6. First note that if x is real, then for any integers n, m

$$\begin{aligned}(x^n)^m = (\underbrace{x \cdots x}_{n \text{ terms}})^m &= \underbrace{\overbrace{(x \cdots x)}^{n \text{ terms}} \cdots (x \cdots x)}_{m \text{ terms}} \\ &= \underbrace{x \cdots x}_{nm \text{ terms}} = \underbrace{\overbrace{(x \cdots x)}^{m \text{ terms}} \cdots (x \cdots x)}_{n \text{ terms}} \\ &= (\underbrace{x \cdots x}_{m \text{ terms}})^n = (x^m)^n,\end{aligned}$$

which means that $(x^n)^m = (x^m)^n$.

a) Since n is a positive integer and $b > 1$ we know that there is a number α such that $\alpha^n = b$ according to Theorem 1.21. Therefore $b^m = (\alpha^n)^m = (\alpha^m)^n$. Uniqueness of Theorem 1.21 gives us $\alpha^m = (b^m)^{1/n}$. Now we write b^p in terms of α,

$$b^p = (\alpha^n)^p = \underbrace{\alpha \cdots \alpha}_{np \text{ terms}} = \underbrace{\alpha \cdots \alpha}_{mq \text{ terms}} = (\alpha^m)^q,$$

where we used the assumption that $mq = np$. Applying Theorem 1.21 again gives us that $\alpha^m = (b^p)^{1/q}$. Thus,

$$(b^m)^{1/n} = \alpha^m = (b^p)^{1/q}.$$

This completes the proof.

b)

$$b^{r+s} = \underbrace{x \cdots x}_{r+s \text{ terms}} = \underbrace{x \cdots x}_{r \text{ terms}} \cdot \underbrace{x \cdots x}_{s \text{ terms}} = b^r b^s.$$

c) If $s < t$ and $1 < b$, then $b^s < b^t$ for any rationals s, t. Therefore $B(r)$ is bounded by b^r since $b^t \in B(r)$ if $t \leq r$. It is clear that $B(r)$ is non-empty. By the least upper-bound property of $\mathbb{R}$ we have that $\alpha = \sup B(r)$ exists. If we assume $b^r < \alpha$, we get a contradiction since b^r is an upper-bound to $B(r)$ and α is supposed to be the *least* upper-bound to B. If we instead assume $\alpha < b^r$, then α cannot be an upper-bound to $B(r)$ since $r \leq r \Rightarrow b^r \in B(r)$ yet $\alpha < b^r$. Hence

$$\alpha = \sup B(r) = b^r.$$

d) Suppose not, then either $b^xb^y < b^{x+y}$ or $b^{x+y} < b^xb^y$. Assume $b^{x+y} < b^xb^y$. Let A be the set which consists of all numbers b^pb^q where p, q are rationals and $p \leq x$, $q \leq y$ holds. It is clear that $b^xb^y = \sup A$ because $b > 1$. Furthermore recall that $b^{x+y} = \sup B(x+y)$. Take any number $b^rb^s \in A$, this means $r \leq x$ and $s \leq y$ and we get that $r + s \leq x + y$. Because r, s are rationals it follows that

$$b^rb^s = b^{r+s} \in B(x + y).$$

But this leads to a contradiction because that would make $b^{x+y} = \sup A$.

Now assume $b^xb^y < b^{x+y}$. Let $b^t \in B(x + y)$, then we know that $t \leq x + y$. For a rational p, $b^p \in B(x)$ if $p \leq x$ and the same is true for $B(y)$ mutatis mutandis. We now show that it is possible to find rationals r, s such that $t \leq r + s \leq x + y$ where $r \leq x$ and $s \leq y$.

Suppose $x < t$ and $y < t$ and WLOG $x \leq y$. Then $t \leq x+y \Rightarrow t-y \leq x$ which means that we can find a rational r such that $t - y \leq r \leq x$ by Theorem 1.20 b). It follows that $s = t - r \leq y$. We know that s is rational since both t and r are. Hence

$$t = r + t - r \leq r + s \leq x + y.$$

In case the assumption that $x < t$ and $y < t$ is not true, then at least one of x or y is greater than or equal to t. WLOG $t \leq y$ and then by Theorem 1.20 b) there exists a rational s such that $t \leq s \leq y$. We have that $t \leq x + y \Rightarrow t - y \leq x$ and by Theorem 1.20 b) we can find a rational r such that $t - y \leq r \leq x$ which implies that $t - r \leq y$. Then

$$t = r + t - r \leq r + y \leq x + y.$$

These are all the cases, which means that for any $t \leq x + y$ we can find two rationals r, s such that $t \leq r + s \leq x + y$ for which $r \leq x$ and $s \leq y$. Therefore $b^r \in B(x)$ and $b^s \in B(y)$ such that $t \leq r + s$. Because $b > 1$ we have

$$b^t \leq b^{r+s} = b^rb^s \leq b^xb^y.$$

This also leads to a contradiction because that would make $b^xb^y = \sup B(x + y)$ contrary to our assumption.

The only alternative remaining is that $b^{x+y} = b^xb^y$ as desired.