Lecture (4)

Mechanics

DR./ IBRAHIM ABADY

3 System of Forces and Resultant-I (Concurrent Forces)

3.1 INTRODUCTION

In this chapter, we will introduce the concept of force, its characteristics and its effect on bodies. When more than one force acts on a body, they constitute a *system of forces*. Various types of systems of forces are possible by the relative orientation of forces within the system. As each such type of force system produces different effects on bodies, they are studied separately. In this chapter, we will discuss the effects of concurrent forces and in the following chapter, the effects of non-concurrent forces.

In Sections 3.5–3.6, we will discuss the methods to determine the resultant of concurrent force systems in a plane and their effects upon bodies. In Section 3.7, we will discuss concurrent spatial force systems.

3.2 FORCE

Newton's first law of motion states that every body continues in its state of rest or of uniform motion in a straight line *unless* it is compelled to change that state by *forces* impressed on it. Hence, **force** can be defined as any *action* of a body on another, which tends to *change* the state of rest or of motion of the other body. There are various types of forces that can act upon a body in a given environment: such as gravitational force acting on a body when it is placed in a gravitational field or push and pull exerted by our hand or tractive force of a locomotive on the train which it is pulling, etc.

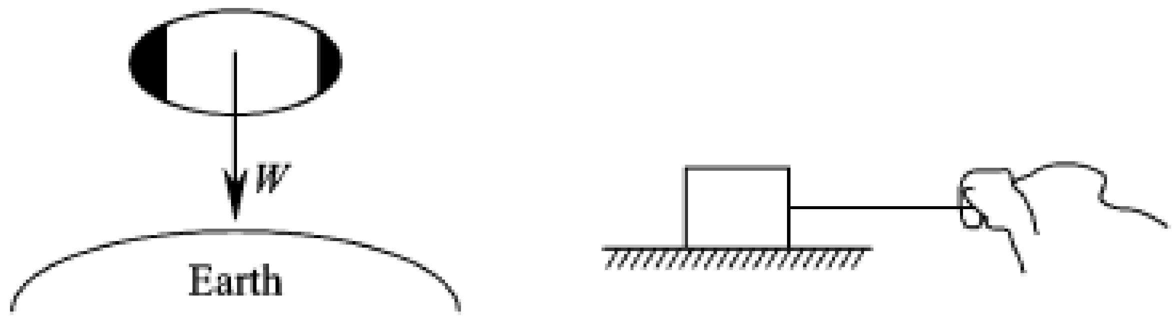


Fig. 3.1 (a) Gravitational force (b) Pull exerted by our hand

To define a force completely, particularly for graphical representation, we must specify its **magnitude**, its **point of application** and its **direction**, which are known as **force characteristics**.

The **magnitude** of a force is obtained by comparing it with a certain *standard force*. Standard force is defined as that force exerted on a standard kilogram body (which is a platinum cylinder preserved at the International Bureau of Weights and Measures at Sevres near Paris) causing uniform acceleration of 1 m/s^2. The magnitude of force can be measured by a spring balance or by a dynamometer. The SI unit of force is given in **newton** (N).

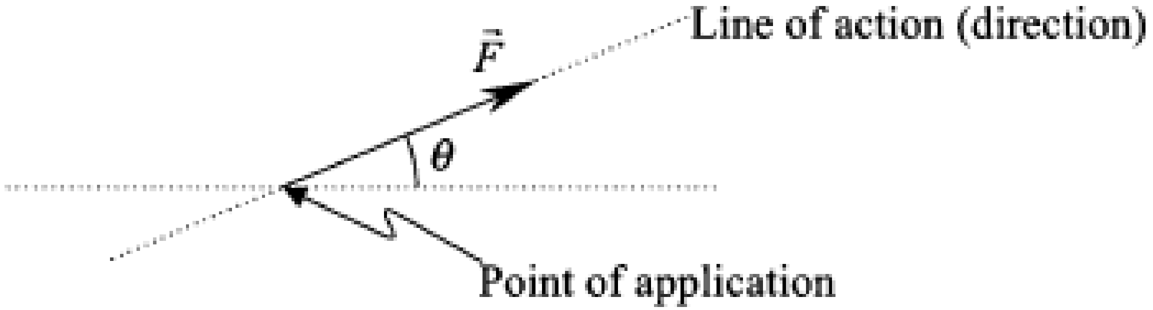
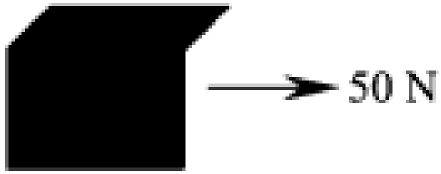


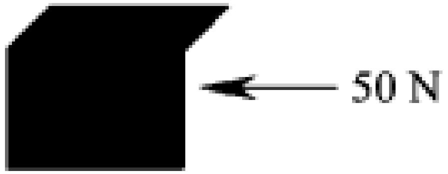
Fig. 3.2 Graphical representation of a force

The **point of application** of a force is that point in the body at which the force can be assumed to be *concentrated*. In reality, forces are never concentrated but are distributed over the entire volume of the body, as in the case of force of gravity or over the entire contact area, as in the case of contact forces. However, for theoretical purposes, these forces may be assumed to be concentrated at a point called the *point of application* of the force without affecting the accuracy of the problem. For example, the weight of a body, even though distributed over its entire volume, is normally applied at its centre of gravity.

The **direction** of a force is that direction in which the acting force *tends to move* the body. It is a straight line passing through the point of application of the force. It is also called the **line of action** of the force. The **sense** of force is indicated by an *arrowhead*. For example, gravity force is always directed towards the centre of the earth or the tension in a string always acts along its length away from the body.

Consider for instance, a block being pulled by a string with a force of 50 N. Then the force acting on the block is represented graphically as shown in Fig. 3.3(a), where its magnitude and sense are also shown. Suppose it is pushed with a force of 50 N. Then it is represented as shown in Fig. 3.3(b). Note that even though this force may be distributed over the entire contact area between the body and the agent causing the motion, it is represented as a concentrated force placed at the centroid of the contact area.





Graphical representation of a force pulling and pushing a block

Fig. 3.3(a) **Fig. 3.3(b)**

From the above discussion, we can readily see that force is a **vector** as it has both **magnitude** and **direction**, and also obeys the **parallelogram law,** which will be explained later in Section 3.6.1.

3.3 EFFECT OF FORCE ON A BODY

An external force acting on a body tends to *deform* the body, causing internal stresses within the body. This internal stress distribution is dependent on the point of application of the force. For example, forces $\vec{F}_1$ and $\vec{F}_2$ applied respectively at ends A and B of a rod cause tension in the rod. The same forces when applied at ends B and A respectively, cause *compression* in the rod.

Hence, in such a case, we must treat the force as a **fixed vector**, i.e., having a *fixed* point of application. Such types of problems are dealt with in the field of strength of materials.

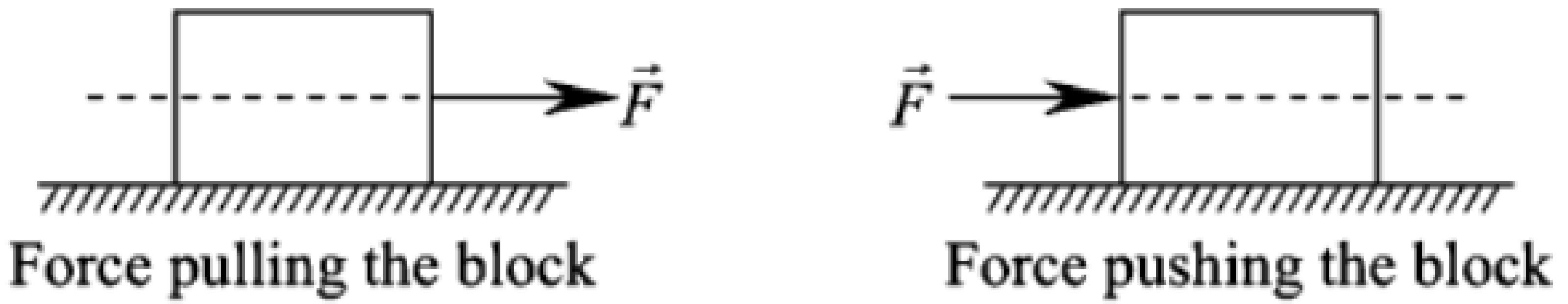


Fig. 3.5

Hence, we can see that forces can be moved to any point along their lines of action on the rigid body. This is known as **principle of transmissibility**. It states that *the conditions of equilibrium or of motion of a rigid body remains unchanged if a force acting at a given point of the rigid body is replaced by a force of the same magnitude and direction, but acting at a different point, provided that the two forces have the same line of action.*

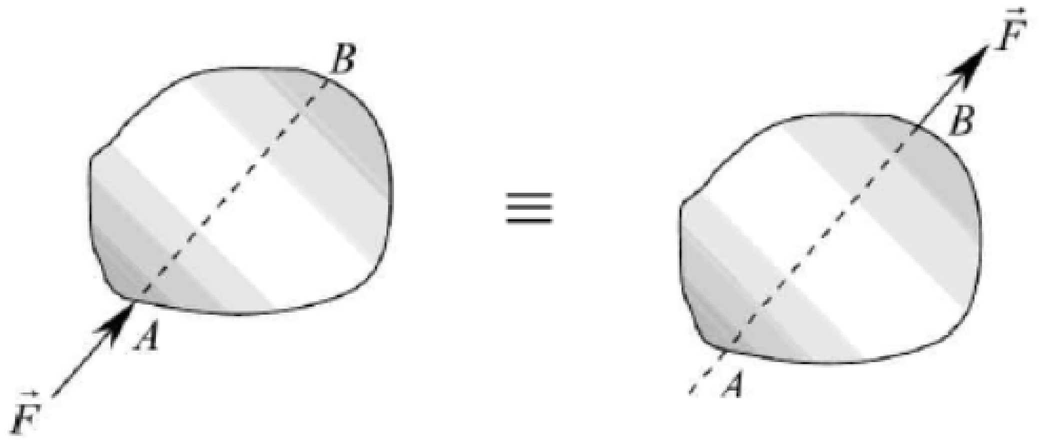
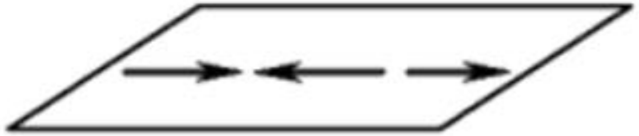
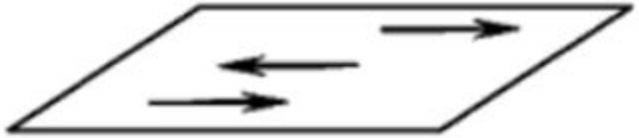
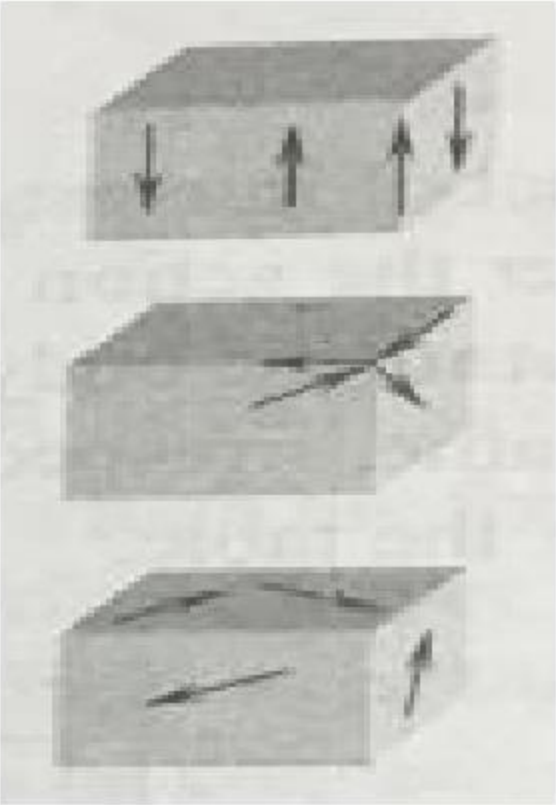


Fig. 3.6 Principle of transmissibility

3.4 SYSTEM OF FORCES

DESCRIPTION	COPLANAR FORCES	FORCES IN SPACE
i) Collinear forces		(Not possible to have)
ii) Parallel forces		
iii) Concurrent forces		
iv) Non-concurrent and non-parallel forces		

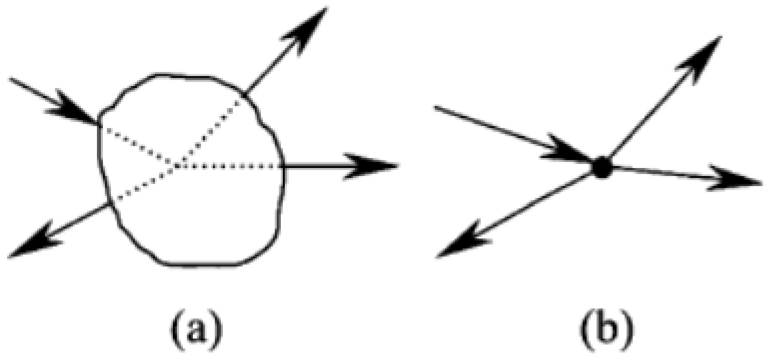


Fig. 3.8 Concurrent forces in a plane Idealization as a particle

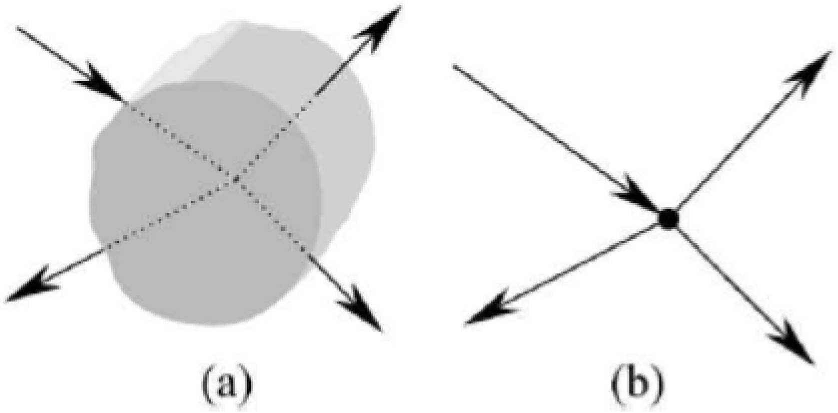


Fig. 3.9 Concurrent forces in space Idealization as a particle

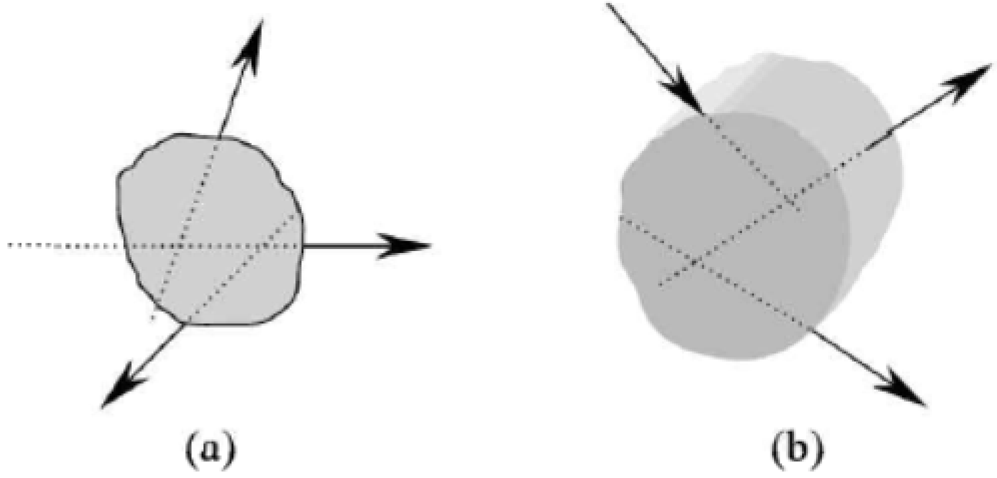


Fig. 3.10 (a) Non-concurrent forces in a plane (b) Non-concurrent forces in space

The system of forces shown in the figure is

(a) coplanar non-concurrent forces

(b) coplanar collinear forces

(c) coplanar concurrent forces

(d) coplanar parallel forces

Collinear forces are those which

(a) are concurrent at a point

(b) are parallel to each other

(c) lie on the same line

(d) act on different planes

Principle of transmissibility can be applied only when the body is treated as

(a) a particle (b) a rigid body (c) deformable (d) a continuum

The weight of a body is a

(a) body force (b) surface force (c) line force (d) reactive force

3.6.1 Parallelogram Law

The parallelogram law states that *when two concurrent forces $\vec{F}_1$ and $\vec{F}_2$ acting on a body are represented by two adjacent sides of a parallelogram, the diagonal passing through their point of concurrency represents the resultant force $\vec{R}$ in magnitude and direction.*

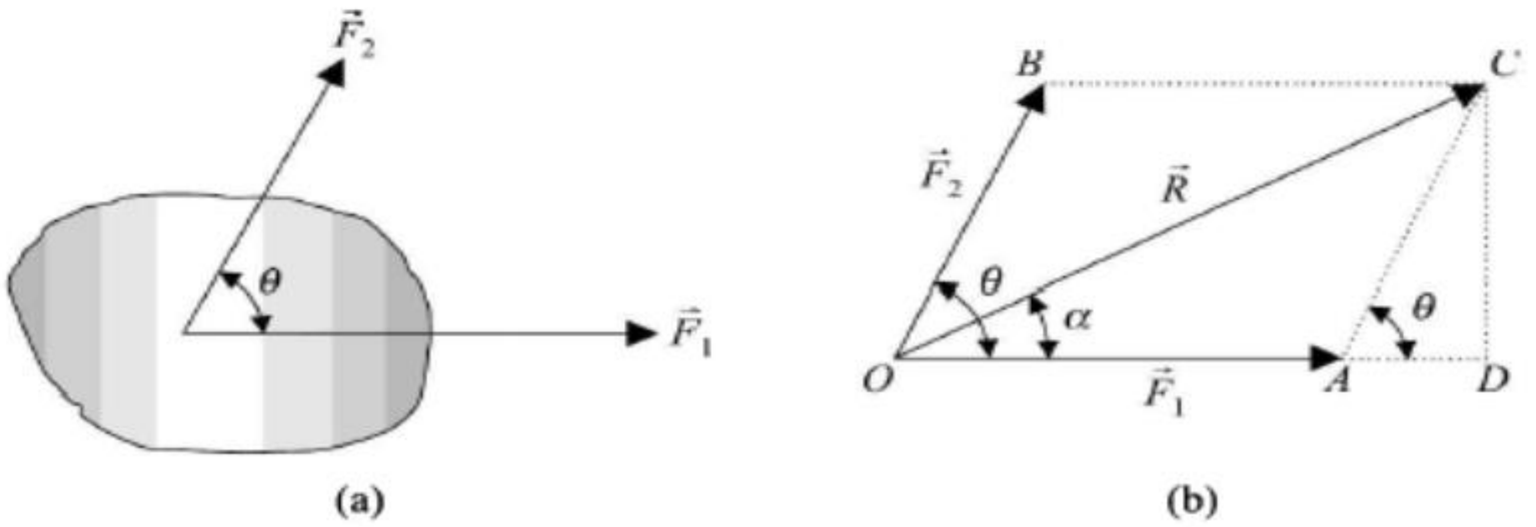


Fig. 3.11 Graphical representation of the parallelogram law

3.6.2 Law of Cosine

The mathematical statement of the parallelogram law is called the **law of cosine**. The magnitude and direction of the resultant can also be determined from Fig. 3.11(b) by trigonometry as follows.

From ΔOCD, we know,

$$OC^2 = (OA + AD)^2 + (CD)^2 \tag{3.1}$$

Hence, the magnitude of the resultant $\vec{R}$ is given by

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta} \tag{3.2}$$

The inclination of $\vec{R}$ with $\vec{F}_1$ is given by

$$\alpha = \tan^{-1}\left[\frac{F_2 \sin\theta}{F_1 + F_2\cos\theta}\right] \tag{3.3}$$

Special Cases :

Case I: If $\theta = 90°$, i.e., if the two forces are perpendicular to each other then

$$R = \sqrt{F_1^2 + F_2^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{F_2}{F_1}\right) \tag{3.4}$$

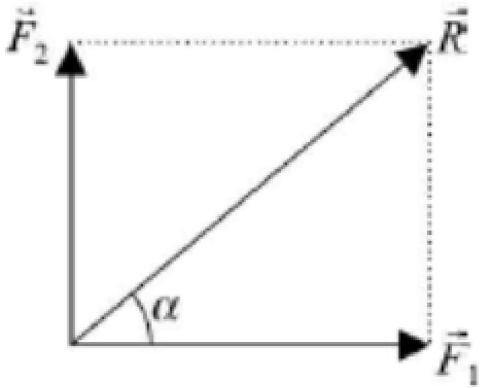


Fig. 3.12(a) Resultant of two concurrent and perpendicular forces

Special Cases :

Case II: If $\theta = 0°$, i.e., if the two forces are collinear and act in the same direction then

$$R = F_1 + F_2 \text{ and } \alpha = 0 \tag{3.5}$$

Fig. 3.12(b) Resultant of two collinear forces in the same direction

Case III: If $\theta = 180°$, i.e., if the two forces are collinear but acting in the opposite direction, where $F_1 > F_2$ then

$$R = F_1 - F_2 \text{ and } \alpha = 0 \tag{3.6}$$

Fig. 3.12(c) Resultant of two collinear forces in the opposite direction

Hence, we see that if two forces are **collinear** then their resultant is given by their **algebraic sum**.

When there are more than two concurrent forces in a force system, the resultant can be found out in steps. First, considering any two forces, the resultant can be determined as explained above. The resultant thus obtained is added on to the next force, and so on, until all given forces are added on by the parallelogram law. Thus, the overall resultant can be obtained.

3.6.4 Sine Law

The mathematical statement of the triangle law is called **sine law**. For a triangle of sides and included angles as shown in Fig. 3.14, sine law can be expressed as

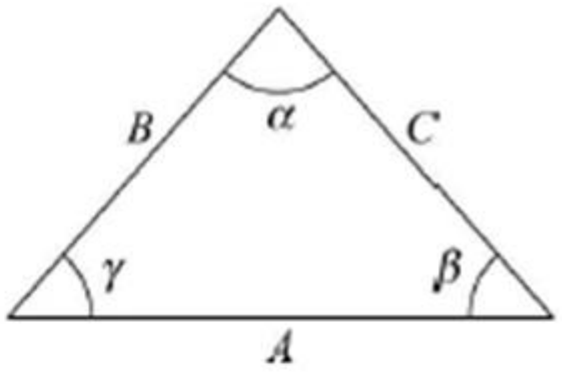


Fig. 3.14

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma} \tag{3.7}$$

where A, B and C are the sides of the triangle and the angles opposite to these sides being respectively α, β and γ.

Applying sine law to we get,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{R}{\sin\gamma} \tag{3.8}$$

Example 3.1 Two forces are applied at the point *A* of a hook support as shown in Fig. 3.16. Determine the magnitude and direction of the resultant force by the trigonometric method using parallelogram law.

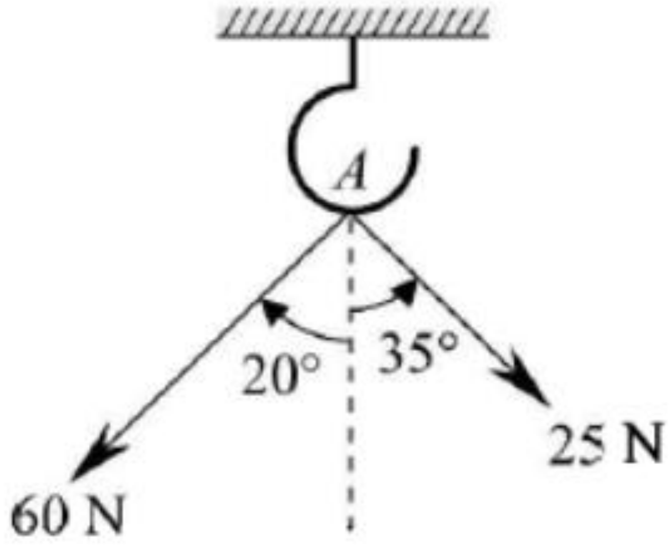


Fig. 3.16

Solution

(i) Parallelogram Law

Let us take the two forces as $|\vec{F_1}| = F_1 = 60$ N, $|\vec{F_2}| = F_2 = 25$ N. The included angle θ between them is $20° + 35° = 55°$. Hence, according to the parallelogram law, the magnitude of the resultant force $\vec{R}$ is given by,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$$

$$= \sqrt{60^2 + 25^2 + 2(60)(25)\cos 55°} = 77.11 \text{ N}$$

The angle made by the resultant $\vec{R}$ with $\vec{F_1}$ is given by,

$$\alpha = \tan^{-1}\left[\frac{F_2 \sin\theta}{F_1 + F_2\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{25 \sin 55°}{60 + 25\cos 55°}\right] = 15.4°$$

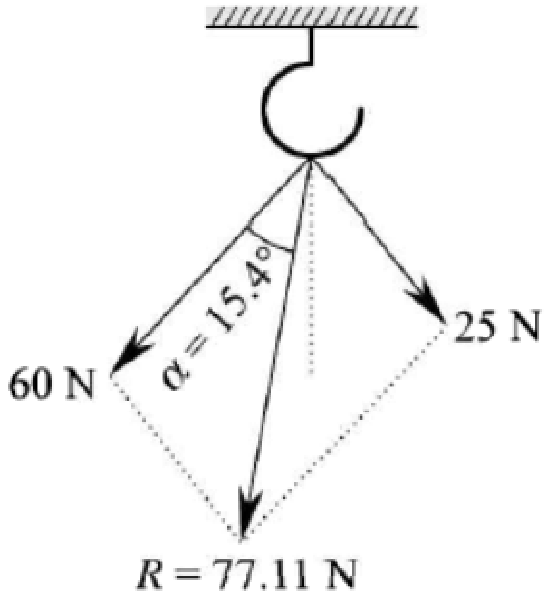


Fig. 3.16(a)

3.6.5 Analytical Method

When there are more than three concurrent forces acting on a particle then graphical and trigonometric methods become tedious to work with. In such cases, we resort to the analytical method as it can be applied to any number of forces.

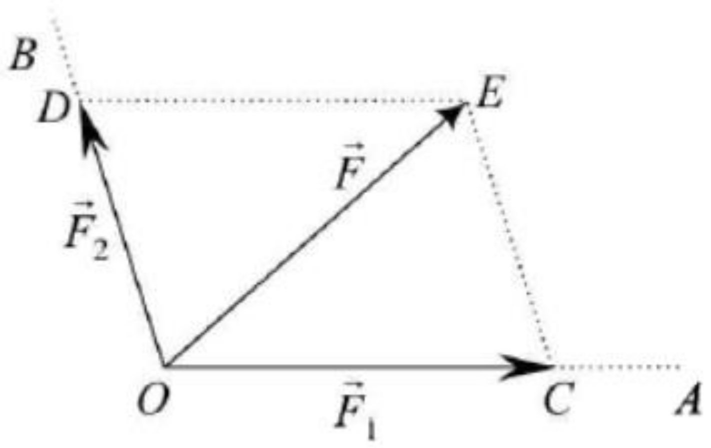


Fig. 3.19 Resolution of a force

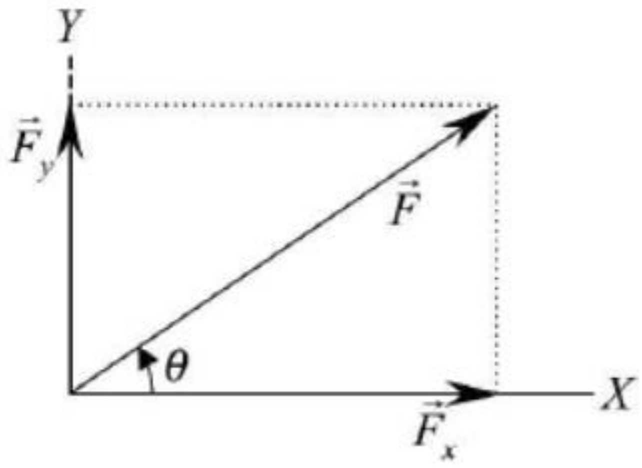


Fig. 3.20 Rectangular components of a force

Rectangular Components of a Force The rectangular components of a force are obtained by drawing lines parallel to the axes or by projecting the force onto X and Y axes as shown in Fig. 3.20. By vector addition, we can write

$$\vec{F} = \vec{F}_x + \vec{F}_y \tag{3.12}$$

where $\vec{F}_x$ and $\vec{F}_y$ are called **vector components** of $\vec{F}$.

Since any vector can be represented as a product of its magnitude and unit vector along its direction, we can also write the force vector as

$$\vec{F} = F_x\vec{i} + F_y\vec{j} \tag{3.13}$$

where F_x and F_y are called the **scalar components** or simply **components** of $\vec{F}$.

If θ is inclination of $\vec{F}$ with respect to the X-axis then the scalar components of $\vec{F}$ are

$$F_x = |\vec{F}| \cos\theta = F \cos\theta \tag{3.14}$$

and

$$F_y = |\vec{F}| \sin\theta = F \sin\theta \tag{3.15}$$

Therefore, force $\vec{F}$ can also be written as

$$\vec{F} = [F \cos\theta]\,\vec{i} + [F \sin\theta]\,\vec{j} \tag{3.16}$$

The magnitude and direction of $\vec{F}$ can be expressed in terms of its components as

$$|\vec{F}| = F = \sqrt{F_x^2 + F_y^2} \quad \text{and} \quad \theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \tag{3.17}$$

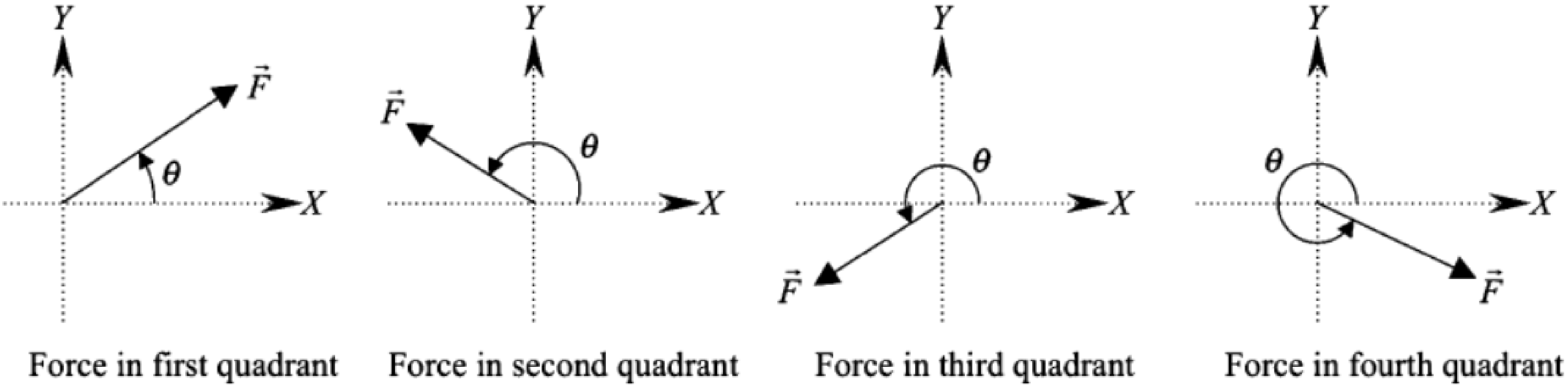


Fig. 3.21

Table 3.1 Sign of Components

Quadrant	θ	*Sign of*	
		x-component	*y-component*
I	0 – 90°	+	+
II	90° – 180°	–	+
III	180° – 270°	–	–
IV	270° – 360°	+	–

Example 3.4 Find the rectangular components of the force shown in Fig. 3.24.

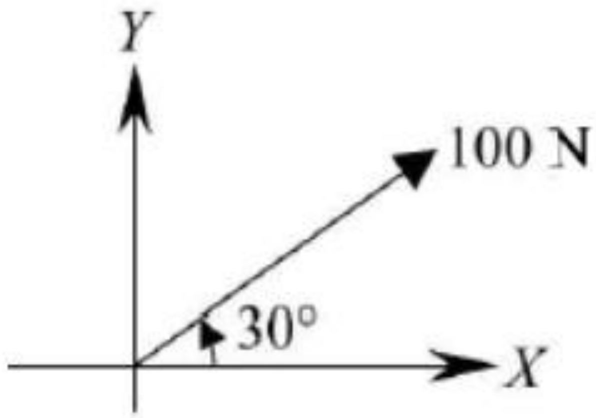


Fig. 3.24

Solution The given force is resolved into rectangular components along X and Y axes

Then the x and y-components of the force are

$$F_x = F\cos\theta$$
$$= 100 \times \cos 30°$$
$$= 86.6\text{ N}$$

$$F_y = F\sin\theta$$
$$= 100 \times \sin 30°$$
$$= 50\text{ N}$$

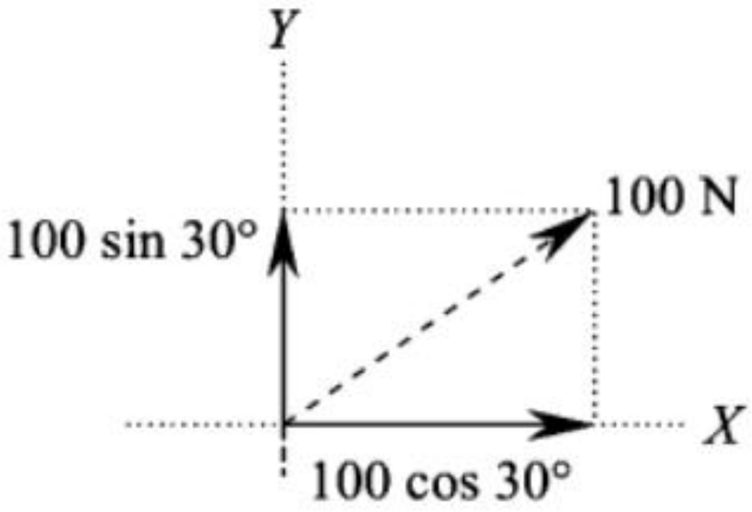


Fig. 3.24(a)

Example 3.5 Find the rectangular components of a force of magnitude 50 N acting as shown in Fig. 3.25.

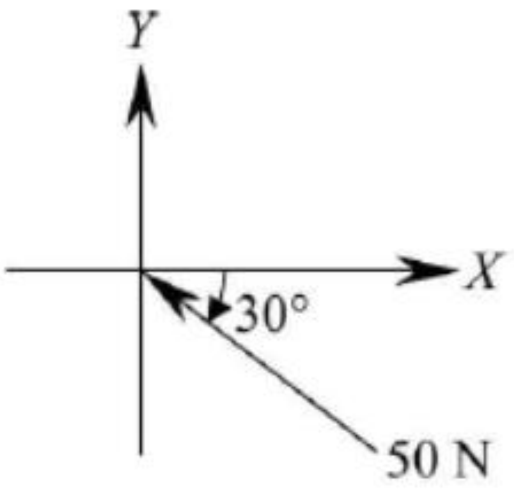


Fig. 3.25

Solution

$$F_x = -F \cos \theta$$
$$= -50 \times \cos 30°$$
$$= -43.3 \text{ N}$$

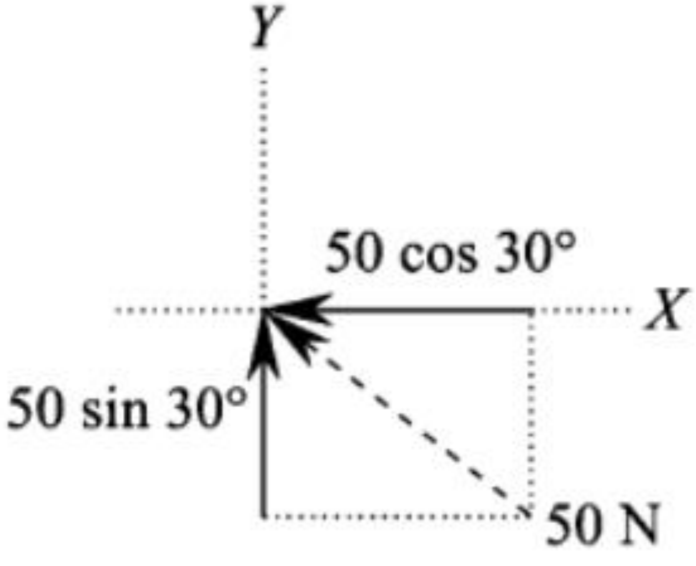


Fig. 3.25(a)

The negative sign indicates that it points along the negative *X*-axis.

$$F_y = F \sin \theta$$
$$= 50 \times \sin 30°$$
$$= 25 \text{ N}$$

Lecture (5)

Mechanics

DR./ IBRAHIM ABADY

Example 3.6 Determine the components of weight [100 N] of a block resting on an inclined plane along the incline and the normal to the incline.

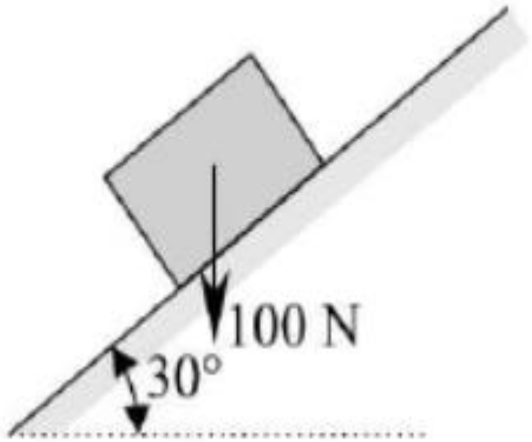
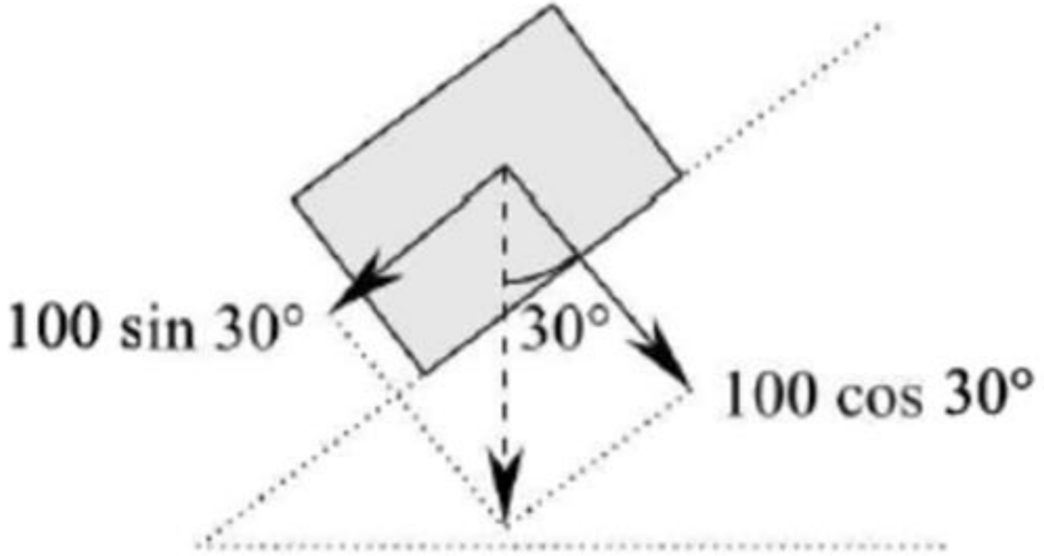


Fig. 3.26

Solution



Hence, the components of the force are

$$F_{\text{normal}} = 100 \cos 30° = 86.6 \text{ N}$$

$$F_{\text{along}} = 100 \sin 30° = 50 \text{ N}$$

Example 3.7

Find the components of a force of magnitude 50 N acting on a block as shown in Fig. 3.27,

(i) along lines parallel and perpendicular to the inclined plane,

(ii) along the horizontal and vertical axes.

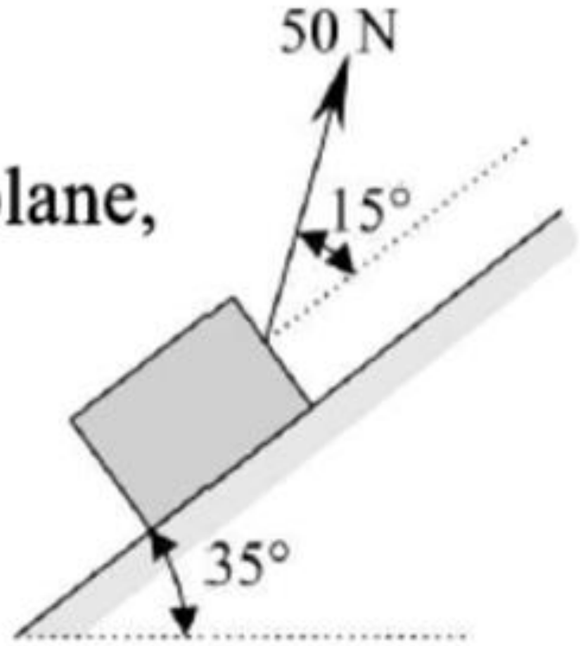
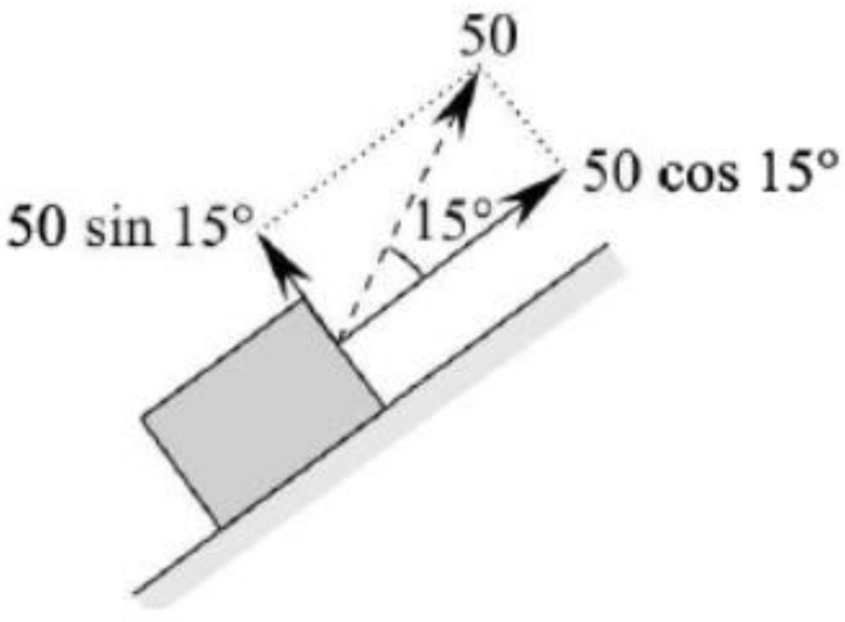


Fig. 3.27

Solution

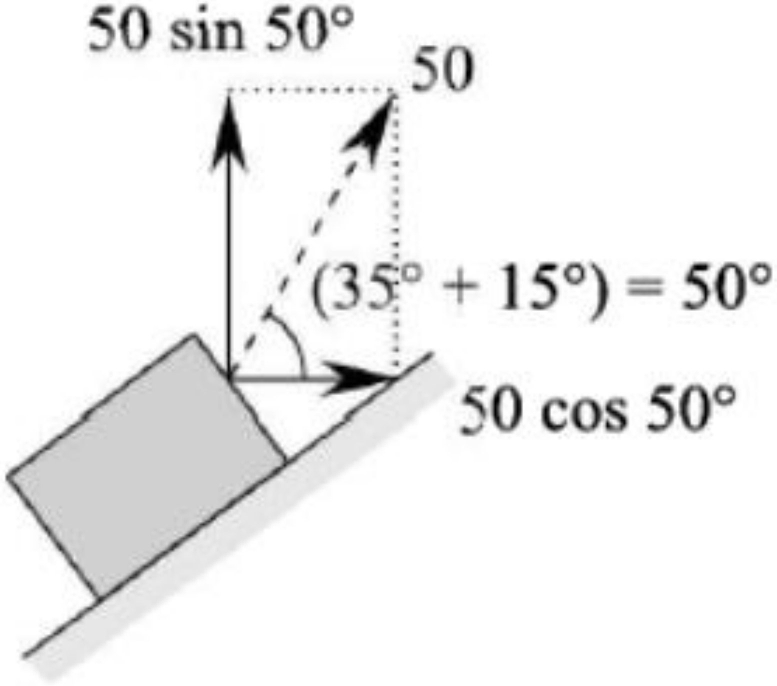
(i) Components of the force parallel and perpendicular to the inclined plane



$$F_{par} = 50 \cos 15° = 48.3 \text{ N}$$

$$F_{per} = 50 \sin 15° = 12.94 \text{ N}$$

(ii) Components of the force along horizontal and vertical axes



$$F_x = 50 \cos 50° = 32.14 \text{ N}$$

$$F_y = 50 \sin 50° = 38.3 \text{ N}$$

Example 3.8 A steel beam is lifted as shown in Fig. 3.28. Determine the components of the tension T in the rope, parallel and perpendicular to the axis of the beam.

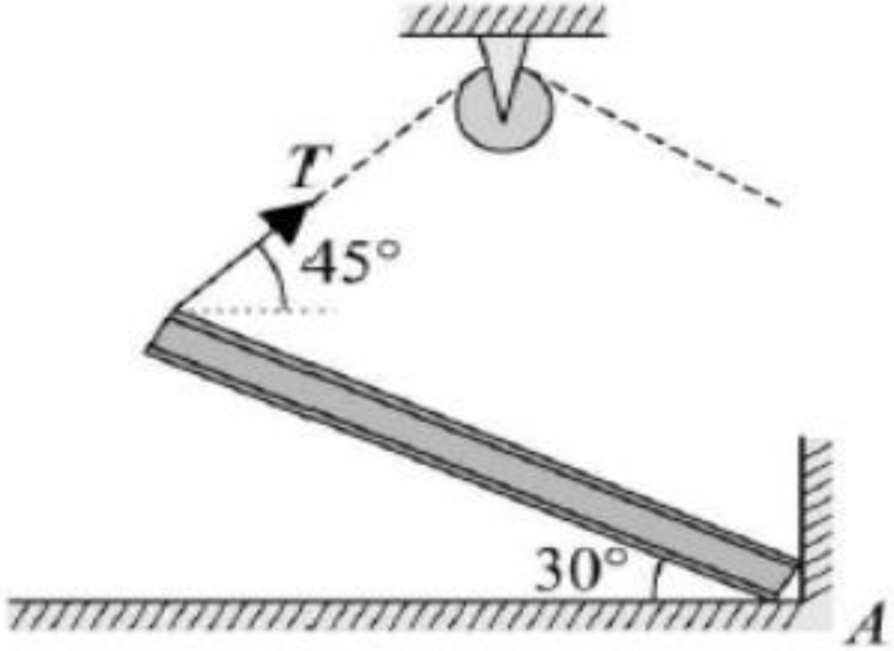
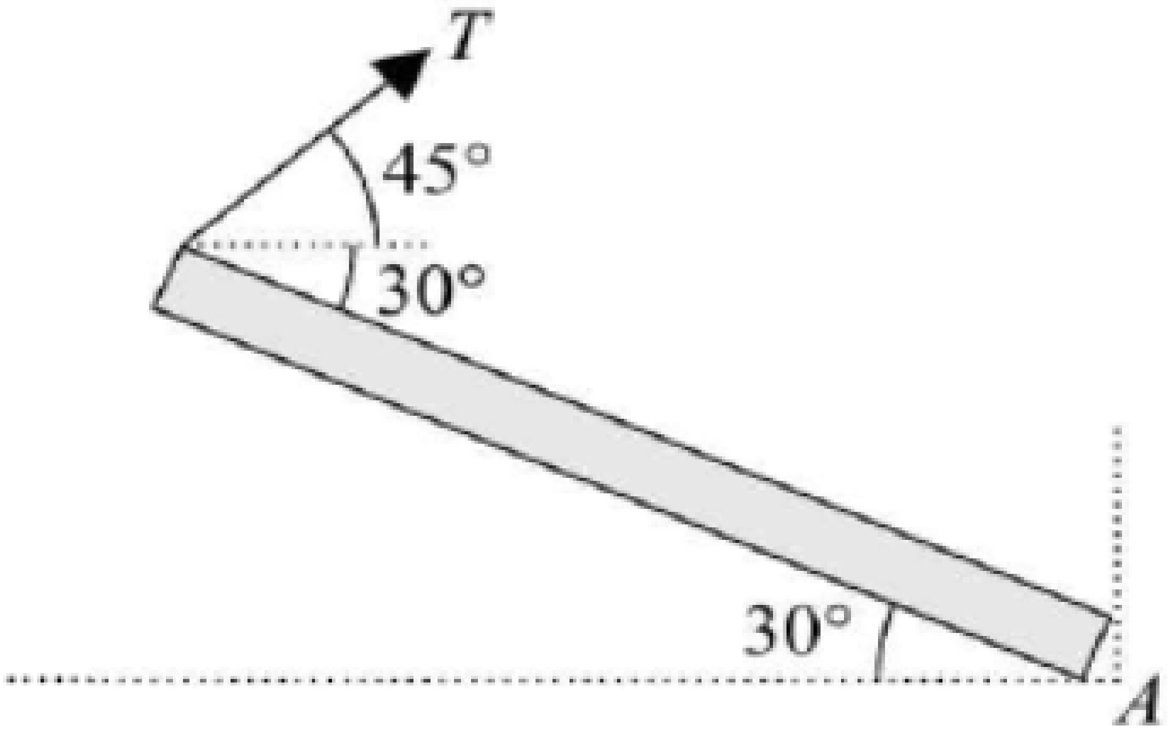


Fig. 3.28

Solution



$$T_{\text{par}} = T\cos 75°$$

$$T_{\text{per}} = T\sin 75°$$

Example 3.9 Find the components of each force shown in Fig. 3.30.

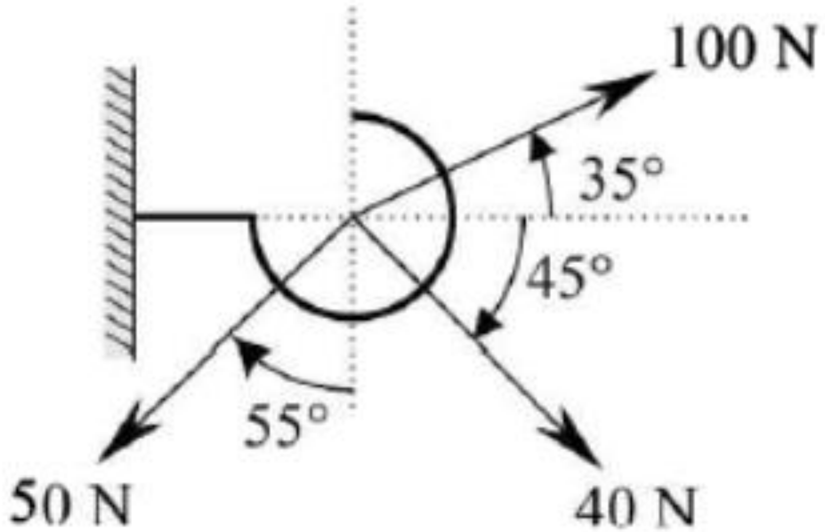


Fig. 3.30

Solution The calculations are summarized in the table below:

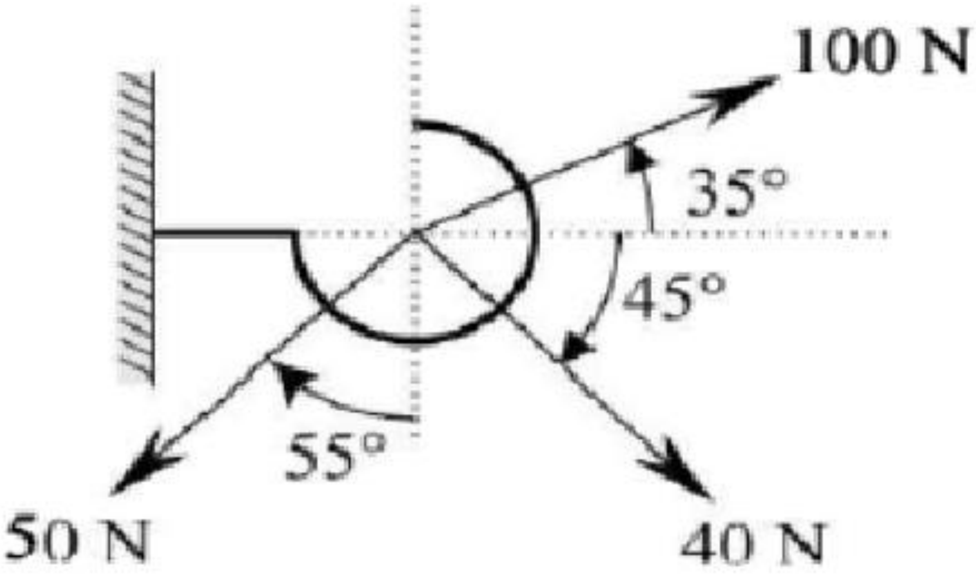


Fig. 3.30

Force	*Magnitude*	*Inclination with respect to X-axis*	*X-component (N)*	*Y-component (N)*
$\vec{F}_1$	100 N	35°	100 cos 35° = 81.92	100 sin 35° = 57.36
$\vec{F}_2$	40 N	45°	40 cos 45° = 28.28	–40 sin 45° = –28.28
$\vec{F}_3$	50 N	35°	–50 cos 35° = –40.96	–50 sin 35° = –28.68

Resultant of Several Concurrent Forces

Consider 'n' number of concurrent forces,

namely, $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ acting on a particle.

Representing each force in the system by its components, we have,

$$\vec{F}_1 = F_{1x}\vec{i} + F_{1y}\vec{j}$$
$$\vec{F}_2 = F_{2x}\vec{i} + F_{2y}\vec{j}$$
$$\vdots \quad \vdots \quad \vdots$$
$$\vec{F}_n = F_{nx}\vec{i} + F_{ny}\vec{j}$$

Therefore, the resultant is obtained by the vector addition of all individual forces:

$$\begin{aligned}\vec{R} &= \vec{F}_1 + \vec{F}_2 + .. + \vec{F}_n \\ &= (F_{1x}\vec{i} + F_{1y}\vec{j}) + (F_{2x}\vec{i} + F_{2y}\vec{j}) + \ldots + (F_{nx}\vec{i} + F_{ny}\vec{j}) \\ &= \sum(F_x)_i\vec{i} + \sum(F_y)_i\vec{j} \end{aligned} \tag{3.18}$$

If R_x and R_y are x and y components of the resultant then

$$R_x\vec{i} + R_y\vec{j} = \sum(F_x)_i\vec{i} + \sum(F_y)_i\vec{j} \tag{3.19}$$

$$R_x = \sum(F_x)_i \text{ and } R_y = \sum(F_y)_i \tag{3.20}$$

The magnitude and direction of the resultant are given by

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{\sum(F_x)_i^2 + \sum(F_y)_i^2} \tag{3.21}$$

$$\tan\alpha = \frac{R_y}{R_x} = \frac{|\sum(F_y)_i|}{|\sum(F_x)_i|} \tag{3.22}$$

Example 3.11 Determine the resultant of four forces concurrent at the origin as shown in Fig. 3.31.

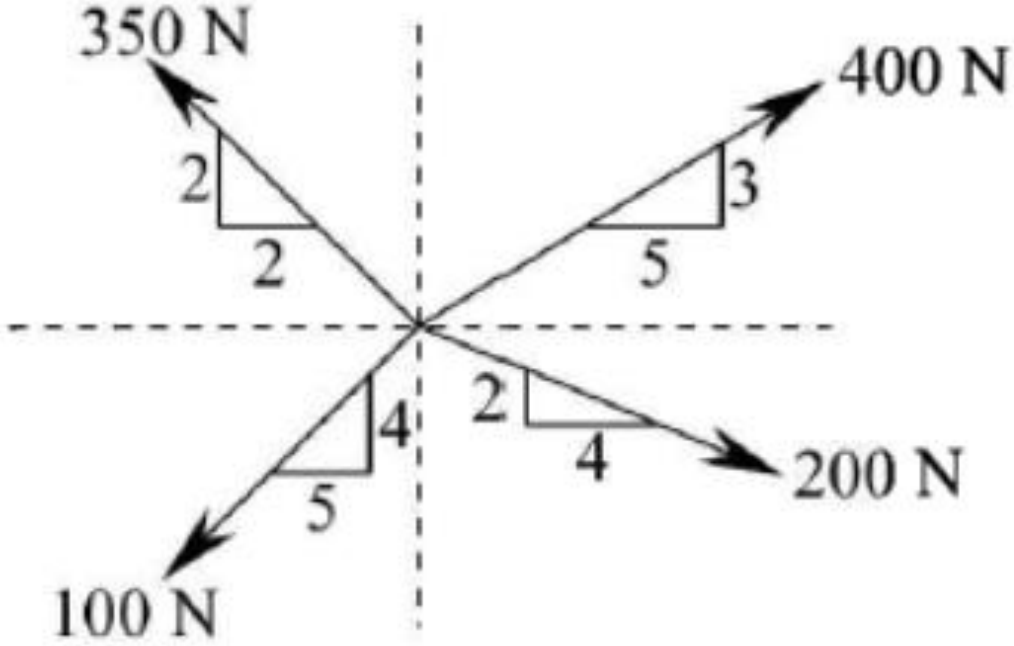


Fig. 3.31

Solution Let us number the forces $\vec{F}_1$ to $\vec{F}_4$, starting from 400 N and moving in the anticlockwise direction.

$$|\vec{F}_1| = F_1 = 400 \text{ N} \qquad \theta_1 = \tan^{-1}(3/5) = 30.96°$$

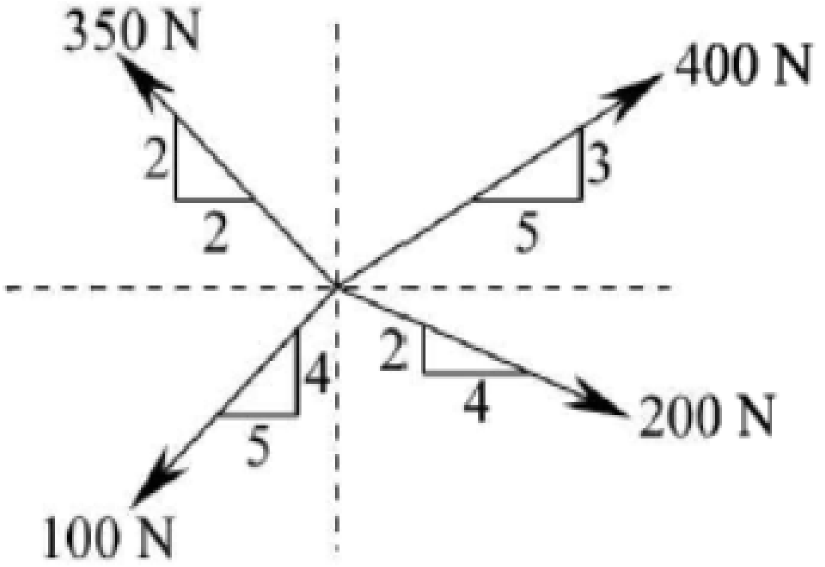


Fig. 3.31

Therefore, its x- and y-components are

$$F_{1x} = F_1 \cos\theta_1 \quad = 400 \times \cos(30.96°) = 343.01 \text{ N}$$

$$F_{1y} = F_1 \sin\theta_1 \quad = 400 \times \sin(30.96°) = 205.78 \text{ N}$$

Similarly, for the other forces, the components can be determined following table:

Force	*Magnitude of force (N)*	*Inclination of force with X-axis*	$(F_x)_i$ N	$(F_y)_i$ N
$\vec{F}_1$	400	30.96°	343.01	205.78
$\vec{F}_2$	350	$\tan^{-1}(2/2) = 45^\circ$	$-350 \cos 45^\circ = -247.49$	$350 \sin 45^\circ = 247.49$
$\vec{F}_3$	100	$\tan^{-1}(4/5) = 38.66^\circ$	$-100 \cos 38.66^\circ = -78.09$	$-100 \sin 38.66^\circ = -62.47$
$\vec{F}_4$	200	$\tan^{-1}(2/4) = 26.57^\circ$	$200 \cos 26.57^\circ = 178.88$	$-200 \sin 26.57^\circ = -89.46$
$\Sigma =$			196.31	301.34

Example 3.12 Find the resultant of a system of forces acting on the block as shown in Fig. 3.32. The 150 N force acts parallel to the incline; the 100 N force acts vertically downwards and the 75 N force acts horizontally. Assume all the forces are concurrent at a point.

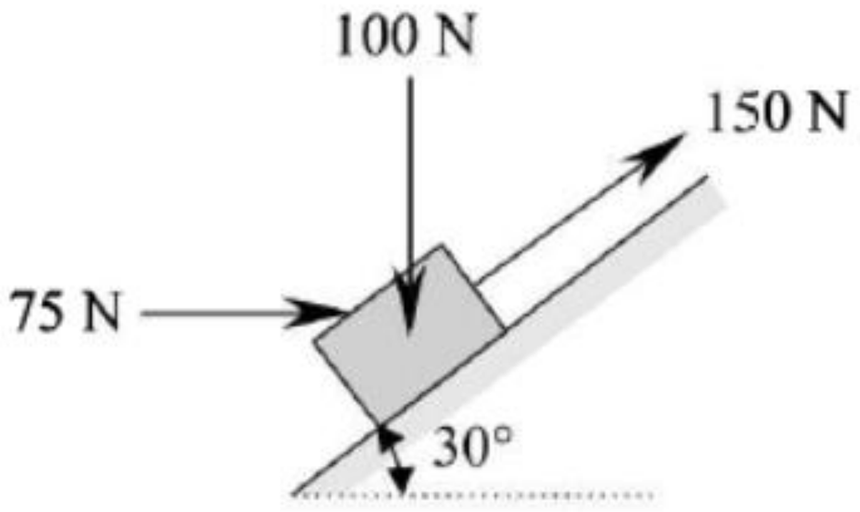


Fig. 3.32

Solution

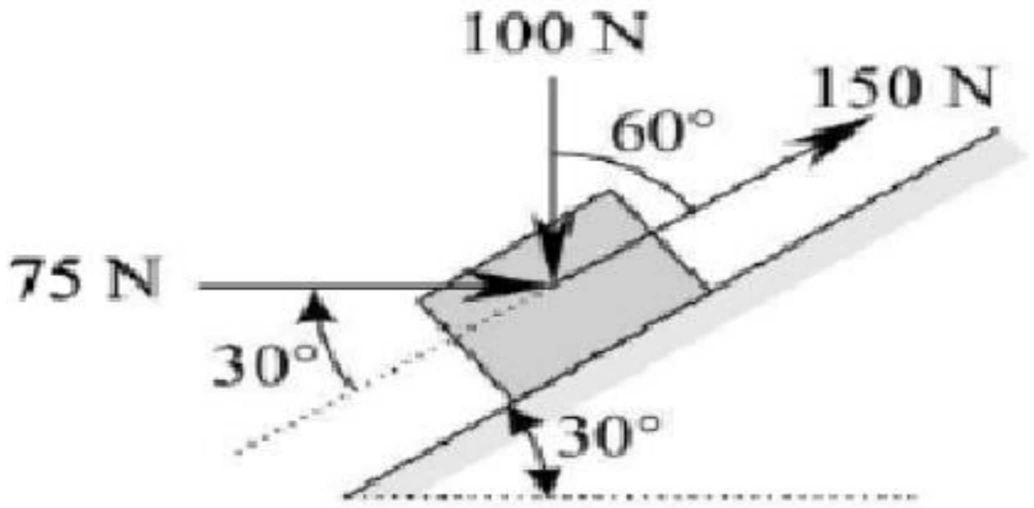


Fig. 3.32(a)

Force	*Magnitude of force (N)*	*Inclination of force with respect to the incline*	$(F_x)_i$ N	$(F_y)_i$ N
$\vec{F}_1$	150	0°	150	0
$\vec{F}_2$	100	60°	–50	–86.6
$\vec{F}_3$	75	30°	64.95	–37.5
$\Sigma =$			164.95	–124.1

Therefore, the magnitude of the resultant is given by

$$R = \sqrt{\sum (F_x)_i^2 + \sum (F_y)_i^2} = \sqrt{(164.95)^2 + (-124.1)^2}$$

$$= 206.42 \text{ N}$$

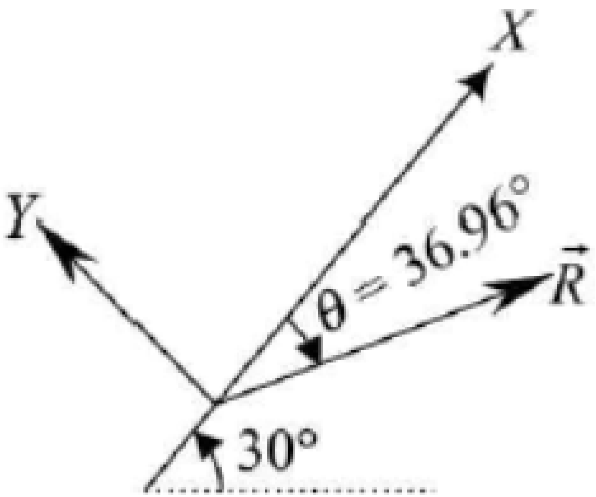


Fig. 3.32(b)

Graphical representation of the resultant force

and its inclination with respect to the inclined plane is

$$\theta = \tan^{-1}\left[\frac{|\sum (F_y)_i|}{|\sum (F_x)_i|}\right] = \tan^{-1}\left[\frac{124.1}{164.95}\right] = 36.96°$$

Example 3.13 The system of forces acting on a block lying on a horizontal plane is shown in Fig. 3.33. Determine the resultant force, if the component of resultant along the *y*-direction is zero. Also, determine the unknown force '*F*.' Assume all the forces are concurrent at a point.

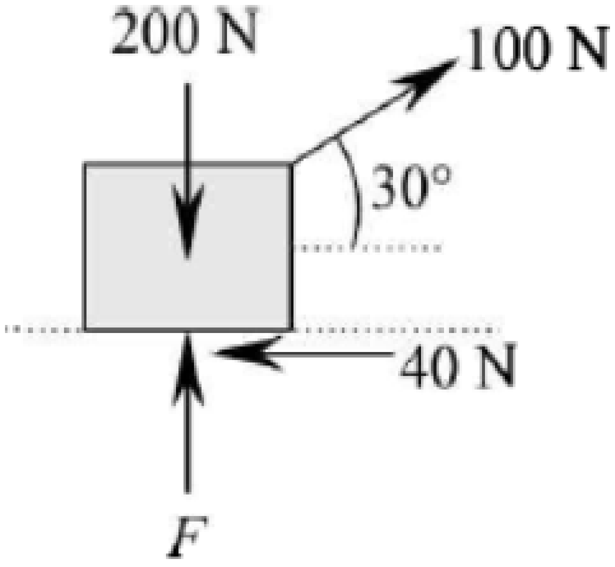
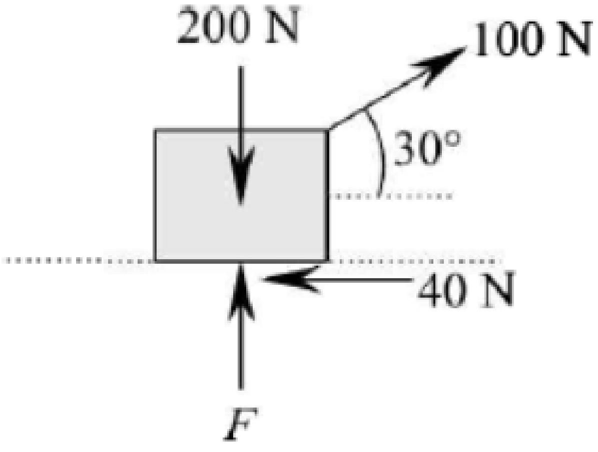


Fig. 3.33

Solution Let us choose the reference axes along and perpendicular to the plane. We see that only the 100 N force is inclined to the axes. Hence, its components along the X and Y axes are respectively:

$$100 \cos 30° = 86.6 \text{ N}$$

$$100 \sin 30° = 50 \text{ N}$$



$$\sum(F_y)_i = F + 100 \sin 30° - 200 \qquad \text{(a)}$$

$$\sum(F_x)_i = 100 \cos 30° - 40 = 46.6 \text{ N} \qquad \text{(b)}$$

As the component of the resultant along the y-direction is $\boxed{\text{zero,}}$

we get from equation (a), $\quad F = 150$ N

Since the component of the resultant along the y-direction is zero,

the resultant of the forces is same as x-component,

$$\text{i.e., } R = \sum(F_x)_i = 46.6 \text{ N.}$$

Example 3.14 Five forces are acting at corner *A* of a regular hexagon as shown in Fig. 3.34. Determine the resultant of the system of forces.

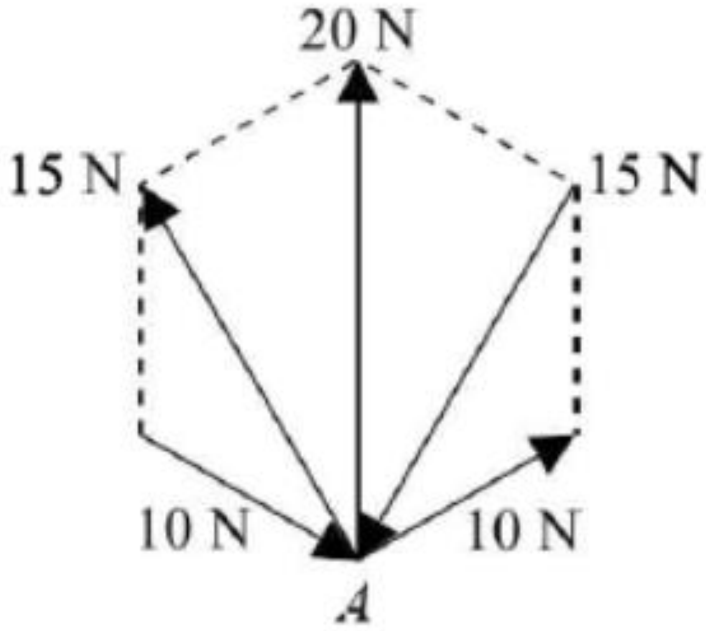


Fig. 3.34

Solution

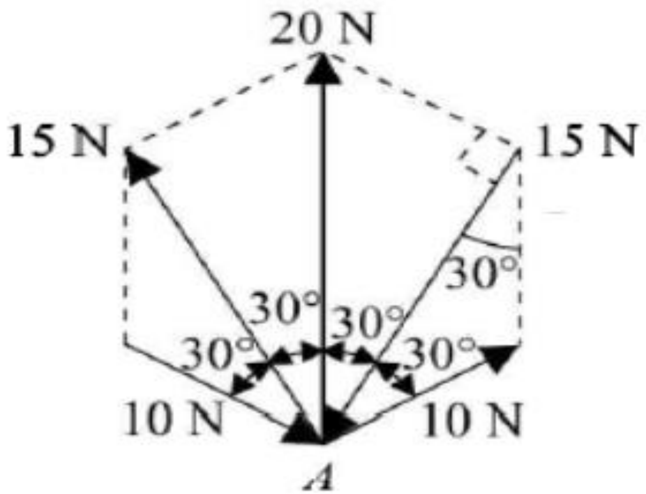
Force	*Magnitude of force (N)*	*Inclination of force with X-axis*	$(F_x)_i$ N	$(F_y)_i$ N
$\vec{F}_1$	10	30°	10 cos 30°	10 sin 30°
$\vec{F}_2$	15	60°	–15 cos 60°	–15 sin 60°
$\vec{F}_3$	20	90°	0	20
$\vec{F}_4$	15	60°	–15 cos 60°	15 sin 60°
$\vec{F}_5$	10	30°	10 cos 30°	–10 sin 30°
$\Sigma =$			2.32	20

Therefore, the magnitude of the resultant is given as

$$R = \sqrt{\sum (F_x)_i^2 + \sum (F_y)_i^2} = 20.13 \text{ N}$$

and its inclination with respect to the X-axis is

$$\alpha = \tan^{-1}\left[\frac{|\sum (F_y)_i|}{|\sum (F_x)_i|}\right] = 83.38°$$



Example 3.15 Determine the resultant of two concurrent forces shown in Fig. 3.35 acting at a gusset plate joint in a truss. What force should be applied to make the resultant zero?

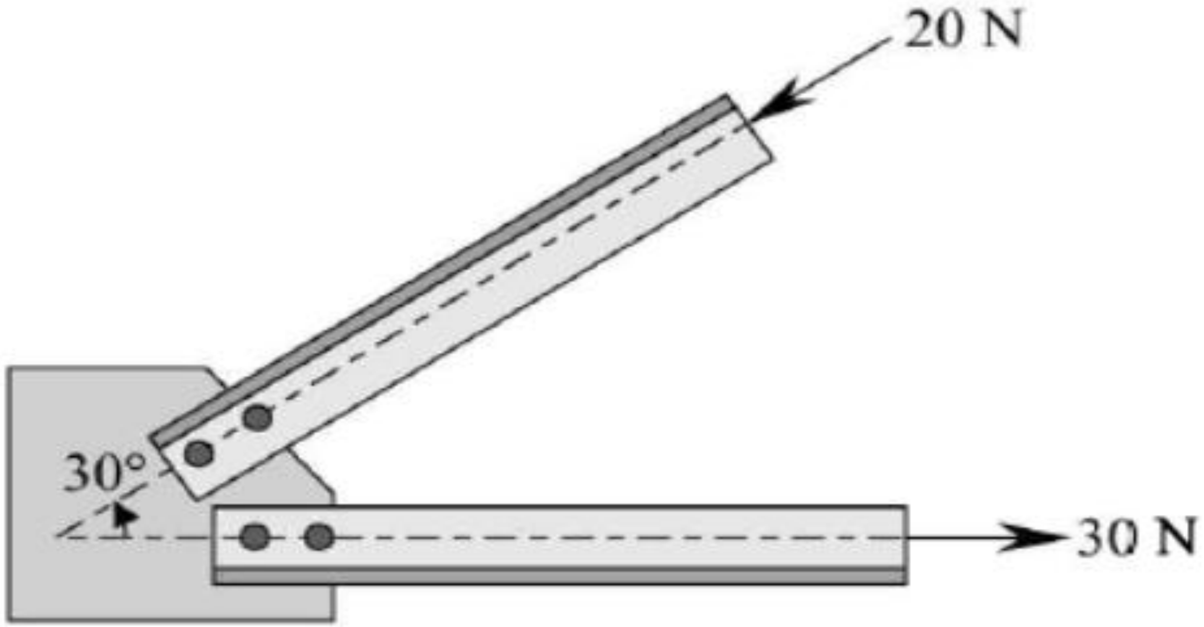


Fig. 3.35

What are trusses?

What are trusses used?

The importance of trusses.

Solution The forces acting along the members can be represented in vector form using rectangular components as

$$\vec{F}_1 = 30\,\vec{i}$$

and $$\vec{F}_2 = -20\cos 30^\circ\,\vec{i} - 20\sin 30^\circ\,\vec{j}$$

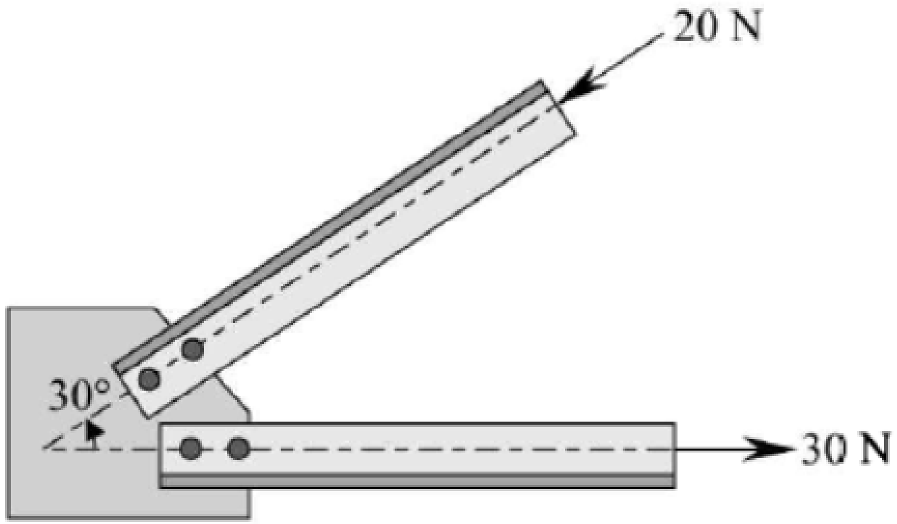


Fig. 3.35

Hence, the resultant of the two forces can be obtained by vector addition:

$$\vec{R} = \vec{F}_1 + \vec{F}_2$$

$$= [30 - 20\cos 30^\circ]\,\vec{i} - [20\sin 30^\circ]\,\vec{j}$$

$$= 12.7\,\vec{i} - 10\,\vec{j}$$

Therefore, the force that must be applied at the point of concurrency to make the resultant zero is

$$-\vec{R} = -12.7\,\vec{i} + 10\,\vec{j}$$

Example 3.16 Show that the resultant of forces $k_1 \overrightarrow{OA}$, $k_2 \overrightarrow{OB}$ is $(k_1 + k_2)\overrightarrow{OC}$, where C is a point on AB such that $k_1 AC = k_2 CB$.

Example 3.16 Show that the resultant of forces $k_1\ \overrightarrow{OA}$, $k_2\ \overrightarrow{OB}$ is $(k_1 + k_2)\overrightarrow{OC}$,

where C is a point on AB

such that $k_1\, AC = k_2\, CB$.

Solution Given: $k_1 AC = k_2 CB$

Therefore, $$k_1 \overrightarrow{AC} = k_2 \overrightarrow{CB}$$

Since $\overrightarrow{AC}$ can be written as $(\overrightarrow{OC} - \overrightarrow{OA})$ and $\overrightarrow{CB}$ can be written as $(\overrightarrow{OB} - \overrightarrow{OC})$,

$$k_1 (\overrightarrow{OC} - \overrightarrow{OA}) = k_2 (\overrightarrow{OB} - \overrightarrow{OC})$$

On rearranging,

$$(k_1 + k_2)\overrightarrow{OC} = k_1 \overrightarrow{OA} + k_2 \overrightarrow{OB}$$

Thus, we can see that the resultant of forces $k_1 \overrightarrow{OA}$ and $k_2 \overrightarrow{OB}$ is $(k_1 + k_2) \overrightarrow{OC}$.

Lecture (6)

Mechanics

DR./ IBRAHIM ABADY

Suez University Faculty of Science Department of Mathematics	
Mid-Term Exam 2023 - 2024	
Course title: Mechanics	Course Code: Mat203
Date: 15/11/2023	Time allowed: 1 Hour

1.	If $\vec{a} = 5\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 4\vec{j} + \vec{k}$, $\vec{c} = 3\vec{i} + \vec{j} + 2\vec{k}$, then find (i) $2\vec{a} + \vec{b} + 3\vec{c}$, (ii) $\vec{a} - 2\vec{b} + \vec{c}$ (iii) $3\vec{a} - \vec{b} - \vec{c}$.
2.	ML^2T^{-2} is the dimension for which of the following physical quantities? (a) Work done (b) Energy (c) Moment of force (d) All of these
3.	Find the direction cosines of the vector: $3\vec{i} + 2\vec{j} + \vec{k}$.
4.	If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{c} = \vec{i} - \vec{j} - \vec{k}$, find $\vec{a} .(\vec{b} \times \vec{c})$ and $\vec{a} \times (\vec{b} \times \vec{c})$.
5.	Given the vectors $\vec{a} = 2\vec{j} + 6\vec{k}$, $\vec{b} = \vec{j} + 3\vec{k}$, and $\vec{c} = 2\vec{i} + \vec{k}$, determine whether they are coplanar or not.

Lecture (7)

Mechanics

DR./ IBRAHIM ABADY

CONCURRENT FORCES IN SPACE

In the previous sections, we discussed the coplanar concurrent forces and the methods to determine their resultants.

In this section, we will discuss concurrent forces in *space*.

Graphical and trigonometric methods become complicated when applied to solve more number of concurrent forces.

Still more complicated do they become when applied to forces in space.

Hence, we employ *only* analytical method to solve forces in space.

Forces in Space

Consider a force $\vec{F}$ in space acting at the origin.

First, resolve the force $\vec{F}$ into components, $\vec{F}_y$ and $\vec{F}_{xz}$ along the vertical plane $OCGF$ [Fig. 3.36(a)].

Then resolve $\vec{F}_{xz}$ into components, $\vec{F}_x$ and $\vec{F}_z$ on the X–Z plane [Fig. 3.36(b)].

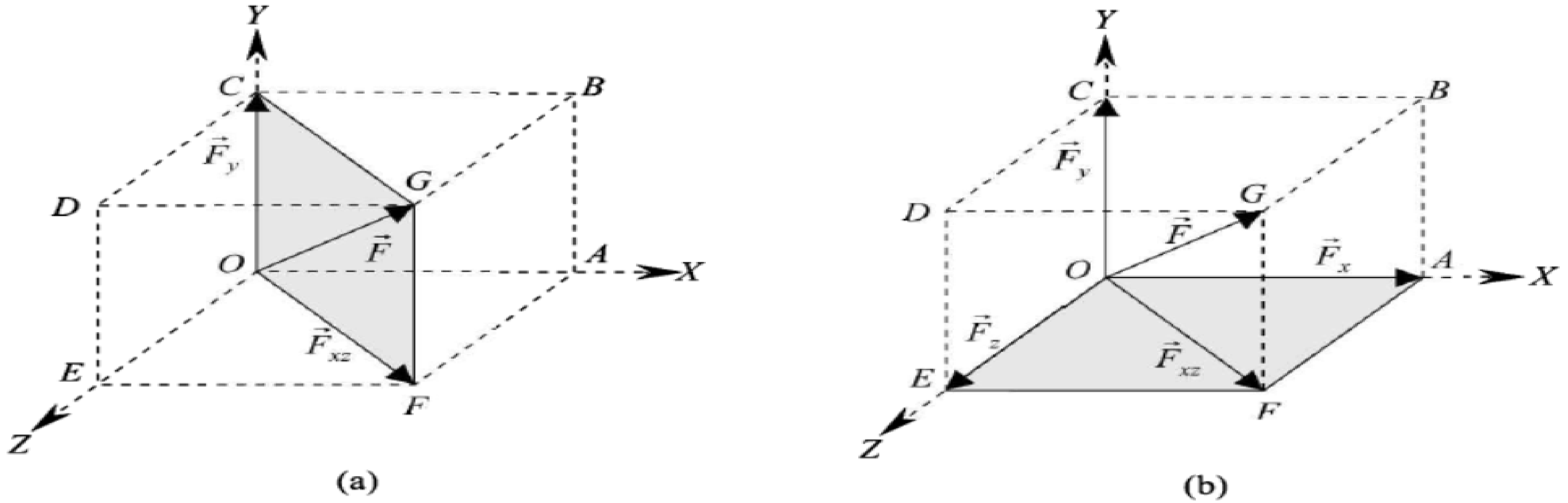


Fig. 3.36 (a) Resolving $\vec{F}$ into $\vec{F}_y$ and $\vec{F}_{xz}$ (b) Resolving $\vec{F}_{xz}$ into $\vec{F}_x$ and $\vec{F}_z$

By vector addition, we can write

$$\vec{F} = \vec{F}_y + \vec{F}_{xz} = \vec{F}_x + \vec{F}_y + \vec{F}_z \qquad (3.23)$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} \qquad (3.24)$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors respectively along X, Y and Z directions.

The magnitude of the force is represented in terms of its components as

$$|\vec{F}| = F = \sqrt{F_x^2 + F_y^2 + F_z^2} \qquad (3.25)$$

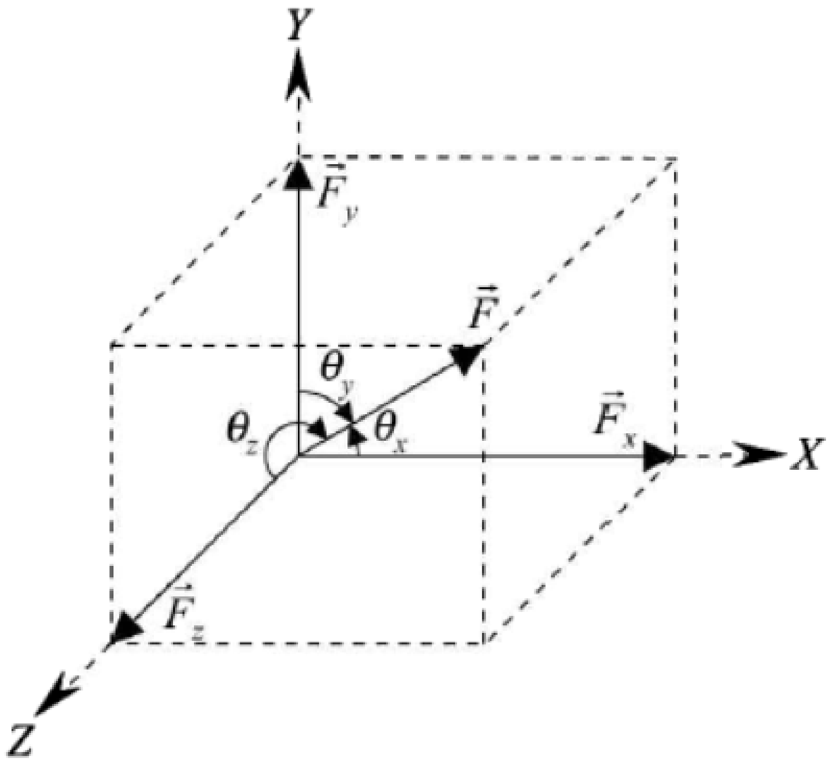


Fig. 3.36(c) Inclination of $\vec{F}$ with respect to *X*, *Y* and *Z* axes

If θ_x, θ_y and θ_z are angles made by $\vec{F}$ with *X*, *Y* and *Z* axes respectively [Fig. 3.36(c)], then Eq. 3.24 can be written as

$$\vec{F} = F\cos\theta_x\vec{i} + F\cos\theta_y\vec{j} + F\cos\theta_z\vec{k}$$

$$= F[\cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}] \qquad (3.26)$$

We know that any vector can be expressed as a product of its magnitude and unit vector along its line of action. Hence, force vector can also be written as

$$\vec{F} = F\hat{n} \tag{3.27}$$

Comparing the above two expressions for force [Eqs 3.26 and 3.27], we can readily see that $\hat{n} = \cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}$ is the unit vector along the line of action of $\vec{F}$. Since the magnitude of $\hat{n}$ is **unity**, we have,

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \tag{3.28}$$

where $\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$ are called **direction cosines** of the force. The direction cosines can be expressed in terms of components of the force as

$$\cos\theta_x = \frac{F_x}{F}, \cos\theta_y = \frac{F_y}{F} \text{ and } \cos\theta_z = \frac{F_z}{F} \tag{3.29}$$

Representation of a Force Passing through any Two Points in Space

In the previous section, we saw how to express a force vector in space passing through the origin.

Hence, in this section, we will see how to express a force vector passing through any two points in space.

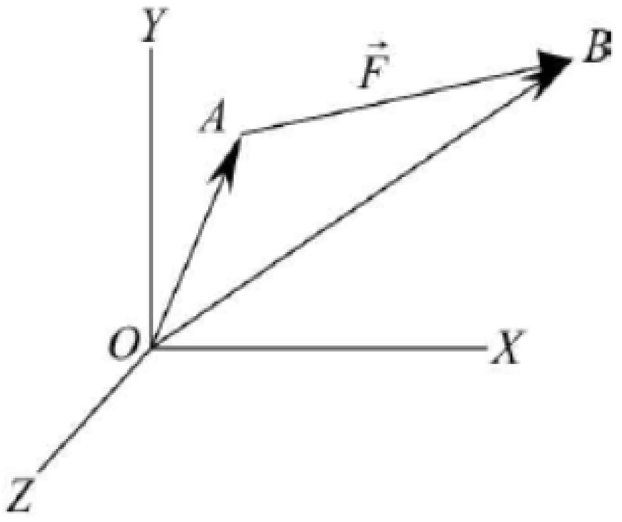


Fig. 3.37 Force passing through any two points in space

Consider a force $\vec{F}$ passing through two points A and B in space, whose respective coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2). Then the position vectors of the points A and B with respect to the origin are

$$\overrightarrow{OA} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \tag{3.30}$$

and

$$\overrightarrow{OB} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \tag{3.31}$$

Hence,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k} \tag{3.32}$$

Then unit vector along AB is given by $\hat{n}_{AB} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$

Hence, the force $\vec{F}$ can be expressed as

$$\vec{F} = F\hat{n}_{AB} \tag{3.33}$$

Example 3.17 Express the force of 100 N passing through the origin and point *A* as shown in Fig. 3.38 in vector form.

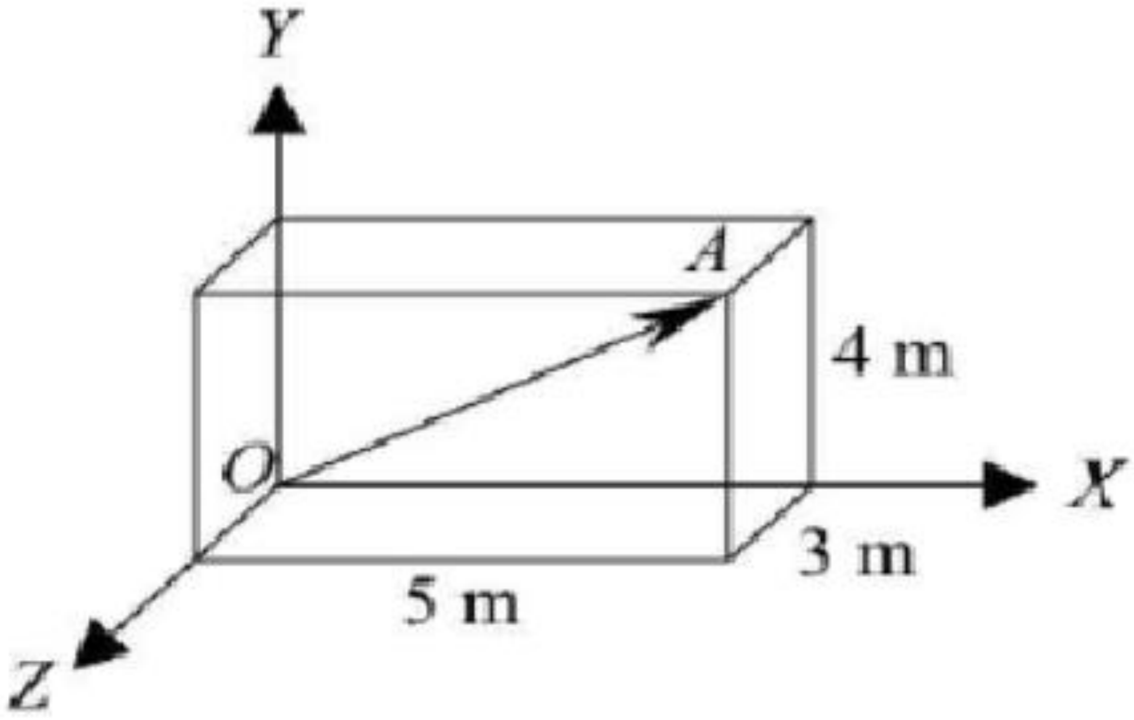


Fig. 3.38

Solution we see that the coordinates of A are (5, 4, 3).

$$\overrightarrow{OA} = 5\vec{i} + 4\vec{j} + 3\vec{k}$$

Then its magnitude is obtained as

$$|\overrightarrow{OA}| = \sqrt{5^2 + 4^2 + 3^2} = \sqrt{50} \text{ m}$$

unit vector along OA is obtained as

$$\hat{n}_{OA} = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} = \frac{5\vec{i} + 4\vec{j} + 3\vec{k}}{\sqrt{50}}$$

Therefore, the force can be expressed in vector form as

$$\vec{F} = F\hat{n}_{OA} = 100\left[\frac{5\vec{i} + 4\vec{j} + 3\vec{k}}{\sqrt{50}}\right] = 70.71 \text{ N}\,\vec{i} + 56.57 \text{ N}\,\vec{j} + 42.43 \text{ N}\,\vec{k}$$

Example 3.18 A force of magnitude 200 N passes through the origin. Its inclinations with respect to the axes are $\theta_x = 30°$, $\theta_y = 75°$. If F_z is positive, then determine the inclination of the force with respect to the Z-axis and express the force in vector form.

Solution We know that the direction cosines are related by the expression

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\Rightarrow \quad \cos^2\theta_z = 1 - \cos^2\theta_x - \cos^2\theta_y$$

$$= 1 - \cos^2(30°) - \cos^2(75°) = 0.183$$

$$\therefore \quad \cos\theta_z = \pm 0.428$$

Since F_z is positive, $\cos\theta_z$ will also be positive. [Note that $\cos\theta_z = F_z/F$.]

$$\therefore \quad \cos\theta_z = 0.428 \quad \Rightarrow \quad \theta_z = 64.66°$$

Therefore, the force can be expressed in vector form as

$$\vec{F} = F\,[\cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}]$$

$$= 200[\cos(30°)\vec{i} + \cos(75°)\vec{j} + \cos(64.66°)\vec{k}]$$

$$= 173.21\text{ N}\,\vec{i} + 51.76\text{ N}\,\vec{j} + 85.6\text{ N}\,\vec{k}$$

Example 3.19 The inclinations of a force passing through the origin are $\theta_y = 55.4°$ and $\theta_z = 67.2°$.

Determine the angle θ_x, if $F_x = -100$ N.

Also, express the force in vector form.

Solution The direction cosines are related by the expression

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\Rightarrow \quad \cos^2\theta_x = 1 - \cos^2\theta_y - \cos^2\theta_z$$

$$= 1 - \cos^2(55.4°) - \cos^2(67.2°) = 0.527$$

$$\therefore \quad \cos\theta_x = \pm 0.726$$

Since F_x is negative, $\cos\theta_x$ will also be negative.

$$\therefore \quad \cos\theta_x = -0.726 \qquad \Rightarrow \qquad \theta_x = 136.55°$$

We know, $\cos\theta_x = \dfrac{F_x}{F}$

$$\therefore \quad F = \frac{F_x}{\cos\theta_x} = \frac{-100}{-0.726} = 137.74 \text{ N}$$

Therefore, the force vector is given as

$$\vec{F} = F[\cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}]$$

$$= 137.74[\cos(136.55°)\vec{i} + \cos(55.4°)\vec{j} + \cos(67.2°)\vec{k}]$$

$$= -100\ \mathrm{N}\ \vec{i} + 78.21\ \mathrm{N}\ \vec{j} + 53.38\ \mathrm{N}\ \vec{k}$$

Example 3.20 The line of action of a force of magnitude 50 N passes through points A and B as shown in Fig. 3.39.

Express the force in vector form.

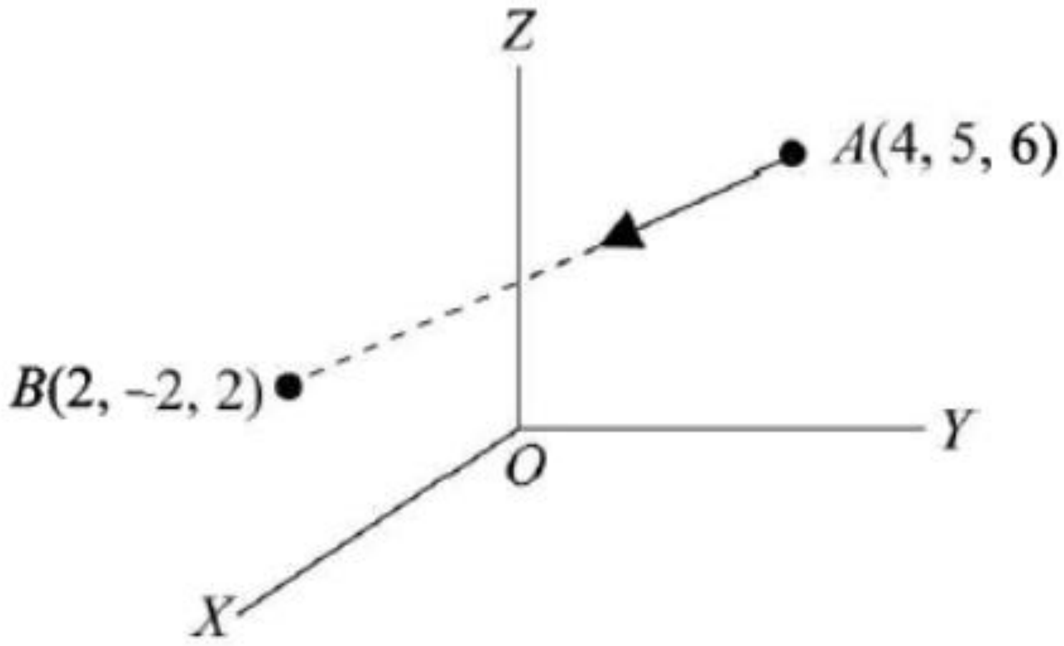
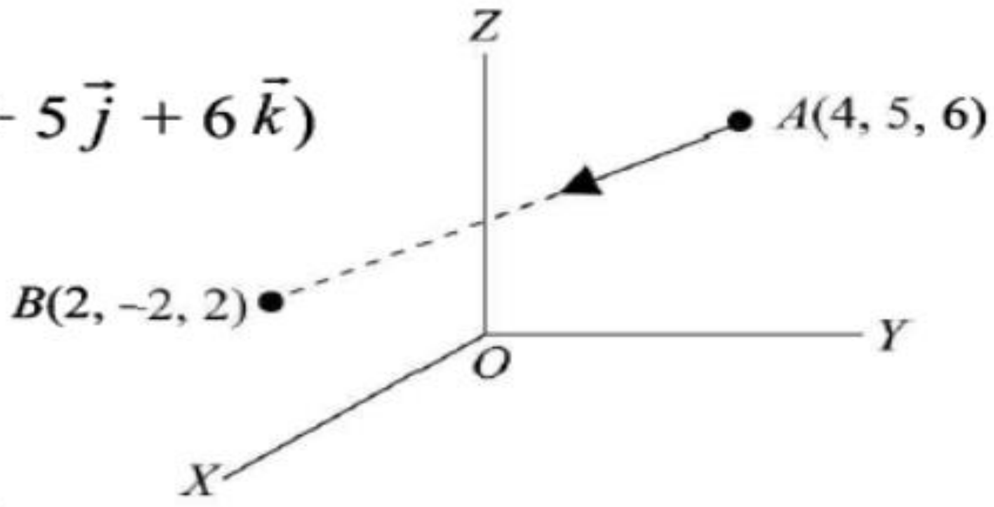


Fig. 3.39

Solution $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (2\vec{i} - 2\vec{j} + 2\vec{k}) - (4\vec{i} + 5\vec{j} + 6\vec{k})$$

$$= -2\vec{i} - 7\vec{j} - 4\vec{k}$$



Therefore, unit vector along $\overrightarrow{AB}$ is

$$\hat{n}_{AB} = \frac{-2\vec{i} - 7\vec{j} - 4\vec{k}}{\sqrt{(-2)^2 + (-7)^2 + (-4)^2}}$$

$$= -0.241\,\vec{i} - 0.843\,\vec{j} - 0.482\,\vec{k}$$

Therefore, the force can be expressed in vector form as

$$\vec{F} = F\hat{n}_{AB}$$

$$= 50[-0.241\,\vec{i} - 0.843\,\vec{j} - 0.482\,\vec{k}]$$

$$= -12.05\text{ N}\,\vec{i} - 42.15\text{ N}\,\vec{j} - 24.1\text{ N}\,\vec{k}$$

Resultant of Several Concurrent Forces in Space

Consider 'n' number of concurrent forces, namely, $\vec{F}_1$, $\vec{F}_2$, ..., $\vec{F}_n$ acting in space.

Representing each force in the system by its components, we have,

$$\vec{F}_1 = F_{1x}\vec{i} + F_{1y}\vec{j} + F_{1z}\vec{k}$$

$$\vec{F}_2 = F_{2x}\vec{i} + F_{2y}\vec{j} + F_{2z}\vec{k}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\vec{F}_n = F_{nx}\vec{i} + F_{ny}\vec{j} + F_{nz}\vec{k}$$

Then their resultant is obtained by vector addition:

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

$$= [F_{1x} + F_{2x} + \ldots + F_{nx}]\,\vec{i}$$

$$+ [F_{1y} + F_{2y} + \ldots + F_{ny}]\,\vec{j}$$

$$+ [F_{1z} + F_{2z} + \ldots + F_{nz}]\,\vec{k}$$

$$= \sum(F_x)_i\,\vec{i} + \sum(F_y)_i\,\vec{j} + \sum(F_z)_i\,\vec{k} \qquad (3.34)$$

If R_x, R_y and R_z are components of the resultant then

$$R_x\vec{i} + R_y\vec{j} + R_z\vec{k} = \sum(F_x)_i\,\vec{i} + \sum(F_y)_i\,\vec{j} + \sum(F_z)_i\,\vec{k}$$

$$R_x = \sum(F_x)_i, \qquad R_y = \sum(F_y)_i \qquad R_z = \sum(F_z)_i \qquad (3.35)$$

The magnitude and direction of the resultant are given by

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{\sum(F_x)_i^2 + \sum(F_y)_i^2 + \sum(F_z)_i^2} \qquad (3.36)$$

$$\cos\theta_x = \frac{R_x}{R} = \frac{\sum(F_x)_i}{R}$$

$$\cos\theta_y = \frac{R_y}{R} = \frac{\sum(F_y)_i}{R}$$

$$\cos\theta_z = \frac{R_z}{R} = \frac{\sum(F_z)_i}{R} \qquad (3.37)$$

Example 3.21 Determine the resultant of a system of three concurrent forces passing through the origin and points (10, –5, 6), (–5, 5, 7) and (6, –4, –3) respectively. The respective magnitudes of the forces are 1500 N, 2500 N and 2000 N.

Solution Let the points through which the forces pass be denoted as A, B and C respectively.

Then their position vectors are given as

$$\overrightarrow{OA} = 10\vec{i} - 5\vec{j} + 6\vec{k} \quad \overrightarrow{OB} = -5\vec{i} + 5\vec{j} + 7\vec{k} \quad \overrightarrow{OC} = 6\vec{i} - 4\vec{j} - 3\vec{k}$$

Then unit vectors along $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ can be determined as follows

$$\hat{n}_{OA} = \frac{10\vec{i} - 5\vec{j} + 6\vec{k}}{\sqrt{(10)^2 + (-5)^2 + (6)^2}} = \frac{10\vec{i} - 5\vec{j} + 6\vec{k}}{\sqrt{161}}$$

$$\hat{n}_{OB} = \frac{-5\vec{i} + 5\vec{j} + 7\vec{k}}{\sqrt{(-5)^2 + (5)^2 + (7)^2}} = \frac{-5\vec{i} + 5\vec{j} + 7\vec{k}}{\sqrt{99}}$$

$$\hat{n}_{OC} = \frac{6\vec{i} - 4\vec{j} - 3\vec{k}}{\sqrt{(6)^2 + (-4)^2 + (-3)^2}} = \frac{6\vec{i} - 4\vec{j} - 3\vec{k}}{\sqrt{61}}$$

Hence, the forces can be expressed in vector form as

$$\vec{F}_{OA} = F_{OA}\,\hat{n}_{OA} = 1500\left(\frac{10\vec{i} - 5\vec{j} + 6\vec{k}}{\sqrt{161}}\right) = 1182.17\,\vec{i} - 591.08\,\vec{j} + 709.3\,\vec{k}$$

$$\vec{F}_{OB} = F_{OB}\,\hat{n}_{OB} = 2500\left(\frac{-5\vec{i} + 5\vec{j} + 7\vec{k}}{\sqrt{99}}\right) = -1256.3\,\vec{i} + 1256.3\,\vec{j} + 1758.82\,\vec{k}$$

$$\vec{F}_{OC} = F_{OC}\,\hat{n}_{OC} = 2000\left(\frac{6\vec{i} - 4\vec{j} - 3\vec{k}}{\sqrt{61}}\right) = 1536.44\,\vec{i} - 1024.3\,\vec{j} - 768.22\,\vec{k}$$

Therefore, the resultant force $\vec{R}$ is given as

$$\vec{R} = \vec{F}_{OA} + \vec{F}_{OB} + \vec{F}_{OC} \quad = 1462.31 \text{ N}\,\vec{i} - 359.08 \text{ N}\,\vec{j} + 1699.9 \text{ N}\,\vec{k}$$

Its magnitude is given by

$$|\vec{R}| = \sqrt{(1462.31)^2 + (-359.08)^2 + (1699.9)^2} \quad = 2270.89 \text{ N}$$

Example 3.22

Determine the resultant of the tension forces acting at point A of the transmission tower. The magnitudes of tensions along cables AB, AC and AD are respectively 1000 N, 2000 N and 1800 N.

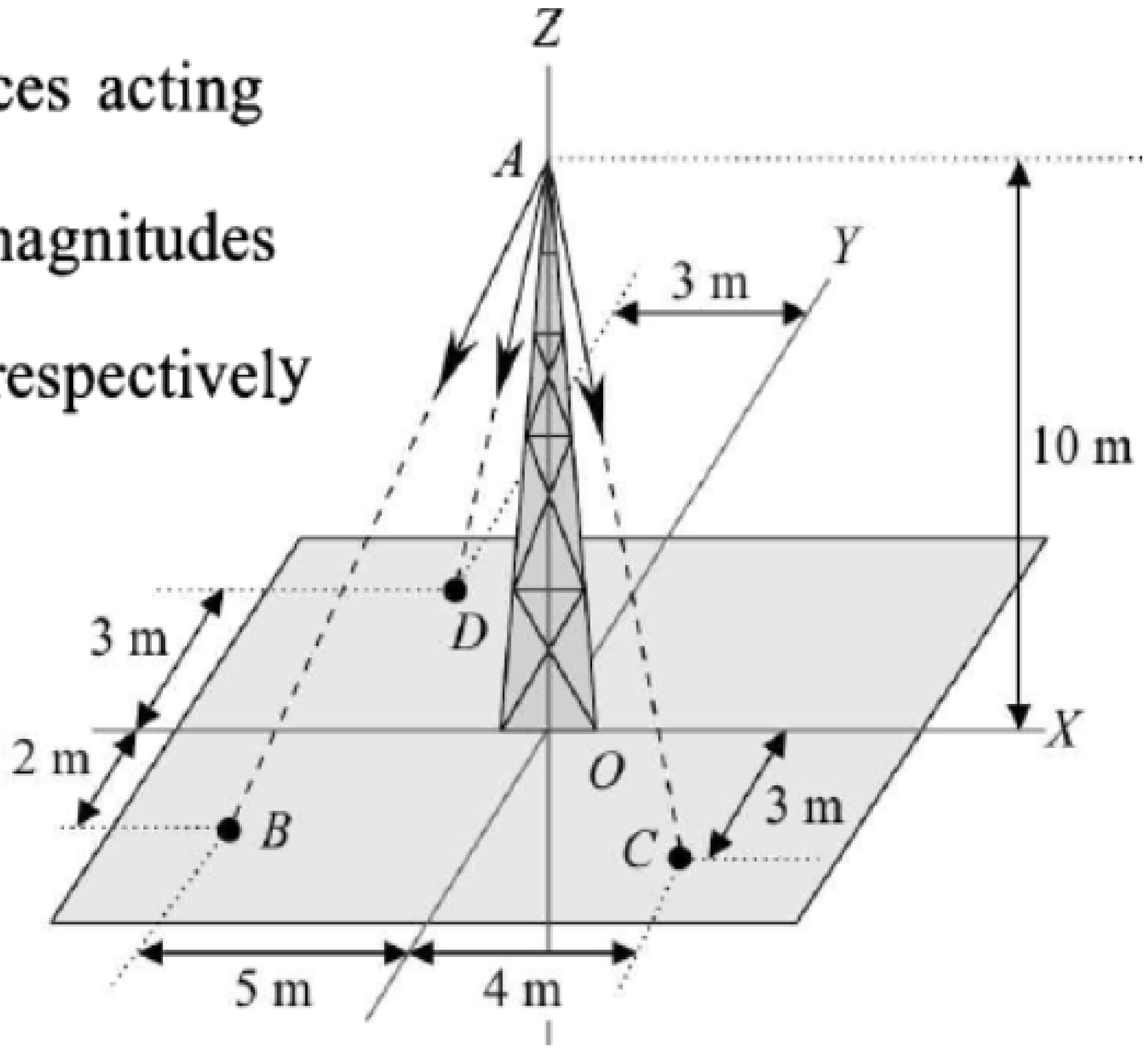


Fig. 3.40

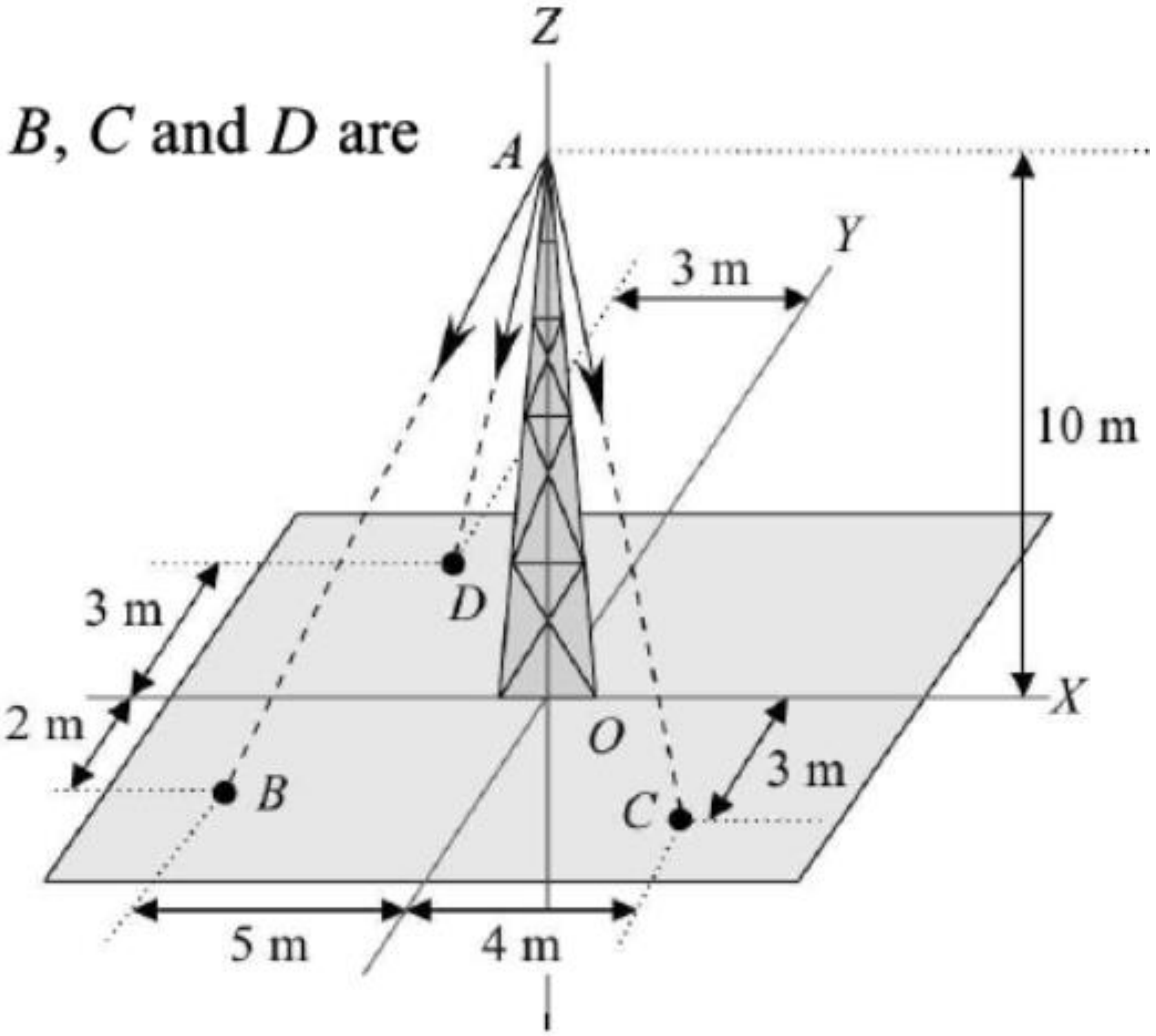
Solution The coordinates of the points A, B, C and D are

$A(0, 0, 10)$ or $\overrightarrow{OA} = 10\vec{k}$

$B(-5, -2, 0)$ or $\overrightarrow{OB} = -5\vec{i} - 2\vec{j}$

$C(4, -3, 0)$ or $\overrightarrow{OC} = 4\vec{i} - 3\vec{j}$

$D(-3, 3, 0)$ or $\overrightarrow{OD} = -3\vec{i} + 3\vec{j}$



Calculation of unit vectors along AB, AC and AD

$$\hat{n}_{AB}: \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -5\vec{i} - 2\vec{j} - 10\vec{k}$$

$$|\overrightarrow{AB}| = \sqrt{(-5)^2 + (-2)^2 + (-10)^2} = \sqrt{129}$$

$$\therefore \quad \hat{n}_{AB} = \frac{-5\vec{i} - 2\vec{j} - 10\vec{k}}{\sqrt{129}}$$

$$\hat{n}_{AC}: \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 4\vec{i} - 3\vec{j} - 10\vec{k}$$

$$|\overrightarrow{AC}| = \sqrt{(4)^2 + (-3)^2 + (-10)^2} = \sqrt{125}$$

$$\therefore \quad \hat{n}_{AC} = \frac{4\vec{i} - 3\vec{j} - 10\vec{k}}{\sqrt{125}}$$

$$\hat{n}_{AD}: \quad \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -3\vec{i} + 3\vec{j} - 10\vec{k}$$

$$|\overrightarrow{AD}| = \sqrt{(-3)^2 + (3)^2 + (-10)^2} = \sqrt{118}$$

$$\therefore \quad \hat{n}_{AD} = \frac{-3\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{118}}$$

Calculation of force vectors

$$\vec{T}_{AB} = T_{AB}\,\hat{n}_{AB} = 1000\left(\frac{-5\vec{i} - 2\vec{j} - 10\vec{k}}{\sqrt{129}}\right) = -440.23\,\vec{i} - 176.09\,\vec{j} - 880.45\,\vec{k}$$

$$\vec{T}_{AC} = T_{AC}\,\hat{n}_{AC} = 2000\left(\frac{4\vec{i} - 3\vec{j} - 10\vec{k}}{\sqrt{125}}\right) = 715.54\,\vec{i} - 536.66\,\vec{j} - 1788.85\,\vec{k}$$

$$\vec{T}_{AD} = T_{AD}\,\hat{n}_{AD} = 1800\left(\frac{-3\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{118}}\right) = -497.11\,\vec{i} + 497.11\,\vec{j} - 1657.03\,\vec{k}$$

Hence, the resultant force is given by

$$\vec{R} = \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} = -221.8\,\vec{i} - 215.64\,\vec{j} - 4326.33\,\vec{k}$$

Its magnitude is

$$R = \sqrt{(-221.8)^2 + (-215.64)^2 + (-4326.33)^2} = 4337.38 \text{ N}$$

and its inclinations with respect to the axes are

$$\theta_x = \cos^{-1}\left[\frac{R_x}{R}\right] = \cos^{-1}\left[\frac{-221.8}{4337.38}\right] = 92.93°$$

$$\theta_y = \cos^{-1}\left[\frac{R_y}{R}\right] = \cos^{-1}\left[\frac{-215.64}{4337.38}\right] = 92.85°$$

$$\theta_z = \cos^{-1}\left[\frac{R_z}{R}\right] = \cos^{-1}\left[\frac{-4326.33}{4337.38}\right] = 175.91°$$