Course Contents

- Logic
- Sets and Set Operations
- Integers, Division and Matrices
- Relations
- Graphs
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1. The Foundations: Logic

- **Mathematical Logic** is a tool for working with compound statements
- **Logic** is the study of correct reasoning
- **Use of logic**
  - **In mathematics:**

    to prove theorems
  - **In computer science:**

    to prove that programs do what they are supposed to do

Section 1.1: Propositional Logic

- Propositional logic: It deals with **propositions**.
- Predicate logic: It deals with **predicates**.

Definition of a Proposition

<u>Definition:</u> A **proposition** (usually denoted by p, q, r, …) is a declarative statement that is either **True** (T) or **False** (F), but not both or somewhere "in between!".

Note: Commands and questions are not propositions.

Examples of Propositions

The following are **all** propositions:

- "It is raining" (In a given situation)
- "Amman is the capital of Jordan"
- "1 + 2 = 3"

But, the following are **NOT** propositions:

- "Who's there?" (Question)
- "La la la la la." (Meaningless)
- "Just do it!" (Command)
- "1 + 2" (Expression with a non-true/false value)
- "1 + 2 = *x*" (Expression with unknown value of *x*)

Operators / Connectives

An **operator** or **connective** combines one or more **operand** expressions into a larger expression. (e.g., "+" in numeric expression.)

- **Unary** operators take 1 operand (e.g. -3);
- **Binary** operators take 2 operands (e.g. 3×4).
- **Propositional** or **Boolean** operators operate on propositions (or their truth values) instead of on numbers.

Some Popular Boolean Operators

Formal Name	Nickname	Arity	Symbol
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

Definition: Let p be a proposition then $\neg p$ is the **negation** of p (Not p , it is not the case that p).

e.g. If p = "London is a city"

then $\neg p$ = "London is **not** a city" or " it is not the case that London is a city"

The **truth table** for NOT:

p	$\neg p$
F	T
T	F

T :≡ True; F :≡ False

":≡" means "is defined as".

Operand column

Result column

The Conjunction Operator

<u>Definition:</u> Let p and q be propositions, the proposition "p **AND** q" denoted by $(p \wedge q)$ is called the **conjunction** of p and q.

e.g. If p = "I will have salad for lunch" and

q = "I will have steak for dinner", then

$p \wedge q$ = "I will have salad for lunch **and** I will have steak for dinner"

Remember: "$\wedge$" points up like an "A", and it means "AND"

Conjunction Truth Table

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.

Operand columns

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

“And”, “But”, “In addition to”, “Moreover”.

The Disjunction Operator

Definition: Let p and q be propositions, the proposition "p **OR** q" denoted by $(p \vee q)$ is called the **disjunction** of p and q.

e.g. p = "My car has a bad engine"

q = "My car has a bad carburetor"

$p \vee q$ = "Either my car has a bad engine **or** my car has a bad carburetor"

Disjunction Truth Table

- Note that $p \vee q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called **inclusive or**, because it **includes** the possibility that both p and q are true.

p	q	$p \vee q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note the differences from AND

Compound Statements

- Let p, q, r be simple statements
- We can form other compound statements, such as
  - $(p \vee q) \wedge r$
  - $p \vee (q \wedge r)$
  - $\neg p \vee \neg q$
  - $(p \vee q) \wedge (\neg r \vee s)$
  - and many others…

Example: Truth Table of $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
F	F	F	F	F
F	F	T	F	F
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

The Exclusive Or Operator

The binary **exclusive-or** operator "⊕" (XOR) combines two propositions to form their logical "exclusive or" (exjunction?).

e.g. p = "I will earn an A in this course"

q = "I will drop this course"

$p \oplus q$ = "I will either earn an A in this course, **or** I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called **exclusive or**, because it **excludes** the possibility that both p and q are true.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note the difference from OR

Natural Language is Ambiguous

Note that English "or" can be ambiguous regarding the "both" case!

"Pat is a singer or
Pat is a writer" $\vee$

"Pat is a man or
Pat is a woman" $\oplus$

Need context to disambiguate the meaning!

For this class, assume "OR" means inclusive.

A Simple Exercise

Let p = "It rained last night",
q = "The sprinklers came on last night" ,
r = "The grass was wet this morning".

Translate each of the following into English:

$\neg p$ = **"It didn't rain last night"**

$r \wedge \neg p$ = **"The grass was wet this morning, and it didn't rain last night"**

$\neg r \vee p \vee q$ = **"Either the grass wasn't wet this morning, or it rained last night, or the sprinklers came on last night"**

Nested Propositional Expressions

- Use parentheses to **group sub-expressions**:
  "I just saw my old **f**riend, and either he's **g**rown or I've **s**hrunk" = $f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
  - $f \wedge g \vee s$ would be ambiguous
- By convention, "$\neg$" takes **precedence** over both "$\wedge$" and "$\vee$". ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$)
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, not $\neg (s \wedge f)$

The Implication Operator

The **implication** $\overbrace{p}^{\boxed{\text{hypothesis}}} \to \overbrace{q}^{\boxed{\text{conclusion}}}$ states that p implies q.

If p is true, then q is true; but if p is not true, then q could be either true or false.

e.g. Let p = "You get 100% on the final"

q = "You will get an A"

$p \to q$ = "If you get 100% on the final, then you will get an A"

Implication Truth Table

- $p \rightarrow q$ is **false** <u>only</u> when p is true but q is **not** true.
- $p \rightarrow q$ does **not** say that p <u>causes</u> q!
- $p \rightarrow q$ does **not** require that p or q <u>**are ever true**</u>!

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	**F**
T	T	T

The <u>only</u> False case!

e.g. "$(1 = 0) \rightarrow$ pigs can fly" is TRUE!

Examples of Implications

- "If this lecture ever ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1 + 1 = 6, then Obama is the president of USA." *True* or *False*?

$P \to Q$ has many forms in English Language:

- "P implies Q "
- "If P, Q "
- "If P, then Q "
- "P only if Q "
- "P is sufficient for Q "
- "Q if P "
- "Q is necessary for P "
- "Q when P "
- "Q whenever P "
- "Q follows from P "

Logical Equivalence

- $\neg p \vee q$ is **logically equivalent** to $p \rightarrow q$

p	q	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$:

- Its **converse** is: $q \rightarrow p$
- Its **inverse** is: $\neg p \rightarrow \neg q$
- Its **contrapositive** is: $\neg q \rightarrow \neg p$

Example of Converse, Inverse, Contrapositive

Write the converse, inverse and contrapositive of the statement "if $x \neq 0$, then John is a programmer"

- Its converse is: "if John is a programmer, then $x \neq 0$"
- Its inverse is: "if $x = 0$, then John is not a programmer"
- Its contrapositive is: "if John is not a programmer, then $x = 0$"

Note: The negation operation ($\neg$) is different from the inverse operation.

Biconditional $\leftrightarrow$ Truth Table

In English:

- “p if and only if q "
- "If p, then q, and **conversely**"
- “p is **sufficient** and **necessary** for q "
- Written $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Translation English Sentences into Logical Expressions

- If you are a **c**omputer science major **or** you are not a **f**reshman, then you can **a**ccess the internet from campus :

  is translated to:

$$(c \vee \neg f) \rightarrow a$$

Precedence of Logical Operators

Operator	Precedence
( )	1
$\neg$	2
$\wedge$, $\vee$	3
$\rightarrow$, $\leftrightarrow$	4
Left to Right	5

Logic and Bit Operations

- Find the bitwise AND, bitwise OR, and bitwise XOR of the bit strings 0110110110 and 1100011101.

	0110110110
	1100011101
	___________
Bitwise AND	0100010100
Bitwise OR	1110111111
Bitwise XOR	1010101011

Truth value	**Bit**
F	0
T	1

Section 1.2: Propositional Equivalence, Tautologies and Contradictions

- A **tautology** is a compound proposition that is always **true.**

  e.g. $p \vee \neg p \equiv \mathbf{T}$

- A **contradiction** is a compound proposition that is always **false.**

  e.g. $p \wedge \neg p \equiv \mathbf{F}$

- Other compound propositions are **contingencies**.

  e.g. $p \rightarrow q$, $p \vee q$

Tautology

Example: $p \to p \vee q$

p	q	$p \vee q$	$p \to p \vee q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

Equivalence Laws $\Leftrightarrow$

- Identity: $p \wedge \mathbf{T} \Leftrightarrow p \quad , \; p \vee \mathbf{F} \Leftrightarrow p$
- Domination: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T} \quad , \; p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- Idempotent: $p \vee p \Leftrightarrow p \quad , \quad p \wedge p \Leftrightarrow p$
- Double negation: $\neg\neg p \Leftrightarrow p$
- Commutative: $p \vee q \Leftrightarrow q \vee p \; , \; p \wedge q \Leftrightarrow q \wedge p$
- Associative: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

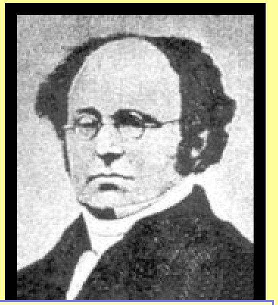
- Distributive: $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$

  $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$
- De Morgan's:

  $\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$

  $\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$
- Trivial tautology/contradiction:

  $p \lor \neg p \Leftrightarrow \mathbf{T}$, $p \land \neg p \Leftrightarrow \mathbf{F}$



Augustus De Morgan (1806-1871)

Implications / Biconditional Rules

- $p \rightarrow q \equiv \neg p \vee q$
- $\neg (p \rightarrow q) \equiv \neg (\neg p \vee q) \equiv p \wedge \neg q$
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (contrapositive)
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $\neg (p \leftrightarrow q) \equiv p \oplus q$

Proving Equivalence via Truth Tables

Example: Prove that $p \lor q$ and $\neg(\neg p \land \neg q)$ are logically equivalent.

p	q	$p \lor q$	$\neg p$	$\neg q$	$\neg p \land \neg q$	$\neg(\neg p \land \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Proving Equivalence using Logic Laws

Example 1. Show that $\neg (P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are logically equivalent.

$\neg (P \vee (\neg P \wedge Q))$

$\equiv \neg P \wedge \neg (\neg P \wedge Q)$ De Morgan

$\equiv \neg P \wedge (\neg(\neg P) \vee \neg Q)$ De Morgan

$\equiv \neg P \wedge (P \vee \neg Q)$ Double negation

$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$ Distributive

$\equiv \mathbf{F} \vee (\neg P \wedge \neg Q)$ Negation

$\equiv (\neg P \wedge \neg Q)$ Identity

Proving Equivalence using Logic Laws

Example 2: Show that $\neg (\neg (P \to Q) \to \neg Q)$ is a contradiction.

$$\neg (\neg (P \to Q) \to \neg Q)$$
$$\equiv \neg (\neg (\neg P \vee Q) \to \neg Q) \text{ Equivalence}$$
$$\equiv \neg ((P \wedge \neg Q) \to \neg Q) \text{ De Morgan}$$
$$\equiv \neg (\neg (P \wedge \neg Q) \vee \neg Q) \text{ Equivalence}$$
$$\equiv \neg (\neg P \vee Q \vee \neg Q) \text{ De Morgan}$$
$$\equiv \neg (\neg P \vee \mathbf{T}) \text{ Trivial Tautology}$$
$$\equiv \neg (\mathbf{T}) \text{ Domination}$$
$$\equiv \mathbf{F} \quad \text{Contradiction}$$

Section 1.3: Predicates and Quantifiers

Predicates: "x is greater than 3" has two parts

First part: x , is a variable.

Second part: "is greater than 3", is a predicate.

"x is greater than 3" can be denoted by the **propositional function** $P(x)$.

$P(x)$: $x > 3$, let $x = 4$, then $P(4)$ is true,

let $x = 1$, then $P(1)$ is false.

Example: If $R(x, y, z) : x + y = z$ then

$R(1, 2, 3)$, $1 + 2 = 3$, is true.

Quantifiers

Quantification $\longrightarrow$ Universal Quantification

$\searrow$ Existential Quantification

Universes of Discourse (U.D) or **Domain** (D):

Collection of all persons, ideas, symbols, …

For every and for some

- Most statements in mathematics and computer science use terms such as *for every* and *for some*.
- For example:
  - *For every* triangle T, the sum of the angles of T is 180 degrees.
  - *For every* integer n, n is less than p, *for some* prime number p.

The Universal Quantifier $\forall$

- $\forall x\, P(x)$: "$P(x)$ is true for all (every) values of x in the universe of discourse".

- Example: What is the truth value of

$$\forall x\,(x^2 \geq x)\,.$$

- If UD is all real numbers, the truth value is false (take $x = 0.5$, this is called a **counterexample**).

- If UD is the set of integers, the truth value is true.

The Existential Quantifier $\exists$

- $\exists x\, Q(x)$: There exists an element x in the universe of discourse such that $Q(x)$ is true.

- Example 1: Let $Q(x)$: $x = x + 1$, Domain is the set of all real numbers:

  - The truth value of $\exists x\, Q(x)$ is false (as the is no real x such that $x = x + 1$).

- Example 2: Let $Q(x)$: $x^2 = x$, Domain is the set of all real numbers:

  - The truth value of $\exists x\, Q(x)$ is true (take $x = 1$).

Important Note

Let $P(x)$: $x^2 \geq x$, Domain is the set $\{0.5, 1, 2, 3\}$.

- $\forall x\, P(x) \equiv P(0.5) \wedge P(1) \wedge P(2) \wedge P(3)$

  $\equiv F \wedge T \wedge T \wedge T$

  $\equiv F$

- $\exists\, x\, P(x) \equiv P(0.5) \vee P(1) \vee P(2) \vee P(3)$

  $\equiv F \vee T \vee T \vee T$

  $\equiv T$

Example

Suppose that the universe of discourse of $P(x, y)$ is $\{1, 2, 3\}$. Write out the following propositions using disjunctions and conjunctions:

- $\exists x\, P(x, 2)$:

  $$\boldsymbol{P(1,2) \vee P(2, 2) \vee P(3, 2)}$$

- $\forall y\, P(3, y)$:

  $$\boldsymbol{P(3,1) \wedge P(3,2) \wedge P(3,3)}$$

Negations

- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$

- Example: Let $P(x)$ is the statement "$x^2 - 1 = 0$", where the domain is the set of real numbers R.

  - The truth value of $\forall x\, P(x)$ is **False**
  - The truth value of $\exists x\, P(x)$ is **True**
  - $\neg \forall x\, P(x) \equiv \exists x\, (x^2 - 1 \neq 0)$, which is **True**
  - $\neg \exists x\, P(x) \equiv \forall x\, (x^2 - 1 \neq 0)$, which is **False**

Example

Suppose that $P(x)$ is the statement "$x + 3 = 4x$" where the domain is the set of integers. Determine the truth values of $\forall x\, P(x)$. Justify your answer.

It is clear that $P(1)$ is True, but $P(x)$ is False for every $x \neq 1$ (take $x = 2$ as a counterexample). Thus, the truth value of $\forall x\, P(x)$ is **False.**

Translation using Predicates and Quantifiers

- “Every student in this class has studied math and C++ course”, Universes of Discourse is the student in this class.

  Translated to: $\forall x\,(M(x) \wedge CPP(x))$

- “For every person *x*, if *x* is a student in this class then *x* has studied math and C++”, UD is all people.

  Translated to: $\forall x\,(S(x) \rightarrow M(x) \wedge CPP(x))$

Translation using Predicates and Quantifiers

- “Some student in this class has studied math and C++ course”, UD is the students in this class.

Translated to: $\exists x\ (M(x) \land CPP(x))$

- But if the UD is all people.

Translated to: $\exists x\ (S(x) \land M(x) \land CPP(x))$

Example

Let $G(x)$, $F(x)$, $Z(x)$, and $M(x)$ be the statements "x is a giraffe“, "x is 15 feet or higher“, "x is in this zoo" and "x belongs to me" respectively. Suppose that the universe of discourse is the set of animals. Express each of the following quantifiers in English:

- $\forall x\ (Z(x) \rightarrow G(x))$

  All animals in this zoo are giraffes

- $\forall x\ (M(x) \rightarrow F(x))$

  I have no animals less than 15 feet high

- $\forall x\ (Z(x) \rightarrow M(x))$

  There are no animals in this zoo that belong to anyone but me

- $\forall x\ (\neg G(x) \rightarrow \neg F(x))$

  No animals, except giraffes, are 15 feet or higher

Summary

- In order to prove the quantified statement $\forall x\, P(x)$ is true
  - It is **not** enough to show that $P(x)$ is true for some $x \in D$
  - You must show that $P(\mathrm{x})$ is true for every $x \in D$
  - You can show that $\exists x\, \neg P(x)$ is false

- In order to prove the universal quantified statement $\forall x\, P(x)$ is false
  - It is enough to exhibit some $x \in D$ for which $P(x)$ is false
  - This x is called the **counterexample** to the statement $\forall x\, P(x)$ is true

Summary

- In order to prove the existential quantified statement $\exists\ x\ Q(x)$ is <u>true</u>
  - It is enough to exhibit some $x \in D$ for which $Q(x)$ is true

- In order to prove the existential quantified statement $\exists\ x\ Q(x)$ is <u>false</u>
  - It is **not** enough to show that $Q(x)$ is false for some $x \in D$
  - You must show that $Q(x)$ is false for every $x \in D$

Free and Bound Variables

- The variable is either **bound** or **free.**

Examples:

- $P(x, y)$, x and y are free variables.
- $\forall x\, P(x, y)$, x is bound and y is free.
- "$P(x)$, where $x = 3$" is another way to bind x.
- $\exists x\,(x + y = 1)$, x is bound and y is free.