Chapter 2

Exercise 2.1

Prove that the empty set is a subset of every set.

Let A be any set, $\emptyset$ be the empty set and assume the opposite. Then there exists at least one element $x \in \emptyset$ such that $x \notin A$. But the empty set does not contain any elements and no such x can therefore be found. This is a contradiction and hence $\emptyset \subset A$ must be true.

■

Exercise 2.2

Prove that the set of all algebraic numbers is countable.

Let $\alpha = (a_0, ..., a_n) \in B_{n+1}$ be an n+1-tuple of integers. Example 2.5 shows that the set of all integers $\mathbb{Z}$ is countable. Using Theorem 2.13 with $A = \mathbb{Z}$ shows that the set of n+1-tuples B_{n+1} is countable.

For particular choice of integers $a_0, ..., a_n$ we may construct an equation of the form

$$a_n z^n + \cdots + a_1 z + a_0 = 0.$$

This polynomial has at most n solutions. Let E_α be the set of all complex numbers z that is a solution to that polynomial. Hence E_α is a finite subset of $\mathcal{A}$ with at most n elements. If S_n is a set such that

$$S_n = \bigcup_{\alpha \in B_{n+1}} E_\alpha$$

then it is the union of a countable collection of finite sets. It follows from Theorem 2.12 that S_n is at most countable. Since the union of a countable collection of at most countable sets is at most countable (follows from Theorem 2.12 again); the union

$$S = \bigcup_{n=1}^{\infty} S_n,$$

is at most countable.

We shall show that $\mathcal{A} \subset S$. For any $z \in \mathcal{A}$ there exist integers $a_0, ..., a_k$ such that $a_k z^k + \cdots + a_1 z + a_0 = 0$. For these integers there is a k+1-tuple $\alpha = (a_0, ..., a_k)$ which is associated with exactly one set E_α. Hence there exists at least one α for which $z \in E_\alpha$ and therefore it follows that $\mathcal{A} \subset S$. This means that $\mathcal{A}$ is at most countable since it is a subset of S.

Lastly we need to show that $\mathcal{A}$ is infinite. The set of rational numbers $\mathbb{Q}$ is countable by the corollary to Theorem 2.13. For any $q \in \mathbb{Q}$ there are integers m, n such that $q = \frac{n}{m}$. Now choose $a_1 = m$ and $a_0 = -n$, then

$$a_1 q + a_0 = m\frac{n}{m} - n = 0,$$

which means that q is algebraic. Since q is arbitrary it follows that $\mathbb{Q} \subset \mathcal{A}$. This shows that $\mathcal{A}$ has an infinite subset and must therefore itself be infinite. $\mathcal{A}$ is an infinite set that is at most countable, and therefore it is countable as desired.

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Exercise 2.3

Prove that there are real numbers which are not algebraic.

Suppose not. Let $\mathcal{A}$ be the set of all algebraic numbers. Since we assume the opposite any real number is algebraic and it follows that $\mathbb{R} \subset \mathcal{A}$.

From Exercise 2.2 we know that $\mathcal{A}$ is countable. By Theorem 2.8 this would entail that $\mathbb{R}$ is countable, since it is an infinite subset of a countable set. But this is a contradiction since Theorem 2.43 shows that $\mathbb{R}$ is uncountable. Hence our assumption is false and there exists real numbers $x \in \mathbb{R}$ such that $x \notin \mathcal{A}$.

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Exercise 2.4

Is the set of all irrational real numbers countable?

Denote the set of irrational numbers by $\mathbb{I}$. According to Theorem 2.13 $\mathbb{Q}$ is countable. Now suppose $\mathbb{I}$ is countable. We already know that

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I},$$

which means that $\mathbb{R}$ is countable by Theorem 2.12. But this is a contradiction since by Theorem 2.43 the set of real numbers $\mathbb{R}$ is uncountable. If $\mathbb{I}$ is assumed to be finite, then that still would make $\mathbb{R}$ countable. Hence, $\mathbb{I}$ must be uncountable.

■

Exercise 2.5

> Construct a bounded set of real numbers with exactly three limit points.

Let k be a natural number and consider the set E_k of numbers $k + \frac{1}{n}$ where $n = 1, 2, 3, \ldots$. Clearly $E_k \subset (k, k+1]$ and is therefore bounded.

We need to show that E_k does not contain any limit points. For any natural numbers n, m we have that

$$d\left(k + \frac{1}{n}, k + \frac{1}{m}\right) = \left|k + \frac{1}{n} - \left(k + \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = d\left(\frac{1}{n}, \frac{1}{m}\right).$$

For a fixed n we want to show that

$$d\left(\frac{1}{n}, \frac{1}{m}\right) \geq d\left(\frac{1}{n}, \frac{1}{n+1}\right),$$

holds with any natural number $m \neq n$.

If $m \geq n+1$, then

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \geq \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right).$$

If $m \leq n-1$, then

$$\begin{aligned} d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| &= \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{n(n-1)} > \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\ &= \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right), \end{aligned}$$

which gives us the result. Hence any neighborhood $N_r\left(k + \frac{1}{n}\right)$ with radius $0 < r < \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$ will contain no other points of E_k except $k + \frac{1}{n}$. This means that $k + \frac{1}{n}$ cannot be a limit point of E_k.

We now show that $k \notin E_k$ is a limit point of E_k. For any radius $r > 0$ we can find a natural number N such that $\frac{1}{N} < r$ by using the Archemdian property of $\mathbb{R}$ (Thm. 1.20)

$$d\left(k, k + \frac{1}{N}\right) = \left|k - \left(k + \frac{1}{N}\right)\right| = \frac{1}{N} < r,$$

which means that $k + \frac{1}{N} \in N_r(k)$.

The desire is to construct a set that is bounded with exactly three limit points. We accomplish this by choosing the set E such that

$$E = E_1 \cup E_2 \cup E_3.$$

Specifically, the limit points to E are $1, 2, 3$ and E_1, E_2, E_3 are all bounded which makes E bounded as well.

■

Exercise 2.6

Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

We first prove that E' is closed. If x is a limit point of E' then for any neighborhood $N_r(x)$ with radius $r > 0$ there exists a $y \in E'$ such that $y \in N_r(x)$. It follows then that the distance is $d(x, y) = h$ for some real number $0 < h < r$. Since $y \in E'$ we know that y is a limit point of E. Hence, we can find a $z \in E$ such that $z \in N_{r-h}(y)$. We have that

$$d(x, z) \leq d(x, y) + d(y, z) < h + r - h = r,$$

which means that $z \in N_r(x)$ so that $x \in E'$.

Next we show that $\overline{E}$ and E have the same limit points. If x is a limit point of $\overline{E}$, then since $\overline{E} = E' \cup E$, it must be the case that x is a limit point of E' or E. Assuming x is a limit point of E leaves us with nothing to prove. So we suppose that x is a limit point of E' alone. We already have established that E' is closed, and therefore $x \in E'$. This directly implies that x is a limit point of E because the members of E' are all limit points of E.

Conversely, let y be a limit point of E. Since $E \subset \overline{E}$, it follows that y is a limit point of $\overline{E}$.

Lastly, we show that E and E' do not always have the same limit points. Let $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ and note that 0 is the only limit point of E. This means that $E' = \{0\}$. We know by Corollary to Theorem 2.20 that E' has no limit points. This shows the result by example.

■

Exercise 2.7

Let $A_1, A_2, A_3, ...$ be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\overline{B}_n = \cup_{i=1}^{n} \overline{A}_i$.

(b) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A}_i$. Show, by an example, that this inclusion can be proper.

(a) We first need to show that for any sets A and B it is true that $(A \cup B)' = A' \cup B'$. If $x \in (A \cup B)'$, then for every neighborhood $N_r(x)$ with a radius $r > 0$ there exists at least one point $p \in N_r(x)$ such that $p \in A \cup B$. Thus, $p \in A$ or $p \in B$ which implies that $x \in A'$ or $x \in B'$ so that $x \in A' \cup B'$.

Suppose now that $x \in A' \cup B'$, then for any neighborhood $N_r(x)$ with a radius $r > 0$ there is a point $p \in N_r(x)$ such that $p \in A$ or $p \in B$. This means that $p \in A \cup B$ and it follows that $x \in (A \cup B)'$.

We now turn to the question at hand. This will be a proof by induction. The base case for $n = 1$ is clearly true. Assume the statement holds for n, then

$$\begin{aligned}
\overline{B}_{n+1} &= \overline{\bigcup_{i=1}^{n+1} A_i} = \overline{\bigcup_{i=1}^{n} A_i \cup A_{n+1}} = \overline{B_n \cup A_{n+1}} \\
&= B_n \cup A_{n+1} \cup \left(B_n \cup A_{n+1}\right)' \\
&= B_n \cup B_n' \cup A_{n+1} \cup A_{n+1}' \\
&= \overline{B}_n \cup \overline{A}_{n+1} = \bigcup_{i=1}^{n} \overline{A}_i \cup \overline{A}_{n+1} \\
&= \bigcup_{i=1}^{n+1} \overline{A}_i,
\end{aligned}$$

where we have used the induction hypothesis in the penultimate equality.

(b) If $x \in \bigcup_{i=1}^{\infty} \overline{A}_i$, then there is an $n \in \mathbb{N}$ such that $x \in \overline{A}_n$. We have that

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} \supset \overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A}_i \supset \overline{A}_n,$$

where we used the result in (a) to get the second equality. This implies that $x \in \overline{B}$. Since x is arbitrary it follows that $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A}_i$ as desired.

We shall conclude by showing that this subset can indeed be proper. Let $A_i = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{i}\right\}$ and note that there exists no $k \in \mathbb{N}$ such that $0 \in A_k$. Thus $0 \notin \bigcup_{i=1}^{\infty} A_i$. Furthermore, each set A_i has no limit points which is why $\overline{A}_i = A_i$, and therefore $\bigcup_{i=1}^{\infty} \overline{A}_i = \bigcup_{i=1}^{\infty} A_i$. It follows then that $0 \notin \bigcup_{i=1}^{\infty} \overline{A}_i$. Now let's consider $\overline{B}$

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\},$$

which implies that $0 \in \overline{B}$. Since $0 \notin \bigcup_{i=1}^{\infty} \overline{A}_i$ we've shown that the inclusion can be proper.

■

Exercise 2.8

> Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

We first show that every point in every open set $E \subset \mathbb{R}^2$ is a limit point of E. Suppose not. Then there exists an open set $E \subset \mathbb{R}^2$ with a point $x \in E$ that is not a limit point of E. It follows that there exists an $r > 0$ such that the neighborhood $N_r(x)$ contains no point of E except x.

Since E is an open set, x must be an interior point. Hence, there is a neighborhood with radius $s > 0$ such that $N_s(x) \subset E$. Neighborhoods in $\mathbb{R}^2$ are non-empty interiors of circles, which means that N_r is circle centered at x.

Consider the cases for r and s:

1. If $r > s$ then N_s is a circle inscribed in a larger circle N_r. This implies $N_s \subset N_r$. Consequently, there exists points $y \in N_s \subset N_r \subset E$ where $y \neq x$. However, this contradicts our assumption that $N_r(x)$ contains no points of E other than x.
2. If $r \leq s$, then N_r is inscribed in N_s. Given that N_s is non-empty circle such that $N_s \subset E$, it follows that N_s contains points of E arbitrarily close to its center point x. In particular, for any distance $0 < d < r$, there are points $y \in E$ where $y \neq x$ such that $y \in N_s$. Since $N_r \subset N_s$, these y must be a point of N_r because $d < r$. This contradicts our assumption that x is not a limit point of E.

Since none of the relations $r < s$, $r = s$ and $r > s$ can be true, such an r cannot exists. We've reached a contradiction and the assumption is wrong. Hence, each point of E is a limit point of E.

Now we show that the case does not hold true in general for closed sets. Consider the set of all natural numbers $\mathbb{N} \subset \mathbb{R}^2$. This set is closed, yet every point in this set is not a limit point of it.

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Exercise 2.9

> Let E° be the set of all interior points of E.
> (a) Prove that E° is always open.
> (b) Prove that E is open if and only if $E^\circ = E$.
> (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
> (d) Prove that the complement of E° is the closure of the complement of E.
> (e) Do E and $\overline{E}$ always have the same interiors?
> (f) Do E and E° always have the same closures?

(a) If $p \in E^\circ$ then p is an interior point of E. This means that there exists some neighborhood $N(p)$ with $r > 0$ such that $N(p) \subset E$. By Theorem 2.19 $N(p)$ is an open set, implying that every point in $N(p)$ is an interior point. Hence, $N(p) \subset E^\circ$ and it follows E° is an open set.

(b) If E is open then for every point $x \in E$ we can find a neighborhood $N(x)$ with $r > 0$ such that $N \subset E$. This implies that every point $x \in E$ is an interior point of E and it is therefore true that $x \in E^\circ$. This shows that $E \subset E^\circ$. Since by construction $E^\circ \subset E$ we have that $E^\circ = E$.

Conversely, if $E^\circ = E$ then it follows from (a) that E is open.

(c) If G is open, then for every point $p \in G$ we can find a neighborhood $N(p)$ with $r > 0$ such that $N(p) \subset G$. Since $G \subset E$, we have that $N(p) \subset E$, which shows that p is an interior point of E. Hence, $p \in E^\circ$. Therefore, $G \subset E^\circ$ since p was arbitrary chosen from G.

(d) Let x be any point in $(E^\circ)^c$. First we show that $(E^\circ)^c \subset \overline{E^c}$. Suppose that $x \notin E$. Then

$$x \in E^c \subset E^c \cup E^{c'} = \overline{E^c}.$$

Now let $x \in E$. Since x is in the complement of E°, we know that x is not an interior point of E. Therefore, for every neighborhood $N(x)$ with radius $r > 0$, we have that $N(x) \not\subset E$. This means that $N(x)$ always has points in E^c, which makes x a limit point of E^c. Thus, $x \in E^{c'} \subset \overline{E^c}$. This shows that $(E^\circ)^c \subset \overline{E^c}$.

Conversely, let $p \in \overline{E^c} = E^c \cup E^{c'}$. Then it is clear that either $p \in E^c$ or $p \in E^{c'}$. Assume $p \in E^c$. Then we know that p is not an interior point to E, so $p \notin E^\circ$, which implies that $p \in (E^\circ)^c$. If $p \in E^{c'}$, then it is a limit point of E^c, and therefore every neighborhood $N(p)$ with radius $r > 0$ have points (other than p) from E^c. Hence, there is no neighborhood such that $N(p) \subset E$ which means that p cannot be an interior point of E. Therefore, $p \in (E^\circ)^c$, and we have now shown that $\overline{E^c} \subset (E^\circ)^c$.

(e) Let $E = (0, 1) \cup (1, 2)$ be a set consisting of line segments in $\mathbb{R}$. Since E is open we have that $E^\circ = E$ by (b). The set of limit points E' is the closed interval $[0, 2]$, so the closure is $\overline{E} = E \cup E' = [0, 2]$. From here we can see that $(\overline{E})^\circ = (0, 2)$, which shows that $E^\circ \neq (\overline{E})^\circ$. Therefore, E and $\overline{E}$ do not always have the same interiors.

(f) Consider the set $E = \left\{\frac{1}{n} \mid n = 1, 2, 3, ...\right\}$ as subset of $\mathbb{R}$. Since 0 is the only limit point of E, we have that its closure is $\overline{E} = E \cup \{0\}$. However, none of the points in E are interior points, and therefore $E^\circ =$

$\emptyset$. This means that $\overline{E^\circ} = \emptyset$ so that $\overline{E} \neq \overline{E^\circ}$. We have shown that E and E° do not always have the same closures.

■

Exercise 2.10

Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

We begin by showing that d is a metric. Conditions 2.15 (a) and (b) are clearly satisfied. We show (c) also is true. Suppose $p \neq q$ so that $d(p,q) = 1$. For any $r \in X$ we have two cases, either r equals one of p, q or neither.

Suppose WLOG that $r = p$. Then we have that $d(p,r) = 0$ and $d(r,q) = 1$. It follows that

$$\underbrace{d(p,q)}_{=1} \leq \underbrace{d(p,r)}_{=0} + \underbrace{d(r,q)}_{=1} = 1.$$

Now let $r \neq p \neq q$ be true. Then $d(p,r) = d(r,q) = 1$, and we can show that

$$\underbrace{d(p,q)}_{=1} < \underbrace{d(p,r)}_{=1} + \underbrace{d(r,q)}_{=1} = 2.$$

Lastly, if $p = q$ then $d(p,q) = 0$ and any point $r \in X$ will satisfy (c) since the distance function is non-negative. This shows that X is a metric with distance function d.

We shall now show that any subset $E \subset X$ is open. Let p be any point in E and consider the neighborhood $N_r(p)$ with radius $r = 1$. Given the metric, this neighborhood contains only p itself so that $N_r(p) \subset E$. Hence, every point in E is an interior point and E is open.

We now show that every subset in X is closed. Let E be any subset of X. Since every subset of X is open, the complement E^c is also open. By Thm. 2.33 E is closed.

Lastly, we show that only finite sets of X are compact (by Definition 2.32). Suppose not, then we have a compact set $K \subset X$ that is infinite. For every $p \in K$ let G_p be the open neighborhood around p with radius $r = 1$. Since every $p \in K$ is associated with an open set G_p, the collection

$\{G_p\}$ is an open cover of K. Because K is compact, there exists a finite number of indices such that

$$K \subset G_{p_1} \cup \cdots \cup G_{p_m}.$$

Every subset G_{p_n} is an open neighborhood around p_n with radius 1. From before we know that these sets only contain a single point, namely p_n. But that is absurd, since it would make K finite.

■

Exercise 2.11

For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned}
d_1(x,y) &= (x-y)^2, \\
d_2(x,y) &= \sqrt{|x-y|}, \\
d_3(x,y) &= \left|x^2 - y^2\right|, \\
d_4(x,y) &= (x-2y), \\
d_5(x,y) &= \frac{|x-y|}{1+|x-y|}.
\end{aligned}$$

Determine, for each of these, whether it is a metric or not.

d_1 Condition 2.15 (c) is not satisfied, which therefore is not a metric. We give an example,

$$d_1(10,0) > d_1(10,4) + d_1(4,0).$$

d_2 Both 2.15 (a) and (b) are clearly true. We show that (c) is also satisfied. Assume not, then there exists points x, y and r such that

$$d_2(x,y) > d_2(x,r) + d_2(r,y),$$

which in this particular case is

$$\sqrt{|x-y|} > \sqrt{|x-r|} + \sqrt{|r-y|}.$$

If $0 < q < p$ then $q^2 < p^2$ for any $p, q \in \mathbb{R}$ so that

$$\begin{aligned}
|x-y| = \sqrt{|x-y|}^2 &> \left(\sqrt{|x-r|} + \sqrt{|r-y|}\right)^2 \\
&= |x-r| + 2\sqrt{|x-r||r-y|} + |r-y| \\
&\geq |x-r| + |r-y|,
\end{aligned}$$

where the last inequality comes from the fact that $\sqrt{|x-r||r-y|} \geq 0$. The above shows that $d(p,q) = |p-q|$ cannot be a metric. But that is a contradiction, since Theorem 1.37 shows that $|p-q|$ satisfies 2.15 (c). Hence, d_2 is a metric.

d_3 Condition 2.15 (a) is not satisfied. We show an example

$$d_3(2,-2) = \left|2^2 - (-2)^2\right| = 4 - 4 = 0.$$

We have found $x \neq y$ such that $d_3(x,y) = 0$. This shows that $d_3(x,y)$ is not a metric.

d_4 We show that d_4 is not a metric since condition (a) of Definition 2.15 is not satisfied. If $x \neq 0$ then

$$d_4(x,x) = |x - 2x| = |x| > 0.$$

d_5 Conditions in (a), (b) are clearly satisfied so we focus on (c). Throughout this exercise WLOG assume $x \leq y$. Suppose $x \leq r \leq y$, we have that

$$\begin{aligned}
d_5(x,y) = \frac{|x-y|}{1+|x-y|} &= \frac{|x-r+r-y|}{1+|x-y|} \\
&\leq \frac{|x-r|}{1+|x-y|} + \frac{|r-y|}{1+|x-y|} \\
&\leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|} \\
&\leq d_5(x,r) + d_5(r,y),
\end{aligned}$$

where we used Theorem 1.37 in the first inequality. Since $x \leq r \leq y$ we know that $|x-r| \leq |x-y|$ and $|r-y| \leq |x-y|$, both of which we used to get the second inequality.

Now assume $x \leq y < r$. First we show that for any $\varepsilon > 0$

$$\frac{|w|}{1+|w|} < \frac{|w|+\varepsilon}{1+|w|+\varepsilon}, \tag{1}$$

holds. The statement is clearly true for $w = 0$. If $|w| > 0$, then

$$\frac{|w|}{1+|w|} = \frac{1}{1+\frac{1}{|w|}} < \frac{1}{1+\frac{1}{|w|+\varepsilon}} = \frac{|w|+\varepsilon}{1+|w|+\varepsilon},$$

which shows that (1) is true.

Since $x \le y < r$, we have that $|x-y| < |x-r|$. This means we can use (1) to get the first inequality below,

$$\begin{aligned} d_5(x,y) = \frac{|x-y|}{1+|x-y|} &< \frac{|x-r|}{1+|x-r|} \\ &\le \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|} \\ &= d_5(x,r) + d_5(r,y). \end{aligned}$$

The last inequality is due to the last term being non-negative. Similar argument can be made for $r < x \le y$ because $|x-y| < |r-y|$ which allows us to use (1) again. This shows that $d_5(x,y)$ is a metric.

■

Exercise 2.12

> Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, ...$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Let $\{G_\alpha\}$ be any open cover of K. Then we know that there is an index α_0 such that $0 \in G_{\alpha_0}$. Since G_{α_0} is an open set, we know that there exists a neighborhood $N_r(0)$ with radius $r > 0$ where $N_r \subset G_{\alpha_0}$.

If $r \ge 1$, then G_{α_0} covers K and there is nothing to prove. Assume therefore that $r < 1$. By Archimedean property of $\mathbb{R}$ we can find positive integers p such that $r \ge 1/p$. Let m be the smallest integer such that $r > 1/m$. It follows that if q is an integer where $q > m$, then $r > 1/q$ so that $1/q \in N_r(0)$. This shows that there are at most $m-1$ points in E that are not in N_r,

$$\frac{1}{m-1}, \frac{1}{m-2}, ..., \frac{1}{2}, 1.$$

Let G_{α_k} denote an open set in the collection $\{G_\alpha\}$ such that $1/k \in G_{\alpha_k}$ $(k = 1, 2, 3, ...)$. Because $\{G_\alpha\}$ is an open cover of K, each of the above $m-1$ points can be associated this way to at least one index (not necessarily distinct) in the collection. Therefore

$$K \subset G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_{m-1}}.$$

We have shown that any open cover of K has a finite sub-cover which implies that K is compact as desired.

■

Exercise 2.13

> Construct a compact set of real numbers whose limit points form a countable set.

Let n be a natural number and construct a sequence $\{x_k\}$ as follows. Define x_1 as the midpoint between $\frac{1}{n+1}$ and $\frac{1}{n}$

$$x_1 = \frac{\frac{1}{n+1} + \frac{1}{n}}{2}.$$

Having chosen $x_1, ..., x_{k-1}$ (for $k = 2, 3, 4, ...$), define x_k as the midpoint between x_{k-1} and $\frac{1}{n}$

$$x_k = \frac{x_{k-1} + \frac{1}{n}}{2}.$$

Let E_n be the set of all numbers generated by the sequence $\{x_k\}$, along with the point $\frac{1}{n}$. For example, if $n = 1$ we have

$$E_1 = \left\{1, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, ...\right\}.$$

Since the sequence $\{x_k\}$ is constructed by repeatedly taking midpoints between the previous term and $\frac{1}{n}$, the sequence will approach the value $\frac{1}{n}$, making it a limit point of E_n.

We can show that any other point $y \in \mathbb{R}$ such that $y \neq \frac{1}{n}$ cannot be a limit point of E_n. Since E_n is a countable set of values that approach $\frac{1}{n}$, there is a number $x_k \neq y$ in E_n that minimises $d(y, x_k)$. Then the neighborhood with $0 < r < d(y, x_k)$ will have no point of E_n (other than possibly y). Hence, $\frac{1}{n}$ is the only limit point so that $E'_n = \left\{\frac{1}{n}\right\}$.

The collection of sets $\{E_n\}$ is disjoint since $E_n \subset \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Let S be the union of this collection along with the point 0,

$$S = \{0\} \cup \bigcup_{n=1}^{\infty} E_n \subset [0, 1].$$

Since S is a union of disjoint sets, the limit points of S is given by

$$S' = \{0\} \cup \bigcup_{n=1}^{\infty} E' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\},$$

which is countable. Furthermore, because $E' \subset E$ and $0 \in S$ we see that $S' \subset S$ so that S is closed. Since S is a subset of the compact interval $[0, 1]$, by Theorem 2.35 S is compact.

As desired, we have constructed a compact set S of real numbers whose limit points S' form a countable set.

■

Exercise 2.14

> Give an example of an open cover of the segment $(0, 1)$ which has no finite sub-cover.

Let G_n be the open set on the form $(1/n, 1)$ for any natural number n. The collection $\{G_n\}$ is an open cover of $(0, 1)$

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (1/n, 1) = (0, 1).$$

Choose finitely many indices $n_1, n_2, ..., n_m$ and let p be the largest index among them. If x is the midpoint between 0 and $\frac{1}{p}$, then $x \notin G_p$. Since $G_{n_k} \subset G_p$ $(k = 1, 2, ..., m)$ we see that

$$x \notin G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_m},$$

yet clearly $x \in (0, 1)$. This shows that the union constructed with indices n_k $(k = 1, 2, ..., m)$ cannot be a sub-cover of $(0, 1)$. The choice of indices were arbitrary, therefore there does not exist a finite sub-cover of $(0, 1)$ using the open cover $\{G_n\}$.

■

Exercise 2.15

> Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}$, for example) if the word "compact" is replaced by "closed" or by "bounded."

We first show that "compact" cannot be replaced by closed. Consider the set of natural numbers starting from the positive integer n and beyond

$$E_n = \{n, n + 1, n + 2, ...\}.$$

Each set in the collection $\{E_n\}$ is closed relative to $\mathbb{R}$ and is unbounded. Since we have that $E_n \supset E_{n+1}$ $(n = 1, 2, 3, ...)$ any intersection of a finite sub-collection of $\{E_n\}$ is non-empty. However, the countable intersection

$$E = \bigcap_{n=1}^{\infty} E_n,$$

is empty. To see this, note that for any natural number m we have that $m \notin E_n$ whenever $m < n$. Hence, there exists no natural number m which belongs to all sets in the collection $\{E_n\}$ and it follows that the intersection is empty.

Now we show that the word "compact" cannot be replaced by "bounded." Let A_n be the set that consists of numbers

$$A_n = \left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots\right\}.$$

This set is bounded by 1 and is not closed since it does not contain its limit point 0. We also have that $A_n \supset A_{n+1}$ for $n = 1, 2, 3, \dots$ and hence any intersection of a finite sub-collection will be non-empty. The countable intersection

$$A = \bigcap_{n=1}^{\infty} A_n,$$

is, however, empty. This can be proven with similar argument as for why the intersection E is empty. We have shown that the word "compact" cannot be replaced by "bounded."

■

Exercise 2.16

> Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

We begin with showing that E is bounded. If $2 < p^2 < 3$ then $1 < |p| < 2$, and indeed we see that E is bounded.

To show that E is closed, consider any rational number $q \notin E$. If $q^2 < 2$ then for any $p \in E$

$$\begin{aligned}\left|q^2 - 2\right| &< \left|p^2 - q^2\right| = |(p+q)(p-q)| \\ &\leq |p+q||p-q| \leq (|p| + |q|)|p-q| \\ &< 4|p-q|,\end{aligned}$$

which shows that $0 < \frac{|q^2-2|}{4} < |p - q|$. Hence, any neighborhood with radius $\frac{|q^2-2|}{4}$ would contain no point of E. If $q^2 > 3$ then

$$\begin{aligned}\left|q^2-3\right| < \left|q^2-p^2\right| &= |(q+p)(q-p)| \\ &\leq (|p|+|q|)|p-q| \\ &< 2|q||p-q|.\end{aligned}$$

Similarly, it follows that any neighborhood around q with radius $\frac{|q^2-3|}{2|q|}$ contains no point of E. This shows that any $q \notin E$ cannot be a limit point of E and hence E must be closed.

To prove that E is not compact in $\mathbb{Q}$, it suffices by Theorem 2.33 to show that E is not compact in $\mathbb{R}$. First, note that E is not closed in $\mathbb{R}$ because it does not contain all of its limit points. In particular, $\sqrt{2}$ is an irrational number that is a limit point of E but not a member of the set. Therefore, by Heine-Borel's theorem (Theorem 2.41) E cannot be compact in $\mathbb{R}$.

Lastly, we answer the question if E is open in $\mathbb{Q}$. Consider the open intervals $A = \left(-\sqrt{3}, -\sqrt{2}\right)$ and $B = \left(\sqrt{2}, \sqrt{3}\right)$ in $\mathbb{R}$. Put $G = A \cup B$ and note that by Theorem 2.24 G is open in $\mathbb{R}$. Since $E = \mathbb{Q} \cap G$, it follows by Theorem 2.30 that E is open in $\mathbb{Q}$.

■

Exercise 2.17

> Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

That E is not countable can be shown using Cantor's diagonal process. Simply exchange 0, 1 with 4, 7 respectively in Theorem 2.14 and its proof. Alternatively, if we can show that E is perfect then by Theorem 2.43 E is uncountable.

E is not dense in $[0,1]$ since 0 is neither a limit point of E, nor a point of E.

Since $E \subset [0,1]$ we know it is a subset of a compact set by Theorem 2.40. According to Theorem 2.35, it suffices to show that E is closed to prove that E is compact. Suppose $x \notin E$ and write the decimal expansion of x using the notation from Section 1.22,

$$x_0 \cdot x_1 x_2 x_3 \ldots$$

Recall that x_0 is the largest integer such that $x_0 \leq x$. We will show that for any $p \in E$ the distance $d = |x-p|$ cannot be less than $2 \cdot \frac{1}{10^{n+1}}$.

Since $x \notin E$ there exists a first integer n such that x_n is neither 4, nor 7. Let p be a point of E that has the same decimal expansion as x up to n

$$p = 0 \cdot p_1 p_2 p_3 ...,$$

so that $p_k = x_k$ for $k = 1, 2, ..., n-1$ and $|x_n - p_n| \geq 1$. It is clear that any other choice of decimals up to n for p would make the distance to x larger. Note that $x_0 = 0$ for otherwise $d \geq 0.222...$ which is larger than $2 \cdot \frac{1}{10^{n+1}}$.

No matter how we choose the rest of the decimals p_i, where $i > n$, the distance $d = |x - p|$ cannot be less than $2 \cdot \frac{1}{10^{n+1}}$. For if $|x_n - p_n| = 1$ and we borrow from it when calculating d, then we see that

$$\left|\min(x_{n+1}, p_{n+1}) + 10 - \max(x_{n+1}, p_{n+1})\right| \geq 3.$$

If we would need to borrow from decimal n+1:th place at some later point during our calculations of d, then the n+1:th decimal place of d would be 2. Since we can borrow at most once from any decimal place when calculating d, we have that

$$d = |x - q| \geq 2 \cdot \frac{1}{10^{n+1}}$$

for any $q \in E$. Therefore, any neighborhood $N(x)$ with radius $r = \frac{1}{10^{n+1}} < d$ would contain no point of E. Hence, E must be closed and it follows by Theorem 2.35 that E is compact.

Now we prove that E is perfect. Since we already have shown that E is closed, we need only to show that no point of E is isolated. For any $q \in E$ let $N(q)$ be some neighborhood around q with radius $r > 0$. Choose any integer $k > 0$ such that $3 \cdot 10^{-k} < r$. Suppose p is a number with the exact same decimal expansion as q except at decimal place k, for which we choose either 4, or 7 such that $p_k \neq q_k$. Clearly, $p \in E$ and we have that

$$|p - q| = |p_k - q_k| \cdot 10^{-k} = 3 \cdot 10^{-k} < r.$$

Thus, q is a limit point of E and we have shown that E is perfect.

■

Exercise 2.18

> Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

Let E be the set from Exercise 2.17. This set is a nonempty perfect set in $\mathbb{R}$ that contains rational numbers. We want to find a number y such that for every $x \in E$ the sum $x + y$ is irrational. Then the set of irrational numbers given by

$$P = \{x + y \mid x \in E\}$$

is perfect. This is due to the fact that the relationships between any points in E are preserved after having moved them all by the same amount y.

We construct a number $y = 0 \cdot y_1 y_2 y_3 ...$ using only the digits 0 and 1. Here we use the notation defined in Section 1.22. The decimals are chosen to be 1s that are spaced by 0s. Specifically, starting with one space of 0 between the first and second 1, then two 0s between the second and third 1, and so on. Below we write the first few digits of the infinite decimal expansion for this number

$$y = 0.101001000100001000001000000100000001000000001000...$$

Intuitively, this number is irrational since it is a non-terminating decimal number that does not contain any repetitions.

We can prove this more formally. Suppose that y is rational. Since y has infinite decimals it must be a repeating decimal. Let i be the first index of the decimal where repetition starts and let ρ be the length of the period. Hence, for any natural number $k \geq i$ it must be true that $x_k = x_{k+m\rho}$ for any positive multiple integer m of ρ, otherwise the decimals would not be repeating.

Since 1 is in the decimal expansion of y, we know that there exists some natural number $j \geq n$ such that $x_j = 1$. We have that

$$x_j = x_{j+\rho} = x_{j+2\rho} = 1.$$

This means that the spacing between 1 is constant ρ, which contradicts the assumption about y and hence y is irrational.

Now that we have shown that our y is irrational, we need only to prove that for any $x \in E$ the number $z = x + y$ is also irrational. Since both x and y are irrational, and cannot be added in such a way to get a terminating decimal expansion, we see that the decimals of z are infinite.

There are two cases we need to verify, whether x is rational or not. If we assume that x is rational, then by Exercise 1.1 the sum $x + y$ is irrational. Assume therefore that x is irrational.

Suppose, to get a contradiction, that the sum $z = x + y$ is rational. Since z is rational with an infinite decimal expansion, it follows that z has a repeating decimal. Let σ be the length of the repetition and z_i the first decimal where the repetition starts.

Take any natural number $k \geq i$ such that $y_k = 1$. Since the number of zeros between the 1s in y are increasing, we know that we can find a natural number b such that $y_{k+b\sigma} = 0$. Now we use the fact that z repeats after the i-th place

$$z_k = x_k + y_k = x_{k+b\sigma} + y_{k+b\sigma}.$$

Plugging in the values for y_k and $y_{k+b\sigma}$ we get the following expression that must be satisfied

$$x_k + 1 = x_{k+b\sigma}. \tag{2}$$

Since $x \in E$ we know that each decimal of x is either 4 or 7. However, the relation (2) cannot be satisfied using any combination of these two digits, and therefore z cannot be a repeating decimal. This contradicts our assumption about z and the sum $z = x + y$ is thus irrational for every $x \in E$.

We have shown that P is a nonempty set of irrational numbers that is perfect as desired.

■

Exercise 2.19

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
(d) Prove that every connected metric space with at least two points is uncountable. *Hint*: Use (c).

(a) If A and B both are closed, then by Theorem 2.27 we have that $\overline{A} = A$ and $\overline{B} = B$. Therefore

$$A \cap \overline{B} = \overline{A} \cap B = A \cap B = \emptyset,$$

since A and B are disjoint.

(b) We prove that the set $A \cap \overline{B}$ is empty. If $p \in B'$, then for every radius $r > 0$ the neighborhood $N_r(p)$ contains points from B. Hence, $N(p) \not\subset A$ and p is not an interior point of A. Because A is open, by definition 2.18 (f) every point of A is an interior point. It follows that $p \notin A$ and therefore $A \cap B' = \emptyset$. Since A and B are disjoint we have that $A \cap \overline{B} = \emptyset$. With the same argument we can establish that $\overline{A} \cap B = \emptyset$. This show that A and B are separated.

(c) By definition 2.18 (a) and Theorem 2.19 A is open. Now consider the complement of B. This set B^c contains all numbers q such that $d(p, q) \leq$

δ. But this is exactly the closure of A, that is $B^c = \overline{A}$. The complement to B is therefore closed and by Theorem 2.23 B is open.

The sets A and B are disjoint. For otherwise there exists some point $q \in A \cap B$ such that $d(p, q) < \delta$ and $d(p, q) > \delta$, which is absurd.

We have shown that A and B are disjoint open sets, hence they are separated using the result in (b).

(d) We prove the contrapositive. Let X be at most countable and arrange the elements in a sequence $\{q_n\}$. If X has less than two points, then X is connected since at least one of the sets A or B will be empty. Hence, assume X has two or more points.

Fix any $p \in X$ and enumerate all distances from this point.

$$\begin{aligned} d(p, q_1) &= d_1 \\ d(p, q_2) &= d_2 \\ &\vdots \end{aligned}$$

Using Cantor's diagonal process we can construct a real number $\delta > 0$ such that $d(p, q_n) \neq \delta$ for all $n \in \mathbb{N}$. We do so in the following manner. Let the n-th decimal of δ be the n-th decimal of d_n that we increment by 1 if d_n's n-th decimal is less than 9, otherwise we decrement by 1. Lastly, we set the integer part of δ to 1.

This procedure will create a non-negative real number $\delta > 0$ that is different to distance d_k in the k-th decimal. Hence, there are no points $q \in X$ such that $d(p, q) = \delta$. Due to this we can write X in terms of the sets $A = \{q \in X \mid d(p, q) < \delta\}$ and $B = \{q \in X \mid d(p, q) > \delta\}$,

$$X = A \cup B.$$

By the result in (c) we have that A and B are separated. It follows therefore that X is not connected.

■

Exercise 2.20

> Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

We prove that closures of connected sets are always connected. Let E be a connected subset of a metric space X. If E is closed, then by Theorem 2.27 we have that $\overline{E} = E$ and it follows that $\overline{E}$ is connected. Assume therefore that E is not closed.

Suppose, for the sake of getting a contradiction, the opposite. Then there exists non-empty separated sets A and B such that $\overline{E} = A \cup B$. Since E is connected, one of the intersections $E \cap A$ or $E \cap B$ must be empty. For otherwise either $A \cap \overline{B}$ or $\overline{A} \cap B$ would be non-empty, because each of these sets would contain parts of the connected set E. And this would contradict the fact that $\overline{E}$ is not connected. Without loss of generalization assume that $E \subset A$.

Now consider the non-empty set B. Since $E \subset A$ and is disjoint with B, there exists at least one point p such that $p \in E' \cap B$. Because $E' \subset \overline{A}$ this would mean that $p \in \overline{A} \cap B$ so that this intersection is non-empty. But this is a contradiction since we assume that $\overline{E}$ is not connected. Hence, closures of connected sets are always connected.

We now turn our attention to interiors of connected sets and show by example that they are not always connected. Let $A = \{(x, y) \in R^2 \mid x > 0\}$ and $B = \{(x, y) \in R^2 \mid x < 0\}$. These are disjoint open sets and are separated by Exercise 19 (b).

If $E = A \cup B \cup \{\mathbf{0}\}$, then E is connected. To see this, note that any partition of E into two subsets F and G would contain points from A. The only way to make sure the intersections $F \cap \overline{G}$ and $\overline{F} \cap G$ are non-empty would be to ensure that $A \cap F$ or $A \cap G$ is empty. This is because A is connected (the same applies for B). Therefore, assume without loss of generalization that $A \subset F$ and $B \subset G$.

Finally, because $\mathbf{0}$ is a limit point to both A and B, no matter where we put this point in F or G, we see that exactly one of the intersections $F \cap \overline{G}$ or $\overline{F} \cap G$ would be non-empty. And so, E is connected.

Now consider the interior of E. The point $\mathbf{0}$ is not an interior point of E and so $E^\circ = A \cup B$. As we know from before A and B are separated. Therefore, interiors of connected sets are not always connected.

■

Exercise 2.21

Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$ and $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.
(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

(a) Assume the opposite, then at least one of the intersections $A_0 \cap \overline{B_0}$ or $\overline{A_0} \cap B_0$ is nonempty. Assume that exists a point $t \in A_0 \cap \overline{B_0}$.

Since $t \in A_0$ it follows that $\mathbf{p}(t) \in A$. Hence, $\mathbf{p}(t) \notin B$ because A and B are separated, and we can conclude that $t \notin B_0$. It must be therefore true that t is a limit point of B_0, that is $t \in A_0 \cap B_0'$.

Let N be a neighborhood around $\mathbf{p}(t)$ with radius $r > 0$. Since t is a limit point of B_0, we can find a point $t' \in B_0$ such that the distance $d(t, t') < \frac{r}{|\mathbf{a}-\mathbf{b}|}$. As for the points $\mathbf{p}(t')$ and $\mathbf{p}(t)$, the distance between them is

$$\begin{aligned} |\mathbf{p}(t') - \mathbf{p}(t)| &= |(1-t')\mathbf{a} + t'\mathbf{b} - (1-t)\mathbf{a} - t\mathbf{b}| \\ &= |(t-t')\mathbf{a} - (t-t')\mathbf{b}| \\ &\le |t'-t| \cdot |\mathbf{a}-\mathbf{b}| \\ &< \frac{r}{|\mathbf{a}-\mathbf{b}|} \cdot |\mathbf{a}-\mathbf{b}| \\ &< r, \end{aligned}$$

which implies that $\mathbf{p}(t)$ is a limit point of B. This means that $\mathbf{p}(t) \in A \cap \overline{B}$. And similarly, it can be shown that $\mathbf{p}(t) \in \overline{A} \cap B$ if $t \in \overline{A_0} \cap B_0$ is nonempty. But this is a contradiction since A and B are separated.

(b) Suppose not, then $\mathbf{p}(t) \in A \cup B$ whenever $t \in (0,1)$. Since A_0 and B_0 are separated, the interval $(0,1)$ belongs to either A_0 or B_0 exclusively. For otherwise, $(0,1)$ would be a connected set that is also a union of two nonempty separated sets $(0,1) \cap A_0$ and $(0,1) \cap B_0$, which is absurd.

If we assume that the intersection $(0,1) \cap A_0$ is empty, then $(0,1) \subset B_0$. Since $0 \in A_0$ and is a limit point of $(0,1)$, we would get that $A_0 \cap \overline{B_0} \neq \emptyset$. But this is a contradiction since A_0 and B_0 are separated.

If we assume that the intersection $(0,1) \cap B_0$ is empty, then similarly $A_0 \cap \overline{B_0} \neq \emptyset$ since $1 \in B_0$ would be a limit point of A_0.

We have exhausted all possibilities and have seen that they lead to contradiction. Hence, our initial assumption must be wrong and therefore there must exist a $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c) Assume the opposite, and let E be a convex subset of $\mathbb{R}^k$. Then there exists two nonempty separated sets A and B such that $E = A \cup B$. Take any two points $\mathbf{a} \in A$ and $\mathbf{b} \in B$, then put

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}.$$

By the result in (b) we know that there is a number $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$, which means that $\mathbf{p}(t_0) \notin E$. But this is a contradiction since E is supposed to be convex and no such t_0 should exist (see Definition 2.17). ■

Exercise 2.22

A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint:* Consider the set of points which have only rational coordinates.

From Theorem 1.20 (b), we know that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$. We shall show that the set of points which have only rational coordinates $\mathbb{Q}^k$ is a countable dense subset of $\mathbb{R}^k$.

That $\mathbb{Q}^k$ is countable follows from Theorem 2.13, simply put $A = \mathbb{Q}$ and B_k will be the set of all points with rational coordinates with k-dimensions.

It remains to show that any point of $\mathbb{R}^k$ is a limit point of $\mathbb{Q}^k$. Fix $\mathbf{x} \in \mathbb{R}^k$ and let N be a neighborhood around $\mathbf{x}$ with some chosen radius $r > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every coordinate $x_i \in \mathbb{R}$ we can choose rational number $p_i \in \mathbb{Q}$ such that the $|p_i - x_i| < \frac{r}{\sqrt{k}}$. Having done so for $1 \le i \le k$, we can construct the rational number

$$\mathbf{p} = (p_1, p_2, ..., p_k) \in \mathbb{Q}^k.$$

Consider now its distance to $\mathbf{x}$

$$|\mathbf{p} - \mathbf{x}| = \sqrt{\sum_{i=0}^{k} (p_i - x_i)^2} \le \sqrt{\sum_{i=0}^{k} |p_i - x_i|^2} < \sqrt{k \cdot \frac{r^2}{k}} = r,$$

which means that $\mathbf{p} \in N$, and therefore $\mathbf{x}$ is a limit point of $\mathbb{Q}^k$. Since the choice of $\mathbf{x}$ was arbitrary, it follows that every point of $\mathbb{R}^k$ is a limit point of $\mathbb{Q}^k$. This proves that $\mathbb{Q}^k$ is a countable dense subset of $\mathbb{R}^k$, and it follows by definition that $\mathbb{R}^k$ is separable.

■

Exercise 2.23

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a sub-collection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Since X is separable, it contains a countable dense subset E. For every $p \in E$ let W_p consists of all neighborhoods with non-negative rational radius $h > 0$

$$W_p = \{N_h(p) \mid h \in \mathbb{Q}, h > 0\}.$$

Each member of W_p is associated with exactly one non-negative rational radius. This means that W_p is countable. By the Corollary to Theorem 2.12 the union

$$V = \bigcup_{p \in E} W_p$$

is also countable. Hence, the members of V can be arranged in a sequence $\{V_\alpha\}$ where each α is a member of some countable set.

It remains to be shown that the collection of open sets $\{V_\alpha\}$ is a base. Let x be any point of X and G be any open subset of X such that $x \in G$. Since x is an interior point of G, we know that there exists a radius $r > 0$ such that the neighborhood $N_r(x) \subset G$.

E is dense in X and therefore we can find a point $p \in E$ such that the distance $d(p, x) = \delta < \frac{r}{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a $h \in \mathbb{Q}$ such that $\delta < h < \frac{r}{2}$ by Theorem 1.20.

For this p and rational radius h there exists a neighborhood V_α in the collection. Because the distance $d(x, p) = \delta < h$ we have that $x \in V_\alpha$. In fact, for any point $y \in V_\alpha$ we have that

$$d(y, x) \leq d(y, p) + d(p, x) < h + \delta < 2 \cdot \frac{r}{2} = r,$$

which implies $y \in N_r(x)$ and therefore $V_\alpha \subset N_r(x)$. Since the choice of x and G were arbitrary it follows that the collection $\{V_\alpha\}$ is a countable base of X.

■

Exercise 2.24

Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, ..., x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, ..., j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = \frac{1}{n}$ $(n = 1, 2, 3, ...)$, and consider the centers of the corresponding neighborhoods.

We begin by showing that the process in the hint must stop after a finite number of steps. Assume the opposite to get a contradiction – then the process never stops and the set of the chosen points

$$A_\delta = \{x_1, x_2, x_3, ...\}$$

is infinite. By construction, it is true that for any two points $x_n, x_m \in A_\delta$ such that $m \neq n$, the distance is $d(x_m, x_n) \geq \delta$. This implies that no point of A_δ is a limit point of A_δ.

Let y be a point in the complement of A_δ and consider its neighborhood $N_r(y)$ with radius $r < \frac{\delta}{2}$. If there exists some $x_i \in A_\delta$ such that $x_i \in N_r(y)$, then for any other point $x_j \in A_\delta$ we have that

$$d(x_i, x_j) \leq d(x_i, y) + d(y, x_j) < \delta/2 + d(y, x_j)$$

which after rearranging

$$d(y, x_j) > d(x_i, x_j) - \delta/2 \geq \delta - \delta/2 = \delta/2.$$

Hence, any neighborhood with center in y with radius less than $d(x_i, y)$ would contain no point of A_δ. But this is absurd, since that would make A_δ an infinite subset of X which has no limit point. Therefore, A_δ must be a finite subset of X.

We shall now show that X can be covered by finitely many neighborhoods of radius δ. Let V_x be the open neighborhood with center in some $x \in A_\delta$ and radius δ. Consider the collection of these neighborhoods

$$V = \{V_x \mid x \in A_\delta\}.$$

That V is finite follows from the fact that A_δ is finite.

If $y \notin A_\delta$ then by construction of A_δ there exists some $x \in A_\delta$ such that $d(x, y) < \delta$. Therefore y belongs to the open neighborhood V_x. This shows that X is covered by the finite collection of open sets V.

We now aim to show that X is separable. Following the same process used to construct A_δ, take $\delta = \frac{1}{n}$ $(n = 1, 2, 3...)$ and construct a sequence of sets $A_{\frac{1}{n}}$. For simplicity of notation, let $E_n = A_{\frac{1}{n}}$ and take the union of these finite sets

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Since each E_n is finite, the union S is at most countable by Theorem 2.12. Moreover, we can conclude that S is countable. Because X is infinite and we can construct infinitely many non-empty sets E_n by increasing n, or if E_n is empty for some n then there exists .

Let $z \in X$ be any point in the complement of S. To prove that S is dense in X it is sufficient to show that z is a limit point of S. Take any neighborhood $N_r(z)$ with center in z for some radius $r > 0$. By the Archimedean property of $\mathbb{R}$ there exist a natural number n such that $r > \frac{1}{n}$.

Since z is in the complement of S it follows that $z \notin E_n$ for any natural number n. Hence, by construction of E_n we can find a point $x \in E_n$ such that $d(x, y) < \frac{1}{n}$. This implies that $x \in N_r(z)$ and therefore z is a limit point of S. Since both z and r were arbitrarily chosen, we can concluded that S is dense in X.

We have shown that S is a countable dense subset of X. This proves that X is separable as desired.

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