

# Two-Stage Scattering Matrix Design with Direct Link

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## I. SYSTEM MODEL IN THE ORIGINAL PAPER

Consider the BD-RIS aided MU-MISO system in [1], where a BS with  $L$  antennas serves  $K$  single-antenna users through a BD-RIS with  $N$  elements. The paper models three channels:

- the direct BS–user channel  $\mathbf{G} \in \mathbb{C}^{L \times K}$ ,
- the BS–RIS channel  $\mathbf{E} \in \mathbb{C}^{N \times L}$ ,
- the RIS–user channel  $\mathbf{H} \in \mathbb{C}^{N \times K}$ .

The BD-RIS is described by an  $N \times N$  scattering matrix  $\Theta$  that is required, for the fully connected architecture, to satisfy the symmetric unitary constraints

$$\Theta = \Theta^T, \Theta^H \Theta = \mathbf{I}_N. \quad (1)$$

Let the effective channel be

$$\mathbf{F}(\Theta) = \mathbf{G}^H + \mathbf{H}^H \Theta \mathbf{E} \quad (2)$$

which is exactly the sum of the direct link and the RIS-reflected link. To obtain a passive beamforming matrix that is good for the later active beamforming step, [1] relaxes the original nonconvex constrained problem to the following Frobenius-norm maximization:

$$\max_{\Theta} f(\Theta) \triangleq \|\mathbf{G}^H + \mathbf{H}^H \Theta \mathbf{E}\|_F^2 \quad (3a)$$

$$\text{s.t. } \|\Theta\|_F^2 \leq N. \quad (3b)$$

This is equation (3) in the paper, written here directly in matrix form. The paper then vectorizes it and reaches the same form. The key observation is that  $f(\Theta)$  is smooth and quadratic in  $\Theta$ .

## II. GRADIENT IN THE ORIGINAL MODEL

Define

$$\mathbf{A}(\Theta) \triangleq \mathbf{G}^H + \mathbf{H}^H \Theta \mathbf{E}. \quad (4)$$

Then

$$f(\Theta) = \|\mathbf{A}(\Theta)\|_F^2 = \text{Tr} (\mathbf{A}(\Theta) \mathbf{A}(\Theta)^H). \quad (5)$$

Because  $\mathbf{A}(\Theta)$  is affine in  $\Theta$ , differentiating at  $\Theta = \mathbf{0}$  is simple. Let us write explicitly

$$\mathbf{A}(\mathbf{0}) = \mathbf{G}^H, \quad \left. \frac{\partial \mathbf{A}}{\partial \Theta} \right|_{\Theta=0} : \Delta \Theta \mapsto \mathbf{H}^H \Delta \Theta \mathbf{E}. \quad (6)$$

Using standard matrix calculus for a function of the form  $\|\mathbf{G}^H + \mathbf{H}^H \Theta \mathbf{E}\|_F^2$ , the gradient at  $\Theta = \mathbf{0}$  is

$$\nabla_{\Theta} f(\mathbf{0}) = \mathbf{H} \mathbf{G}^H \mathbf{E}^H, \quad (7)$$

which is exactly the expression the paper gives as equation (6) [1].

The paper then chooses the direction

$$\mathbf{D}_0 = \nabla_{\Theta} f(\mathbf{0}) = \mathbf{H} \mathbf{G}^H \mathbf{E}^H \quad (8)$$

and rescales it to satisfy the sphere constraint  $\|\Theta\|_F^2 \leq N$ . Finally, because the actual feasible set for a fully connected BD-RIS is the symmetric unitary set, the paper projects this matrix onto that set by the symmetric-unitary projection:

$$\Theta = \text{symuni}(\mathbf{H} \mathbf{G}^H \mathbf{E}^H). \quad (9)$$

This is the form reported as equation (15) in the paper [1].

### III. MODIFIED SYSTEM: NO DIRECT BS–USER CHANNEL

Now suppose the direct channel is no longer present. That is, the term  $\mathbf{G}^H$  in (??) is removed and the users are served solely through the cascaded BS–RIS–user path. The effective channel reduces to

$$\mathbf{F}_{\text{no-dir}}(\Theta) = \mathbf{H}^H \Theta \mathbf{E}. \quad (10)$$

A natural analogue of the relaxed problem (3) is then

$$\max_{\Theta} f_{\text{no-dir}}(\Theta) \triangleq \|\mathbf{H}^H \Theta \mathbf{E}\|_F^2 \quad (11a)$$

$$\text{s.t. } \|\Theta\|_F^2 \leq N. \quad (11b)$$

#### A. Why the old gradient cannot be used

If we differentiate (11b) at the same point  $\Theta_0 = \mathbf{0}$ , we get

$$\mathbf{F}_{\text{no-dir}}(\mathbf{0}) = \mathbf{0} \Rightarrow f_{\text{no-dir}}(\mathbf{0}) = 0. \quad (12)$$

Let us compute its gradient. Define

$$\mathbf{B}(\Theta) \triangleq \mathbf{H}^H \Theta \mathbf{E}, \quad (13)$$

so that

$$f_{\text{no-dir}}(\Theta) = \text{Tr}(\mathbf{B}(\Theta) \mathbf{B}(\Theta)^H). \quad (14)$$

The differential is

$$df_{\text{no-dir}} = 2 \text{Re} \left\{ \text{Tr}(\mathbf{B}(\Theta)^H \mathbf{H}^H d\Theta \mathbf{E}) \right\}. \quad (15)$$

At  $\Theta = \mathbf{0}$  we have  $\mathbf{B}(\mathbf{0}) = \mathbf{0}$ , so the whole differential vanishes:

$$\nabla_{\Theta} f_{\text{no-dir}}(\Theta)|_{\Theta=0} = \mathbf{0}_{N \times N}. \quad (16)$$

This is the key difference from the original model. Once we remove  $\mathbf{G}$ , the origin becomes a flat point and no longer yields a usable direction.

#### B. Choosing a feasible initial point

A simple way to recover a meaningful direction is to evaluate the gradient at a *feasible* point, not at the origin. A natural feasible point for the fully connected BD-RIS is

$$\Theta_0 = \mathbf{I}_N, \quad (17)$$

which is symmetric and unitary. Evaluate  $f_{\text{no-dir}}$  at  $\Theta_0$ :

$$f_{\text{no-dir}}(\mathbf{I}_N) = \|\mathbf{H}^H \mathbf{E}\|_F^2. \quad (18)$$

Now differentiate  $f_{\text{no-dir}}(\Theta)$  at  $\Theta = \mathbf{I}_N$ . Using

$$\mathbf{B}(\Theta) = \mathbf{H}^H \Theta \mathbf{E}, \quad d\mathbf{B} = \mathbf{H}^H (d\Theta) \mathbf{E}, \quad (19)$$

we have

$$df_{\text{no-dir}} = 2 \text{Re} \left\{ \text{Tr}(\mathbf{B}(\Theta)^H \mathbf{H}^H (d\Theta) \mathbf{E}) \right\} \quad (20)$$

$$= 2 \text{Re} \left\{ \text{Tr}(\mathbf{E}^H \Theta^H \mathbf{H} \mathbf{H}^H (d\Theta) \mathbf{E}) \right\}. \quad (21)$$

At  $\Theta = \mathbf{I}_N$ , this becomes

$$df_{\text{no-dir}}|_{\Theta=\mathbf{I}_N} = 2 \text{Re} \left\{ \text{Tr}(\mathbf{E}^H \mathbf{H} \mathbf{H}^H (d\Theta) \mathbf{E}) \right\}. \quad (22)$$

By cyclic property of the trace,

$$\text{Tr}(\mathbf{E}^H \mathbf{H} \mathbf{H}^H (d\Theta) \mathbf{E}) = \text{Tr}(\mathbf{H} \mathbf{H}^H \mathbf{E} \mathbf{E}^H (d\Theta)). \quad (23)$$

Therefore the gradient at  $\Theta = \mathbf{I}_N$  is

$$\nabla_{\Theta} f_{\text{no-dir}}(\Theta)|_{\Theta=\mathbf{I}_N} = 2 \mathbf{H} \mathbf{H}^H \mathbf{E} \mathbf{E}^H. \quad (24)$$

The scalar factor 2 is irrelevant for the projection step (the paper itself notes that scaling does not change the symmetric-unitary projection [1]. Hence the direction is

$$\mathbf{Z} \triangleq \mathbf{H} \mathbf{H}^H \mathbf{E} \mathbf{E}^H. \quad (25)$$

### C. Low-complexity solution without the direct link

The paper's logic for the original model was:

- 1) form a matrix that is proportional to the gradient at the chosen point;
- 2) scale it to lie on the sphere  $\|\Theta\|_F^2 = N$ ;
- 3) project it onto the symmetric unitary set by  $\text{symuni}(\cdot)$ ;
- 4) use the resulting  $\Theta$  in the second stage.

We can carry out the same logic here. Since  $\text{symuni}(\cdot)$  is homogeneous (i.e.  $\text{symuni}(\rho \mathbf{Z}) = \text{symuni}(\mathbf{Z})$  for any  $\rho \neq 0$ , see Lemma 1 in the paper [1]), we can skip the explicit scaling and directly write the low-complexity passive beamforming matrix as

$$\Theta_{\text{no-dir}} = \text{symuni}(\mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H). \quad (26)$$

This is the exact analogue of equation (15) in [1], with

$$\mathbf{H}\mathbf{G}^H\mathbf{E}^H \text{ replaced by } \mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H. \quad (27)$$

## IV. GROUP-CONNECTED CASE

The paper extends the fully connected solution to the group-connected case by first forcing the matrix to be block diagonal with  $N_g \times N_g$  blocks, and then applying  $\text{symuni}(\cdot)$  on each block, see its equations (16)–(17) [1]. If we denote by

$$\text{blkdiag}_{N_g}(\mathbf{Z}) \quad (28)$$

the operation that zeroes all off-group entries of  $\mathbf{Z}$  (i.e. the Hadamard product with a block mask), then the group-connected analogue of (26) is

$$\Theta_{\text{no-dir}}^{(\text{group})} = \text{diag}(\text{symuni}(\mathbf{Z}_1), \dots, \text{symuni}(\mathbf{Z}_G)), \quad (29)$$

where

$$\text{blkdiag}_{N_g}(\mathbf{H}\mathbf{H}^H\mathbf{E}\mathbf{E}^H) = \text{diag}(\mathbf{Z}_1, \dots, \mathbf{Z}_G), \quad (30)$$

$G = N/N_g$ , and each  $\mathbf{Z}_g \in \mathbb{C}^{N_g \times N_g}$ . This is exactly what MATLAB function `group_symuni` does: it masks the matrix to a block-diagonal one and then calls `symuni` on the result.

## V. DISCUSSION

The essential mathematical difference between the original paper's setting and the “no-direct-link” setting is this:

- With  $\mathbf{G} \neq 0$ , the objective has a nonzero linear term at  $\Theta = 0$ , which immediately produces the rank- $N$  matrix  $\mathbf{H}\mathbf{G}^H\mathbf{E}^H$  as a “good” direction.
- With  $\mathbf{G} = 0$ , the objective is purely quadratic in  $\Theta$  and the origin becomes flat. To preserve the same low-complexity flavour, we move to a feasible point (the identity) and take the gradient there, which gives a matrix with the same structure but with channel correlation terms only.

Because the symmetric-unitary projection is a nearest-point projection in Frobenius norm onto the feasible BD-RIS set, using (26) is consistent with the logic of [1]: we take the dominant information embedded in the channels and project it to the physically realisable scattering matrix.

## REFERENCES

- [1] T. Fang and Y. Mao, “A low-complexity beamforming design for beyond-diagonal ris aided multi-user networks,” *IEEE Communications Letters*, vol. 28, no. 1, pp. 203–207, 2024.