9. We check that this order on $\mathbb{C}$ satisfies properties 1.5 (i) and (ii). Clearly (i) holds using the new order relation together with the structre of $\mathbb{R}$. Suppose $z < w$ and let $x = e + fi$ such that $w < x$. Since $z < w$, then $a < c$ or $a = c$ and $b < d$. The same two cases holds for w, x since $w < x$.

- If $z < w$ such that $a \leq c$ and $w < x$ such that $c < e$, then $a \leq c < e \Rightarrow z < x$.

- If $z < w$ such that $a < c$ and $w < x$ such that $c = e$, then $a < c = e \Rightarrow z < x$.

- If $z < w$ such that $a = c$, $b < d$ and $w < x$ such that $c = e$, $d < f$, then $b < d < f \Rightarrow z < x$.

Now suppose this ordered set has the least-upper-bound property. Let $E \subset \mathbb{C}$ be the set of all numbers v such that $\mathrm{Re}(v) < \mathrm{Re}(z)$. It is clear that E is non-empty and bounded above. Then $\alpha = \sup E$ exists.

If $\alpha \in E$, then $\mathrm{Re}(\alpha) < \mathrm{Re}(z)$. Let u be a complex number such that $\mathrm{Re}(u) = (\mathrm{Re}(\alpha) + \mathrm{Re}(z))/2$. Then

$$\mathrm{Re}(\alpha) = \frac{\mathrm{Re}(\alpha) + \mathrm{Re}(\alpha)}{2} < \frac{\mathrm{Re}(\alpha) + \mathrm{Re}(z)}{2} < \mathrm{Re}(z),$$

which implies $u \in E$ yet $\alpha < u$. Since α is an upper-bound to E this means that $\alpha \notin E$.

In that case, $\forall v \in E$ it is true that $\mathrm{Re}(v) < \mathrm{Re}(\alpha)$. Now choose $\beta = \mathrm{Re}(\alpha) + i(\mathrm{Im}(\alpha) - 1)$. Since $\mathrm{Re}(\beta) = \mathrm{Re}(\alpha)$, if $v \in E$ then $v < \beta$. Hence β is an upper-bound to E. Because β is chosen such that $\mathrm{Re}(\beta) = \mathrm{Re}(\alpha)$ while $\mathrm{Im}(\beta) < \mathrm{Im}(\alpha)$, we have that $\beta < \alpha$. But this is a contradiction since α is assumed to be the *least* upper-bound to E. This shows that α cannot exist and hence the initial assumption of least upper-bound property is false.

$\square$