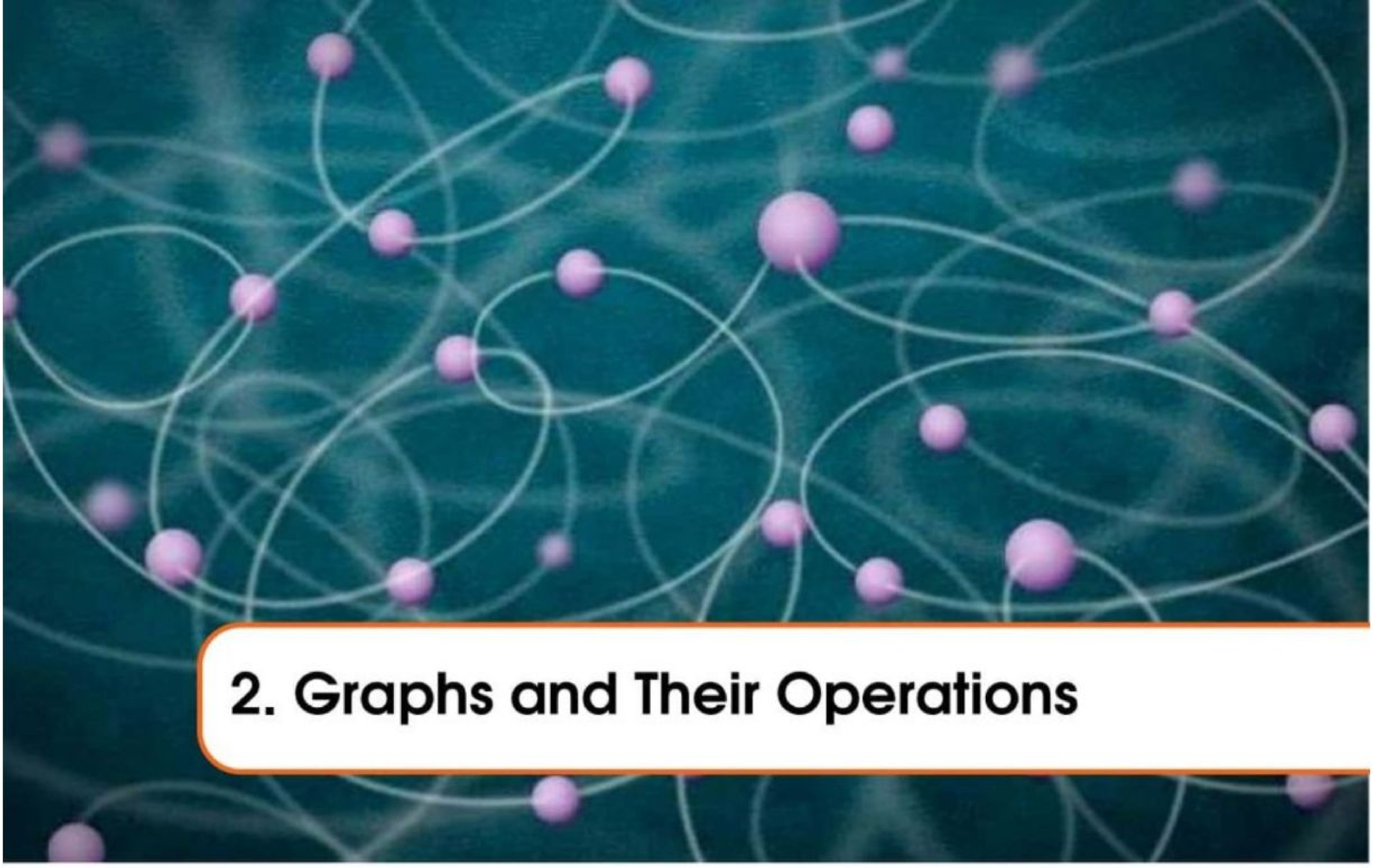

2. Graphs and Their Operations

We have already seen that the notion of subgraphs can be defined for any graphs as similar to the definition of subsets to sets under consideration. Similar to the definitions of basic set operations, we can define the corresponding basic operations for graphs also. In addition to these fundamental graph operations, there are some other new and useful operations are also defined on graphs. In this chapter, we discuss some basic graph operation.

2.1 Union, Intersection and Ringsum of Graphs

Definition 2.1.1 — Union of Graphs. The *union* of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \cup G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

Definition 2.1.2 — Intersection of Graphs. The *intersection* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \cap G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Definition 2.1.3 — Ringsum of Graphs. The *ringsum* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \oplus G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \oplus E(G_2)$, where $\oplus$ is the symmetric difference (XOR Operation) of two sets.

Figure 2.1 illustrates the union, intersection and ringsum of two given graphs.

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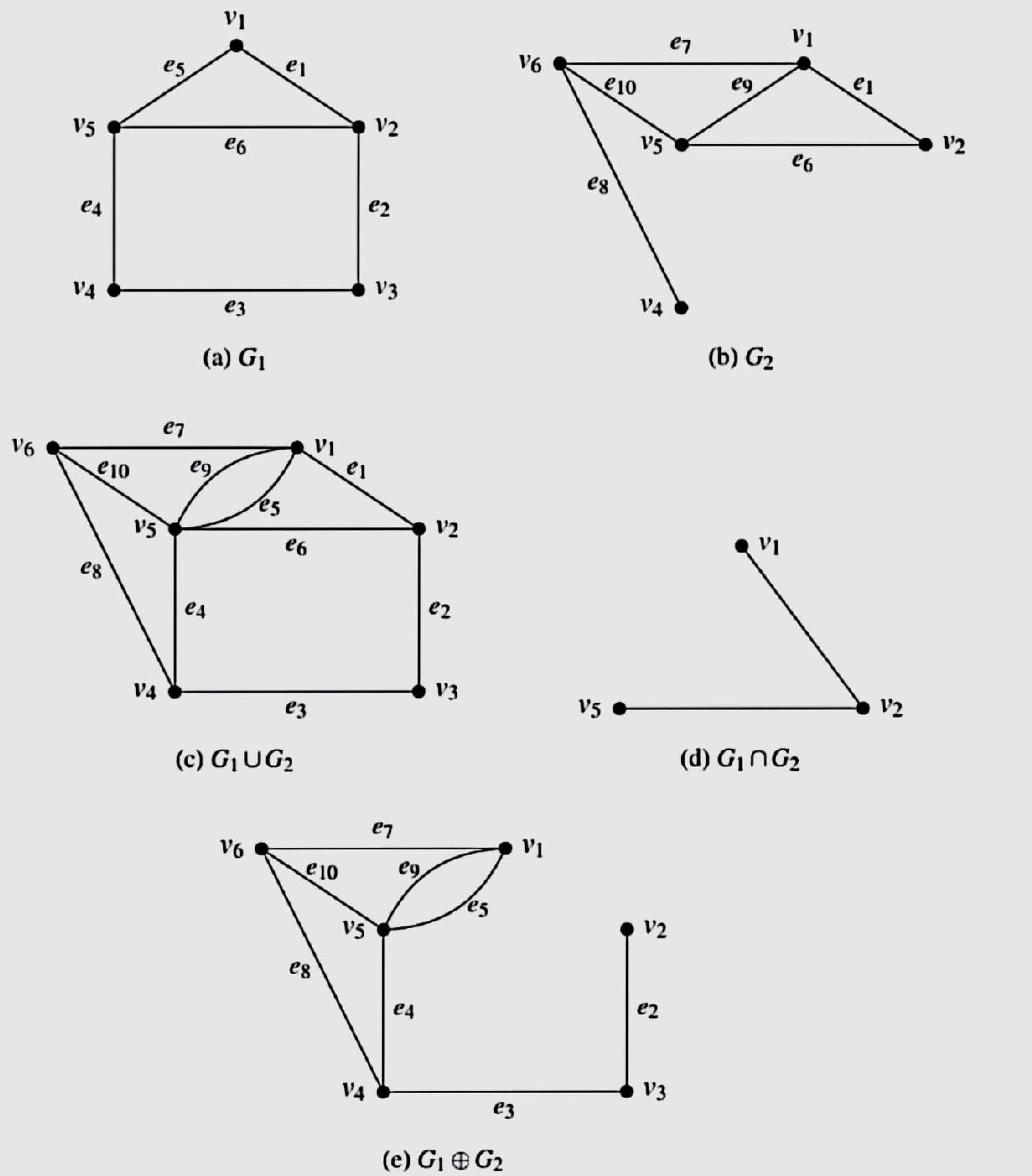


Figure 2.1: Illustrations to graph operations

R

1. The union, intersection and ringsum operations of graphs are commutative. That is, $G_1 \cup G_2 = G_2 \cup G_1, G_1 \cap G_2 = G_2 \cap G_1$ and $G_1 \oplus G_2 = G_2 \oplus G_1$.
2. If G_1 and G_2 are edge-disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$.
3. If G_1 and G_2 are vertex-disjoint, then $G_1 \oplus G_2$ is empty.
4. For any graph G, $G \cap G = G \cup G$ and $G \oplus G$ is a null graph.

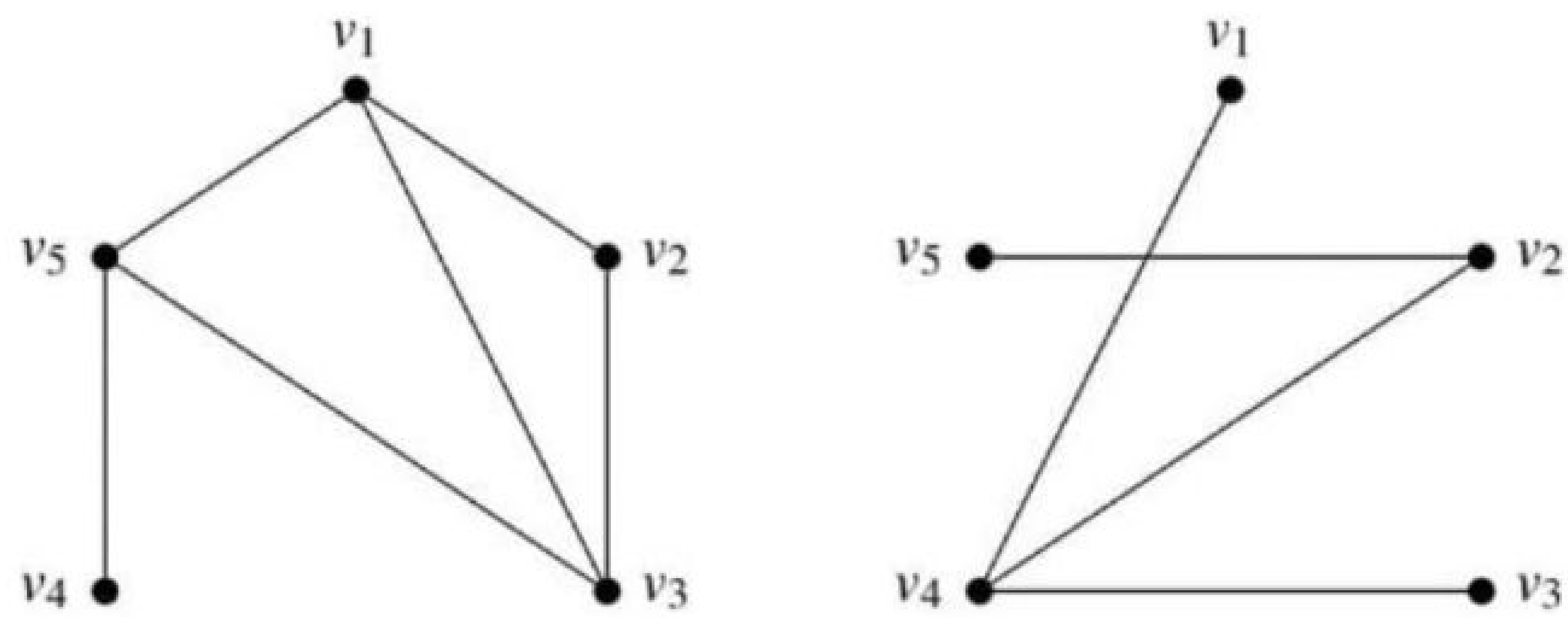
2.2 Complement of Graphs

Definition 2.2.1 — Complement of Graphs. The *complement* or *inverse* of a graph G, denoted by $\bar{G}$ is a graph with $V(G) = V(\bar{G})$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

R Note that for a graph G and its complement $\bar{G}$, we have

(i) $G_1 \cup \bar{G} = K_n$;
(ii) $V(G) = V(\bar{G})$;
(iii) $E(G) \cup E(\bar{G}) = E(K_n)$;
(iv) $|E(G)| + |E(\bar{G})| = |E(K_n)| = \binom{n}{2}$.

A graph and its complement are illustrated below.



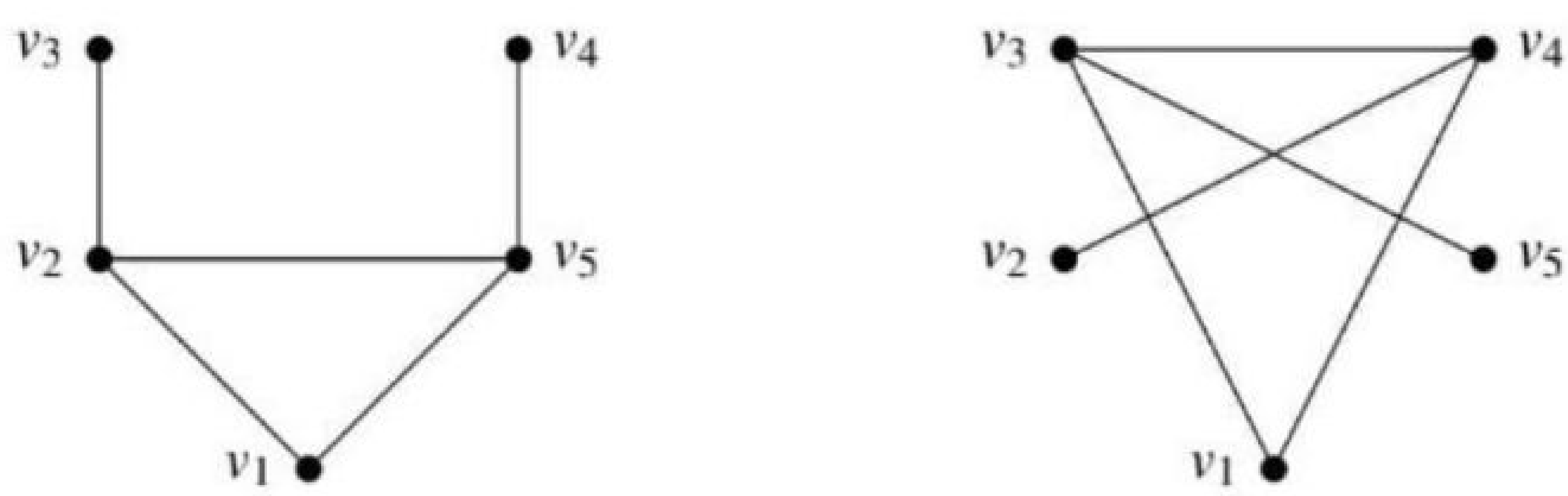
(a) G (b) $\bar{G}$

Figure 2.2: A graph and its complement

2.2.1 Self-Complementary Graphs

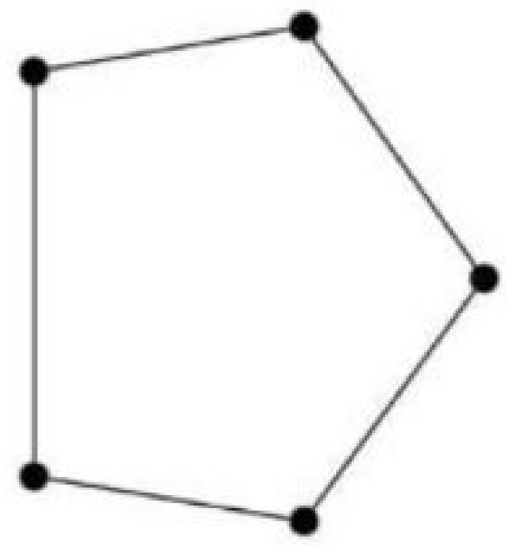
Definition 2.2.2 — Self-Complementary Graphs. A graph G is said to be *self-complementary* if G is isomorphic to its complement. If G is self complementary, then $|E(G)| = |E(\bar{G})| = \frac{1}{2}|E(K_n)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

The following are two examples of self complementary graphs.

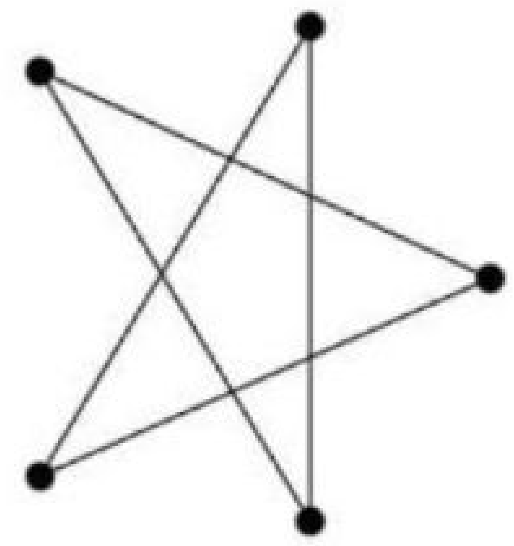


(a) Bull graph G. (b) $\bar{G}$.

Figure 2.3: Example of self-complementary graphs



(a) G_2



(b) $\bar{G}_2$.

Figure 2.4: Example of self-complementary graphs

Problem 2.1 For any self-complementary graph G of order n, show that $n \equiv 0, 1 (\text{mod } 4)$.
Solution: For self-complementary graphs, we have
(i) $V(G) = V(\bar{G})$;
(ii) $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$;
(iii) $|E(G)| = |E(\bar{G})|$.
Therefore, $|E(G)| = |E(\bar{G})| = \frac{n(n-1)}{4}$. This implies, 4 divides either n or $n-1$. That is, for self-complementary graphs of order n, we have $n \equiv 0, 1 (\text{mod } 4)$. ■

(Note that we say $a \equiv b (\text{mod } n)$, which is read as "$a$ is congruent to b modulo n", if $a - b$ is completely divisible by n).

2.3 Join of Graphs

Definition 2.3.1 The *join* of two graphs G and H, denoted by $G + H$ is the graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

In other words, the join of two graphs G and H is defined as the graph in which every edge of the first graph is adjacent to all vertices of the second graph.

Figure 2.5 illustrates the join of two graphs P_3 and P_4 and Figure 2.6 illustrates the join of two graphs C_5 and P_2.

Figure 2.5: The join of the paths P_4 and P_3.

2.4 Deletion and Fusion

Definition 2.4.1 — Edge Deletion in Graphs. If e is an edge of G, then $G - e$ is the graph obtained by removing the edge of G. The subgraph of G thus obtained is called an *edge-deleted subgraph* of G. Clearly, $G - e$ is a spanning subgraph of G.

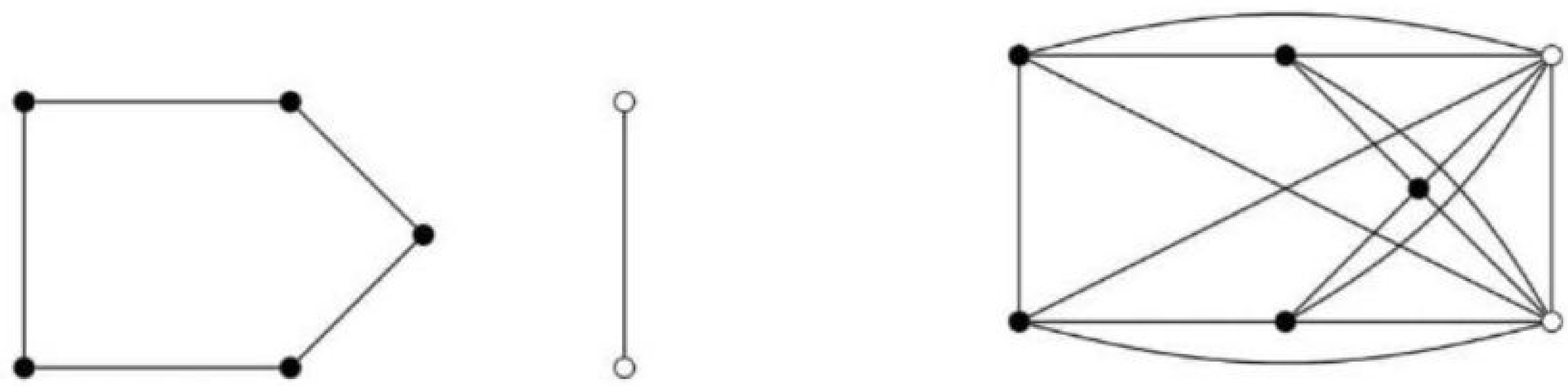
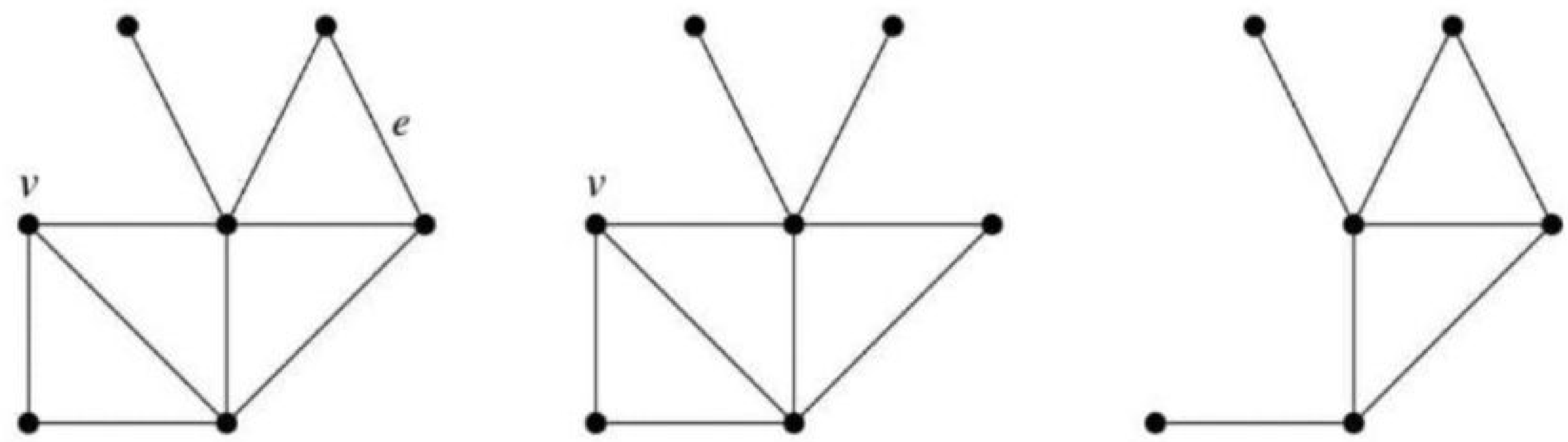

(a) C_5 (b) P_2 (c) $C_5 + P_2$.

Figure 2.6: The join of the cycle C_5 and the path P_2.

Similarly, vertex-deleted subgraph of a graph is defined as follows:

Definition 2.4.2 — Vertex Deletion in Graphs. If v is a vertex of G, then $G - v$ is the graph obtained by removing the vertex v and all edges G that are incident on v. The subgraph of G thus obtained is called an *vertex-deleted subgraph* of G. Clearly, $G - v$ will not be a spanning subgraph of G.

Figure 2.7 illustrates the edge deletion and the vertex deletion of a graph G.



(a) G (b) $G - e$ (c) $G - v$

Figure 2.7: Illustrations to edge deletion and vertex deletion

Definition 2.4.3 — Fusion of Vertices. A pair of vertices u and v are said to be *fused* (or *merged* or *identified*) together if the two vertices are together replaced by a single vertex w such that every edge incident with either u or v is incident with the new vertex w (see Figure 2.8).

Note that the fusion of two vertices does not alter the number of edges, but reduces the number of vertices by 1.

2.4.1 Edge Contraction

Definition 2.4.4 — Edge Contraction in Graphs. An *edge contraction* of a graph G is an operation which removes an edge from a graph while simultaneously merging its two end vertices that it previously joined. Vertex fusion is a less restrictive form of this operation.

A graph obtained by contracting an edge e of a graph G is denoted by $G \circ e$.

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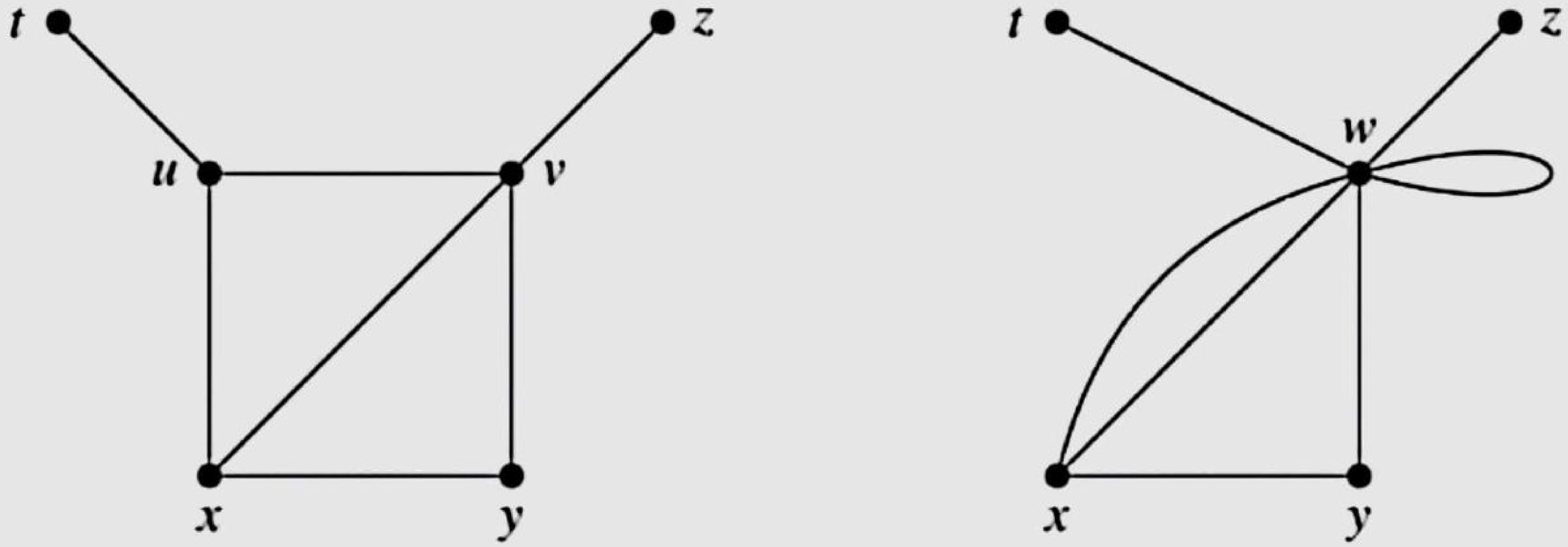
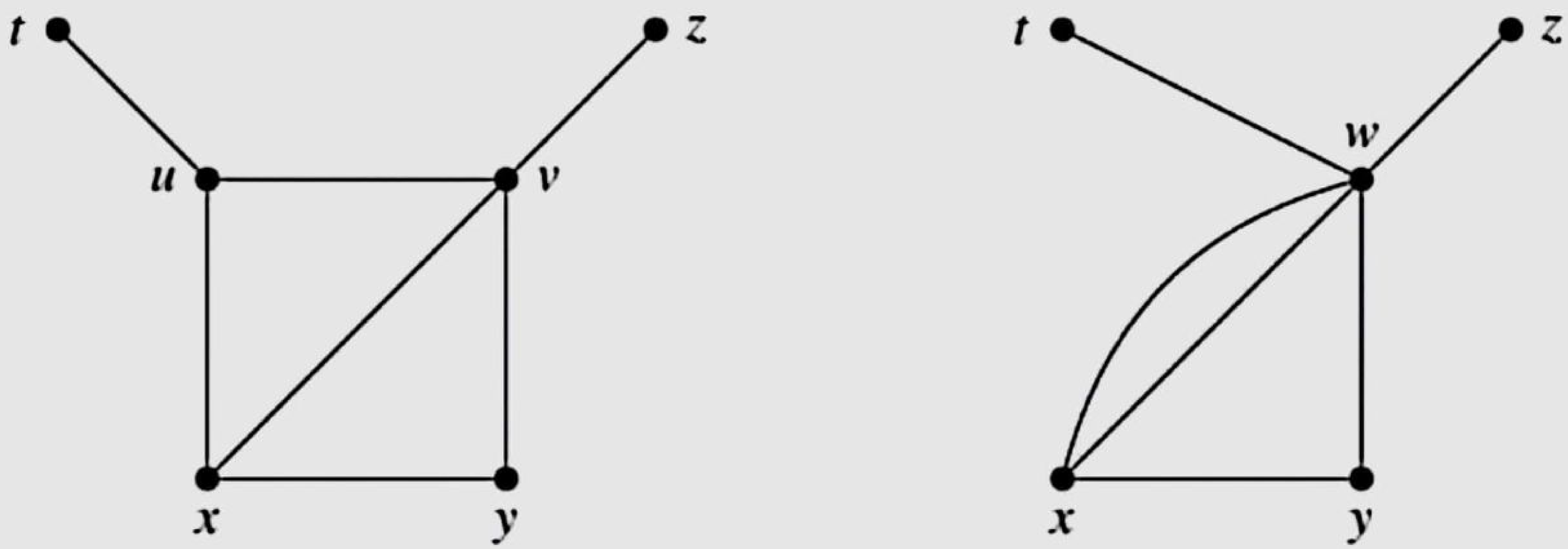


(a) G (b) $G - e$

Figure 2.8: Illustrations to fusion of two vertices



(a) G (b) $G \circ uv$

Figure 2.9: Illustrations to edge contraction of a graph.