5. Let l be a lower bound to A. Then $\forall x \in A$ it follows that $l \leq x$. Therefore we have $-x \leq -l$, which shows that $-A$ is bounded above by $-l$. The set of real numbers $\mathbb{R}$ has the least upper-bound property, thus

$$\beta = \sup(-A).$$

For any $x \in A$ we know that $-x \in -A$. Since $-A$ is bounded above we have $-x \leq \beta$ which implies $-\beta \leq x$. It follows that $-\beta$ is a lower-bound to A.

Let us now show that $-\beta = \inf A$. Suppose there exists another lower-bound γ to A such that $-\beta < \gamma$. If this is the case then $-\beta$ cannot be $\inf A$. Since γ is a lower-bound to A we know that $\gamma \leq x$ for any $x \in A$ which implies $-x \leq -\gamma$. Hence for any $-x \in -A$ we have that $-x \leq -\gamma$. We have now found $-\gamma$ to be an upper-bound to $-A$. Because $-\beta < \gamma$ implies that $-\gamma < \beta$ holds, we have found that γ is an upper-bound to A which is smaller that β. But this is a contradiction since $\beta = \sup -A$ and no such γ could exists. Therefore $-\beta = \inf A$ and

$$\inf A = -\sup(-A),$$

as desired.

$\square$