

## Chapter 2

### Exercise 1

Prove that the empty set is a subset of every set.

Let  $A$  be any set,  $\emptyset$  be the empty set and assume the opposite. Then there exists at least one element  $x \in \emptyset$  such that  $x \notin A$ . But the empty set does not contain any elements and no such  $x$  can therefore be found. This is a contradiction and hence  $\emptyset \subset A$  must be true. ■

### Exercise 2

Prove that the set of all algebraic numbers is countable.

Let  $\alpha = (a_0, \dots, a_n) \in B_{n+1}$  be an  $n+1$ -tuple of integers. Example 2.5 shows that the set of all integers  $\mathbb{Z}$  is countable. Using Theorem 2.13 with  $A = \mathbb{Z}$  shows that the set of  $n+1$ -tuples  $B_{n+1}$  is countable.

For particular choice of integers  $a_0, \dots, a_n$  we may construct an equation of the form

$$a_n z^n + \dots + a_1 z + a_0 = 0.$$

This polynomial has at most  $n$  solutions. Let  $E_\alpha$  be the set of all complex numbers  $z$  that is a solution to that polynomial. Hence  $E_\alpha$  is a finite subset of  $\mathcal{A}$  with at most  $n$  elements. If  $S_n$  is a set such that

$$S_n = \bigcup_{\alpha \in B_{n+1}} E_\alpha$$

then it is the union of a countable collection of finite sets. It follows from Theorem 2.12 that  $S_n$  is at most countable. Since the union of a countable collection of at most countable sets is at most countable (follows from Theorem 2.12 again); the union

$$S = \bigcup_{n=1}^{\infty} S_n,$$

is at most countable.

We shall show that  $\mathcal{A} \subset S$ . For any  $z \in \mathcal{A}$  there exist integers  $a_0, \dots, a_k$  such that  $a_k z^k + \dots + a_1 z + a_0 = 0$ . For these integers there is a  $k+1$ -tuple  $\alpha = (a_0, \dots, a_k)$  which is associated with exactly one set  $E_\alpha$ . Hence there exists at least one  $\alpha$  for which  $z \in E_\alpha$  and therefore it

follows that  $\mathcal{A} \subset S$ . This means that  $\mathcal{A}$  is at most countable since it is a subset of  $S$ .

Lastly we need to show that  $\mathcal{A}$  is infinite. The set of rational numbers  $\mathbb{Q}$  is countable by the corollary to Theorem 2.13. For any  $q \in \mathbb{Q}$  there are integers  $m, n$  such that  $q = \frac{n}{m}$ . Now choose  $a_1 = m$  and  $a_0 = -n$ , then

$$a_1 q + a_0 = m \frac{n}{m} - n = 0,$$

which means that  $q$  is algebraic. Since  $q$  is arbitrary it follows that  $\mathbb{Q} \subset \mathcal{A}$ . This shows that  $\mathcal{A}$  has an infinite subset and must therefore itself be infinite.  $\mathcal{A}$  is an infinite set that is at most countable, and therefore it is countable as desired. ■

### Exercise 3

Prove that there are real numbers which are not algebraic.

Suppose not. Let  $\mathcal{A}$  be the set of all algebraic numbers. Since we assume the opposite any real number is algebraic and it follows that  $\mathbb{R} \subset \mathcal{A}$ .

From Exercise 2.2 we know that  $\mathcal{A}$  is countable. By Theorem 2.8 this would entail that  $\mathbb{R}$  is countable, since it is an infinite subset of a countable set. But this is a contradiction since Theorem 2.43 shows that  $\mathbb{R}$  is uncountable. Hence our assumption is false and there exists real numbers  $x \in \mathbb{R}$  such that  $x \notin \mathcal{A}$ . ■

### Exercise 4

Is the set of all irrational real numbers countable?

Denote the set of irrational numbers by  $\mathbb{I}$ . According to Theorem 2.13  $\mathbb{Q}$  is countable. Now suppose  $\mathbb{I}$  is countable. We already know that

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I},$$

which means that  $\mathbb{R}$  is countable by Theorem 2.12. But this is a contradiction since by Theorem 2.43 the set of real numbers  $\mathbb{R}$  is uncountable. If  $\mathbb{I}$  is assumed to be finite, then that still would make  $\mathbb{R}$  countable. Hence,  $\mathbb{I}$  must be uncountable. ■

### Exercise 5

Construct a bounded set of real numbers with exactly three limit points.

Let  $k$  be a natural number and consider the set  $E_k$  of numbers  $k + \frac{1}{n}$  where  $n = 1, 2, 3, \dots$ . Clearly  $E_k \subset (k, k + 1]$  and is therefore bounded.

We need to show that  $E_k$  does not contain any limit points. For any natural numbers  $n, m$  we have that

$$d\left(k + \frac{1}{n}, k + \frac{1}{m}\right) = \left|k + \frac{1}{n} - \left(k + \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = d\left(\frac{1}{n}, \frac{1}{m}\right).$$

For a fixed  $n$  we want to show that

$$d\left(\frac{1}{n}, \frac{1}{m}\right) \geq d\left(\frac{1}{n}, \frac{1}{n+1}\right),$$

holds with any natural number  $m \neq n$ .

If  $m \geq n + 1$ , then

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \geq \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right).$$

If  $m \leq n - 1$ , then

$$\begin{aligned} d\left(\frac{1}{n}, \frac{1}{m}\right) &= \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{n(n-1)} > \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\ &= \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right), \end{aligned}$$

which gives us the result. Hence any neighborhood  $N_r\left(k + \frac{1}{n}\right)$  with radius  $0 < r < \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$  will contain no other points of  $E_k$  except  $k + \frac{1}{n}$ . This means that  $k + \frac{1}{n}$  cannot be a limit point of  $E_k$ .

We now show that  $k \notin E_k$  is a limit point of  $E_k$ . For any radius  $r > 0$  we can find a natural number  $N$  such that  $\frac{1}{N} < r$  by using the Archimedean property of  $\mathbb{R}$  (Thm. 1.20)

$$d\left(k, k + \frac{1}{N}\right) = \left|k - \left(k + \frac{1}{N}\right)\right| = \frac{1}{N} < r,$$

which means that  $k + \frac{1}{N} \in N_r(k)$ .

The desire is to construct a set that is bounded with exactly three limit points. We accomplish this by choosing the set  $E$  such that

$$E = E_1 \cup E_2 \cup E_3.$$

Specifically, the limit points to  $E$  are  $1, 2, 3$  and  $E_1, E_2, E_3$  are all bounded which makes  $E$  bounded as well. ■

### Exercise 6

Prove that  $E'$  is closed. Prove that  $E$  and  $\overline{E}$  have the same limit points. Do  $E$  and  $E'$  always have the same limit points?

We first prove that  $E'$  is closed. Suppose not, then there exists a limit point  $x$  of  $E'$  such that  $x \notin E'$ . For any neighborhood  $N_r(x)$  with radius  $r > 0$  there is a  $y \in E'$  such that  $y \in N_r(x)$ . Since  $y \in E'$  we know that  $y$  is a limit point of  $E$ . Hence, we can find a  $z \in E$  such that  $z \in N_{r-h}(y)$  with  $0 < h < r$ . We have that

$$d(z, x) \leq d(z, y) + d(y, x) < r - h + h = r,$$

which means that  $z \in N_r(x)$ . Since  $r > 0$  is arbitrary it follows that  $x$  is a limit point to  $E$ . But this is a contradiction since it would imply  $x \in E'$ . Therefore our assumption is wrong and  $E'$  must be closed.

Next we show that  $\overline{E}$  and  $E$  have the same limit points. If  $x$  is a limit point of  $\overline{E}$ , then since  $\overline{E} = E' \cup E$ , it must be the case that  $x$  is a limit point of  $E'$  or  $E$ . Assuming  $x$  is a limit point of  $E$  leaves us with nothing to prove. So we suppose that  $x$  is a limit point of  $E'$  alone. We already have established that  $E'$  is closed, and therefore  $x \in E'$ . This in turn directly implies that  $x$  is a limit point of  $E$  because the members of  $E'$  are all limit points of  $E$ .

Conversely, if  $y$  is a limit point of  $E$  then we know that  $y \in E'$ . Since  $\overline{E} = E' \cup E$  it follows that  $y \in \overline{E}$  and is a limit point of  $\overline{E}$ .

Lastly, we show that  $E$  and  $E'$  do not always have the same limit points. Let  $E = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$  and note  $0$  is the only limit point of  $E$ . This means that  $E' = \{0\}$ . We know by Corollary to Thm. 2.20 that  $E'$  has no limit points. This shows the result by example. ■

### Exercise 7

Let  $A_1, A_2, A_3, \dots$  be subsets of a metric space.

- If  $B_n = \cup_{i=1}^n A_i$ , prove that  $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$ .
- If  $B = \cup_{i=1}^\infty A_i$ , prove that  $\overline{B} \supset \cup_{i=1}^\infty \overline{A_i}$ .

(a) We first need to show that for any sets  $A$  and  $B$  it is true that  $(A \cup B)' = A' \cup B'$ . If  $x \in (A \cup B)'$ , then for every neighborhood  $N_r(x)$  with a radius  $r > 0$  there exists at least one point  $p \in N_r(x)$  such that  $p \in A \cup B$ . Thus,  $p \in A$  or  $p \in B$  which implies that  $x \in A'$  or  $x \in B'$  so that  $x \in A' \cup B'$ .

Suppose now that  $x \in A' \cup B'$ , then for any neighborhood  $N_r(x)$  with a radius  $r > 0$  there is a point  $p \in N_r(x)$  such that  $p \in A$  or  $p \in B$ . This means that  $p \in A \cup B$  and it follows that  $x \in (A \cup B)'$ .

We now turn to the question at hand. This will be a proof by induction. The base case for  $n = 1$  is clearly true. Assume the statement holds for  $n$ , then

$$\begin{aligned}\overline{B_{n+1}} &= \overline{\bigcup_{i=1}^{n+1} A_i} = \overline{\bigcup_{i=1}^n A_i \cup A_{n+1}} = \overline{B_n \cup A_{n+1}} \\ &= B_n \cup A_{n+1} \cup (B_n \cup A_{n+1})' \\ &= B_n \cup B'_n \cup A_{n+1} \cup A'_{n+1} \\ &= \overline{B_n} \cup \overline{A_{n+1}} = \bigcup_{i=1}^n \overline{A_i} \cup \overline{A_{n+1}} \\ &= \bigcup_{i=1}^{n+1} \overline{A_i},\end{aligned}$$

where we have used the induction hypothesis in the penultimate equality.

(b) If  $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$ , then there is an  $n \in \mathbb{N}$  such that  $x \in \overline{A_n}$ . We have that

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} \supset \overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i} \supset \overline{A_n},$$

where we used the result in (a) to get the second equality. This implies that  $x \in \overline{B}$ . Since  $x$  is arbitrary it follows that  $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$  as desired.

We shall conclude by showing that this subset can indeed be proper. Let  $A_i = \{\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i}\}$  and note that there exists no  $k \in \mathbb{N}$  such that  $0 \in A_k$ . Thus  $0 \notin \bigcup_{i=1}^{\infty} A_i$ . Furthermore, each set  $A_i$  has no limit points which is why  $\overline{A_i} = A_i$ , and therefore  $\bigcup_{i=1}^{\infty} \overline{A_i} = \bigcup_{i=1}^{\infty} A_i$ . It follows then that  $0 \notin \bigcup_{i=1}^{\infty} \overline{A_i}$ . Now let's consider  $\overline{B}$

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\left\{\frac{1}{2}, \frac{1}{3}, \dots\right\}} = \left\{0, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

which implies that  $0 \in \overline{B}$ . Since  $0 \notin \bigcup_{i=1}^{\infty} \overline{A_i}$  we've shown that the inclusion can be proper. ■

### Exercise 8

Is every point of every open set  $E \subset \mathbb{R}^2$  a limit point of  $E$ ? Answer the same question for closed sets in  $\mathbb{R}^2$ .

We first show that every point in every open set  $E \subset \mathbb{R}^2$  is a limit point of  $E$ . Suppose not. Then there exists an open set  $E \subset \mathbb{R}^2$  with a point  $x \in E$  that is not a limit point of  $E$ . It follows that there exists an  $r > 0$  such that the neighborhood  $N_r(x)$  contains no point of  $E$  except  $x$ .

Since  $E$  is an open set,  $x$  must be an interior point. Hence, there is a neighborhood with radius  $s > 0$  such that  $N_s(x) \subset E$ . Neighborhoods in  $\mathbb{R}^2$  are non-empty interiors of circles, which makes  $N_s$  and  $N_r$  circles centered at  $x$ .

Consider the cases for  $r$  and  $s$ :

1. If  $r > s$  then  $N_s$  is a circle inscribed in a larger circle  $N_r$  which means that  $N_s \subset N_r$ . It follows that there exists points  $y \in N_s \subset N_r \subset E$  where  $y \neq x$ . But this is impossible due to our assumption.
2. If  $r \leq s$  then  $N_r$  is inscribed in  $N_s$ . Since  $N_s \subset E$  that would make  $N_r \subset E$  which is also impossible given our assumption.

Since none of the relations  $r < s$ ,  $r = s$  and  $r > s$  can be true, such an  $r$  cannot exist. We've reached a contradiction and the assumption is wrong. Hence, each point of  $E$  is a limit point of  $E$ .

Now we show that the case does not hold true in general for closed sets. Consider the set of all natural numbers  $\mathbb{N} \subset \mathbb{R}^2$ . This set is closed, yet every point in this set is not a limit point of it. ■

### Exercise 9

Let  $E^\circ$  be the set of all interior points of  $E$ .

- (a) Prove that  $E^\circ$  is always open.
- (b) Prove that  $E$  is open if and only if  $E^\circ = E$ .
- (c) If  $G \subset E$  and  $G$  is open, prove that  $G \subset E^\circ$ .
- (d) Prove that the complement of  $E^\circ$  is the closure of the complement of  $E$ .
- (e) Do  $E$  and  $\overline{E}$  always have the same interiors?
- (f) Do  $E$  and  $E^\circ$  always have the same closures?

(a) If  $p \in E^\circ$  then  $p$  is an interior point of  $E$ . This means that there exists some neighborhood  $N(p)$  with  $r > 0$  such that  $N(p) \subset E$ . By Theorem 2.19  $N(p)$  is an open set, implying that every point in  $N(p)$  is an interior point. Hence,  $N(p) \subset E^\circ$  and it follows  $E^\circ$  is an open set.

(b) If  $E$  is open then for every point  $x \in E$  we can find a neighborhood  $N(x)$  with  $r > 0$  such that  $N \subset E$ . This implies that every point  $x \in E$  is an interior point of  $E$  and it is therefore true that  $x \in E^\circ$ . This shows that  $E \subset E^\circ$ . Since by construction  $E^\circ \subset E$  we have that  $E^\circ = E$ .

Conversely, if  $E^\circ = E$  then all points in  $E$  are interior points of  $E$ . This is true since  $E^\circ$  is the set of all interior points of  $E$ . Thus, by definition 2.18 (f),  $E$  is an open set.

(c) If  $G$  is open, then for every point  $p \in G$  we can find a neighborhood  $N(p)$  with  $r > 0$  such that  $N(p) \subset G$ . Since  $G \subset E$ , we have that  $N(p) \subset E$ , which shows that  $p$  is an interior point of  $E$ . Hence,  $p \in E^\circ$ . Therefore,  $G \subset E^\circ$  since  $p$  was arbitrary chosen from  $G$ .

(d) Let  $x$  be any point in  $(E^\circ)^c$ . First we show that  $(E^\circ)^c \subset \overline{E^c}$ . Suppose that  $x \notin E$ . Then

$$x \in E^c \subset E^c \cup E^{c'} = \overline{E^c}.$$

Now let  $x \in E$ . Since  $x$  is in the complement of  $E^\circ$ , we know that  $x$  is not an interior point of  $E$ . Therefore, for every neighborhood  $N(x)$  with radius  $r > 0$ , we have that  $N(x) \not\subset E$ . This means that  $N(x)$  always has points in  $E^c$ , which makes  $x$  a limit point of  $E^c$ . Thus,  $x \in E^{c'} \subset \overline{E^c}$ . This shows that  $(E^\circ)^c \subset \overline{E^c}$ .

Conversely, let  $p \in \overline{E^c} = E^c \cup E^{c'}$ . Then it is clear that either  $p \in E^c$  or  $p \in E^{c'}$ . Assume  $p \in E^c$ . Then we know that  $p$  is not an interior point to  $E$ , so  $p \notin E^\circ$ , which implies that  $p \in (E^\circ)^c$ . If  $p \in E^{c'}$ , then it is a limit point of  $E^c$ , and therefore every neighborhood  $N(p)$  with radius  $r > 0$  have points (other than  $p$ ) from  $E^c$ . Hence, there is no neighborhood such that  $N(p) \subset E$  which means that  $p$  cannot be an interior point of  $E$ . Therefore,  $p \in (E^\circ)^c$ , and we have now shown that  $\overline{E^c} \subset (E^\circ)^c$ .

(e) Let  $E = (0, 1) \cup (1, 2)$  be a set consisting of line segments in  $\mathbb{R}$ . Since  $E$  is open we have that  $E^\circ = E$  by (b). The set of limit points of  $E$  are  $\{0, 1, 2\}$ , so the closure  $\overline{E}$  is the closed interval  $[0, 2]$ . From here we can see that  $(\overline{E})^\circ = (0, 2)$ , which shows that  $E^\circ \neq (\overline{E})^\circ$ . Therefore,  $E$  and  $\overline{E}$  do not always have the same interiors.

(f) Consider the set  $E = \{\frac{1}{n} \mid n = 1, 2, 3, \dots\}$  as subset of  $\mathbb{R}$ . Since 0 is the only limit point of  $E$ , we have that its closure is  $\overline{E} = E \cup \{0\}$ . However, none of the points in  $E$  are interior points, and therefore  $E^\circ = \emptyset$ . This means that  $\overline{E^\circ} = \emptyset$ . We have shown that  $\overline{E} \neq \overline{E^\circ}$  as desired. ■

### Exercise 10

Let  $X$  be an infinite set. For  $p \in X$  and  $q \in X$ , define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

We begin by showing that  $d$  is a metric. Conditions 2.15 (a) and (b) are clearly satisfied. We show (c) also is true. Suppose  $p \neq q$  so that  $d(p, q) = 1$ . For any  $r \in X$  we have two cases, either  $r$  equals one of  $p, q$  or neither.

Suppose WLOG that  $r = p$ . Then we have that  $d(p, r) = 0$  and  $d(r, q) = 1$ . It follows that

$$\underbrace{d(p, q)}_{=1} \leq \underbrace{d(p, r)}_{=0} + \underbrace{d(r, q)}_{=1} = 1.$$

Now let  $r \neq p \neq q$  be true. Then  $d(p, r) = d(r, q) = 1$ , and we can show that

$$\underbrace{d(p, q)}_{=1} < \underbrace{d(p, r)}_{=1} + \underbrace{d(r, q)}_{=1} = 2.$$

Lastly, if  $p = q$  then  $d(p, q) = 0$  and any point  $r \in X$  will satisfy (c) since the distance function is non-negative. This shows that  $X$  is a metric with distance function  $d$ .

We shall now show that any subset  $E \subset X$  is open. Let  $p$  be any point in  $E$  and consider the neighborhood  $N_r(p)$  with radius  $r = 1$ . Given the metric, this neighborhood contains only  $p$  itself so that  $N_r(p) \subset E$ . Hence, every point in  $E$  is an interior point and  $E$  is open.

We now show that every subset in  $X$  is closed. Let  $E$  be any subset of  $X$ . Since every subset of  $X$  is open, the complement  $E^c$  is also open. By Thm. 2.33  $E$  is closed.



Lastly, we show that only finite sets of  $X$  are compact (by Definition 2.32). Suppose not, then we have a compact set  $K \subset X$  that is infinite. For every  $p \in K$  let  $G_p$  be the open neighborhood around  $p$  with radius  $r = 1$ . Since every  $p \in K$  is associated with an open set  $G_p$ , the collection  $\{G_p\}$  is an open cover of  $K$ . Because  $K$  is compact, there exists a finite number of indices such that

$$K \subset G_{p_1} \cup \cdots \cup G_{p_m}.$$

Every subset  $G_{p_n}$  is an open neighborhood around  $p_n$  with radius 1. From before we know that these sets only contain a single point, namely  $p_n$ . But that is absurd, since it would make  $K$  finite. ■

### Exercise 11

For  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ , define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= (x - 2y), \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

$d_1$  Condition 2.15 (c) is not satisfied, which therefore is not a metric. We give an example,

$$d_1(10, 0) > d_1(10, 4) + d_1(4, 0).$$

$d_2$  Both 2.15 (a) and (b) are clearly true. We show that (c) is also satisfied. Assume not, then there exists points  $x, y$  and  $r$  such that

$$d_2(x, y) > d_2(x, r) + d_2(r, y),$$

which in this particular case is

$$\sqrt{|x - y|} > \sqrt{|x - r|} + \sqrt{|r - y|}.$$

If  $0 < q < p$  then  $q^2 < p^2$  for any  $p, q \in \mathbb{R}$  so that

$$\begin{aligned}
|x - y| &= \sqrt{|x - y|}^2 > \left( \sqrt{|x - r|} + \sqrt{|r - y|} \right)^2 \\
&= |x - r| + 2\sqrt{|x - r||r - y|} + |r - y| \\
&\geq |x - r| + |r - y|,
\end{aligned}$$

where the last inequality comes from the fact that  $\sqrt{|x - r||r - y|} \geq 0$ . The above shows that  $d(p, q) = |p - q|$  cannot be a metric. But that is a contradiction, since Theorem 1.37 shows that  $|p - q|$  satisfies 2.15 (c). Hence,  $d_2$  is a metric.

$d_3$  Condition 2.15 (a) is not satisfied. We show an example

$$d_3(2, -2) = |2^2 - (-2)^2| = 4 - 4 = 0.$$

We have found  $x \neq y$  such that  $d_3(x, y) = 0$ . This shows that  $d_3(x, y)$  is not a metric.

$d_4$  We show that  $d_4$  is not a metric since condition (a) of Definition 2.15 is not satisfied. If  $x \neq 0$  then

$$d_4(x, y) = |x - 2x| = |x| > 0.$$

$d_5$ ) Conditions in (a), (b) are clearly satisfied so we focus on (c). Throughout this exercise WLOG assume  $x \leq y$ . Suppose  $x \leq r \leq y$ , we have that

$$\begin{aligned}
d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} = \frac{|x - r + r - y|}{1 + |x - y|} \\
&\leq \frac{|x - r|}{1 + |x - y|} + \frac{|r - y|}{1 + |x - y|} \\
&\leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \\
&\leq d_5(x, r) + d_5(r, y),
\end{aligned}$$

where we used Theorem 1.37 in the first inequality. Since  $x \leq r \leq y$  we know that  $|x - r| \leq |x - y|$  and  $|r - y| \leq |x - y|$ , both of which we used to get the second inequality.

Now assume  $x \leq y < r$ . First we show that for any  $\varepsilon > 0$

$$\frac{|w|}{1 + |w|} < \frac{|w| + \varepsilon}{1 + |w| + \varepsilon}, \quad (1)$$

holds. The statement is clearly true for  $w = 0$ . If  $|w| > 0$ , then

$$\frac{|w|}{1+|w|} = \frac{1}{1+\frac{1}{|w|}} < \frac{1}{1+\frac{1}{|w|+\varepsilon}} = \frac{|w|+\varepsilon}{1+|w|+\varepsilon},$$

which shows that (1) is true.

Since  $x \leq y < r$ , we have that  $|x - y| < |x - r|$ . This means we can use (1) to get the first inequality below,

$$\begin{aligned} d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} < \frac{|x - r|}{1 + |x - r|} \\ &\leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \\ &= d_5(x, r) + d_5(r, y). \end{aligned}$$

The last inequality is due to the last term being non-negative. Similar argument can be made for  $r < x \leq y$  because  $|x - y| < |r - y|$  which allows us to use (1) again. This shows that  $d_5(x, y)$  is a metric. ■

## Exercise 12

Let  $K \subset \mathbb{R}$  consist of 0 and the numbers  $1/n$ , for  $n = 1, 2, 3, \dots$ . Prove that  $K$  is compact directly from the definition (without using the Heine-Borel theorem).

Let  $\{G_\alpha\}$  be any open cover of  $K$ . Then we know that there is an index  $\alpha_0$  such that  $0 \in G_{\alpha_0}$ . Since  $G_{\alpha_0}$  is an open set, we know that there exists a neighborhood  $N_r(0)$  with radius  $r > 0$  where  $N_r \subset G_{\alpha_0}$ .

If  $r \geq 1$ , then  $G_{\alpha_0}$  covers  $K$  and there is nothing to prove. Assume therefore that  $r < 1$ . By Archimedean property of  $\mathbb{R}$  we can find positive integers  $p$  such that  $r \geq 1/p$ . Let  $m$  be the smallest integer such that  $r > 1/m$ . It follows that if  $q$  is an integer where  $q > m$ , then  $r > 1/q$  so that  $1/q \in N_r(0)$ . This shows that there are at most  $m - 1$  points in  $E$  that are not in  $N_r$ ,

$$\frac{1}{m-1}, \frac{1}{m-2}, \dots, \frac{1}{2}, 1.$$

Let  $G_{\alpha_k}$  denote an open set in the collection  $\{G_\alpha\}$  such that  $1/k \in G_{\alpha_k}$  ( $k = 1, 2, 3, \dots$ ). Because  $\{G_\alpha\}$  is an open cover of  $K$ , each of the above  $m - 1$  points can be associated this way to at least one index in the collection. Therefore

$$K \subset G_{\alpha_0} \cup G_1 \cup \dots \cup G_{\alpha_{m-1}}.$$

We have shown that any open cover of  $K$  has a finite sub-cover which implies that  $K$  is compact as desired. ■

### Exercise 13

Construct a compact set of real numbers whose limit points form a countable set.

Let  $n$  be a natural number and construct a sequence  $\{x_k\}$  as follows. Define  $x_1$  as the midpoint between  $\frac{1}{n+1}$  and  $\frac{1}{n}$

$$x_1 = \frac{\frac{1}{n+1} + \frac{1}{n}}{2}.$$

Having chosen  $x_1, \dots, x_{k-1}$  (for  $k = 2, 3, 4, \dots$ ), define  $x_k$  as the midpoint between  $x_{k-1}$  and  $\frac{1}{n}$

$$x_k = \frac{x_{k-1} + \frac{1}{n}}{2}.$$

Let  $E_n$  be the set of all numbers generated by the sequence  $\{x_k\}$ , along with the point  $\frac{1}{n}$ . For example, if  $n = 1$  we have

$$E_1 = \left\{1, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \dots\right\}.$$

Since the sequence  $\{x_k\}$  is constructed by repeatedly taking midpoints between the previous term and  $\frac{1}{n}$ , the sequence will approach the value  $\frac{1}{n}$ , making it a limit point of  $E_n$ .

We can show that any other point  $y \in \mathbb{R}$  such that  $y \neq \frac{1}{n}$  cannot be a limit point of  $E_n$ . Since  $E_n$  is a countable set of values that approach  $\frac{1}{n}$ , there is a number  $x_k \neq y$  in  $E_n$  that minimises  $d(y, x_k)$ . Then the neighborhood with  $0 < r < d(y, x_k)$  will have no point of  $E_n$  (other than possibly  $y$ ). Hence,  $\frac{1}{n}$  is the only limit point so that  $E'_n = \{\frac{1}{n}\}$ .

The collection of sets  $\{E_n\}$  is disjoint since  $E_n \subset (\frac{1}{n+1}, \frac{1}{n}]$ . Let  $S$  be the union of this collection along with the point 0,

$$S = \{0\} \cup \bigcup_{n=1}^{\infty} E_n \subset [0, 1].$$

Since  $S$  is a union of disjoint sets, the limit points of  $S$  is given by

$$S' = \{0\} \cup \bigcup_{n=1}^{\infty} E'_n = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

which is countable. Furthermore, because  $E' \subset E$  and  $0 \in S$  we see that  $S' \subset S$  so that  $S$  is closed. Since  $S$  is a subset of the compact interval  $[0, 1]$ , by Theorem 2.35  $S$  is compact.

We have constructed a compact set  $S$  of real numbers whose limit points  $S'$  form a countable set. This concludes the proof. ■