuous r.v. with this property.
- The exponential distribution is "memoryless." The distribution of the residual time until the next arrival is also exponential with the same mean as the original distribution.

Memoryless Property of the Exponential Distribution



$$P[\tilde{t} \le t+t \mid \tilde{t} > t] = \frac{P[t < \tilde{t} \le t+t]}{P[\tilde{t} > t]}$$

$$= \frac{P[\tilde{t} \le t+t] - P[\tilde{t} \le t]}{P[\tilde{t} > t]}$$

$$= \frac{1-e^{-l.(t+t)} - (1-e^{-l.t})}{1-(1-e^{-l.t})}$$

$$= 1-e^{-l.t}$$

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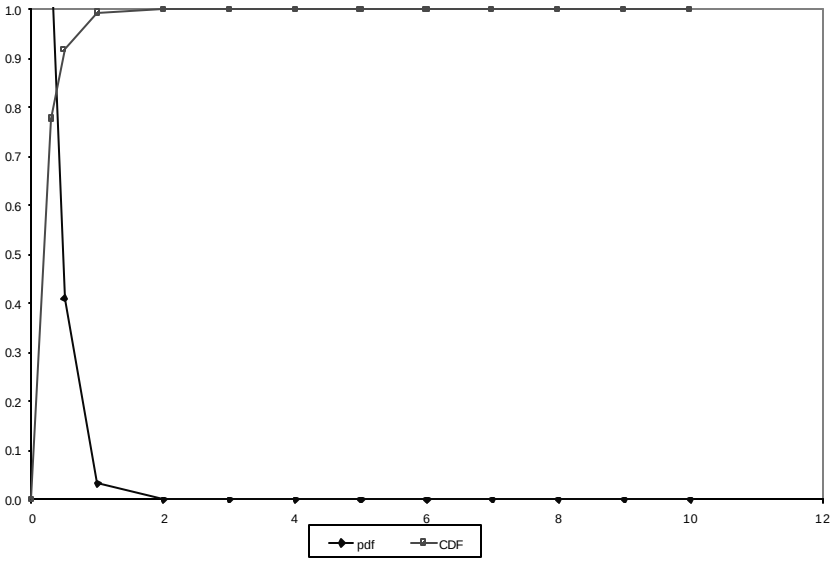
Exponential Distribution



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Pareto Distribution

- A case of a heavy-tailed distribution.
- The probability of large values is not negligible.

$$f_X(x) = \frac{a}{x^{1+a}} \qquad a > 0 \quad , \quad x \geq 1$$

$$F_X(x) = 1 - \frac{1}{x^a} \qquad a > 0 \quad , \quad x \geq 1$$

- Mean: $\frac{a}{a-1} \qquad a > 1$
- Variance: $\frac{a}{(a-1)^2(a-2)} \qquad a > 2$

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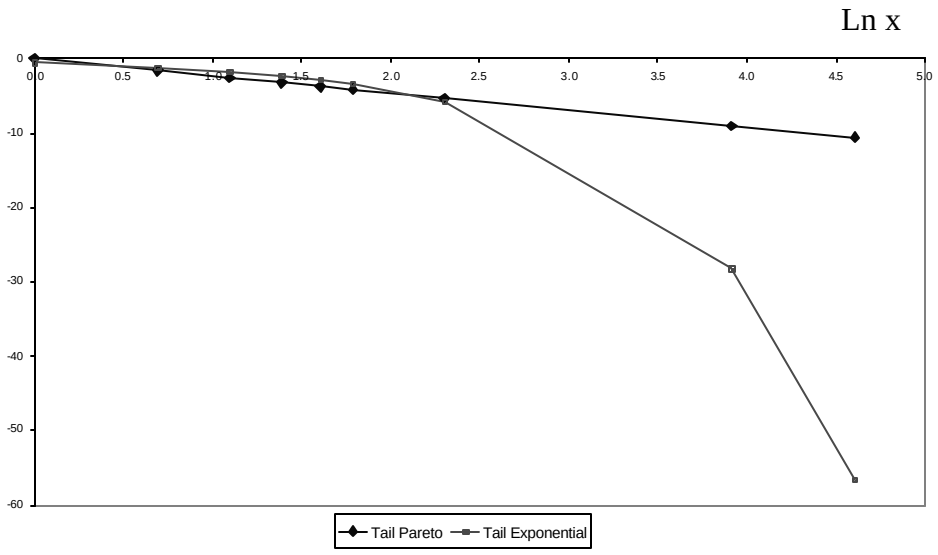
Tail of the Pareto and Exponential Distributions



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Sample Statistics

Sample Statistics

- Sample Mean

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Sample Variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2$$

File	File Size (KB)
1	2.3
2	3.7
3	10.4
4	2.2
5	7.3
6	102.0
7	2.9
8	4.0
9	30.0
10	1.2
11	3.4
12	20.0
13	3.5
14	9.0
15	2.8

$\overline{x} = 13.65$

$s^2 = 659.59$

$s = 25.68$

Confidence Intervals

- When analyzing measurements one cannot make a definite statement such as “the average measured response time is 0.65 sec.”
- What we can say is something to the effect of “with 90% confidence, the measured response time is in the interval (0.62,0.68).”
- The interval (0.62, 0.68) is the confidence interval and 90% is the confidence level.

Confidence Interval Estimation of the Mean

- Known population standard deviation.
- Unknown population standard deviation:
  - Large samples (n > 30): sample standard deviation is a good estimate for population standard deviation. OK to use normal distribution.

Central Limit Theorem

- If the observations in a sample are independent and come from the same population that has mean μ and standard deviation σ then the sample mean for **large** samples has a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$.

$$\bar{x} \sim N(m, s/\sqrt{n})$$

- The standard deviation of the sample mean is called the *standard error.*

Computing Confidence Intervals

- For large samples (n > 30).

  A 100 (1- α)% confidence interval for the population mean is

$$(\bar{x} - z_{1-a/2}\, s/\sqrt{n},\ \bar{x} + z_{1-a/2}\, s/\sqrt{n})$$

  where $z_{1-a/2}$ is the (1-α/2)-quantile of a N(0,1)

$$z_{1-a/2} = \text{NORMSINV}(1-a/2)$$

  MS Excel function

Example of Confidence Interval

File	Resp Time	(xi-xbar)*(xi-xbar)
1	0.650	0.00037636
2	0.540	0.00820836
3	0.620	0.00011236
4	0.570	0.00367236
5	0.620	0.00011236
6	0.680	0.00244036
7	0.590	0.00164836
8	0.550	0.00649636
9	0.625	0.00003136
10	0.675	0.00197136
11	0.645	0.00020736
12	0.673	0.00179776
13	0.667	0.00132496
14	0.700	0.00481636
15	0.654	0.00054756
xbar	0.6306	0.03376360
s2	0.002411686	
s	0.049108917	

	Interval	
90%	0.609742	0.651458
95%	0.605773	0.655427
99%	0.597937	0.663263

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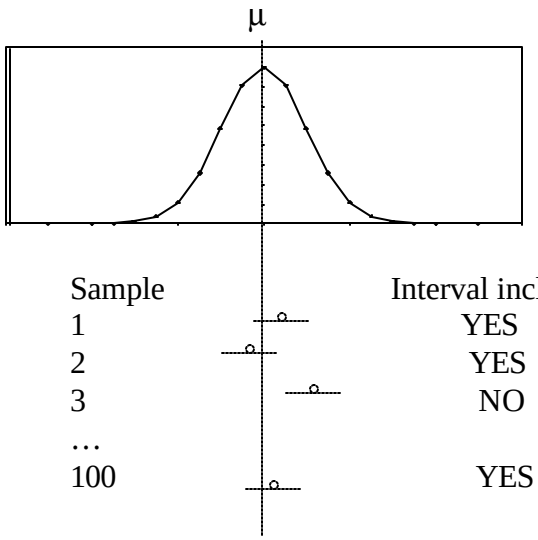

100 (1 – α) of the 100 samples include the population mean μ.

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Determining Sample Size

- Large samples imply high confidence.
- Large samples require more data collection effort.
- How to determine the sample size *n* to estimate the population parameter with accuracy *r*% and confidence level of *100 (1-a)%*?

Determining the Sample Size for the Mean

- Perform a set of measurements to estimate the sample mean and the sample variance.
- Determine the sample to obtain proper accuracy as follows:

$$\bar{x} \pm z \frac{s}{\sqrt{n}} = \bar{x} \pm \frac{\bar{x} r}{100}$$

$$\Rightarrow n = \left(\frac{100 z s}{r \bar{x}} \right)^2$$

Determining the Sample Size for the Mean

- A preliminary test shows that the sample mean of the response time is 5 sec and the sample standard deviation is 1.5. How many repetitions are needed to get the response time within 2% accuracy at 95% confidence level?

$r = 2 \qquad \bar{x} = 5 \qquad s = 1.5$

$z = 1.96$

$$n = \left(\frac{100 \times 1.96 \times 1.5}{2 \times 5} \right)^2 = 864.36$$

865 repetitions would be Needed!

Determining the Sample Size for the Mean

Accuracy (r)	Confidence Level (1-alpha)	X	S	Sample size
1	0.95	5	0.8	984
2	0.95	5	0.8	246
5	0.95	5	0.8	40
1	0.9	5	0.8	693
2	0.9	5	0.8	174
5	0.9	5	0.8	28