Chapter 2

Exercise 1

Prove that the empty set is a subset of every set.

Let A be any set, $\emptyset$ be the empty set and assume the opposite. Then there exists at least one element $x \in \emptyset$ such that $x \notin A$. But the empty set does not contain any elements and no such x can therefore be found. This is a contradiction and hence $\emptyset \subset A$ must be true.

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Exercise 2

Prove that the set of all algebraic numbers is countable.

Let $\alpha = (a_0, ..., a_n) \in B_{n+1}$ be an n+1-tuple of integers. Example 2.5 shows that the set of all integers $\mathbb{Z}$ is countable. Using Theorem 2.13 with $A = \mathbb{Z}$ shows that the set of n+1-tuples B_{n+1} is countable.

For particular choice of integers $a_0, ..., a_n$ we may construct an equation of the form

$$a_n z^n + \cdots + a_1 z + a_0 = 0.$$

This polynomial has at most n solutions. Let E_α be the set of all complex numbers z that is a solution to that polynomial. Hence E_α is a finite subset of $\mathcal{A}$ with at most n elements. If S_n is a set such that

$$S_n = \bigcup_{\alpha \in B_{n+1}} E_\alpha$$

then it is the union of a countable collection of finite sets. It follows from Theorem 2.12 that S_n is at most countable. Since the union of a countable collection of at most countable sets is at most countable (follows from Theorem 2.12 again); the union

$$S = \bigcup_{n=1}^{\infty} S_n,$$

is at most countable.

We shall show that $\mathcal{A} \subset S$. For any $z \in \mathcal{A}$ there exist integers $a_0, ..., a_k$ such that $a_k z^k + \cdots + a_1 z + a_0 = 0$. For these integers there is a k+1-tuple $\alpha = (a_0, ..., a_k)$ which is associated with exactly one set E_α. Hence there exists at least one α for which $z \in E_\alpha$ and therefore it follows that $\mathcal{A} \subset S$. This means that $\mathcal{A}$ is at most countable since it is a subset of S.

Lastly we need to show that $\mathcal{A}$ is infinite. The set of rational numbers $\mathbb{Q}$ is countable by the corollary to Theorem 2.13. For any $q \in \mathbb{Q}$ there are integers m, n such that $q = \frac{n}{m}$. Now choose $a_1 = m$ and $a_0 = -n$, then

$$a_1 q + a_0 = m\frac{n}{m} - n = 0,$$

which means that q is algebraic. Since q is arbitrary it follows that $\mathbb{Q} \subset \mathcal{A}$. This shows that $\mathcal{A}$ has an infinite subset and must therefore itself be infinite. $\mathcal{A}$ is an infinite set that is at most countable, and therefore it is countable as desired.

$\blacksquare$

Exercise 3

Prove that there are real numbers which are not algebraic.

Suppose not. Let $\mathcal{A}$ be the set of all algebraic numbers. Since we assume the opposite any real number is algebraic and it follows that $\mathbb{R} \subset \mathcal{A}$.

From Exercise 2.2 we know that $\mathcal{A}$ is countable. By Theorem 2.8 this would entail that $\mathbb{R}$ is countable, since it is an infinite subset of a countable set. But this is a contradiction since Theorem 2.43 shows that $\mathbb{R}$ is uncountable. Hence our assumption is false and there exists real numbers $x \in \mathbb{R}$ such that $x \notin \mathcal{A}$.

$\blacksquare$

Exercise 4

Is the set of all irrational real numbers countable?

Denote the set of irrational numbers by $\mathbb{I}$. According to Theorem 2.13 $\mathbb{Q}$ is countable. Now suppose $\mathbb{I}$ is countable. We already know that

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I},$$

which means that $\mathbb{R}$ is countable by Theorem 2.12. But this is a contradiction since by Theorem 2.43 the set of real numbers $\mathbb{R}$ is uncountable. If $\mathbb{I}$ is assumed to be finite, then that still would make $\mathbb{R}$ countable. Hence, $\mathbb{I}$ must be uncountable.

$\blacksquare$

Exercise 5

Construct a bounded set of real numbers with exactly three limit points.

Let k be a natural number and consider the set E_k of numbers $k + \frac{1}{n}$ where $n = 1, 2, 3, \dots$. Clearly $E_k \subset (k, k+1]$ and is therefore bounded.

We need to show that E_k does not contain any limit points. For any natural numbers n, m we have that

$$d\left(k + \frac{1}{n}, k + \frac{1}{m}\right) = \left|k + \frac{1}{n} - \left(k + \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = d\left(\frac{1}{n}, \frac{1}{m}\right).$$

For a fixed n we want to show that

$$d\left(\frac{1}{n}, \frac{1}{m}\right) \geq d\left(\frac{1}{n}, \frac{1}{n+1}\right),$$

holds with any natural number $m \neq n$.

If $m \geq n+1$, then

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \geq \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right).$$

If $m \leq n-1$, then

$$\begin{aligned} d\left(\frac{1}{n}, \frac{1}{m}\right) &= \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{n(n-1)} > \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\ &= \left|\frac{1}{n} - \frac{1}{n+1}\right| = d\left(\frac{1}{n}, \frac{1}{n+1}\right), \end{aligned}$$

which gives us the result. Hence any neighborhood $N_r\left(k + \frac{1}{n}\right)$ with radius $0 < r < \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$ will contain no other points of E_k except $k + \frac{1}{n}$. This means that $k + \frac{1}{n}$ cannot be a limit point of E_k.

We now show that $k \notin E_k$ is a limit point of E_k. For any radius $r > 0$ we can find a natural number N such that $\frac{1}{N} < r$ by using the Archemdian property of $\mathbb{R}$ (Thm. 1.20)

$$d\left(k, k + \frac{1}{N}\right) = \left|k - \left(k + \frac{1}{N}\right)\right| = \frac{1}{N} < r,$$

which means that $k + \frac{1}{N} \in N_r(k)$.

The desire is to construct a set that is bounded with exactly three limit points. We accomplish this by choosing the set E such that

$$E = E_1 \cup E_2 \cup E_3.$$

Specifically, the limit points to E are $1, 2, 3$ and E_1, E_2, E_3 are all bounded which makes E bounded as well.

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Exercise 6

> Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

We first prove that E' is closed. Suppose not, then there exists a limit point x of E' such that $x \notin E'$. For any neighborhood $N_r(x)$ with radius $r > 0$ there is a $y \in E'$ such that $y \in N_r(x)$. Since $y \in E'$ we know that y is a limit point of E. Hence, we can find a $z \in E$ such that $z \in N_{r-h}(y)$ with $0 < h < r$. We have that

$$d(z, x) \leq d(z, y) + d(y, x) < r - h + h = r,$$

which means that $z \in N_r(x)$. Since $r > 0$ is arbitrary it follows that x is a limit point to E. But this is a contradiction since it would imply $x \in E'$. Therefore our assumption is wrong and E' must be closed.

Next we show that $\overline{E}$ and E have the same limit points. If x is a limit point of $\overline{E}$, then since $\overline{E} = E' \cup E$, it must be the case that x is a limit point of E' or E. Assuming x is a limit point of E leaves us with nothing to prove. So we suppose that x is a limit point of E' alone. We already have established that E' is closed, and therefore $x \in E'$. This in turn directly implies that x is a limit point of E because the members of E' are all limit points of E.

Conversely, if y is a limit point of E then we know that $y \in E'$. Since $\overline{E} = E' \cup E$ it follows that $y \in \overline{E}$ and is a limit point of $\overline{E}$.

Lastly, we show that E and E' do not always have the same limit points. Let $E = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ and note 0 is the only limit point of E. This means that $E' = \{0\}$. We know by Corollary to Thm. 2.20 that E' has no limit points. This shows the result by example.

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Exercise 7

> Let $A_1, A_2, A_3, ...$ be subsets of a metric space.
> (a) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\overline{B}_n = \cup_{i=1}^{n} \overline{A}_i$.
> (b) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A}_i$.

(a) We first need to show that for any sets A and B it is true that $(A \cup B)' = A' \cup B'$. If $x \in (A \cup B)'$, then for every neighborhood $N_r(x)$ with a radius $r > 0$ there exists at least one point $p \in N_r(x)$ such that $p \in A \cup B$. Thus, $p \in A$ or $p \in B$ which implies that $x \in A'$ or $x \in B'$ so that $x \in A' \cup B'$.

Suppose now that $x \in A' \cup B'$, then for any neighborhood $N_r(x)$ with a radius $r > 0$ there is a point $p \in N_r(x)$ such that $p \in A$ or $p \in B$. This means that $p \in A \cup B$ and it follows that $x \in (A \cup B)'$.

We now turn to the question at hand. This will be a proof by induction. The base case for $n = 1$ is clearly true. Assume the statement holds for n, then

$$\begin{aligned} \overline{B}_{n+1} &= \overline{\bigcup_{i=1}^{n+1} A_i} = \overline{\bigcup_{i=1}^{n} A_i \cup A_{n+1}} = \overline{B_n \cup A_{n+1}} \\ &= B_n \cup A_{n+1} \cup (B_n \cup A_{n+1})' \\ &= B_n \cup B_n' \cup A_{n+1} \cup A_{n+1}' \\ &= \overline{B}_n \cup \overline{A}_{n+1} = \bigcup_{i=1}^{n} \overline{A}_i \cup \overline{A}_{n+1} \\ &= \bigcup_{i=1}^{n+1} \overline{A}_i, \end{aligned}$$

where we have used the induction hypothesis in the penultimate equality.

(b) If $x \in \bigcup_{i=1}^{\infty} \overline{A}_i$, then there is an $n \in \mathbb{N}$ such that $x \in \overline{A}_n$. We have that

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} \supset \overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A}_i \supset \overline{A}_n,$$

where we used the result in (a) to get the second equality. This implies that $x \in \overline{B}$. Since x is arbitrary it follows that $\bigcup_{i=1}^{\infty} \overline{A}_i \subset \overline{B}$ as desired.

We shall conclude by showing that this subset can indeed be proper. Let $A_i = \left\{\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i}\right\}$ and note that there exists no $k \in \mathbb{N}$ such that $0 \in A_k$. Thus $0 \notin \bigcup_{i=1}^{\infty} A_i$. Furthermore, each set A_i has no limit points which is why $\overline{A}_i = A_i$, and therefore $\bigcup_{i=1}^{\infty} \overline{A}_i = \bigcup_{i=1}^{\infty} A_i$. It follows then that $0 \notin \bigcup_{i=1}^{\infty} \overline{A}_i$. Now let's consider $\overline{B}$

$$\overline{B} = \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\left\{\frac{1}{2}, \frac{1}{3}, \dots\right\}} = \left\{0, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

which implies that $0 \in \overline{B}$. Since $0 \notin \bigcup_{i=1}^{\infty} \overline{A_i}$ we've shown that the inclusion can be proper.

$\blacksquare$

Exercise 8

Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

We first show that every point in every open set $E \subset \mathbb{R}^2$ is a limit point of E. Suppose not. Then there exists an open set $E \subset \mathbb{R}^2$ with a point $x \in E$ that is not a limit point of E. It follows that there exists an $r > 0$ such that the neighborhood $N_r(x)$ contains no point of E except x.

Since E is an open set, x must be an interior point. Hence, there is a neighborhood with radius $s > 0$ such that $N_s(x) \subset E$. Neighborhoods in $\mathbb{R}^2$ are non-empty interiors of circles, which makes N_s and N_r circles centered at x.

Consider the cases for r and x:

1. If $r > s$ then N_s is a circle inscribed in a larger circle N_r which means that $N_s \subset N_r$. It follows that there exists points $y \in N_s \subset N_r \subset E$ where $y \neq x$. But this is impossible due to our assumption.

2. If $r \leq s$ then N_r is inscribed in N_s. Since $N_s \subset E$ that would make $N_r \subset E$ which is also impossible given our assumption.

Since none of the relations $r < s$, $r = s$ and $r > s$ can be true, such an r cannot exists. We've reached a contradiction and the assumption is wrong. Hence, each point of E is a limit point of E.

Now we show that the case does not hold true in general for closed sets. Consider the set of all natural numbers $\mathbb{N} \subset \mathbb{R}^2$. This set is closed, yet every point in this set is not a limit point of it.

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Exercise 9

Let E° be the set of all interior points of E.
(a) Prove that E° is always open.
(b) Prove that E is open if and only if $E^\circ = E$.
(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
(d) Prove that the complement of E° is the closure of the complement of E.
(e) Do E and $\overline{E}$ always have the same interiors?
(f) Do E and E° always have the same closures?

(a) If $p \in E^\circ$ then p is an interior point of E. This means that there exists some neighborhood $N(p)$ with $r > 0$ such that $N(p) \subset E$. By Theorem 2.19 $N(p)$ is an open set, implying that every point in $N(p)$ is an interior point. Hence, $N(p) \subset E^\circ$ and it follows E° is an open set.

(b) If E is open then for every point $x \in E$ we can find a neighborhood $N(x)$ with $r > 0$ such that $N \subset E$. This implies that every point $x \in E$ is an interior point of E and it is therefore true that $x \in E^\circ$. This shows that $E \subset E^\circ$. Since by construction $E^\circ \subset E$ we have that $E^\circ = E$.

Conversely, if $E^\circ = E$ then all points in E are interior points of E. This is true since E° is the set of all interior points of E. Thus, by definition 2.18 (f), E is an open set.

(c) If G is open, then for every point $p \in G$ we can find a neighborhood $N(p)$ with $r > 0$ such that $N(p) \subset G$. Since $G \subset E$, we have that $N(p) \subset E$, which shows that p is an interior point of E. Hence, $p \in E^\circ$. Therefore, $G \subset E^\circ$ since p was arbitrary chosen from G.

(d) Let x be any point in $(E^\circ)^c$. First we show that $(E^\circ)^c \subset \overline{E^c}$. Suppose that $x \notin E$. Then

$$x \in E^c \subset E^c \cup E^{c'} = \overline{E^c}.$$

Now let $x \in E$. Since x is in the complement of E°, we know that x is not an interior point of E. Therefore, for every neighborhood $N(x)$ with radius $r > 0$, we have that $N(x) \not\subset E$. This means that $N(x)$ always has points in E^c, which makes x a limit point of E^c. Thus, $x \in E^{c'} \subset \overline{E^c}$. This shows that $(E^\circ)^c \subset \overline{E^c}$.

Conversely, let $p \in \overline{E^c} = E^c \cup E^{c'}$. Then it is clear that either $p \in E^c$ or $p \in E^{c'}$. Assume $p \in E^c$. Then we know that p is not an interior point to E, so $p \notin E^\circ$, which implies that $p \in (E^\circ)^c$. If $p \in E^{c'}$, then it is a limit point of E^c, and therefore every neighborhood $N(p)$ with radius $r > 0$ have points (other than p) from E^c. Hence, there is no neighborhood such that $N(p) \subset E$ which means that p cannot be an interior point of E. Therefore, $p \in (E^\circ)^c$, and we have now shown that $\overline{E^c} \subset (E^\circ)^c$.

(e) Let $E = (0,1) \cup (1,2)$ be a set consisting of line segments in $\mathbb{R}$. Since E is open we have that $E^\circ = E$ by (b). The set of limit points of E are $\{0,1,2\}$, so the closure $\overline{E}$ is the closed interval $[0,2]$. From here we can see that $\left(\overline{E}\right)^\circ = (0,2)$, which shows that $E^\circ \neq \left(\overline{E}\right)^\circ$. Therefore, E and $\overline{E}$ do not always have the same interiors.

(f) Consider the set $E = \left\{ \frac{1}{n} \mid n = 1, 2, 3, ... \right\}$ as subset of $\mathbb{R}$. Since 0 is the only limit point of E, we have that its closure is $\overline{E} = E \cup \{0\}$. However, none of the points in E are interior points, and therefore $E^\circ = \emptyset$. This means that $\overline{E^\circ} = \emptyset$. We have shown that $\overline{E} \neq \overline{E^\circ}$ as desired.

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Exercise 10

Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{(if } p \neq q) \\ 0 & \text{(if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

We begin by showing that d is a metric. Conditions 2.15 (a) and (b) are clearly satisfied. We show (c) also is true. Suppose $p \neq q$ so that $d(p,q) = 1$. For any $r \in X$ we have two cases, either r equals one of p, q or neither.

Suppose WLOG that $r = p$. Then we have that $d(p,r) = 0$ and $d(r,q) = 1$. It follows that

$$\underbrace{d(p,q)}_{=1} \leq \underbrace{d(p,r)}_{=0} + \underbrace{d(r,q)}_{=1} = 1.$$

Now let $r \neq p \neq q$ be true. Then $d(p,r) = d(r,q) = 1$, and we can show that

$$\underbrace{d(p,q)}_{=1} < \underbrace{d(p,r)}_{=1} + \underbrace{d(r,q)}_{=1} = 2.$$

Lastly, if $p = q$ then $d(p,q) = 0$ and any point $r \in X$ will satisfy (c) since the distance function is non-negative. This shows that X is a metric with distance function d.

We shall now show that any subset $E \subset X$ is open. Let p be any point in E and consider the neighborhood $N_r(p)$ with radius $r = 1$. Given the metric, this neighborhood contains only p itself so that $N_r(p) \subset E$. Hence, every point in E is an interior point and E is open.

We now show that every subset in X is closed. Let E be any subset of X. Since every subset of X is open, the complement E^c is also open. By Thm. 2.33 E is closed.

Lastly, we show that only finite sets of X are compact (by Definition 2.32). Suppose not, then we have a compact set $K \subset X$ that is infinite. For every $p \in K$ let G_p be the open neighborhood around p with radius $r = 1$. Since every $p \in K$ is associated with an open set G_p, the collection $\left\{G_p\right\}$ is an open cover of K. Because K is compact, there exists a finite number of indices such that

$$K \subset G_{p_1} \cup \cdots \cup G_{p_m}.$$

Every subset G_{p_n} is an open neighborhood around p_n with radius 1. From before we know that these sets only contain a single point, namely p_n. But that is absurd, since it would make K finite.

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Exercise 11

For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= \left|x^2 - y^2\right|, \\ d_4(x,y) &= (x-2y), \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

d_1 Condition 2.15 (c) is not satisfied, which therefore is not a metric. We give an example,

$$d_1(10,0) > d_1(10,4) + d_1(4,0).$$

d_2 Both 2.15 (a) and (b) are clearly true. We show that (c) is also satisfied. Assume not, then there exists points x, y and r such that

$$d_2(x,y) > d_2(x,r) + d_2(r,y),$$

which in this particular case is

$$\sqrt{|x-y|} > \sqrt{|x-r|} + \sqrt{|r-y|}.$$

If $0 < q < p$ then $q^2 < p^2$ for any $p, q \in \mathbb{R}$ so that

$$
\begin{aligned}
|x-y| = \sqrt{|x-y|}^2 &> \left(\sqrt{|x-r|} + \sqrt{|r-y|}\right)^2 \\
&= |x-r| + 2\sqrt{|x-r||r-y|} + |r-y| \\
&\geq |x-r| + |r-y|,
\end{aligned}
$$

where the last inequality comes from the fact that $\sqrt{|x-r||r-y|} \geq 0$. The above shows that $d(p,q) = |p-q|$ cannot be a metric. But that is a contradiction, since Theorem 1.37 shows that $|p-q|$ satisfies 2.15 (c). Hence, d_2 is a metric.

d_3 Condition 2.15 (a) is not satisfied. We show an example

$$
d_3(2,-2) = \left|2^2 - (-2)^2\right| = 4 - 4 = 0.
$$

We have found $x \neq y$ such that $d_3(x,y) = 0$. This shows that $d_3(x,y)$ is not a metric.

d_4 We show that d_4 is not a metric since condition (a) of Definition 2.15 is not satisfied. If $x \neq 0$ then

$$
d_4(x,y) = |x - 2x| = |x| > 0.
$$

d_5) Conditions in (a), (b) are clearly satisfied so we focus on (c). Throughout this exercise WLOG assume $x \leq y$. Suppose $x \leq r \leq y$, we have that

$$
\begin{aligned}
d_5(x,y) = \frac{|x-y|}{1+|x-y|} &= \frac{|x-r+r-y|}{1+|x-y|} \\
&\leq \frac{|x-r|}{1+|x-y|} + \frac{|r-y|}{1+|x-y|} \\
&\leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|} \\
&\leq d_5(x,r) + d_5(r,y),
\end{aligned}
$$

where we used Theorem 1.37 in the first inequality. Since $x \leq r \leq y$ we know that $|x-r| \leq |x-y|$ and $|r-y| \leq |x-y|$, both of which we used to get the second inequality.

Now assume $x \leq y < r$. First we show that for any $\varepsilon > 0$

$$
\frac{|w|}{1+|w|} < \frac{|w|+\varepsilon}{1+|w|+\varepsilon}, \tag{1}
$$

holds. The statement is clearly true for $w = 0$. If $|w| > 0$, then

$$\frac{|w|}{1+|w|} = \frac{1}{1+\frac{1}{|w|}} < \frac{1}{1+\frac{1}{|w|+\varepsilon}} = \frac{|w|+\varepsilon}{1+|w|+\varepsilon},$$

which shows that (1) is true.

Since $x \leq y < r$, we have that $|x-y| < |x-r|$. This means we can use (1) to get the first inequality below,

$$\begin{aligned} d_5(x,y) &= \frac{|x-y|}{1+|x-y|} < \frac{|x-r|}{1+|x-r|} \\ &\leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|} \\ &= d_5(x,r) + d_5(r,y). \end{aligned}$$

The last inequality is due to the last term being non-negative. Similar argument can be made for $r < x \leq y$ because $|x-y| < |r-y|$ which allows us to use (1) again. This shows that $d_5(x,y)$ is a metric.

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Exercise 12

> Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Let $\{G_\alpha\}$ be any open cover of K. Then we know that there is an index α_0 such that $0 \in G_{\alpha_0}$. Since G_{α_0} is an open set, we know that there exists a neighborhood $N_r(0)$ with radius $r > 0$ where $N_r \subset G_{\alpha_0}$.

If $r \geq 1$, then G_{α_0} covers K and there is nothing to prove. Assume therefore that $r < 1$. By Archimedean property of $\mathbb{R}$ we can find positive integers p such that $r \geq 1/p$. Let m be the smallest integer such that $r > 1/m$. It follows that if q is an integer where $q > m$, then $r > 1/q$ so that $1/q \in N_r(0)$. This shows that there are at most $m-1$ points in E that are not in N_r,

$$\frac{1}{m-1}, \frac{1}{m-2}, \ldots, \frac{1}{2}, 1.$$

Let G_{α_k} denote an open set in the collection $\{G_\alpha\}$ such that $1/k \in G_{\alpha_k}$ $(k = 1, 2, 3, \ldots)$. Because $\{G_\alpha\}$ is an open cover of K, each of the above $m-1$ points can be associated this way to at least one index in the collection. Therefore

$$K \subset G_{\alpha_0} \cup G_1 \cup \cdots \cup G_{\alpha_{m-1}}.$$

We have shown that any open cover of K has a finite sub-cover which implies that K is compact as desired.

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Exercise 13

> Construct a compact set of real numbers whose limit points form a countable set.

Let n be a natural number and construct a sequence $\{x_k\}$ as follows. Define x_1 as the midpoint between $\frac{1}{n+1}$ and $\frac{1}{n}$

$$x_1 = \frac{\frac{1}{n+1} + \frac{1}{n}}{2}.$$

Having chosen $x_1, ..., x_{k-1}$ (for $k = 2, 3, 4, ...$), define x_k as the midpoint between x_{k-1} and $\frac{1}{n}$

$$x_k = \frac{x_{k-1} + \frac{1}{n}}{2}.$$

Let E_n be the set of all numbers generated by the sequence $\{x_k\}$, along with the point $\frac{1}{n}$. For example, if $n = 1$ we have

$$E_1 = \left\{1, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, ...\right\}.$$

Since the sequence $\{x_k\}$ is constructed by repeatedly taking midpoints between the previous term and $\frac{1}{n}$, the sequence will approach the value $\frac{1}{n}$, making it a limit point of E_n.

We can show that any other point $y \in \mathbb{R}$ such that $y \neq \frac{1}{n}$ cannot be a limit point of E_n. Since E_n is a countable set of values that approach $\frac{1}{n}$, there is a number $x_k \neq y$ in E_n that minimises $d(y, x_k)$. Then the neighborhood with $0 < r < d(y, x_k)$ will have no point of E_n (other than possibly y). Hence, $\frac{1}{n}$ is the only limit point so that $E_n' = \left\{\frac{1}{n}\right\}$.

The collection of sets $\{E_n\}$ is disjoint since $E_n \subset \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Let S be the union of this collection along with the point 0,

$$S = \{0\} \cup \bigcup_{n=1}^{\infty} E_n \subset [0, 1].$$

Since S is a union of disjoint sets, the limit points of S is given by

$$S' = \{0\} \cup \bigcup_{n=1}^{\infty} E' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\},$$

which is countable. Furthermore, because $E' \subset E$ and $0 \in S$ we see that $S' \subset S$ so that S is closed. Since S is a subset of the compact interval $[0, 1]$, by Theorem 2.35 S is compact.

We have now constructed a compact set S of real numbers whose limit points S' form a countable set. This concludes the proof.

■