7. We follow the outline in the book.

a) Proof by induction. The case for $n = 1$ is clearly true. For $n + 1$ we will use the fact that for a geometric series $\sum_{j=0}^{k-1} b^j = \frac{b^k-1}{b-1}$ is true for any $b > 1$. We have that

$$
\begin{aligned}
b^{n+1} - 1 = (b-1)\sum_{k=0}^{n} b^k &= (b-1)\left(\sum_{k=0}^{n-1} b^k + b^n\right) \\
&= b^n - 1 + (b-1)b^n \geq n(b-1) + (b-1)b^n \\
&\geq n(b-1) + (b-1) = (n+1)(b-1),
\end{aligned}
$$

where we have used the induction step for the first inequality and $b > 1 \Rightarrow b^n > 1$ for the second.

b) In a) we choose $b^{1/n}$. Then

$$
(b^{1/n})^n - 1 \geq n(b^{1/n} - 1) \;\Rightarrow\; b - 1 \geq n(b^{1/n} - 1).
$$

c) The result in b) together with the fact that $n > (b-1)/(t-1)$ gives us

$$
\begin{aligned}
b - 1 \geq n(b^{1/n} - 1) > \frac{b-1}{t-1}(b^{1/n} - 1) \Rightarrow \\
t - 1 > b^{1/n} - 1 \;\Rightarrow\; b^{1/n} < t.
\end{aligned}
$$

d) Since $b^w < y$ we have that $1 < y \cdot b^{-w} = t$. Therefore we can use the result in c) for sufficently large n

$$
b^{1/n} < t = y \cdot b^{-w} \Rightarrow b^{w+(1/n)} < y.
$$

e) We assume $b^w > y$ which implies $t = y^{-1} \cdot b^w > 1$. Now we can use c) for sufficently large n

$$
b^{1/n} < t = y^{-1} \cdot b^w \Rightarrow y < b^{w-(1/n)}.
$$

f) The set $A \subset \mathbb{R}$ is non-empty. To see this notice that $y > 0$ so that we can make b^w arbitrarily close to 0 by choosing w to be negative integers since $b > 1$. Therefore we can always find a w for which $0 < b^w < y$.

Since A is a non-empty set bounded above by y, we can use the least upper-bound property of $\mathbb{R}$ to show that $x = \sup A$ exists. Now A

is an ordered-set for which we know that only one relation ($<$, $>$, $=$) between b^x and y holds.

Assume $b^x < y$. Then according to the result in d) we have that

$$b^{x+(1/n)} < y.$$

This means that $x \in A$ and $x+(1/n) \in A$. Since $x < x+(1/n)$ we know that x cannot be an upper-bound to A. But this is a contradiction since $x = \sup A$.

Now we assume $b^x > y$. The result in e) gives us that

$$b^{x-(1/n)} > y.$$

This means that $x - (1/n)$ is an upper-bound to A where $x - (1/n) < x$. But then x cannot be the *least* upper-bound to A, which is a contradiction since $x = \sup A$.

Thus $b^x = y$ must be true.

g) Suppose not, then there exists numbers $x \neq x'$ such that $b^x = b^{x'}$. We have that

$$1 = \frac{b^x}{b^{x'}} = b^{x-x'}.$$

Since $b > 1$ the only way to get $b^{x-x'} = 1$ is if $x - x' = 0$. But this is a contradiction since we assumed $x \neq x'$. This completes the proof.