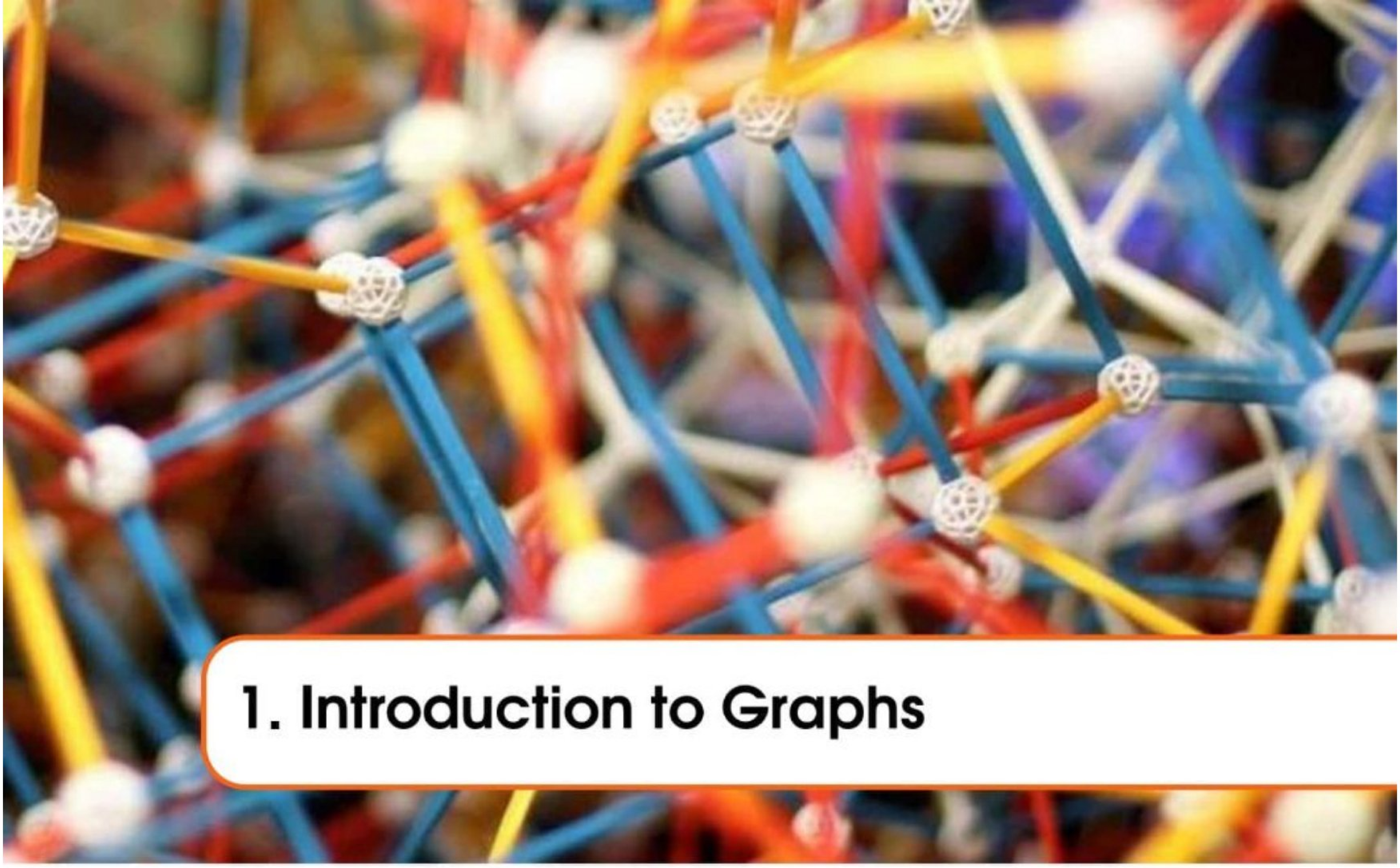

1. Introduction to Graphs

Graph Theory is a well-known area of discrete mathematics which deals with the study of graphs. A graph may be considered as a mathematical structure that is used for modelling the pairwise relations between objects.

Graph Theory has many theoretical developments and applications not only to different branches of mathematics, but also to various other fields of basic sciences, technology, social sciences, computer science etc. Graphs are widely used as efficient tools to model many types of practical and real-world problems in physical, biological, social and information systems. Graph-theoretical models and methods are based on mathematical combinatorics and related fields.

1.1 Basic Definitions

Definition 1.1.1 — Graph. A *graph* G can be considered as an ordered triple (V, E, ψ), where

(i) $V = \{v_1, v_2, v_3, \ldots\}$ is called the *vertex set* of G and the elements of V are called the *vertices* (or *points* or *nodes*);

(ii) $E = \{e_1, e_2, e_3, \ldots\}$ is the called the *edge set* of G and the elements of E are called *edges* (or *lines* or *arcs*); and

(iii) ψ is called the *adjacency relation*, defined by $\psi : E \to V \times V$, which defines the association between each edge with the vertex pairs of G.

Usually, the graph is denoted as $G = (V, E)$. The vertex set and edge set of a graph G are also written as $V(G)$ and $E(G)$ respectively.

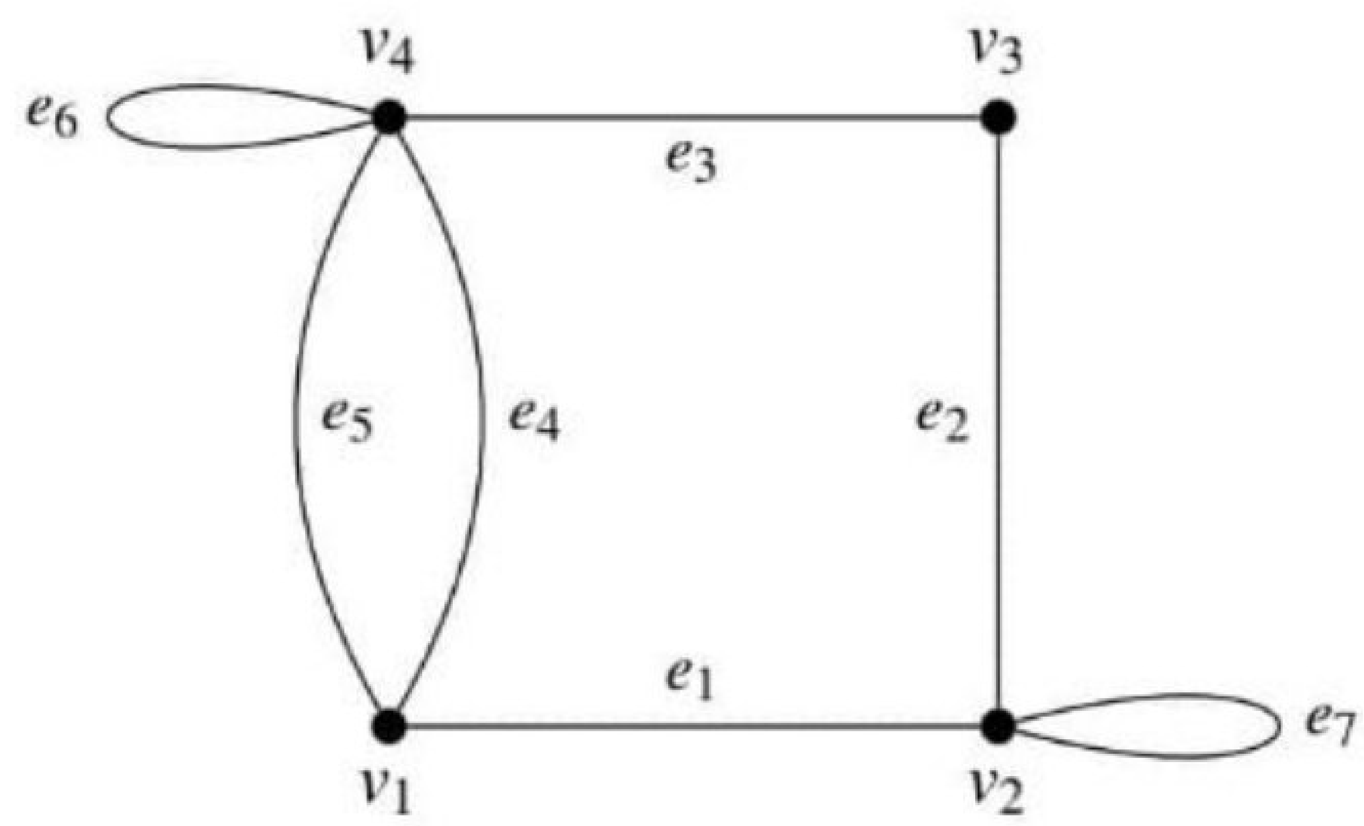

Figure 1.1: An example of a graph

If two vertices u and v are the (two) end points of an edge e, then we represent this edge by uv or vu. If $e = uv$ is an edge of a graph G, then we say that u and v are *adjacent vertices* in G and that *e joins* u and v. In such cases, we also say that u and v are adjacent to each other.

Given an edge $e = uv$, the vertex u and the edge e are said to be *incident with* each other and so are v and e. Two edges e_i and e_j are said to be *adjacent edges* if they are incident with a common vertex.

Definition 1.1.2 — Order and Size of a Graph. The *order* of a graph G, denoted by $\nu(G)$, is the number of its vertices and the *size* of G, denoted by $\varepsilon(G)$, is the number of its edges.

A graph with p-vertices and q-edges is called a (p,q)*-graph*. The $(1,0)$-graph is called a *trivial graph*. That is, a trivial graph is a graph with a single vertex. A graph without edges is called an *empty graph* or a *null graph*. The following figure illustrates a null graph of order 5.

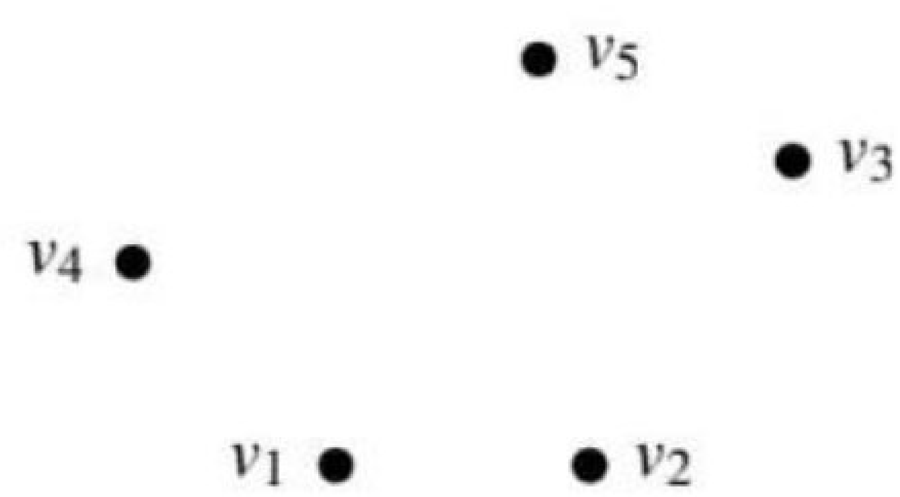

Figure 1.2: Null graph of order 5.

Definition 1.1.3 — Finite and Infinite Graphs. A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph*. Otherwise, it is an *infinite graph*.

Definition 1.1.4 — Self-loop. An edge of a graph that joins a node to itself is called *loop* or a *self-loop*. That is, a loop is an edge uv, where $u = v$.

Definition 1.1.5 — Parallel Edges. The edges connecting the same pair of vertices are called *multiple edges* or *parallel edges*.

In Figure 1.2, the edges e_6 and e_7 are loops and the edges e_4 and e_5 are parallel edges.

Definition 1.1.6 — Simple Graphs and Multigraphs. A graph G which does not have loops or parallel edges is called a *simple graph*. A graph which is not simple is generally called a *multigraph*.

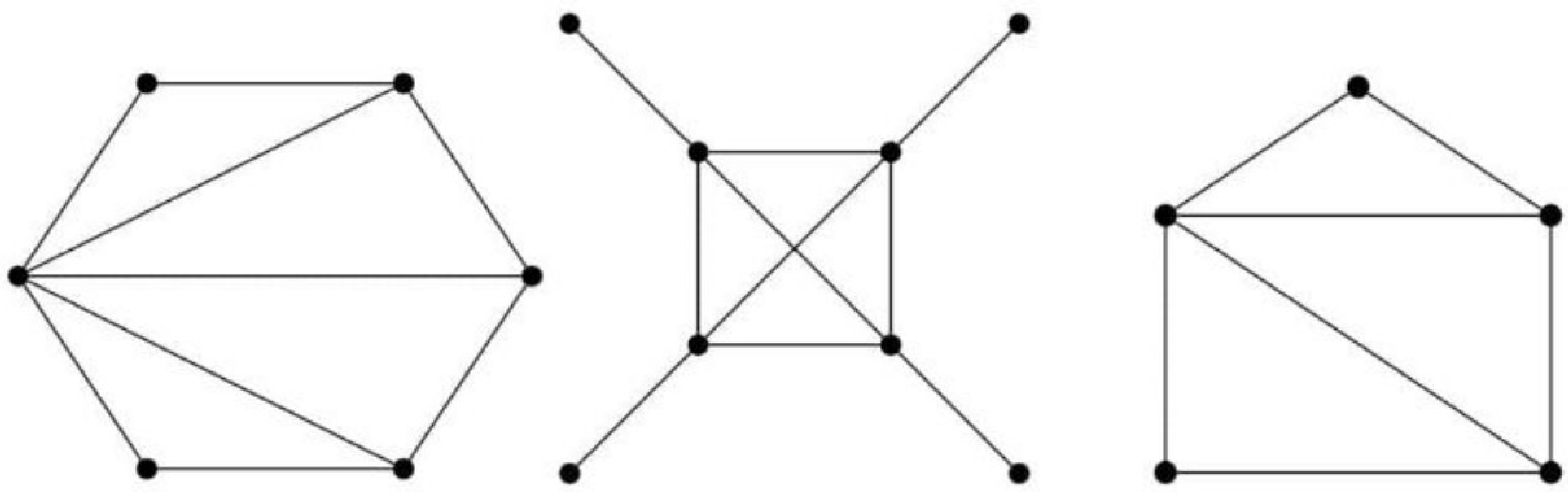

Figure 1.3: Some examples of simple graphs

1.2 Degrees and Degree Sequences in Graphs

Definition 1.2.1 — Degree of a vertex. The number of edges incident on a vertex v, with self-loops counted twice, is called the *degree* of the vertex v and is denoted by $deg_G(v)$ or $deg(v)$ or simply $d(v)$.

Definition 1.2.2 — Isolated vertex. A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are those with zero degree.

Definition 1.2.3 — Pendant vertex. A vertex of degree 1, is called a *pendent vertex* or an end vertex.

Definition 1.2.4 — Internal vertex. A vertex, which is neither a pendent vertex nor an isolated vertex, is called an *internal vertex* or an *intermediate vertex*.

Definition 1.2.5 — Minimum and Maximum Degree of a Graph. The *maximum degree of a graph* G, denoted by $\Delta(G)$, is defined to be $\Delta(G) = \max\{d(v) : v \in V(G)\}$. Similarly, the *minimum degree of a graph* G, denoted by $\delta(G)$, is defined to be $\delta(G) = \min\{d(v) : v \in V(G)\}$. Note that for any vertex v in G, we have $\delta(G) \leq d(v) \leq \Delta(G)$.

The following theorem is a relation between the sum of degrees of vertices in a graph G and the size of G.

Theorem 1.2.1 In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. That is, $\sum_{v \in V(G)} d(v) = 2\varepsilon$.

Proof. Let $S = \sum_{v \in V(G)} d(v)$. Notice that in counting S, we count each edge exactly twice. That is, every edge contributes degree 1 each to both of its end vertices and a loop provides degree 2 to the vertex it incidents with. Hence 2 to the sum of degrees of vertices in G. Thus, $S = 2|E| = 2\varepsilon$. ■

The above theorem is usually called the *first theorem on graph theory*. It is also known as the *hand shaking lemma*. The following two theorems are immediate consequences of the above theorem.

Theorem 1.2.2 For any graph G, $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$.

Proof. By Theorem-1, we have $2\varepsilon = \sum_{v \in V(G)} d(v)$. Therefore, note that $\frac{2|E|}{|V|} = \frac{\sum d(v)}{|V|}$, the average degree of G. Therefore, $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$. ■

Theorem 1.2.3 For any graph G, the number of odd degree vertices is always even.

Proof. Let $S = \sum_{v \in V(G)} d(v)$. By Theorem 1.2.1, we have $S = 2\varepsilon$ and hence S is always even. Let V_1 be the set of all odd degree vertices and V_2 be the set of all even degree vertices in G. Now, let $S_1 = \sum_{v \in V_1} d(v)$ and $S_2 = \sum_{v \in V_2} d(v)$. Note that S_2, being the sum of even integers, is also an even integer.

We also note that $S = S_1 + S_2$ (since V_1 and V_2 are disjoint sets and $V_1 \cup V_2 = V$). Therefore, $S_1 = S - S_2$. Being the difference between two even integers, S_1 is also an even integer. Since V_1 is a set of odd degree vertices, S_1 is even only when the number of elements in V_1 is even. That is, the number of odd degree vertices in G is even, completing the proof. ■

Definition 1.2.6 — Degree Sequence. The *degree sequence* of a graph of order n is the n-term sequence (usually written in descending order) of the vertex degrees. In Figure-1, $\delta(G) = 2$, $\Delta(G) = 5$ and the degree sequence of G is $(5,4,3,2)$.

Definition 1.2.7 — Graphical Sequence. An integer sequence is said to be *graphical* if it is the degree sequence of some graphs. A graph G is said to be the *graphical realisation* of an integer sequence S if S the degree sequence of G.

Problem 1.1 Is the sequence $S = \langle 5,4,3,3,2,2,2,1,1,1,1 \rangle$ graphical? Justify your answer.
Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, $\sum a_i = 25$, not an even number. Therefore, the given sequence is not graphical. ■

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Problem 1.2 Is the sequence $S = \langle 9,9,8,7,7,6,6,5,5\rangle$ graphical? Justify your answer.

Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, $\sum a_i = 62$, an even number. But note that the maximum degree that a vertex can attain in a graph of order n is $n-1$. If S were graphical, the corresponding graph would have been a graph on 10 vertices and have $\Delta(G) = 9$. Therefore, the given sequence is not graphical. ■

Problem 1.3 Is the sequence $S = \langle 9,8,7,6,6,5,5,4,3,3,2,2\rangle$ graphical? Justify your answer.

Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, we have $\sum a_i = 60$, an even number. Also, note that the all elements in the sequence are less than the number of elements in that sequence.

Therefore, the given sequence is graphical and the corresponding graph is drawn below. ■

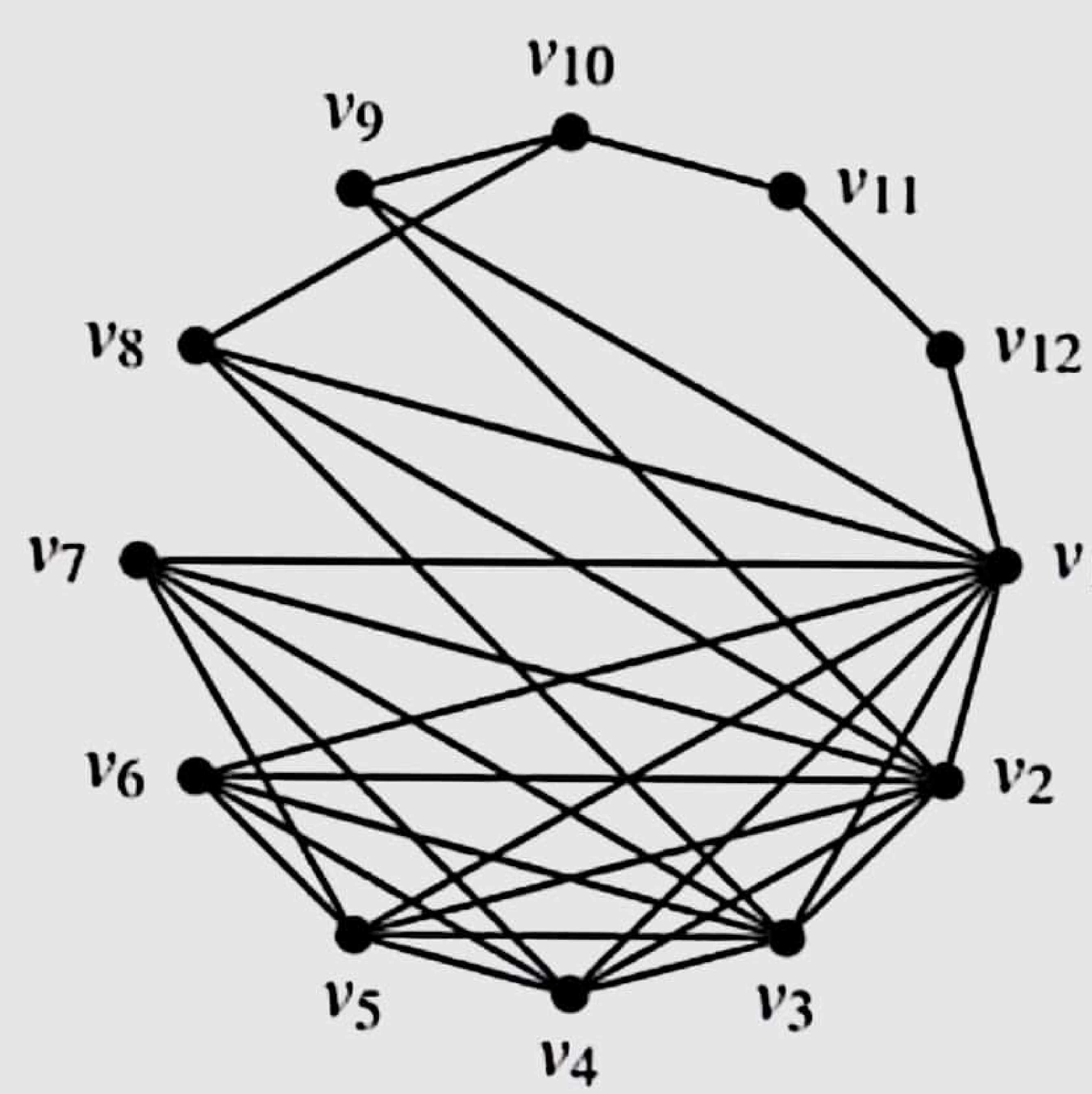

Figure 1.4: Graphical realisation of the degree sequence S.