AOU
الجامعة العربية المفتوحة
Arab Open University

Arab Open University
Faculty of Computer Studies

MT131 - Discrete Mathematics

2. Sets, Functions, Sequences and Summation

Sets

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A set is an unordered collection of objects.
  - The students in this class.
  - The chairs in this room.
- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$
- There must be an underlying Universal set U, either specifically stated or understood.
- **For example:**
  - $A = \{1, 3, 5, 7\}$; $3 \in A$, $2 \notin A$
  - $C = \{x \mid x = n^2 + 1, n \text{ is an integer}, 0 \leq n \leq 10\}$

Describing a Set: Roster Method

- $S = \{a, b, c, d\}$
- Order not important

  $S = \{a, b, c, d\} = \{b, c, a, d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$
- Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a, b, c, d, \ldots, z\}$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a, e, i, o, u\}$

- Set of all odd positive integers less than 10:

  $O = \{1, 3, 5, 7, 9\}$

- Set of all positive integers less than 100:

  $S = \{1, 2, 3, \ldots, 99\}$

- Set of all integers less than 0:

  $L = \{\ldots, -3, -2, -1\}$

Some Important Sets

- **N** = *natural numbers* = {0, 1, 2, 3, …}
- **Z** = *integers* = {…, -3, -2, -1, 0, 1, 2, 3, …}
- $\mathbf{Z}^+$ = *positive integers* = {1, 2, 3, …}
- **R** = set of *real numbers*
- $\mathbf{R}^+$ = set of *positive real numbers*
- **C** = set of *complex numbers*
- **Q** = set of rational numbers

Describing a Set: Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $T = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid P(x)\}$$

- Example: $S = \{x \mid \text{Prime}(x)\}$
- Positive rational numbers:

  $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$

Describing a Set: Interval Notation

$[a, b] = \{x \mid a \leq x \leq b\}$

$[a, b) = \{x \mid a \leq x < b\}$

$(a, b] = \{x \mid a < x \leq b\}$

$(a, b) = \{x \mid a < x < b\}$

Closed interval $[a, b]$

Open interval (a, b)

Examples

Use the set builder notation to describe the sets

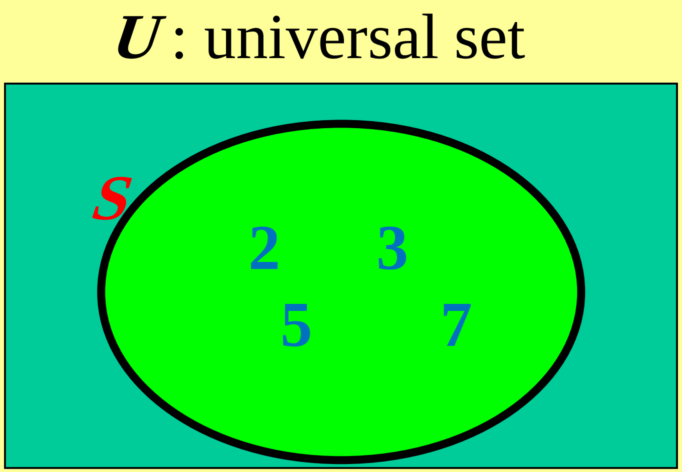
$$A = \{1, 4, 9, 16, 25, \ldots\}$$

$$B = \{a, d, e, h, m, o\}$$

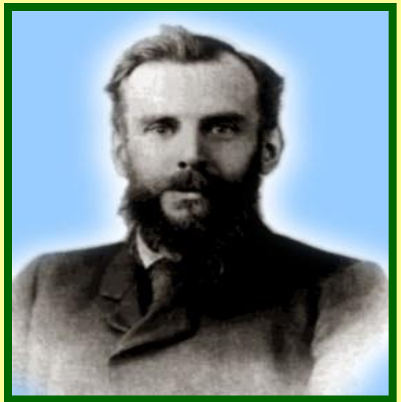
$$A = \{\ x \mid x = n^2\ ,\ n \text{ is positive integer}\}$$

$$B = \{\ x \mid x \text{ is a letter of the word mohamed}\}$$

Describing a Set: Venn Diagrams



Prime < 10



John Venn
1834-1923

Membership in Sets and the Empty Set

- $x \in S$: x is an element or member of the set S.

  e.g. $3 \in \mathrm{N}$

  "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"
- $\varnothing$ (null or the empty set) is the unique set that contains no elements.
- $\varnothing = \{\}$
- Empty set $\varnothing$ does not equal the singleton set $\{\varnothing\}$

  $$\varnothing \neq \{\varnothing\}$$

Definition of Set Equality

- Two sets are declared to be equal if and only if they contain <u>exactly the same</u> elements.

i.e. A and B are equal if and only if

$$\forall x\,(x \in A \leftrightarrow x \in B)$$

Example: **(Order and repetition do not matter)**

$$\{1, 3, 5\} = \{3, 5, 1\} = \{1, 3, 3, 3, 5, 5, 5\}$$

Subset Relation

- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.

$$S \subseteq T \Leftrightarrow \forall x\,(x \in S \rightarrow x \in T)$$

- $\varnothing \subseteq S$ & $S \subseteq S$
- If $S \subseteq T$ is true and $T \subseteq S$ is true then $S = T$,

$$\forall x\ (x \in S \leftrightarrow x \in T)$$

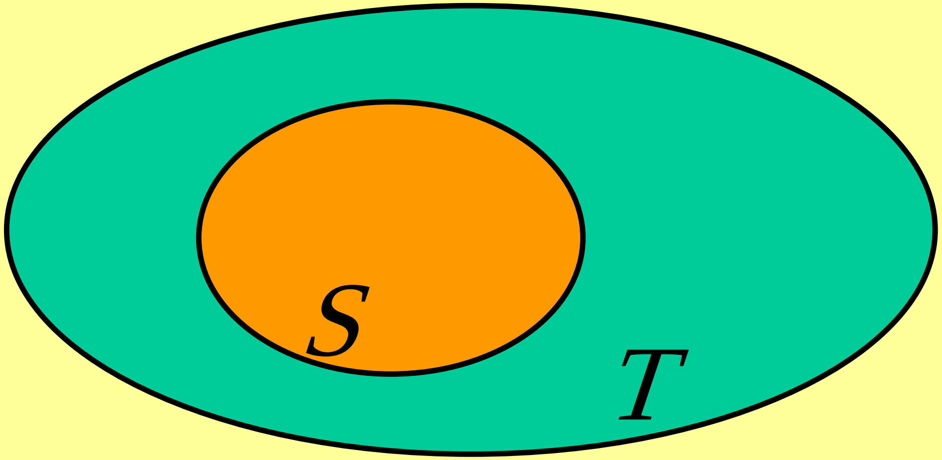
- $\neg(S \subseteq T)$ means $S \not\subseteq T$

i.e. $\exists\, x\ (x \in S \wedge x \notin T)$

Proper Subset

- $S \subset T$ (S is a proper subset of T) means every element of S is also an element of T, but $S \neq T$.

Venn Diagram equivalent of $S \subset T$



e.g. $\{1, 2\} \subset \{1, 2, 3\}$

Sets Are Objects, Too!

- The objects that are elements of a set may themselves be sets.

e.g. let $S = \{x \mid x \subseteq \{1,2,3\}\}$

then $S = \{\emptyset,$

$\{1\}, \{2\}, \{3\},$

$\{1,2\}, \{1,3\}, \{2,3\},$

$\{1,2,3\}\}$

- Note that $1 \neq \{1\} \neq \{\{1\}\}$

Cardinality and Finiteness

- $|S|$ or card(S) (read the cardinality of S) is a measure of how many different elements S has.

e.g. $|\varnothing| = |\{\ \}| = 0$

$|\{1, 2, 3, 5\}| = 4$

$|\{a, b, c\}| = 3$

$|\{\{1, 2, 3\},\{4, 5\}\}| = 2$

The Power Set Operation

- The power set $P(S)$ of a set S is the set of all subsets of S.

e.g.

- $P(\{a\}) = \{\varnothing, \{a\}\}$
- $P(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- $P(\{1, 2, 3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- $P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$
- $P(\varnothing) = \{\varnothing\}$

Note: If a set has n elements then $P(S)$ has 2^n elements.

Ordered n-tuples

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element and so on.
- These are like sets, except that duplicates matter, and the order makes a difference.

e.g. (2, 5, 6, 7) is a 4-tuple.

- **Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.**
- **Note that 2-tuples are called ordered pairs.**

Cartesian Products of Sets

- The Cartesian product of any two sets A and B is defined by

$$A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}.$$

e.g. $\{a, b\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

- Note that for finite A and B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is not commutative: $A \times B \neq B \times A$.

e.g. $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$

Cartesian Products of Sets

- The Cartesian products of the sets A_1, A_2, …, A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples (a_1, a_2, …, a_n) where a_i belongs to A_i for $i = 1, \ldots, n$.

Example: What is $A \times B \times C$ where $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$

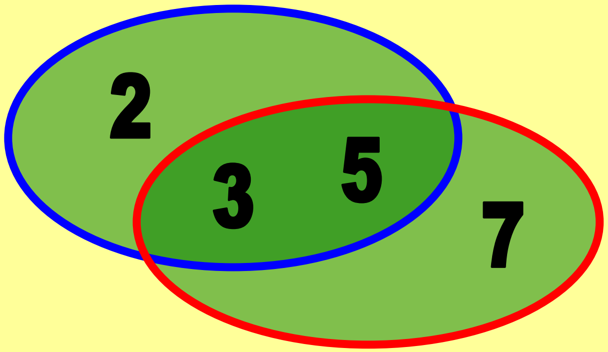
Solution: $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

Set Operations
The Union Operator $\cup$

- For any two sets A and B, $A \cup B$ is the set containing all elements that are either in A, or in B or in both.
- Formally:

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

$$\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\} = \{2, 3, 5, 7\}$$

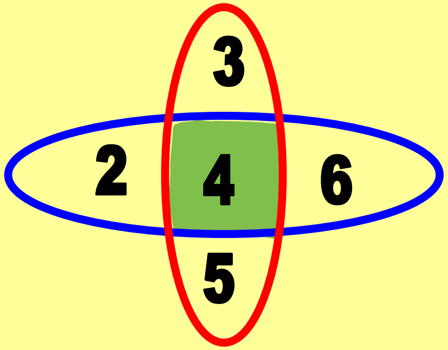


The Intersection Operator $\cap$

- For any two sets A and B, their intersection $A \cap B$ is the set containing all elements that are in both A and in B.
- Formally:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

- $\{a, b, c\} \cap \{2, 3\} = \varnothing$ disjoint
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \{4\}$



Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example:

$\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

$\{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$

$|\{1, 2, 3, 4, 5\}| = 3 + 4 - 2 = 5$

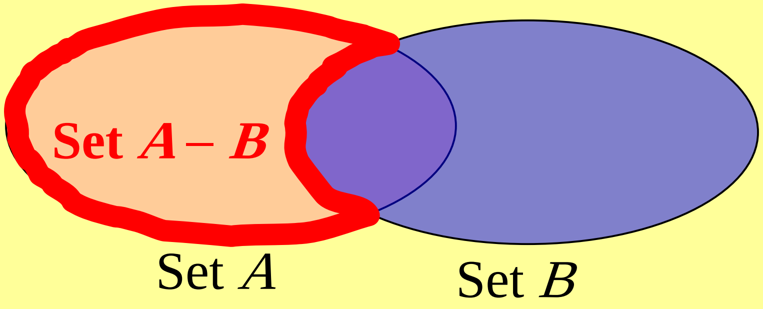
Set Difference

- For any two sets A and B, the difference of A and B, written $A - B$, is the set of all elements that are in A but not in B.

$$A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$$
$$= \{x \mid \neg(x \in A \rightarrow x \in B)\}$$

$A - B = A \cap \overline{B}$ is called the complement of B with respect to A.

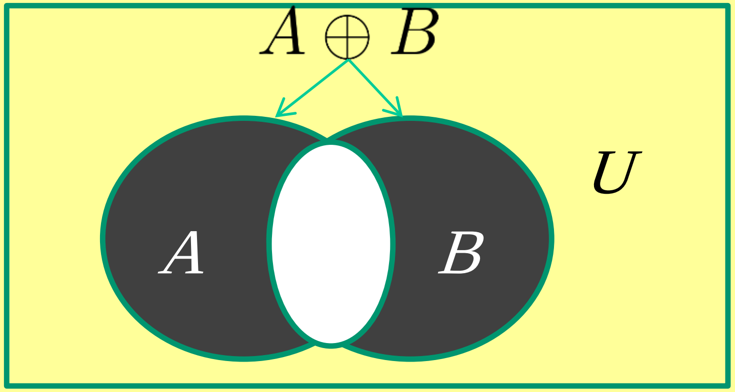
e.g. $\{1, 2, 3, 4, 5, 6\} - \{2, 3, 5, 7, 9, 11\} = \{1, 4, 6\}$



Symmetric Difference

- For any two sets A and B, the symmetric difference of A and B, written $A \oplus B$, is the set of all elements that are in A but not in B or in B but not in A.

$$A \oplus B :\equiv \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$
$$= (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$



e.g. $\{1, 2, 3, 4, 5, 6\} \oplus \{2, 3, 5, 7, 9, 11\} = \{1, 4, 6, 7, 9, 11\}$

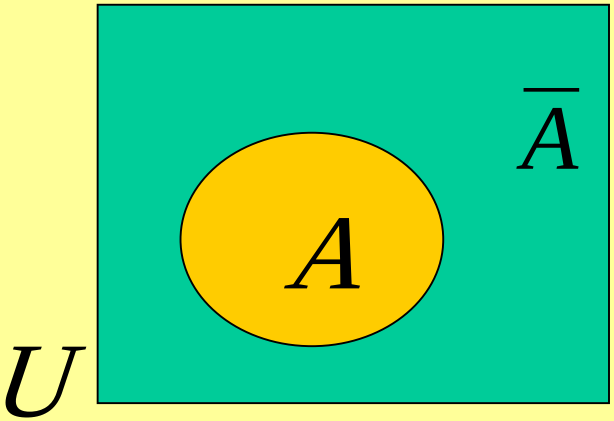
Set Complements

- U: Universe of Discourse

  $\overline{A}$: For any set $A \subseteq U$, the complement of A,

  i.e. it is $U - A$.

$$\overline{A} = \{x \mid x \notin A\}$$

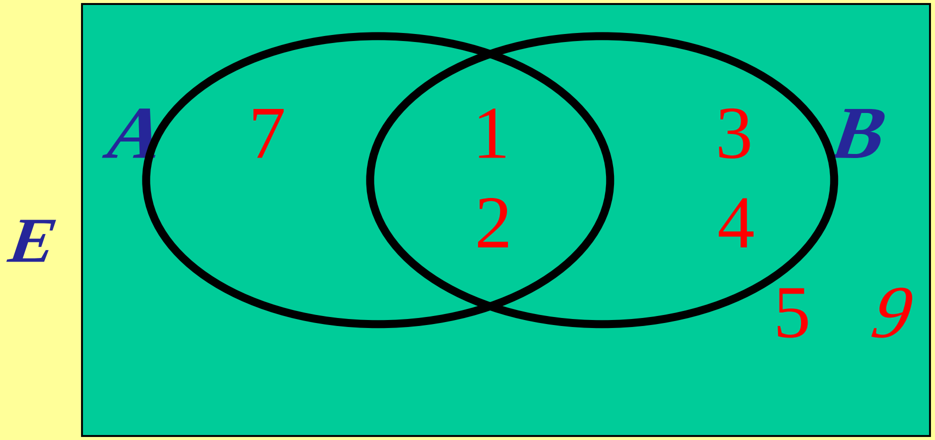


e.g. If $U = \mathbf{N}$,

$$\overline{\{3,5\}} = \{0,1,2,4,6,7, \ldots\}$$

Example

Let A and B are two subsets of a set E such that $A \cap B = \{1, 2\}$, $|A| = 3$, $|B| = 4$, $\overline{A} = \{3, 4, 5, 9\}$ and $\overline{B} = \{5, 7, 9\}$. Find the sets A, B and E.



$A = \{1, 2, 7\}$, $B = \{1, 2, 3, 4\}$,

$E = \{1, 2, 3, 4, 5, 7, 9\}$

Example

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$.

- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A \cap B = \{4, 5\}$
- $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
- $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
- $A - B = \{1, 2, 3\}$
- $B - A = \{6, 7, 8\}$
- $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

Set Identities

- Identity: $A \cup \varnothing = A = A \cap U$
- Domination: $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distribution: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
  $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proving Set Identities

To prove statements about sets of the form $E_1 = E_2$, where the Es are set expressions, there are three useful techniques:

1. Proving $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
2. Using set builder notation and logical equivalences.
3. Using set identities.
4. Using Membership table

Example: Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Method 1: Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$

Assume $x \in \overline{A \cap B}$

$\Leftrightarrow x \notin A \cap B$ by the definition of the complement

$\Leftrightarrow \neg\, x \in A \cap B$ by the definition of the negation

$\Leftrightarrow \neg\, ((x \in A) \wedge (x \in B))$ by the definition of intersection

$\Leftrightarrow \neg\, (x \in A) \vee \neg\, (x \in B)$ by De Morgan's law

$\Leftrightarrow x \notin A \vee x \notin B$ by the definition of negation

$\Leftrightarrow x \in \overline{A} \vee x \in \overline{B}$ by the definition of the complement

$\Leftrightarrow x \in \overline{A} \cup \overline{B}$ by the definition of union

Method 2: Set Builder Notation

Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$

$$= \{x \mid \neg(x \in A \cap B)\}$$

$$= \{x \mid \neg(x \in A \wedge x \in B)\}$$

$$= \{x \mid \neg x \in A \vee \neg x \in B)\}$$

$$= \{x \mid x \notin A \vee x \notin B)\}$$

$$= \{x \mid x \in \overline{A} \vee x \notin \overline{B})\}$$

$$= \{x \mid x \in \overline{A} \cup \overline{B})\} = \overline{A} \cup \overline{B}$$

Method 3 : Using Set Identities

Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{De Morgan's law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{De Morgan's law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{Commutative law} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{Commutative law}
\end{aligned}
$$

Method 3 : Using Set Identities

Show that $(B \cup C) - A = (B - A) \cup (C - A)$.

$$
\begin{aligned}
(B \cup C) - A &= (B \cup C) \cap \overline{A} \\
&= (B \cap \overline{A}) \cup (C \cap \overline{A}) \\
&= (B - A) \cup (C - A).
\end{aligned}
$$

Method 4 : Membership Tables

Construct a membership table to show that the distributive law holds: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	**0**	0	0	**0**
0	0	1	0	**0**	0	1	**0**
0	1	0	0	**0**	1	0	**0**
0	1	1	1	**1**	1	1	**1**
1	0	0	0	**1**	1	1	**1**
1	0	1	0	**1**	1	1	**1**
1	1	0	0	**1**	1	1	**1**
1	1	1	1	**1**	1	1	**1**

Computer Representation of Sets

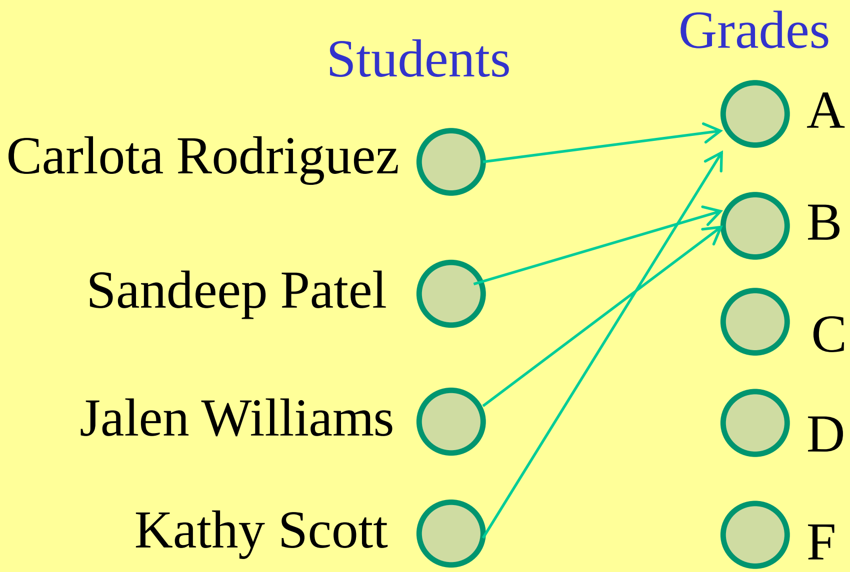
- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The bit string (of length $|U| = 10$) that represents the set $A = \{1, 3, 5, 6, 9\}$ has a one in the first, third, fifth, sixth, and ninth position, and zero elsewhere. It is

1 0 1 0 1 1 0 0 1 0.

Functions

Definition: Let A and B be nonempty sets. A function f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called mappings or transformations.



Functions

- A function $f: A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x \left[x \in A \to \exists y [y \in B \wedge (x, y) \in f] \right]$$

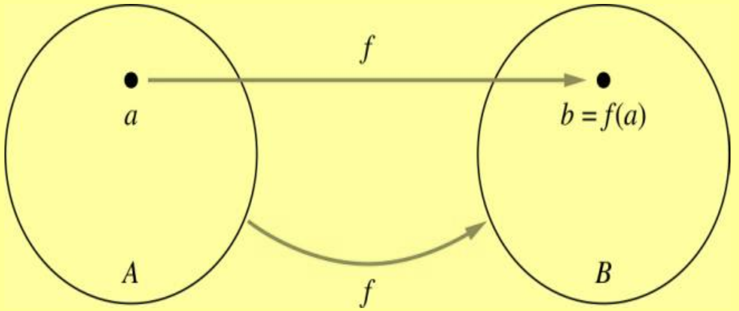
and

$$\forall x, y_1, y_2 \left[[(x, y_1) \in f \wedge (x, y_2) \in f] \to y_1 = y_2 \right]$$

Functions

Given a function $f: A \rightarrow B$

- We say f maps A to B or f is a mapping from A to B.
- A is called the domain of f.
- B is called the codomain of f.
- If $f(a) = b$,
  - then b is called the image of a under f.
  - a is called the preimage of b.
- The range of f is the set of all images of points in A under f. We denote it by $f(A)$.
- Two functions are equal when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment

  Students and grades example

- A formula.

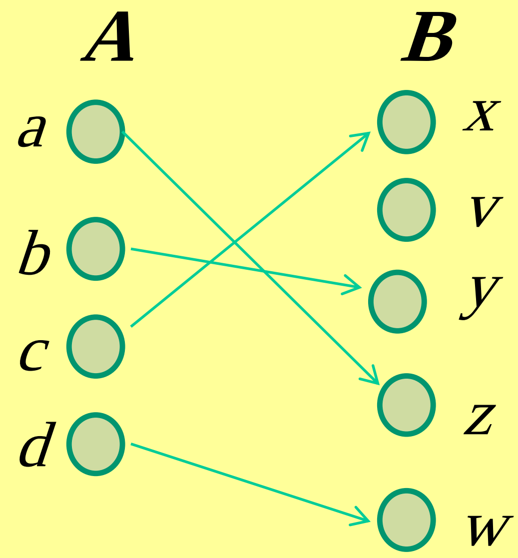
  $f(x) = x + 1$

- A computer program

  A Java program that when given an integer n, produces the n^{th} Fibonacci Number.

Injections

Definition: A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an injection if it is one-to-one.

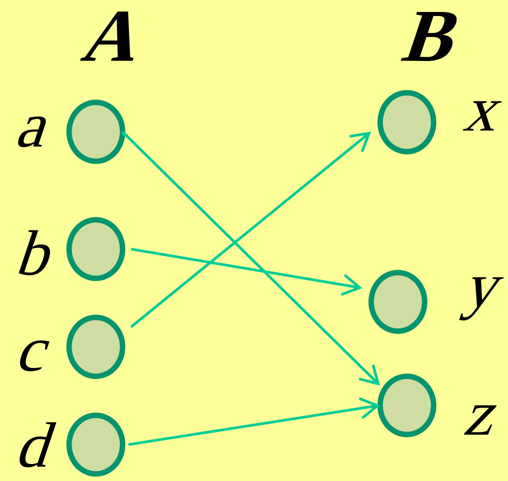


e.g. The function $f(x) = x - 1$ from $\mathbf{Z}$ to $\mathbf{Z}$ is one-to-one since

$$f(a) = f(b)$$
$$\to a - 1 = b - 1$$
$$\to a = b$$

Surjections

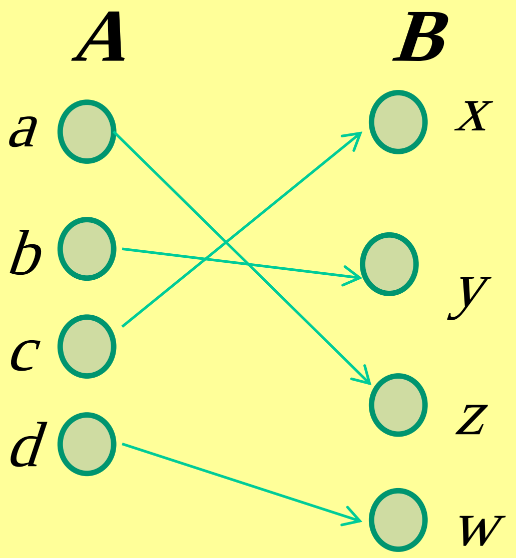
Definition: A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.



e.g. The function $f(x) = x - 1$ from $\mathbf{Z}$ to $\mathbf{Z}$ is onto since for all $b \in \mathbf{Z}$, there is $a \in \mathbf{Z}$ such that $f(a) = b$ $(a = b + 1)$

Bijections

Definition: A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto (surjective and injective).



e.g. The function $f(x) = x - 1$ from $\mathbf{Z}$ to $\mathbf{Z}$ is one-to-one and onto, hence bijective

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1, 2, 3, 4\}$, f would not be onto.

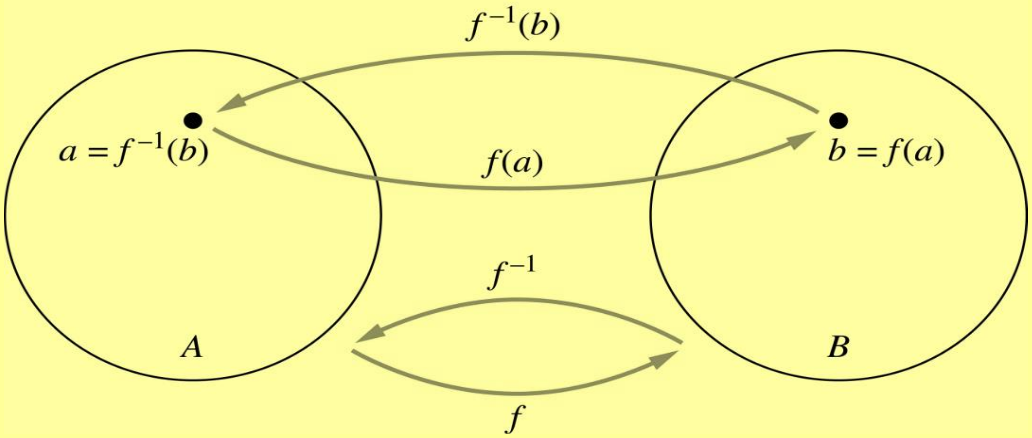
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because, for example, there is no integer x with $x^2 = -1$.

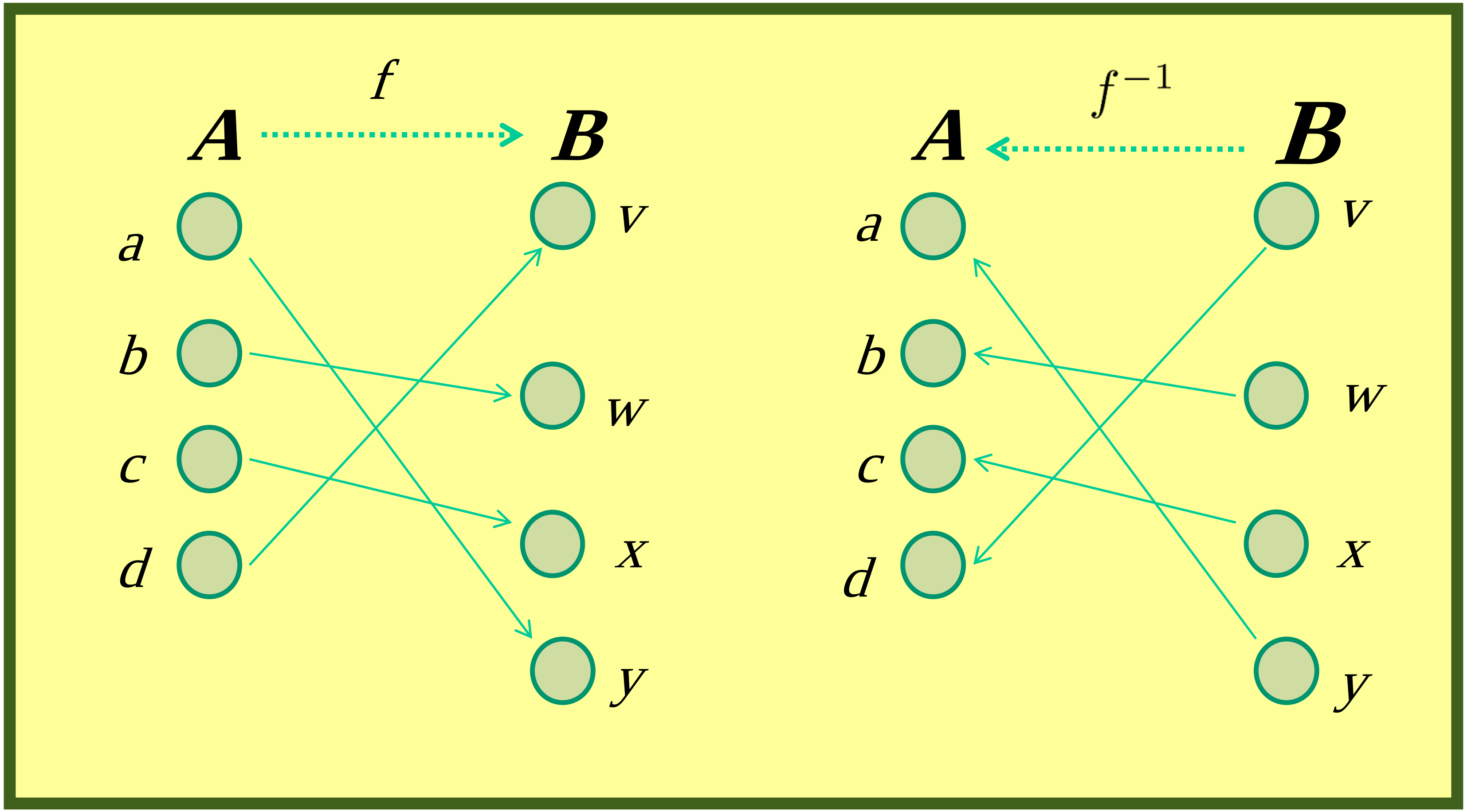
Inverse Functions

Definition: Let f be a bijection from A to B. Then the inverse of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless f is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Example 2: Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

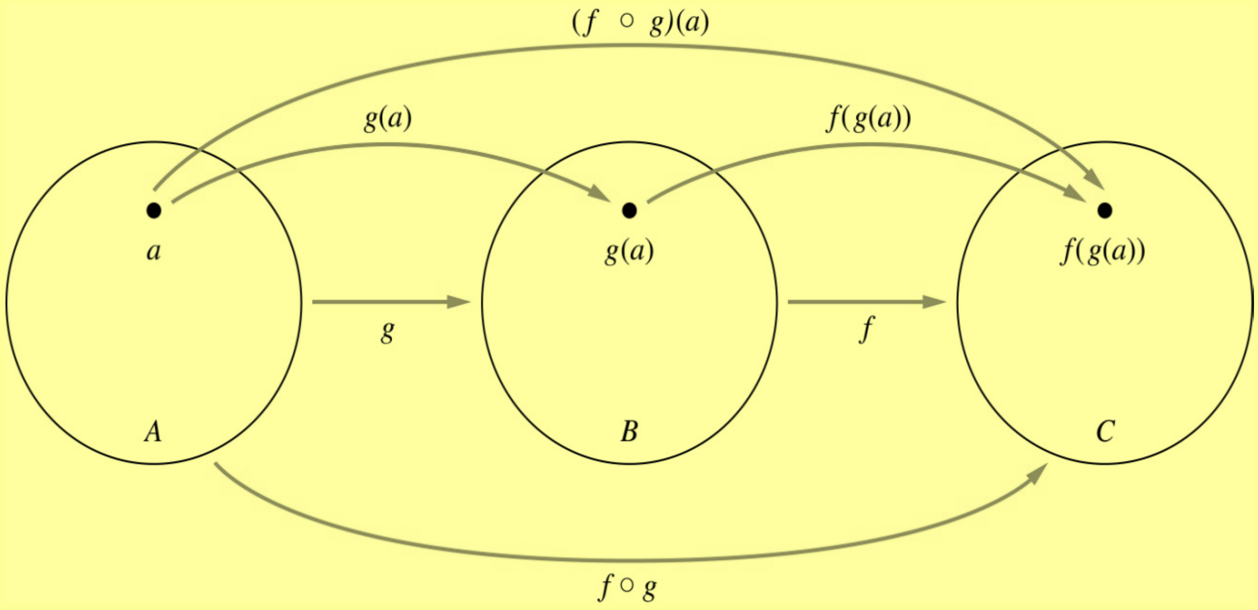
Solution: The function f is invertible because it is a one-to-one correspondence ($f(a) = f(b) \rightarrow a + 1 = b + 1 \rightarrow a = b$). The inverse function f^{-1} reverses the correspondence so

$$f^{-1}(y) = y - 1.$$

Composition

Definition: Let $f: B \rightarrow C$, $g: A \rightarrow B$. The composition of f *with* g, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



Composition

Example 1: Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$$f \circ g\,(a) = f(g(a)) = f(b) = 2.$$

$$f \circ g\,(b) = f(g(b)) = f(c) = 1.$$

$$f \circ g\,(c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Example 2: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2 \qquad g(f(x)) = 2x^2 + 1$$

Composition

Example 3: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g and also the composition of g and f?

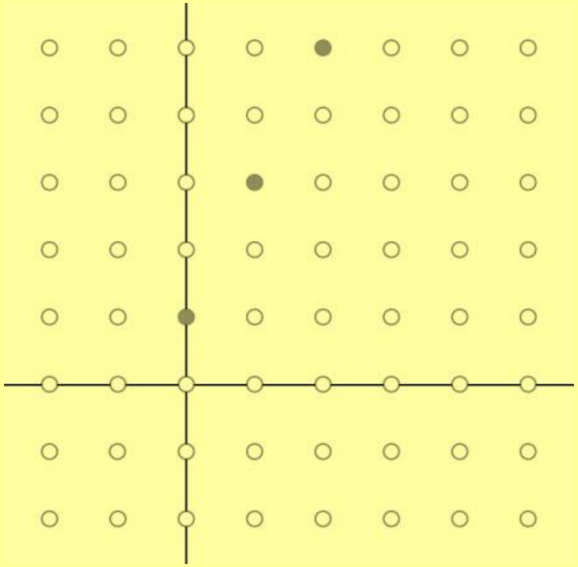
Solution:

$$f \circ g\,(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

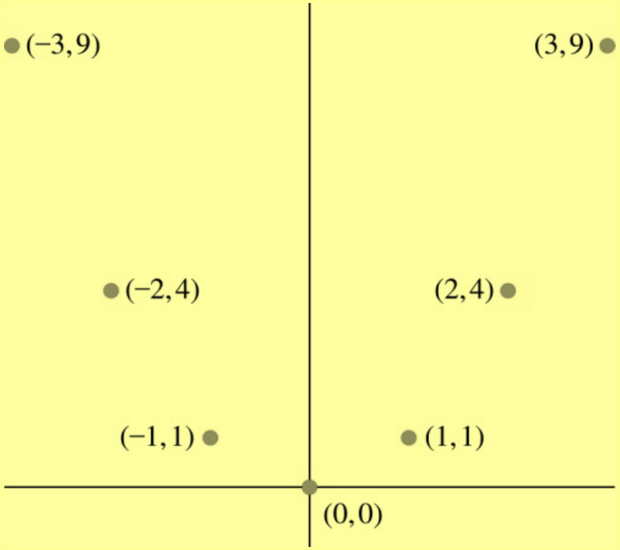
$$g \circ f\,(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Graphs of Functions

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from **N** to **N**



Graph of $f(x) = x^2$ from **Z** to **Z**

Important Functions

(a) The floor function, denoted $f(x) = \lfloor x \rfloor$

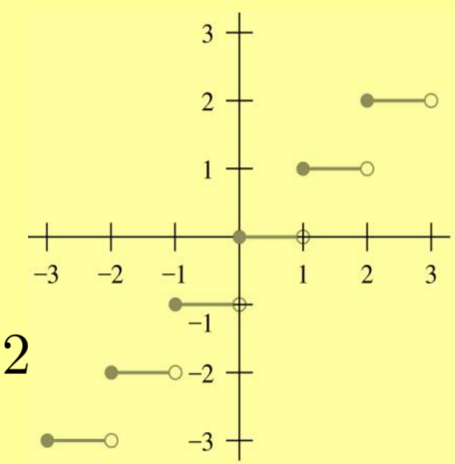
is the largest integer less than or equal to *x*.

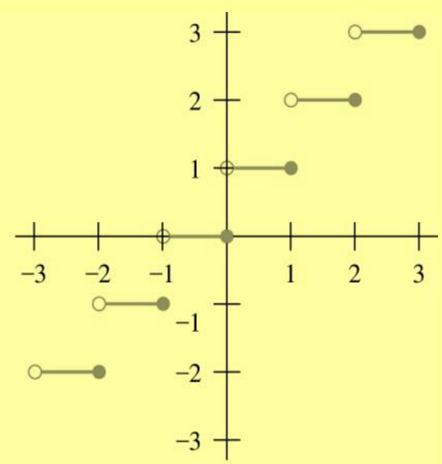
(b) The ceiling function, denoted $f(x) = \lceil x \rceil$

is the smallest integer greater than or equal to *x*.

$\lfloor 3.5 \rfloor = 3$

$\lfloor -1.5 \rfloor = -2$





$\lceil 3.5 \rceil = 4$

$\lceil -1.5 \rceil = -1$

Graphs of floor and ceiling from **R** to **Z**

Sequences and Summations

- Sequences are ordered lists of elements.
- Sequences used in discrete mathematics in many ways. For example, they can be used to represent solutions to certain counting problems.
- Sequences are also an important data structure in computer science.
- We will often need to work with sums of terms of sequences in our study of discrete mathematics

Sequences

Definition 1 A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Geometric Sequence

Definition 2 A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

EXAMPLE 2 The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1, 2 and 5, and 6 and 1/3, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$$1, -1, 1, -1, 1, \ldots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with

$$2, 10, 50, 250, 1250, \ldots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

Arithmetic Sequence

Definition 3 An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

EXAMPLE 3 The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3, respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with

$$-1, 3, 7, 11, \ldots,$$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

$$7, 4, 1, -2, \ldots.$$

Summations

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

Examples

EXAMPLE What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

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EXAMPLE What is the value of $\sum_{k=4}^{8} (-1)^k$?

Solution: We have

$$\begin{aligned}\sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

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Summation Formulas

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$ (Distributive law.)

$$\sum_x f(x) + g(x) = \left(\sum_x f(x)\right) + \sum_x g(x)$$ (Application of commut-ativity.)

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$ (Series splitting.)

Example

Find $\sum_{k=50}^{100} k^2$?

$$\sum_{k=1}^{100} k^2 = (\sum_{k=1}^{49} k^2) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = (\sum_{k=1}^{100} k^2) - \sum_{k=1}^{49} k^2$$

$$= \frac{100.101.201}{6} - \frac{49.50.99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925$$

Extra Practice (Book Page 177-178)

29. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ **b)** $\sum_{j=0}^{4}(-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8}(2^{j+1}-2^j)$

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a) $\sum_{j \in S} j$ **b)** $\sum_{j \in S} j^2$

c) $\sum_{j \in S}(1/j)$ **d)** $\sum_{j \in S} 1$

31. What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3 \cdot 2^j$ **b)** $\sum_{j=1}^{8} 2^j$

c) $\sum_{j=2}^{8}(-3)^j$ **d)** $\sum_{j=0}^{8} 2 \cdot (-3)^j$

32. Find the value of each of these sums.

a) $\sum_{j=0}^{8}(1+(-1)^j)$ **b)** $\sum_{j=0}^{8}(3^j-2^j)$

c) $\sum_{j=0}^{8}(2 \cdot 3^j + 3 \cdot 2^j)$ **d)** $\sum_{j=0}^{8}(2^{j+1}-2^j)$

39. Find $\sum_{k=100}^{200} k$. (Use Table 2.)

40. Find $\sum_{k=99}^{200} k^3$. (Use Table 2.)

41. Find $\sum_{k=10}^{20} k^2(k-3)$. (Use Table 2.)