Exercise 1.13: First we show that $\mathrm{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ using induction. It is clear that $\mathrm{Fib}(1) = (\varphi - \psi)/\sqrt{5} = 1$. Now we show that this holds for $k+1$

$$\begin{aligned}\mathrm{Fib}(k+1) = \mathrm{Fib}(k) + \mathrm{Fib}(k-1) &= \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} \\ &= \frac{(\varphi+1)\varphi^{k-1} - (\psi+1)\psi^{k-1}}{\sqrt{5}} = \frac{\varphi^2\varphi^{k-1} - \psi^2\psi^{k-1}}{\sqrt{5}} \\ &= \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}},\end{aligned}$$

where used the fact that both φ and ψ satisfy $x^2 = x + 1$. Using the result above, we get that

$$\begin{aligned}\left|\frac{\varphi^n}{\sqrt{5}} - \mathrm{Fib}(n)\right| = \left|\frac{\psi^n}{\sqrt{5}}\right| = \left|\frac{(1-\sqrt{5})^n}{2^n\sqrt{5}}\right| &< \frac{1}{2} \cdot \left|\frac{1-\sqrt{5}}{2}\right|^n < \frac{1}{2} \cdot \left|\frac{1-3}{2}\right|^n \\ &\leq \frac{1}{2}.\end{aligned}$$

Suppose there exists another integer m that is closer to $\varphi^n/\sqrt{5}$. Then

$$\begin{aligned}\left|\mathrm{Fib}(n) - m\right| &= \left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}} + \frac{\varphi^n}{\sqrt{5}} - m\right| \\ &\leq \left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}}\right| + \left|\frac{\varphi^n}{\sqrt{5}} - m\right| \\ &< \frac{1}{2} + \frac{1}{2} = 1,\end{aligned}$$

which is a contradiction. Since $\mathrm{Fib}(n) \neq m$ and both are integers their difference must be strictly greater than 1.

□