Nonlinear Equations

- A nonlinear equation in one or more variables is a nonlinear function from R^n to R. We assume that we have a real nonlinear function in several variables *f(x)*. for example:

$$f: R^2 \to R \text{ such that } f(x,y) = 2x^2 + 3y^2 = 1 \text{ or}$$
$$f(x,y) = 2x^2 + 3y^2 - 1 = 0$$

$$f: R^3 \to R \text{ such that } f(x,y,z) = x^2 - y^2 + z^2 - 2xz - yz + 4x = -12$$
$$\text{or } f(x,y,z) = x^2 - y^2 + z^2 - 2xz - yz + 4x + 12 = 0$$

Any nonlinear equation can be solved numerically with or without using differentiation.

1- Bisection Method

Theorem

An equation $f(x)=0$, *where* $f(x)$ is a real continuous function, has at least one root between x_l *and* x_u *if* $f(x_l)\, f(x_u) < 0$.

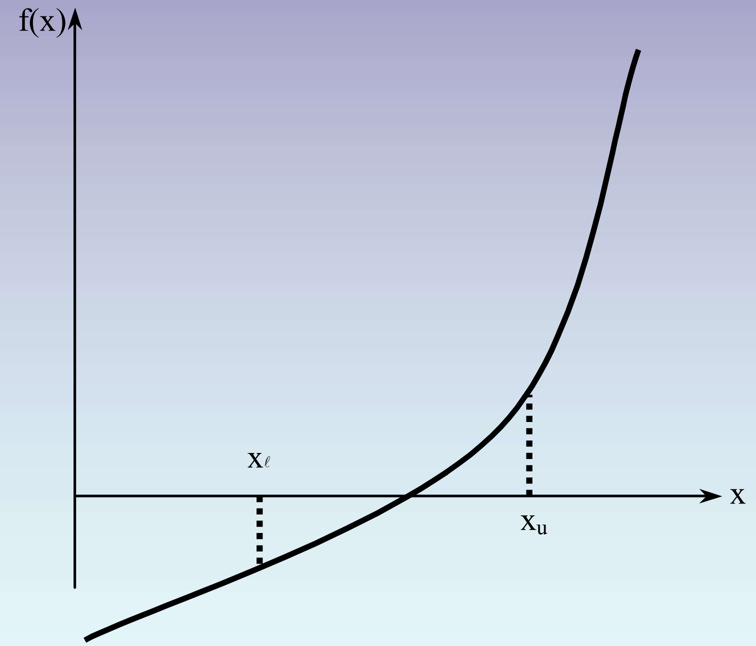


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

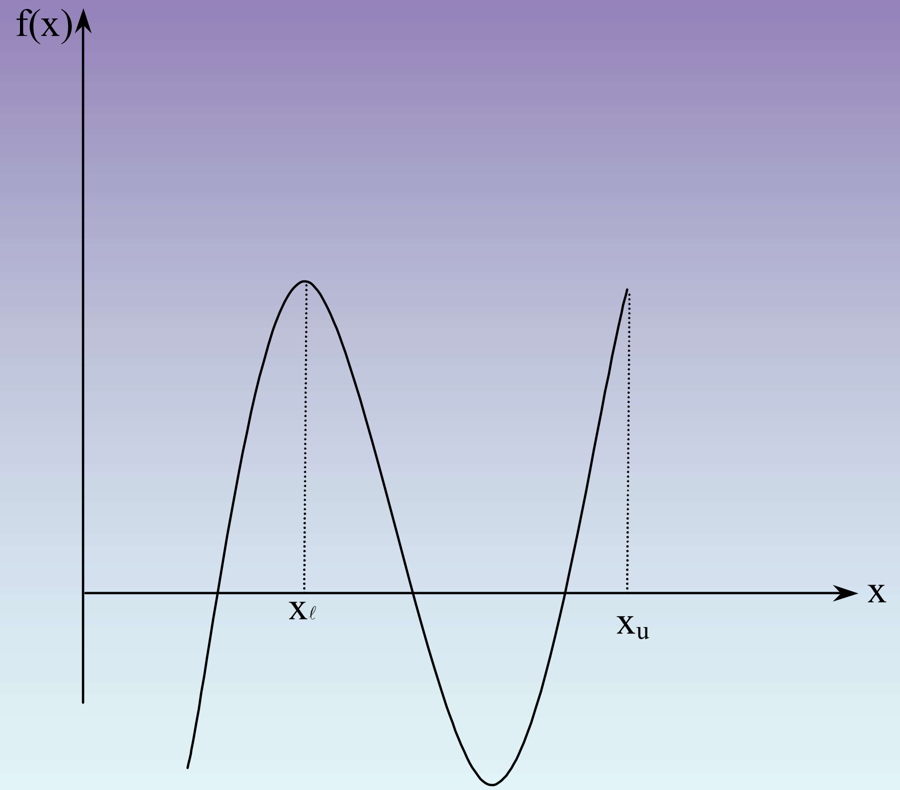


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

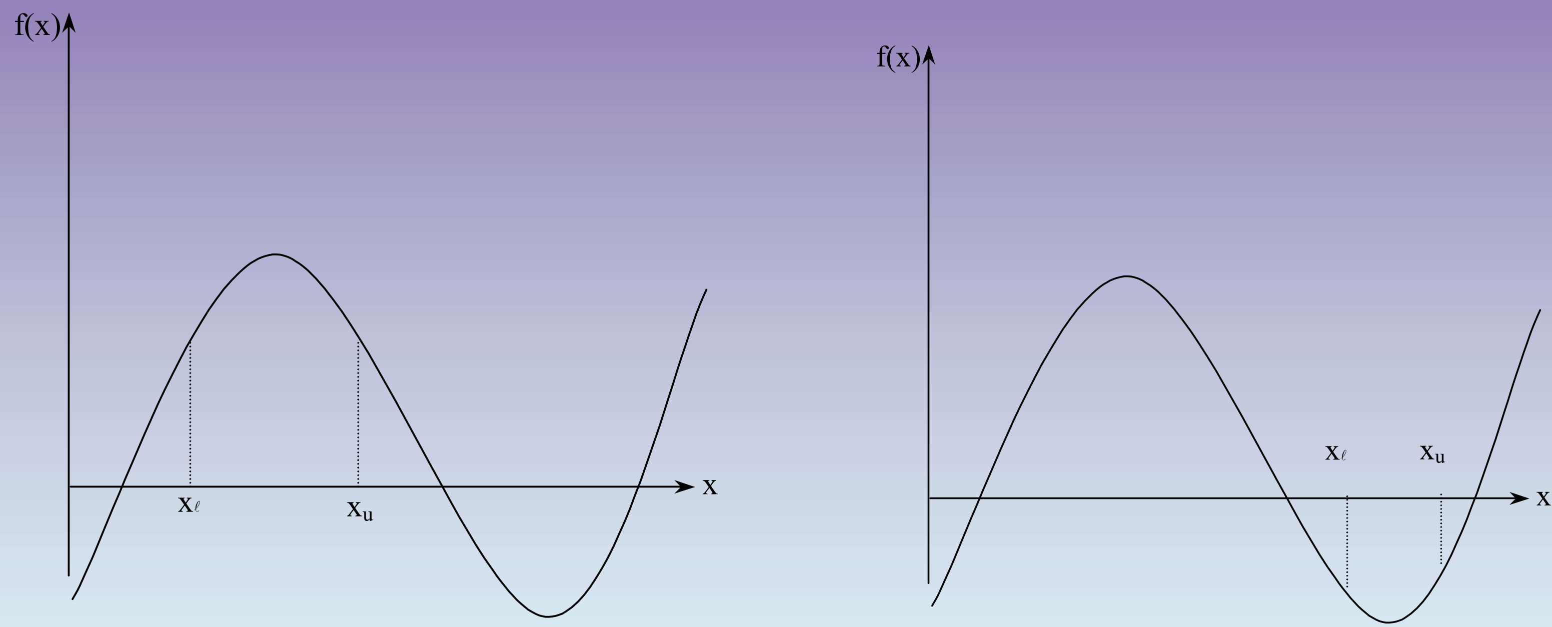


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

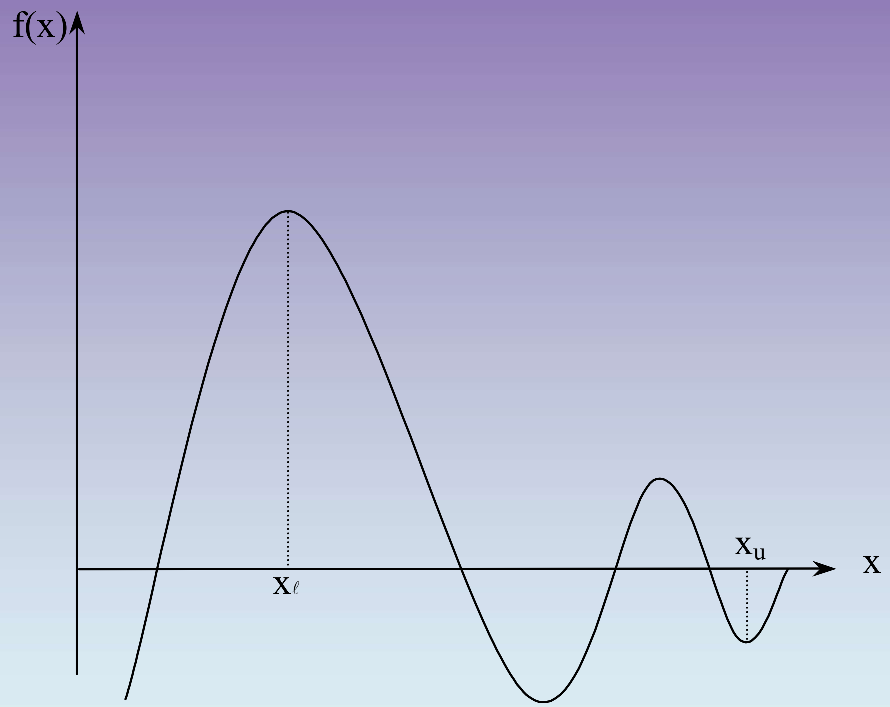


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_l *and* x_u as two guesses for the root such that $f(x_l)\,f(x_u) < 0$. This was demonstrated in Figure 1.

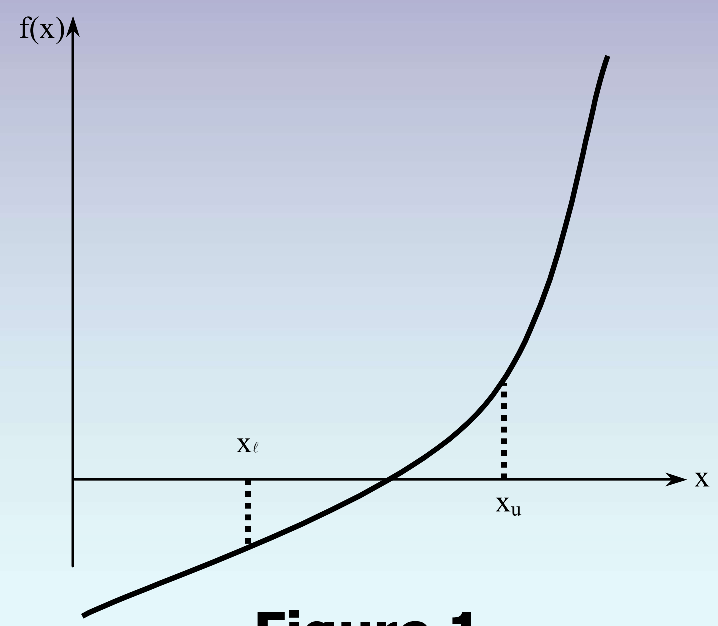


Figure 1

Step 2

Estimate the root, x_m of the equation $f(x) = 0$ as the mid point between x_ℓ *and* x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

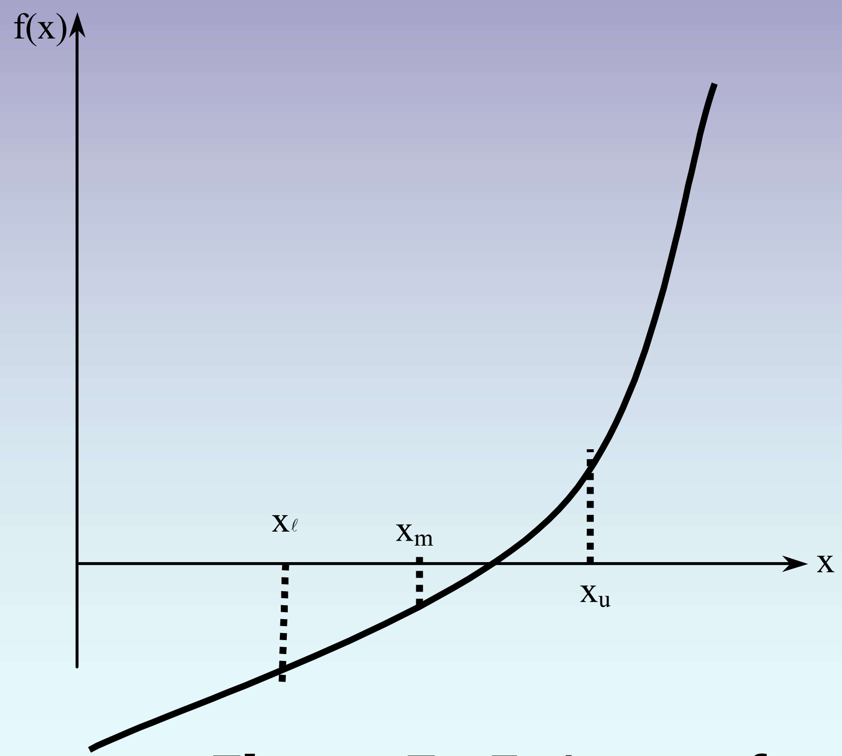


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_l *and* x_m; *then* $x_l = x_l$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m *and* x_u; *then* $x_l = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

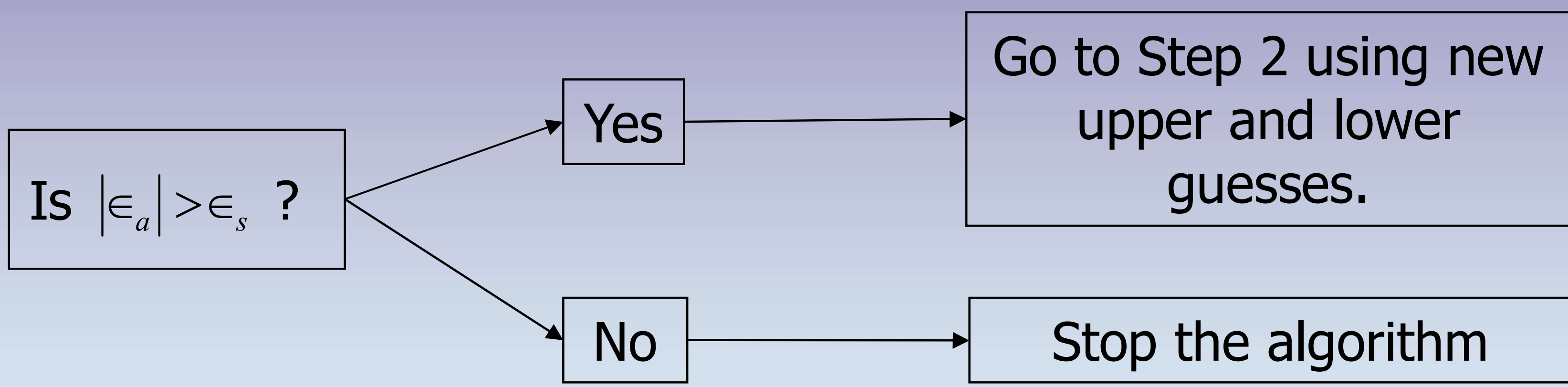
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is <u>more than the maximum number of iterations allowed.</u> If so, one needs to terminate the algorithm and notify the user about it.

Example 1

Use the bisection method of finding roots of equation

$$x^3 - 0.165x^2 + 3.993\times10^{-4} = 0 \qquad 0 \le x \le 0.11$$

a) Conduct three iterations to estimate the root of the above equation.

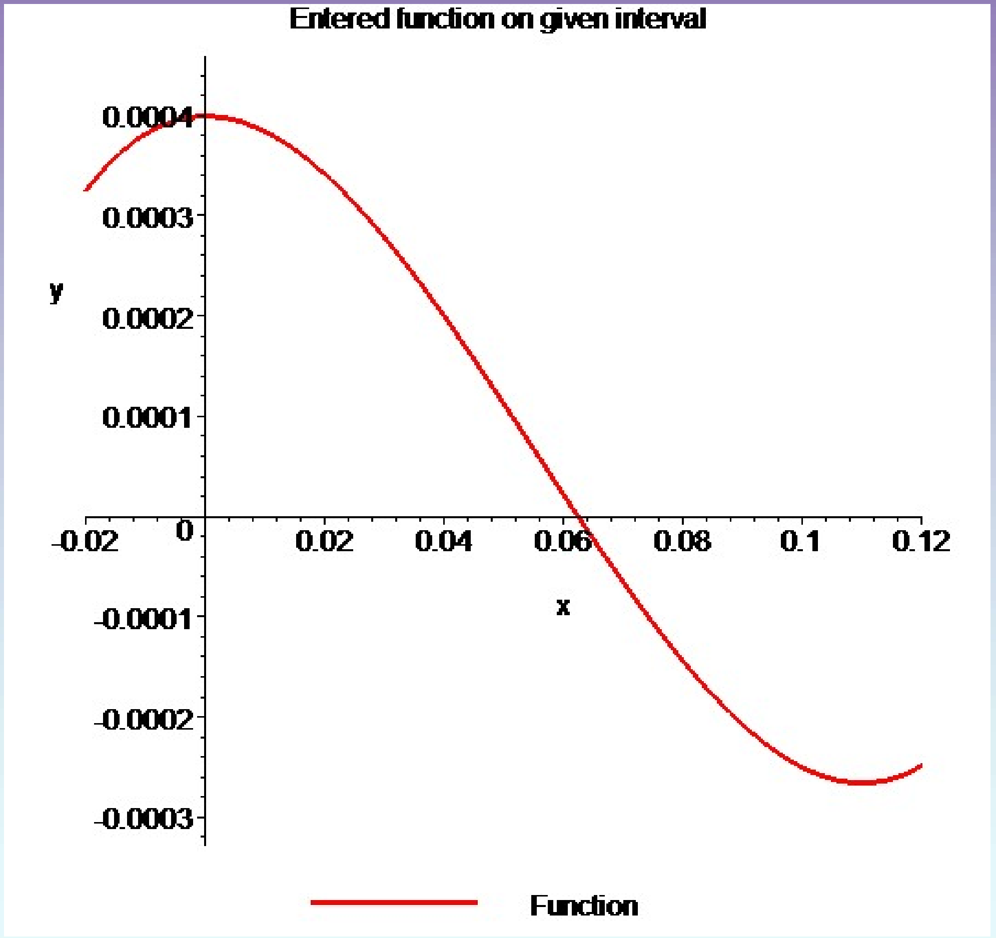
b) Find the absolute relative approximate error at the end of each iteration.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Let us assume

$$x_\ell = 0.00$$

$$x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u .

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = \left(3.993\times10^{-4}\right)\left(-2.662\times10^{-4}\right) < 0$$

So there is at least on root between x_ℓ and x_u, that is between 0 and 0.11

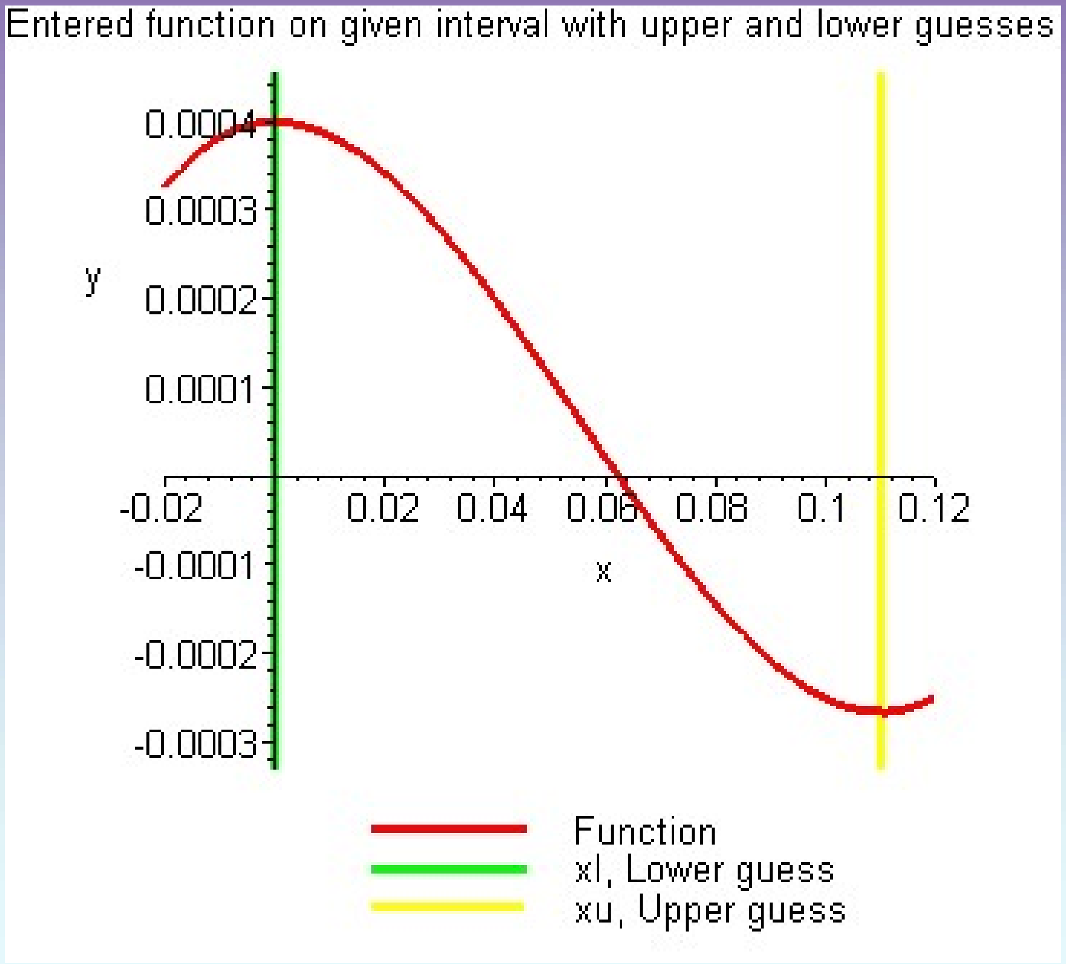


Figure 8 Graph demonstrating sign change between initial limits

Iteration 1

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0+0.11}{2} = 0.055$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = \left(3.993\times10^{-4}\right)\left(6.655\times10^{-5}\right) > 0$$

Hence the root is bracketed between x_m *and* x_u, that is, between 0.055 and 0.11. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $\left|\in_a\right|$ cannot be calculated as we do not have a previous approximation.

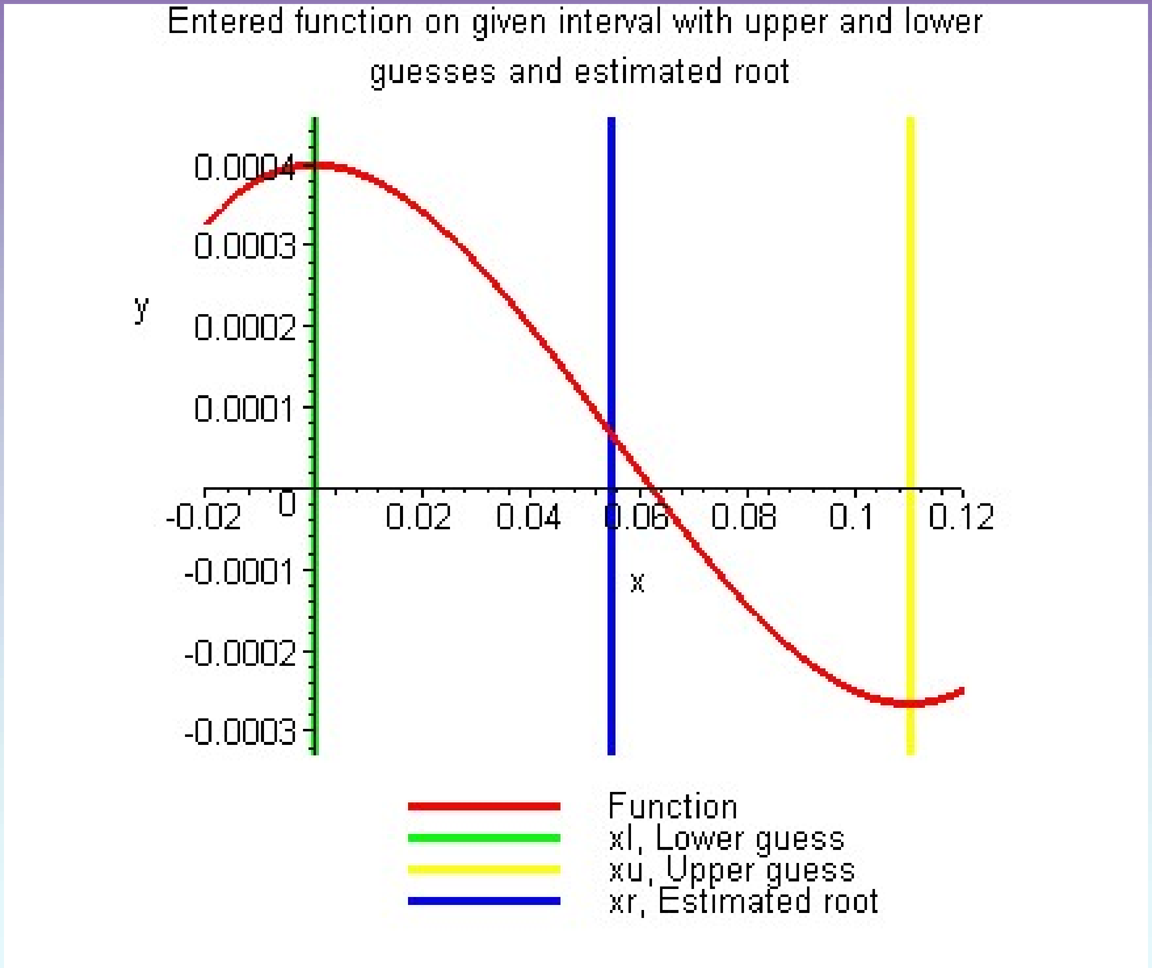


Figure 9 Estimate of the root for Iteration 1

Iteration 2

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times10^{-4} = -1.622\times10^{-4}$$
$$f(x_l)f(x_m) = f(0.055)f(0.0825) = \left(-1.622\times10^{-4}\right)\left(6.655\times10^{-5}\right) < 0$$

Hence the root is bracketed between x_l *and* x_m, that is, between 0.055 and 0.0825. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.0825$$

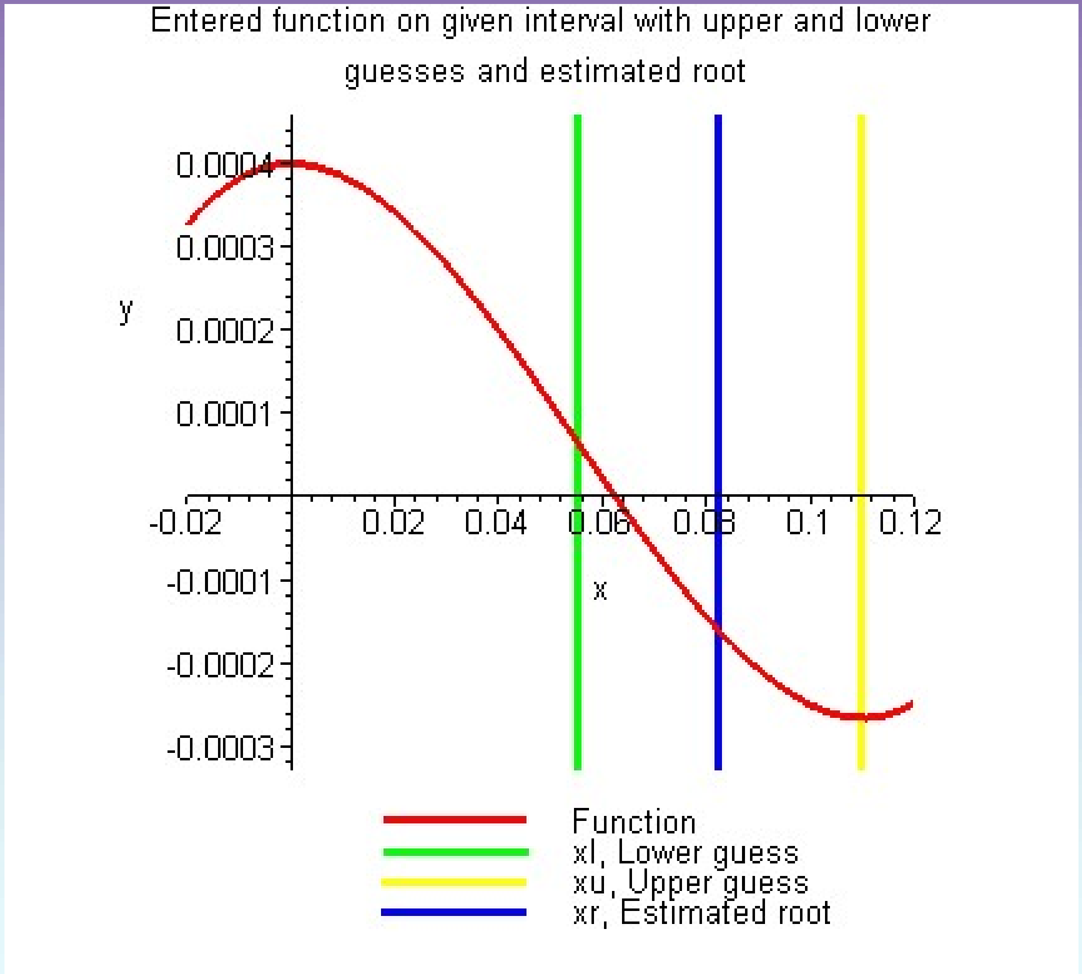


Figure 10 Estimate of the root for Iteration 2

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.333\%$$

None of the significant digits are at least correct in the estimate root of $x_m = 0.0825$ because the absolute relative approximate error is greater than 5%.

Iteration 3

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.0825}{2} = 0.06875$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$

$$f(x_l)f(x_m) = f(0.055)f(0.06875) = (6.655 \times 10^{-5})(-5.563 \times 10^{-5}) < 0$$

Hence the root is bracketed between x_l *and* x_m, that is, between 0.055 and 0.06875. So, the lower and upper limits of the new bracket are $x_l = 0.055,\ x_u = 0.06875$

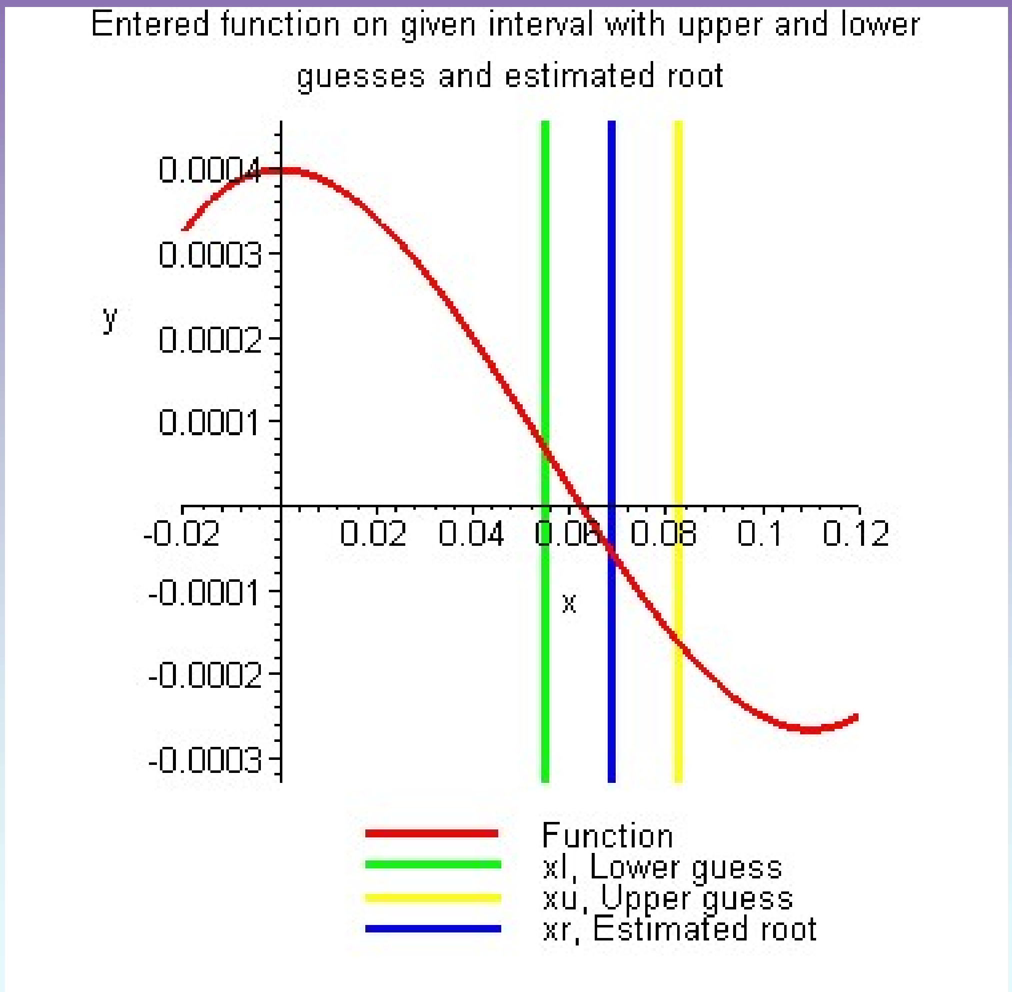


Figure 11 Estimate of the root for Iteration 3

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100$$

$$= 20\%$$

Still, none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

Seven more iterations were conducted, and these iterations are shown in Table 1.

Table 1 Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert \in_a \rvert \%$	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

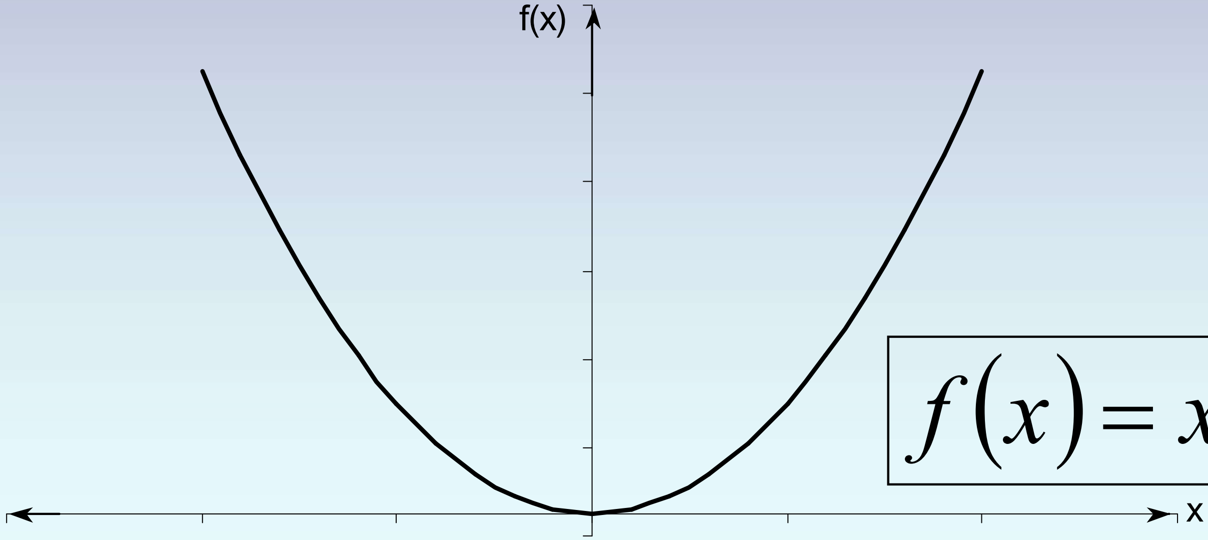
Advantages

- Always convergent
- The root bracket gets halved تخفض للنصف with each iteration - guaranteed.

Drawbacks

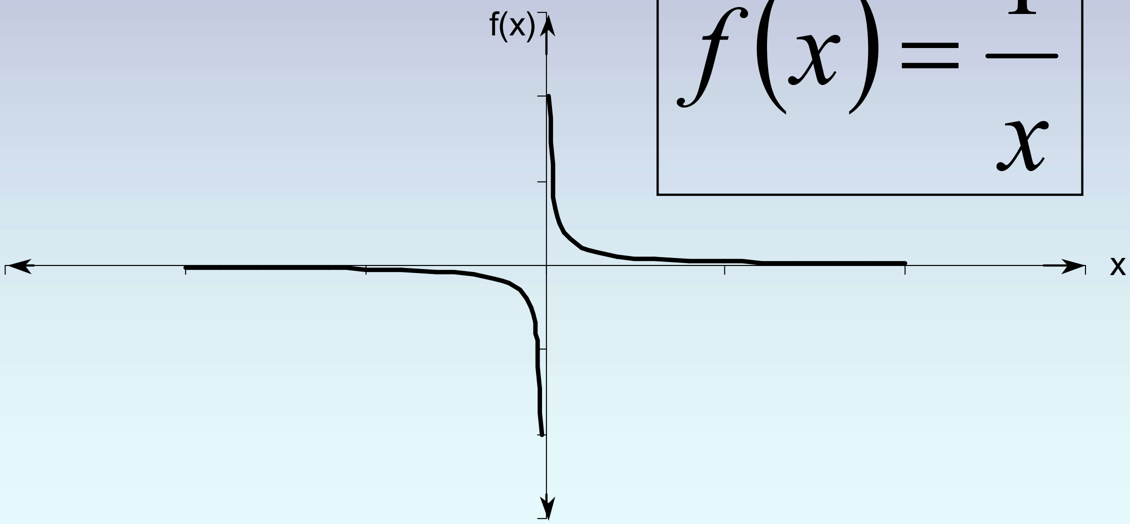
- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



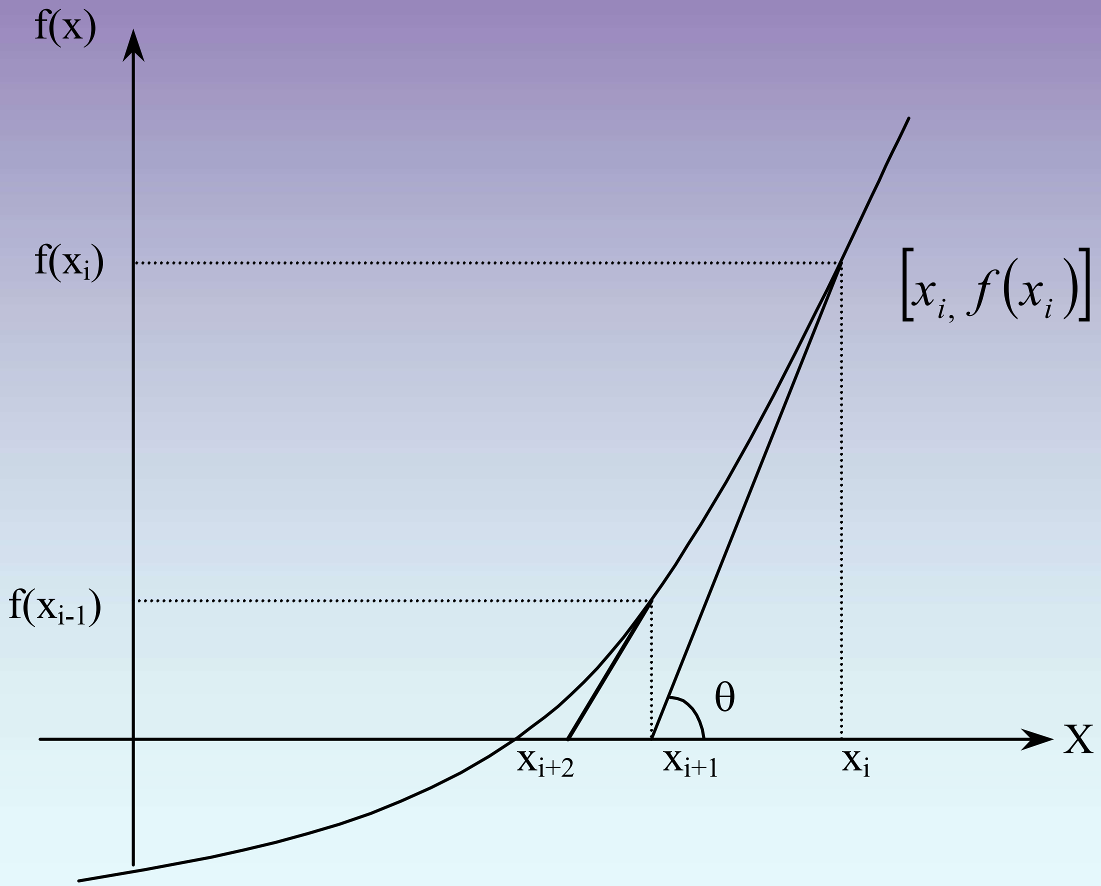
$$f(x) = x^2$$

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$

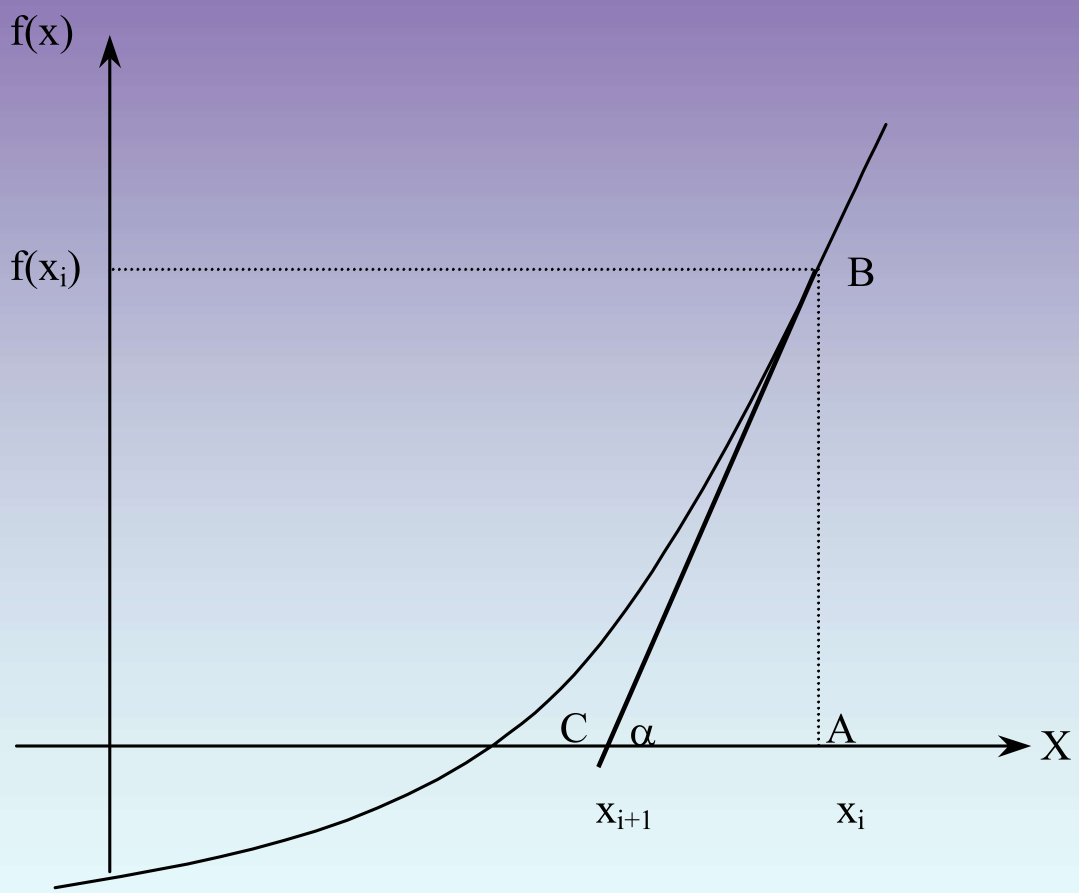
2-Newton-Raphson Method



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

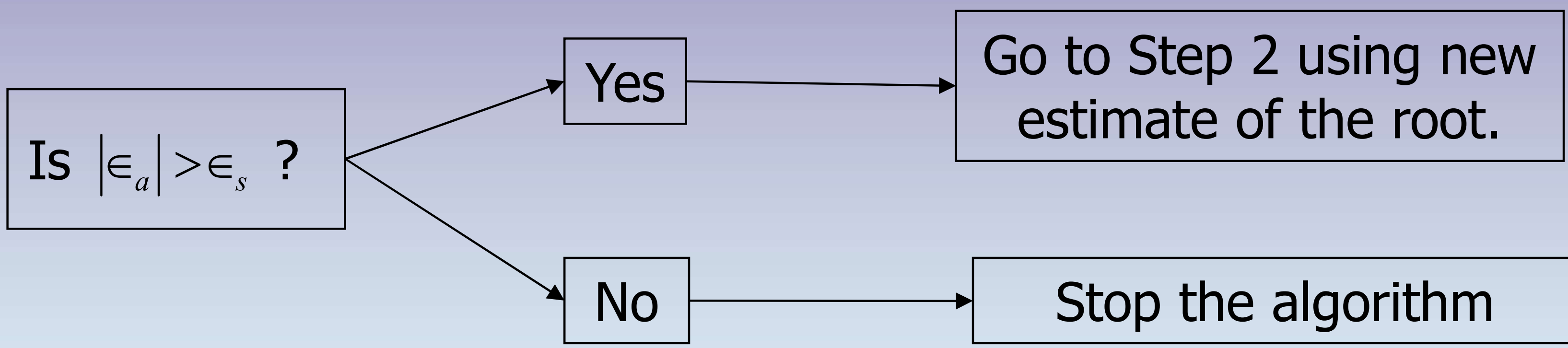
Step 3

Find the absolute relative approximate error $\left|\in_a\right|$ as

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Example 1

Use Newton-Raphson method of finding roots of equation

$$x^3 - 0.165x^2 + 3.993\times10^{-4} = 0$$

$$0 \le x \le 0.11$$

a) Conduct three iterations to estimate the root of the above equation.

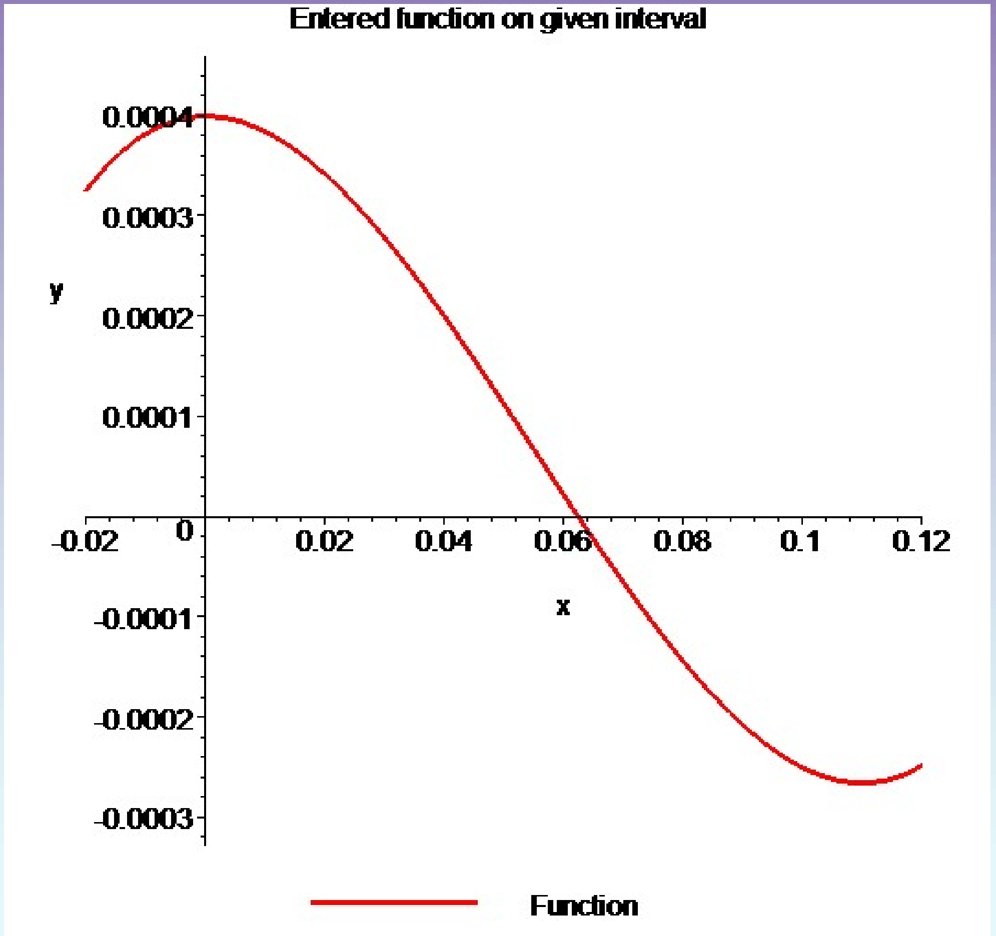
b) Find the absolute relative approximate error at the end of each iteration.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Solve for $f'(x)$

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of $f(x)=0$ is $x_0 = 0.05\text{m}$.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993\times 10^{-4}}{3(0.05)^2 - 0.33(0.05)}$$

$$= 0.05 - \frac{1.118\times 10^{-4}}{-9\times 10^{-3}}$$

$$= 0.05 - (-0.01242)$$

$$= 0.06242$$

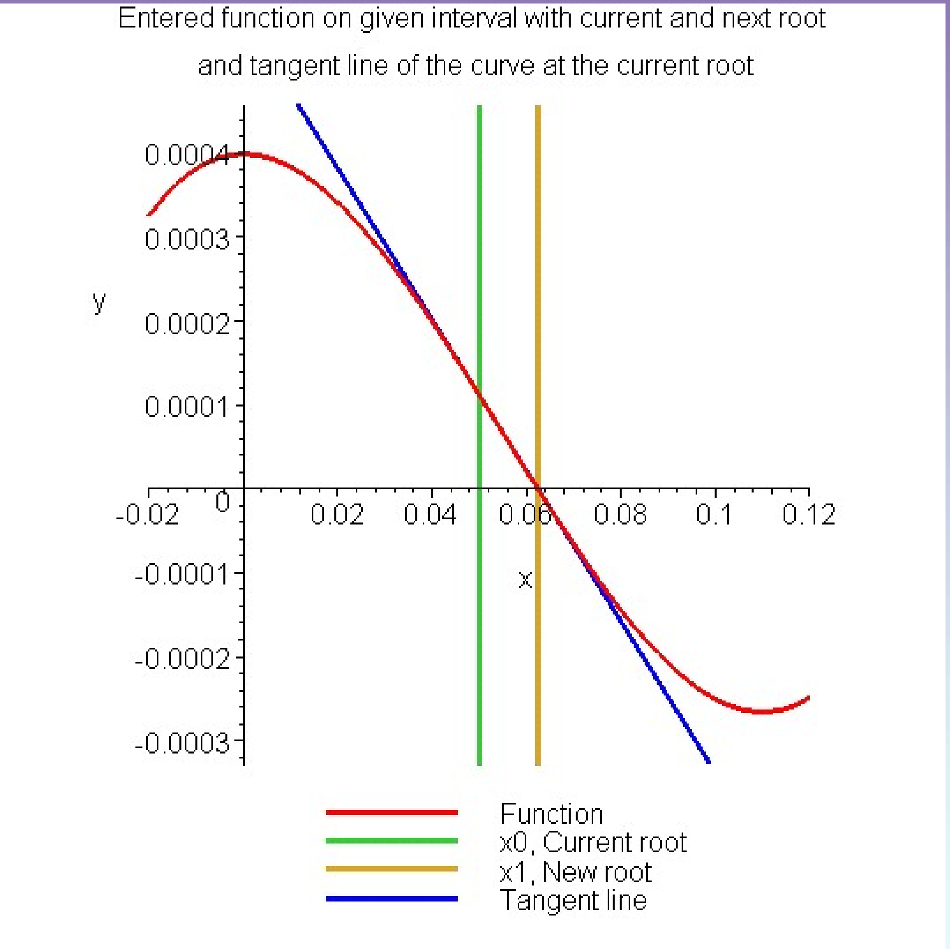


Figure 5 Estimate of the root for the first iteration.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06242 - 0.05}{0.06242}\right| \times 100$$

$$= 19.90\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digits to be correct in your result.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993 \times 10^{-4}}{3(0.06242)^2 - 0.33(0.06242)}$$

$$= 0.06242 - \frac{-3.97781 \times 10^{-7}}{-8.90973 \times 10^{-3}}$$

$$= 0.06242 - \left(4.4646 \times 10^{-5}\right)$$

$$= 0.06238$$

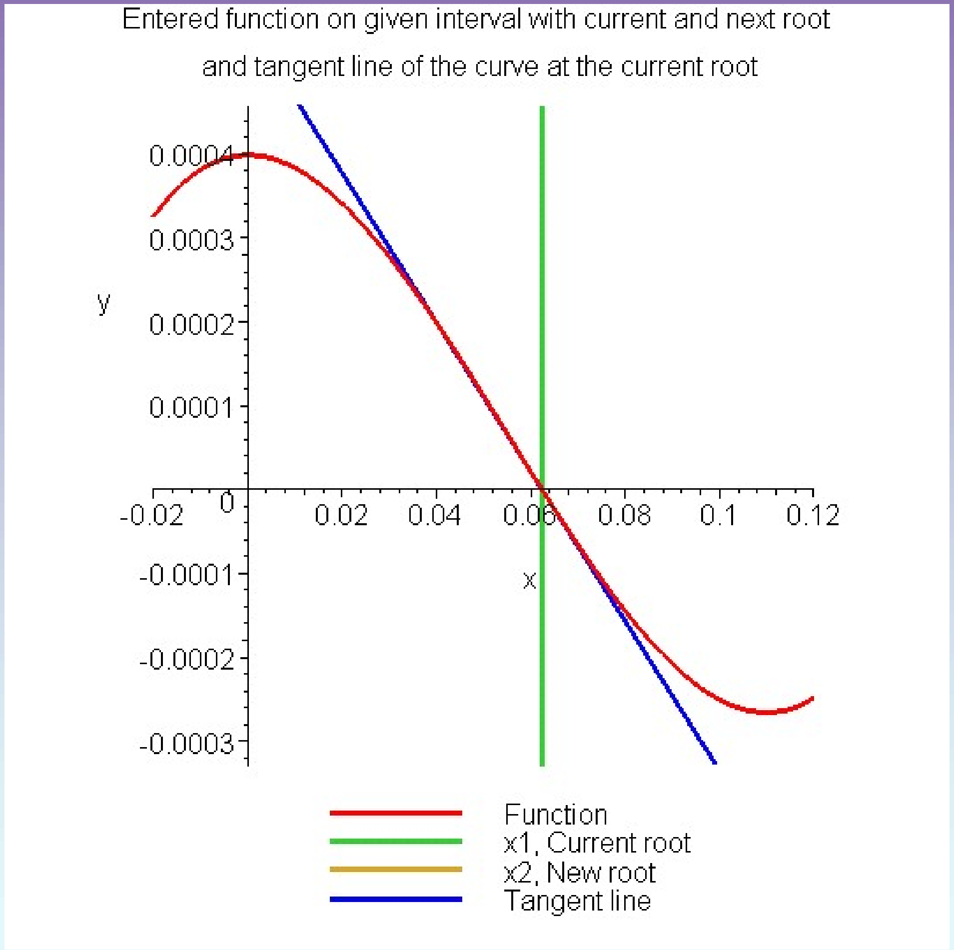


Figure 6 Estimate of the root for the Iteration 2.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06242}{0.06238}\right| \times 100$$

$$= 0.0716\ \%$$

The maximum value of m for which $\left|\in_a\right| \leq 0.5 \times 10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993 \times 10^{-4}}{3(0.06238)^2 - 0.33(0.06238)}$$

$$= 0.06238 - \frac{4.44 \times 10^{-11}}{-8.91171 \times 10^{-3}}$$

$$= 0.06238 - \left(-4.9822 \times 10^{-9}\right)$$

$$= 0.06238$$

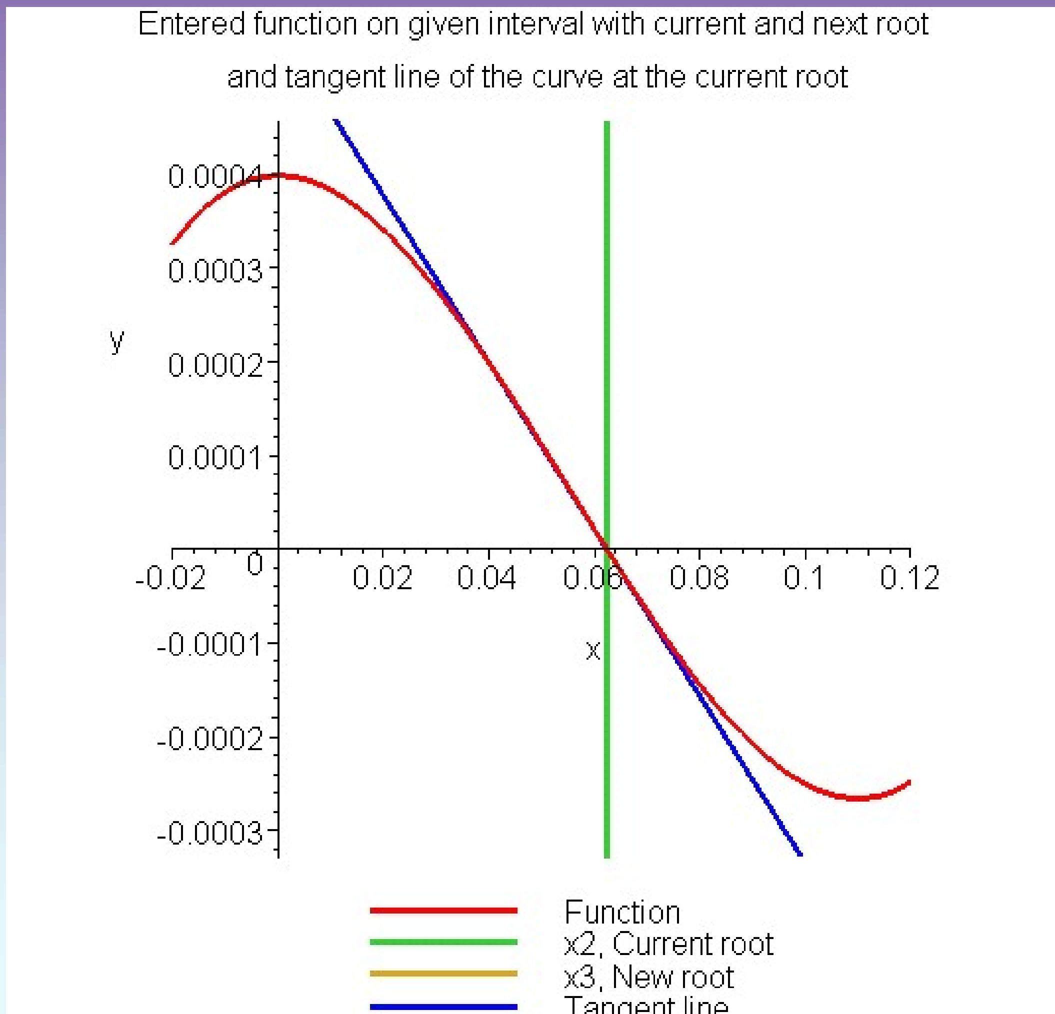


Figure 7 Estimate of the root for the Iteration 3.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06238}{0.06238}\right| \times 100$$

$$= 0\%$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through all the calculations.

Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Drawbacks

1. <u>Divergence at inflection points</u>
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method.

For example, to find the root of the equation $f(x)=(x-1)^3+0.512=0$.

The Newton-Raphson method reduces to $x_{i+1}=x_i-\dfrac{\left(x_i^3-1\right)^3+0.512}{3\left(x_i-1\right)^2}$.

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x=1$.

Eventually after 12 more iterations the root converges to the exact value of $x=0.2$.

Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

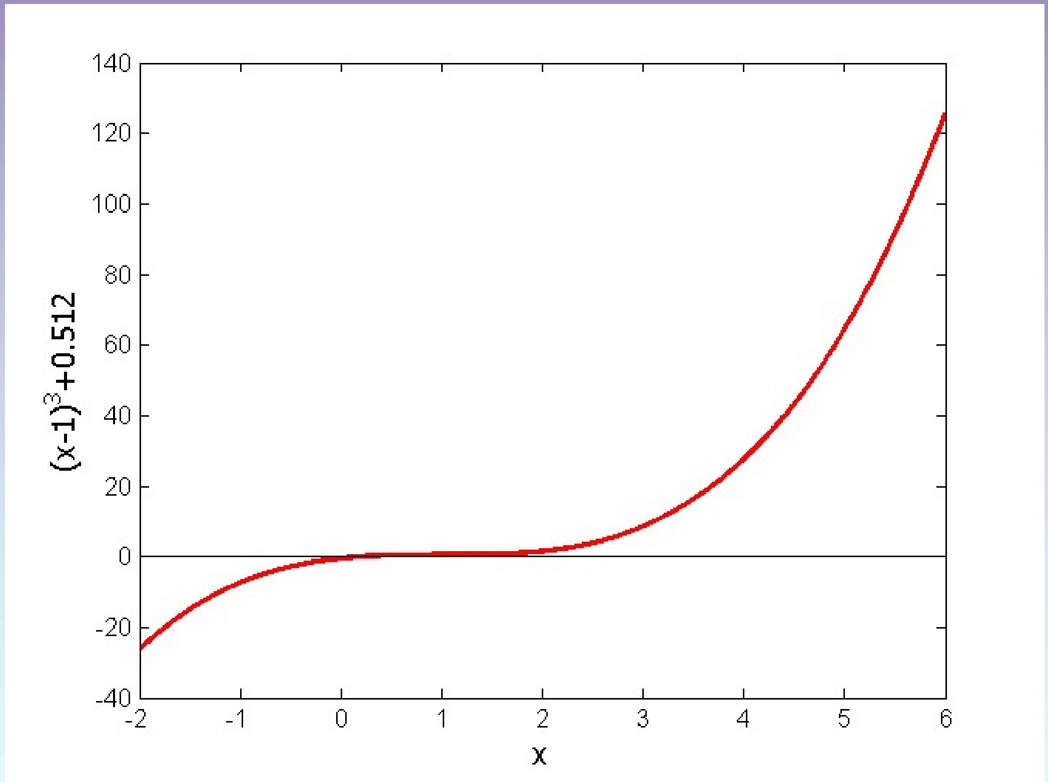


Figure 8 Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

Drawbacks – Division by Zero

2. <u>Division by zero</u>
   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

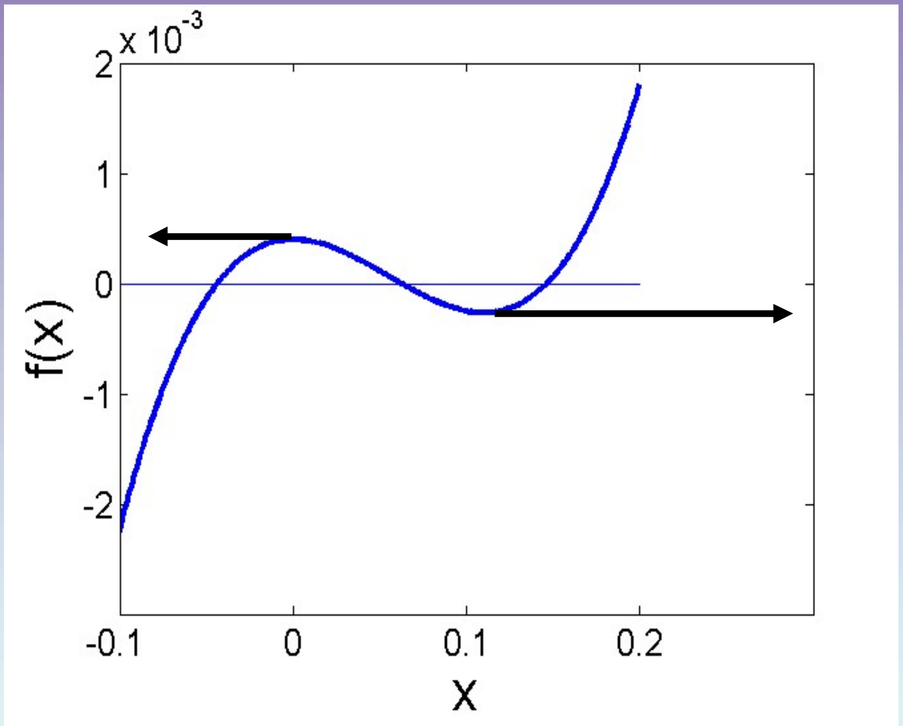


Figure 9 Pitfall of division by zero or near a zero number

Drawbacks – Root Jumping

4. <u>Root Jumping</u>
In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example
$$f(x) = \sin x = 0$$

Choose
$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$

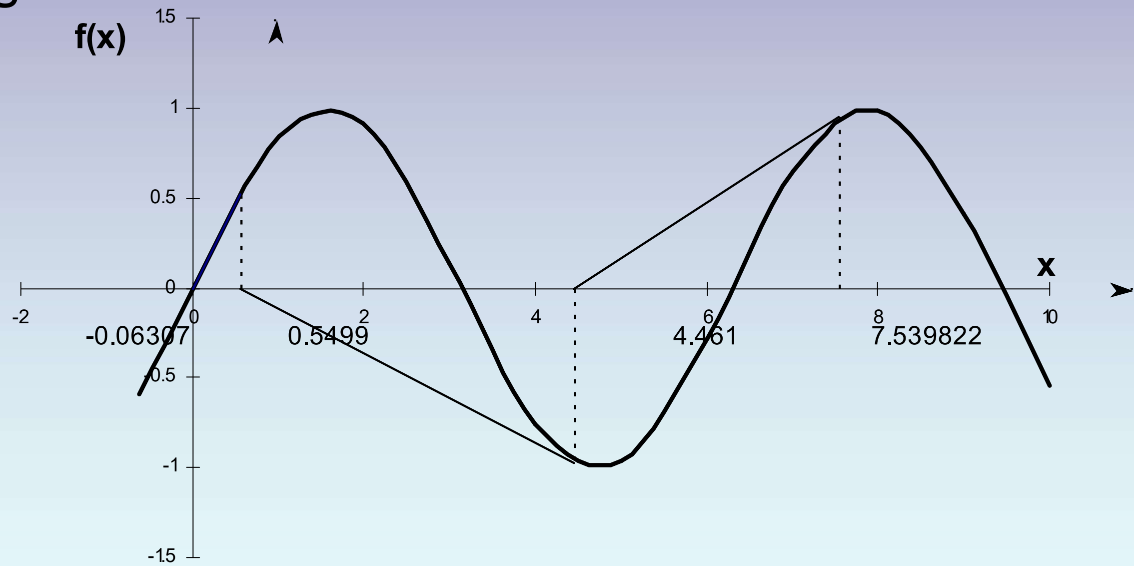


Figure 11 Root jumping from intended location of root for $f(x) = \sin x = 0$

3-Secant Method

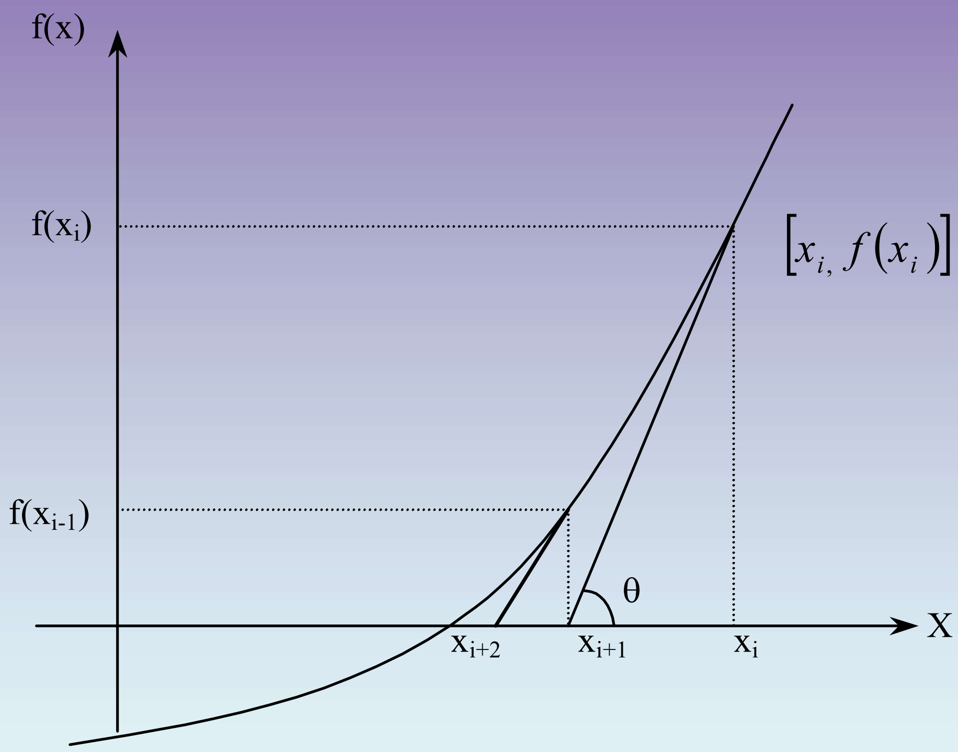


Figure 1 Geometrical illustration of the Newton-Raphson method.

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Algorithm for Secant Method

Step 1

Calculate the next estimate of the root from two initial guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 2

Find if the absolute relative approximate error is greater than the prespecified relative error tolerance.

If so, go back to step 1, else stop the algorithm.

Also check if the number of iterations has exceeded the maximum number of iterations.

Example 1

Use the Secant method of finding roots of equation

$$x^3 - 0.165x^2 + 3.993\times10^{-4} = 0 \qquad 0 \le x \le 0.11$$

a) Conduct three iterations to estimate the root of the above equation.

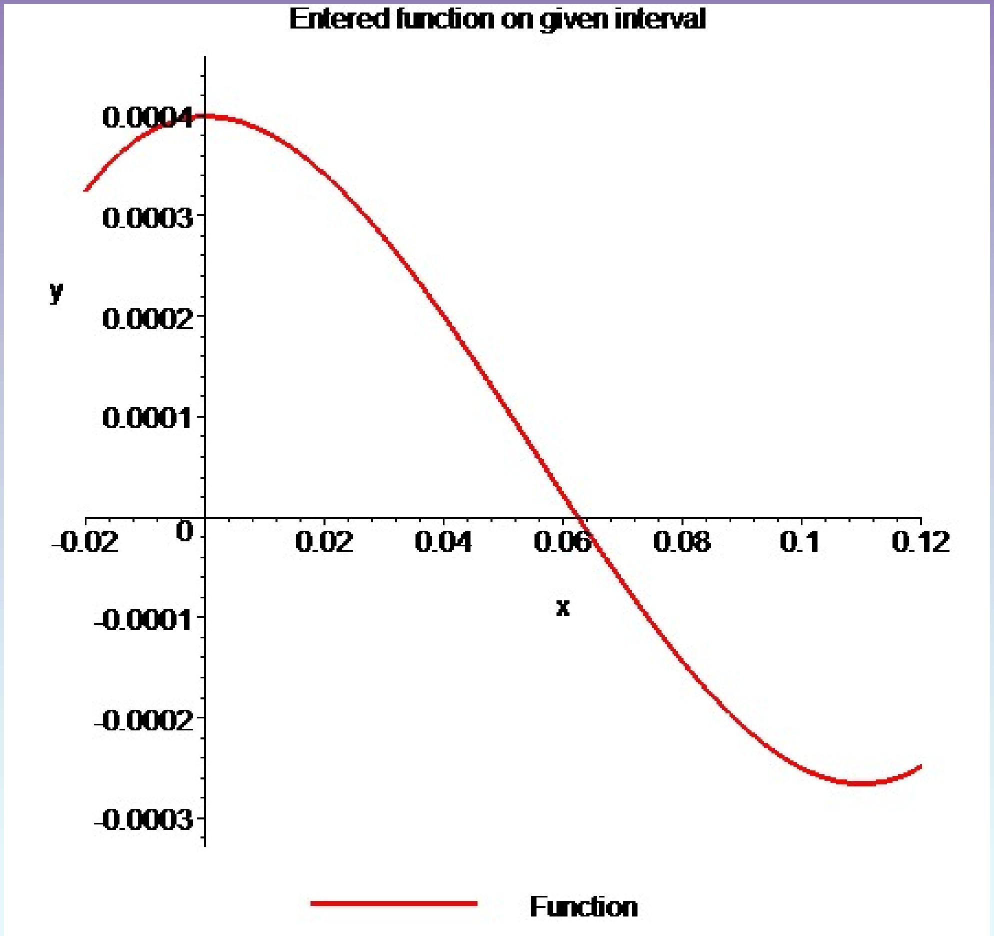
b) Find the absolute relative approximate error at the end of each iteration.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05$.

<u>Iteration 1</u>

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= 0.05 - \frac{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times 10^{-4}\right)(0.05 - 0.02)}{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times 10^{-4}\right) - \left(0.02^3 - 0.165(0.02)^2 + 3.993\times 10^{-4}\right)}$$

$$= 0.06461$$

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 1 is

$$\left|\in_a\right| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06461 - 0.05}{0.06461}\right| \times 100$$

$$= 22.62\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for one significant digits to be correct in your result.

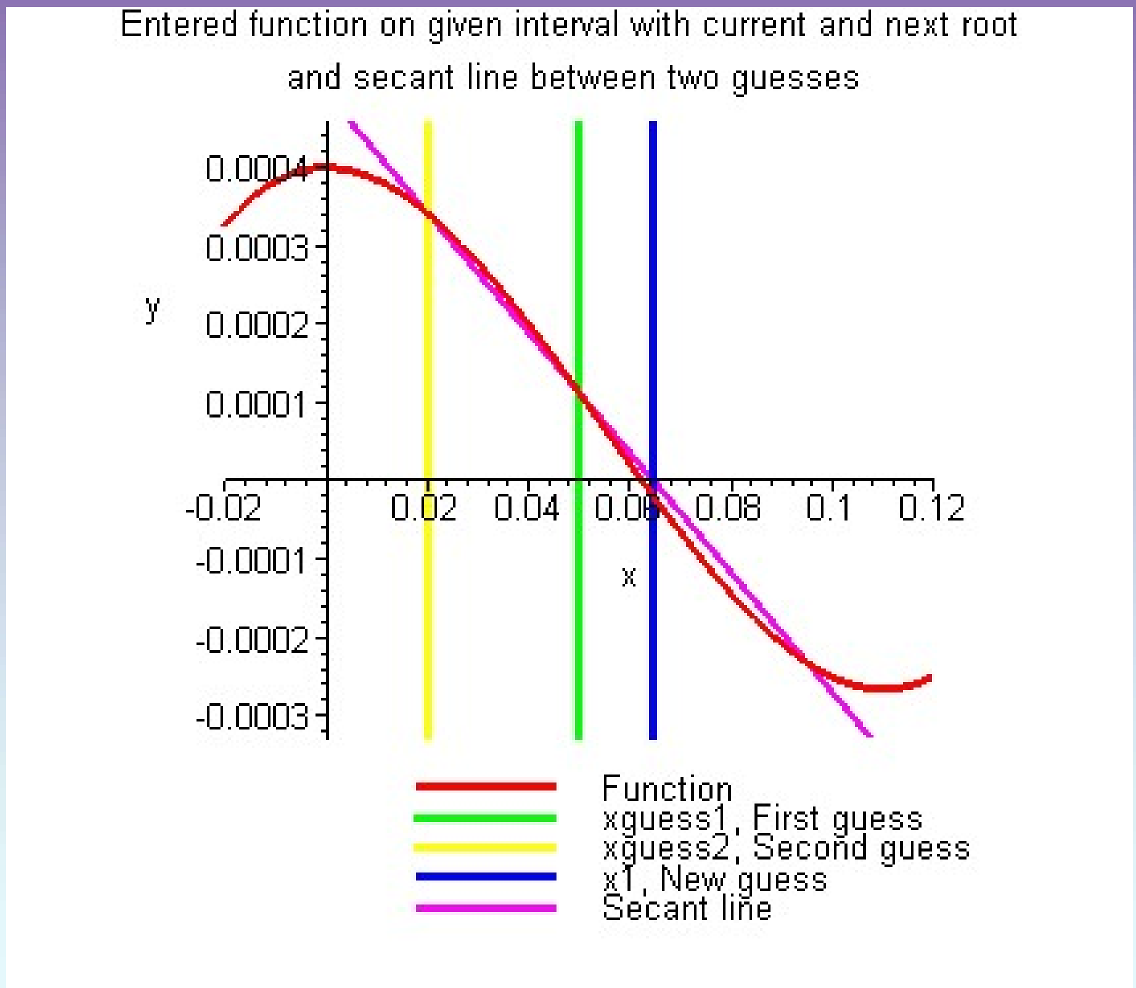


Figure 5 Graph of results of Iteration 1.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 0.06461 - \frac{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)(0.06461 - 0.05)}{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06241$$

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06241 - 0.06461}{0.06241}\right| \times 100$$

$$= 3.525\%$$

The number of significant digits at least correct is 1, as you need an absolute relative approximate error of 5% or less.

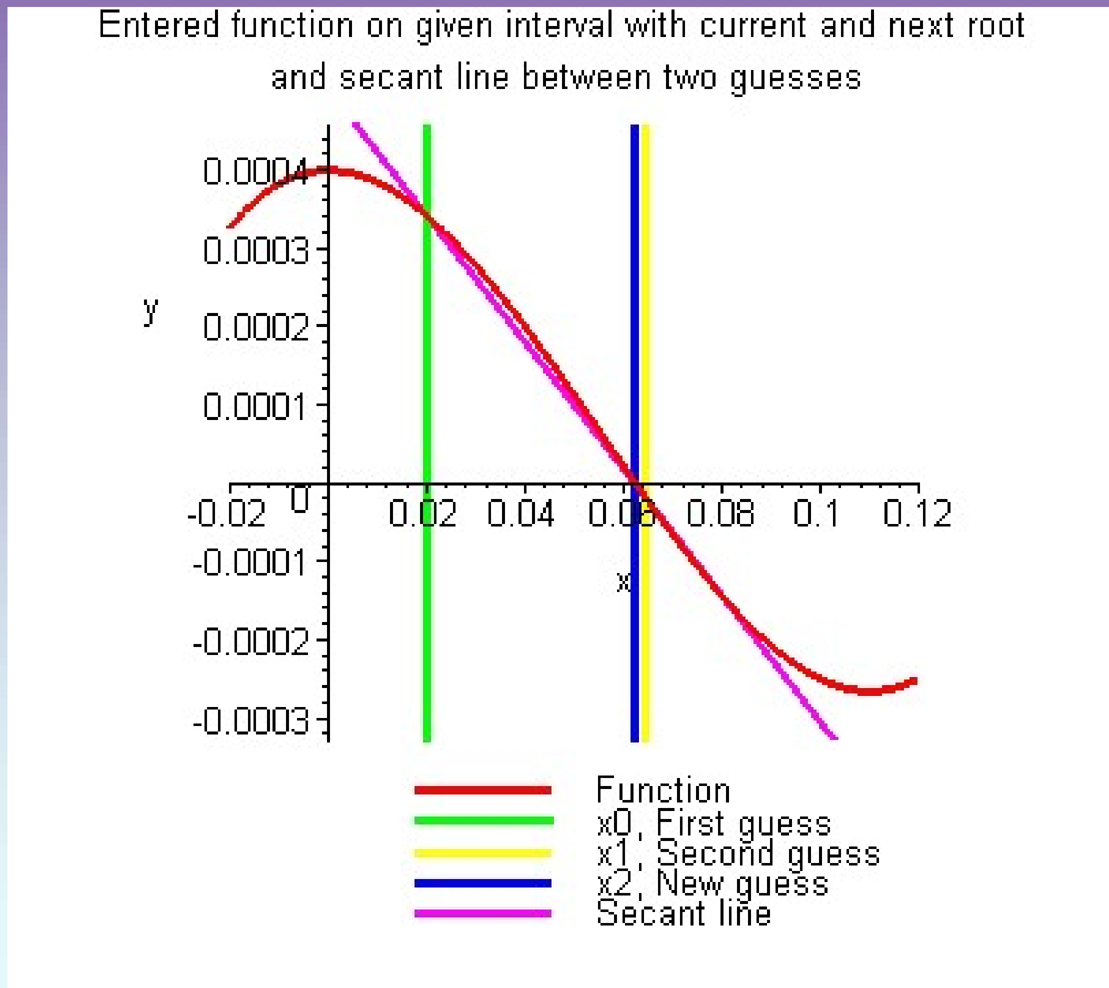


Figure 6 Graph of results of Iteration 2.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 0.06241 - \frac{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right)(0.06241 - 0.06461)}{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06238$$

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06241}{0.06238}\right| \times 100$$

$$= 0.0595\%$$

The number of significant digits at least correct is 5, as you need an absolute relative approximate error of 0.5% or less.

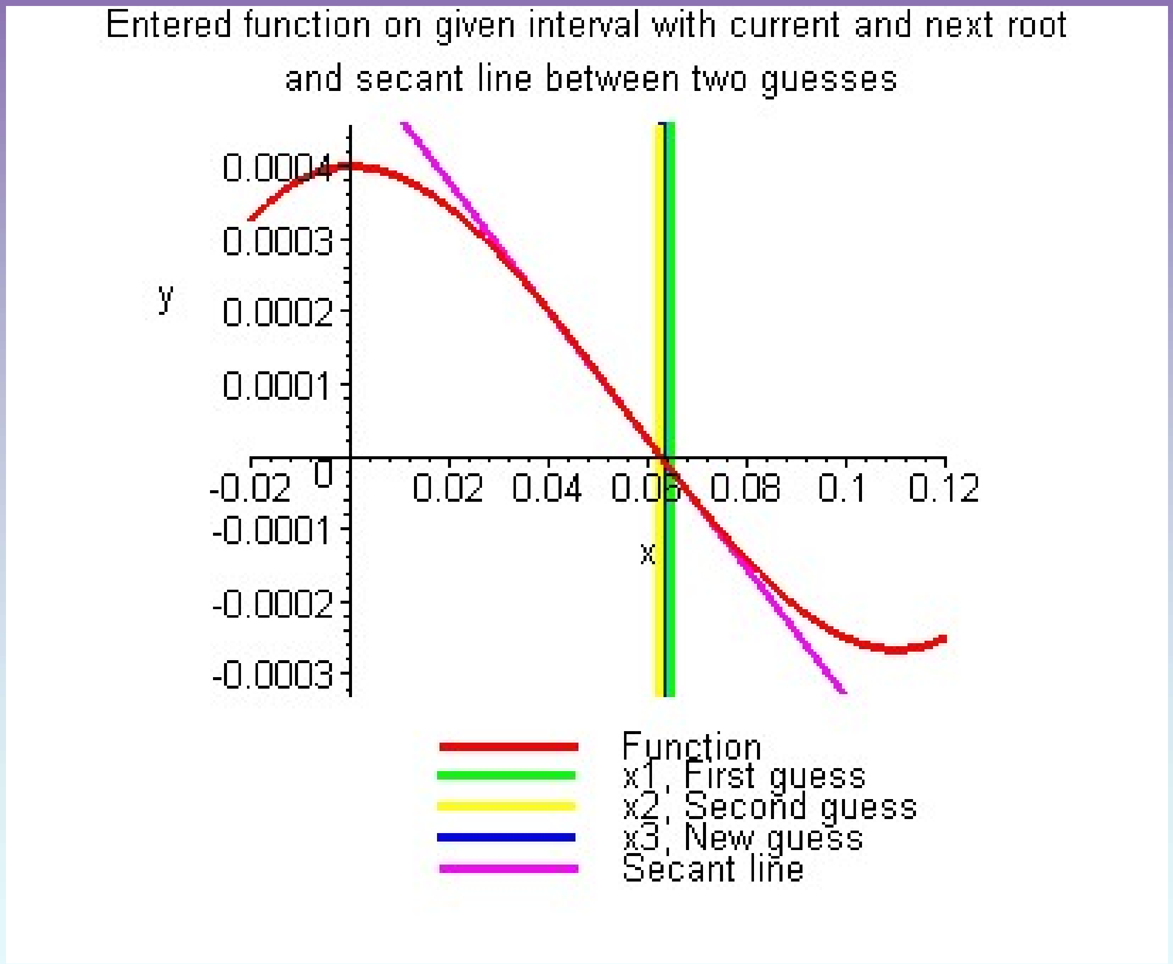
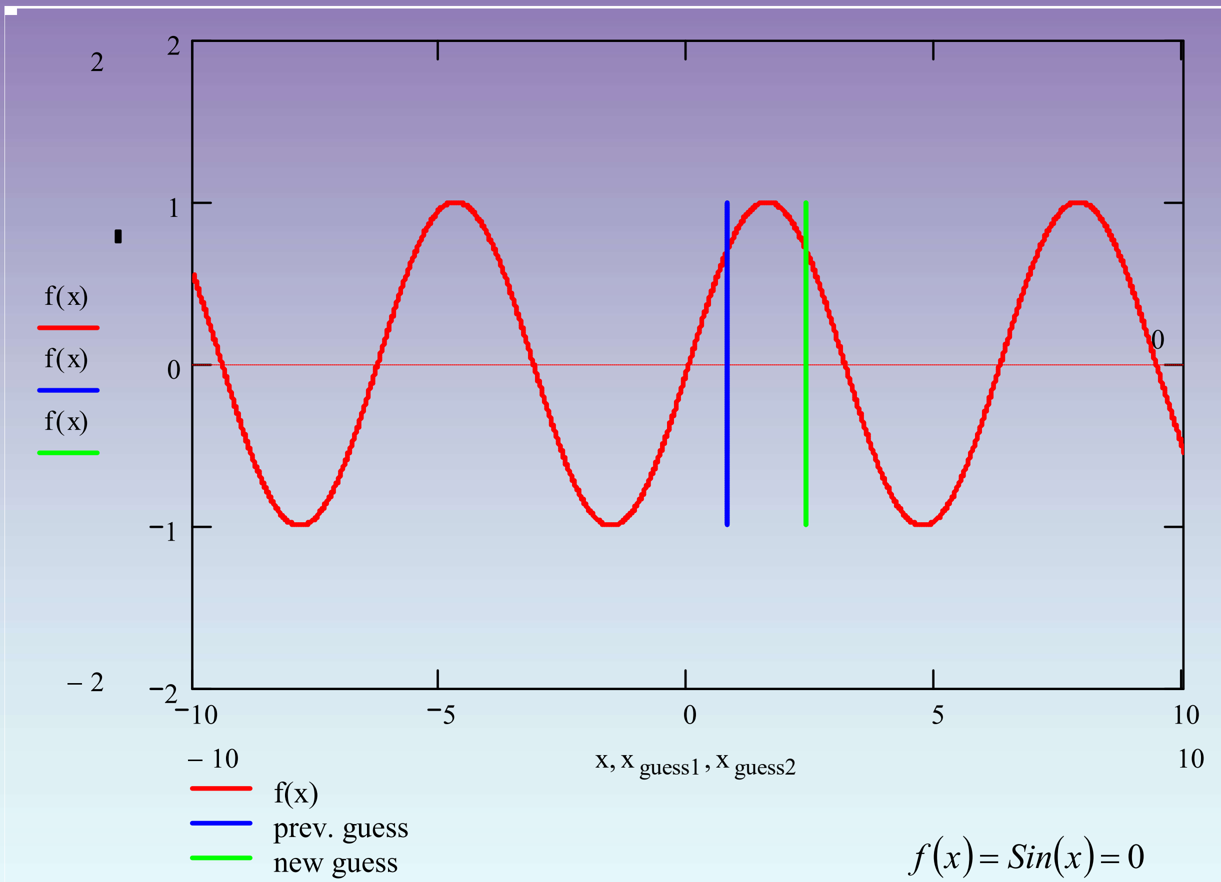


Figure 7 Graph of results of Iteration 3.

Advantages

- Converges fast, if it converges
- Requires two guesses that do not need to bracket the root

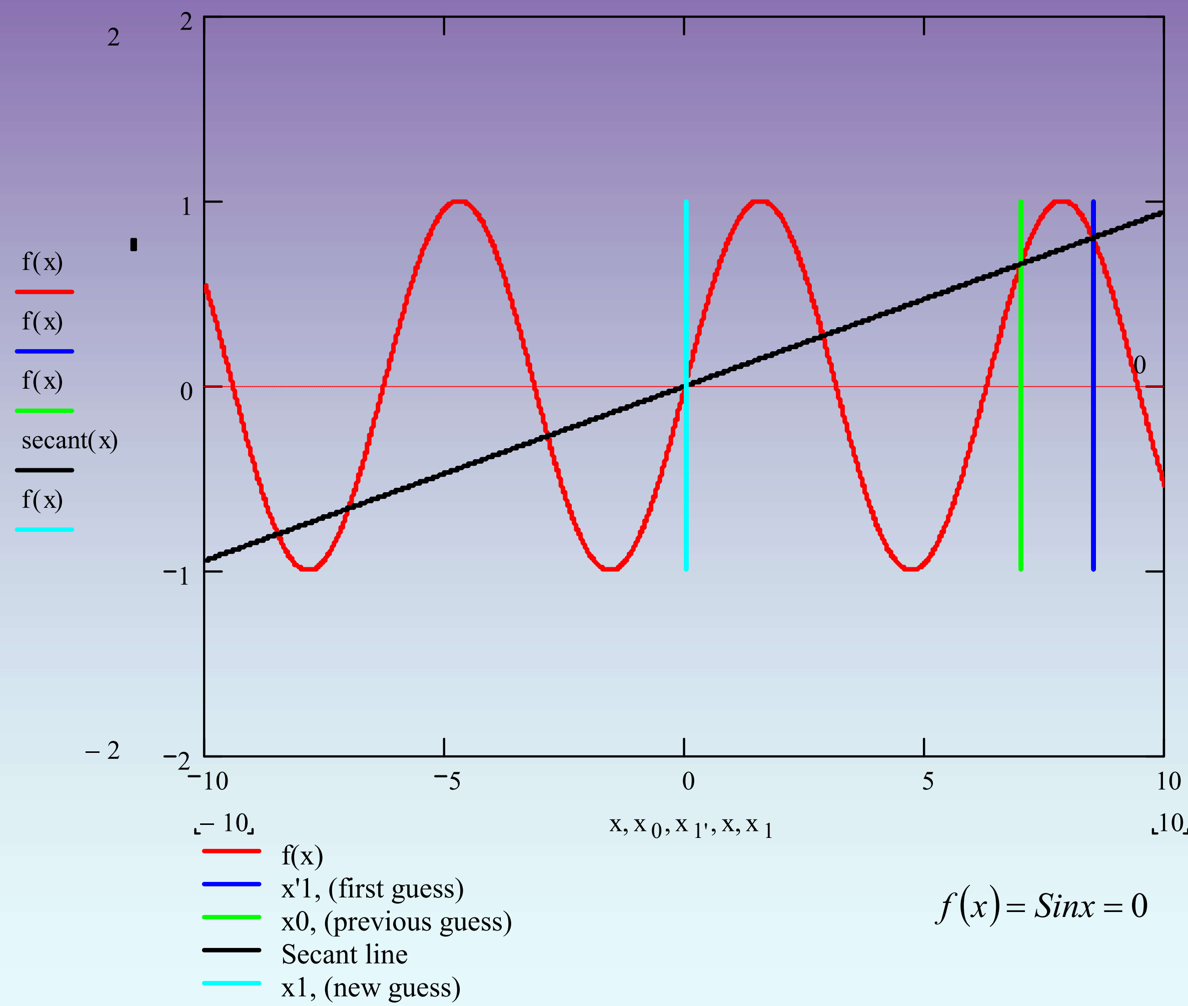
Drawbacks



$$f(x) = Sin(x) = 0$$

Division by zero

Drawbacks



$$f(x) = Sinx = 0$$

Root Jumping