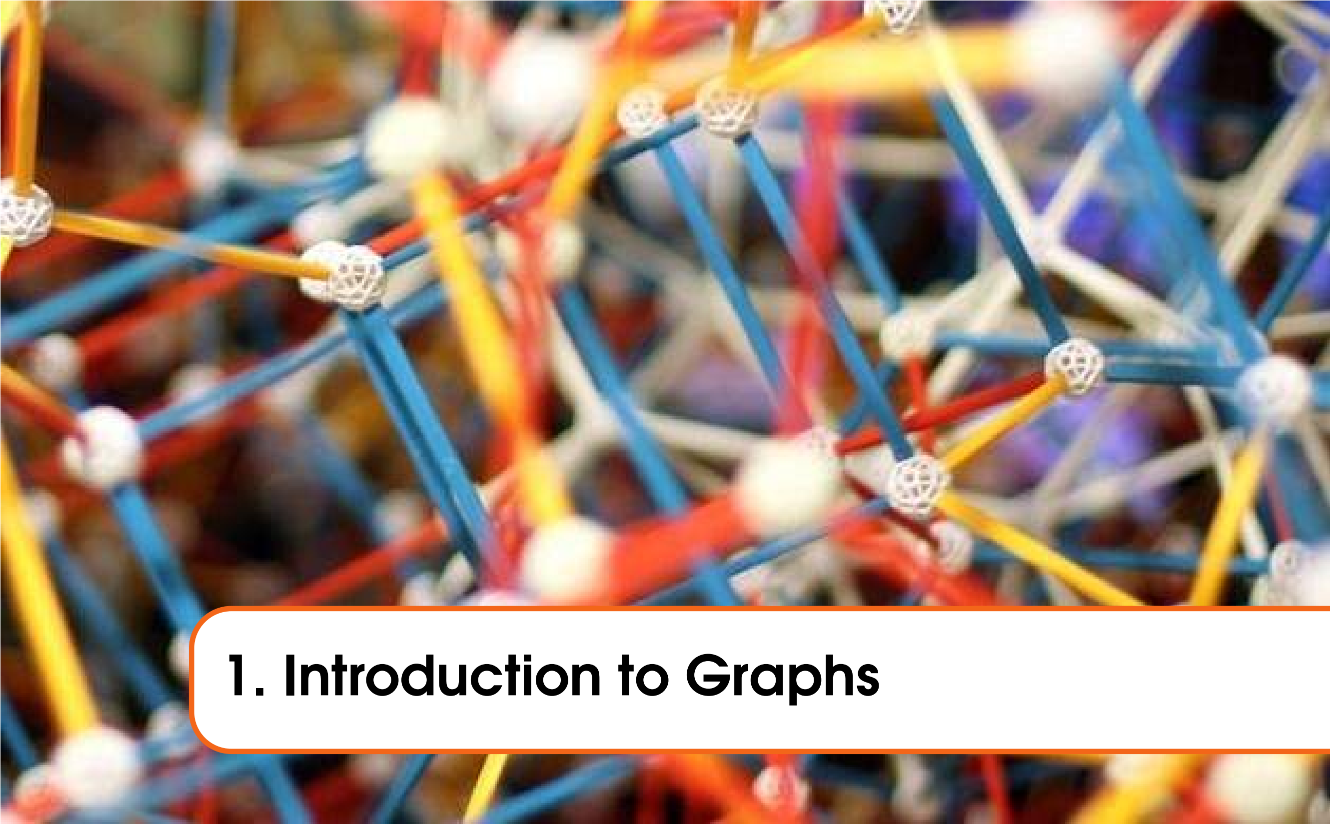


1. Introduction to Graphs

Graph Theory is a well-known area of discrete mathematics which deals with the study of graphs. A graph may be considered as a mathematical structure that is used for modelling the pairwise relations between objects.

Graph Theory has many theoretical developments and applications not only to different branches of mathematics, but also to various other fields of basic sciences, technology, social sciences, computer science etc. Graphs are widely used as efficient tools to model many types of practical and real-world problems in physical, biological, social and information systems. Graph-theoretical models and methods are based on mathematical combinatorics and related fields.

1.1 Basic Definitions

Definition 1.1.1 — Graph. A *graph* G can be considered as an ordered triple (V,E,ψ), where

(i) $V = \{v_1, v_2, v_3, \ldots\}$ is called the *vertex set* of G and the elements of V are called the *vertices* (or *points* or *nodes*);

(ii) $E = \{e_1, e_2, e_3, \ldots\}$ is the called the *edge set* of G and the elements of E are called *edges* (or *lines* or *arcs*); and

(iii) ψ is called the *adjacency relation*, defined by $\psi : E \to V \times V$, which defines the association between each edge with the vertex pairs of G.

Usually, the graph is denoted as $G = (V,E)$. The vertex set and edge set of a graph G are also written as $V(G)$ and $E(G)$ respectively.

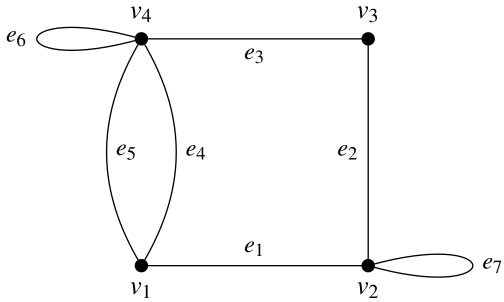


Figure 1.1: An example of a graph

If two vertices u and v are the (two) end points of an edge e, then we represent this edge by uv or vu. If $e = uv$ is an edge of a graph G, then we say that u and v are *adjacent vertices* in G and that e *joins* u and v. In such cases, we also say that u and v are adjacent to each other.

Given an edge $e = uv$, the vertex u and the edge e are said to be *incident with* each other and so are v and e. Two edges e_i and e_j are said to be *adjacent edges* if they are incident with a common vertex.

Definition 1.1.2 — Order and Size of a Graph. The *order* of a graph G, denoted by $\nu(G)$, is the number of its vertices and the *size* of G, denoted by $\varepsilon(G)$, is the number of its edges.

A graph with p-vertices and q-edges is called a (p,q)*-graph*. The $(1,0)$-graph is called a *trivial graph*. That is, a trivial graph is a graph with a single vertex. A graph without edges is called an *empty graph* or a *null graph*. The following figure illustrates a null graph of order 5.

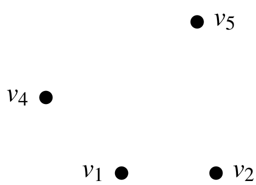


Figure 1.2: Null graph of order 5.

Definition 1.1.3 — Finite and Infinite Graphs. A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph*. Otherwise, it is an *infinite graph*.

Definition 1.1.4 — Self-loop. An edge of a graph that joins a node to itself is called *loop* or a *self-loop*. That is, a loop is an edge uv, where $u = v$.

Definition 1.1.5 — Parallel Edges. The edges connecting the same pair of vertices are called *multiple edges* or *parallel edges*.

In Figure 1.2, the edges e_6 and e_7 are loops and the edges e_4 and e_5 are parallel edges.

Definition 1.1.6 — Simple Graphs and Multigraphs. A graph G which does not have loops or parallel edges is called a *simple graph*. A graph which is not simple is generally called a *multigraph*.

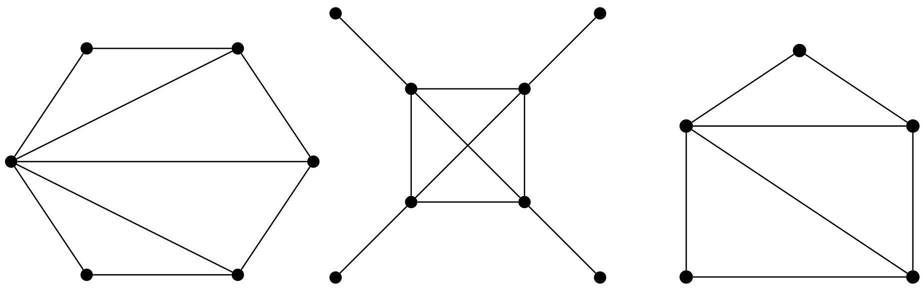


Figure 1.3: Some examples of simple graphs

1.2 Degrees and Degree Sequences in Graphs

Definition 1.2.1 — Degree of a vertex. The number of edges incident on a vertex v, with self-loops counted twice, is called the *degree* of the vertex v and is denoted by $deg_G(v)$ or $deg(v)$ or simply $d(v)$.

Definition 1.2.2 — Isolated vertex. A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are those with zero degree.

Definition 1.2.3 — Pendant vertex. A vertex of degree 1, is called a *pendent vertex* or an end vertex.

Definition 1.2.4 — Internal vertex. A vertex, which is neither a pendent vertex nor an isolated vertex, is called an *internal vertex* or an *intermediate vertex*.

Definition 1.2.5 — Minimum and Maximum Degree of a Graph. The *maximum degree of a graph* G, denoted by $\Delta(G)$, is defined to be $\Delta(G) = \max\{d(v) : v \in V(G)\}$. Similarly, the *minimum degree of a graph* G, denoted by $\delta(G)$, is defined to be $\delta(G) = \min\{d(v) : v \in V(G)\}$. Note that for any vertex v in G, we have $\delta(G) \leq d(v) \leq \Delta(G)$.

The following theorem is a relation between the sum of degrees of vertices in a graph G and the size of G.

Theorem 1.2.1 In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. That is, $\sum_{v \in V(G)} d(v) = 2\varepsilon$.

Proof. Let $S = \sum_{v \in V(G)} d(v)$. Notice that in counting S, we count each edge exactly twice. That is, every edge contributes degree 1 each to both of its end vertices and a loop provides degree 2 to the vertex it incidents with. Hence 2 to the sum of degrees of vertices in G. Thus, $S = 2|E| = 2\varepsilon$. ■

The above theorem is usually called the *first theorem on graph theory*. It is also known as the *hand shaking lemma*. The following two theorems are immediate consequences of the above theorem.

Theorem 1.2.2 For any graph G, $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$.

Proof. By Theorem-1, we have $2\varepsilon = \sum_{v \in V(G)} d(v)$. Therefore, note that $\frac{2|E|}{|V|} = \frac{\sum d(v)}{|V|}$, the average degree of G. Therefore, $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$. ■

Theorem 1.2.3 For any graph G, the number of odd degree vertices is always even.

Proof. Let $S = \sum_{v \in V(G)} d(v)$. By Theorem 1.2.1, we have $S = 2\varepsilon$ and hence S is always even. Let V_1 be the set of all odd degree vertices and V_2 be the set of all even degree vertices in G. Now, let $S_1 = \sum_{v \in V_1} d(v)$ and $S_2 = \sum_{v \in V_2} d(v)$. Note that S_2, being the sum of even integers, is also an even integer.

We also note that $S = S_1 + S_2$ (since V_1 and V_2 are disjoint sets and $V_1 \cup V_2 = V$). Therefore, $S_1 = S - S_2$. Being the difference between two even integers, S_1 is also an even integer. Since V_1 is a set of odd degree vertices, S_1 is even only when the number of elements in V_1 is even. That is, the number of odd degree vertices in G is even, completing the proof. ■

Definition 1.2.6 — Degree Sequence. The *degree sequence* of a graph of order n is the n-term sequence (usually written in descending order) of the vertex degrees. In Figure-1, $\delta(G) = 2$, $\Delta(G) = 5$ and the degree sequence of G is $(5,4,3,2)$.

Definition 1.2.7 — Graphical Sequence. An integer sequence is said to be *graphical* if it is the degree sequence of some graphs. A graph G is said to be the *graphical realisation* of an integer sequence S if S the degree sequence of G.

Problem 1.1 Is the sequence $S = \langle 5,4,3,3,2,2,2,1,1,1,1 \rangle$ graphical? Justify your answer.
Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, $\sum a_i = 25$, not an even number. Therefore, the given sequence is not graphical. ■

Problem 1.2 Is the sequence $S = \langle 9,9,8,7,7,6,6,5,5 \rangle$ graphical? Justify your answer.

Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, $\sum a_i = 62$, an even number. But note that the maximum degree that a vertex can attain in a graph of order n is $n-1$. If S were graphical, the corresponding graph would have been a graph on 10 vertices and have $\Delta(G) = 9$. Therefore, the given sequence is not graphical. ■

Problem 1.3 Is the sequence $S = \langle 9,8,7,6,6,5,5,4,3,3,2,2 \rangle$ graphical? Justify your answer.

Solution: The sequence $S = \langle a_i \rangle$ is graphical if every element of S is the degree of some vertex in a graph. For any graph, we know that $\sum_{v \in V(G)} d(v) = 2|E|$, an even integer. Here, we have $\sum a_i = 60$, an even number. Also, note that the all elements in the sequence are less than the number of elements in that sequence.

Therefore, the given sequence is graphical and the corresponding graph is drawn below. ■

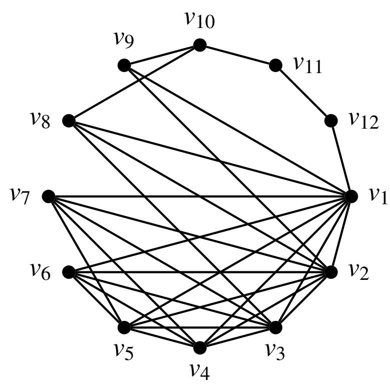


Figure 1.4: Graphical realisation of the degree sequence S.

1.2.1 Neighbourhoods

Definition 1.2.8 — Neighbourhood of a Vertex. The *neighbourhood* (or *open neighbourhood*) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v. That is, $N(v) = \{x \in V : vx \in E\}$. The *closed neighbourhood* of a vertex v, denoted by $N[v]$, is simply the set $N(v) \cup \{v\}$.

Then, for any vertex v in a graph G, we have $d_G(v) = |N(v)|$. A special case is a loop that connects a vertex to itself; if such an edge exists, the vertex is said to belong to its own neighbourhood.

Given a set S of vertices, we define the neighbourhood of S, denoted by $N(S)$, to be the union of the neighbourhoods of the vertices in S. Similarly, the closed neighbourhood of S, denoted by $N[S]$, is defined to be $S \cup N(S)$.

Neighbourhoods are widely used to represent graphs in computer algorithms, in terms of the adjacency list and adjacency matrix representations. Neighbourhoods are also used in the clustering coefficient of graphs, which is a measure of the average density of its neighbourhoods. In addition, many important classes of graphs may be defined by properties of their neighbourhoods, or by symmetries that relate neighbourhoods to each other.

1.3 Subgraphs and Spanning Subgraphs

Definition 1.3.1 — Subgraph of a Graph. A graph $H(V_1, E_1)$ is said to be a *subgraph* of a graph $G(V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Definition 1.3.2 — Spanning Subgraph of a Graph. A graph $H(V_1, E_1)$ is said to be a *spanning subgraph* of a graph $G(V, E)$ if $V_1 = V$ and $E_1 \subseteq E$.

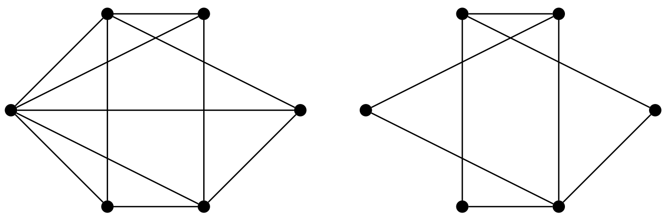


Figure 1.5: Examples of Subgraphs

In the above figure, the second graph is a spanning subgraph of the first graph, while the third graph is a subgraph of the first graph.

1.3.1 Induced Subgraphs

Definition 1.3.3 — Induced Subgraph. Suppose that V' be a subset of the vertex set V of a graph G. Then, the subgraph of G whose vertex set is V' and whose edge set is the set of edges of G that have both end vertices in V' is denoted by $G[V]$ or $\langle V \rangle$ called an *induced subgraph* of G.

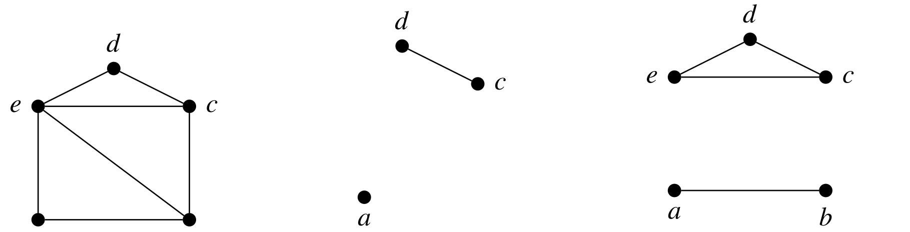
Definition 1.3.4 — Edge-Induced Subgraph. Suppose that E' be a subset of the edge set V of a graph G. Then, the subgraph of G whose edge set is E' and whose vertex set is the set of end vertices of the edges in E' is denoted by $G[E]$ or $\langle E \rangle$ called an *edge-induced subgraph* of G.

Figure 1.6 depicts an induced subgraph and an edge induced subgraph of a given graph.

1.4 Fundamental Graph Classes

1.4.1 Complete Graphs

Definition 1.4.1 — Complete Graphs. A *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph



(a) The graph G

(b) The induced subgraph $G[a,c,d]$

(c) The induced subgraph $G[ab,cd,ce,de]$

Figure 1.6: Induced and edge-induced subgraphs of a graph G.

on n vertices is denoted by K_n.

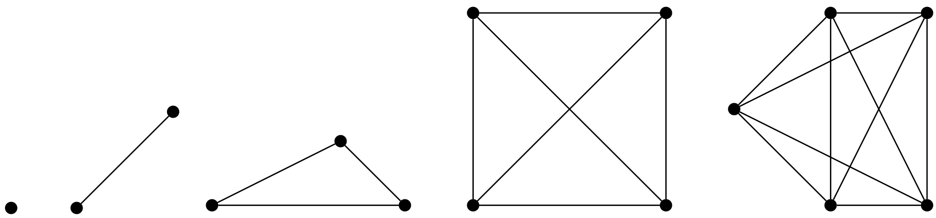


Figure 1.7: First few complete graphs

Problem 1.4 Show that a complete graph K_n has $\frac{n(n-1)}{2}$ edges.
Solution: Note that any two vertices in a complete graph are adjacent to each other. Hence, the number of edges in a complete graph is equal to the number of distinct pairs of vertices in it. Therefore, the number of such pairs of vertices in K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$. That is, the number of edges in K_n is $\frac{n(n-1)}{2}$. ■

We can write an alternate solution to this problem as follows:
Solution: Note that every vertex in a complete graph K_n is adjacent to all other $n-1$ vertices in K_n. That is, $d(v) = n-1$ for all vertices in K_n. Since K_n has n vertices, we have $\sum \lim_{v \in V(K_n)} d(v) = n(n-1)$. Therefore, by the first theorem on graph theory, we have $2|E(K_n)| = n(n-1)$. That is, the number of edges in K_n is $\frac{n(n-1)}{2}$. ■

Problem 1.5 Show that the size of every graph of order n is at most $\frac{n(n-1)}{2}$.
Solution: Note that every graph on n vertices is a spanning subgraph of the complete graph K_n. Therefore, $E(G) \subseteq E(K_n)$. That is, $|E(G)| \leq |E(K_n)| = \frac{n(n-1)}{2}$. That is, any graph of order n can have at most $\frac{n(n-1)}{2}$ edges. ■

1.4.2 Bipartite Graphs

Definition 1.4.2 — Bipartite Graphs. A graph G is said to be a *bipartite graph* if its vertex set V can be partitioned into two sets, say V_1 and V_2, such that no two vertices in the same partition can be adjacent. Here, the pair (V_1, V_2) is called the *bipartition* of G.

Figure 1.8 gives some examples of bipartite graphs. In all these graphs, the white vertices belong to the same partition, say V_1 and the black vertices belong to the other partition, say V_2.

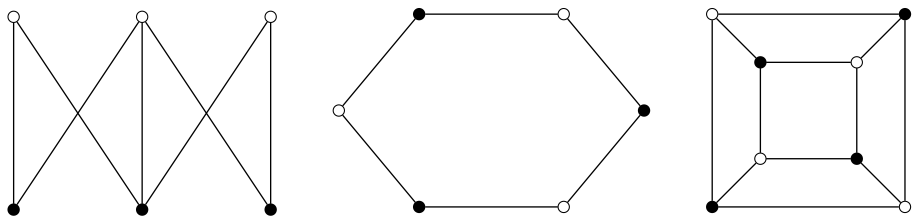


Figure 1.8: Examples of bipartite graphs

Definition 1.4.3 — Complete Bipartite Graphs. A bipartite graph G is said to be a *complete bipartite graph* if every vertex of one partition is adjacent to every vertex of the other. A complete bipartite graph with bipartition (X,Y) is denoted by $K_{|X|,|Y|}$ or $K_{a,b}$, where $a = |X|, b = |Y|$.

The following graphs are also some examples of complete bipartite graphs. In these examples also, the vertices in the same partition have the same colour.

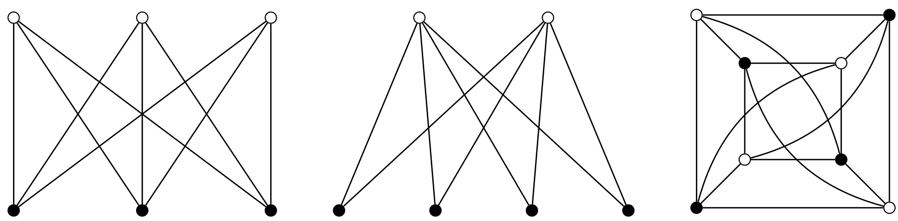


Figure 1.9: Examples of complete bipartite graphs

Problem 1.6 Show that a complete bipartite graph $K_{a,b}$ has ab vertices.
Solution: Let $K_{a,b}$ be a complete bipartite graph with bipartition (X,Y). Note that all a vertices in X have the same degree b and all b vertices in Y have the same degree a. Therefore, $\sum\limits_{v \in V(K_{a,b})} d(v) = ab + ba = 2ab$. By the first theorem on graph theory, we have $2|E(K_{a,b})| = 2ab$. That is, $|E(K_{a,b})| = ab$. ■

Theorem 1.4.1 The complete graph K_n can be expressed as the union of k bipartite graphs if and only if $n \leq 2^k$.

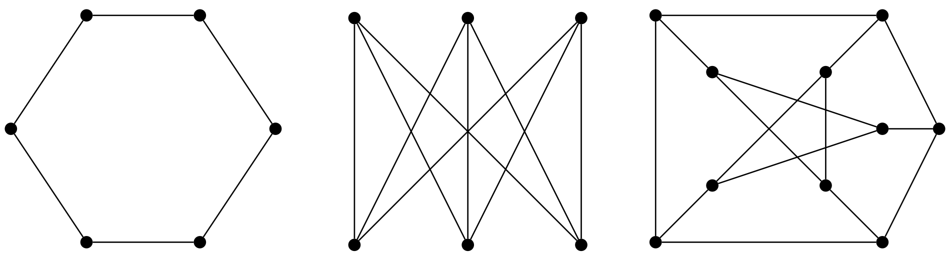
Proof. First assume that K_n can be expressed as the union of k bipartite graphs. We use the method of induction on k. First let $k = 1$. Note that K_n contains triangle K_3 (and K_3 is not bipartite) except for $n \leq 2$. Therefore, the result is true for $k = 1$.

Now assume that $k > 1$ and the result holds for all complete graphs having fewer than k complete bipartite components. Now assume that $K_n = G_1 \cup G_2 \cup \ldots, \cup G_k$, where each G_i is bipartite. Partition the vertex set V into two components such that the graph G_k has no edge within X or within Y. The union of other $k-1$ bipartite subgraphs must cover the complete subgraphs induced by X and Y. Then, by Induction hypothesis, we have $|X| \leq 2^{k-1}$ and $|YX| \leq 2^{k-1}$. Therefore, $n = |X| + |Y| \leq 2^{k-1} + 2^{k-1} = 2^k$. Therefore, the necessary part follows by induction. ■

1.4.3 Regular Graphs

Definition 1.4.4 — Regular Graphs. A graph G is said to be a *regular graph* if all its vertices have the same degree. A graph G is said to be a *k-regular graph* if $d(v) = k \; \forall \; v \in V(G)$. Every complete graph is an $(n-1)$-regular graph.

The degree of all vertices in each partition of a complete bipartite graph is the same. Hence, the complete bipartite graphs are also called *biregular graphs*. Note that, for the complete bipartite graph $K_{|X|,|Y|}$, we have $d_X(v) = |Y|$ and $d_Y(v) = |X|$.



(a) A 2-regular graph (b) A 3-regular graph (c) Petersen Graph

Figure 1.10: Examples of regular graphs

1.5 Isomorphic Graphs

Definition 1.5.1 — Isomorphism of Two Graphs. An *isomorphism* of two graphs G and H is a bijective function $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

That is, two graphs G and H are said to be isomorphic if

(i) $|V(G)| = |V(H)|$,

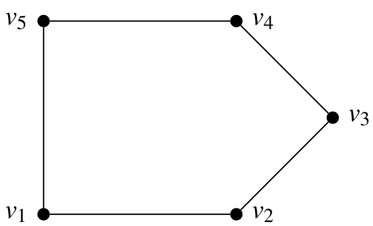
(ii) $|E(G)| = |E(H)|$,

(iii) $v_iv_j \in E(G) \implies f(v_i)f(v_j) \in E(H)$.

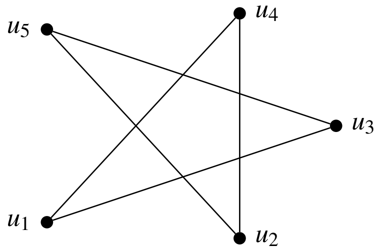
This bijection is commonly described as *edge-preserving bijection*.

If an isomorphism exists between two graphs, then the graphs are called *isomorphic graphs* and denoted as $G \simeq H$ or $G \cong H$.

For example, consider the graphs given in Figure 1.11.



(a) G_1.



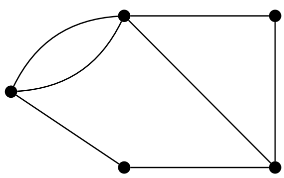
(b) G_2.

Figure 1.11: Examples of isomorphic graphs

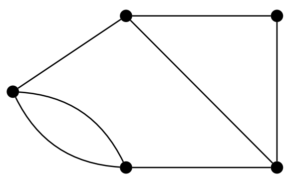
In the above graphs, we can define an isomorphism f from the first graph to the second graph such that $f(v_1) = u_1, f(v_2) = u_3, f(v_3) = u_5, f(v_4) = u_2$ and $f(v_5) = u_4$. Hence, these two graphs are isomorphic.

1.6 Exercises

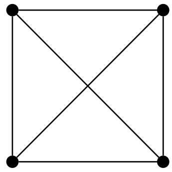
1. Show that every loop-less graph G has a bipartite subgraph with at least $\frac{\varepsilon}{2}$ edges.
2. Verify whether graph isomorphism is an equivalence relation?
3. For $k > 0$, show that a k-regular bipartite graph has the same number of vertices in each of its partite sets.
4. Show that every simple graph on n vertices subgraph of K_n.
5. Show that every subgraph of a bipartite graph is bipartite.
6. Verify whether the integer sequences $(7,6,5,4,3,3,2)$ and $(6,6,5,4,3,3,1)$ are graphical.
7. Show that if G is simple and connected but not complete, then G has three vertices u, v and w such that $uv, vw \in E(G)$, but $uw \notin E$.
8. Show that every induced subgraph of a complete graph K_n is also a complete subgraph.
9. If G is an r-regular graph, then show that r divides the size of G.
10. Show that every subgraph of a bipartite graph is bipartite.
11. If G is an r-regular graph and r is odd, then show that $\frac{\varepsilon}{r}$ is an even integer.
12. Let G be a graph in which there is no pair of adjacent edges. What can you say about the degree of the vertices in G?
13. Check whether the following pairs of graphs are isomorphic? Justify your answer.



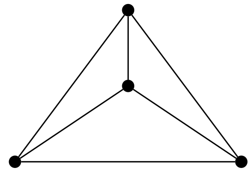
(a) G



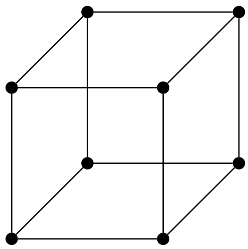
(b) H



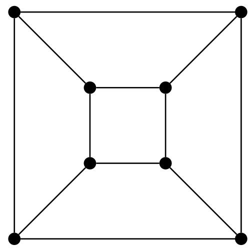
(a) G



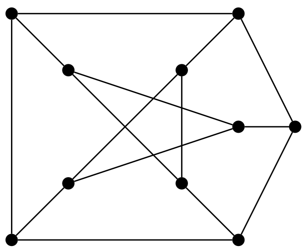
(b) H



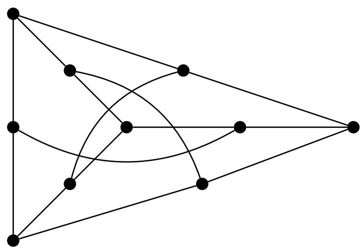
(a) G



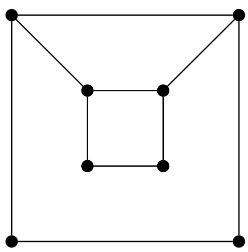
(b) H



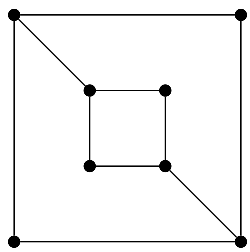
(a) G



(b) H



(a) G

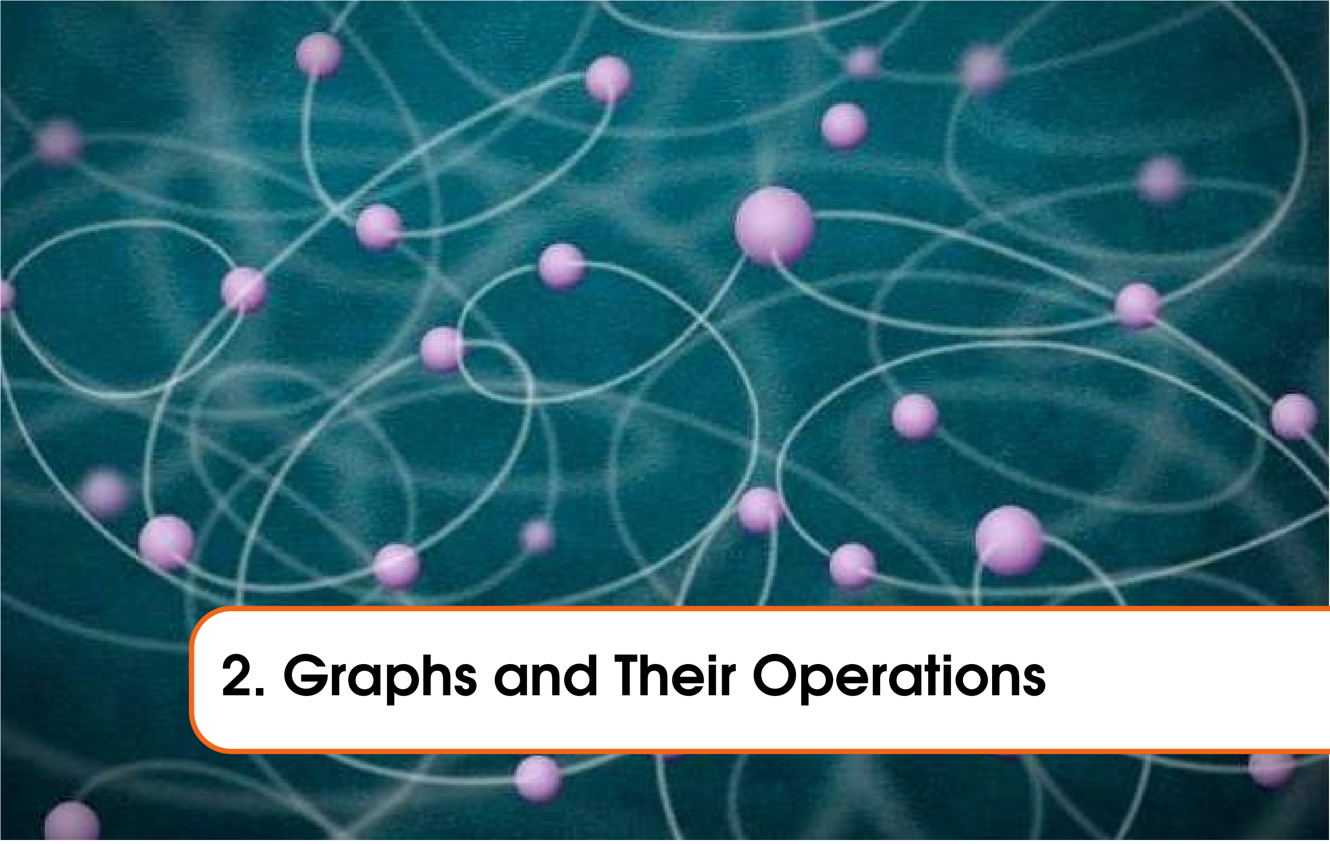


(b) H

14. Let G be a graph with n vertices and e edges and let m be the smallest positive integer such that $m \geq \frac{2e}{n}$. Prove that G has a vertex of degree at least m.
15. Prove that it is impossible to have a group of nine people at a party such that each one knows exactly five of the others in the group.
16. Let G be a graph with n vertices, t of which have degree k and the others have degree $k+1$. Prove that $t = (k+1)n - 2e$, where e is the number of edges in G.
17. Let G be a k-regular graph, where k is an odd number. Prove that the number of edges in G is a multiple of k.
18. Let G be a graph with n vertices and exactly $n-1$ edges. Prove that G has either a

vertex of degree 1 or an isolated vertex.

19. What is the smallest integer n such that the complete K_n has at least 500 edges?
20. Prove that there is no simple graph with six vertices, one of which has degree 2, two have degree 3, three have degree 4 and the remaining vertex has degree 5.
21. Prove that there is no simple graph on four vertices, three of which have degree 3 and the remaining vertex has degree 1.
22. Let G be a simple regular graph with n vertices and 24 edges. Find all possible values of n and give examples of G in each case.



2. Graphs and Their Operations

We have already seen that the notion of subgraphs can be defined for any graphs as similar to the definition of subsets to sets under consideration. Similar to the definitions of basic set operations, we can define the corresponding basic operations for graphs also. In addition to these fundamental graph operations, there are some other new and useful operations are also defined on graphs. In this chapter, we discuss some basic graph operation.

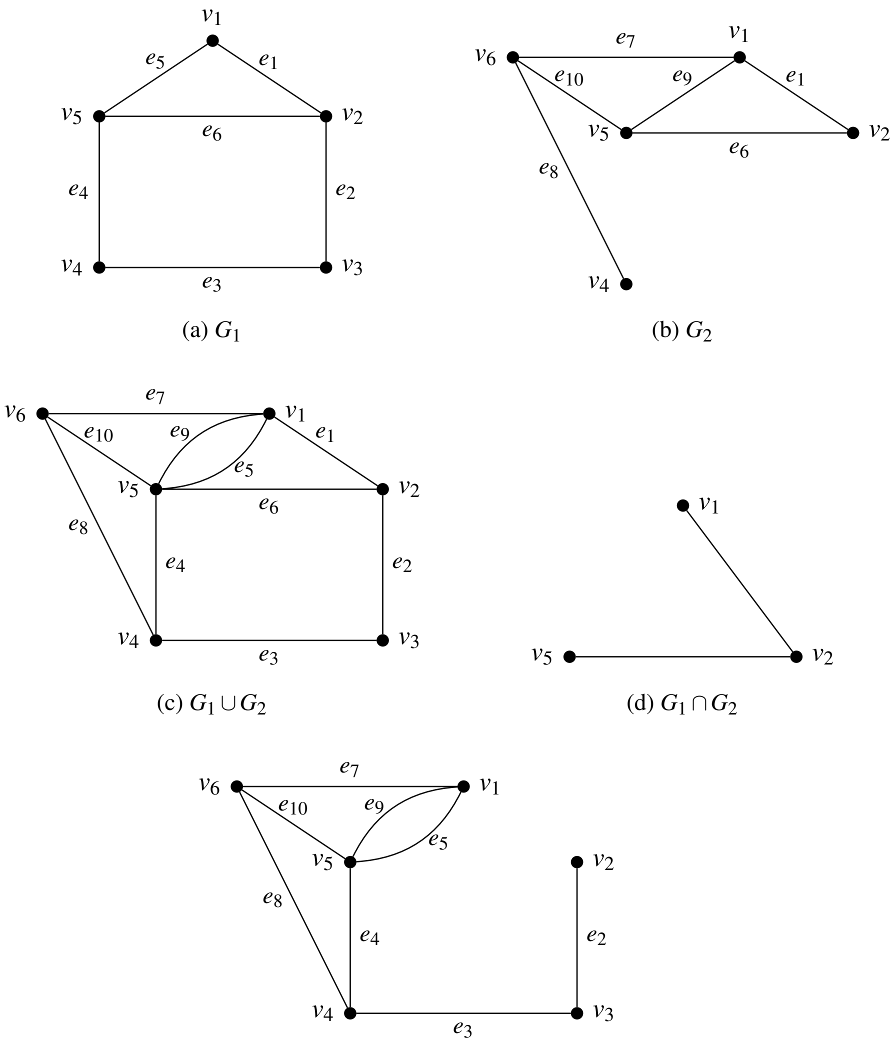
2.1 Union, Intersection and Ringsum of Graphs

Definition 2.1.1 — Union of Graphs. The *union* of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \cup G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

Definition 2.1.2 — Intersection of Graphs. The *intersection* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \cap G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Definition 2.1.3 — Ringsum of Graphs. The *ringsum* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \oplus G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \oplus E(G_2)$, where $\oplus$ is the symmetric difference (XOR Operation) of two sets.

Figure 2.1 illustrates the union, intersection and ringsum of two given graphs.



(a) G_1 (b) G_2

(c) $G_1 \cup G_2$ (d) $G_1 \cap G_2$

(e) $G_1 \oplus G_2$

Figure 2.1: Illustrations to graph operations

R

1. The union, intersection and ringsum operations of graphs are commutative. That is, $G_1 \cup G_2 = G_2 \cup G_1, G_1 \cap G_2 = G_2 \cap G_1$ and $G_1 \oplus G_2 = G_2 \oplus G_1$.
2. If G_1 and G_2 are edge-disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$.
3. If G_1 and G_2 are vertex-disjoint, then $G_1 \oplus G_2$ is empty.
4. For any graph G, $G \cap G = G \cup G$ and $G \oplus G$ is a null graph.

Definition 2.1.4 — Decomposition of a Graph. A graph G is said to be *decomposed* into two subgraphs G_1 and G_2 , if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a null graph.

2.2 Complement of Graphs

Definition 2.2.1 — Complement of Graphs. The *complement* or *inverse* of a graph G, denoted by $\bar{G}$ is a graph with $V(G) = V(\bar{G})$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

R Note that for a graph G and its complement $\bar{G}$, we have

(i) $G_1 \cup \bar{G} = K_n$;
(ii) $V(G) = V(\bar{G})$;
(iii) $E(G) \cup E(\bar{G}) = E(K_n)$;
(iv) $|E(G)| + |E(\bar{G})| = |E(K_n)| = \binom{n}{2}$.

A graph and its complement are illustrated below.

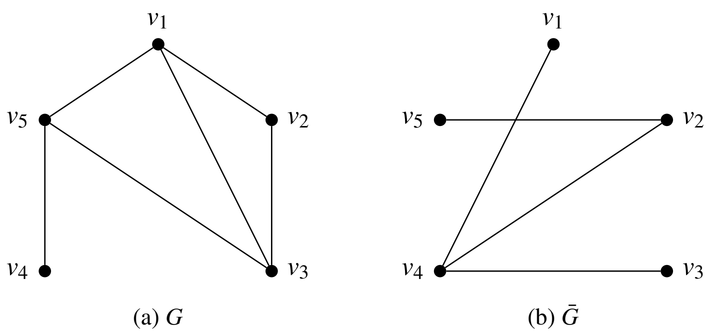


Figure 2.2: A graph and its complement

2.2.1 Self-Complementary Graphs

Definition 2.2.2 — Self-Complementary Graphs. A graph G is said to be *self-complementary* if G is isomorphic to its complement. If G is self complementary, then $|E(G)| = |E(\bar{G})| = \frac{1}{2}|E(K_n)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

The following are two examples of self complementary graphs.

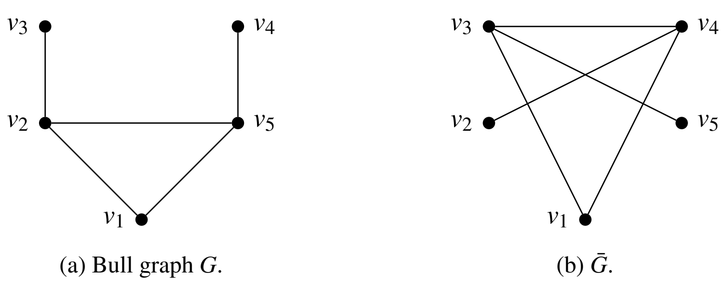
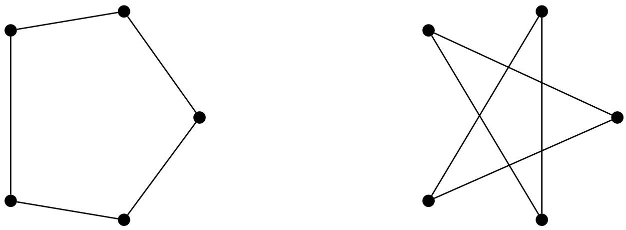


Figure 2.3: Example of self-complementary graphs



(a) G_2 (b) G_2.

Figure 2.4: Example of self-complementary graphs

Problem 2.1 For any self-complementary graph G of order n, show that $n \equiv 0, 1 (\text{mod } 4)$.
Solution: For self-complementary graphs, we have

(i) $V(G) = V(\bar{G})$;
(ii) $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$;
(iii) $|E(G)| = |E(\bar{G})|$.

Therefore, $|E(G)| = |E(\bar{G})| = \frac{n(n-1)}{4}$. This implies, 4 divides either n or $n-1$. That is, for self-complementary graphs of order n, we have $n \equiv 0, 1 (\text{mod } 4)$. ■

(Note that we say $a \equiv b (\text{mod } n)$, which is read as "$a$ is congruent to b modulo n", if $a-b$ is completely divisible by n).

2.3 Join of Graphs

Definition 2.3.1 The *join* of two graphs G and H, denoted by $G+H$ is the graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

In other words, the join of two graphs G and H is defined as the graph in which every edge of the first graph is adjacent to all vertices of the second graph.

Figure 2.5 illustrates the join of two graphs P_3 and P_4 and Figure 2.6 illustrates the join of two graphs C_5 and P_2.

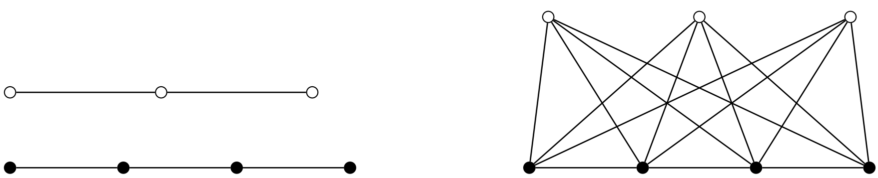
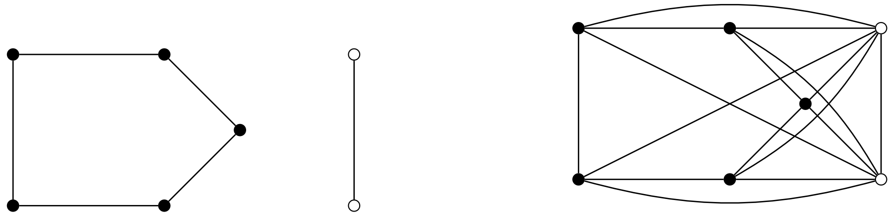


Figure 2.5: The join of the paths P_4 and P_3.

2.4 Deletion and Fusion

Definition 2.4.1 — Edge Deletion in Graphs. If e is an edge of G, then $G-e$ is the graph obtained by removing the edge of G. The subgraph of G thus obtained is called an *edge-deleted subgraph* of G. Clearly, $G-e$ is a spanning subgraph of G.



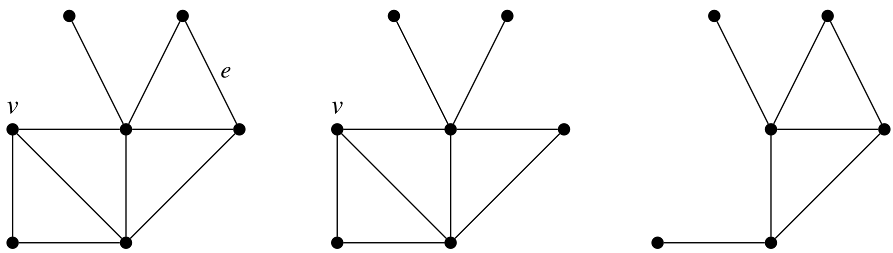
(a) C_5 (b) P_2 (c) $C_5 + P_2$.

Figure 2.6: The join of the cycle C_5 and the path P_2.

Similarly, vertex-deleted subgraph of a graph is defined as follows:

Definition 2.4.2 — Vertex Deletion in Graphs. If v is a vertex of G, then $G - v$ is the graph obtained by removing the vertex v and all edges G that are incident on v. The subgraph of G thus obtained is called an *vertex-deleted subgraph* of G. Clearly, $G - v$ will not be a spanning subgraph of G.

Figure 2.7 illustrates the edge deletion and the vertex deletion of a graph G.



(a) G (b) $G - e$ (c) $G - v$

Figure 2.7: Illustrations to edge deletion and vertex deletion

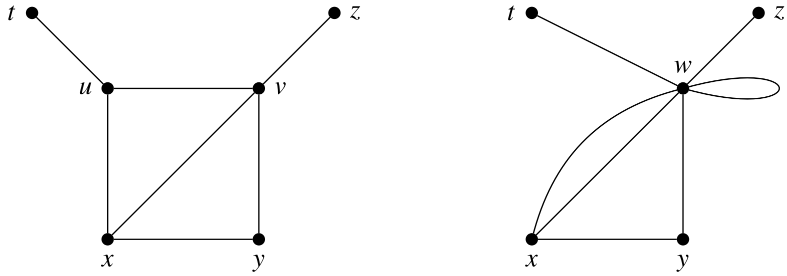
Definition 2.4.3 — Fusion of Vertices. A pair of vertices u and v are said to be *fused* (or *merged* or *identified*) together if the two vertices are together replaced by a single vertex w such that every edge incident with either u or v is incident with the new vertex w (see Figure 2.8).

Note that the fusion of two vertices does not alter the number of edges, but reduces the number of vertices by 1.

2.4.1 Edge Contraction

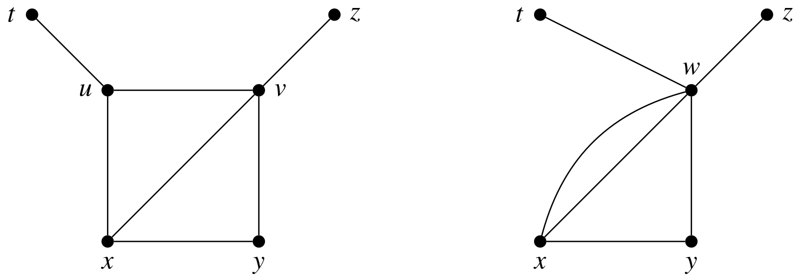
Definition 2.4.4 — Edge Contraction in Graphs. An *edge contraction* of a graph G is an operation which removes an edge from a graph while simultaneously merging its two end vertices that it previously joined. Vertex fusion is a less restrictive form of this operation.

A graph obtained by contracting an edge e of a graph G is denoted by $G \circ e$.



(a) G (b) $G-e$

Figure 2.8: Illustrations to fusion of two vertices



(a) G (b) $G \circ uv$

Figure 2.9: Illustrations to edge contraction of a graph.

2.5 Subdivision and Smoothing

2.5.1 Subdivision of a Graph

Definition 2.5.1 — Subdivision of an Edge. Let $e = uv$ be an arbitrary edge in G. The *subdivision* of the edge e yields a path of length 2 with end vertices u and v with a new internal vertex w (That is, the edge $e = uv$ is replaced by two new edges, uw and wv).

(a) G (b) G

Figure 2.10: Subdivision of an edge

Definition 2.5.2 — Subdivision of a Graph. A *subdivision* of a graph G (also known as an *expansion* of G) is a graph resulting from the subdivision of (some or all) edges in G (see 2.11). The newly introduced vertices in the subdivisions are represented by white vertices.

Definition 2.5.3 — Homeomorphic Graphs. Two graphs are said to be *homeomorphic* if both can be obtained by the same graph by subdivisions of edges.

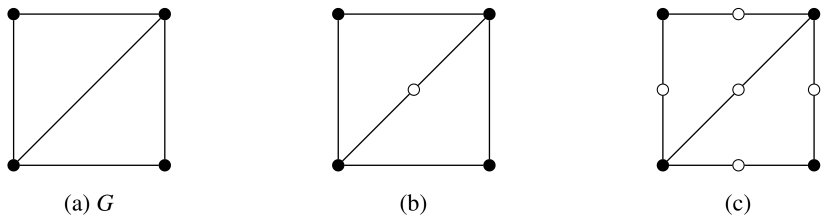


Figure 2.11: Illustrations to Subdivision of graphs

In Figure 2.11, the second and third graphs are homeomorphic, as they are obtained by subdividing the edges of the first graph in the figure.

2.5.2 Smoothing a Vertex

Definition 2.5.4 — Smoothing Vertices in Graphs. The reverse operation, *smoothing out* or *smoothing* a vertex w of degree 2 with regards to the pair of edges (e_i, e_j) incident on w, removes w and replaces the pair of edges (e_i, e_j) containing w with a new edge e that connects the other endpoints of the pair (e_i, e_j) (see the illustration).

Figure 2.12: Smoothing of the vertex w

Smoothing of a vertex of a graph G is also called an elementary transformation of G.

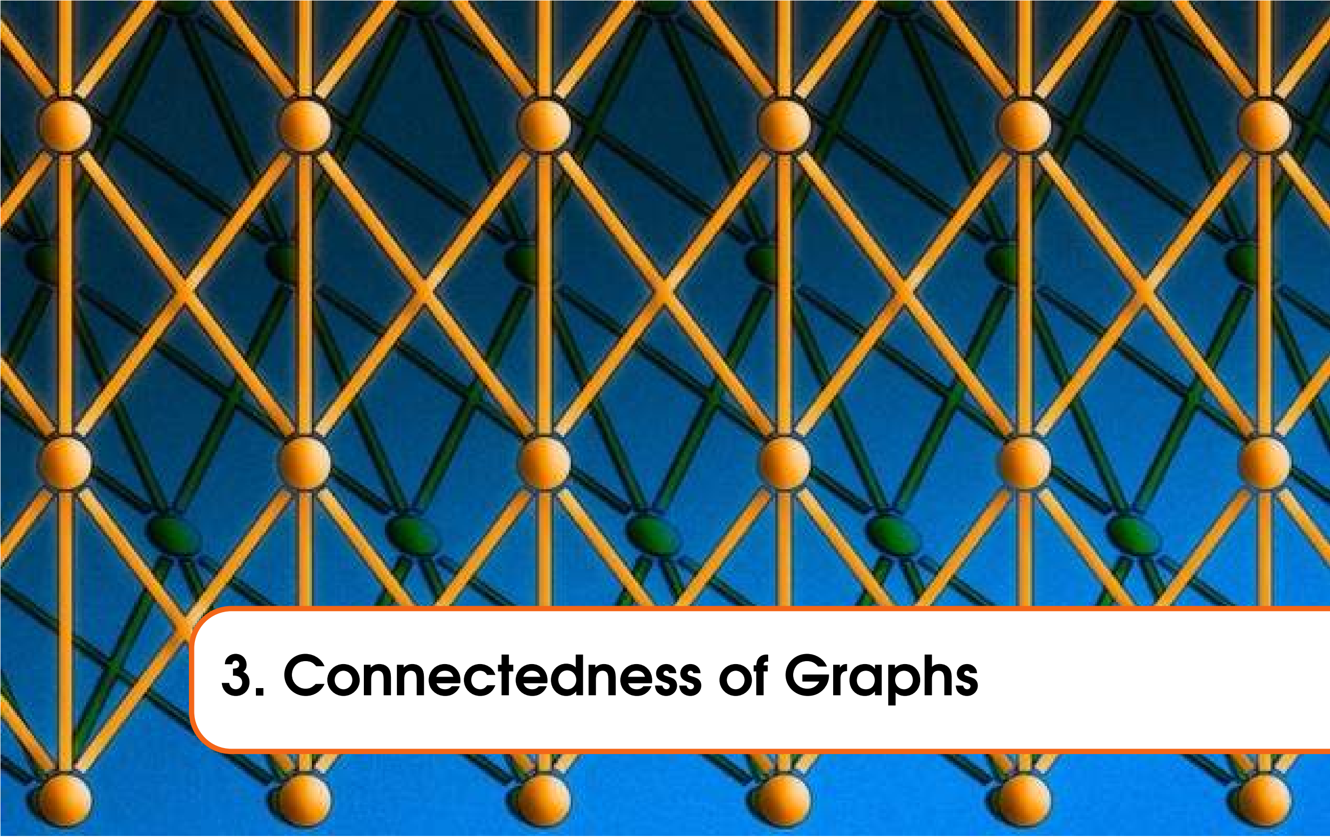
Problem 2.2 Show that a graph obtained by subdividing all edges of a graph G is a bipartite graph.

Solution: Let H be the subdivision of G. Let $X = V(G)$ and Y be the newly introduced vertices during subdivision. Clearly, $X \cup Y = V(H)$. Note that adjacency is not defined among the vertices of Y. When we subdivide an edge uv in G, then the edge uv will be removed and hence u and v becomes non-adjacent in H. Therefore, no two vertices in X can be adjacent in H. Thus, H is bipartite. ■

2.6 Exercises

1. Show that the complement of a complete bipartite graph is the disjoint union of two complete graphs.
2. The isomorphic image of a graph walk W is a walk of the same length.
3. For any graphs G and H, the ringsum $G \oplus H$ is empty if and only if $E(G) = E(H)$.
4. Show that the ringsum of two edge-disjoint collections of circuits is a collection of circuits.
5. For any graph G with six vertices, then G or its complement $\bar{G}$ contains a triangle.

6. Every graph G contains a bipartite spanning subgraph whose size is at least half the size of G.
7. Any graph G has a regular supergraph H of degree $\Delta(G)$ such that G is an induced subgraph of H.
8. how that if a self-complementary graph contains pendent vertex, then it must have at least another pendent vertex.
9. Draw all the non-isomorphic self complementary graphs on four vertices.
10. Prove that a graph with n vertices $(n > 2)$ cannot be bipartite if it has more than $\frac{n^2}{4}$ edges.
11. Verify whether the join of two bipartite graphs is bipartite. Justify your answer.
12. What is the order and size of the join of two graphs?
13. Does the join of two graphs hold commutativity? Illustrate with examples.



3. Connectedness of Graphs

3.1 Paths, Cycles and Distances in Graphs

Definition 3.1.1 — Walks. A *walk* in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a *walk* is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a *closed walk*.

A walk can end on the same vertex on which it began or on a different vertex. A walk can travel over any edge and any vertex any number of times.

Definition 3.1.2 — Trails and Tours. A *trail* is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A *tour* is a trail that begins and ends on the same vertex.

Definition 3.1.3 — Paths and Cycles. A *path* is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A *cycle* or a *circuit* is a path that begins and ends on the same vertex.

Definition 3.1.4 — Length of Paths and Cycles. The *length* of a walk or circuit or path or cycle is the number of edges in it.

A path of order n is denoted by P_n and a cycle of order n is denoted by C_n. Every edge of G can be considered as a path of length 1. Note that the length of a path on n vertices is $n-1$.

A cycle having odd length is usually called an *odd cycle* and a cycle having even length is called an *even cycle*.

Definition 3.1.5 — Distance between two vertices. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$ or simply $d(u,v)$, is the length (number of edges) of a *shortest path* (also called a *graph geodesic*) connecting them. This distance is also known as the *geodesic distance*.

Definition 3.1.6 — Eccentricity of a Vertex. The *eccentricity* of a vertex v, denoted by $\varepsilon(v)$, is the greatest geodesic distance between v and any other vertex. It can be thought of as how far a vertex is from the vertex most distant from it in the graph.

Definition 3.1.7 — Radius of a Graph. The *radius* r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = \min\limits_{v\in V(G)} \varepsilon(v)$.

Definition 3.1.8 — Diameter of a Graph. The *diameter* of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = \max\limits_{v\in V(G)} \varepsilon(v)$.

Here, note that the diameter of a graph need not be twice its radius unlike in geometry. We can even see many graphs having same radius and diameter. Complete graphs are examples of the graphs with radius equals to diameter.

Definition 3.1.9 — Center of a Graph. A *center* of a graph G is a vertex of G whose eccentricity equal to the radius of G.

Definition 3.1.10 — Peripheral Vertex of a Graph. A *peripheral vertex* in a graph of diameter d is one that is distance d from some other vertex. That is, a peripheral vertex is a a vertex that achieves the diameter. More formally, a vertex v of G is peripheral vertex of a graph G, if $\varepsilon(v) = d$.

For a general graph, there may be several centers and a center is not necessarily on a diameter.

The distances between vertices in the above graph are given in Table 3.1. Note that a vertex v_i is represented by i in the table (to save the space).

Note that the radius of G is given by $r(G) = \min\{\varepsilon(v)\} = 4$ and the diameter of G is given by $diam(G) = \max\{\varepsilon(v)\} = 6$ and all eight central vertices are represented by white vertices in Figure 3.1.

Definition 3.1.11 — Geodetic Graph. A graph in which any two vertices are connected by a unique shortest path is called a *geodetic graph*.

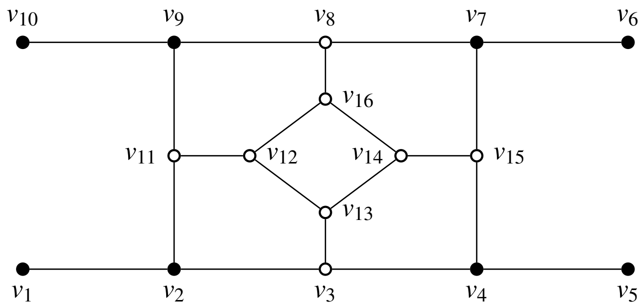


Figure 3.1: A graph with eight centers.

v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ε
1	0	1	2	3	4	6	5	4	3	4	2	3	3	4	4	4	6
2	1	0	1	2	3	5	4	3	2	3	1	2	2	3	3	3	5
3	2	1	0	1	2	4	3	4	3	4	2	2	1	1	2	3	4
4	3	2	1	0	1	3	2	3	4	5	5	3	2	2	1	3	5
5	4	3	2	1	0	4	3	4	5	6	4	4	3	3	2	4	6
6	6	5	4	3	4	0	1	2	3	4	4	4	4	3	2	3	6
7	5	4	3	2	3	1	0	1	2	3	3	3	3	2	1	2	5
8	4	3	4	3	4	2	1	0	1	2	2	2	3	2	2	1	4
9	3	2	3	4	5	3	2	1	0	1	1	2	3	3	3	2	5
10	4	3	4	5	6	4	3	2	1	0	2	3	4	4	4	3	6
11	2	1	2	3	4	4	3	2	1	1	0	1	2	3	4	2	4
12	3	2	2	3	4	4	2	1	2	3	1	0	1	2	3	1	4
13	3	2	1	2	3	4	3	3	3	4	2	1	0	1	2	2	4
14	4	3	2	2	3	3	2	2	3	4	3	2	1	0	1	1	4
15	4	3	2	1	2	2	1	2	3	4	4	3	2	1	0	2	4
16	4	3	3	3	4	3	2	1	2	3	2	3	2	1	2	0	4

Table 3.1: Eccentricities of vertices of the graph in Figure 3.1.

Theorem 3.1.1 If G is a simple graph with $diam(G) \geq 3$, then $diam(\bar{G}) \leq 3$.

Proof. If $diam(G) \geq 3$, then there exist at least two non-adjacent vertices u and v in G such that u and v have no common neighbours in G. Hence, every vertex x in $G - \{u,v\}$ is non-adjacent to u or v or both in G. This makes x adjacent to u or v or both in $\bar{G}$. Moreover, $uv \in E(\bar{G})$. So, for every pair of vertices x,y, there is an x,y path of length at most 3 in $\bar{G}$ through the edge uv. Hence, $diam(\bar{G}) \leq 3$. ∎

3.2 Connected Graphs

Definition 3.2.1 — Connectedness in a Graph. Two vertices u and v are said to be *connected* if there exists a path between them. If there is a path between two vertices u and v, then u is said to be *reachable* from v and vice versa. A graph G is said to be *connected* if there exist paths between any two vertices in G.

Definition 3.2.2 — Component of a Graph. A connected *component* or simply, a *component* of a graph G is a maximal connected subgraph of G.

Each vertex belongs to exactly one connected component, as does each edge. A connected graph has only one component.

A graph having more than one component is a *disconnected graph* (In other words, a disconnected graph is a graph which is not connected). The number of components of a graph G is denoted by $\omega(G)$.

In view of the above notions, the following theorem characterises bipartite graphs.

Theorem 3.2.1 A connected graph G is bipartite if and only if G has no odd cycles.

Proof. Suppose that G is a bipartite graph with bipartition (X,Y). Assume for contradiction that there exists a cycle $v_1, v_2, v_3, \ldots\ldots, v_k, v_1$ in G with k odd. Without loss of generality, we may additionally assume that $v_1 \in X$. Since G is bipartite, $v_2 \in Y$, $v_3 \in X$, $v_4 \in Y$ and so on. That is, $v_i \in X$ for odd values of i and $v_i \in Y$ for even values of i. Therefore, $v_k \in X$. But, then the edge $v_k, v_1 \in E$ is an edge with both endpoints in X, which contradicts the fact that G is bipartite. Hence, a bipartite graph G has no odd cycles.

Conversely, assume that G is a graph with no odd cycles. Let $d(u,v)$ denote the distance between two vertices u and v in G. Pick an arbitrary vertex $u \in V$ and define $X = \{x \in V(G) : d(x,u) \text{ is even}\}$. Clearly, $u \in X$ as $d(u,u) = 0$. Now, define another $Y = \{y \in V(G) : d(u,y) \text{ is odd}\}$. That is, $Y = V - X$. If possible, assume that there exists an edge $vw \in E(G)$ such that $v,w \in X$ (or $v,w \in Y$). Then, by construction $d(u,v)$ and $d(u,w)$ are both even (or odd). Let $P(u,w)$ and $P(u,v)$ be the shortest paths connecting u to w, and u to v respectively. Then, the cycle given by $P(u,w) \cup \{wv\} \cup P(v,u)$ has odd length $1 + d(u,w) + d(u,v)$, which is a contradiction. Therefore, no such edge wv may exist and G is bipartite. ■

Theorem 3.2.2 A graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in the subset V_2.

Proof. Suppose that such a partitioning exists. Consider two arbitrary vertices u and v of G, such that $u \in V_1$ and $v \in V_2$. No path can exist between vertices u and v; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, assume that G is a disconnected graph. Consider a vertex u in G. Let V_1 be the set of all vertices that are joined by paths to u. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any vertex in V_2 by an edge. Hence, we get the required partition. ■

Theorem 3.2.3 If a graph has exactly has two vertices of odd degree, then there exists a path joining these two vertices.

Proof. Let G be a graph with two vertices v_1 and v_2 of odd degree and all other vertices of even degree. Then, by Theorem 1.2.3, both of them should lie in the same component of G. Since every component of G must be connected, there must be a path between v_1 and v_2. ■

Theorem 3.2.4 Let G be a graph with n vertices and k components. Then, G has at most $\frac{1}{2}(n-k)(n-k+1)$ edges.

Proof. Let G be a graph with n vertices and k components. Let the number of vertices in each of the k components of G be $n_1, n_2, \ldots, n_k$ respectively. Then we have,

$$n_1 + n_2 + \ldots + n_k = n; \; n_i \geq 1 \tag{3.1}$$

First, note that any connected graph on n vertices must have at least $n-1$ edges. The proof of the theorem is based on the inequality $\sum\limits_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$, which can be proved as follows.

$$\begin{aligned}
\sum_{i=1}^{k}(n_i - 1) &= n - k \\
\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 &= (n-k)^2 \\
\sum_{i=1}^{k}(n_i^2 - 2n_i) + k + \textit{non-negative cross terms} &= n^2 + k^2 - 2nk \\
\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k &\leq n^2 + k^2 - 2nk \\
\sum_{i=1}^{k} n_i^2 - 2n + k &\leq n^2 + k^2 - 2nk \\
\sum_{i=1}^{k} n_i^2 &\leq n^2 + k^2 - 2nk + 2n - k \\
\therefore \sum_{i=1}^{k} n_i^2 &\leq n^2 - (k-1)(2n-k).
\end{aligned}$$

Hence, we have

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \tag{3.2}$$

Now, note that the number edges in K_n is $\frac{n(n-1)}{2}$. Hence, the maximum number of edges in i-th component of G (which is a simple connected graph) is $\frac{n_i(n_i-1)}{2}$. Therefore, the maximum number of edges in G is

$$\begin{aligned}
\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} &= \sum_{i=1}^{k} \frac{n_i^2-n_i}{2} \\
&= \frac{1}{2}\sum_{i=1}^{k}(n_i^2-n_i) \\
&= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] \\
&\leq \frac{1}{2}\left[n^2-(k-1)(2n-k)-n\right] \text{ (By Eq. (3.1) and Ineq. (3.2))} \\
&= \frac{1}{2}\left[n^2-2nk+k^2+2n-k-n\right] \\
&= \frac{1}{2}\left[n^2-2nk+k^2+n-k\right] \\
&= \frac{1}{2}\left[(n-k)^2+(n-k)\right] \\
&= \frac{1}{2}(n-k)(n-k+1).
\end{aligned}$$

■

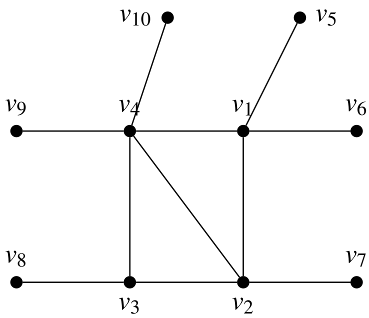
Problem 3.1 Show that an acyclic graph on n vertices and k components has $n-k$ edges.
Solution. The solution follows directly from the first part of the above theorem.
Problem 3.2 Show that every graph on n vertices having more than $\frac{(n-1)(n-2)}{2}$ edges is connected.
Solution. Consider the complete graph K_n and v be an arbitrary vertex of K_n. Now remove all $n-1$ edges of K_n incident on v so that it becomes disconnected with K_{n-1} as one component and the isolated vertex v as the second component. Clearly, this disconnected graph has $\frac{(n-1)(n-2)}{2}$ edges (all of which belong to the first component). Since all pairs of vertices in the first component K_{n-1} are any adjacent to each other, any new edge drawn must be joining a vertex in K_{n-1} and the isolated vertex v, making the revised graph connected. ■

3.3 Edge Deleted and Vertex Deleted Subgraphs

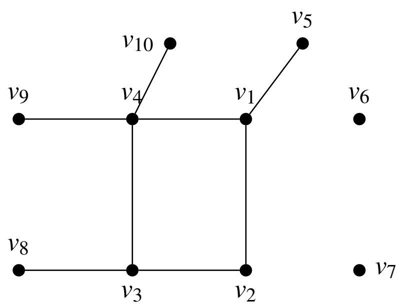
Definition 3.3.1 — Edge Deleted Subgraphs. Let $G(V,E)$ be a graph and $F \subseteq E$ be a set of edges of G. Then, the graph obtained by deleting F from G, denoted by $G-F$, is the subgraph of G obtained from G by removing all edges in F. Note that $V(G-F) = V(G)$. That is, $G-F = (V, E-F)$.

Note that any edge deleted subgraph of a graph G is a spanning subgraph of G.

Definition 3.3.2 — Vertex Deleted Subgraphs. Let $W \subseteq V(G)$ be a set of vertices of G. Then the graph obtained by deleting W from G, denoted by $G-W$, is the subgraph of G



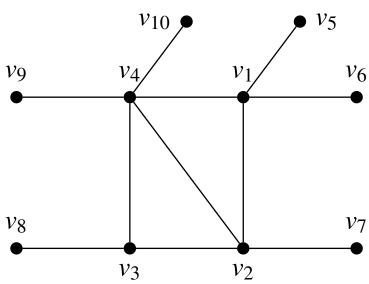
(a) G



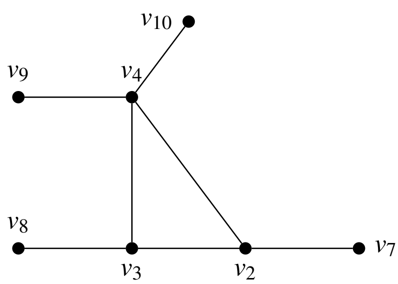
(b) $G' = G - \{v_1v_6, v_2v_4, v_2v_7\}$.

Figure 3.2: A graph and its edge deleted subgraph.

obtained from G by removing all vertices in W and all edges incident to those vertices. See Figure 3.3 for illustration of a vertex-deleted subgraph of a given graph.



(a) G



(b) $G' = G - \{v_1, v_5, v_6\}$.

Figure 3.3: A graph and its edge deleted subgraph.

Cut-Edges and Cut-Vertices

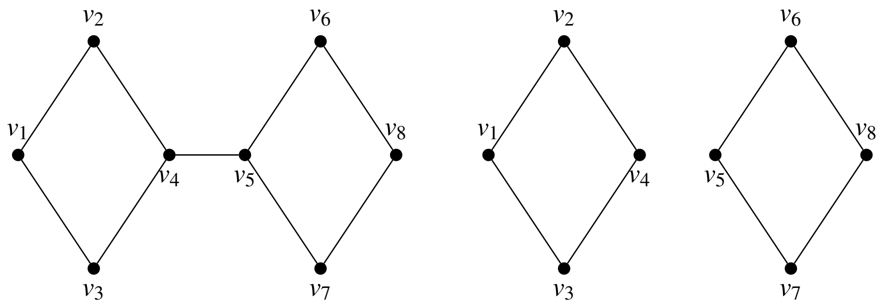
Definition 3.3.3 — Cut-Edge. An edge e of a graph G is said to be a *cut-edge* or a *bridge* of G if $G - e$ is disconnected.

In the above graph G, the edge v_4v_5 is a cut-edge, since $G - v_4v_5$ is a disconnected graph.

The following is a necessary and sufficient condition for an edge of a graph G to be a cut edge of G.

Theorem 3.3.1 An edge e of a graph G is a cut-edge of G if and only if it is not contained in any cycle of G.

Proof. Let $e = uv$ be a cut edge of G. Then, the vertices u and v must be in different components of $G - e$. If possible, let e is contained in cycle C in G. Then, $C - e$ is a path between u and v in $G - e$, a contradiction to the fact that u and v are in different components of $G - e$. Therefore, e can not be in any cycle of G.

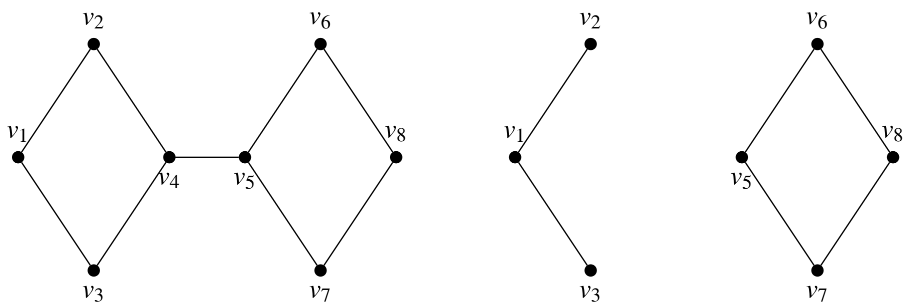


(a) Connected graph G (b) $G \circ uv$

Figure 3.4: Disconnected graph $G - v_4v_5$

Conversely, assume that e is not in any cycle of G. Then, there is no (u, v)-path other than e. Therefore, u and v are in different components of $G - e$. That is, $G - e$ is disconnected and hence e is a cut-edge of G. ■

Definition 3.3.4 — Cut-Vertex. A vertex v of a graph G is said to be a *cut-vertex* of G if $G - v$ is disconnected.



(a) Connected graph G (b) $G \circ uv$

Figure 3.5: disconnected graph $G - v_4$

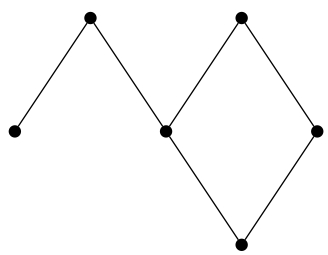
In graph G, v_4 is a cut-vertex as $G - v_4$ is a disconnected graph. Similarly, v_5 is also a cut-vertex of G.

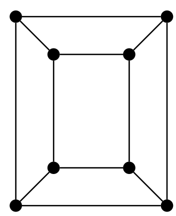
Since removal of any pendent vertex will not disconnect a given graph, every cut-vertex will have degree greater than or equal to 2. But, note that every vertex v, with $d(v) \geq 2$ need not be a cut-vertex.

3.4 Exercises

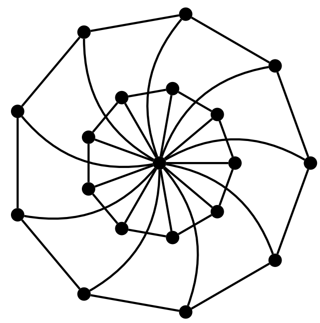
1. Show that every uv-walk contains a uv-path.
2. Show that every closed walk contains a cycle.

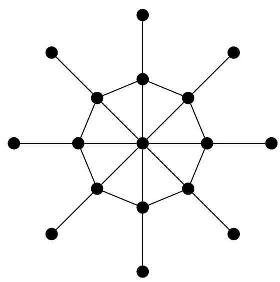
3. Show that every graph with n vertices and k edges, $n > k$ has $n-k$ components.
4. If every vertex of a graph G has degree greater than or equal to 2, then G has some cycles.
5. If G is a simple graph with $d(v) \geq k$, $\forall\, v \in V(G)$, then G contains a path of length at least k. If $k \geq 2$, then G contains a cycle of length $k+1$.
6. Show that if G is simple and $\delta(G) \geq k$, then G has a path of length k.
7. If a connected graph G is decomposed into two subgraphs G_1 and G_2, then show that there must be at least one common vertex between G_1 and G_2.
8. If we remove an edge e from a graph G and $G-e$ is still connected, then show that e lies along some cycle of G.
9. If the intersection of two paths is a disconnected graph, then show that the union of the two paths has at least one circuit.
10. If P_1 and P_2 are two different paths between two given vertices of a graph G, then show that $P_1 \oplus P_2$ is a circuit or a set of circuits in G.
11. Show that the complement of a complete bipartite graph is the disjoint union of two complete graphs.
12. For a simple graph G, with n vertices, if $\delta(G) = \frac{n-1}{2}$, then G is connected.
13. Show that any two longest paths in a connected graph have a vertex in common.
14. For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.
15. Determine the maximum number of edges in a bipartite subgraph of the Petersen graph.
16. If H is a subgraph of G, then show that $d_G(u,v) \leq d_H(u,v)$.
17. Prove that if a connected graph G has equal order and size, then G is a cycle.
18. Show that eccentricities of adjacent vertices differ by at most 1.
19. Prove that if a graph has more edges than vertices then it must possess at least one cycle.
20. If the intersection of two paths is a disconnected graph, then show that the union of the two paths has at least one cycle.
21. The radius and diameter of a graph are related as $rad(G) \leq diam(G) \leq 2r(G)$.

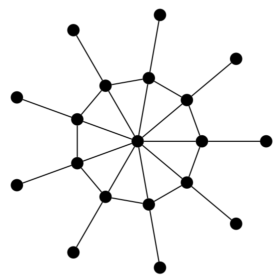


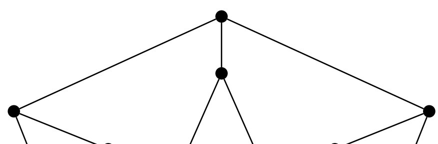


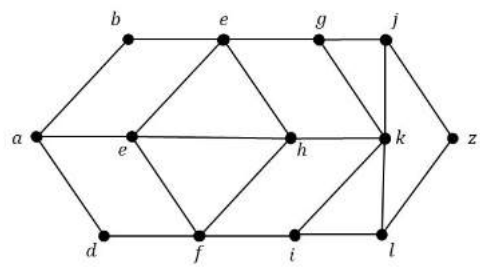
22. Find the eccentricity of the vertices and the radius, the diameter and center(s) of the following graphs:







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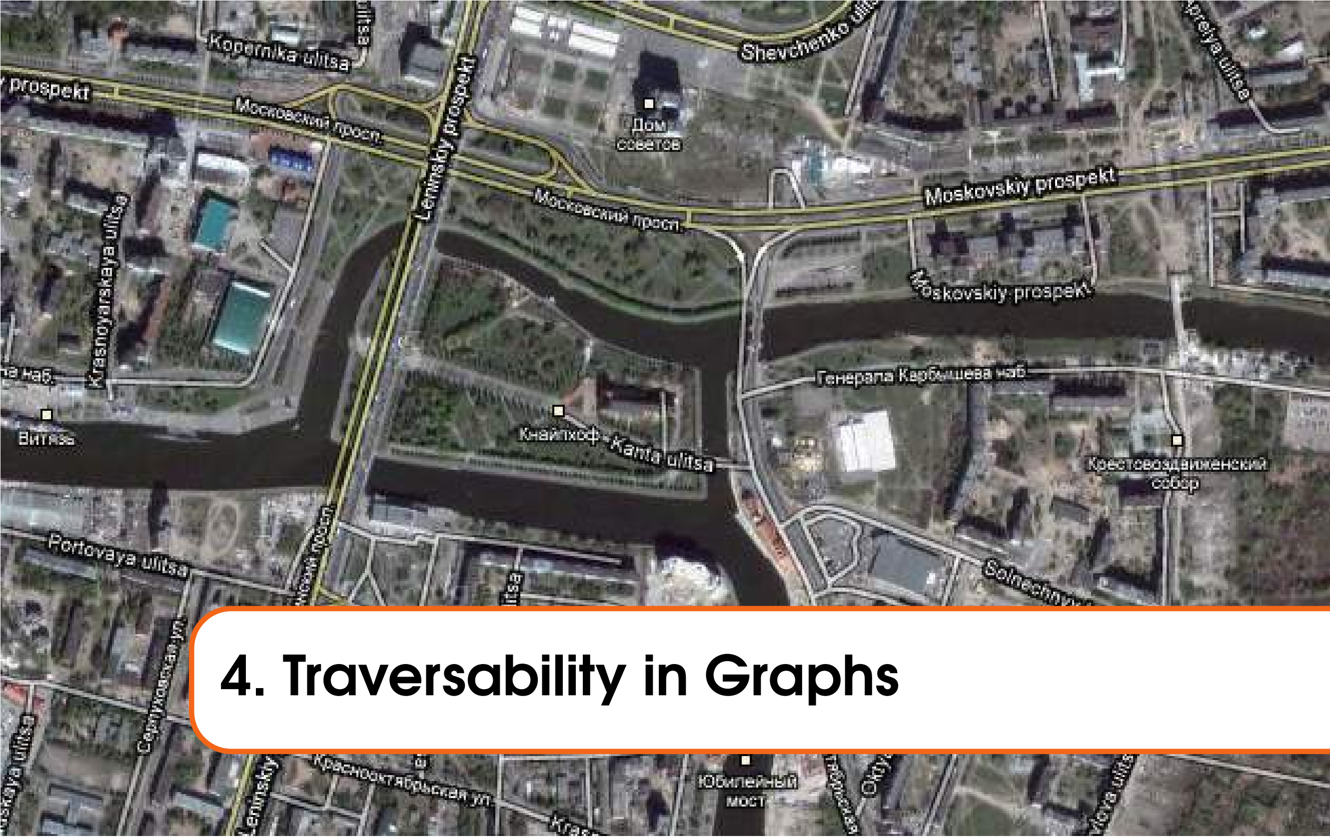
TRAVERSABILITY IN GRAPHS & DIGRAPHS



4. Traversability in Graphs

4.1 Königsberg Seven Bridge Problem

The city of *Königsberg* in Prussia (now Kaliningrad, Russia) was situated on either sides of the *Pregel River* and included two large islands which were connected to each other and the mainlands by *seven bridges* (see the below picture).

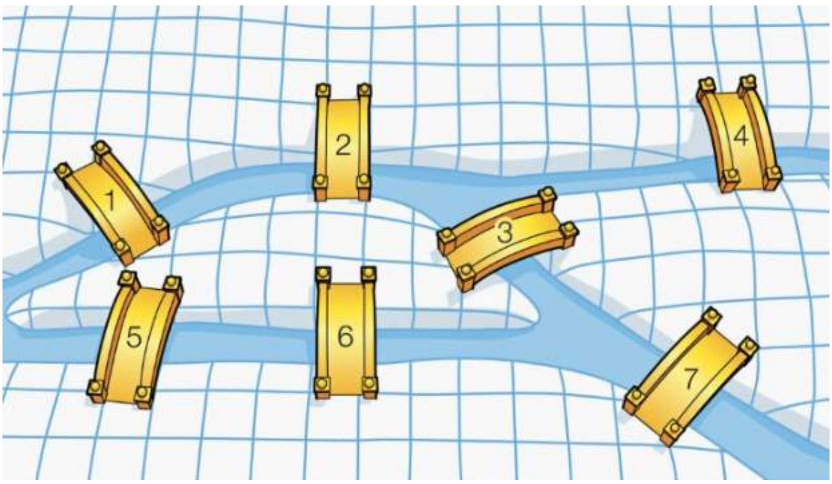


Figure 4.1: Königsberg's seven bridge problem.

The problem was to devise a walk through the city that would cross each bridge once and only once, subject to the following conditions:

(i) The islands could only be reached by the bridges;
(ii) Every bridge once accessed must be crossed to its other end;
(iii) The starting and ending points of the walk are the same.

The Königsberg seven bridge problem was instrumental to the origination of Graph Theory as a branch of modern Mathematics.

In 1736, a Swiss Mathematician *Leonard Euler* introduced a graphical model to this problem by representing each land area by a vertex and each bridge by an edge connecting corresponding vertices (see the following figure).

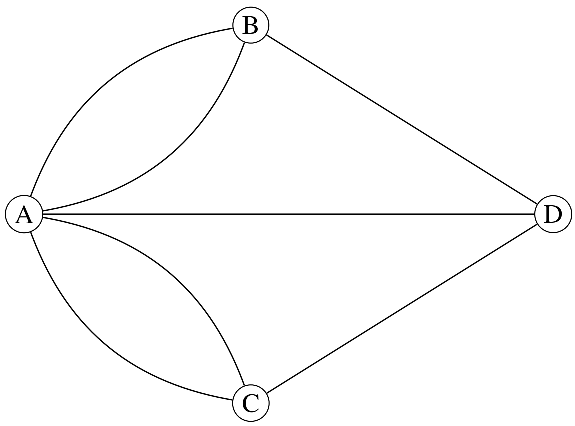


Figure 4.2: Graphical representation of seven bridge problem

Using this graphical model, Euler proved that no such walk (trail) exists.

4.2 Eulerian Graphs

Definition 4.2.1 — Traversable Graph. An *Eulerian trail* or *Euler walk* in an undirected graph is a walk that uses each edge exactly once. If an Euler trail exists in a given graph G, then G is called a *traversable graph* or a *semi-Eulerian graph*.

Definition 4.2.2 — Eulerian Graph. An *Eulerian cycle* or *Eulerian circuit* or *Euler tour* in an undirected graph is a cycle that uses each edge exactly once. If such an Euler cycle exists in the graph concerned, then the graph is called an *Eulerian graph* or a *unicursal graph*.

The following theorem characterises the class of Eulerian graphs:

Theorem 4.2.1 — Euler Theorem. A connected graph G is Eulerian if and only if every vertex in G is of even degree.

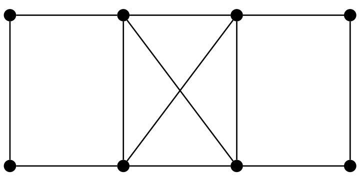
Proof. If G is Eulerian, then there is an Euler circuit, say P, in G. Every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So, every edge is accounted for and there are no repeats. Thus every degree must be even.

Conversely, let us assume that each vertex of G has even degree. We need to show that G is Eulerian. We prove the result by induction on the number of edges of G. Let us start with a vertex $v_0 \in V(G)$. As G is connected, there exists a vertex $v_1 \in V(G)$ that is adjacent to v_0. Since G is a simple graph and $d(v_i) \geq 2$, for each vertex $v_i \in V(G)$, there exists a vertex $v2 \in V(G)$, that is adjacent to v_1 with $v_2 \neq v_0$. Similarly, there exists a vertex $v_3 \in V(G)$, that is adjacent to v_2 with $v_3 \neq v_1$. Note that either $v_3 = v_0$, in which case, we have a circuit $v_0v_1v_2v_0$ or else one can proceed as above to get a vertex $v_4 \in V(G)$ and so on. As the number of vertices is finite, the process of getting a new vertex will finally end with a vertex v_i being adjacent to a vertex v_k, for some i, $0 \leq i \leq k-2$. Hence, $v_i - v_{i+1} - v_{i+2} - \ldots - v_k - v_i$ forms a circuit, say C, in G.

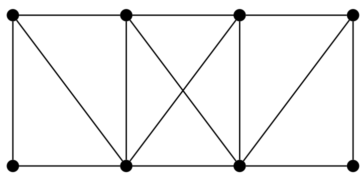
If C contains every edge of G, then C gives rise to a closed Eulerian trail and we are done. So, let us assume that $E(C)$ is a proper subset of $E(G)$. Now, consider the graph G_1 that is obtained by removing all the edges in C from G. Then, G_1 may be a disconnected graph but each vertex of G_1 still has even degree. Hence, we can do the same process explained above to G_1 also to get a closed Eulerian trail, say C_1. As each component of G_1 has at least one vertex in common with C, if C_1 contains all edges of G_1, then $C \cup C_1$ is a closed Euler trail in G. If not, let G_2 be the graph obtained by removing the edges of C_1 from G_1. (That is, $G_2 = G_1 - E(C \cup C_1)$).

Since G is a finite graph, we can proceed to find out a finite number of cycles only. Let the process of finding cycles, as explained above, ends after a finite number of steps, say r. Then, the reduced graph $G_r = G_{r-1} - E(C_{r-1}) = G - E(C \cup C_1 \cup C_{r-1})$ will be an empty graph (null graph). Then, $C \cup C_1 \cup C_2 \ldots \cup C_{r-1}$ is a closed Euler trail in G. Therefore, G is Eulerian. This completes the proof. ∎

Illustrations to an Eulerian graph and a non-Eulerian graph are given in Figure 4.3.



(a) An Eulerian graph.



(b) A non-Eulerian graph

Figure 4.3: Examples of Eulerian and non-Eulerian graphs

In the first graph in Figure 4.3, every vertex has even degree and hence by Theorem 4.2.1, it is Eulerian. In the second graph, some vertices have odd degree and hence it is not Eulerian.

Note: In an Euler graph, it can be noted that every edge of G is contained in exactly one cycle of G. Hence, we have the following Theorem.

Theorem 4.2.2 A connected graph G is Eulerian if and only if it can be decomposed into edge-disjoint cycles.

Proof. Assume that G can be decomposed into edge-disjoint cycles. Since the degree of every vertex in a cycle is 2, the degree of every vertex in G is two or multiples of 2. That is, all vertices in G are even degree vertices. Then, by Theorem 4.2.1, G is Eulerian.

Converse part is exactly the same as that of Theorem 4.2.1. ■

Theorem 4.2.3 A connected graph G is traversable if and only if it has exactly two odd degree vertices.

Proof. In a traversable graph, there must be an Euler trail. The starting vertex and terminal vertex need not be the same. Therefore, these two vertices can have odd degrees. Remaining part of the theorem is exactly as in the proof of Theorem 4.2.1. ■

4.3 Chinese Postman Problem

In his job, a postman picks up mail at the post office, delivers it, and then returns to the post office. He must, of course, cover each street in his area at least once. Subject to this condition, he wishes to choose his route in such a way that walks as little as possible. This problem is known as the Chinese postman problem, since it was first considered by a Chinese mathematician, Guan in 1960.

We refer to the street system as a weighted graph (G,w) whose vertices represent the intersections of the streets, whose edges represent the streets (one-way or two-way) and the weight represents the distance between two intersections, of course, a positive real number. A closed walk that covers each edge at least once in G is called a *postman tour*. Clearly, the Chinese postman problem is just that of finding a minimum-weight postman tour. We will refer to such a postman tour as an optimal tour.

An algorithm for finding an optimal Chinese postman route is as follows:

S-1 : List all odd vertices.
S-2 : List all possible pairings of odd vertices.
S-3 : For each pairing find the edges that connect the vertices with the minimum weight.
S-4 : Find the pairings such that the sum of the weights is minimised.
S-5 : On the original graph add the edges that have been found in Step 4.
S-6 : The length of an optimal Chinese postman route is the sum of all the edges added to the total found in Step 4.
S-7 : A route corresponding to this minimum weight can then be easily found.

■ **Example 4.1** Consider the following weighted graph:

1. The odd vertices are A and H; There is only one way of pairing these odd vertices, namely AH;
2. The shortest way of joining A to H is using the path $\{AB,BF,FH\}$, a total length of 160;

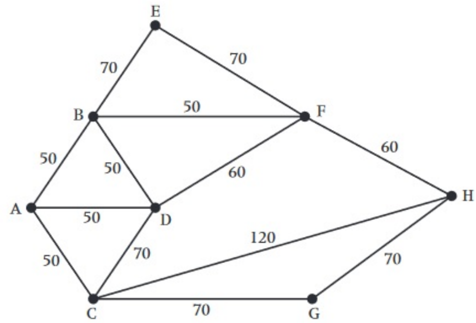


Figure 4.4: An example for Chinese Postman Problem

3. Draw these edges onto the original network.
4. The length of the optimal Chinese postman route is the sum of all the edges in the original network, which is $840m$, plus the answer found in Step 4, which is $160m$. Hence the length of the optimal Chinese postman route is $1000m$.
5. One possible route corresponding to this length is $ADCGHCABDFBEFHFBA$, but many other possible routes of the same minimum length can be found.

■

4.4 Hamiltonian Graphs

Definition 4.4.1 — Traceable Graphs. A *Hamiltonian path* (or *traceable path*) is a path in an undirected (or directed graph) that visits each vertex exactly once. A graph that contains a Hamiltonian path is called a *traceable graph*.

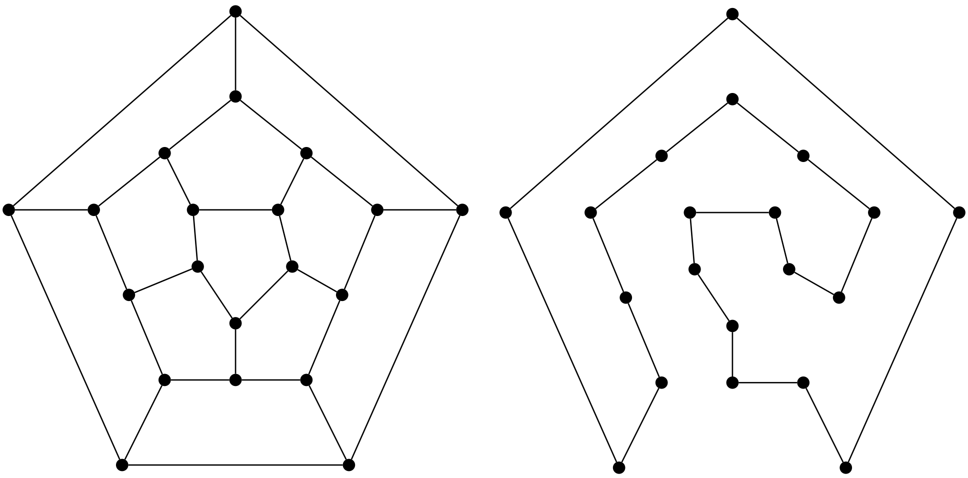
Definition 4.4.2 — Hamiltonian Graphs. A *Hamiltonian cycle*, or a *Hamiltonian circuit*, or a *vertex tour* or a *graph cycle* is a cycle that visits each vertex exactly once (except for the vertex that is both the start and end, which is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Hamiltonian graphs are named after the famous mathematician *William Rowan Hamilton* who invented the Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron.

A necessary and sufficient condition for a graph to be a Hamiltonian is still to be determined. But there are a few sufficient conditions for certain graphs to be Hamiltonian. The following theorem is one of those results.

Theorem 4.4.1 — Dirac's Theorem. Every graph G with $n \geq 3$ vertices and minimum degree $\delta(G) \geq \frac{n}{2}$ has a Hamilton cycle.

Proof. Suppose that $G = (V,E)$ satisfies the hypotheses of the theorem. Then G is connected, since otherwise the degree of any vertex in a smallest component C of G would be at most

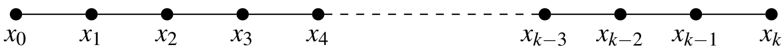


(a) (b)

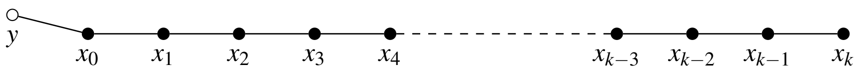
Figure 4.5: Dodecahedron and a Hamilton cycle in it.

$|C|-1<\frac{n}{2}$, contradicting the hypothesis $\delta(G)\geq\frac{n}{2}$.

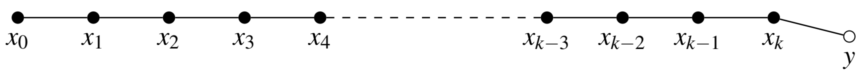
Let $P=x_0x_1\dots x_k$ be a longest path in G, as seen in the figure given below:



Note that the length of P is k. Since P cannot be extended to a longer path, all the neighbours of x_0 lie on P. Assume the contrary. Let y be an adjacent vertex of x_0 which is not in P. Then, the path $P'=yx_0,x_2\dots x_k$ is a path of length $k+1$ (see the graph given below), contradicting the hypothesis that P is the longest path in G.



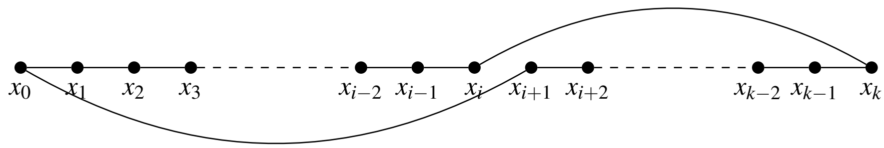
Similarly, we note that and all the neighbours of x_k will also lie on P, unless we reach at a contradiction as mentioned above (see the graph given below).



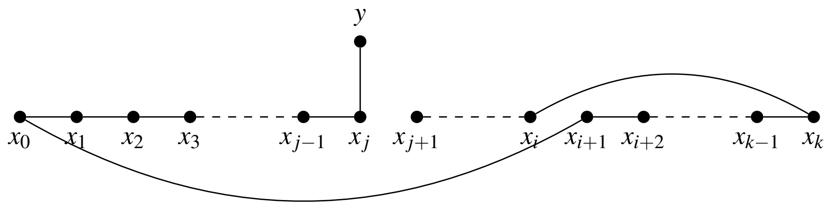
Hence, at least $\frac{n}{2}$ of the vertices $x_0,\dots,x_{k-1}$ are adjacent to x_k, and at least $\frac{n}{2}$ of the vertices $x_1,\dots,x_k$ are adjacent to x_0. Another way of saying the second part of the last sentence is: at least $\frac{n}{2}$ of the vertices $x_i\in\{x_0,\dots,x_{k-1}\}$ are such that $x_0x_{i+1}\in E$. Combining both statements and using the pigeon-hole principle, we see that there is some x_i with $0\leq i\leq k-1$, $x_ix_k\in E$ and $x_0x_{i+1}\in E$.

Consider the cycle $C=x_0x_{i+1}x_{i+2}\dots x_{k-1}x_kx_ix_{i-1}\dots x_1x_0$ as given in the following graph.

We claim that the above cycle C is a Hamilton cycle of G. Otherwise, since G is connected, there would be some vertex x_j of C adjacent to a vertex y not in C, so that $e=x_jy\in E$. But



then we could attach e to a path ending in x_j containing k edges of C, constructing a path in G longer than P (see the graph given below), which is a contradiction to the hypothesis that P is the longest path in G.



Therefore, C must cover all vertices of G and hence it is a Hamiltonian cycle in G. This completes the proof. ∎

Theorem 4.4.2 — Ore's Theorem. Let G be a graph with n vertices and let u and v be non-adjacent vertices in G such that $d(u)+d(v) \geq n$. Let $G+uv$ denote the super graph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

Proof. Let G be a graph with n vertices and suppose u and v are non-adjacent vertices in G such that $d(u)+d(v) \geq n$. Let $G' = G+uv$ be the super graph of G obtained by adding the edge uv. Note that, except for u and v, $d_G(x) = d_{G'}(x) \forall x \in V(G)$.

Let G be Hamiltonian. The only difference between G and G' is the edge uv. Then, obviously G' is also Hamiltonian as a Hamilton cycle in G will be a Hamilton cycle in G' as well.

Conversely, let G' be Hamiltonian. We have to show that G is Hamiltonian. Assume the contrary. Then, by (contrapositive of) Dirac's Theorem, we have $\delta(u) < \frac{n}{2}$ and $\delta(u) < \frac{n}{2}$ and hence we have $d(u)+d(v) < n$, which contradicts the hypothesis that $d(u)+d(v) \geq n$. Hence G is Hamiltonian. ∎

The following theorem determines the number of edge-disjoint Hamilton cycles in a complete graph K_n, where n is odd.

Theorem 4.4.3 In a complete graph K_n, where $n \geq 3$ is odd, there are $\frac{n-1}{2}$ edge-disjoint Hamilton cycles.

Proof. Note that a complete graph has $\frac{n(n-1)}{2}$ edges and a hamilton cycle in K_n contains only n edges. Therefore, the maximum number of edge-disjoint hamilton cycles is $\frac{n-1}{2}$.

Now, assume that $n \geq 3$ and is odd. Construct a subgraph G of K_n as explained below:

The vertex v_1 is placed at the centre of a circle and the remaining $n-1$ vertices are placed on the circle, at equal distances along the circle such that the angle made at the centre by two points is $\frac{360}{n-1}$ degrees. The vertices with odd suffixes are placed along the upper half of the circle and the vertices with even suffixes are placed along the lower half circle. Then, draw edges v_iv_{i+1}, where $1 \leq i \leq n$, with the meaning that $v_{n+1} = v_1$, are drawn as shown in Figure 4.7.

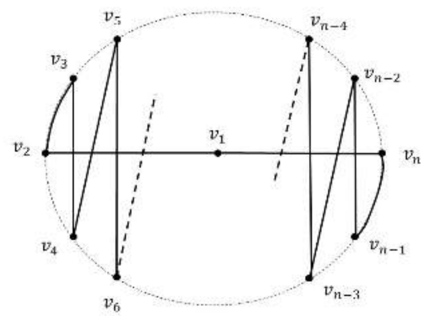


Figure 4.6: A Hamilton cycle G of K_n.

Clearly, the reduced graph G_1 is a cycle covering all vertices of K_n. That is If we rotate the vertices along the curve for $\frac{360}{n-1}$ degrees, we get another Hamilton subgraph G_2 of K_n, which has no common edges with G_1. (see figure 4.7).

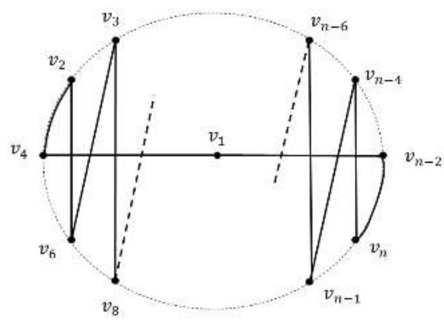


Figure 4.7: Another Hamilton cycle G_2 of K_n.

In a similar way, rotate the polygonal pattern clockwise by $\frac{360}{(n-1)}$ degrees. After $(n-1)$-th rotation, all vertices will be exactly as in Figure 4.6. Therefore, $n-1$ rotations are valid. But, it can be noted that the cycle G_i obtained after the i-th rotation and the cycle $G_{\frac{n-1}{2}+i}$ are isomorphic graphs, because all vertices in the upper half cycle in G_i will be in the lower half cycle in $G_{\frac{n-1}{2}+i}$ and vice versa, in the same order(see Figure 4.8).

That is, we have now that there are $\frac{n-1}{2}$ distinct such non-isomorphic edge-disjoint cycles

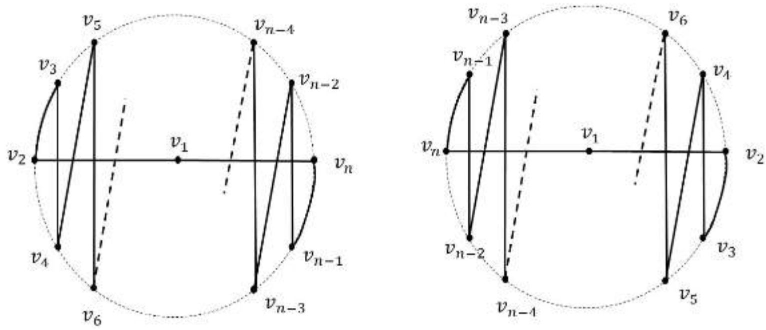


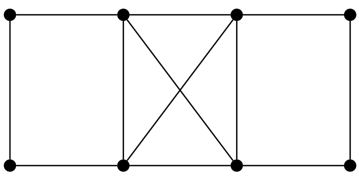
Figure 4.8: Isomorphic Hamilton cycles G_1 and $G_{\frac{n-1}{2}+1}$ of K_n.

in K_n. Hence, the number of edge-disjoint Hamilton cycles is $\frac{n-1}{2}$. ■

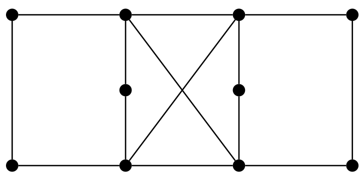
4.5 Some Illustrations

We can find out graphs, which are either Eulerian or Hamiltonian or simultaneously both, whereas some graph are neither Eulerian nor Hamiltonian. We note that the dodecahedron is an example for a Hamiltonian graph which is not Eulerian (see Figure 4.5a).

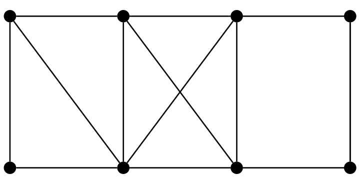
Let us now examine some examples all possible types of graphs in this category.



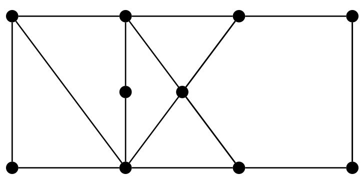
(a) A graph which is both Eulerian and Hamiltonian



(b) A graph which is Eulerian, but not Hamiltonian.



(c) A graph which is Hamiltonian, but not Eulerian.



(d) A graph which is neither Eulerian nor Hamiltonian.

Figure 4.9: Traversability in graphs

4.6 Weighted Graphs

Definition 4.6.1 — Weighted Graphs. A **weighted graph** is a graph G in which each edge e has been assigned a real number $w(e)$, called the *weight* (or *length*) of the edge e.

Figure 4.10 illustrates a weighted graph:

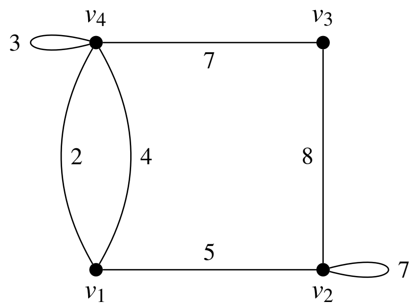


Figure 4.10: An example of a weighted graph

If H is a subgraph of a weighted graph, the weight $w(H)$ of H is the sum of the weights $w(e_1)+w(e_2)+\ldots+w(e_k)$ where $\{e_1,e_2,\ldots,e_k\}$ is the set of edges of H.

Many optimisation problems amount to finding, in a suitable weighted graph, a certain type of subgraph with minimum (or maximum) weight.

4.7 Travelling Salesman's Problem

Suppose a travelling salesman's territory includes several towns with roads connecting certain pairs of these towns. As a part of his work, he has to visit each town. For this, he needs to plan a round trip in such a way that he can visit each of the towns exactly once.

We represent the salesman's territory by a weighted graph G where the vertices correspond to the towns and two vertices are joined by a weighted edge if and only if there is a road connecting the corresponding towns which does not pass through any of the other towns, the edge's weight representing the length of the road between the towns.

Then, the problem reduces to check whether the graph G is a Hamiltonian graph and to construct a Hamiltonian cycle of minimum weight (or length) if G is Hamiltonian. This problem is known as the *Travelling Salesman Problem*.

It is sometimes difficult to determine if a graph is Hamiltonian as there is no easy characterisation of Hamiltonian graphs. Moreover, for a given a weighted graph G which is Hamiltonian there is no easy or efficient algorithm for finding an optimal circuit in G, in general. These facts make our problem difficult.

To find out an optimal Hamilton cycle, we use the following algorithm with an assumption that the given graph G is a weighted complete graph.

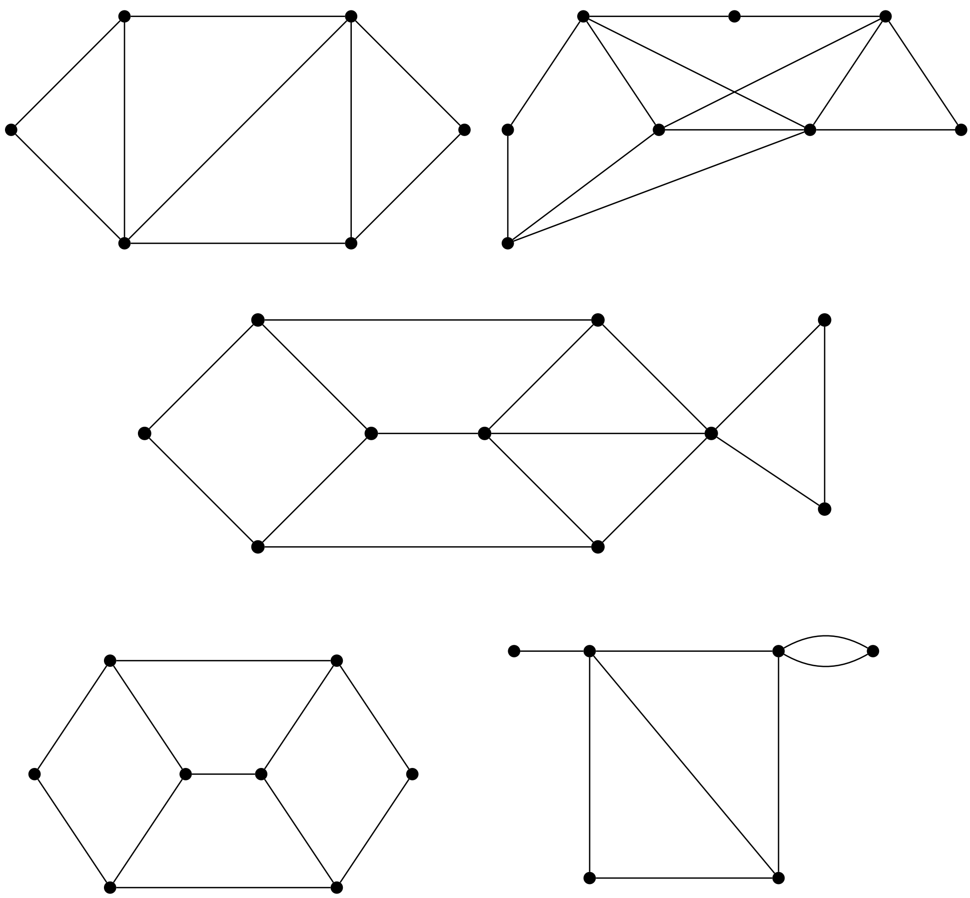
Two Optimal Algorithm

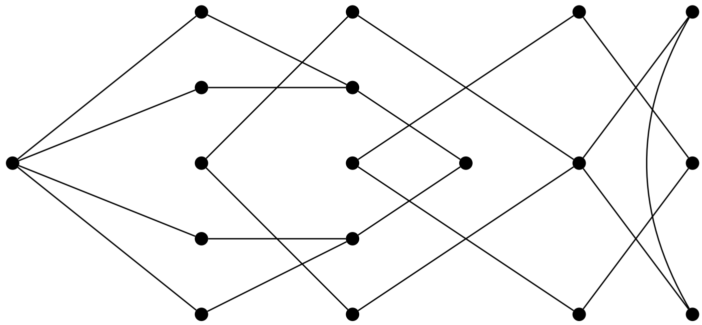
1. Let $C = v_1v_2, v_3\ldots, v_nv_1$ be any Hamiltonian cycle of the weighted graph G and let w be the weight of C. That is, $w = w(v_1v_2)+w(v_2,v_3)+\ldots+w(v_nv_1)$.

2. Set $i = 1$.
3. Set $j = i+2$.
4. Let $C_{ij} = v_1v_2v_2v_3\ldots v_iv_jv_{j-1}\ldots v_{i+1}v_{j+1}v_{j+2}\ldots v_nv_1$ be the Hamiltonian cycle and let w_{ij} denote the weight of C_{ij}, so that $w_{ij} = w - w(v_iv_{i+1}) - w(v_jv_{j+1}) + w(v_iv_j) + w(v_{i+1}v_{j+1})$.
   If $w_{ij} < w$, (that is, if $w(v_iv_j) + w(v_{i+1}v_{j+1}) < w(v_iv_{i+1}) + w(v_jv_{j+1})$), then replace C by C_{ij} and w by w_{ij} and return to Step 1, taking the sequence of vertices $v_1v_2v_3\ldots v_nv_1$ as given by our new C.
5. Set $j = j+1$. If $j < n$, do Step 4. Otherwise, set $i = i+1$. If $i < n-2$, do Step 3. Otherwise, stop.

4.8 Exercises

1. Draw a graph that has a Hamilton path, but not a Hamilton Cycle.





2. Show that if G is Eulerian,then every block (see Chapter 7 for the notion of blocks) of G is Eulerian.
3. Show that a Hamilton path of a graph G, if exists, is the longest path in G.
4. Show that if G is a self-complementary graph, then G has a Hamilton path.
5. Show that every complete graph $K_n; n \geq 3$ is Hamiltonian.
6. Verify whether Petersen graph is Eulerian. Justify your answer.
7. Verify whether Petersen graph is Hamiltonian. Justify your answer.
8. Verify whether the following graphs are Eulerian and Hamiltonian. Justify your answer.
9. Show that every even graph (a graph without odd degree vertices) can be decomposed into cycles.
10. Show that if either
    (a) G is not 2-connected, or
    (b) G is bipartite with bipartition (X,Y) where $|X| \leq |Y|$.
    Then G is non-Hamiltonian.
11. Characterise all simple Euler graphs having an Euler tour which is also a Hamiltonian cycle.
12. Let G be a Hamiltonian graph. Show that G does not have a cut vertex.
13. There are n guests at a dinner party, where $n \geq 4$. Any two of these guests know, between them, all the other $n-2$. Prove that the guests can be seated round a circular table so that each one is sitting between two people they know.
14. Let G be a simple k-regular graph, with $2k-1$ vertices. Prove that G is Hamiltonian.

III

TREES

6. Trees

Definition 6.0.1 — Tree. A graph G is called a *tree* if it is connected and has no cycles. That is, a tree is a connected acyclic (circuitless) graph.

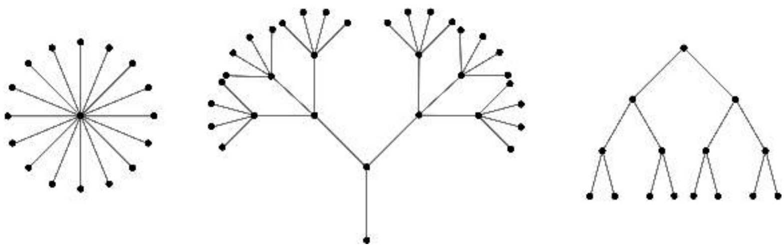


Figure 6.1: Examples of trees

Definition 6.0.2 — Tree. An acyclic graph may possibly be a disconnected graph whose components are trees. Such graphs are called **forests**.

6.1 Properties of Trees

Theorem 6.1.1 A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Proof. Let G be a graph and let there be exactly one path between every pair of vertices in G. So G is connected. If G contains a cycle, say between vertices u and v, then there are two distinct paths between u and v, which is a contradiction to the hypothesis. Hence, G is connected and is without cycles, therefore it is a tree.

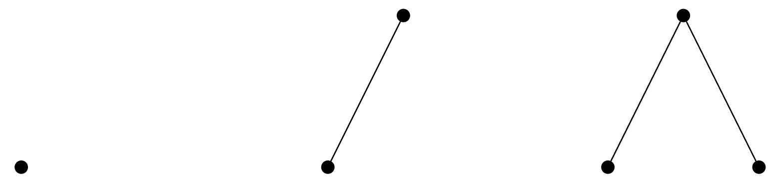
Conversely, let G be a tree. Since G is connected, there is at least one path between every pair of vertices in G. Let there be two distinct paths, say P and P' between two vertices u and v of G. Then, the union of $P \cup P'$ contains a cycle which contradicts the fact that G is a tree. Hence, there is exactly one path between every pair of vertices of a tree. ■

Then, by Definition 3.1.11, we have the following result:

Theorem 6.1.2 All trees are geodetic graphs.

Theorem 6.1.3 A tree with n vertices has $n-1$ edges.

Proof. We prove the result by using mathematical induction on n, the number of vertices. The result is obviously true for $n = 1,2,3$. See illustrations in Figure 6.2.



(a) (b) (c)

Figure 6.2: Trees with $n = 1,2,3$.

Let the result be true for all trees with fewer than n vertices. Let T be a tree with n vertices and let e be an edge with end vertices u and v. So, the only path between u and v is e. Therefore, deletion of e from T disconnects T.

Now, $T-e$ consists of exactly two components T_1 and T_2 say, and as there were no cycles to begin with, each component is a tree. Let n_1 and n_2 be the number of vertices in T_1 and T_2 respectively. Then, note that $n_1 + n_2 = n$. Also, $n_1 < n$ and $n_2 < n$. Thus, by induction hypothesis, the number of edges in T_1 and T_2 are respectively $n_1 - 1$ and $n_2 - 1$. Hence, the number of edges in T is $n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$. ■

Theorem 6.1.4 Any connected graph with n vertices and $n-1$ edges is a tree.

Proof. Let G be a connected graph with n vertices and $n-1$ edges. We show that G contains no cycles. Assume to the contrary that G contains cycles. Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph H is a tree. As H has n vertices, so the number of edges in H is $n-1$. Now, the number of edges in G is greater than the number of edges in H. That is, $n-1 > n-1$, which is not possible. Hence, G has no cycles and therefore is a tree. ■

Theorem 6.1.5 Every edge of a tree is a cut-edge of G.

Proof. Since a tree T is an acyclic graph, no edge of T is contained in a cycle. Therefore, by Theorem 3.3.1, every edge of T is a cut-edge. ■

A graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

The following theorem is another characterization of trees.

Theorem 6.1.6 A graph is a tree if and only if it is minimally connected.

Proof. Let the graph G be minimally connected. Then, G has no cycles and therefore is a tree. Conversely, let G be a tree. Then, G contains no cycles and deletion of any edge from G disconnects the graph. Hence, G is minimally connected. ■

Theorem 6.1.7 A graph G with n vertices, $n-1$ edges and no cycles is connected.

Proof. Let G be a graph without cycles with n vertices and $n-1$ edges. We have to prove that G is connected. Assume that G is disconnected. So G consists of two or more components and each component is also without cycles. We assume without loss of generality that G has two components, say G_1 and G_2. Add an edge e between a vertex u in G_1 and a vertex v in G_2. Since there is no path between u and v in G, adding e did not create a cycle. Thus $G \cup \{e\}$ is a connected graph (tree) of n vertices, having n edges and no cycles. This contradicts the fact that a tree with n vertices has $n-1$ edges. Hence, G is connected. ■

Theorem 6.1.8 Any tree with at least two vertices has at least two pendant vertices.

Proof. Let the number of vertices in a given tree T be n, where $(n > 1)$. So the number of edges in T is $n-1$. Therefore, the degree sum of the tree is $2(n-1)$ (by the first theorem of graph theory). This degree sum is to be divided among the n vertices. Since a tree is connected it cannot have a vertex of zero degree. Each vertex contributes at least 1 to the above sum. Thus, there must be at least two vertices of degree exactly 1. That is, every tree must have at least two pendant vertices. ■

Theorem 6.1.9 Let G be a graph on n vertices. Then, the following statements are equivalent:

(i) G is a tree.
(ii) G is connected and has $n-1$ edges.
(iii) G is acyclic (circuitless) and has $n-1$ edges.
(iv) There exists exactly one path between every pair of vertices in G.
(v) G is a minimally connected graph.

Proof. The equivalence of these conditions can be established using the results $(i) \implies (ii), (ii) \implies (iii), (iii) \implies (iv), (iv) \implies (v)$ and $(v) \implies (i)$.

Part-$(i) \implies (ii)$: This part states that if G is a tree on n vertices, then G is connected and has $n-1$ edges. Since G is a tree, clearly, by definition of a tree it is connected. The remaining part follows from the result that every tree on n vertices has $n-1$ vertices.

Part-$(ii) \implies (ii)$: This part states that if G is connected and has $n-1$ edges, then G is acyclic and has $n-1$ edges. Clearly, This result follows from the result that a connected graph on n vertices and $n-1$ edges is acyclic.

Part-$(iii) \implies (iv)$: This part states that if G is an acyclic graph on n vertices and has $n-1$ edges, then there exists exactly one path between every pair of vertices in G. By a previous theorem, we have an acyclic graph G on n vertices and $n-1$ edges is connected. Therefore, G is a tree. Hence, by our first theorem, there exists exactly one path between every pair of vertices in G.

Part-$(iv) \implies (v)$: This part states that if there exists exactly one path between every pair of vertices in G, then G is minimally connected. Assume that every pair of vertices in G is connected by a unique path.

Let u and v be any two vertices in G and P be the unique (u,v)-path in G. Let e be any edge in this path P. If we remove the edge from P, then there will be no (u,v)-path in $G-e$. That is, $G-e$ is disconnected. Therefore, G is minimally connected.

Part-$(v) \implies (i)$: This part states that if G is minimally connected, then G is a tree. Clearly, G is connected as it is minimally connected. Since G is minimally connected, removal of any edge makes G disconnected. That is, every edge of G is a cut edge of G. That is, no edge of G is contained in a cycle in G. Therefore, G is acyclic and hence is a tree. ■

Theorem 6.1.10 A vertex v in a tree is a cut-vertex of T if and only if $d(v) \geq 2$.

Proof. Let v be a cut-vertex of a tree T. Since, no pendant vertex of a graph can be its cut-vertex, clearly we have $d(v) \geq 2$.

Let v be a vertex of a tree T such that $d(v) \geq 2$. Then v is called an *internal vertex* (or *intermediate vertex*) of T. Since $d(v) \geq 2$, there are two at least two neighbours for v in T. Let u and w be two neighbours of v. Then, $u-v-w$ is a $(u-w)$-path in G. By Theorem-1, we have the path $u-v-w$ is the unique $(u-w)$-path in G. Therefore, $T-v$ is disconnected and u and w are in different components of T. Therefore, v is a cut-vertex of T. This completes the proof. ■

6.2 Distances in Trees

Definition 6.2.1 — Metric. A *metric* on a set A is a function $d : A \times A \to [0, \infty)$, where $[0, \infty)$ is the set of non-negative real numbers and for all $x, y, z \in A$, the following conditions are satisfied:

1. $d(x,y) \geq 0$ (non-negativity or separation axiom);
2. $d(x,y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles);

3. $d(x,y) = d(y,x)$ (symmetry);
4. $d(x,z) \leq d(x,y) + d(y,z)$ (sub-additivity or triangle inequality).

Conditions 1 and 2, are together called a *positive-definite function*.

A metric is sometimes called the *distance function*.

In view of the definition of a metric, we have

Theorem 6.2.1 The distance between vertices of a connected graph is a metric.

Definition 6.2.2 — Center of a graph. A vertex in a graph G with minimum eccentricity is called the *center* of G.

Theorem 6.2.2 Every tree has either one or two centers.

Proof. The maximum distance, $\max d(v, v_i)$ from a given vertex v to any other vertex occurs only when v_i is a pendant vertex. With this observation, let T be a tree having more than two vertices. Tree T has two or more pendant vertices.

Deleting all the pendant vertices from T, the resulting graph T' is again a tree. The removal of all pendant vertices from T uniformly reduces the eccentricities of the remaining vertices (vertices in T') by one. Therefore, the centers of T are also the centers of T'. From T', we remove all pendant vertices and get another tree T''. Continuing this process, we either get a vertex, which is a center of T, or an edge whose end vertices are the two centers of T. ■

6.3 Degree Sequences in Trees

Theorem 6.3.1 The sequence $\langle d_i \rangle; 1 \leq i \leq n$ of positive integers is a degree sequence of a tree if and only if (i) $d_i \geq 1$ for all $i, 1 \leq i \leq n$ and (ii) $\sum d_i = 2n-2$.

Proof. Since a tree has no isolated vertex, we have $d_i \geq 1$ for all i. Also, $\sum d_i = 2|E| = 2(n-1)$, as a tree with n vertices has $n-1$ edges.

We use induction on n to prove the converse part. For $n = 2$, the sequence is $\{1,1\}$ and is obviously the degree sequence of K_2 . Suppose the claim is true for all positive sequences of length less than n. Let $\langle d_i \rangle$ be the non-decreasing positive sequence of n terms, satisfying conditions (i) and (ii). Then $d_1 = 1$ and $d_n > 1$.

Now, consider the sequence $D' = \{d_2, d_3, \ldots, d_{n-1}, d_n - 1\}$, which is a sequence of length $n-1$. Obviously in D', we have $d_i \geq 1$ and $\sum d_i = d_2 + d_3 + \ldots + d_{n-1} + d_n - 1 = d_1 + d_2 + d_3 + \ldots + d_{n-1} + d_n - 1 - 1 = 2n - 2 - 2 = 2(n-1) - 2$ (because $d_1 = 1$). So D' satisfies conditions (i) and (ii) and by induction hypothesis, there is a tree T_0 realising D'. In T_0, add a new vertex and join it to the vertex having degree $d_n - 1$ to get a tree T. Therefore, the degree sequence of T is $\{d_1, d_2, \ldots, dn\}$. This completes the proof. ■

Theorem 6.3.2 Let T be a tree with k edges. If G is a graph whose minimum degree satisfies $\delta(G) \geq k$, then G contains T as a subgraph. In other words, G contains every tree of order at most $\delta(G)+1$ as a subgraph.

Proof. We use induction on k. If $k=0$, then $T=K_1$ and it is clear that K_1 is a subgraph of any graph. Further, if $k=1$, then $T=K_2$, and K_2 is a subgraph of any graph whose minimum degree is one.

Assume that the result is true for all trees with $k-1$ edges ($k \geq 2$) and consider a tree T with exactly k edges. We know that T contains at least two pendant vertices. Let v be one of them and let w be the vertex that is adjacent to v.

Consider the graph $T-v$. Since $T-v$ has $k-1$ edges, the induction hypothesis applies, so $T-v$ is a subgraph of G. We can think of $T-v$ as actually sitting inside G (meaning w is a vertex of G, too).

Since G contains at least $k+1$ vertices, and $T-v$ contains k vertices, there exist vertices of G that are not a part of the subgraph $T-v$. Further, since the degree of w in G is at least k, there must be a vertex u not in $T-v$ that is adjacent to w. The subgraph $T-v$ together with u forms the tree T as a subgraph of G. ■

6.4 On Counting Trees

A *labelled graph* is a graph, each of whose vertices (or edges) is assigned a unique name ($v_1, v_2, v_3, \ldots$ or $A, B, C, \ldots$) or labels $(1,2,3,\ldots)$.

The distinct vertex labelled trees on 4 vertices are given in Figure 6.3.

The distinct unlabelled trees on 4 vertices are given in Figure 6.4.

6.5 Spanning Trees

Definition 6.5.1 — Spanning Tree. A *spanning tree* of a connected graph G is a tree containing all the vertices of G. A *spanning tree* of a graph is a maximal tree subgraph of that graph. A spanning tree of a graph G is sometimes called the *skeleton* or the *scaffold graph*.

Theorem 6.5.1 Every connected graph G has a spanning tree.

Proof. Let G be a connected graph on n vertices. Pick an arbitrary edge of G and name it e_1. If e_1 belongs to a cycle of G, then delete it from G. (Else, leave it unchanged and pick it another one). Let $G_1 = G - e_1$. Now, choose an edge e_2 of G_1. If e_2 belongs to a cycle of G_1, then remove e_2 from G_1. Proceed this step until all cycles in G are removed iteratively. Since G is a finite graph the procedure terminates after a finite number of times. At this stage, we get a subgraph T of G, none of whose edges belong to cycles. Therefore, T is a connected acyclic subgraph of G on n vertices and hence is a spanning tree of G, completing the proof. ■

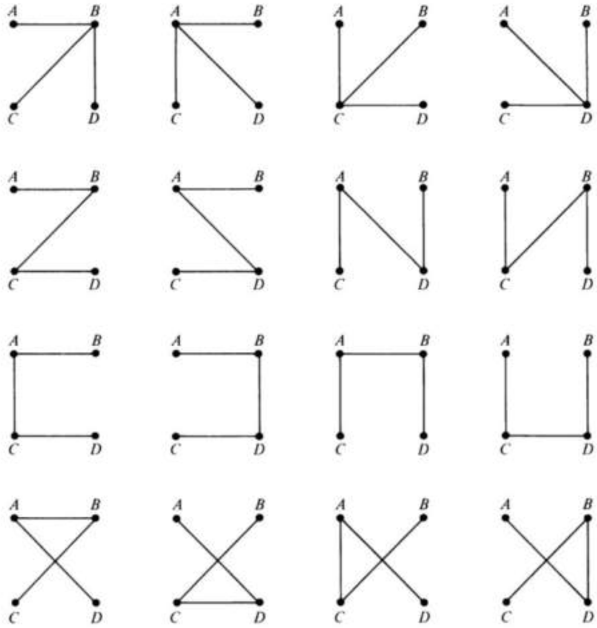


Figure 6.3: Distinct labelled trees on 4 vertices

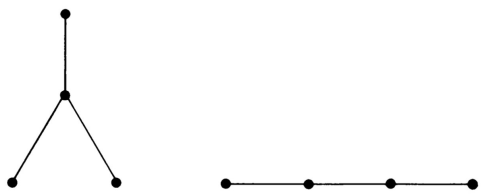


Figure 6.4: Distinct unlabelled trees on 4 vertices

Definition 6.5.2 — Branches and Chords of Graphs. Let T be a spanning tree of a given graph G. Then, every edge of T is called a *branch* of T. An edge of G that is not in a spanning tree of G is called a *chord* (or a *tie* or a *link*).

Note that the branches and chords are defined in terms of a given spanning tree.

Theorem 6.5.2 Show that every graph with n vertices and ε edges has $n-1$ branches and $\varepsilon - n + 1$ chords.

Proof. Let G be a graph with n vertices and ε edges and let T be a spanning tree of G. Then, by Theorem 6.1.3, T has $n-1$ edges. Therefore, the number of branches is $n-1$. The number chords in G with respect to T is $|E(G)| - |E(T)| = \varepsilon - (n-1) = \varepsilon - n + 1$. ■

The set of all chords of a tree T is called a *chord set* or a *co-tree* or a *tie set* and is usually denoted by $\bar{T}$. Therefore, we have $T \cup \bar{T} = G$.