1.2.1 Neighbourhoods

Definition 1.2.8 — Neighbourhood of a Vertex. The *neighbourhood* (or *open neighbourhood*) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v. That is, $N(v) = \{x \in V : vx \in E\}$. The *closed neighbourhood* of a vertex v, denoted by $N[v]$, is simply the set $N(v) \cup \{v\}$.

Then, for any vertex v in a graph G, we have $d_G(v) = |N(v)|$. A special case is a loop that connects a vertex to itself; if such an edge exists, the vertex is said to belong to its own neighbourhood.

Given a set S of vertices, we define the neighbourhood of S, denoted by $N(S)$, to be the union of the neighbourhoods of the vertices in S. Similarly, the closed neighbourhood of S, denoted by $N[S]$, is defined to be $S \cup N(S)$.

Neighbourhoods are widely used to represent graphs in computer algorithms, in terms of the adjacency list and adjacency matrix representations. Neighbourhoods are also used in the clustering coefficient of graphs, which is a measure of the average density of its

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neighbourhoods. In addition, many important classes of graphs may be defined by properties of their neighbourhoods, or by symmetries that relate neighbourhoods to each other.

1.3 Subgraphs and Spanning Subgraphs

Definition 1.3.1 — Subgraph of a Graph. A graph $H(V_1,E_1)$ is said to be a *subgraph* of a graph $G(V,E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Definition 1.3.2 — Spanning Subgraph of a Graph. A graph $H(V_1,E_1)$ is said to be a *spanning subgraph* of a graph $G(V,E)$ if $V_1 = V$ and $E_1 \subseteq E$.

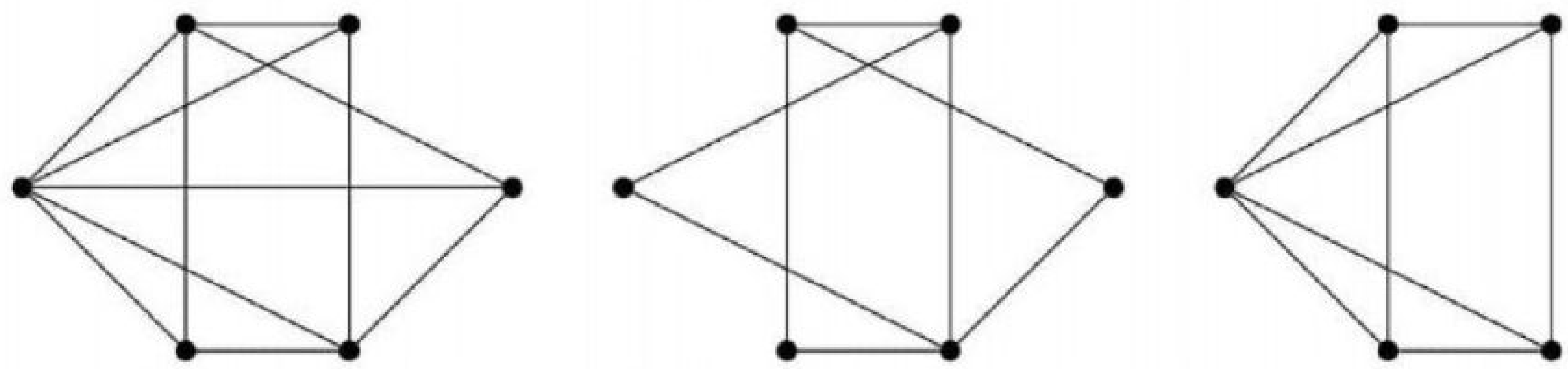

Figure 1.5: Examples of Subgraphs

In the above figure, the second graph is a spanning subgraph of the first graph, while the third graph is a subgraph of the first graph.

1.3.1 Induced Subgraphs

Definition 1.3.3 — Induced Subgraph. Suppose that V' be a subset of the vertex set V of a graph G. Then, the subgraph of G whose vertex set is V' and whose edge set is the set of edges of G that have both end vertices in V' is denoted by $G[V]$ or $\langle V \rangle$ called an *induced subgraph* of G.

Definition 1.3.4 — Edge-Induced Subgraph. Suppose that E' be a subset of the edge set V of a graph G. Then, the subgraph of G whose edge set is E' and whose vertex set is the set of end vertices of the edges in E' is denoted by $G[E]$ or $\langle E \rangle$ called an *edge-induced subgraph* of G.

Figure 1.6 depicts an induced subgraph and an edge induced subgraph of a given graph.

1.4 Fundamental Graph Classes

1.4.1 Complete Graphs

Definition 1.4.1 — Complete Graphs. A *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph

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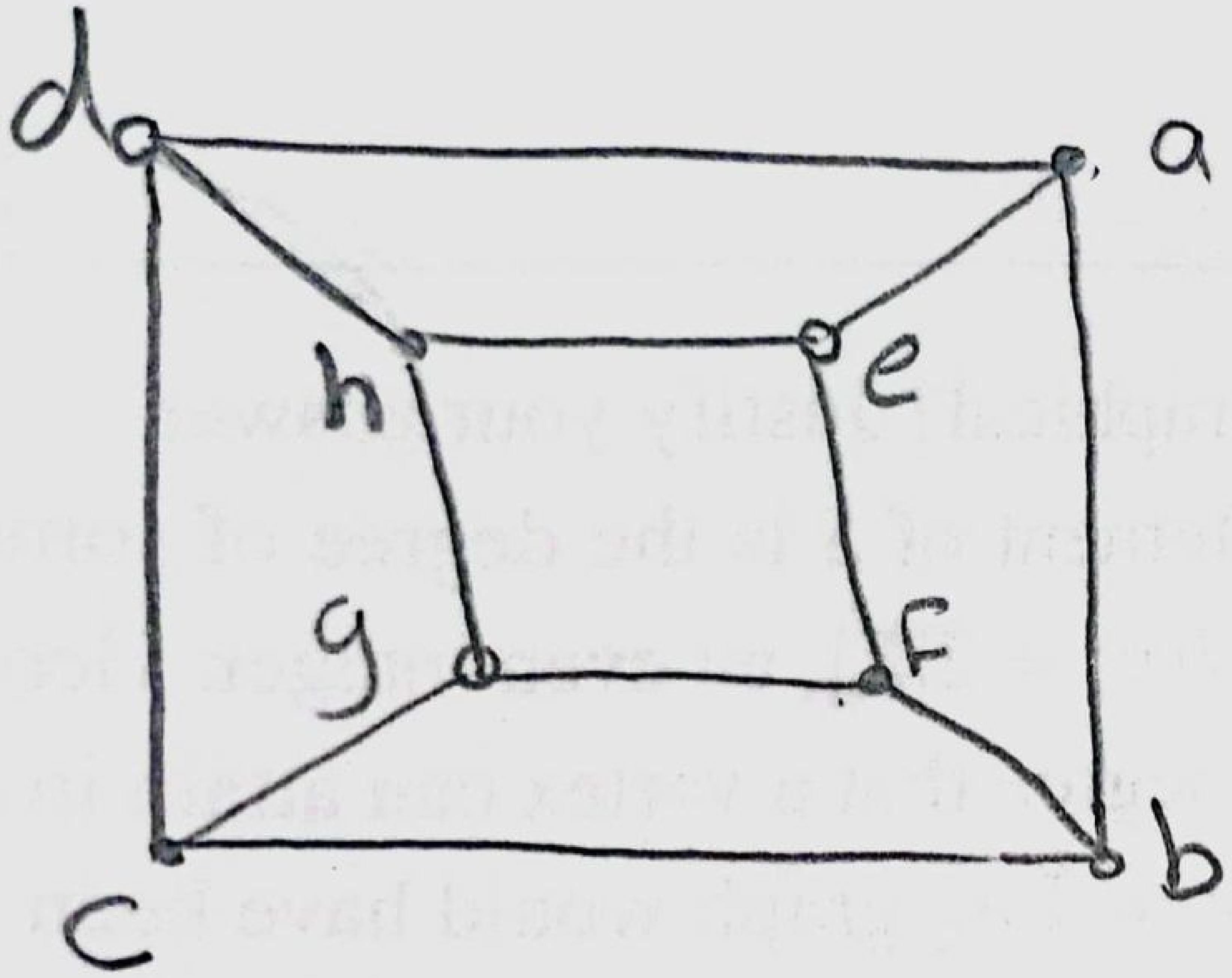
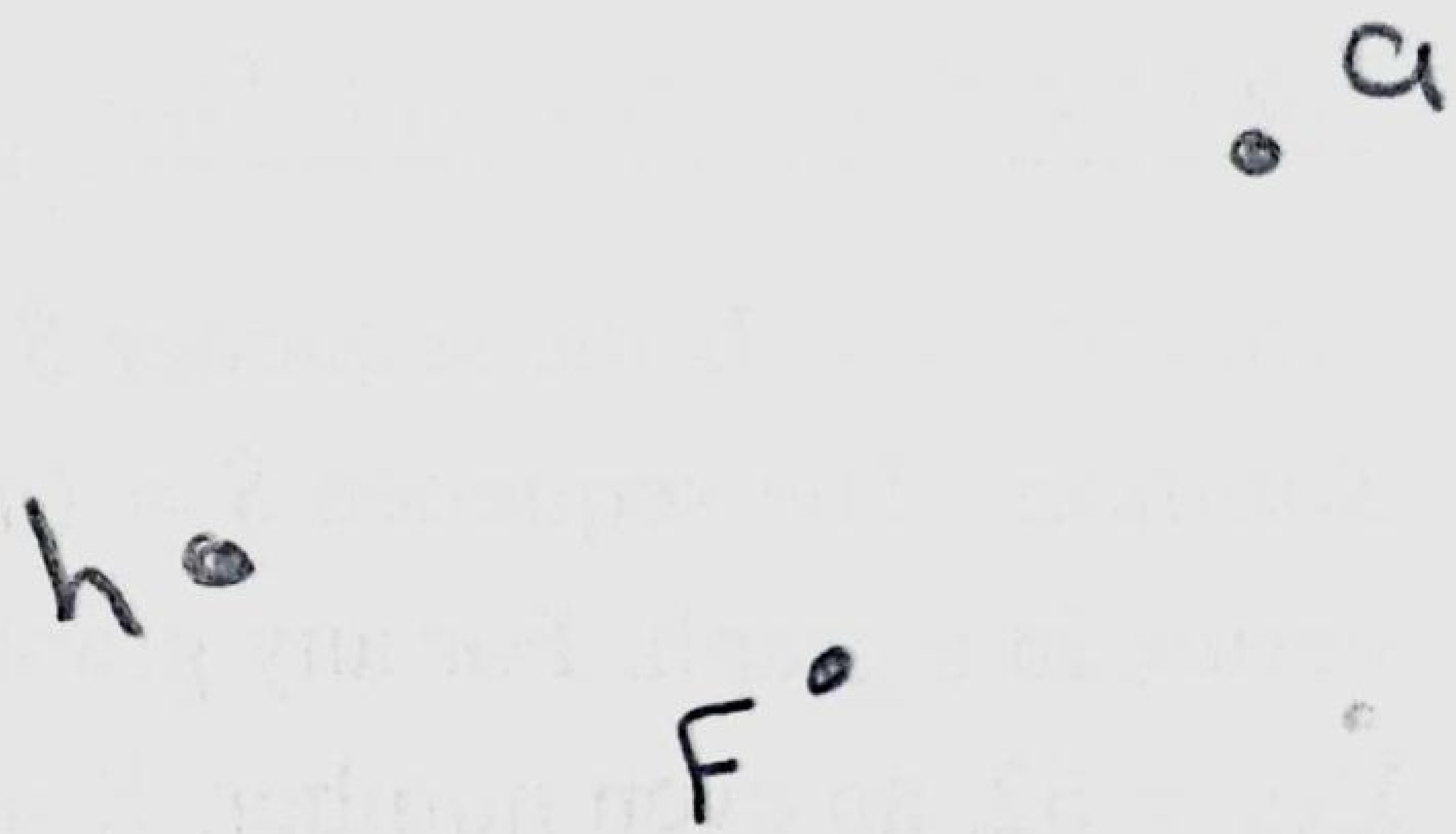
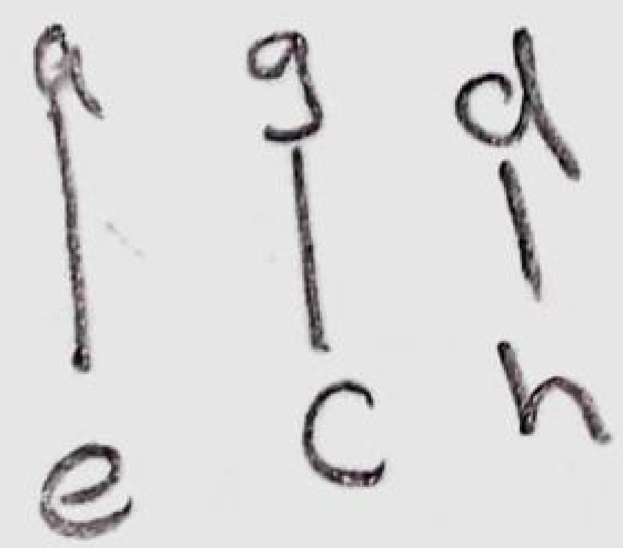


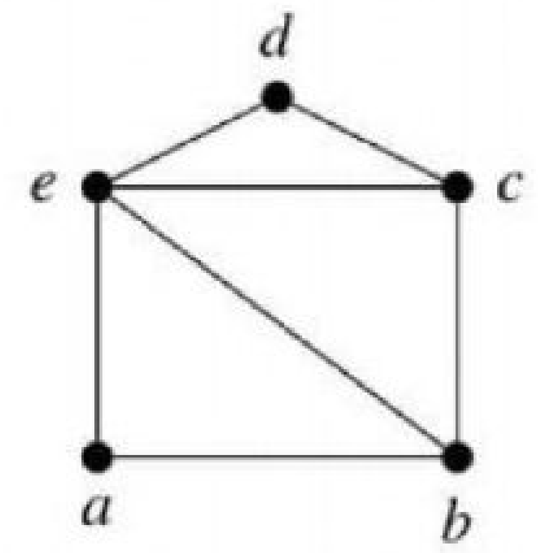
$\boxed{1}$ G [a, h, F]



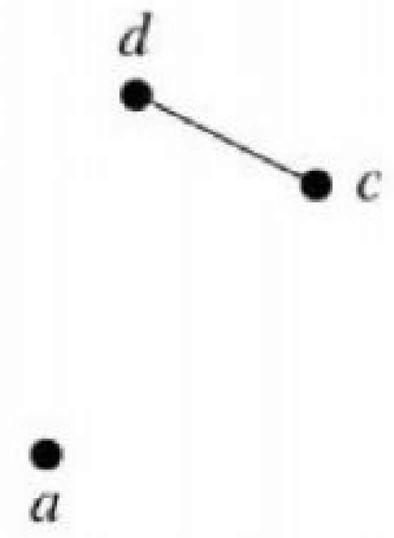
(null graph)

$\boxed{2}$ G [ae, gc, dh]

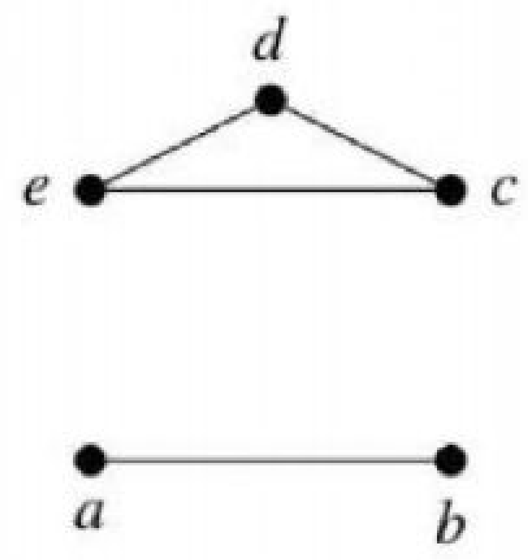




(a) The graph G



(b) The induced subgraph $G[a,c,d]$



(c) The induced subgraph $G[ab,cd,ce,de]$

Figure 1.6: Induced and edge-induced subgraphs of a graph G.

on n vertices is denoted by K_n.

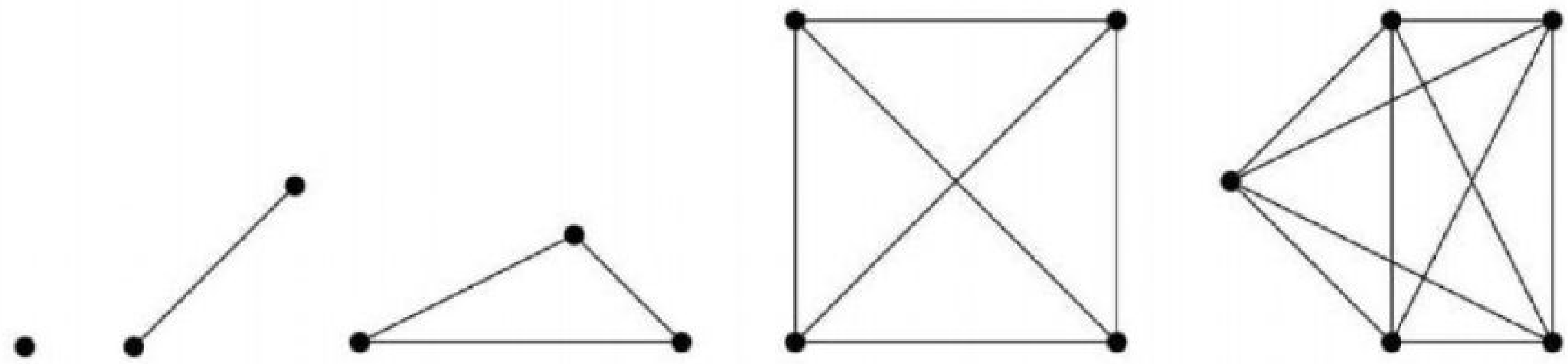


Figure 1.7: First few complete graphs

Problem 1.4 Show that a complete graph K_n has $\frac{n(n-1)}{2}$ edges.

Solution: Note that any two vertices in a complete graph are adjacent to each other. Hence, the number of edges in a complete graph is equal to the number of distinct pairs of vertices in it. Therefore, the number of such pairs of vertices in K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$. That is, the number of edges in K_n is $\frac{n(n-1)}{2}$. ■

We can write an alternate solution to this problem as follows:

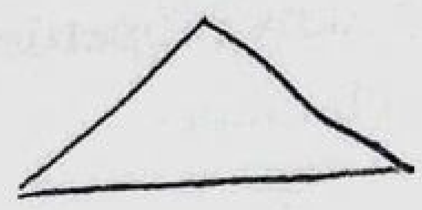
Solution: Note that every vertex in a complete graph K_n is adjacent to all other $n-1$ vertices in K_n. That is, $d(v) = n-1$ for all vertices in K_n. Since K_n has n vertices, we have $\sum \lim_{v \in V(K_n)} d(v) = n(n-1)$. Therefore, by the first theorem on graph theory, we have $2|E(K_n)| = n(n-1)$. That is, the number of edges in K_n is $\frac{n(n-1)}{2}$. ■

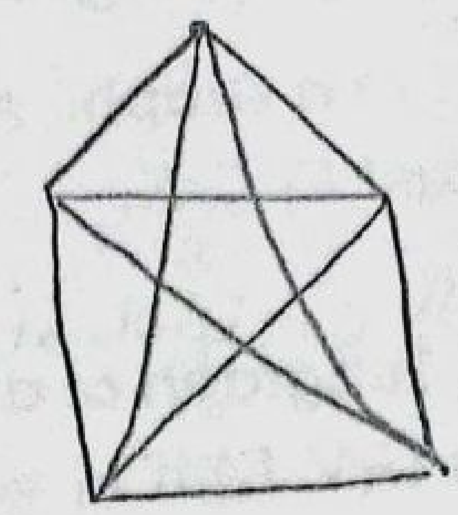
Problem 1.5 Show that the size of every graph of order n is at most $\frac{n(n-1)}{2}$.

Solution: Note that every graph on n vertices is a spanning subgraph of the complete graph K_n. Therefore, $E(G) \subseteq E(K_n)$. That is, $|E(G)| \leq |E(K_n)| = \frac{n(n-1)}{2}$. That is, any graph of order n can have at most $\frac{n(n-1)}{2}$ edges. ■

1.4.2 Bipartite Graphs

Definition 1.4.2 — Bipartite Graphs. A graph G is said to be a *bipartite graph* if its vertex set V can be partitioned into two sets, say V_1 and V_2, such that no two vertices in

K_3  K_2  K_1

$K_5 \longrightarrow$ 

$\downarrow$

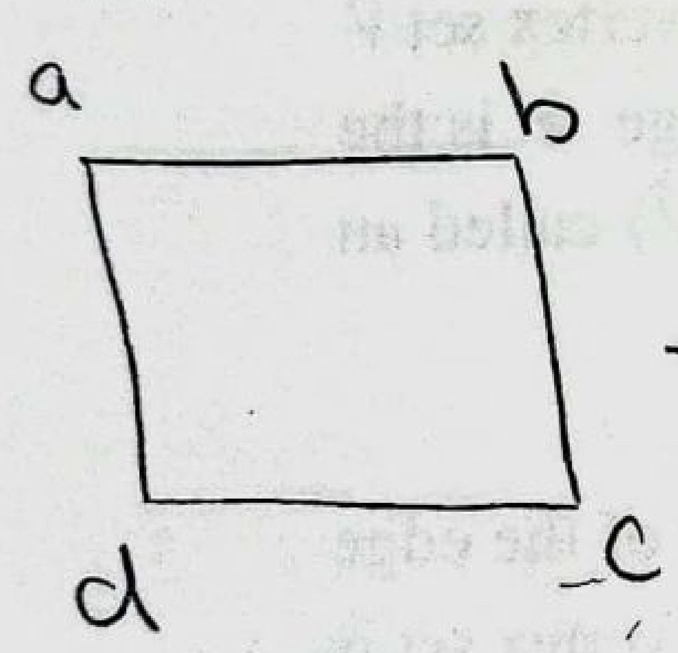
$$\sum d = 2E$$

$$(n-1) + (n-1) + (n-1) + (n-1) + (n-1) = 2E$$

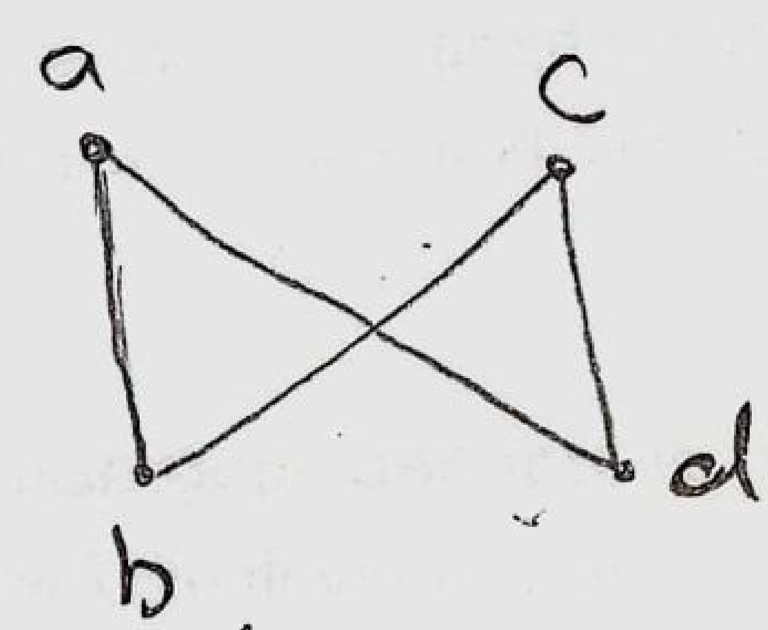
$$n(n-1) = 2E \longrightarrow \boxed{E = \frac{n(n-1)}{2}}$$

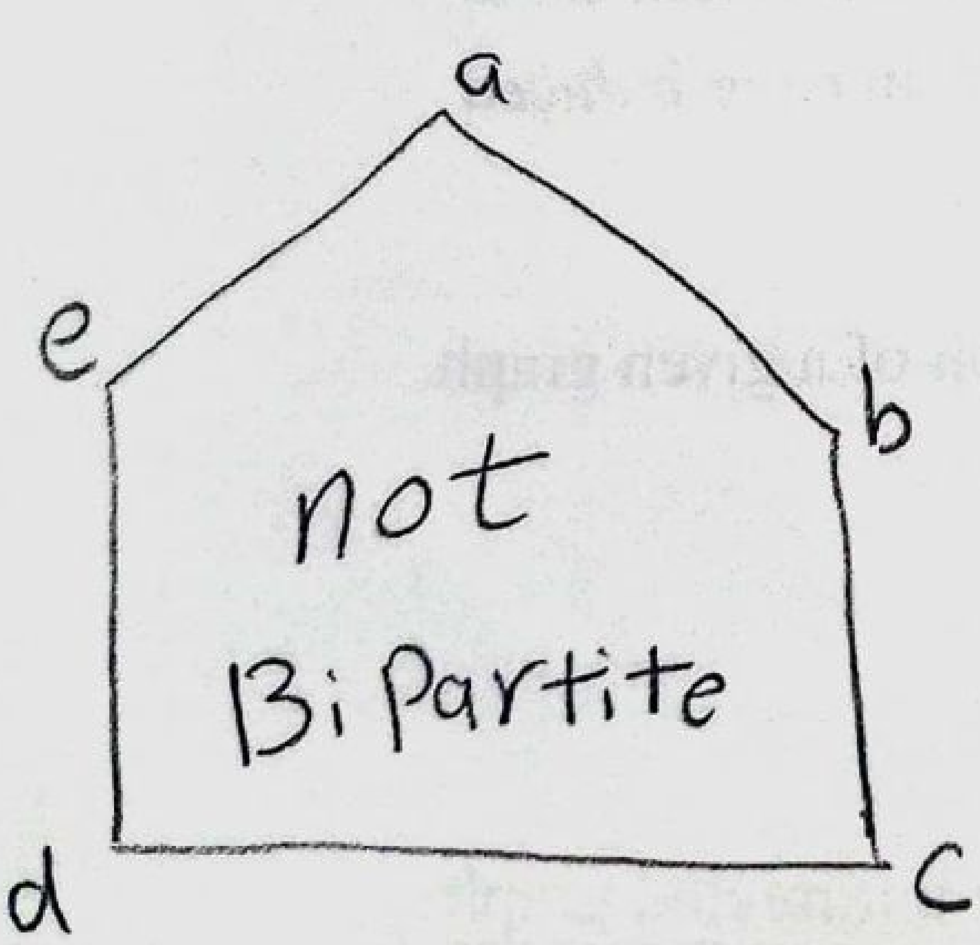
* every complete graph is $(n-1)$ regular graph

$K_5 \rightarrow$ 4-regular graph

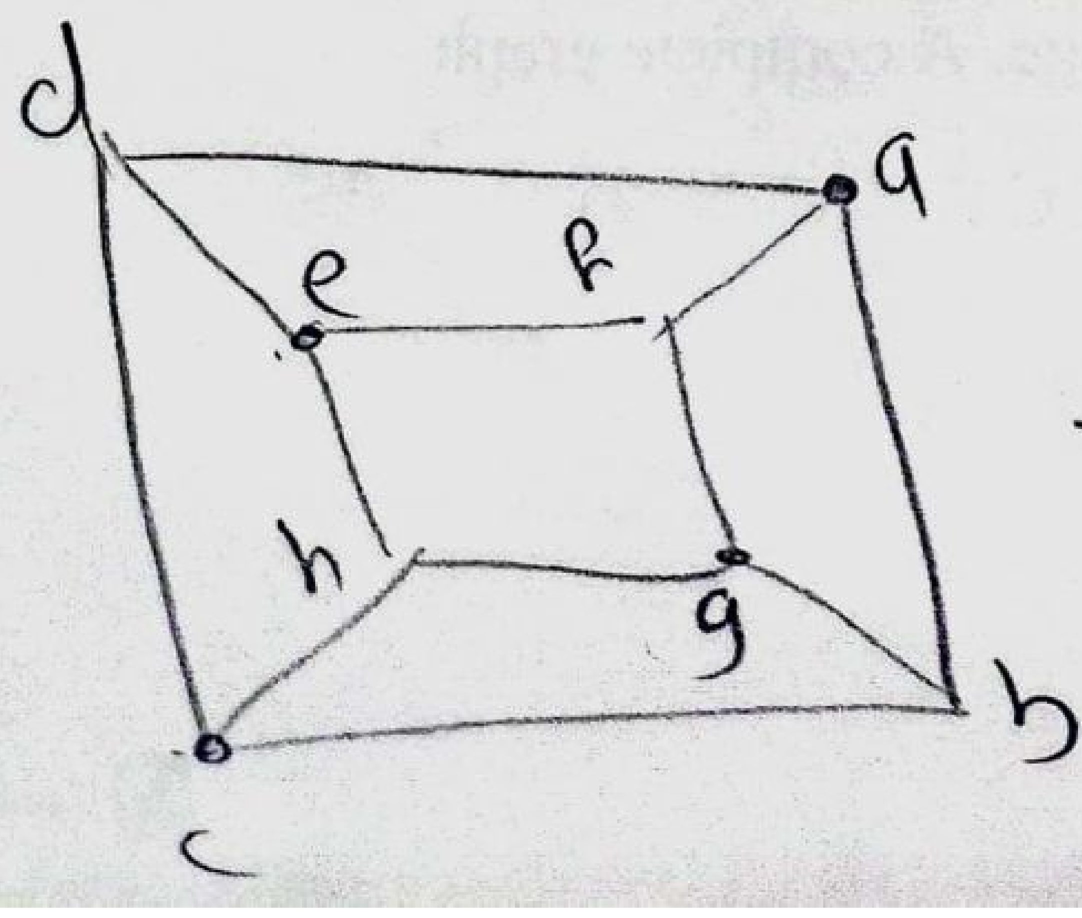


$\xrightarrow{\text{BiPartite}}$

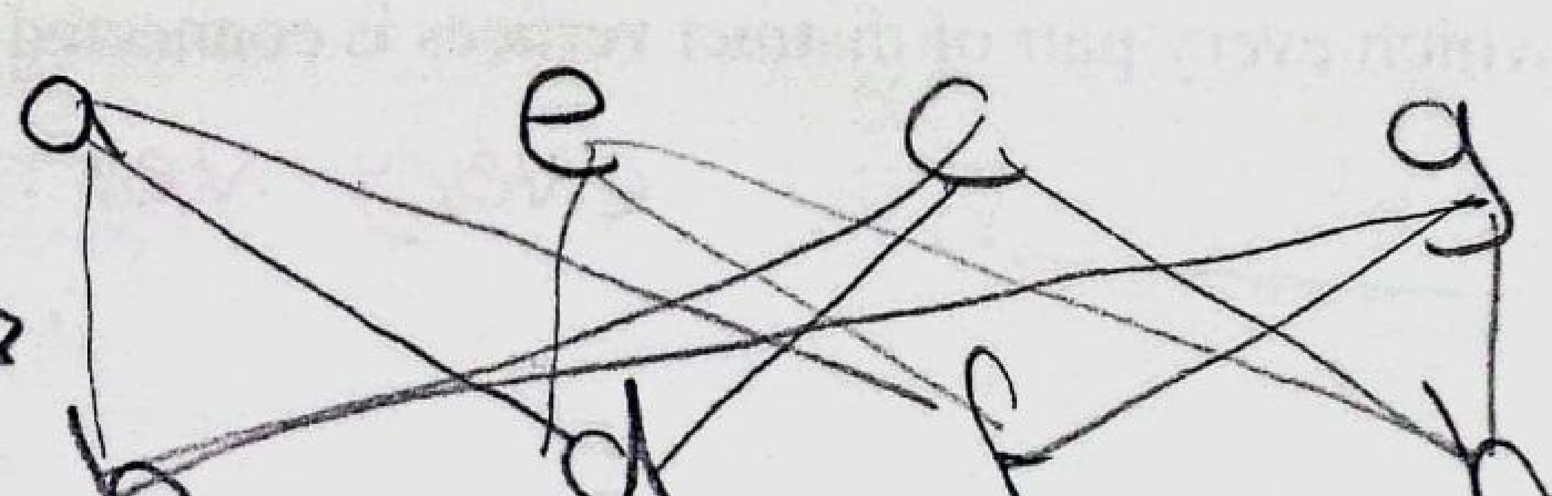




$\xrightarrow{\text{BiPartite}}$ X



$\longrightarrow$



the same partition can be adjacent. Here, the pair (V_1, V_2) is called the *bipartition* of G.

Figure 1.8 gives some examples of bipartite graphs. In all these graphs, the white vertices belong to the same partition, say V_1 and the black vertices belong to the other partition, say V_2.

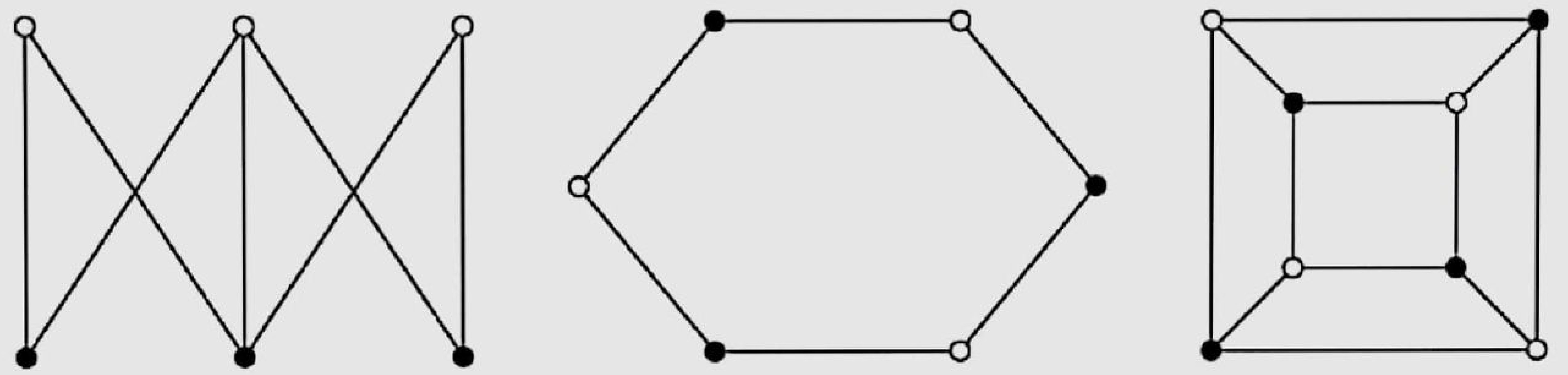


Figure 1.8: Examples of bipartite graphs

Definition 1.4.3 — Complete Bipartite Graphs. A bipartite graph G is said to be a *complete bipartite graph* if every vertex of one partition is adjacent to every vertex of the other. A complete bipartite graph with bipartition (X,Y) is denoted by $K_{|X|,|Y|}$ or $K_{a,b}$, where $a = |X|, b = |Y|$.

The following graphs are also some examples of complete bipartite graphs. In these examples also, the vertices in the same partition have the same colour.

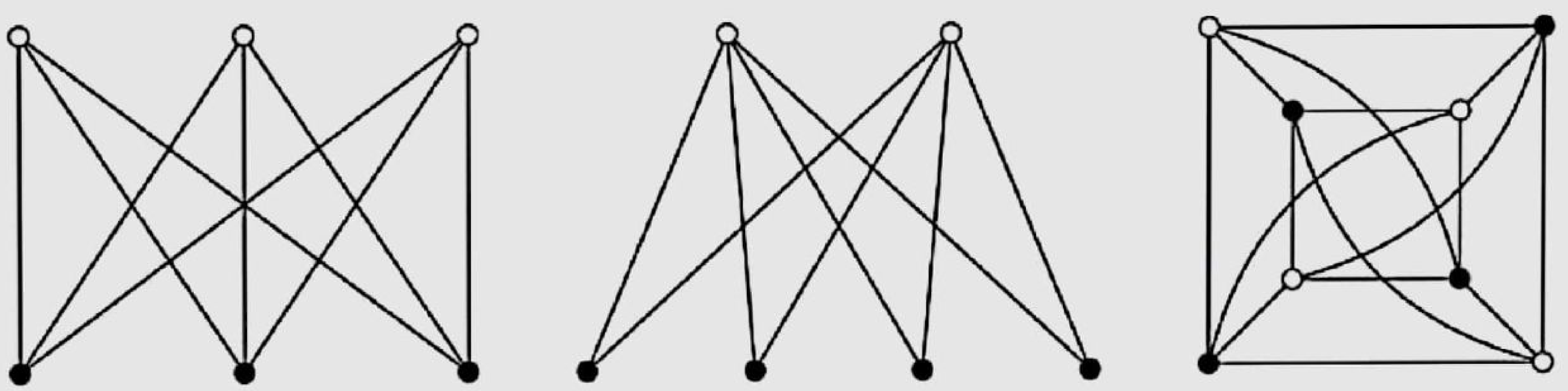
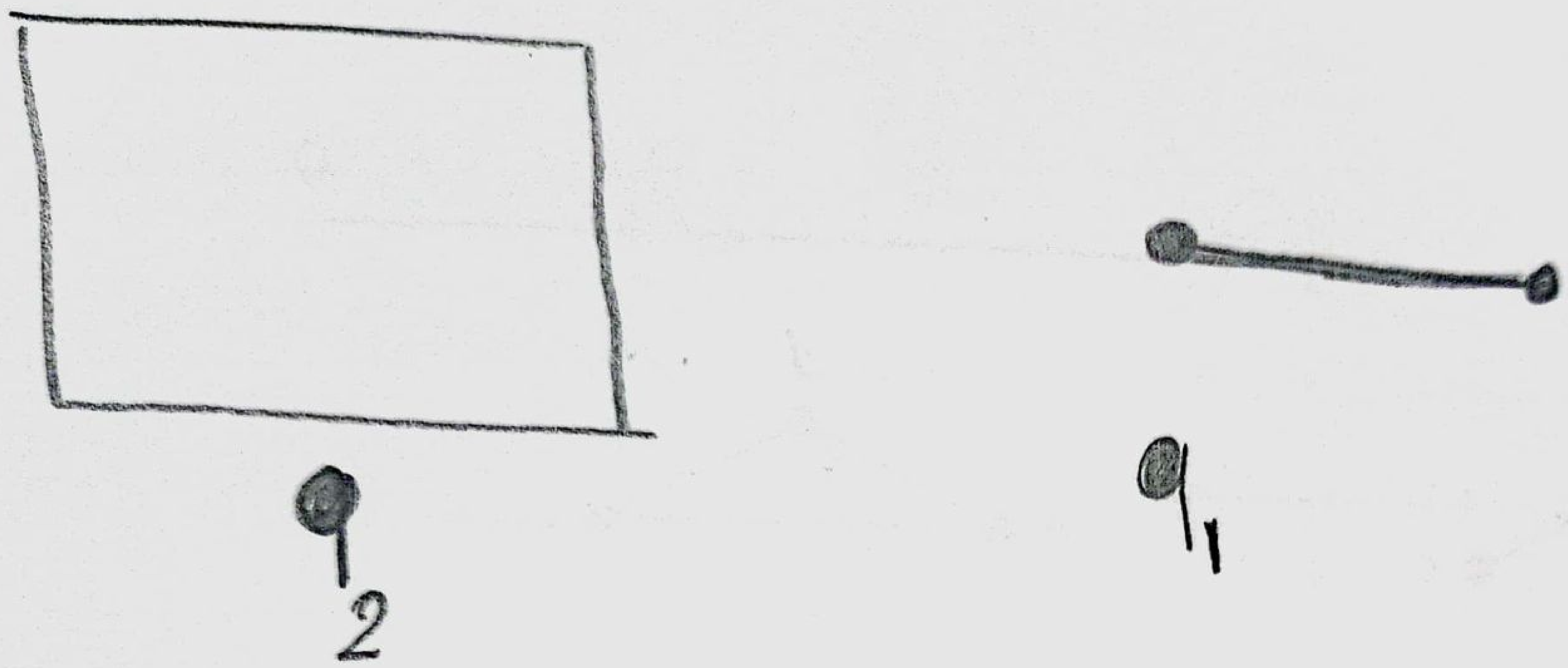
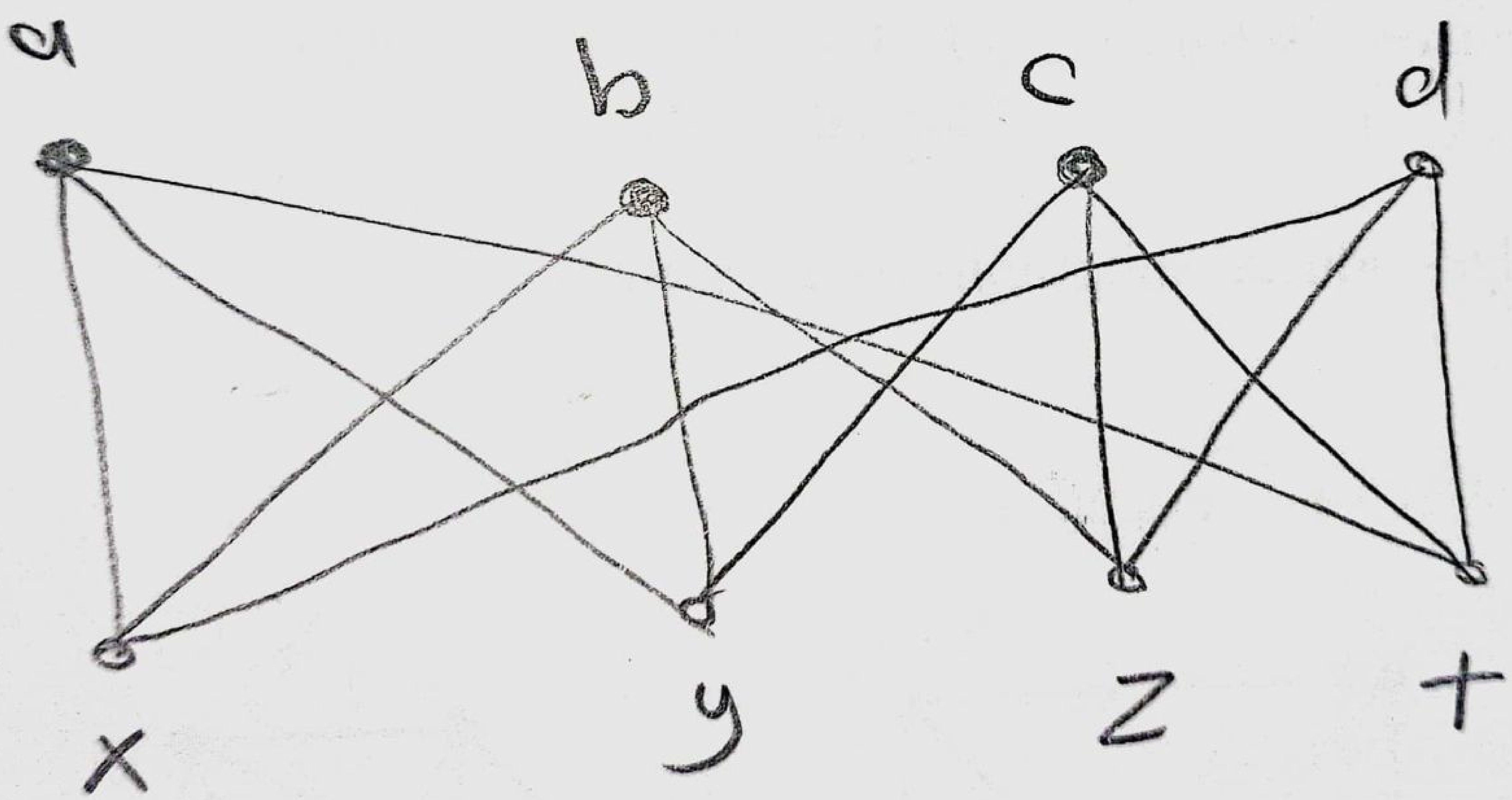


Figure 1.9: Examples of complete bipartite graphs

Problem 1.6 Show that a complete bipartite graph $K_{a,b}$ has ab vertices.

Solution: Let $K_{a,b}$ be a complete bipartite graph with bipartition (X,Y). Note that all a vertices in X have the same degree b and all b vertices in Y have the same degree a. Therefore, $\sum\limits_{v \in V(K_{a,b})} d(v) = ab + ba = 2ab$. By the first theorem on graph theory, we have $2|E(K_{a,b})| = 2ab$. That is, $|E(K_{a,b})| = ab$. ■



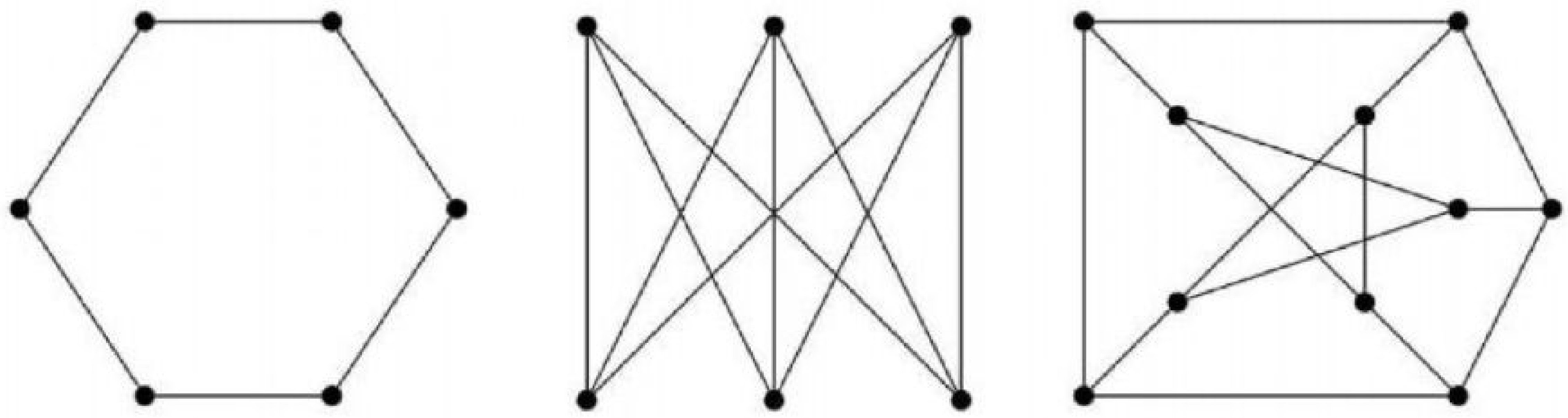

a
b
c
d
x
y
z
t

Now assume that $k > 1$ and the result holds for all complete graphs having fewer than k complete bipartite components. Now assume that $K_n = G_1 \cup G_2 \cup \ldots, \cup G_k$, where each G_i is bipartite. Partition the vertex set V into two components such that the graph G_k has no edge within X or within Y. The union of other $k-1$ bipartite subgraphs must cover the complete subgraphs induced by X and Y. Then, by Induction hypothesis, we have $|X| \leq 2^{k-1}$ and $|YX| \leq 2^{k-1}$. Therefore, $n = |X| + |Y| \leq 2^{k-1} + 2^{k-1} = 2^k$. Therefore, the necessary part follows by induction. ■

1.4.3 Regular Graphs

Definition 1.4.4 — Regular Graphs. A graph G is said to be a *regular graph* if all its vertices have the same degree. A graph G is said to be a *k-regular graph* if $d(v) = k \;\forall\; v \in V(G)$. Every complete graph is an $(n-1)$-regular graph.

The degree of all vertices in each partition of a complete bipartite graph is the same. Hence, the complete bipartite graphs are also called *biregular graphs*. Note that, for the complete bipartite graph $K_{|X|,|Y|}$, we have $d_X(v) = |Y|$ and $d_Y(v) = |X|$.



(a) A 2-regular graph (b) A 3-regular graph (c) Petersen Graph

Figure 1.10: Examples of regular graphs

1.5 Isomorphic Graphs

Definition 1.5.1 — Isomorphism of Two Graphs. An *isomorphism* of two graphs G and H is a bijective function $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

That is, two graphs G and H are said to be isomorphic if

(i) $|V(G)| = |V(H)|$,

(ii) $|E(G)| = |E(H)|$,

(iii) $v_iv_j \in E(G) \implies f(v_i)f(v_j) \in E(H)$.

This bijection is commonly described as *edge-preserving bijection*.

If an isomorphism exists between two graphs, then the graphs are called *isomorphic graphs* and denoted as $G \simeq H$ or $G \cong H$.

For example, consider the graphs given in Figure 1.11.

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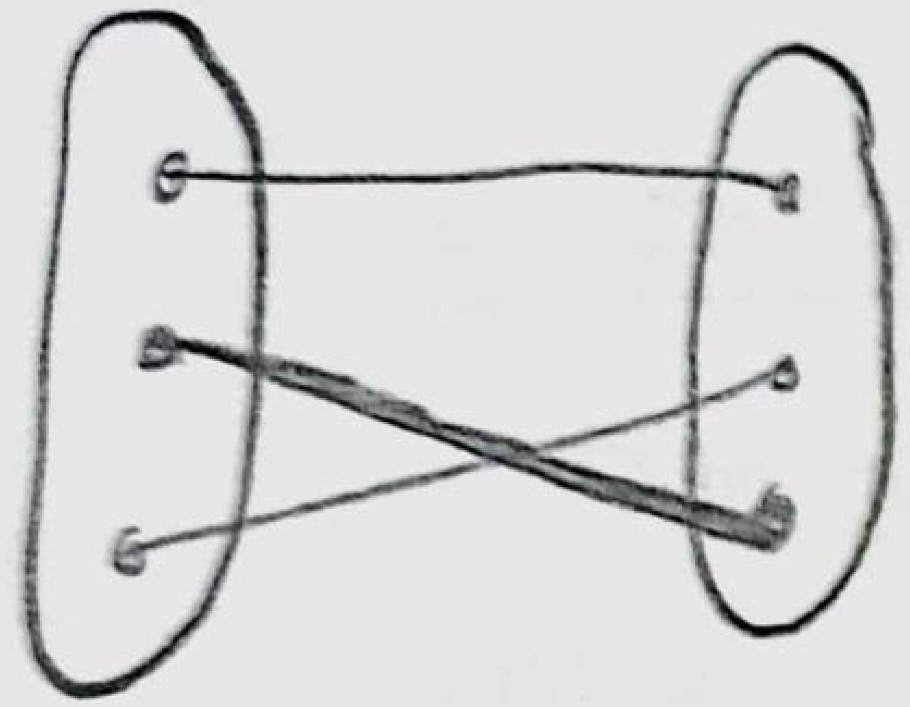
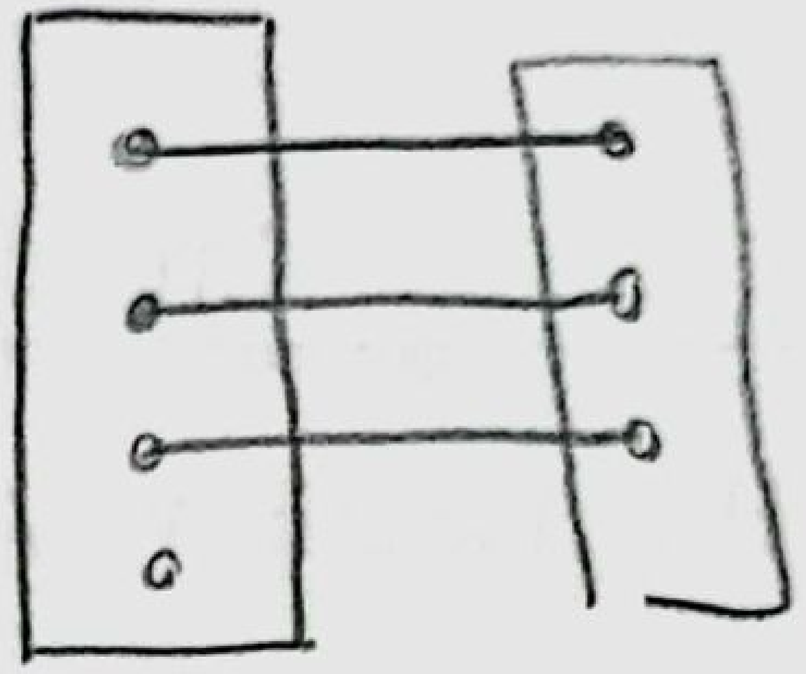
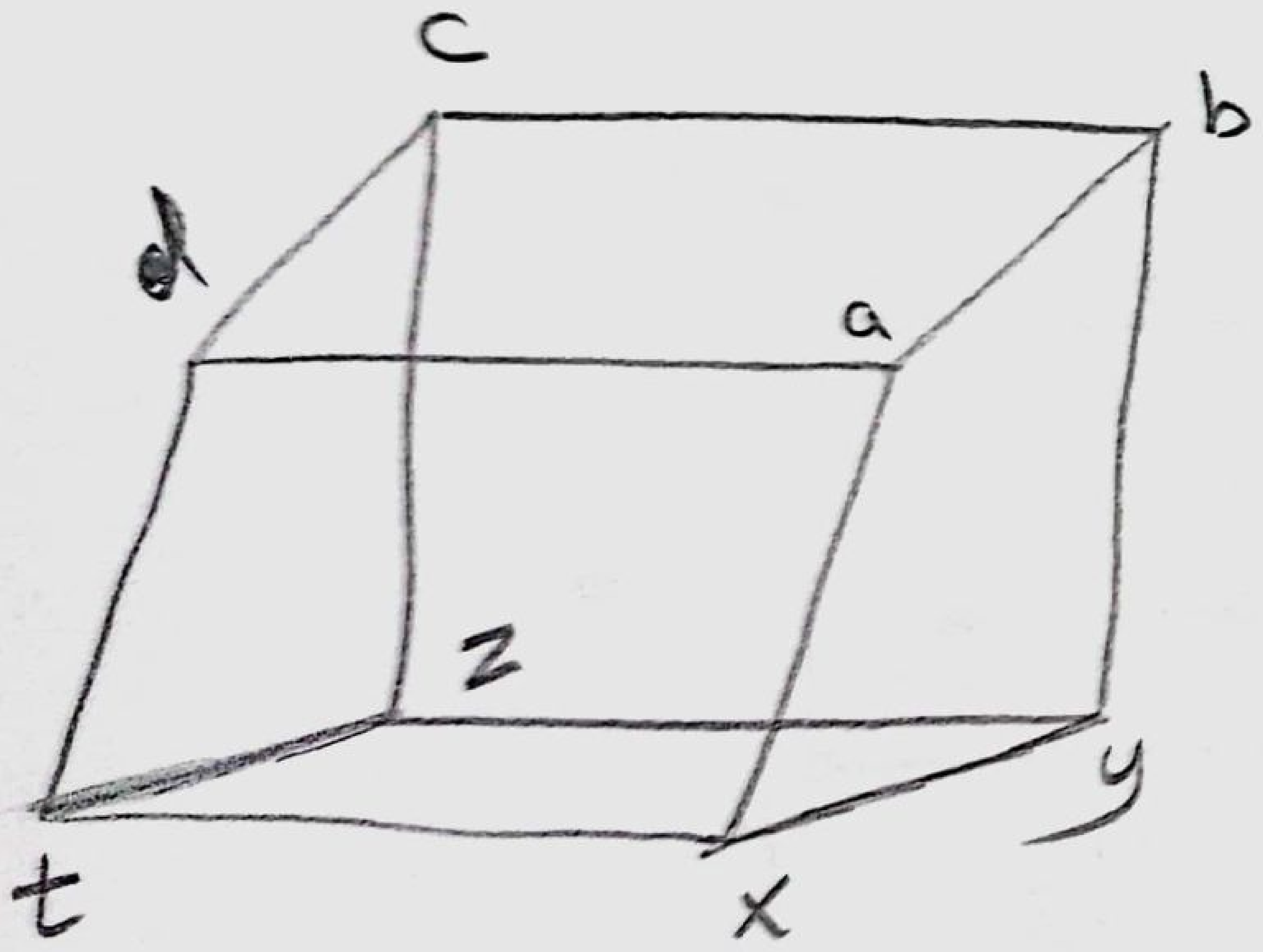
fun. one to one



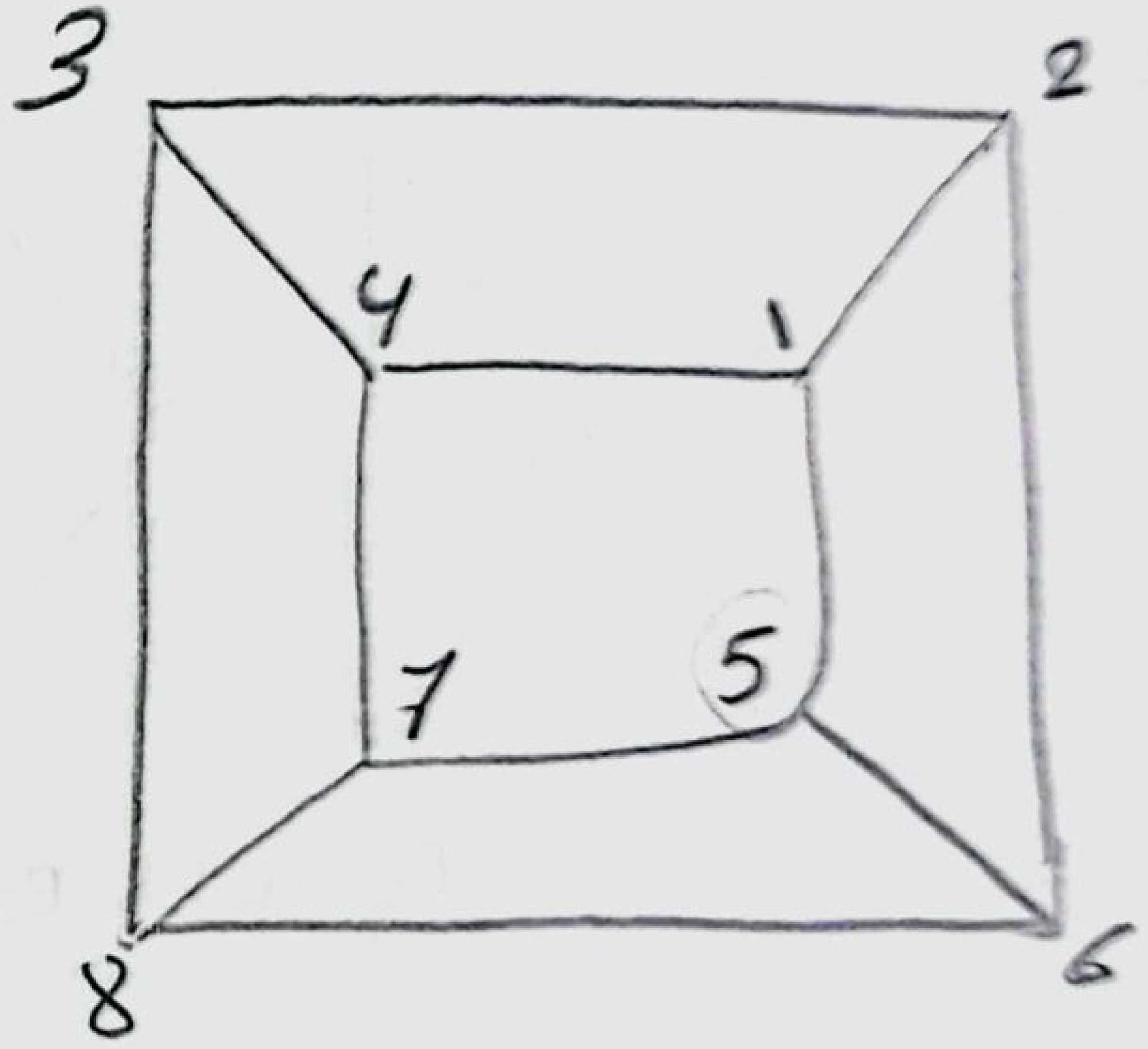
one to one
but not onto



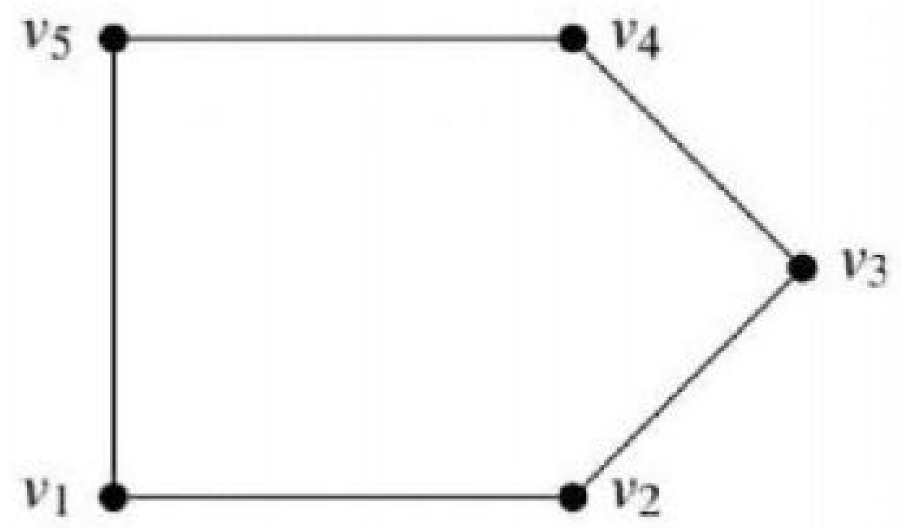
احادية
وليست
فوقية



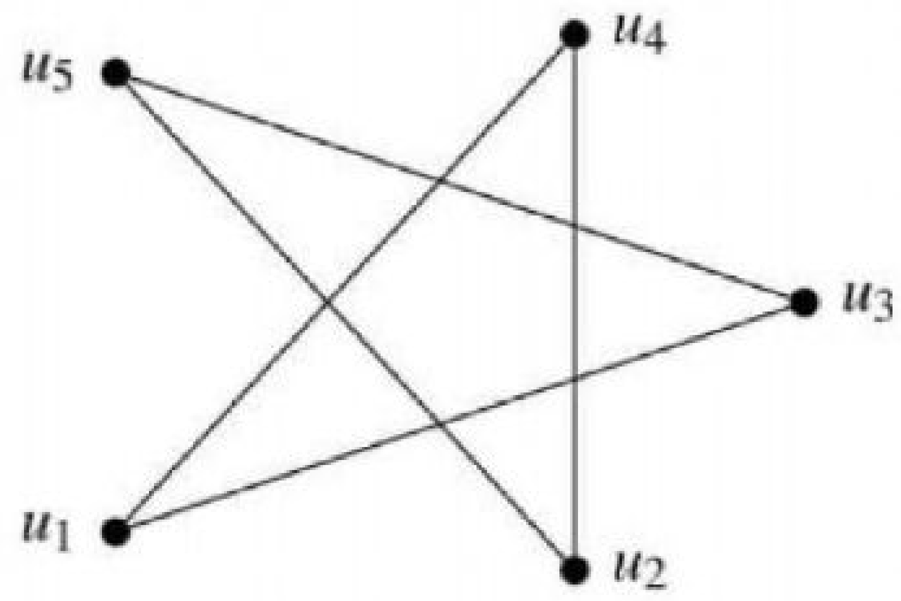
f(a) = 1
f(b) = 2
f(c) = 3
f(d) = 4
f(y) = 6.
f(x) = 5
f(z) = 8
f(t) = 7



f(ax) = 15
f(cz) = 38



(a) G_1.



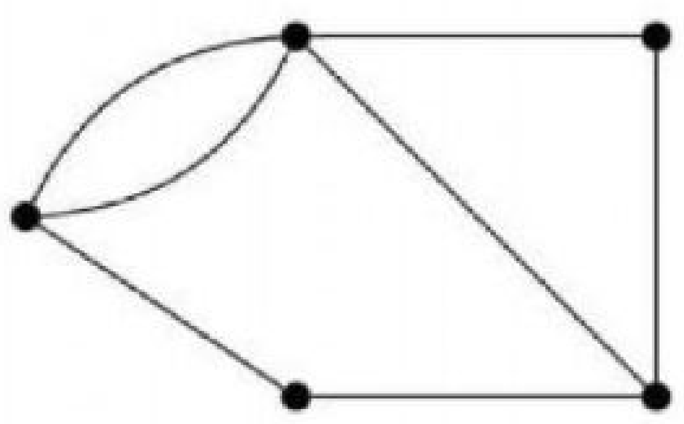
(b) G_2.

Figure 1.11: Examples of isomorphic graphs

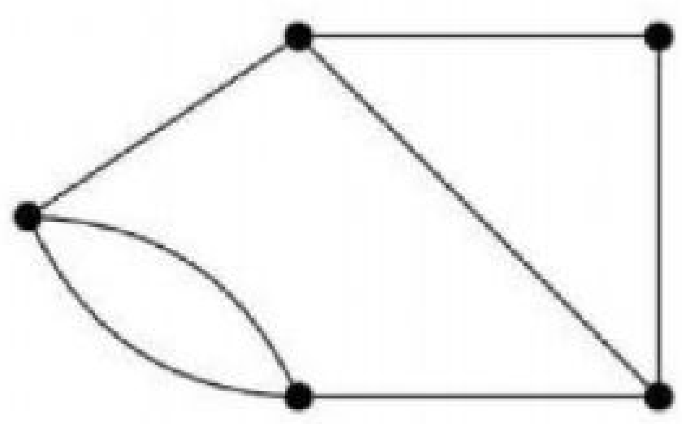
In the above graphs, we can define an isomorphism f from the first graph to the second graph such that $f(v_1) = u_1, f(v_2) = u_3, f(v_3) = u_5, f(v_4) = u_2$ and $f(v_5) = u_4$. Hence, these two graphs are isomorphic.

1.6 Exercises

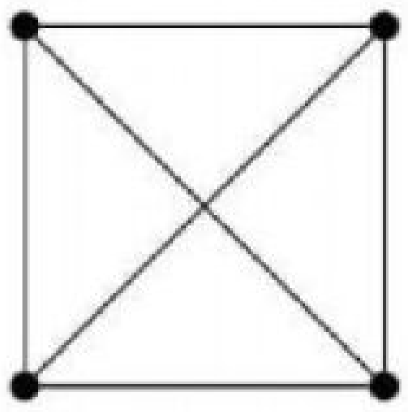
1. Show that every loop-less graph G has a bipartite subgraph with at least $\frac{\varepsilon}{2}$ edges.
2. Verify whether graph isomorphism is an equivalence relation?
3. For $k > 0$, show that a k-regular bipartite graph has the same number of vertices in each of its partite sets.
4. Show that every simple graph on n vertices subgraph of K_n.
5. Show that every subgraph of a bipartite graph is bipartite.
6. Verify whether the integer sequences $(7,6,5,4,3,3,2)$ and $(6,6,5,4,3,3,1)$ are graphical.
7. Show that if G is simple and connected but not complete, then G has three vertices u, v and w such that $uv, vw \in E(G)$, but $uw \notin E$.
8. Show that every induced subgraph of a complete graph K_n is also a complete subgraph.
9. If G is an r-regular graph, then show that r divides the size of G.
10. Show that every subgraph of a bipartite graph is bipartite.
11. If G is an r-regular graph and r is odd, then show that $\frac{\varepsilon}{r}$ is an even integer.
12. Let G be a graph in which there is no pair of adjacent edges. What can you say about the degree of the vertices in G?
13. Check whether the following pairs of graphs are isomorphic? Justify your answer.



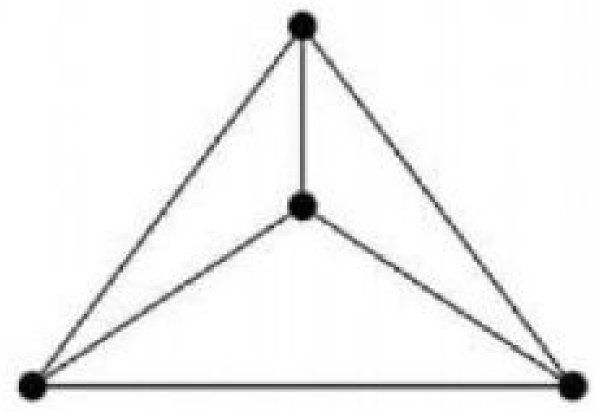
(a) G



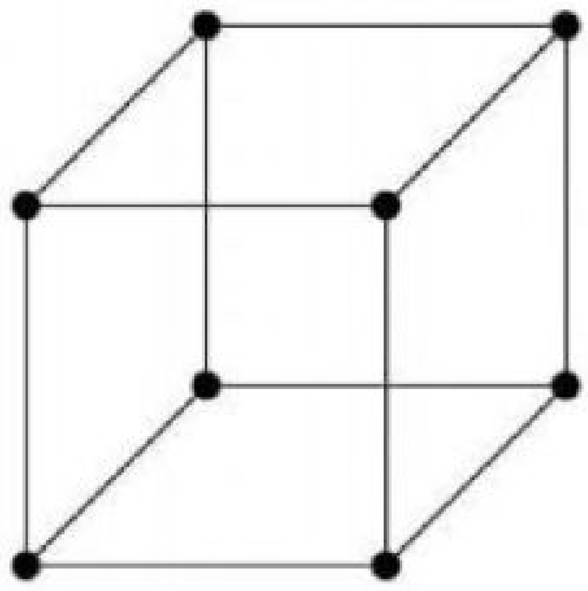
(b) H



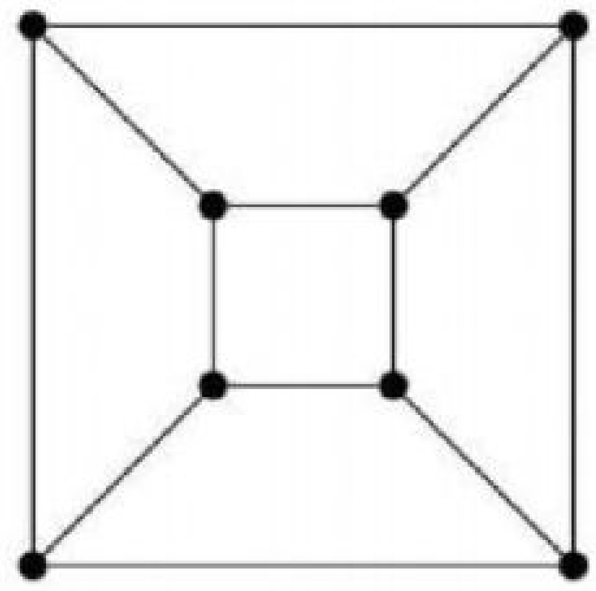
(a) G



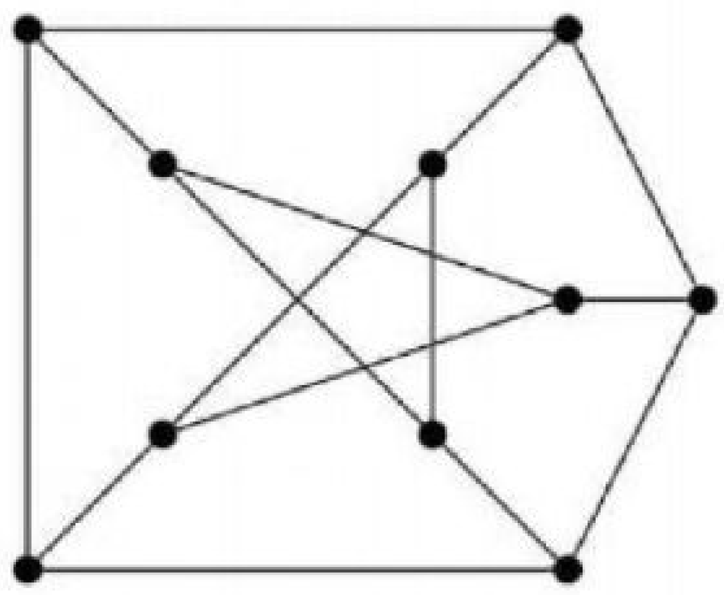
(b) H



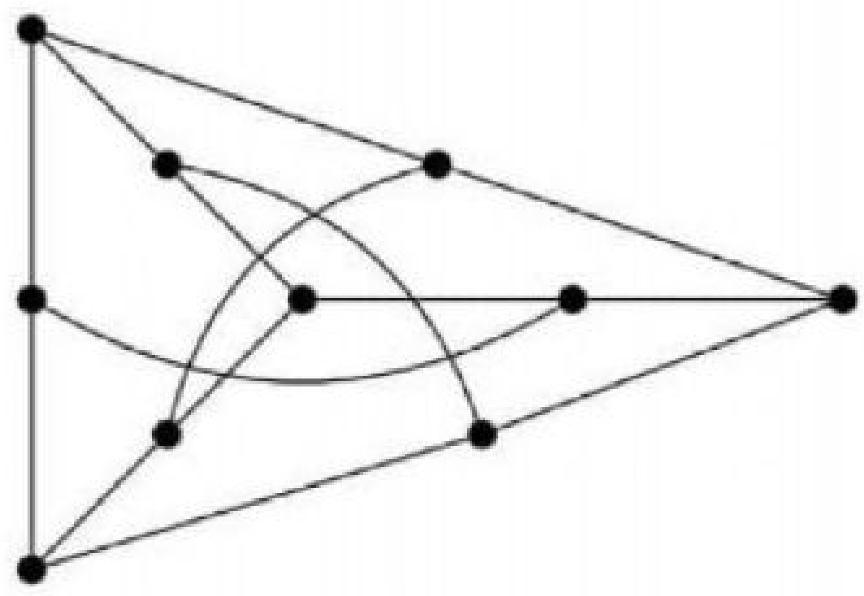
(a) G



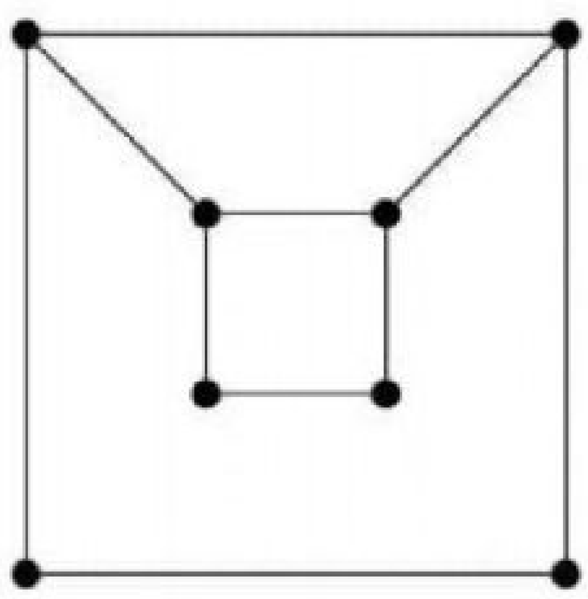
(b) H



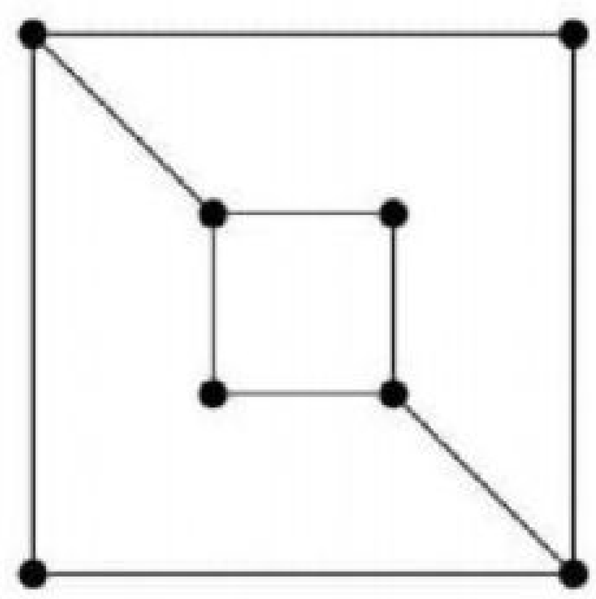
(a) G



(b) H



(a) G



(b) H

14. Let G be a graph with n vertices and e edges and let m be the smallest positive integer such that $m \geq \frac{2e}{n}$. Prove that G has a vertex of degree at least m.
15. Prove that it is impossible to have a group of nine people at a party such that each one knows exactly five of the others in the group.
16. Let G be a graph with n vertices, t of which have degree k and the others have degree $k+1$. Prove that $t = (k+1)n - 2e$, where e is the number of edges in G.
17. Let G be a k-regular graph, where k is an odd number. Prove that the number of edges in G is a multiple of k.
18. Let G be a graph with n vertices and exactly $n-1$ edges. Prove that G has either a

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vertex of degree 1 or an isolated vertex.

19. What is the smallest integer n such that the complete K_n has at least 500 edges?
20. Prove that there is no simple graph with six vertices, one of which has degree 2, two have degree 3, three have degree 4 and the remaining vertex has degree 5.
21. Prove that there is no simple graph on four vertices, three of which have degree 3 and the remaining vertex has degree 1.
22. Let G be a simple regular graph with n vertices and 24 edges. Find all possible values of n and give examples of G in each case.

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