Two-Stage Scattering Matrix Design with Direct Link

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I. System Model in the Original Paper

Consider the BD-RIS aided MU-MISO system in [1], where a BS with L antennas serves K single-antenna users through a BD-RIS with N elements. The paper models three channels:

- the direct BS–user channel $\mathbf{G} \in \mathbb{C}^{L\times K}$,
- the BS–RIS channel $\mathbf{E} \in \mathbb{C}^{N\times L}$,
- the RIS–user channel $\mathbf{H} \in \mathbb{C}^{N\times K}$.

The BD-RIS is described by an $N \times N$ scattering matrix $\mathbf{\Theta}$ that is required, for the fully connected architecture, to satisfy the symmetric unitary constraints

$$\mathbf{\Theta} = \mathbf{\Theta}^T,\ \mathbf{\Theta}^H\mathbf{\Theta} = \mathbf{I}_N. \tag{1}$$

Let the effective channel be

$$\mathbf{F}(\mathbf{\Theta}) = \mathbf{G}^H + \mathbf{H}^H\mathbf{\Theta}\mathbf{E} \tag{2}$$

which is exactly the sum of the direct link and the RIS-reflected link. To obtain a passive beamforming matrix that is good for the later active beamforming step, [1] relaxes the original nonconvex constrained problem to the following Frobenius-norm maximization:

$$\max_{\mathbf{\Theta}} \quad f(\mathbf{\Theta}) \triangleq \left\|\mathbf{G}^H + \mathbf{H}^H\mathbf{\Theta}\mathbf{E}\right\|_F^2 \tag{3a}$$

$$\text{s.t.} \quad \|\mathbf{\Theta}\|_F^2 \le N. \tag{3b}$$

This is equation (3) in the paper, written here directly in matrix form. The paper then vectorizes it and reaches the same form. The key observation is that $f(\mathbf{\Theta})$ is smooth and quadratic in $\mathbf{\Theta}$.

II. Gradient in the original model

Define

$$\mathbf{A}(\mathbf{\Theta}) \triangleq \mathbf{G}^H + \mathbf{H}^H\mathbf{\Theta}\mathbf{E}. \tag{4}$$

Then

$$f(\mathbf{\Theta}) = \|\mathbf{A}(\mathbf{\Theta})\|_F^2 = \operatorname{Tr}\left(\mathbf{A}(\mathbf{\Theta})\mathbf{A}(\mathbf{\Theta})^H\right). \tag{5}$$

Because $\mathbf{A}(\mathbf{\Theta})$ is affine in $\mathbf{\Theta}$, differentiating at $\mathbf{\Theta} = \mathbf{0}$ is simple. Let us write explicitly

$$\mathbf{A}(\mathbf{0}) = \mathbf{G}^H, \qquad \left.\frac{\partial \mathbf{A}}{\partial \mathbf{\Theta}}\right|_{\mathbf{\Theta}=\mathbf{0}} : \Delta\mathbf{\Theta} \mapsto \mathbf{H}^H\Delta\mathbf{\Theta}\mathbf{E}. \tag{6}$$

Using standard matrix calculus for a function of the form $\|\mathbf{G}^H + \mathbf{H}^H\mathbf{\Theta}\mathbf{E}\|_F^2$, the gradient at $\mathbf{\Theta} = \mathbf{0}$ is

$$\nabla_{\mathbf{\Theta}} f(\mathbf{0}) = \mathbf{H}\mathbf{G}^H\mathbf{E}^H, \tag{7}$$

which is exactly the expression the paper gives as equation (6) [1].

The paper then chooses the direction

$$\mathbf{D}_0 = \nabla_{\mathbf{\Theta}} f(\mathbf{0}) = \mathbf{H}\mathbf{G}^H\mathbf{E}^H \tag{8}$$

and rescales it to satisfy the sphere constraint $\|\mathbf{\Theta}\|_F^2 \le N$. Finally, because the actual feasible set for a fully connected BD-RIS is the symmetric unitary set, the paper projects this matrix onto that set by the symmetric-unitary projection:

$$\mathbf{\Theta} = \operatorname{symuni}\big(\mathbf{H}\mathbf{G}^H\mathbf{E}^H\big). \tag{9}$$

This is the form reported as equation (15) in the paper [1].

III. Modified system: no direct BS–user channel

Now suppose the direct channel is no longer present. That is, the term $\mathbf{G}^H$ in (**??**) is removed and the users are served solely through the cascaded BS–RIS–user path. The effective channel reduces to

$$\mathbf{F}_{\text{no-dir}}(\boldsymbol{\Theta}) = \mathbf{H}^H \boldsymbol{\Theta} \mathbf{E}. \tag{10}$$

A natural analogue of the relaxed problem (3) is then

$$\max_{\boldsymbol{\Theta}} \quad f_{\text{no-dir}}(\boldsymbol{\Theta}) \triangleq \left\| \mathbf{H}^H \boldsymbol{\Theta} \mathbf{E} \right\|_F^2 \tag{11a}$$

$$\text{s.t.} \quad \|\boldsymbol{\Theta}\|_F^2 \leq N. \tag{11b}$$

A. Why the old gradient cannot be used

If we differentiate (11b) at the same point $\boldsymbol{\Theta}_0 = \mathbf{0}$, we get

$$\mathbf{F}_{\text{no-dir}}(\mathbf{0}) = \mathbf{0} \quad \Rightarrow \quad f_{\text{no-dir}}(\mathbf{0}) = 0. \tag{12}$$

Let us compute its gradient. Define

$$\mathbf{B}(\boldsymbol{\Theta}) \triangleq \mathbf{H}^H \boldsymbol{\Theta} \mathbf{E}, \tag{13}$$

so that

$$f_{\text{no-dir}}(\boldsymbol{\Theta}) = \operatorname{Tr}\big(\mathbf{B}(\boldsymbol{\Theta}) \mathbf{B}(\boldsymbol{\Theta})^H\big). \tag{14}$$

The differential is

$$\mathrm{d} f_{\text{no-dir}} = 2 \operatorname{Re}\left\{ \operatorname{Tr}\big(\mathbf{B}(\boldsymbol{\Theta})^H \, \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}\big) \right\}. \tag{15}$$

At $\boldsymbol{\Theta} = \mathbf{0}$ we have $\mathbf{B}(\mathbf{0}) = \mathbf{0}$, so the whole differential vanishes:

$$\nabla_{\boldsymbol{\Theta}} f_{\text{no-dir}}(\boldsymbol{\Theta})|_{\boldsymbol{\Theta}=\mathbf{0}} = \mathbf{0}_{N \times N}. \tag{16}$$

This is the key difference from the original model. Once we remove $\mathbf{G}$, the origin becomes a flat point and no longer yields a usable direction.

B. Choosing a feasible initial point

A simple way to recover a meaningful direction is to evaluate the gradient at a *feasible* point, not at the origin. A natural feasible point for the fully connected BD-RIS is

$$\boldsymbol{\Theta}_0 = \mathbf{I}_N, \tag{17}$$

which is symmetric and unitary. Evaluate $f_{\text{no-dir}}$ at $\boldsymbol{\Theta}_0$:

$$f_{\text{no-dir}}(\mathbf{I}_N) = \|\mathbf{H}^H \mathbf{E}\|_F^2. \tag{18}$$

Now differentiate $f_{\text{no-dir}}(\boldsymbol{\Theta})$ at $\boldsymbol{\Theta} = \mathbf{I}_N$. Using

$$\mathbf{B}(\boldsymbol{\Theta}) = \mathbf{H}^H \boldsymbol{\Theta} \mathbf{E}, \qquad \mathrm{d}\mathbf{B} = \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}, \tag{19}$$

we have

$$\mathrm{d} f_{\text{no-dir}} = 2 \operatorname{Re}\left\{ \operatorname{Tr}\big(\mathbf{B}(\boldsymbol{\Theta})^H \, \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}\big) \right\} \tag{20}$$

$$= 2 \operatorname{Re}\left\{ \operatorname{Tr}\big(\mathbf{E}^H \boldsymbol{\Theta}^H \mathbf{H} \, \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}\big) \right\}. \tag{21}$$

At $\boldsymbol{\Theta} = \mathbf{I}_N$, this becomes

$$\mathrm{d} f_{\text{no-dir}}|_{\boldsymbol{\Theta}=\mathbf{I}_N} = 2 \operatorname{Re}\left\{ \operatorname{Tr}\big(\mathbf{E}^H \mathbf{H} \, \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}\big) \right\}. \tag{22}$$

By cyclic property of the trace,

$$\operatorname{Tr}\big(\mathbf{E}^H \mathbf{H} \, \mathbf{H}^H (\mathrm{d}\boldsymbol{\Theta}) \mathbf{E}\big) = \operatorname{Tr}\big(\mathbf{H}\mathbf{H}^H \mathbf{E}\mathbf{E}^H (\mathrm{d}\boldsymbol{\Theta})\big). \tag{23}$$

Therefore the gradient at $\boldsymbol{\Theta} = \mathbf{I}_N$ is

$$\nabla_{\boldsymbol{\Theta}} f_{\text{no-dir}}(\boldsymbol{\Theta})|_{\boldsymbol{\Theta}=\mathbf{I}_N} = 2 \, \mathbf{H}\mathbf{H}^H \mathbf{E}\mathbf{E}^H. \tag{24}$$

The scalar factor 2 is irrelevant for the projection step (the paper itself notes that scaling does not change the symmetric-unitary projection [1]. Hence the direction is

$$\mathbf{Z} \triangleq \mathbf{H}\mathbf{H}^H \mathbf{E}\mathbf{E}^H. \tag{25}$$

C. *Low-complexity solution without the direct link*

The paper's logic for the original model was:

1) form a matrix that is proportional to the gradient at the chosen point;
2) scale it to lie on the sphere $\|\boldsymbol{\Theta}\|_F^2 = N$;
3) project it onto the symmetric unitary set by $\mathrm{symuni}(\cdot)$;
4) use the resulting $\boldsymbol{\Theta}$ in the second stage.

We can carry out the same logic here. Since $\mathrm{symuni}(\cdot)$ is homogeneous (i.e. $\mathrm{symuni}(\rho\mathbf{Z}) = \mathrm{symuni}(\mathbf{Z})$ for any $\rho \neq 0$, see Lemma 1 in the paper [1]), we can skip the explicit scaling and directly write the low-complexity passive beamforming matrix as

$$\boldsymbol{\Theta}_{\text{no-dir}} = \mathrm{symuni}\big(\mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H\big). \tag{26}$$

This is the exact analogue of equation (15) in [1], with

$$\mathbf{H}\mathbf{G}^H\mathbf{E}^H \quad \text{replaced by} \quad \mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H. \tag{27}$$

IV. Group-connected case

The paper extends the fully connected solution to the group-connected case by first forcing the matrix to be block diagonal with $N_g \times N_g$ blocks, and then applying $\mathrm{symuni}(\cdot)$ on each block, see its equations (16)–(17) [1]. If we denote by

$$\mathrm{blkdiag}_{N_g}(\mathbf{Z}) \tag{28}$$

the operation that zeroes all off-group entries of $\mathbf{Z}$ (i.e. the Hadamard product with a block mask), then the group-connected analogue of (26) is

$$\boldsymbol{\Theta}_{\text{no-dir}}^{(\text{group})} = \mathrm{diag}\big(\mathrm{symuni}(\mathbf{Z}_1), \ldots, \mathrm{symuni}(\mathbf{Z}_G)\big), \tag{29}$$

where

$$\mathrm{blkdiag}_{N_g}\big(\mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H\big) = \mathrm{diag}(\mathbf{Z}_1, \ldots, \mathbf{Z}_G), \tag{30}$$

$G = N/N_g$, and each $\mathbf{Z}_g \in \mathbb{C}^{N_g \times N_g}$. This is exactly what MATLAB function `group_symuni` does: it masks the matrix to a block-diagonal one and then calls `symuni` on the result. The result of the simulations is shown below:

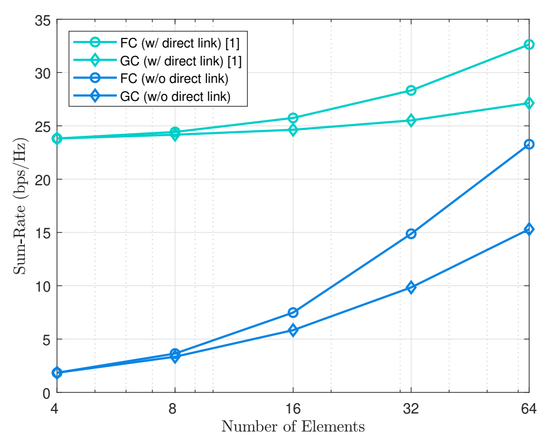

Fig. 1. Comparison of sum-rate across fully-connected "FC", group-connected "GC" BD-RIS architectures when the direct link is present versus when it is not.

V. Discussion

The essential mathematical difference between the original paper's setting and the "no-direct-link" setting is this:

- With $\mathbf{G} \neq \mathbf{0}$, the objective has a nonzero linear term at $\mathbf{\Theta} = \mathbf{0}$, which immediately produces the rank-N matrix $\mathbf{H}\mathbf{G}^H\mathbf{E}^H$ as a "good" direction.
- With $\mathbf{G} = \mathbf{0}$, the objective is purely quadratic in $\mathbf{\Theta}$ and the origin becomes flat. To preserve the same low-complexity flavour, we move to a feasible point (the identity) and take the gradient there, which gives a matrix with the same structure but with channel correlation terms only.

Because the symmetric-unitary projection is a nearest-point projection in Frobenius norm onto the feasible BD-RIS set, using (26) is consistent with the logic of [1]: we take the dominant information embedded in the channels and project it to the physically realisable scattering matrix.

References

[1] T. Fang and Y. Mao, "A low-complexity beamforming design for beyond-diagonal ris aided multi-user networks," *IEEE Communications Letters*, vol. 28, no. 1, pp. 203–207, 2024.