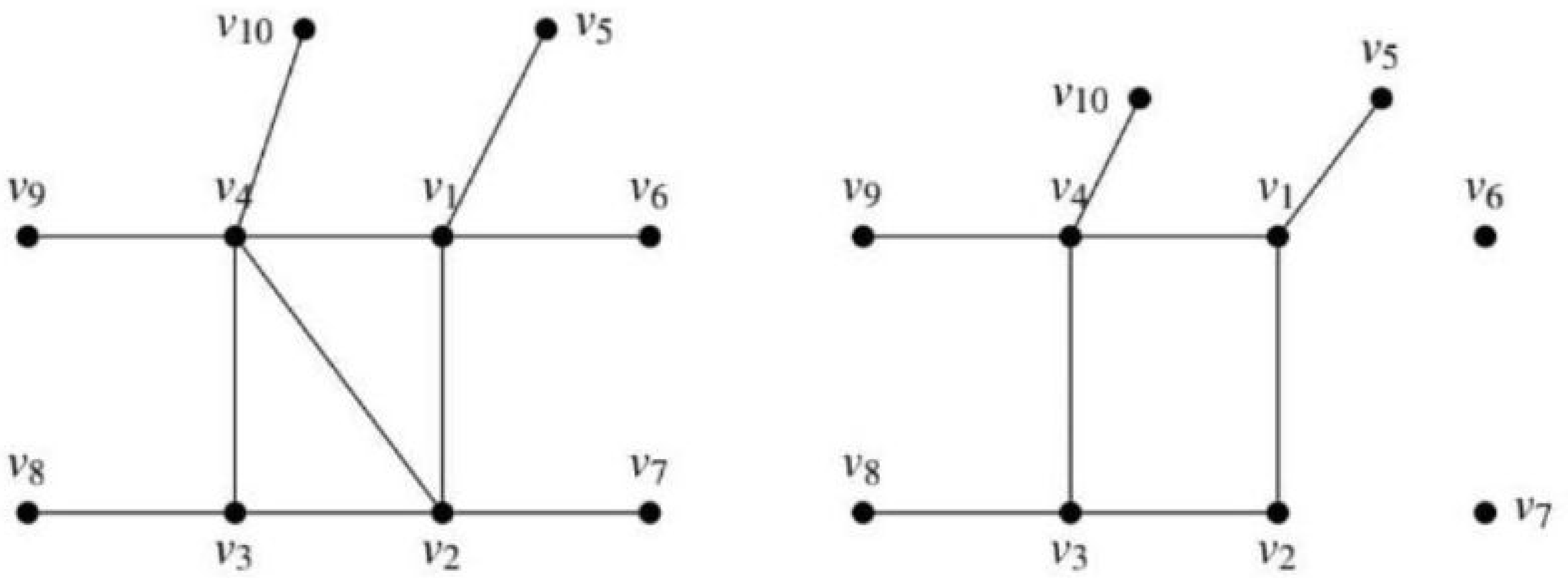
3.3 Edge Deleted and Vertex Deleted Subgraphs

Definition 3.3.1 — Edge Deleted Subgraphs. Let $G(V,E)$ be a graph and $F \subseteq E$ be a set of edges of G. Then, the graph obtained by deleting F from G, denoted by $G-F$, is the subgraph of G obtained from G by removing all edges in F. Note that $V(G-F) = V(G)$. That is, $G-F = (V, E-F)$.

Note that any edge deleted subgraph of a graph G is a spanning subgraph of G.

Definition 3.3.2 — Vertex Deleted Subgraphs. Let $W \subseteq V(G)$ be a set of vertices of G. Then the graph obtained by deleting W from G, denoted by $G-W$, is the subgraph of G

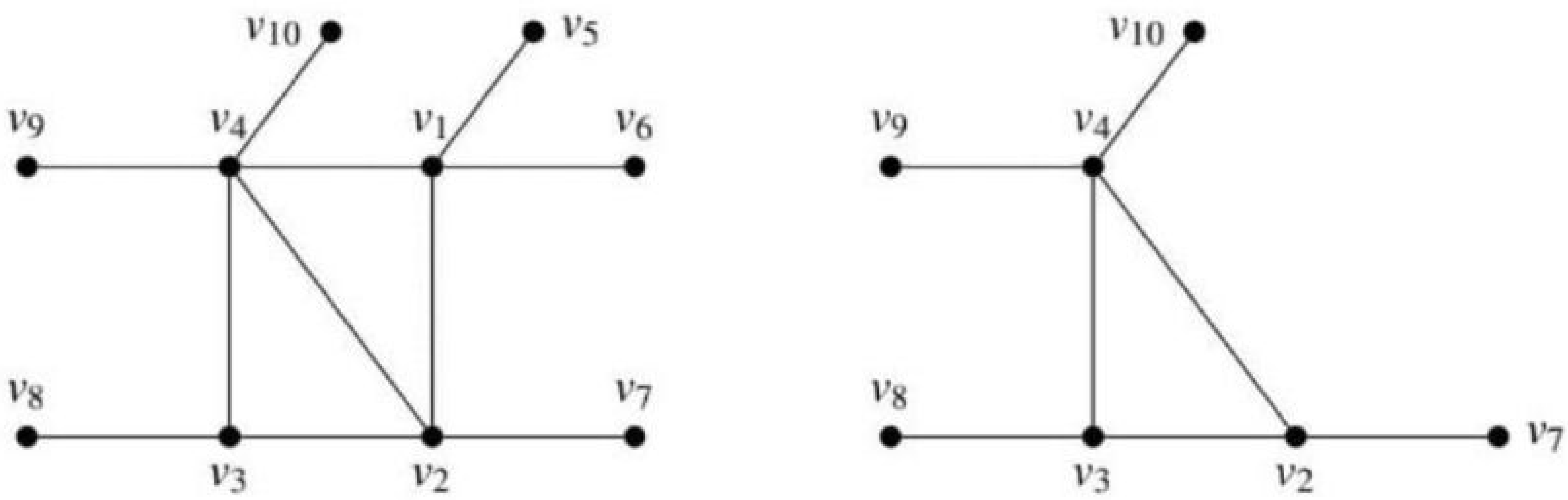


(a) G (b) $G' = G - \{v_1v_6, v_2v_4, v_2v_7\}$.

Figure 3.2: A graph and its edge deleted subgraph.

obtained from G by removing all vertices in W and all edges incident to those vertices.

See Figure 3.3 for illustration of a vertex-deleted subgraph of a given graph.



(a) G (b) $G' = G - \{v_1, v_5, v_6\}$.

Figure 3.3: A graph and its edge deleted subgraph.

Cut-Edges and Cut-Vertices

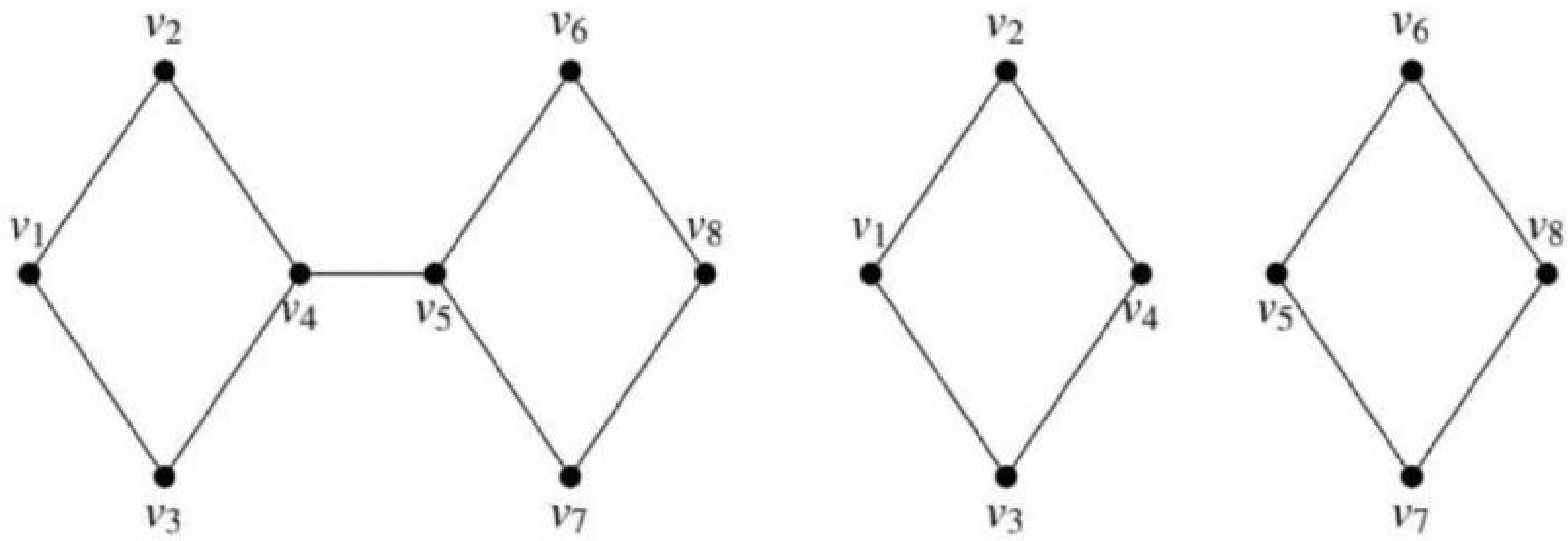
Definition 3.3.3 — Cut-Edge. An edge e of a graph G is said to be a *cut-edge* or a *bridge* of G if $G - e$ is disconnected.

In the above graph G, the edge v_4v_5 is a cut-edge, since $G - v_4v_5$ is a disconnected graph.

The following is a necessary and sufficient condition for an edge of a graph G to be a cut edge of G.

Theorem 3.3.1 An edge e of a graph G is a cut-edge of G if and only if it is not contained in any cycle of G.

Proof. Let $e = uv$ be a cut edge of G. Then, the vertices u and v must be in different components of $G - e$. If possible, let e is contained in cycle C in G. Then, $C - e$ is a path between u and v in $G - e$, a contradiction to the fact that u and v are in different components of $G - e$. Therefore, e can not be in any cycle of G.

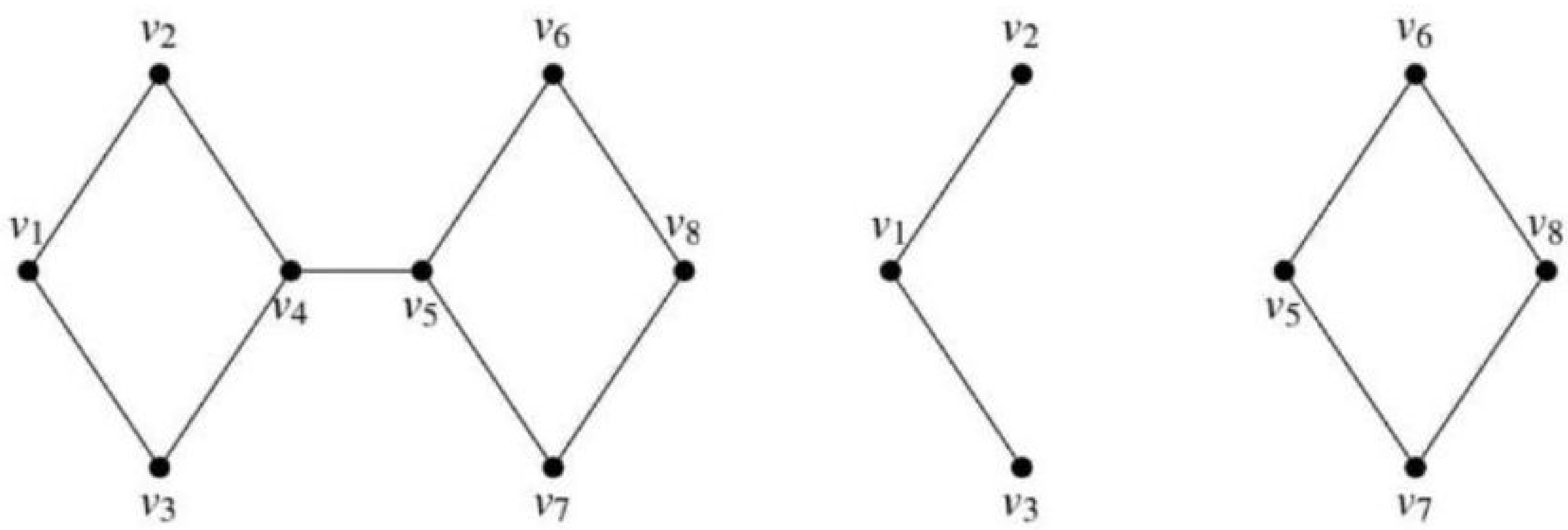


(a) Connected graph G (b) $G \circ uv$

Figure 3.4: Disconnected graph $G - v_4v_5$

Conversely, assume that e is not in any cycle of G. Then, there is no (u, v)-path other than e. Therefore, u and v are in different components of $G - e$. That is, $G - e$ is disconnected and hence e is a cut-edge of G. ■

Definition 3.3.4 — Cut-Vertex. A vertex v of a graph G is said to be a *cut-vertex* of G if $G - v$ is disconnected.



(a) Connected graph G (b) $G \circ uv$

Figure 3.5: disconnected graph $G - v_4$

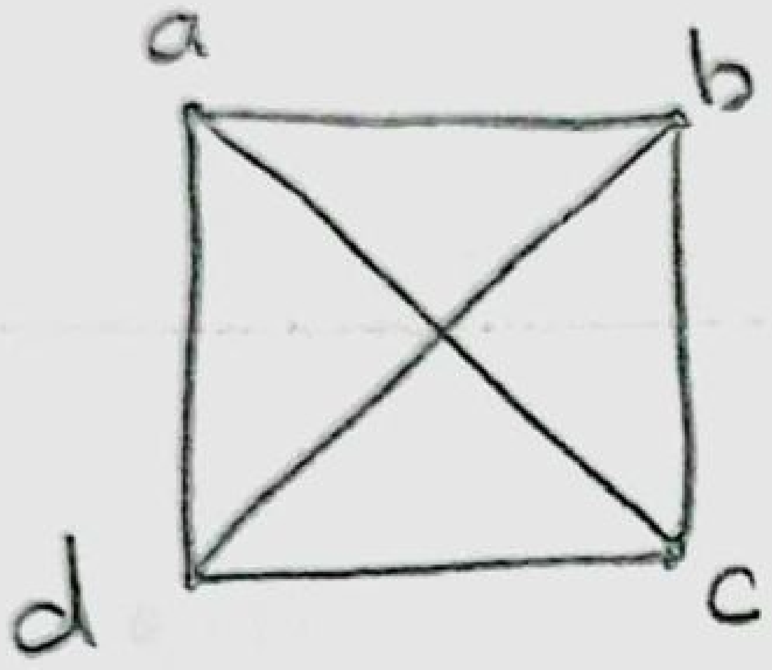
In graph G, v_4 is a cut-vertex as $G - v_4$ is a disconnected graph. Similarly, v_5 is also a cut-vertex of G.

Since removal of any pendent vertex will not disconnect a given graph, every cut-vertex will have degree greater than or equal to 2. But, note that every vertex v, with $d(v) \geq 2$ need not be a cut-vertex.

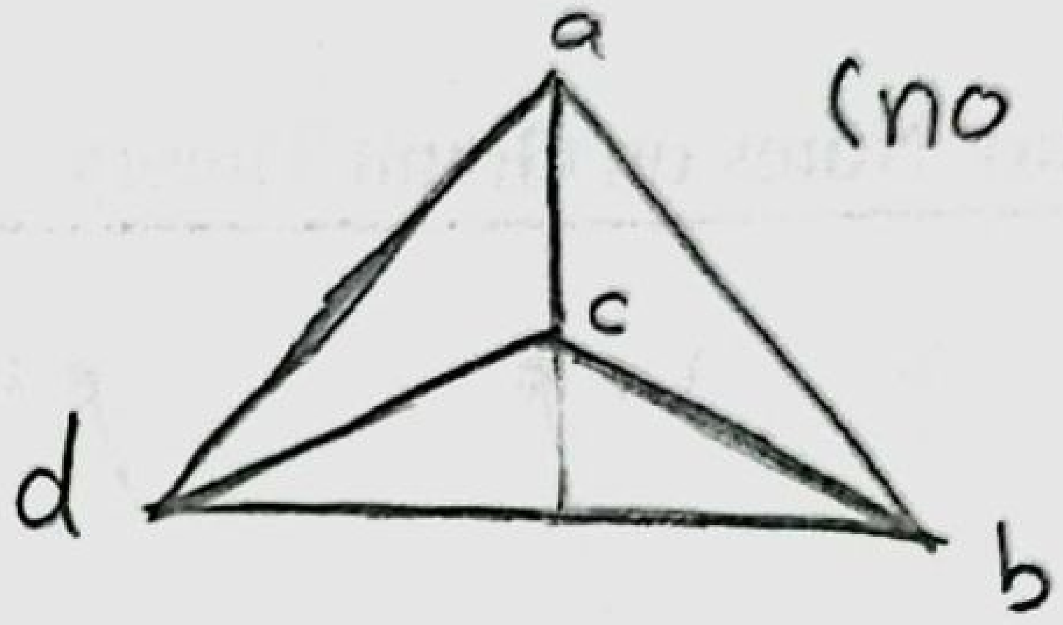
3.4 Exercises

1. Show that every uv-walk contains a uv-path.
2. Show that every closed walk contains a cycle.

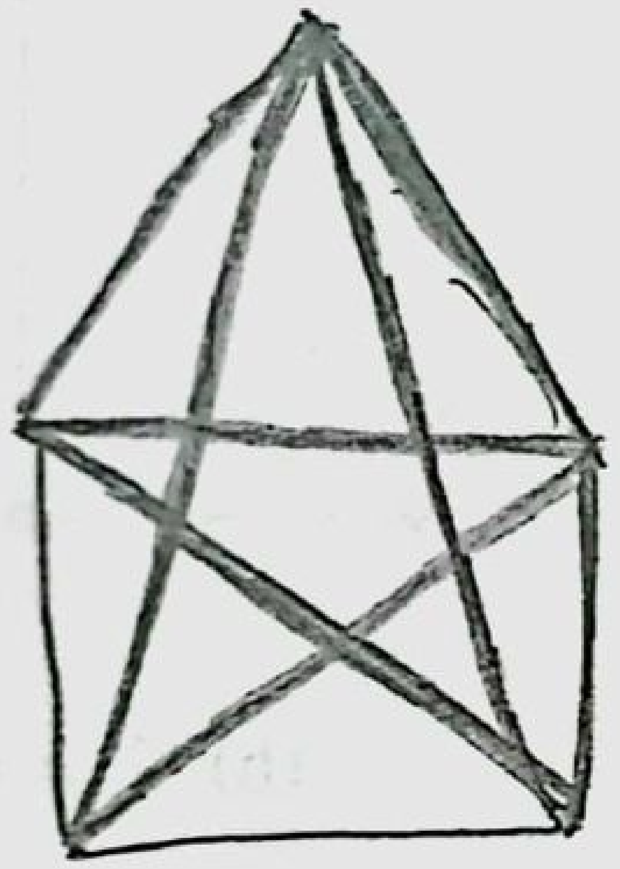
$K_4 \rightarrow$



cycle (no cut edge) (no cut vertex)



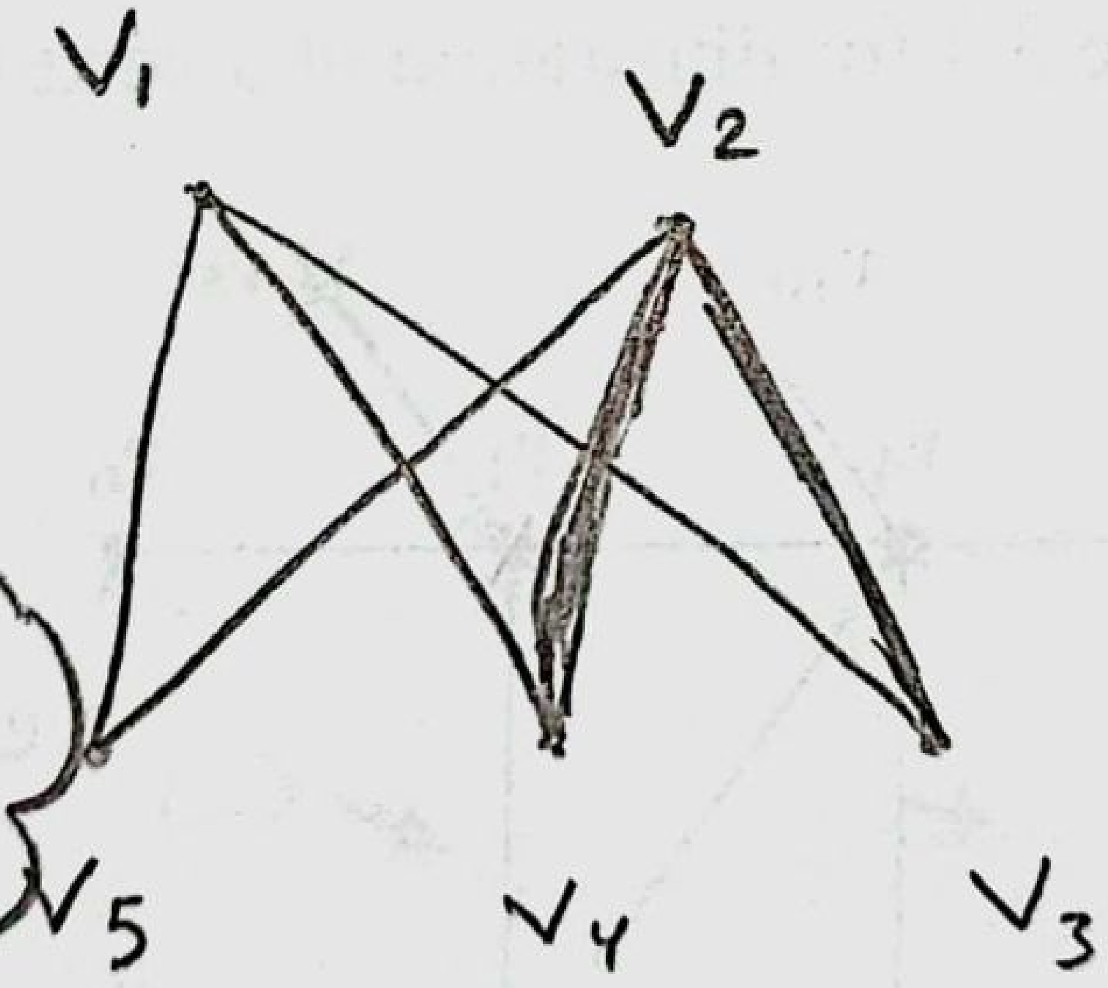
$K_5 \longrightarrow$



(no cutedge)

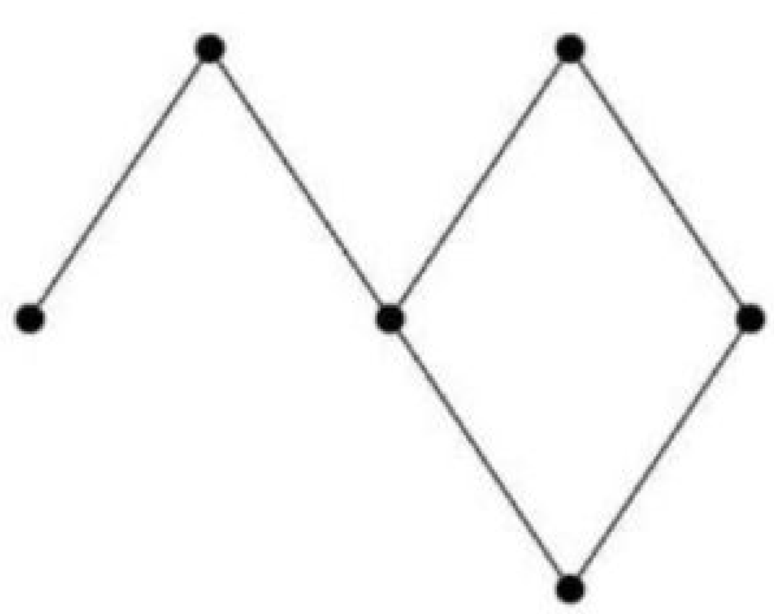
$K_n \rightarrow$ no cut edge
no cut vertex

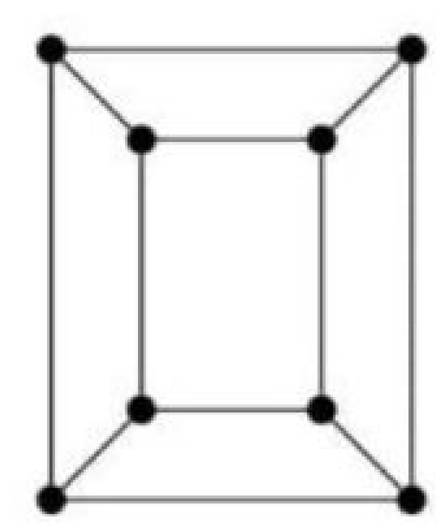
$K_{2,3}$



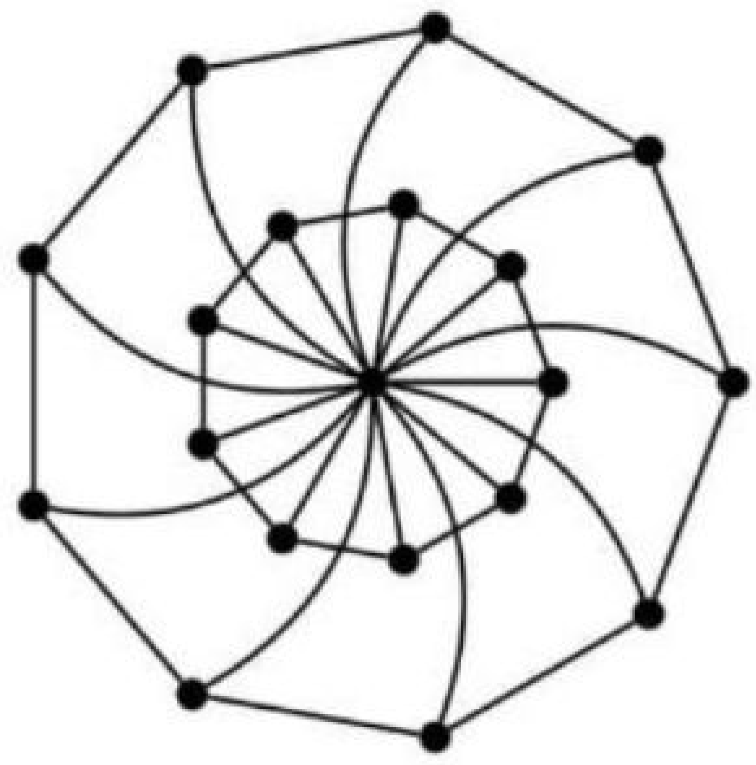
$K_{n,m} \rightarrow$ no cut vertex
no cut edge

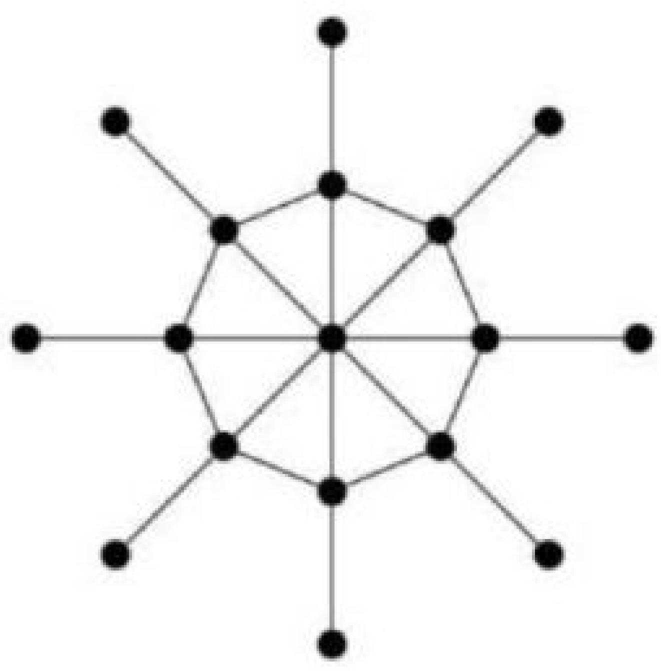
3. Show that every graph with n vertices and k edges, $n > k$ has $n-k$ components.
4. If every vertex of a graph G has degree greater than or equal to 2, then G has some cycles.
5. If G is a simple graph with $d(v) \geq k$, $\forall v \in V(G)$, then G contains a path of length at least k. If $k \geq 2$, then G contains a cycle of length $k+1$.
6. Show that if G is simple and $\delta(G) \geq k$, then G has a path of length k.
7. If a connected graph G is decomposed into two subgraphs G_1 and G_2, then show that there must be at least one common vertex between G_1 and G_2.
8. If we remove an edge e from a graph G and $G-e$ is still connected, then show that e lies along some cycle of G.
9. If the intersection of two paths is a disconnected graph, then show that the union of the two paths has at least one circuit.
10. If P_1 and P_2 are two different paths between two given vertices of a graph G, then show that $P_1 \oplus P_2$ is a circuit or a set of circuits in G.
11. Show that the complement of a complete bipartite graph is the disjoint union of two complete graphs.
12. For a simple graph G, with n vertices, if $\delta(G) = \frac{n-1}{2}$, then G is connected.
13. Show that any two longest paths in a connected graph have a vertex in common.
14. For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.
15. Determine the maximum number of edges in a bipartite subgraph of the Petersen graph.
16. If H is a subgraph of G, then show that $d_G(u,v) \leq d_H(u,v)$.
17. Prove that if a connected graph G has equal order and size, then G is a cycle.
18. Show that eccentricities of adjacent vertices differ by at most 1.
19. Prove that if a graph has more edges than vertices then it must possess at least one cycle.
20. If the intersection of two paths is a disconnected graph, then show that the union of the two paths has at least one cycle.
21. The radius and diameter of a graph are related as $rad(G) \leq diam(G) \leq 2r(G)$.

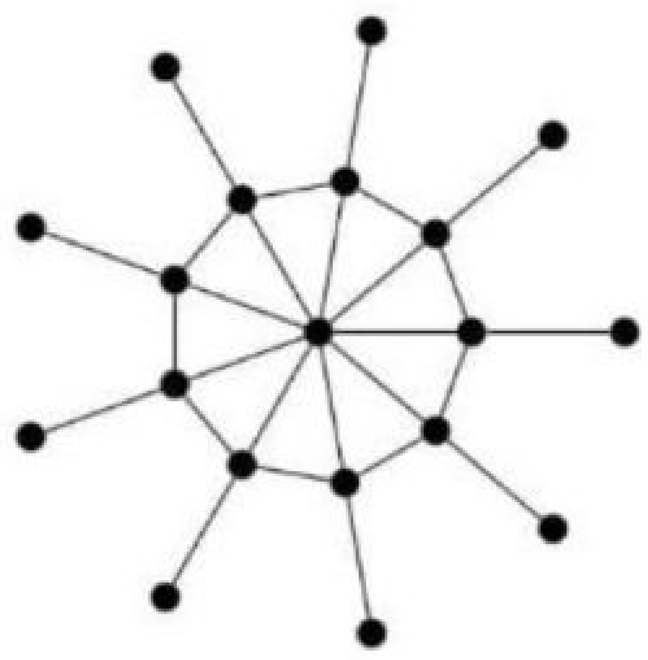


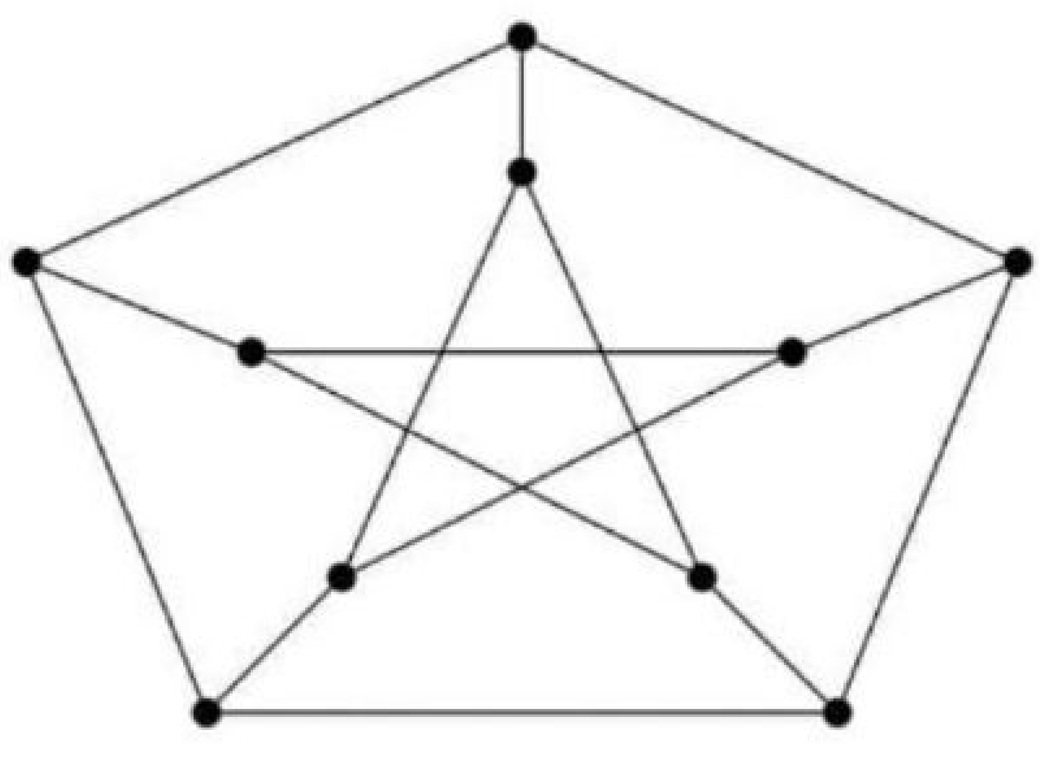


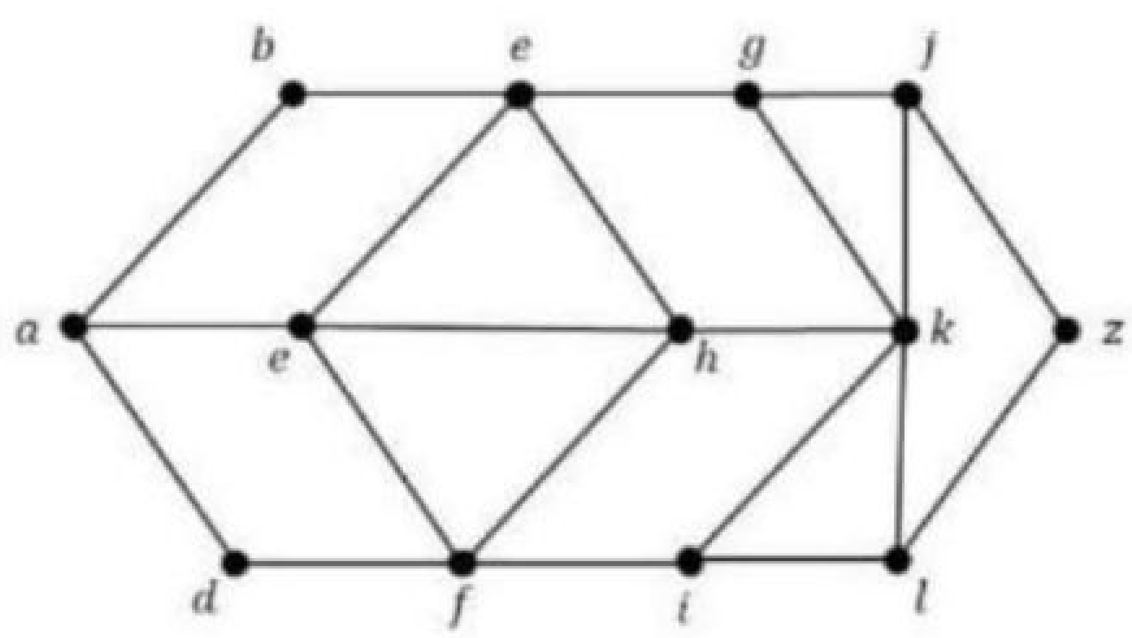
22. Find the eccentricity of the vertices and the radius, the diameter and center(s) of the following graphs:











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