Elementary Differential Equations With Applications

By

Prof. Hamed Moustafa EL-Sherbiny

Elementary Differential Equations

With Applications

By

Prof. Hamed Moustafa EL-Sherbiny

Contents

Chapter 3

3- Differential equations of the first order and the higher degrees

3-1 Equations can be solved in p

3-2 Equations can be solved in x

3-3 Equations can be solved in y

3-4 Clairaut's equation

Chapter 4

4- Homogenous linear differential equations of higher order with constant coefficients

4-1 Auxiliary (Characteristic) equation and classification

4-2 the roots are real and distinct

4-3 some of the roots are real and identical

4-4 some of the roots are complex

Chapter 5

5-1 Introduction

5-2 The differential operator method

5-3 undetermined coefficients

5-4 Variation of parameters

Chapter 1

Introduction

Introduction

The words differential and equations certainly suggest solving some kind of equations that contain derivatives. So it is: in fact tells the complete story about the course that you are about to begin.

But before you start solving anything you must learn some of the basic definitions and terminology of the subject .This is what section 1-1 is all about.

1-1 Basic definitions and terminology

Definitions 1.1 <u>differential equation</u>

An equation containing the derivatives or differentials of one or more dependent variables with respect to one or more independent variables is said to be a differential equation (DE).

If the number of independent variables is only one, the differential equation is ordinary differential equation (ODE), otherwise is said ti be partial differential equation (PDE) if there are more than one independent variables.

Definition 1.2 <u>Order of the differential equation</u>

The order of the highest order in a differential equation is called the order of the equation.

For example the equation $y'' + x\,y' = e^x$ is of order two.

Definition 1.3 <u>Degree of the differential equation</u>

The Degree of a differential equation is the higher power of the derivative of the highest order in the differential equation after making the equation not contain any roots.

Examples:

1- The differential equation

$\left(\frac{d^2y}{dx^2}\right)^2 + 2\left(\frac{dy}{dx}\right)^3 + 5y^8 = x^3$ Is of second order and third degree

2- the equation :

$$z_{xx} + 3z_{xy} = x + y$$

Is a partial differential equation of order two and degree one

3- the differential equation on the form

$$y = 2xy' + \frac{c}{y'}$$

Is a differential equation of the first order and second degree where if we multiply both sides of the equation by y' then, the equation reduces to:

$$y\,y' = 2x(y')^2 + C$$

Or

$$y\,y' - 2x(y')^2 = C$$

Definition 1.4 **<u>Linear differential equations</u>**

A linear differential equation is any differential equation that can be written in the following form:

$$a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots + a_1\, y'(t) + a_0 y(t) = g(t) \qquad (1)$$

Where the coefficients

$$a_n(t)\ ,a_{n-1}(t)\ ,\dots\dots\dots,a_1(t),a_0(t)\ and\ g(t)$$

Are continuous functions

It is clear that the differential equation is linear if it is of degree one, otherwise it is nonlinear.

Definition 1.5 <u>Solution of a differential equation.</u>

Any function $f(x)$ defined on some interval I which when substituted into a differential equation reduces the equation to identity, is said ti be a solution of the equation on the interval I

Example: The function $y = \frac{x^4}{16}$ is a solution of the equation $y' = x\sqrt{y}$

And the function $y = x\,e^x$ is a solution of $y'' - 2y' + y = 0$ *(the proof is left to the student)*

Once we defined the solution of the differential equation, we turn to review the different types of solutions:

(i) General solution:

The general solution is the solution that satisfy the differential equation and contains arbitrary constants.

(ii) Particular solution:

The special solution is the solution that satisfy the differential equation, does not contain arbitrary constants, and obtained from the general solution by assigning values to the parameters.

(iii) Singular solution:

The singular solution is the solution that satisfy the differential equation, does not contain arbitrary constants, and cannot obtained from the general solution by assigning values to the parameters.

(iv) Explicit and implicit solution

An explicit solution is any solution that given in the form $y = y(t)$

An implicit solution is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions or actual implicit / explicit solutions.

Once we know the definition of the solution of the differential equation and know that the general solution is a function contains arbitrary constants, the question that arise here is that if we have a function with arbitrary constants can we derive the differential equation that can be satisfied by this function ? .This is the aim of the next section

1-2 Formation of the differential equation

The technique herein is by eliminating the arbitrary constants from the given general solution making use of differentiation of the curve many times equal to a number of constants and the resulting equation should be of order equal to the number of constants.

To clarify this, suppose that we have the following relationship between the variables x, y in the following form

$$F(x, y, c) = 0 \qquad (1)$$

Equation (1) represents a set of plane curves with one parameter c and with the process of differentiation with respect to x we obtain

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\left(\frac{\partial F}{\partial x}\right) / \left(\frac{\partial F}{\partial y}\right) \qquad (2)$$

The equation (2) contains at its right side a function in the variables x,y and parameter c and by eliminating constant c between equations (1) and (2) we obtain an equation containing x, y, y'

$$\emptyset\ (x, y, y') = 0 \qquad (3)$$

Which is a differential equation of the first order corresponds to the set of curves (1). (Since the set of curves contains only one constant)

The geometric meaning of the differential equation (3) is that gives a relationship between the coordinates of any point in the level and the direction of the group curve (1) passing through this point.

Example 1:

Form the differential equation of the curve equation

$$(x-\alpha)^2 + (y-\beta)^2 = 0 \qquad (1)$$

The parameters in the equation are absolute α, β only and c is constant

<u>Solution</u>

$$(x-\alpha) + (y-\beta)\, y' = 0 \quad (2)$$

$$\Rightarrow 1 + \ (y-\beta)\, y' + {y'}^2 = 0 \qquad (3)$$

$$\Rightarrow (y-\beta) = \ -\frac{1+{y'}^2}{y'} \qquad (4)$$

By substitution from equation (4) in equation (2) we get the equation

$$(x-\alpha)=\frac{y'(1+y'^2)}{y'} \qquad (5)$$

From equations (4), (5) in equation (1) we obtain the differential equation of the curve

Example 2:

Derive the ODE that corresponds the curve $y(x) = a\,e^{bx}$

<u>Solution</u>

Differentiating the curve twice with respect to x, we have:

$$y' = a\,be^{bx} \quad and \qquad y'' = a\,b^2\,e^{bx}$$

It is clear that:

$$\frac{y''}{y'} = b = \frac{y'}{y}$$

Thus, we have,

$$y\,y'' - y'^2 = 0$$

<u>Exercises</u>

(i): Specify the order and degree for each of the following differential equations:

1- $y'' + 2xy' - y = e^x$

2- $y' + 3y'^3 + 5y = 0$

3- $5\left(\frac{dy}{dx}\right)^5 - \left(\frac{d^2y}{dx^2}\right) + 3y = x$

4- $1 + y'^2 = 2y'$

(ii): Derive the differential equations that correspond to each of the following the general solutions:

1- $y = a\ sinx + b\ cosx$

2- $y = c_1e^{3x} + c_2e^{2x}$

1-3 Review on the chapter

We classify a differential equation by its type: ordinary or partial; by its order and by whether it is linear or nonlinear.

A solution of a differential equation is any function having a sufficient number of derivatives that satisfies the equation identically on some interval.

When, we have n-parameter family of solutions called general solution of certain differential equation, one can derive this equation by differentiating the equation n time and eliminate the n-th arbitrary constants and the resulting equation should be of n-th order

Once we know how to form a differential equation from a given parameter family of solutions, we are now in a position to solve some differential equations. We begin with first order differential equations. We shall see that the technique for solving it depends on what kind of first order equation it is.

Thus there many methods of solution; what works for one kind does not necessarily apply to another kind of equation.

This is the aim of the next chapter

Chapter 2

First order differential equations

2-1 Preliminary Theory

We are often interested in solving a first-order differential equation:

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Subject to the initial condition $y(x_0) = \ y_0$ where x_0 is a number in an interval I and y_0 is an arbitrary real number. The equation (1) under this condition is called initial value problem (IVP). Two fundamental questions arise in considering an initial value problem. Does the solution of the problem exist? And if a solution exists, is it unique?

It is often desirable to know before tackling the (IVP) whether the solution exists and when it does, whether it is the only solution of the problem. The following theorem due to Picard gives conditions that are sufficient to guarantee that the solution exists and is unique

Theorem 2-1 (Existence of a unique solution)

Let R be a rectangular region in the xy-plane defined by $a < x < b$, c < y< d that contains the point (x_0 , y_0) in its interior. If $f(x, y)$

And $\frac{\partial f}{\partial y}$ are continuous on R, then there an interval I centered at x_0 and a unique function $y (x)$ defined on I satisfying the initial value problem above.

Example 1:

Discuss the existence and uniqueness theorem on the (IVP):

$$\frac{dy}{dx} = x\sqrt{y} \quad , \qquad y(0) = 0$$

Solution

It is clear that it possesses at least two solutions whose graphs pass through (0,0) (precisely $y = 0$, $y = \frac{x^4}{16}$), one since the functions $f(x, y) = x\sqrt{y}$, $\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}$ are continuous in the upper half-plane defined by $y_0 > 0$

Say for example (0,1), there is some interval around x_0 on which the differential equation has a unique solution.

Example 2:

Use the existence and uniqueness theorem to prove that the equation:

$\frac{dy}{dx} = x^2 + y^2$ possess a unique solueion.

Solution

We observe that $f(x,) = x^2 + y^2$, $\frac{\partial f}{\partial y} = 2y$ are continuous throughout out the center xy- plane. It can be further shown that the interval is $(-\infty, \infty)$ and the unique solution exists on the interval.

<u>2-2 Separable Differential Equations</u>

A separable differential equation is any differential equation that can be written in the following form:

$$\frac{dy}{dx} = \frac{F(x)}{G(y)}$$

Note that in order for a differential equation to be separable all the $y's$ in the differential equation must be in one side multiplied by dy and all the $x's$ in the differential equation must be on the other multiplied be dx, i.e, can be written in the form:

$$G(y)dy = F(x)dx$$

$$\int G(y)\,dy = \int F(x)\,dx$$

Example 1:

Solve the initial value problem:

$$\frac{dy}{dx} = -\frac{x}{y} \qquad , \quad y(4) = 3$$

Solution

From $ydy = -xdx$, Thus $\int y\,dy = -\int x\,dx$

Then $y^2 + x^2 = c^2$, but using the initial condition, then $c^2 = 25$ and the solution is

$$y^2 + x^2 = 25$$

Example 2:

Solve the following differential equation:

$$2x(y^2 + 1)dx - 2y(x^2 + 1)dy = 0$$

Solution

It is clear that this differential equation is separable. So, let's separate the variables and then integrate both sides.

By dividing on the term $(y^2+1)(x^2+1)$ we get

$$\frac{2x}{(x^2+1)}dx - \frac{2y}{(y^2+1)}dy = 0$$

By integrating both sides we get

$$\ln(x^2+1) - \ln(y^2+1) = c$$

Where c is the arbitrary constant we can put in the form $\ln c_1$ and get

$$\frac{x^2+1}{y^2+1} = c_1$$

This form represents the general solution of the required differential equation and can be abbreviated as follows

$$(x^2+1) = c_1\,(y^2+1)$$

The equation either represents ellipse or hyperbola and this depends on the value of the constant

There are some equations which cannot be separable but can be reduced to separable differential equations. These equations are in the form:

$$\frac{dy}{dx} = f\,(ax+by+c), \quad a, b, c \text{ are constants}$$

For the first moment one can think that the variables cannot be separated so we take the substitution

This implies that

$$\frac{dz}{dx} = a + b\ \frac{dy}{dx} = a + b\ f(z)$$

In the last equation the variables can be separated as follows

$dx = \frac{dz}{a+bf(z)}$. Thus we can get x in terms of $z = ax + by + c$

Example 3:

Solve the differential equation

$$\frac{dy}{dx} = (x + y - 1)^2$$

Solution

Let $= x + y - 1$, $\frac{dz}{dx} = 1 + \frac{dy}{dx} = 1 + z^2$, thus the variables can be separated as: $dx = \frac{dz}{1+z^2}$ Thus $x + c = tan^{-1}z$

i.e, $x + c = tan^{-1}(x + y - 1)$

Example 4:

Solve the following differential equation.

$$\frac{dy}{dx} = 6y^2x$$

Solution

Let's separate the variables in the differential equation

$$\frac{dy}{y^2} = 6x\ dx$$

And the integrating both sides, we get

$$\int \frac{dy}{y^2} = \int 6x\, dx \quad \Rightarrow \quad -\frac{1}{y} = 3x^2 + c$$

This is the general solution of the differential equation

Exercises

Find the general solution for each of the following differential equations:

1- $y\, y' = x(y+1)$

2- $y' = x\sqrt{1-y^2}$

3- $y' = -\frac{x}{y}$

4- $(1+x)dy - y\, dx = 0$

5- $y' = \cos^2 x \cos y$

6- $\csc y\, dx + \sec^2 x\, dy = 0$

7- $y' = e^{-x-y}$

8- $y' = (y-1)(y-2)$

9- $y' = xy^2 + y^2 + xy + y$

10- $\frac{dy}{dx} = y^2 - 9$

Further examples of differential equations that can be reduced to separable variables

Example 4:

Solve the following differential equation

$$(x+y)dx - dy = 0$$

Solution

This equation is in a form whose variables cannot be separated directly but placed $u = x + y$ we can separate the resulting equation variables because

$$du = dx + dy \quad \Rightarrow \quad dy = du - dx$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + u$$

Hence, $dx = \frac{du}{1+u}$ and integrating both sides yields

$$x + c = \ln(1+u) = \ln(x+y+1)$$

And this is the general solution of the differential equation

Example 5:

Solve the following differential equation

$$y' = (x+y)^2$$

Solution

Similar to that used in example 2

Let $= x + y$, $\frac{dz}{dx} = 1 + \frac{dy}{dx}$ = $1 + z^2$, thus the variables can be separated as: $dx = \frac{dz}{1+z^2}$ Thus $x + c = tan^{-1}z$

i.e, $x + c = tan^{-1}(x+y)$ which can be put in the form:

$$x + y = \tan(x+c)\,, \qquad c \;\; constant$$

<u>Exercises</u>

Find the general solution for each of the following differential equations:

1- $(2x - y)dx + (4x - 2y + 3)dy = 0$

2- $(2x + y + 1)dx + (4x + 2y + 3)dy = 0$

3- $(2x + 3y - 3)dx + (4x + 6y - 5)dy = 0$

4- $(2x - 6y + 3)dx - (x - 3y + 1)dy = 0$

2-3 Homogenous Differential Equations

The differential equation of the form

$$P(x,y)dx + Q(x,y)dy = 0 \qquad (1)$$

Is said to be homogenous if both functions P, Q are homogenous of the same degree, that is

$$P(x,y) = x^n\, F\left(\frac{y}{x}\right)$$

$$Q(x,y) = x^n G\left(\frac{y}{x}\right)$$

In this way equation (1) becomes the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Let now $z = \frac{y}{x}$ or $y = x\,z$, then

$\frac{dy}{dx} = z + x\frac{dz}{dx}$ or $f(z) - z = \frac{dz}{dx}$ and the variables can now be separated and

have:

$$dx = \frac{dz}{F(z) - z}$$

Thus the variables are separate and x can be found in terms of z and finally z can be substituted by x,y

<u>Examples of homogenous equations:</u>

Example 1:

Fine the general solution for the following differential equation

$$(y^2 + x^2)dx - 2xy\,dy = 0$$

Solution

Where the two functions

$$P(x, y) = y^2 + x^2 \quad , \quad Q(x, y) = -2xy$$

Both are homogenous from the degree 2 and thus the equation is homogenous so

Let $y = x\,z$ then $dy = x\,dz + z\,dx$

In the original equation we get

$$(1 - z^2)dx - 2xz\,dz = 0$$

By separating variables we find that

$$\frac{2z}{1 - z^2}dz = \frac{dx}{x}$$

By integration, we get

$$\int \frac{-2z}{1-z^2}\,dz = -\int \frac{dx}{x} + \ln c$$

$$\Longrightarrow \ln(1-z^2) = -\ln x + \ln c$$

$$\Longrightarrow \quad 1 - z^2 = \frac{c}{x} \quad \Rightarrow \quad z^2 = 1 - \frac{c}{x}$$

since $z = \frac{y}{x}$ we find that the general solution of the differential equation is

$$y^2 = x^2 - xc$$

Example 2:

Find the general solution for the following differential equation

$$\left(2y\, e^{\frac{y}{x}} - x\right)\frac{dy}{dx} + 2x + y = 0$$

Solution

Where the two functions

$$P(x,y) = 2y\, e^{\frac{y}{x}} - x \qquad and \qquad Q(x,y) = 2x + y$$

Both are homogenous why?

Let $y = x\,z$ then

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

In the original equation we get

$$(2z\,e^{z} - 1)\left(z + x\frac{dz}{dx}\right) + 2 + z = 0$$

$$\Rightarrow z + x\frac{dz}{dx} = -\frac{2+z}{2z\,e^{z} - 1}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{2+z}{1-2z\,e^{z}} - z$$

$$\Rightarrow x\frac{dz}{dx} = \frac{2(1+z^{2}e^{z})}{1-2z\,e^{z}}$$

$$\Rightarrow \int \frac{1-2z\,e^{z}}{2(1+z^{2}e^{z})}dz = \int \frac{dx}{x}$$

$$\Rightarrow -\int \frac{-e^{-z}+2z}{e^{-z}+z^{2}}dz = 2\int \frac{dx}{x}$$

$$\Rightarrow 2\ln x = -\ln(e^{-z}+z^{2}) + \ln c$$

$$\Rightarrow 2\ln x = \ln\frac{c}{e^{-z}+z^{2}}$$

$$\Rightarrow x^{2} = \frac{c}{e^{-z}+z^{2}}$$

$$y^{2} + x^{2}e^{\frac{x}{y}} = c$$

<u>Important note</u>

There are different types of differential equations with linear coefficients that are not homogeneous for the first glance but can be reduced to homogeneous equations. These linear equations have the following form

$$(a_1x + b_1y + c_1)\, dx + (a_2x + b_2y + c_2)\, dy = 0 \qquad (1)$$

This equation has two situations, either the two straight lines are intersected or the two lines are parallel

***First*: if the two lines are intersected:**

This case means that $a_1b_2 \neq a_2b_1$ this means that the equations of two lines are

$$a_1x + b_1y + c_1 = 0 \quad , \quad a_2x + b_2y + c_2 = 0$$

Intersect at the point (u, v) so, we put

$$x = X + u\,, \quad y = Y + v$$

$$\Rightarrow dx = dX\,, \quad dy = dY$$

Using this to substitution in the equation (1), the equation yields:

$$(a_1X + b_1Y)dX + (a_2X + b_2Y)dY = 0$$

This is a homogenous differential equation that can be solved as above.

***Second*: if the two lines are parallel:**

This case means that $a_1 b_2 = a_2 b_1$ this means that the equations of two lines are

$$a_1 x + b_1 y + c_1 = 0 \ , \quad a_2 x + b_2 y + c_2 = 0$$

Are parallel and

$$a_1 x + b_1 y = \alpha(a_2 x + b_2 y)$$

Then equation (1) becomes the next form

$$(a_1 x + b_1 y + c_1)dx + (\alpha(a_2 x + b_2 y) + c_2)dy \ = 0$$

So, we use

$$u = a_1 x + b_1 y \ \Rightarrow \quad du = a_1 dx + b_1 dy$$

$$\Rightarrow dy = \frac{1}{b_1}(du - a_1 dx)$$

The pervious equation turns to take the form:

$$(u + c_1)dx + (\alpha u + c_2)\left(\frac{du - a_1 dx}{b_1}\right) = 0$$

$$\Rightarrow [\ b_1(u + c_1) - a_1(\alpha u + c_2)]dx + (\alpha u + c_2)du = 0$$

And the solution can be given as above

<u>Example 3:</u>

Find the general solution for the following differential equation:

$$\frac{dy}{dx}=\frac{4x-y+7}{2x+y-1}$$

Solution

Where the two lines intersect why? We want to find the intersect point.

$$4x-y+7=0\,,2x+y-1=0$$

We find that point of intersection between them is $(u,v)=(-1,3)$, so we put

$$x=X-1\ \ ,\ \ y=Y+3$$

$$\Rightarrow dx=dX\,,\quad dy=dY$$

In equation (1) we find that

$$\frac{dY}{dX}=\frac{4X-Y}{2X+Y}\qquad (2)$$

Equation (2) is homogenous and can be solved as follows:

Put $Y=X\,Z$, and then

$$\frac{dY}{dX}=Z+X\frac{dZ}{dX}$$

By substituting in the equation (2) we get

$$Z+X\frac{dZ}{dX}=\frac{4-z}{2+z}\quad\Rightarrow X\frac{dZ}{dX}=\frac{-(Z^2+3Z-4)}{2+Z}$$

$$\Rightarrow\ \int\frac{dX}{X}=-\int\frac{(2+Z)dZ}{Z^2+3Z-4}$$

$$\Rightarrow \ln X = -\frac{1}{5}\int\left(\frac{2}{Z+4}+\frac{3}{Z-1}\right)\,dZ$$

$$\Rightarrow\ -5\ln X = 2\ln(Z+4)+3\ln(Z-1)+\ln C$$

$$\Rightarrow \frac{1}{X^5} = (Z+4)^2(Z-1)^3C$$

$$\frac{1}{X^5} = \left(\frac{Y}{X}+4\right)^2\left(\frac{Y}{X}-1\right)^3 C$$

$$\Rightarrow 1 = (Y+4X)^2\,(Y-X)^3C$$

$$\Rightarrow 1 = (y+4x+1)^2\,(y-x-4)^3\,C$$

Exercises

Find the general solution for each of the following differential equations:

1- $y' = \frac{x^2+y^2}{x\,y}$

2- $(3x^2y+y^3)\,dx+(x^3+3xy^2)\,dy=0$

3- $y' = \frac{x-y}{x+y}$

4- $(x^2 + y^2)\, dx - x\, y\, dy = 0$

5- $x\, y' - y = x\, e^{\frac{y}{x}}$

2-4 Exact differential equations

Definition

The differential equation:

$$P(x, y)\, dx + Q(x, y)\, dy = 0$$

Is said to be exact if and only if the left side of the equation can put in fully differential component of a function U n i.e,

$$dU = P(x, y)\, dx + Q(x, y) dy$$

In other words, the equation $U(x, y)$ is the same

$$\frac{\partial U}{\partial x} = P(x, y) \quad , \quad \frac{\partial U}{\partial y} = Q(x, y)$$

In this case, the general solution is $U(x, y) = c$

Theorem

The necessary and sufficient condition for the equation:

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

To be exact is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Where P , Q have the first partial derivatives, the condition is necessary and sufficient for the equation to be exact.

Examples for exact differential equations

Example 1:

Find the general solution for the following equation

$$(2x\cos y + 3x^2\,y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$$

Solution

We first look at whether this equation is exact or not

$$P = 2x\cos y + 3x^2\, y \quad , \quad Q = x^3 - x^2 \sin y - y$$

$$\frac{\partial P}{\partial y} = -2x\sin y + 3x^2 \quad , \quad \frac{\partial Q}{\partial x} = 3x^2 - 2x\sin y$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial u}{\partial y} = -x^2 \sin y + x^3 + \emptyset'(y)$$

In comparison with equation (ii) we find that

$$\emptyset'(y) = -y \;\Rightarrow\; \emptyset(y) = -\frac{1}{2}y^2$$

Substituting this value in equation (iii) yields:

$$u = x^2\cos y + x^3 y - \frac{y^2}{2}$$

Since the general solution is $u = c$ so the general solution is

$$x^2 \cos y + x^3\, y - \frac{y^2}{2} = c$$

Example 2:

Find the general solution for the following differential equation:

$$(2y^2 + 4xy - x^2)\, dx + (2x^2 + 4\, xy - y^2)\, dy = 0$$

Solution

We first look at whether this equation is exact or not

$$P = 2y^2 + 4xy - x^2 \quad , \quad Q = 2x^2 + 4xy - y^2$$

$$\frac{\partial P}{\partial y} = 4y + 4x\, , \quad \frac{\partial Q}{\partial x} = 4x + 4y$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

So the equation is exact and there exist a function u such that

$$\frac{\partial u}{\partial x} = P(x, y) = 2y^2 + 4xy - x^2 \qquad (i)$$

$$\frac{\partial u}{\partial y} = Q(x, y) = 2x^2 + 4xy - y^2 \qquad (ii)$$

By integrating the equation (i) we find that

$$u = \int (2y^2 + 4xy - x^2)\, dx + \emptyset(y)$$

$$= 2xy^2 + 2x^2y - \frac{x^3}{3} + \emptyset(y) \qquad (iii)$$

Then

$$\frac{\partial u}{\partial y} = 4xy + 2x^2 + \emptyset'(y)$$

From equation (ii) we find that

$$\emptyset'(y) = -y^2 \;\Rightarrow\; \emptyset(y) = -\frac{1}{3}y^3$$

To compensate for this value in equation (iii) we find that

$$u = 2x^2y + +2xy^2 - \frac{x^3}{3} - \frac{y^3}{3}$$

Since the general solution is $u = c$ so the general solution is

$$2x^2y + 2xy^2 - \frac{x^3}{3} - \frac{y^3}{3} = c$$

<u>Exercises</u>

Find the general solution for each of the following differential equation:

1-$(xy \cos xy + \sin xy)\, dx + (x^2 \cos xy + e^y)\, dy = 0$

2- $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)\, dy = 0$

3- $(2xy\cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0$

4- $(\cos x \cos y - \cot x)\, dx - \sin xy \;\; dy = 0$

5- $(2xy - \tan y)\, dx + (2y + x^2 + 1)\, dy = 0$

6- $(2xy - 9x^2)\, dx + (2y + x^2 + 1)dy = 0$

7- $(2xy^2 + 4)\, dx + 2(x^2y - 3)\, dy = 0$

8- $\left(\frac{2xy}{x^2+1} - 2x\right)\, dx + (\ln(x^2 + 1) - 2)\, dy = 0$

9- $(3y^2\, e^{3xy} - 1)\, dx + (2\, y\, e^{3xy} + 3xy^2\, e^{3xy})dy = 0$

Incomplete equations (integral factor):

Consider the equation

$$M\, dx + N\, dy = 0$$

And assume that it is not complete (exact).We are looking for a function that can reduce the equation into complete (exact) dy multiplying it in both sides if the equation. Let us denote this function by $\mu\,(x, y)$ and call it integrating factor. This is difficult to find.So we restrict our search to the integrating factor as a function of one independent variable x or y, only.

Integrating factor as a function of x only:

Let $\mu(x)$ be the integrating factor and multiplying it in the incomplete equation to reduce it to the complete one gives:

$$\mu M\, dx + \mu N\, dy = 0$$

As it is now complete equation, then:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \text{ Implies } \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x}$$

Separating the variables we have:

$$\frac{\partial \mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

Since, the left hand side is a function of x only, then $\mu(x)$ exists if the right hand side is a function of x only and by integration its value is:

$$\mu(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx}$$

By similar way one can establish the condition for the existence of the integral factor of y only, and has the form:

$$\mu(y) = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy}$$

Examples of incomplete equations (integral factor):

Example 1:

Find the general solution for the following differential equation

$$(2x^2 + y)\,dx + (x^2y - x)\,dy = 0$$

Solution

We first look at whether this equation is exact or not

$$M = 2x^2 + y \;,\; N = x^2y - x$$

$$\frac{\partial M}{\partial y} = 1 \;\;,\;\; \frac{\partial N}{\partial x} = 2xy - 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So the equation is not exact so we must test the theory and look for the integral factor

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2 - 2xy = -2(xy - 1)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{2}{x}$$

$$\Longrightarrow \mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = \frac{1}{x^2}$$

Since the integral factor is a function in x only and after calculating its value, we must multiply the original equation in the integral factor.

$$(2+yx^{-2})\,dx+(y-x^{-1})\,dy=0$$

This resulting equation is a complete equation why? Its general solution takes shape

$$2x+\frac{1}{2}y^2-\frac{y}{x}=c$$

Example 2:

Find the general solution for the following differential equation

$$xy^3\,dx+(x^2y^2-1)\,dy=0$$

Solution

We first look at whether this equation is exact or not

$$M=xy^3\ \ ,\quad N=x^2y^2-1$$

$$\frac{\partial M}{\partial y} = 3xy^2 \quad , \quad \frac{\partial N}{\partial x} = 2xy^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So the equation is not exact so we must test the theory and look for the integral factor

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = xy^2$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{xy^2}{x^2y^2 - 1}$$

Is a function of x and y but

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{xy^2}{xy^3} = -\frac{1}{y}$$

Is a function of y only, then:

$$\mu(y) = e^{-\int \frac{dy}{y} = \frac{1}{y}}$$

So we must multiply the original equation in the integral factor we find that

$$xy^2\, dx + (x^2y - y^{-1})\, dy = 0$$

This resulting equation is a complete equation why? Its general solution takes the form:

$$\frac{x^2y^2}{2} - \ln y = c$$

Exercises

Find the general solution for each of the following differential equations:

1- $(xy + y - 1)\,dx + x\,dy = 0$

2- $2(2y^2 + 5xy - 2y + 4)\,dx + x\,(2x + 2y - 1)\,dy = 0$

3- $(xy + y^2)\,dx + (xy - x^2)\,dy = 0$

4- $\left(3y + 3e^x y^{\frac{2}{3}}\right)\,dx + x\,dy = 0$

5- $(\sin y + x^2 + 2x)\,dx + \cos y\;dy = 0$

2-6 Linear differential equations:

The general form of linear differential equations of the first order and the first degree is

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1)$$

Where p, q are functions of x only.

Assuming that the integral factor is $\mu(x)$ by multiplying the equation (1) in the integral factor, it becomes a complete equation on the next form.

$$\mu(x)\, dy + \mu(x)\{p(x)y - q(x)\}\, dx = 0 \quad (2)$$

Using the conditions of the exact equation we find that

$$\frac{\partial}{\partial y}\{\mu(x)\{p(x)y - q(x)\}\} = \frac{\partial}{\partial x}\, \mu(x)$$

$$\Rightarrow\ \mu(x)p(x) = \frac{\partial}{\partial x}\mu(x)$$

$$\Rightarrow\ \int \frac{\partial \mu}{\mu} = \int p(x)\, dx$$

$$\Rightarrow \ln \mu = \int p(x)\ dx$$

$$\Longrightarrow \quad \mu = e^{\int p(x)\,dx}$$

Then equation (2) becomes the form

$$\Longrightarrow \; e^{\int p(x)\,dx}\,dy + e^{\int p(x)\,dx}\,\{p(x)\,y - q(x)\}\,dx = 0$$

$$\Longrightarrow \; e^{\int p(x)\,dx}\,dy \; - e^{\int p(x)\,dx} p(x).\,y.\,dx = e^{\int p(x)\,dx} q(x)\,dx$$

$$\Longrightarrow \; \frac{d}{dx}\left\{e^{\int p(x)\,dx}.\,y\right\} = e^{\int p(x)\;dx}\;q(x)$$

$$\Longrightarrow \; e^{\int p(x)\,dx}.\,y = \int e^{\int p(x)\;dx}\;q(x)\,dx$$

$$\Longrightarrow \; y = e^{-\int p(x)\,dx}.\int e^{\int p(x)\;dx}\;q(x)\;dx + c$$

The final form is the general solution of the linear equation of the form (1).

Examples of linear differential equations:

Example 1:

Find the general solution for the following differential equation

$$\frac{dy}{dx}+\frac{2}{x}y=x^2$$

Solution

This equation is linear where

$$p(x)=\frac{2}{x}\ ,\ \ q(x)=x^2$$

$$\mu=e^{\int p(x)\,dx}$$

$$\therefore\ \ \mu=e^{\int\frac{2}{x}\,dx}\ \ =e^{2\int\frac{dx}{x}}=e^{2\ln x}$$

So be the general solution to the form

$$\mu.y=\int\mu.q(x)\ dx+c$$

$$\therefore x^2y=\int x^2.x^2\ dx+c$$

$$\therefore x^2y=\frac{x^5}{5}+c\ \Longrightarrow y=\frac{x^5}{5}+\frac{c}{x^2}$$

Example 2:

Find the general solution for the following differential equation

$$\frac{dy}{dx} - \frac{3}{x}y = \frac{1}{x^2}$$

Solution

This equation is linear where $p(x) = -\frac{3}{x}$, $q(x) = \frac{1}{x^2}$

$$\mu(x) = e^{\int p(x)\ dx} = e^{-3\int \frac{dx}{x}} = \frac{1}{x^3}$$

And the general solution is given from the relation

$$\mu . y = \int \mu . q(x)\ dx + c$$

$$\therefore \frac{1}{x^3}\ y = \int \frac{1}{x^3} . \frac{1}{x^2}\ dx + c$$

$$\therefore x^{-3}y = -\frac{1}{4x^4} + c$$

$$y = -\frac{1}{4x} + cx^3$$

Exercises

Find the general solution for each of the following differential equations:

1- $(1 + y^2)\,dx - \left(\sqrt{1 + y^2}\sin y - xy\right)dy = 0$

2- $x\,y' + y - e^x = 0$

3- $y' - y\tan x = \sec x$

4- $\frac{dy}{dx} + 2xy = 4x$

5- $(1 + x^2)\,y' + 4xy = x$

6- $2y' - y = 4\sin 3x$

2-6 The Bernolli equation:

The general form of the Bernolli differential equation of the first order and the first degree is

$$\frac{dy}{dx} + p(x)y = q(x)\, y^{\propto} \qquad (1)$$

Where $\propto \neq 0$, $\propto \neq 1$ and p, q functions of x only. And called the Bernoulli equation of y. to solve this equation we divide the equation on $y^{\propto}$

$$y^{-\propto}\frac{dy}{dx} + p(x)\, y^{1-\propto} = q(x) \qquad (2)$$

Using substitution

$$z = y^{1-\propto} \Rightarrow \frac{dz}{dx} = (1-\propto) y^{-\propto} \frac{dy}{dx}$$

By multiplying the equation (2) in the term $(1-\propto)$, it turns into the following equation

$$\frac{dz}{dx} + (1-\propto) p(x)\, z = (1-\propto)\, q(x)$$

Which is a linear equation in z can be solved as above.

Example 1:

Find the general solution for the following differential equation

$$\frac{dy}{dx} + \frac{2}{x}\, y = x\, y^2 \qquad (1)$$

Solution

This equation is the Bernolli equation with $\propto = 2$ and using the substitution

$$z = y^{1-\propto} = y^{-1} \quad \Rightarrow \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}$$

By multiplying the equation (1) in the term $-y^{-2}$, it turns into the following equation

$$\frac{dz}{dx} - \frac{2}{x}\, z = -x$$

This equation is a linear equation in z and can be solved where

$$p(x) = -\frac{2}{x}\ ,\quad q(x) = -x$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{-2\int \frac{dx}{x}} = \frac{1}{x^2}$$

$$\mu\, z = \int \mu\, q(x)\, dx\ +\ c$$

Thus $\frac{z}{x^2} = -\int \frac{x}{x^2}\, dx + c = -\ln x + c$

i.e, $\frac{1}{yx^2} = -\ln x + c$ and the general solution is:

$$y\, x^2 = \frac{1}{-\ln x + c}$$

Example 2:

Find the general solution for the following differential equation

$$2xy\frac{dy}{dx} - y^2 + x = 0$$

Solution

This equation can be rewritten in the form:

$$\frac{dy}{dx} - \frac{1}{2x}y = -\frac{1}{2}y^{-1}$$

Which is Benolli equation with $\propto = -1$, put

$$z = y^{1-\propto} = y^2 \quad \Rightarrow \frac{dz}{dx} = 2y\frac{dy}{dx}$$

By multiplying the equation (1) in the term $2y$, it turns into the following equation.

$$\frac{dz}{dx} - \frac{1}{x}z = -1$$

Which is linear equation in z

$$\mu\,(x) = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\mu\ z = \int \mu(x)\,q(x)\ \ dx\ + c$$

$$\frac{z}{x} = \int \frac{-\,dx}{x}\,dx\ + \ln c$$

$$\frac{z}{x} = \ln\frac{c}{x} \text{ thus } z = x\ln\frac{c}{x}$$

And finally the general solution is $y^2 = x\ln\frac{c}{x}$

Exercises

Find the general solution for each of the following differential equations:

1- $y\,dx + \left(x - \frac{1}{2}x^2 y\right) dx = 0$

2- $3x\,y' = y(1 + x\sin x - 3y^2 \sin x)$

3- $y' + y = y^2(\cos x - \sin x)$

4- $\frac{dy}{dx} - \frac{3}{x}y = x^4\, y^{1/3}$

5- $y' + y = xy^3$

6- $y' = (x^2y^2 + 2y)/x$

2-7 Riccati differential equation:

The general form of Riccati differential equation is

$$\frac{dy}{dx} = p(x)\, y^2 + q(x)\, y + f(x) \qquad (1)$$

Where p, q, f are a functions of x only. And we can solve it if we know special solution to the equation as $y = y_1$. Using substitution

$$y = y_1 + \frac{1}{z}$$

Equation (1) can be reduced to a linear equation that can be easily solved as above.

Example 1:

Prove that $y = x$ is a solution to the following differential equation and then find its general solution

$$2x^2\frac{dy}{dx} = (x-1)(y^2-x^2) + 2xy \qquad (1)$$

Solution

Since $y = x$ is a solution to the differential equation so this solution satisfy the equation.

$$y = x \implies \frac{dy}{dx} = 1$$

In equation (1) we find that two sides are equal, i.e.
$y = x$, a solution of equation (1) which representing the Riccati formula then using substitution.

$$y = y_1 + \frac{1}{z} \Rightarrow y = x + \frac{1}{z} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{z^2}\frac{dz}{dx}$$

Equation (1) becomes the following equation

$$\frac{dz}{dx} + z = \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{x}\right)$$

A linear equation in z can be solved where

$$p(x) = 1, \;\; q(x) = \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{x}\right)$$

$$\mu = e^{\int p(x)\, dx} = e^{\int dx} = e^{x}$$

So there is a general solution to the linear equation in the form

$$\mu . z = \int \mu . q(x)\, dx + c$$

$$\therefore\ e^{x}\, z = \frac{1}{2}\int \left(\frac{e^{x}}{x^{2}} - \frac{e^{x}}{x}\right) dx\ + c$$

$$\therefore e^{x}\, z = \frac{1}{2}\left\{-\frac{e}{x} + \int \frac{e^{x}}{x}\, dx - \int \frac{e^{x}}{x}\, dx\right\} + c = \frac{1}{2}\left\{-\frac{e^{x}}{x}\right\} + c$$

$$\Longrightarrow\ z = -\frac{1}{2x} + c\, e^{-x}$$

Exercises

Prove that $y = x$ is a solution to the following differential equation and then find its general solution.

$$1 - \frac{dy}{dx} + xy^{2} - (2x^{2} + 1)y + x^{3} + x - 1 = 0$$

$2 - \ 2x^2y' = 2xy + (y^2 - x^2)(x\cot x - 1)$

$3 - \quad y' = (y - x)(y + x) + !$

$4 - \ (x^2 + \alpha)\dfrac{dy}{dx} + 2y^2 - 3xy - \alpha = 0$

Chapter 3

Differential equations of the first order and higher degrees

3-1 Equations solvable in p

3-2 Equations solvable in x

3-3 Equations solvable in y

3-4 Clairaut's equation

Chapter 3

Differential equations of the first order and higher degrees

In this chapter, we will study the differential equations the first order and higher degree, which has the general form:

$$(\frac{dy}{dx})^n + a_1(\frac{dy}{dx})^{(n-1)} + a_2(\frac{dy}{dx})^{n(n-2)} + \cdots \ldots + a_n = 0$$

Let $p = \frac{dy}{dx}$ then the equation turns to:

$$p^n + a_1 p^{(n-1)} + a_2 p^{(n-2)} + \cdots \ldots \ldots + a_n = 0 \qquad (1)$$

Case 1: equations can be solved in p

Case 2: equations can be solved in x

Case 3: equations can be solved in y

3-1 Equations can be solved in p

In this case, the differential equation reduces to a polynomial of degree n represented by (1) which can be analyzed into linear factors as follows

$$(P - k_1)\,(P - k_2)(P - k_3) \ldots .\,(P - k_n) = 0 \qquad (2)$$

That: $p = F_1(x, y)$, $p = F_2(x, y)$,.........., $p = F_n(x, y)$ (3)

Where the set of the equations (3) can be solved and the solutions are on the form

$$g_1(x, y, c) = 0\,, g_2(x, y, c) = 0\,, \ldots \ldots \ldots \ldots\,, g_n(x, y, c) = 0 \qquad (4)$$

Thus, the general solution of equation (1) is in form

$$g_1(x, y, c), g_2(x, y, c), \dots\dots\dots\dots, g_n(x, y, c) \qquad (5)$$

Example 1:

Find the general solution for the following differential equation

$$p^2 - p - 6 = 0$$

Solution

The previous equation can be written on the form

$$(p - 3)(p + 2) = 0$$

So we find that either $(p - 3) = 0$ then $p = 3$ or $p = -2$ and that

$$p = 3 \implies \frac{dy}{dx} = 3 \implies y = 3x + c$$

$$p = -2 \implies \frac{dy}{dx} = -2 \implies y = -2x + c$$

So the general solution to the form

$$(y - 3x + c)(y + 2x + c) = 0$$

Example 2

Find the general solution for the following differential equation

$$x^2p^2 + xyp - 6y^2 = 0$$

Solution

The previous equation can be written on the form

$$(xp - 2y)(xp + 3y) = 0$$

So we find that either $(xp - 2y) = 0$ then $(xp + 3y) = 0$

$$p = \frac{2y}{x} \Longrightarrow \frac{dy}{dx} = \frac{2y}{x} \Longrightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

$$\Longrightarrow \ln y = 2\ln x + \ln c \Longrightarrow \ln y = \ln x^2c \Longrightarrow y = x^2c$$

and

$$p = -\frac{3y}{x} \Longrightarrow \frac{dy}{dx} = -\frac{3y}{x} \Longrightarrow \frac{dy}{y} = -3\frac{dx}{x}$$

$$\Longrightarrow \ln y = -3\ln x + \ln c \Longrightarrow \ln y = \ln\frac{c}{x^3} \Longrightarrow y = \frac{c}{x^3}$$

Thus, the general solution possess the form:

$$(y - x^2c)\left(y - \frac{c}{x^3}\right) = 0$$

3-2 Equations can be solved in x:

In this case, the differential equation is the form

$$x = f(y, p) \qquad (1)$$

And be making the differential for them w.r.t y to be transformed into

$$\frac{\partial x}{\partial y} = \frac{1}{p} = \frac{\partial f}{\partial y} \pm \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} \quad then \;\; \frac{1}{p} = F\left(y, p, \frac{\partial p}{\partial y}\right) \qquad (2)$$

Where equation (2) is a differential equation of the first order and the first degree can be solved and get y in terms of p and then substitute in (1) to get x in terms of p by eliminating p between x and y we obtain the explicit solution $y(x)$

Example 1:

Find the general solution for the following differential equation:

$$x = p^3 \qquad (1)$$

Solution

Equation (1) of the resolvable type in x and the differential procedure for it with respect to y is converted to

$$\frac{dx}{dy} = 3p^2 \frac{dp}{dy} \implies \frac{1}{p} = 3p^2 \frac{dp}{dy}$$

$$\implies \int dy = \int 3\, p^3\, dp \implies y = \frac{3}{4}\, p^4 + c \qquad (2)$$

By deleting p between equation (1) and equation (2) we obtain the Cartesian solution in the form

$$y = \frac{3}{4}\, x^{\frac{4}{3}} + c$$

Equation (1) and (2) called the solution in parametric form

Example 2

Find the general solution for the following differential equation

$$x = 4(p^3 + p) \qquad (1)$$

Solution

Equation (1) of the resolvable type in x and the differential procedure for it with respect to y is converted to:

$$\frac{dx}{dy} = 4\left(3p^2\frac{dp}{dy} + \frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1}{p} = (4 + 12p^2)\frac{dp}{dy}$$

$$\Rightarrow \int dy = \int (4p + 12p^3)\, dp$$

$$\Rightarrow y = 2p^2 + 3p^4 + c \qquad (2)$$

Equations (1), (2) represent the parametric solution.

3-3 Equations can be solved in y:

In this case, the differential equation is in the form

$$y = f(x, p) \quad (1)$$

And by making a differential for it with respect to x it turns to

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}$$

$$\Longrightarrow \quad p = F\left(x, p, \frac{dp}{dx}\right) \qquad (2)$$

Where equation (2) is a differential equation of the first order and the first degree can be solved in p and eliminate p between the resulting equation and equation (1) we get the explicit solution y(x) If we cannot eliminate p between the equations, we leave the solution in the parametric form.

Example 1

Find the general solution for the following differential equation

$$y = 2p(x - 1) \qquad (1)$$

Solution

Equation (1) of the resolvable type in y and the differential procedure for it with respect to x is converted to

$$\frac{dy}{dx} = 2p + 2(x - 1)\frac{dp}{dx}$$

$$\Longrightarrow \quad p = 2p + 2(x - 1)\frac{dp}{dx}$$

$$-p = 2(x-1)\frac{dp}{dx}$$

$$\Rightarrow \int \frac{dx}{x-1} = -2\int \frac{dp}{p}$$

$$\Rightarrow \quad x-1 = \frac{c}{p^2} \qquad (2)$$

By substitution on the value of x in equation (1) we find that

$$y = \frac{2c}{p} \qquad (3)$$

The equations (2), (3) represent the parametric solutions of equation (1).

Example 2

Find the general solution for the following differential equation.

$$y = 2p^3 + \frac{1}{p} \qquad (1)$$

Solution

Equation (1) of the resolvable type in y and the differential procedure for it with respect to x is converted to

$$\frac{dy}{dx} = (6p^2 - p^{-2})\frac{dp}{dx}$$

$$\Rightarrow \quad p = (6p^2 - p^{-2})\frac{dp}{dx}$$

$$\Rightarrow \quad \int dx = \int (6p - p^{-3})dp$$

$$\Rightarrow \quad x = 3p^2 + \frac{p^{-2}}{2} + c \qquad (2)$$

The equation (2) represent the parametric solution of equation (1).

3-4 Clairaut's Equation

The general form of this equation is

$$y = px + f(p)$$

And it is form the type of solution in y

Example 3

Find the general solution and the singular (odd) solution of the differential equation.

$$y = px - p^2 \qquad (1)$$

Solution

Equation (1) is the Clairaut's equation and the differential procedure for it with respect to x is converted to

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - 2p\frac{2p}{dx}$$

$$\Rightarrow \quad p = p + (x - 2p)\frac{dp}{dx}$$

$$(x - 2p)\frac{dp}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} = 0$$

$$dp = 0 \qquad \Rightarrow \qquad p = c$$

So the general solution of the equation (1) shall be on the form

$$y = cx - c^2$$

And to find singular solution we put

$$(x - 2p) = 0$$

So be

$$x = 2p \qquad (2)$$

By substitution on the value of x in equation (1) we find that

$$y = 2p^2 - p^2 \quad \Rightarrow \quad y = p^2 \qquad (3)$$

The equations (2) and (3) are parametric solution of equation (1).

To find a Cartesian solution, we substitute the value of p from equation (2) in equation (3) then

$$\Rightarrow y = p^2 = \frac{x^2}{4} \Rightarrow 4y = x^2$$

Which is parabola equation.

Exercises

Find the general solution for each of the following differential equations

$1 -\quad x = p^2(y+1)$

$2 -\quad p^2 - 2xp + 1 = 0$

$3 -\quad y = xp^2 + p$

$4 -\quad y = (1+x^2)p\ \tan^{-1} x$

Find the general solution and the odd solution for each of the following differential equations

$1-\qquad y = px + \dfrac{1}{p}$

$2-\qquad y = px + p^3$

Chapter 4

Homogenous linear differential equations of higher order

With constant coefficients

4-1 Auxiliary (characteristic) equation and classification

4-2 All the characteristic roots are real and distinct

4-3 some of the characteristic roots are real and identical

4-4 some of the characteristic roots are complex

Chapter 4

Homogenous linear differential equations of higher order With constant coefficients

4-1 Auxiliary (characteristic) equation and classification

The general form of the homogenous linear differential equation of the nth-order with constant coefficients is

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \;\dots\dots\dots + a_1 y' + a_0 y = 0 \qquad (1)$$

Where a_n, a_{n-1} , ……… , a_1 , a_0 are constant, it can be simplified to the form

$$\left(a_n \frac{d^n}{d\,x^n} + a_{n-1} \frac{d^{n-1}}{d\,x^{n-1}} + \cdots .\, a_1\, y' + a_0\right) y = 0 \qquad (2)$$

Which can be put in the form

$$L_n(x)\, y = 0 \qquad (3)$$

Which is called the differential operator.

To solve equation (1) or (2), assume that $y = e^{mx}$ satisfies the equation, thus we have

$$e^{mx}\,(a_n m^n + a_{n-1} m^{n-1} + \ldots\ldots + a_1 m + a_0) = 0$$

Which the auxiliary (characteristic) equation:

$$(a_n m^n + a_{n-1} m^{n-1} + \ldots\ldots + a_1 m + a_0) = 0 \qquad (4)$$

Equation (4) of a polynomial of degree n, so it has n roots which may be equal, repeated or complex.

In the following study we well study each of these cases and find the general solution in each case.

4-2 all the roots are real and distinct

If the roots of equation (3) are real and different, that is

$$m_1 , m_2 , m_3 , \dots\dots\dots , m_n$$

Are real where

$$m_1 \neq m_2 \neq m_3 \neq \dots\dots\dots \neq m_n$$

We have n solution

$$y_1 = e^{m_1 x} , \; y_2 = e^{m_2 x}, \dots\dots, y_n = e^{m_n x}$$

And the general solution is a linear combination of these solutions. i.e.

$$y(x) = c_1 y_1 + c_2 y_2 + \cdots \dots . + c_n y_n$$

Example 1

Find the general solution of the differential equation

$$2y'' + 5y' + 2y = 0$$

Solution

Assume that the solution is $y = e^{mx}$ therefore to be

$$y' = me^{mx} \ , \ y'' = m^2 e^{mx}$$

Thus the given equation reduces the auxiliary equation:

$$2m^2 + 5m + 2 = 0$$

$$\therefore (2m+1)(m+2) = 0$$

$$\Longrightarrow m_1 = -2 \ , \ m_2 = -\frac{1}{2}$$

Since the two roots are real and different, there are two solutions.

$$y_1 = A_1 e^{-2x} \ , \ y_2 = A_2 e^{-\frac{1}{2}x}$$

Where A_1 , A_2 are constants, the general solution is the sum of the two solutions

$$y = y_1 + y_2 = A_1 e^{-2x} + A_2 e^{-\frac{1}{2}x}$$

Example 2

Find the general solution of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0$$

Solution

Assume that the solution is $y = e^{mx}$ therefore we have

$$y' = me^{mx} \quad , \quad y'' = m^2 e^{mx} \quad , \quad y''' = m^3 \, e^{mx}$$

In the given equation we get the auxiliary equation

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$(m-1)(m-2)(m-3) = 0$$

$$\Longrightarrow m_1 = 1 \, , m_2 = 2 \, , m_3 = 3$$

Since the two roots are real and different, there are two solutions

$$y_1 = A_1 e^{x} \, , y_2 = A_2 e^{2x} \, , y_3 = A_3 e^{3x}$$

Where A_1 , A_2 , A_3 are constants, the general solution is the sum of the solutions

$$y = y_1 + y_2 + y_3 = A_1 e^{x} + A_2 e^{2x} + A_3 e^{3x}$$

<u>4-3 Some of the roots are real and identical</u>

If the roots of equation (3) are real and equal (duplicate) i.e.

$$m_1 , m_2 , m_3 , \dots \dots \dots \ , m_n \text{ real roots where}$$

$$m_1 = m_2 = m_3 = \dots \dots \dots = m_n$$

We have n solutions on the form

$$y_1 = A_1 \, e^{mx} \ , \ y_2 = A_2 \, xe^{mx} \ , \dots \dots \dots \ , y_n = A_n x^{n-1} e^{mx}$$

Where $A_1 , A_2 , \dots \dots \dots \ , A_n$ constants. The general solution is the sum of these solutions

$$y = y_1 + y_2 + y_3 + \dots \dots \dots + y_n$$

$$y = (A_1 + A_2 x + A_3 x^2 + \dots \dots \dots + A_n \, x^{n-1}) \, e^{mx}$$

Example 1

Find the general solution of the differential equation

$$y'' + 4y' + 4y = 0$$

Solution

Assume that the solution is $y = e^{mx}$ therefor to be

$$y' = me^{mx} \ , \ y'' = m^2 e^{mx}$$

In the given equation we get the auxiliary equation

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0 \ \Rightarrow \ m_1 = m_2 = -2$$

Since the two roots are real and equal, there are two solutions

$$y_1 = A_1 e^{-2x} \ , \ y_2 = A_2 x \, e^{-2x}$$

$$y = y_1 + y_2 = (A_1 + A_2) e^{-2x}$$

Example 2

Find the general solution of the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

Solution

Assume that the solution is $y = e^{mx}$ therefore to be

$$y' = me^{mx}\ ,\ \ y'' = m^2e^{mx}\ \ \ ,\ \ y''' = m^3e^{mx}$$

In the given equation we get the auxiliary equation

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m^3 - 1) - 3(m^2 - m) = 0$$

$$\Longrightarrow (m - 1)(m^2 + m + 1) - 3m(m - 1) = 0$$

$$\Longrightarrow (m - 1)(m^2 - 2m + 1) = 0\ \ \Longrightarrow (m - 1)^3 = 0$$

$$\Longrightarrow\ \ m_1 = m_2 = m_3 = 1$$

Since the roots are real and equal, there are three solutions

$$y_1 = A_1e^x\ ,\ \ y_2 = A_2xe^x\ \ ,\ \ y_3 = A_3x^2e^x$$

Where $A_1, A_2\ , A_3$ are constants, the general solution is the sum of the solutions

$$y = y_1 + y_2 + y_3 = (A_1 + A_2x + A_3x^2)e^x$$

<u>4-4 Some of the roots are complex</u>

If the roots of equations (3) are not real, i.e. contain a complex roots which in the form

$$m_1 = \alpha + i\beta\ ,\ \ m_2 = \alpha - i\beta$$

We have two solutions on the form

$$y_1 = A_1 e^{(\alpha+i\beta)x} \ , \ \ y_2 = A_2 e^{(\alpha-i\beta)x}$$

Where A_1, A_2 are constants, the general solution is the sum of the two solutions

$$y = y_1 + y_2 = e^{x\alpha}(A_1 \cos \beta x + A_2 \sin \beta x)$$

Example 1

Find the general solution of the differential equation

$$y'' - 6y' + 19y = 0$$

Solution

Assume that the solution is $y = Ae^{mx}$ therefore to be

$$y' = me^{mx} \,, \qquad y'' = m^2 e^{mx}$$

In the given equation we get the auxiliary equation

$$m^2 - 6m = 13 = 0 \qquad \Longrightarrow \qquad m = 3 \pm 2i$$

Since the two roots are not real, there are two solutions

$$y_1 = A_1\, e^{(3+2i)x} \ , \ y_2 = A_2 e^{(3-2i)x}$$

Where A_1, A_2 are constants, the general solution is the sum of the two solutions

$$y = y_1 + y_2 = e^{3x}\left(A_1 e^{2ix} + A_2 e^{-2ix}\right)$$

$$y = e^{3x}(A_1 \cos 2x + A_2 \sin 2x\ )$$

Example 2

Find the general solution of the differential equation

$$y'' - 2y' + 10y = 0$$

Solution

Assume that the solution is $y = Ae^{mx}$ therefore to be

$$y' = me^{mx} \quad , \quad y'' = m^2 e^{mx}$$

In the given equation we get the auxiliary equation

$$m^2 - 2m + 10 = 0$$

Can only be analyzed using the law

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \implies m = 1 \pm 3i$$

Since the two roots are not real, there are two solutions

$$y_1 = A_1 e^{(1+3i)x} \ , \ \ y_2 = A_2 \ e^{(2-3i)x}$$

Where A_1 , A_2 are constants, the general solution is the sum of the two solutions

$$y = y_1 + y_2 = e^x\left(A_1 e^{3ix} + A_2 e^{-3ix}\right)$$

$$y = e^x(A_1 \cos 3x + A_2 \sin 3x)$$

Note 1

Some cases include only the first and second cases, some of which include only the first and third cases, some of which only include the second and third cases, some of which include the first, second and third cases.

Note 2

We have to know that all the solutions that corresponds to the roots of the characteristic equations for any given differential equations are all independent solutions

Exercises

Find the general solution for each of the following differential equations

$1 -\quad y'' - y' - 2y = 0$

$2 -\quad y'' + 2y' + 4y = 0$

$3 -\quad y'' - 3y' + 2y = 0$

$4 -\quad y^{(4)} + 13y'' + 36y = 0$

$5 -\quad y'' - 8y = 0$

$6 -\quad y'' - 6y' + 25y = 0$

$7 -\quad y'' + 27y = 0$

$8 - \quad y^{(4)} - 2y' - 3y = 0$

<u>Chapter 5</u>

Non-Homogenous linear differential Equations with constant coefficients

5-1 Introduction

5-2 The differential operator method

5-3 Undetermined coefficient method

5-4 Variation of parameters method

<u>Chapter 5</u>

Non-Homogenous linear differential Equations with constant coefficients

5-1 Introduction

The general form of the Non-Homogenous linear differential equation with constant coefficients is

$$a_n\, y^{(n)} + a_{n-1} y^{(n-1)} + \;\dots\dots\dots + a_1 y' + a_0 y = g(x) \qquad (1)$$

Where a_n , a_{n-1} , ……… , a_1 , a_0 are constants, $g(x) \neq 0$

Which can be simplified to the form

$$\left(a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \;\dots\dots\dots + a_1 \frac{d}{dx} + a_0\right) y = g(x) \quad (2)$$

Or to the form

$$L_n(x) y = g(x) \qquad (3)$$

Where

$$L_n(x) = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \;\dots\dots\dots + a_1 \frac{d}{dx} + a_0$$

Which is called the differential operator

Theorem 4-1

The general solution of the equation (1) has the form

$$y(x) = y_c(x) + y_p(x)$$

Where

$y_{c(x)}$ is the solution of the homogenous equation (3), as in the previous chapter.

$y_p(x)$ is the particular solution of equation (3).

To find the particular solution we follow some of the following methods

1- The operators method

2-Undetermined coefficients method

3- Variation of Parameters method

5-2 The operator method

In this section, let us denote by D to the derivative $\frac{d}{dx}$, then the general form of the linear non-homogenous differential equation is:

$$\left(a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \ \dots\dots\dots + a_1 \frac{d}{dx} + a_0\right) y = g(x)$$

$$f(D)y = g(x) \qquad (4)$$

Where $f(D)$ is called the differential operator. This operator has some important properties listed as follows:

$$1- \quad D[y_1(x)+y_2(x)] = Dy_1(x)+Dy_2(x)$$

$$2- \quad D^n(D^m y) = D^m(D^n y)$$

$$3- \quad D(cy) = cDy \quad ,c = constant$$

$$4- \ f(D)[ay_1(x)+by_2(x)] = af(D)y_1(x)+bf(D)y_{2(x)} \ ,a,b \ are \ constants$$

The operator also satisfies the following properties:

$$(i) f(D)e^{kx} = f(k)e^{kx}$$
$$(ii) f(D^2)\sin kx = f(-k^2)\sin kx$$

$$(iii) f(D^2)\cos kx = f(-k^2)\cos kx$$

$$(iv) \ \ f(D)e^{kx}.v(x) = e^{kx} f(D+k)v(x)$$

Since we are interested here to establish particular solution for equation (4) we act by the inverse differential operator $\frac{1}{f(D)}$ of equation (4), i.e, we have $y = \frac{1}{f(D)}.g(x)$. So, we have to know the properties of $\frac{1}{f(D)}$ which are listed below:

$$(i)\ \frac{1}{f(D)} e^{kx} = \frac{1}{f(k)}\ e^{kx}\ , \qquad f(k) \neq 0$$

$$(ii)\ \frac{1}{f(D^2)} \sin kx\ = \frac{1}{f(-k^2)} \sin kx \quad , \ f(-k^2) \neq 0$$

$$(iii)\ \frac{1}{f(D^2)} \cos kx\ = \frac{1}{f(-k^2)} \cos kx \quad , \ f(-k^2) \neq 0$$

$$(iv)\ \frac{1}{f(D)} e^{kx}. v(x) = e^{kx} \frac{1}{f(D+k))}\ v(x)$$

Using these properties we illustrate the technique by the following examples.

Example 1

Find the general solution of the differential equation

$$y'' + 6y' + 9y = 50\ e^{2x} \qquad (1)$$

Solution

First we find $y_c(x)$ as a solution of the homogenous equation

$$y'' + 6y' + 0y = 0$$

To find the general solution of this linear homogenous equation we put

$$y = e^{mx}$$

That leads to

$$m^2 + 6m + 9 = 0$$

$$\Longrightarrow (m+3)^2 = 0 \quad \Longrightarrow m_1 = m_2 = -3$$

$$\Longrightarrow y_c(x) = (A_1 + A_2 x)e^{-3x} \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for equation (1) that is,

The equation (1) can be written on the form

$$(D^2 + 6D + 9)\, y_p(x) = 50\, e^{2x}$$

$$\Longrightarrow y_p(x) = \frac{1}{(D^2 + 6D + 9)}\, 50e^{2x} = \frac{50}{(D+3)^2}\, e^{2x}$$

$$= \frac{50}{(2+3)^2} e^{2x} = 2e^{2x}$$

From the equations (2),(3) the general solution is

$$y(x) = y_c(x) + y_p(x) = (A_1 + A_2 x)e^{-3x} + 2e^{2x}$$

Example 2

Find the general solution of the differential equation

$$y^{//} - y = x^3 e^x$$

Solution

First, we find $y_c(x)$ as a solution of the homogenous equation

$$y^{//} - y = 0$$

The auxiliary equation for it is

$$m^2 - 1 = 0$$

$$\Longrightarrow (m-1)(m+1) = 0 \quad \Longrightarrow m_1 = 1\,, \;\; m_2 = -1$$

$$\Longrightarrow \;\; y_c(x) = A_1 e^x + A_2 e^{-x}$$

Second, we find $y_p(x)$ is a special solutions for equation (1) that is, the equation (1) can be written on the form

$$(D^2 - 1) y_p(x) = x^3 e^x$$

$$\Longrightarrow y_p(x) = \frac{1}{(D^2 - 1)}\{x^3 e^x\} = \frac{e^x}{(D+1)^2 - 1}\{x^3\}$$

$$= \frac{e^x}{D^2 + 2D}\{x^3\} = \frac{e^x}{D+2}\left\{\frac{1}{D}x^3\right\}$$

$$= \frac{e^x}{D+2}\left\{\frac{x^4}{4}\right\} = \frac{e^x}{8}\left(1 + \frac{D}{2}\right)^{-1}\{x^4\}$$

$$= \frac{e^x}{8}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \frac{D^4}{16} - \ \dots\dots\dots\right)\{x^4\}$$

$$= \frac{e^x}{8}\left(x^4 - 2x^3 + 3x^2 - 3x + \frac{3}{2}\right)$$

$$\Longrightarrow y_p(x) = \frac{e^x}{8}\left(x^4 - 2x^3 + 3x^2 - 3x + \frac{3}{2}\right) \qquad (3)$$

From the equations (2),(3) the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = A_1e^x + A_2e^{-x} + \frac{e^x}{8}\left(x^4 - 2x^3 + 3x^2 - 3x + \frac{3}{2}\right)$$

Example 3

Find the general solution of the differential equation

$$y' + y = -5\sin 2x \qquad (1)$$

Solution

First, we find $y_c(x)$ is a solution of the homogenous equation

$$y' + y = 0$$

The auxiliary equation for it is

$$m + 1 = 0 \quad \Rightarrow m = -1$$

$$\Rightarrow y_c(x) = Ae^{-x} \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for the equation (1) that is, the equation (1) can be written on the form

$$(D + 1)y_p(x) = -5\sin 2x$$

$$\Rightarrow y_p(x) = \frac{1}{(D+1)}\{-5\sin 2x\}$$

$$= \frac{-5(D-1)}{(D^2-1)}\sin 2x$$

$$= \frac{-5(D-1)}{(-4-1)}\sin 2x$$

$$= 2\cos 2x - \sin 2x \quad (3)$$

From the equations (2),(3) the general solution is

$$y(x) = y_c(x) + y_p(x)$$
$$= Ae^{-x} + 2\cos 2x - \sin 2x$$

Example 4

Find the general solution of the differential equation

$$y^{//} + 9y = \sin 3x$$

Solution

First, we find $y_c(x)$ is a solution of the homogenous equation:

$$y^{//} + 9y = 0$$

The auxiliary equation for it is

$$m^2 + 9 = 0 \quad \Rightarrow \quad m = \pm 3i$$

$$\Rightarrow y_c(x) = A_1 \sin 3x + A_2 \cos 3x \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for equation (1) that is, the equation (1) can be written on the form

$$(D^2 + 9)y_p(x) = \sin 3x$$

$$\Longrightarrow y_p(x) = \frac{1}{(D^2+9)}\{\sin 3x\} \quad , \quad F(-3^2) = 0$$

$$e^{3ix} = \cos 3x + i \sin 3x \quad , e^{-3ix} = \cos 3x - i \sin 3x$$

$$\therefore \cos 3x = \mathrm{Re}\left(e^{3ix}\right) \ , \ \sin 3x = \mathrm{Im}\left(e^{3ix}\right)$$

$$\therefore y_p(x) = \frac{1}{(D^2+9)}\{\sin 3x\} = \mathrm{Im}\frac{1}{(D-3i)(D+3i)}e^{3ix}$$

$$= \mathrm{Im}\frac{1}{6i(D-3i)}e^{3ix}$$

$$= -\frac{x}{6}\cos 3x \qquad (WHY?)$$

$$\Longrightarrow \quad y_p(x) = -\frac{x}{6}\cos 3x \qquad (3)$$

From the equations (2),(3) the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= A_1 \sin 3x + A_2 \cos 3x \pm \frac{x}{6}\cos 3x$$

Exercises

Find the general solution for each of the following differential equations

$1 -$ $\quad y'' + y' - 12y = x^2 e^x$

$2 - \quad y' - y = x^2 - x + 1$

$3 - \quad y'' - 4y' + 3y = x^3$

$4 - \quad y'' + 8y' + 25y = 48\cos x - 16\sin x$

5-3 Undetermined coefficients method

There are several cases to use this method depends on the form of g(x) in equation 1

Case 1:

If the function g(x) in equation (1) is in a polynomial form, of degree s, i.e. equation (1) has the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{/} + a_o y = A_s x^s + A_s x^{s-1} + \cdots + A_0$$

Then the particular solution must be :

(i) a polynomial of the same degree s if $a_0 \neq 0$, i.e.,

$$y_p = B_s x^s + b_s x^{s-1} + \cdots + B_0$$

(ii) a polynomial of the same degree s multiplied by x if $a_0 = 0 \; and \; a_1 \neq 0$

And substitute y_p in the given equation and by comparing the coefficients of the different porers of x one can obtain the coefficients B_0 , B_1 , B_2 , ...

For example, if $g(x) = x^3 + 2$ we assume that

$$y_p(x) = ax^3 + bx^2 + cx + d$$

Case 2:

If the function $g(x)$ in equation (1) is an exponential function multiplied by a constant.

For example $(x) = 2e^{3x}$, we assume that

$$y_p(x) = Ae^{3x}$$

We find the value of the constant and then put $y_p(x)$ in its final form.

Case 3:

If the function $g(x)$ in equation (1) is a triangular function form as

$$g(x) = \sin bx \quad or \quad g(x) = \cos bx$$

Or both, we assume that

$$y_p(x) = A\sin bx + B\cos bx$$

And we find the value of constants and then be $y_p(x)$ in their final form

Note:

• The previous cases can be generalized, i.e., the function can come in two or three cases.

For example, if $g(x) = x^2 e^{3x}$ we assume that

$$y_p(x) = (ax^2 + bx + c)e^{3x}$$

• We use this method only in the cases described and cannot be generalized on a larger scale

Example 1

Find the general solution of the differential equation

$$y'' - y = x^3 + 3x \qquad (1)$$

Solution

First, we find $y_c(x)$ is a solution of the homogenous equation:

$$y'' - y = 0$$

The auxiliary equation for it is:

$$m^2 - 1 = 0 \implies m = \pm 1$$

$$\implies y_c(x) = A_1 e^x + A_2 e^{-x} \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for the equation (1) using the method of Undetermined coefficients, by assuming that

$$y_p(x) = ax^3 + bx^2 + cx + d$$

$$\implies y_p'(x) = 3ax^2 + 2bx + c$$

$$\implies y_p''(x) = 6ax + 2b$$

In equation (1) and by comparing the coefficients of the two sides we obtain the values of the constants and the special solution in the for

$$y_p(x) = -x^3 - 9x \qquad (3)$$

From the equation (2), (3) the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= A_1 e^x + A_2 e^{-x} - x^3 - 9x$$

Example 2

Find the general solution of the differential equation

$$y'' - y' - 2y = e^{x} \qquad (1)$$

Solution

First, we find $y_c(x)$ is a solution of the homogenous equation:

$$m^2 - m - 2 = 0 \implies (m-2)(m+1) = 0$$

$$\implies m_1 = 2\,,\; m_2 = -1$$

$$\implies y_c(x) = A_1\, e^{2x} + A_2 e^{-x} \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for the equation (1) using the method of Undetermined coefficients, by assuming that

$$y_p(x) = ae^{3x} \implies y_p'(x) = 3ae^{3x}\;,\; y_p''(x) = 9ae^{3x}$$

In equation (1) and by comparing the coefficients of the two sides we obtain the values of the constant $a = \frac{1}{4}$ and the special solution in the form

$$y_p(x) = \frac{1}{4}e^{3x} \qquad (3)$$

From the equation (2), (3) the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= A_1 e^{2x} + A_2 e^{-x} + \frac{e^{3x}}{4}$$

Example 3

Find the general solution of the differential equation

$$y'' - y' - 2y = \sin 2x \qquad (1)$$

Solution

First, we find $y_c(x)$ is a solution of the homogenous equation:

$$y'' - y' - 2y = 0$$

The auxiliary equation for it is

$$m^2 - m - 2 = 0 \quad \Rightarrow \quad (m-2)(m+1) = 0$$

$$\Rightarrow m_1 = 2, \;\; m_2 = -1$$

$$y_c(x) = A_1 e^{2x} + A_2 e^{-x} \qquad (2)$$

Second, we find $y_p(x)$ is a special solution for the equation (1) using the method of Undetermined coefficients, by assuming that

$$y_p(x) = a \sin 2x + b \cos 2x$$

$$\Longrightarrow y_p'(x) = 2a \cos 2x - 2b \sin 2x$$

$$\Longrightarrow y_p''(x) = -4a \sin 2x - 4b \cos 2x$$

In equation (1) and by comparing the coefficients of the two sides we obtain the values of the constants

$$a = -\frac{3}{20}, \quad b = \frac{1}{20}$$

And the special solution in the form

$$y_p(x) = \frac{1}{20}(\cos 2x - 3 \sin 2x) \qquad (3)$$

From the equation (2), (3) the general solution can be established.

<u>Exercises</u>

Find the general solution for each of the following differential equations.

$$1- \quad y'' + y' - 12y = x^2 e^x$$

$$2- \quad y' - y = x^2 - x + 1$$

$$3- \quad y'' - 4y' + 3y = x^3$$

$$4- \quad y'' + 8y' + 25y = 48\cos x - 16\sin x$$

<u>5-4: Variation of Parameters method</u>

Consider the higher differential equation in the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + + \cdots a_1 y^{(n)} + a_0 y \ = f(x) \qquad (1)$$

Since we know from above the solution of the corresponding homogeneous differential equation which has the form.

$$y_p(x) = \sum_{1}^{n} c_i y_i \quad , (c_i, i = 1,2,3, \dots . n \ are \ constants) \qquad (2)$$

This is the particular solution of the differential equation (1), to find the special solution we assume that

$$y_1(x) = \sum_{1}^{n} c_i(x) y_i \qquad (3)$$

Satisfies (1), to substitute in (!), we have

$$y_1' = \sum_{1}^{n} c_i(x)\, y_i' + \sum_{1}^{n} c_i'(x)\, y_i$$

And choose

$$\sum_{1}^{n} c_i'(x)\, y_i = 0 \quad (4)$$

Similarly,

$$y_1'' = \sum_{1}^{n} c_i(x)\, y_i'' + \sum_{1}^{n} c_i'(x)\, y_i'$$

And choose

$$\sum_{1}^{n} c_i'(x)\, y_i' = 0 \qquad (5)$$

And so on (n-1) times in each time we choose the second sum be zero, i.e.

$$y_1^{(n)} = \sum_{1}^{n} c_i'(x)\, y_i^{n-2} = 0 \qquad (6)$$

And this yields,

$$y_1^{(n)} = \sum_1^n c_i(x)\, y_i^{(n)} + \sum_1^n c_i'(x)\;\; y_i^{(n-1)}$$

By substituting by all these values in equation (1), we have

$$\sum_1^n c_i'(x)\, y_i^{(n-1)} = f(x) \qquad\qquad (7)$$

So, it is clear that we have n equations in n unknowns $c_i\ (i = 1,2, \dots \dots \dots\ \ , n)$, solving it and integrating each one we get all $c_i\ (i = 1,2, \dots \dots \dots\ \ , n)$

This method can be illustrated by the following

Example 1

Find the general solution of the differential equation

$$y' + y = \sec x \qquad (1)$$

Solution

First we solve the homogenous equation

$$y'' + y = 0 \qquad\qquad (2)$$

The auxiliary equation is $m^2 + 1 = 0$, i.e. $m = \pm i$ and the general solution of the equation (2) is

$$y_c = c_1 \cos x + c_2 \sin x \quad , c_1 \; and \; c_2 \quad are \; constants$$

Now, suppose that $y = c_1(x) \, cos \, x + c_2(x) \sin x$ is a particular solution of (1), then

$$y' = -c_1(x) \sin x + c_2(x) \cos x + c_1{}' cosx + c_2{}' sinx$$

Let

$$c_1' \cos x + c_2' \sin x = 0 \qquad (3)$$

Similarly

$$y'' = -c_1(x) \sin x + c_2(x) \cos x + c_1' \cos x + c_2' \sin x$$

Let

$$c_1' \cos x + c_2' \sin x = 0 \quad (3)$$

Similarly,

$$y'' = -c_1(x) \cos x - c_2(x) \sin x - c_1' \sin x + c_2' \cos x$$

Substituting in (1) we have

$$-c_1' \sin x + c_2' \cos x \sec x \qquad (4)$$

Solving equation (3) and (4) together yields,

$$c_1' = -\tan x \quad then \quad c_1 = \ln|\cos x| \;\; and \;\; c_2' = 1 \quad , then \; c_2 = x$$

And then the particular solution of (1) is

$$y_p = (\ln|\cos x|)\cos x + x\sin x$$

And the general solution of (1) is, finally

$$y(x) = c_1\cos x + c_2\sin x + (\ln|\cos x|)\cos x + x\sin x$$

Exercises

$1-\quad y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

$2-\quad y'' - y = \dfrac{2}{1+e^x}$

$3-\quad y'' + y = \tan x$

$4-\quad y'' + y = \cot x$

Chapter 6

Homogeneous order differential equations

With variables coefficients

6-1 Intoduction

6-2 Homogenous Euler differential equation

6-3 Non-Homogenous Euler Differential

6-1 Intoduction

The general linear Non-Homogenous differential equation has the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots . + a_1 y' + a_0 y = f(x) \qquad (1)$$

Where $a_n(x), a_{n-1}(x), \ldots\ a_0(x)$, $f(x)$ are continuous functions and $f(x) \neq 0$. if $f(x) = 0$, then equation (1) is called higher order differential with variables coefficients.

We restrict our study on certain type of coefficients (of Euler type) in both homogenous and Non-Homogenous cases.

6-2 Homogenous Euler Equation

Homogenous Euler differential equation of the n-th order has the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots . + a_1 x\, y' + a_0 y = 0$$

To solve this equation, we use the substitution

$$x = e^t \,, \;\; y = e^{kt} = x^k$$

We illustrate this technique by the following examples.

Example 1

Find the general solution of the differential equation

$$x^2y'' - xy' - 3y = 0$$

Solution

This equation represents the Euler Equation of homogenous differential so we assume that the solution is

$$y = e^{kt} \;\; , \quad x = e^t \quad \Longrightarrow \quad y = x^k$$

$$\therefore \;\; y' = kx^{k-1} \,, \;\; y'' = k(k-1)x^{k-2}$$

In the given equation we get the auxiliary equation

$$k(k-1) - k - 3 = 0 \quad \Longrightarrow \quad k^2 - 2k - 3 = 0$$

$$\Longrightarrow \quad (k-3)(k+1) = 0 \quad \Longrightarrow \quad k_1 = 3, \;\; k_2 = -1$$

Since the two roots are real and different, there are two solutions

$$y_1 = A_1 e^{k_1 t} = A_1 x^{k_1} = A_1 x^3$$

$$y_2 = A_2 e^{k_2 t} = A_2 x^{k_2} = A_2 x^{-1}$$

Where A_1 , A_2 are constants and the general solution is the sum of the two solutions.

$$y = A_1 x^3 + A_2 x^{-1}$$

Example 2

Find the general solution of the differential equation

$$x^2 y'' + 4xy' + 4y = 0$$

Solution

This equation represents the Euler equation of homogenous differential so we assume that the solution is

$$y = e^{kt} \;\; , \;\; x = e^t \;\; \Longrightarrow \;\; y = x^k$$

$$\therefore y' = k\, x^{k-1} \;\; , \;\; y' = k(k-1)\, x^{k-2}$$

In the given equation we get the auxiliary equation

$$k(k-1) + 4k + 4 = 0 \quad \Longrightarrow \quad k^2 + 3k + 4 = 0$$

$$\Longrightarrow k = \frac{1}{2}\left(-3 \pm \sqrt{7}i\right)$$

Since the two roots are imaginary, there are two solutions

$$y_1 = A_1\, e^{\frac{1}{2}(-3+\sqrt{7}i)} \,,\;\; y_2 = A_2 e^{\frac{1}{2}(-3-\sqrt{7}i)}$$

Where A_1 , A_2 are constants and the general solution is the sum of the two solutions.

$$y = y_1 + y_2 = A_1\, e^{\frac{1}{2}(-3+\sqrt{7}i)} + A_2 e^{\frac{1}{2}(-3-\sqrt{7}i)}$$

$$= e^{-\frac{3t}{2}} \left(A_1 \cos \frac{\sqrt{7}}{2} i + A_2 \sin \frac{\sqrt{7}}{2} i \right)$$

$$= x^{-\frac{3}{2}} \left(A_1 \cos \left(\frac{\sqrt{7}}{2} \ln x \right) + A_2 \sin \left(\frac{\sqrt{7}}{2} \ln x \right) \right)$$

Example 3

Find the general solution of the differential eaution

$$x^2 y' - 5xy' + 9y = 0$$

Solution

This equation represents the Euler equation of homogenous differential so we assume that the solution is

$$y = e^{kt} \;\;,\;\;\; x = e^t \;\; \Longrightarrow \;\;\; y = x^k$$

$$\therefore\, y' = k\, x^{k-1} \;\;,\;\; y' = k(k-1)\, x^{k-2}$$

In the given equation we get the auxiliary equation

$$k(k-1) - 5k + 9 = 0 \implies k^2 - 6k + 9 = 0$$

$$\implies (k-3)^2 = 0 \qquad \implies \qquad k_1 = k_2 = 3$$

Since the two roots are equal, there are two solutions

$$y_1 = A_1 e^{3t} = A_1 x^3$$

$$y_2 = A_2\, t\, e^{3t} = A_2 x^3 \ln x$$

Where A_1, A_2 are constants and the general solution is the sum of the two solutions

$$y = y_1 + y_2 = A_1 x^3 + A_2 x^3 \ln x = (A_1 + A_2 \ln x)\, x^3$$

Exercises

Find the general solution of each of the following differential equations

$1 -\quad 3x^2\, y' - 2xy' - 12y = 0$

$2 -\quad x^2 y' + 7xy' + 9y = 0$

$3 - \quad x^2y' - xy' + y = 0$

6-3 Non-Homogenous Euler differential equation

The non-homogenous Euler differential equation of n-th order in the form

$$a_n\, x^n y^{(n)} + a_{n-1}x^{n-1}\, y^{(n-1)} + \ \ldots\ldots\ldots + a_1xy' + a_0y = g(x)$$

To solve this kind of equations we use the substitution

$$x = e^t \quad or \qquad t = \ln x$$

This substitution coverts the differential equation with variable coefficients into a corresponding differential equation with constant coefficients as follows

$$\theta = \frac{d}{dt} \qquad , \quad D = \frac{d}{dx}$$

$$\Longrightarrow\ \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}\ \Longrightarrow x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\Longrightarrow x\, Dy = \theta y\ \Longrightarrow xD = \theta \qquad\qquad (3)$$

Also find that

$$D^2 = \frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dt}\frac{dt}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dt}.\frac{1}{x}\right) = \frac{d}{dt}.\frac{-1}{x^2} + \frac{1}{x}\frac{d}{dx}\left(\frac{d}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{d}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{d}{dt}\right).\frac{dt}{dx}$$

$$\Longrightarrow D^2 = \frac{-1}{x^2}\theta + \frac{1}{x^2}\theta^2$$

$$\Longrightarrow x^2 D^2 = \theta(\theta - 1) \quad (4)$$

From relations (3), (4) we find that

$$x^3 D^3 = \theta(\theta - 1)(\theta - 2)$$

$$\therefore \quad x^n D^n = \theta(\theta - 1)(\theta - 2) \ldots\ldots\ldots \big(\theta - (n-1)\big)$$

By substitution in the Euler equation we find it transformed from a formula with variable coefficients to a formula with constant coefficients that can be solved as we will explain in the following examples.

Example 1

Find the general solution of the differential equation

$$x^2 y'' - xy' - 3y = 4x^2$$

Solution

This equation represents the Euler differential equation of Non-Homogenous and can be written in the form

$$(x^2D^2 - xD - 3)y = 4x^2$$

So we use substitution $x = e^t$ and assume that $\theta = \frac{d}{dt}$ then

$$xD = \theta \quad , \quad x^2D^2 = \theta(\theta - 1)$$

In the given equation we get the equation

$$[\theta(\theta - 1) - \theta - 3]y = 4e^{2t}$$

$$\Longrightarrow (\theta^2 - 2\theta - 3)y = 4e^{2t}$$

This is a second order differential equation with constant coefficients and for solution

First, we find $y(x)$ is a solution of homogenous equation

$$(\theta^2 - 2\theta - 3)y = 0$$

Where the auxiliary equation

$$k^2 - 2k - 3 = 0$$

$$\Longrightarrow (k-3)(k+1)=0 \quad \Longrightarrow \quad k_1=3\,,\ \ k_2=-1$$

Since the two roots are real and different, there are two solutions

$$y_1 = A_1\, e^{k_1 t} = A_1 e^{3t},\ \ y_2 = A_2 e^{k_2 t} \ = A_2\, e^{-t}$$

Where $A_1\,, A_2$ are constants and the solution is the sum of the two solutions.

$$y_c(x) = A_1 e^{3t} + A_2\, e^{-t}$$

Second, we find $y_p(x)$ the solution of the Non-Homogenous equation.

$$(\theta^2 - 2\theta - 3)y_p = 4e^{2t}$$

$$y_p = \frac{1}{(\theta^2 - 2\theta - 3)}\{4e^{2t}\}$$

$$= \frac{4e^{2t}}{(2^2 - 2(2) - 3)} = -\frac{4}{3}e^{2t}$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= A_1 e^{3t} + A_2\, e^{-t} - \frac{4}{3}e^{2t}$$

$$= A_1 x^3 + A_2 x^{-1} - \frac{4}{3}x^2$$

Example 2

Find the general solution of the differential equation

$$x^2 y^{//} + 4xy' - 4y = \ln x$$

Solution

This equation represents the Euler differential equation of Non-Homogenous and can be written on the form

$$(x^2 D^2 + 4xD - 4)y = \ln x$$

So we use substitution $x = e^t$ and assume that $\theta = \frac{d}{dt}$ then

$$xD = \theta \,, \;\; x^2 D^2 = \theta(\theta - 1)$$

In the given equation we get the equation

$$[\theta(\theta - 1) + 4\theta - 4]y = t$$

$$\Longrightarrow (\theta^2 + 3\theta - 4)y = t$$

This is a second order differential equation with constant coefficients and for solution

First, we find $y(x)$ is a solution of homogenous equation

$$(\theta^2 + 3\theta - 4)y = 0$$

Where the auxiliary equation

$$k^2 + 3k - 4 = 0$$

$$\Longrightarrow (k-1)(k+4) = 0 \quad \Longrightarrow \quad k_1 = 1 \quad , \quad k_2 = -4$$

Since the two roots are real and different, there are two solutions.

$$y_1 = A_1 e^{k_1 t} = A_1 e^{t} \ , \ \ y_2 = A_2 e^{k_2 t} = A_2 e^{-4t}$$

Where A_1 , A_2 are constants and the solution in the sum of the two solutions

$$y_c(x) = A_1 e^{t} + A_2 e^{-4t}$$

Second, we find $y_p(x)$ the solution of the Non-Homogenous equation.

$$(\theta^2 + 3\theta - 4)y_p = t$$

$$y_p = \frac{1}{(\theta^2 + \theta - 4)}\{t\} = \frac{1}{(\theta - 1)(\theta + 4)}\{t\}$$

$$= \frac{1}{-4(1-\theta)\left(1 + \frac{\theta}{4}\right)}\{t\} = -\frac{1}{4}(1-\theta)^{-1}\left(1 + \frac{\theta}{4}\right)^{-1}\{t\}$$

$$= -\frac{1}{4}(1 + \theta + \theta^2 + \cdots)\left(1 - \frac{\theta}{4} + \left(\frac{\theta}{4}\right)^2 + \cdots\right)\{t\}$$

$$= -\frac{1}{4}\left(1 + \frac{3\theta}{4} + \frac{13}{16}\theta^2 + \cdots\right)\{t\} = -\frac{1}{4}\left(t + \frac{3}{4}\right)$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = y_c(x) + y_p(x)$$

$$= A_1 e^t + A_2 e^{-4t} - \frac{1}{4}\left(t + \frac{3}{4}\right)$$

$$= A_1 x + A_2 x^{-4} - \frac{1}{4}\left(\ln x + \frac{3}{4}\right)$$

Exercises

Find the general solution for each of the following differential equations

$1 -\ \ 3x^2y' - 2xy' - 12y = x^3$

$2 -\ \ x^2y' + 7xy' + 0y = 2\ln x$

$3 -\ \ x^2y' - xy' + y = 3(\ln x)^2 + 2\ln x$

4- $x^2y' - xy' + y = x\ln x$

Chapter 7

Power series solution of a differential equations

7-1 Taylor method

7-2 Frobinious method

Introduction

Power series solution of a differential equations

All the methods discussed above for solving the differential equations cannot solve all the differential equations such as differential equations of variable coefficients and cannot reduced to differential equations with constants coefficients.

To solve such differential equations we have to look for infinite series solutions. There are two ways for doing this:

7-1 Taylor method

It is known that the Tailor expansion of any function $y(x)$ around any point $x = a$ is:

$$y(x) = y(a) + \frac{y^{/}(a)}{1\,!}\,(x-a) + \frac{y^{//}(a)}{2\,!}(x-a)^2 + \cdots + \frac{y^{(n)}(a)}{n\,!}(x-a)^n + \cdots \quad (1)$$

Note: If $a = 0$ the expansion is called Macklorein expansion and has

The form:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!} + \cdots \qquad (2)$$

So, if we have the differential equation

$$y'' + \varphi(x)\, y' + \propto (x) y = 0 \qquad (3)$$

The, by iterated differentiation of $y(x)$ to determine $y^{(n)}(\mathrm{a}), \mathrm{n} = 0,1,2,..$ and substituting $x = a$ in (1) one can establish the solution of (3) in the Taylor expansion form around $x = a$

Example 1

Solve the differential equation

$$y'' = x^2 - y^2 \qquad (4)$$

Around $x = 0$, where $y(0) = y'(0) = 1$ $\qquad (5)$

Solution

Putting $x = 0$ in both sides of equation (4), we have

$$y''(0) = -y^2(0) = -1$$

Differentiating (4) with respect to x we have:

$$y'''(x) = 2x - 2y(x)\, y'(x) \qquad (6)$$

Putting $x = 0$ in both sides of (6), we have:

$$y'''(0) = -2(-1) = 2$$

Similarly, differentiating (6) with respect to x, we have:

$$y^{(4)}(x) = -2y'(x) - 2y(x)\, y''(x) \qquad (7)$$

Putting $x = 0$ in both sides of equation (7), we have:

$$y^{(4)}(0) = 2$$

By repeating these steps, one can get

$$y^{(5)}(0) = 10 \ , \quad y^{(6)}(0) = 5 \ \dots\dots\dots..$$

Substituting, now these values in the Macklorein form we obtain the solution of (4) in the form:

$$y(x) = -1 - \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \frac{10}{5!}x^5 + \cdots \quad (8)$$

Note: It may be easy to differentiate both sides of the differential equation (3) n times by using Leibenz theorem and get a relation between $\boldsymbol{y^{(n+2)}(x), y^{(n+1)}(x) and\ y^{n}(x)}$ then putting $\boldsymbol{x = a}$in this relation to get similar relation between $\boldsymbol{y^{(n+2)}(a), y^{(n+1)}(a) and\ y^{n}(a)}$ and choosing $\boldsymbol{n = 0}$, $\boldsymbol{1}$, $\boldsymbol{2}$, $\boldsymbol{3}$, … one can obtain $\boldsymbol{y''(a), y'''(a), y^{(4)}(a)}$, … in terms of $\boldsymbol{y(a) and\ y'(a)}$, then substituting these values in Taylor form or Macklorein form the solution of (3) established.

Example 2

Solve the problem

$$y''(x) + xy(x) = 0 \qquad , \quad y(0) = a \ \ and\ y'(0) = b \qquad (9)$$

Solution

Differentiating in (9) n times (by using Leibenz theorem), we have

$$y^{(n+2)}(x) + xy^{n}(x) + ny^{(n-1)}(x) = 0$$

At $x = 0$, this relation reduces to:

$$y^{(n+2)}(x) + xy^{(n)}(x) + ny^{(n-1)}(x) = 0$$

At $x = 0$, this relation reduces to:

$$y^{(n+2)}(0) = ny^{(n-1)}(0) \qquad (10)$$

Since $y(0) = a \quad and \quad y'(0) = b$ and using (10), we have

$$n = 0 \;\rightarrow\; y''(0) = 0$$

$$n = 1 \;\uparrow\; y'''(0) = -y(0) = -a$$

$$n = 2 \rightarrow \; y^{(4)}(0) = -2y^{/}(0) = \; -2b$$

$$n = 3 \rightarrow \; y^{(5)}(0) = -3y^{//}(0) = \; 0 \quad ,$$

$$n = 4 \rightarrow \; y^{(6)}(0) = -4y^{///}(0) = \; 8a \;\; ,....$$

Then the solution has the form

$$y(x) = a + b - \frac{8a}{3!}x^3 - \frac{2b}{4!}x^4 + \frac{8a}{6!}x^6 + \cdots$$

It is clear from the above that the Taylor and Macklorien methods is not always practical because it may be difficult in some problems. However, we can establish a technique depends upon the fact that the solution of the given problems is in an infinite series form:

$$y(x) = x^v(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots) = \sum_0^\infty a_n x^{n+v} \qquad (11)$$

By substituting in any second order differential equation, one can obtain a relation between the coefficients and then find all the coefficients in terms of a_0 and a_1.

Example

Solve the problem

$$xy'' + y' + my = 0, \quad m\ is\ real\ constant$$

Solution

Assume that the solution in the form

$$y(x) = \sum_0^\infty a_n\, x^{n+v} \ , \ \ then$$

$$y'(x) = \sum_0^\infty a_n\,(n+v)\,x^{n+v} \quad and \quad y''(x) = \sum_0^\infty a_n\,(n+v-1)\,x^{n+v}$$

Substituting by these values in the equation, we have:

$$\sum_0^\infty (n+v)^2\, a_n\, x^{n+v-1} + m \sum_0^\infty a_n\, x^{n+v} = 0$$

Comparing the coefficients of x^{n+v} in both sides, we obtain the relation between the coefficients in the form:

$$a_{n+1} = -\frac{m}{(n+v+1)^2}\ a_n \ , i.e.$$

$$a_{n+1} = -\frac{m}{(n+1)^2}\ a_n \ \ at \ \ v = 0$$

Thus,

$$a_{n+1} = \frac{-m}{(n+1)^2} \cdot \frac{-m}{(n)^2} \cdot \frac{-m}{(n-1)^2}, \dots \dots \dots \ , \frac{-m}{(1)^2} a_0 = \frac{(-m)^n}{(n!)^2} a_0$$

Hence, the solution of the given equation has the form:

$$y(x) = a_0 \sum_{0}^{\infty} \frac{(-m)^n}{(n!)^2}\ x^n$$

(In complete solution why?)

<u>Important definitions</u>

Definition 1 (analytic function)

The function $f(x)$ is said to be analytic at $x = 0$ if and only if $f^{/}(x)$ has definite value at all points around $= 0$. Or if $f^{/}(x)$ can be expressed as a polynomial in x.

Definition 2 (Ordinary point)

The point $x = 0$ is said to be an ordinary point of the equation:

$$y'' + \varphi(x)\, y' + \propto (x)\, y = 0 \qquad (3)$$

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad and$$

$$\propto (x) = b_0 + b\, x + b_2 x^2 + b_3 x^3 + \cdots$$

<u>Note,</u> if the point is not ordinary it is said to be singular.

Definition 3 (regular singular point)

The singular point is said to be regular singular if the two series:

$$x\, \varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad and$$

$$x^2 \propto (x) = b_0 + b\, x + b_2 x^2 + b_3 x^3 + \cdots$$

Are convergent around $x = 0$

<u>7-2 Frobinious method</u>

This method is used to solve the problem around ordinary point or regular singular point, so consider the problem

$$y'' + \varphi(x)\, y' {+}\propto (x)\, y = 0$$

Assume the solution in the form

$$y(x) = \sum_{0}^{\infty} a_n\, x^{n+c}, \quad then$$

$$y'(x) = \sum_{0}^{\infty} a_n(n+c)\, x^{n+c-1}\,, \quad y''(x)$$

$$= \sum_{0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2}$$

Multiplying both sides of equation (3) by x^2 yields:

$$x^2 y'' + x^2\varphi(x)\, y' + x^2 \propto (x)\, y = 0 \qquad (13)$$

By substituting the values of (12) in equation (13), we have:

$$\sum_{0}^{\infty} a_n(n+c)(n+\nu-1)x^{n+c} + \sum_{0}^{\infty} \alpha_n x^n a_n(n+c)x^{n+c}\,,$$

$$+\sum_{0}^{\infty} \boldsymbol{\beta_n} x^n a_n x^{n+c} = 0 \qquad (14)$$

Comparing the coefficients of the least power of, i.e. coefficients of x^{ν} in both sides we obtain

$$\nu(\nu-1)+\ \alpha_0 c+\beta_0=0\text{ , i.e.}$$

$$\nu^2+(a_0-1)c+\boldsymbol{\beta_0}\ \ =0$$

The last equation has two roots in. Hence three cases arises:

First: the difference between the two roots is incorrect number. In this case one of the roots produce a solution y_1 and the other root produces another solution y_2. Then the general solution is $y(x)= y_1+y_2$

First: the difference between the two roots is integer

Third: The two are identical

Example 2

Using the power series, find the solution of:

$$y''+2xy'-y=0 \qquad (15)$$

Around $x=0$

Solution

Assume the solution of the equation has the form:

$$y(x)=\sum_{n=0}^{\infty} a_n\, x^n$$

Then

$$y'(x) = \sum_{n=2}^{\infty} n\, a_n\, x^{n-2} \qquad , \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2}$$

Substituting in (15) we have:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} - \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Comparing the coefficients of x^n in both sides, we obtain the relation:

$$(n+2)(n+1)a_{n+2} + 2n\, a_n - na_n = 0$$

$$(n+2)\,(n+1)\, a_{n+2} = (1-2n)a_n$$

$$a_{n+2} = \frac{(1-2n)}{(n+2)(n+1)} a_n$$

In this recurrence relation, we choose $n = 1{,}2{,}3, \ldots.$

$$n = 0 \quad \rightarrow \quad a_2 = \frac{1}{2!} a_0$$

$$n = 2 \quad \rightarrow \quad a_4 = -\frac{3}{4.3} a_2 = -\frac{3}{4!} a_0$$

$$n = 4 \quad \rightarrow \quad a_6 = -\frac{7}{6.5} a_4 = \frac{21}{6!} a_0$$

Similarly

$$n = 1 \quad \rightarrow \quad a_3 = -\frac{1}{3.2} a_1 = -\frac{1}{3!} a_1$$

$$n = 3 \quad \rightarrow \quad a_5 = -\frac{5}{5.4}\, a_3 = \frac{5}{5!} a_1$$

$$n = 5 \quad \rightarrow \quad a_7 = -\frac{9}{7.6} a_5 = -\frac{45}{7!} a_1$$

By substituting all these values in $y(x)$ we have

$$y(x) = a_0 \left[1 + \frac{1}{2!} x^2 - \frac{3}{4!} x^4 + \frac{21}{6!} x^6 + \cdots \right] + a_1 \left[x - \frac{1}{3!} x^3 + \frac{5}{5!} x^5 - \frac{45}{7!} x^7 + \cdots \right]$$

Example 3

Solve the differential equation:

$$(1 - x^2) y'' - 2xy' + m(m+1)y = 0 \quad (16)$$

Solution

Assume that the solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (17)$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n$$

By substituting these values in the given equation, we have

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n - \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^{n+2}$$
$$- 2\sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^{n+1} + m(m+1) \sum_{n=0}^{\infty} a_n x^n = 0$$

Comparing the coefficients of x^n in both sides, we obtain the relation:

$$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2n\, a_n + m(m+1)a_n = 0$$

$$(n+1)(n+2)a_{n+2} = (n^2 - n + 2n - m^2 - m)a_n$$

$$\therefore\ (n+1)(n+2)\, a_{n+2} = (n^2 + n - m^2 - m)a_n$$

$$\therefore\ (n+1)(n+2)a_{n+2} = \big((n^2 - m^2) + (n-m)\big)a_n$$

$$\therefore\ (n+1)(n+2)a_{n+2} = \big((n-m)(n+m) + (n-m)\big)a_n$$

$$\therefore\ (n+1)(n+2)a_{n+2} = (n-m)(m+m-1)a_n$$

$$\therefore\ a_{n+2} = \frac{(n-m)(n+m+1)}{(n+1)(n+2)}a_n$$

Choosing $n = 0,2,4,\ldots$

$$\therefore a_2 = \frac{-m\,(m+1)}{2.1} a_0$$

$$\therefore a_4 = \frac{(2-m)(m+3)}{4.3} a_2 = \frac{m(m-2)(m+1)(m+3)}{4!} a_0$$

Similarly, choosing $n = 1,3,5, \dots$ yields

$$\therefore a_3 = \frac{-(m-1)(m+2)}{3.2} a_1$$

$$\therefore a_5 = \frac{(3-m)(m+4)}{5.4} a_3 = \frac{(m-3)(m-1)(m+2)(m+4)}{5!} a_1$$

Finally, substituting these coefficients in (17), we obtain the solution in the form:

$$y(x) = a_0 \left[1 - \frac{m(m+1)}{2!} x^2 + \frac{(m-2)m(m+1)(m+3)}{4!} x^4 - \cdots \right]$$
$$+ \left[x - \frac{(m-1)(m+2)}{3!} x^3 \right.$$
$$\left. + \frac{(m-3)(m-1)(m+2)(m+4)}{5!} x^5 - \cdots \right]$$

Where

$$y_1(x) = 1 - \frac{m(m+1)}{2!} x^2 + \frac{(m-2)\,m\,(m+1)\,(m+3)}{4!} x^4 - \cdots$$

$$y_2(x) = x - \frac{(m-1)(m+2)}{3!} x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!} x^5 - \cdots$$

And the solution is complete.

Example 4

By using power series solution solve

$$4xy'' + 2y' + y = 0$$

Solution

The point $x = 0$ is regular singular point

$$y(x) = \sum_{n=0}^{\infty} a_n \, x^{n+c}$$

$$\therefore y'(x) = \sum_{n=0}^{\infty} a_n \, (n+c) \, x^{n+c-1}$$

$$\therefore y''(x) = \sum_{n=0}^{\infty} a_n \, (n+c)(n+c-1) \, x^{n+c-2}$$

We obtain

$$4\sum_{n=0}^{\infty}(n+c)(n+c-1)\,a_n\,x^{n+c-1}+2\sum_{n=0}^{\infty}(n+c)a_n\,x^{n+c-1}$$

$$+\sum_{n=0}^{\infty}a_n\,x^{n+c}=0$$

This leads to

$$2\sum_{n=0}^{\infty}(n+c)(2n+2c-1)\,a_n\,x^{n+c-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+c-1}=0$$

Comparing the coefficients of x^{c-1} in both sides, we have

$$2c(2c-1)a_0=0$$

This implies that:

$$c_2=\frac{1}{2}\quad,\quad c_1=0$$

And comparing the coefficients of x^{n+c} in both sides, we have

$$(2(n+c+1)(2n+2c+1)a_{n+1}=-a_n$$

Case 1:-

$c=0$ in (2), leads to:

$$a_{n+1}=-\frac{1}{(2n+2)(2n+1)}a_n \qquad (3)$$

At $n=0$, we have:

$$a_1 = -\frac{1}{2}a_0$$

At $n = 1$, we have:

$$a_2 = -\frac{1}{4.3}a_1 = \frac{1}{4!}a_0$$

At $n = 2$, we have:

$$a_3 = -\frac{1}{6.5}a_2 = -\frac{1}{6!}a_0$$

Thus, the solution in this case at $c = 0$ has the form:

$$y_1(x) = a_0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right] = a_0 \cos\sqrt{x}$$

Case 2:-

$c = \frac{1}{2}$ in this case we have the relation:

$$a_{n+1} = -\frac{1}{(2n+3)(2n+2)}a_n \qquad (4)$$

If $n = 0$ (4) reduces to

$$a_1 = -\frac{1}{3!}a_0$$

If $n = 1$ (4) reduces to

$$a_2 = -\frac{1}{5.4}a_1 = \frac{1}{5!}a_0$$

If $n = 2$ (4) reduces to

$$a_3 = -\frac{1}{7.6}a_2 = -\frac{1}{7!}a_0$$

Thus, the solution in this case at $c = \frac{1}{2}$ has the form:

$$y_2(x) = a_0\sqrt{x}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right]$$

$$y_2(x) = a_0\left[\sqrt{x} - \frac{x^{\frac{3}{2}}}{3!} + \frac{x^{\frac{5}{2}}}{5!} - \frac{x^{\frac{7}{2}}}{7!} + \cdots\right] = a_0 \sin\sqrt{x}$$

So, the general solution of (1) takes the form:

$$y(x) = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}$$

Exercises

In terms of infinite series, find the solution of the following

1) $2x(x-1)y'' + 3(x-1)y' - y = 0$

2) $2xy'' + 5(1+2x)y' + 5y = 0$

3) $2x^2(x+1)y'' + x(7x-1)y' + y = 0$

Chapter 8

Applications lf Ordinary Differential Equations

8-1 Growth and Decay

8-2 Half-Life

8-3 Carbon Dating

8-4 Series circuits

8-5 Orthogonal trajectories

8-1 Growth and Decay

The initial value problem

$$\frac{dx}{dt} = kx \quad , \quad x(t_0) = x_0 \qquad\qquad (1)$$

Where k is a constant of proportionality, occurs in many physical theories involving either growth or decay. For example in biology it is often observed that the rate at which certain bacteria grow is proportional to a number of bacteria present at any time. Over short time, the population of small animals, such as rodents, rabbits or any others can be predicted fairly accurately by a solution of (1). In physics an IVP such as (1) provides a model for approximating the amount of a substance that is disintegrating or decaying through radioactivity.

In chemistry the amount of substance remaining during certain reactions is also described by (1).

The constant k in (1) is either positive or negative and can be determined from the solution of the problem using subsequent measurement of x at time $t_1 > \ t_0$

Example 1

A culture initially N_0 number of bacteria. At $t = 1$ hour the number of bacteria is measured to be $\left(\frac{3}{2}\right) N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

Solution

The IVP that represented this problem is

$$\frac{dN}{dt} = kN \ , \ \ N(t_0) = N_0 \qquad (2)$$

Solving this first order separable differential equation

$$\frac{dN}{N} = k\ dt \ \ i.e. \ln|N| = k\ t + \ln|c|$$

Thus the solution has the form

$$N(t) = c\ e^{kt}$$

$$at \ \ t = 0\ yields \ \ N(0) = N_0 = c$$

Thus, the solution reduces to

$$N(t) = N_0\, e^{kt}$$

$$at\ t = 1\ we\ have\ \ \frac{3}{2}N_0 = N_0\, e^{k} = \frac{3}{2}$$

$$Hence,\ \ k = \ln\frac{3}{2} = 0.4055$$

To find the time at which the bacteria have tripled we have

$$3N_0 = N_0\, e^{0.4055\, t}\ that\ is\ \ 0.4055\, t = \ln 3$$

$$and\ also\ \ t = \frac{\ln 3}{0.4055} \cong 2.71\ hours$$

8-2 Half-Life

The half-life is simply the time it takes for one half of the atoms in an initial amount S_0 to disintegrate or transmute, into the atom of another element. The longer the half-life of a substance, the more stable it is.

Example 2

A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount S_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution

Let $S(t)$ denote the amount of plutonium remaining at any time. As in the last example the IVP that represented this problem is

$$\frac{dS}{dt} = kS \quad , \quad S(t_0) = S_0 \qquad\qquad (3)$$

And its solution has the form $\quad S(t) = S_0\, e^{kt}$

If 0.043 % of the atom of S_0 have disintegrated the 99.957 % of the substance remains. To find k we use 0.99957 $S_0 = S(15)$ of the substance remains. To find k

$$0.99947\, S_0 = S_0\, e^{15k}$$

So $15\, k = \ln 0.99957 \quad or \quad k = \frac{\ln 0.99957}{15} = -0.00002867$

Now, the required half-life is the corresponding value of time for which $S(t) = \frac{S_0}{2}$. Solving for t gives

$\frac{S_0}{2} = S_0\, e^{-0.00002867}$ This reduces to

$$-0.00002684\, t = \ln\frac{1}{2} = -\ln 2 \ \ , i.e.$$

$$t = \frac{\ln 2}{0.00002867} \cong 24.180\, years$$

8-3 Carbon Dating

The theory of carbon dating is based on the fact that the isotope carbon 14 (C-14) is produced in the atmosphere by the action of cosmic radiation on nitrogen. The ratio of the amount of C-14to ordinary carbon in the atmosphere appears to be constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When an organism dies, the absorption of C-14 by either breathing or eating ceases. Thus by comparing the proportionate amount of C-14 present, say, in a fossil with the constant ratio found in the atmosphere, it is possible to obtain a reasonable of its age.

The method is based on the knowledge that the half-life of C-14 is approximately 5600 years.

Example 3

A fossilized bone is found to contain $\frac{1}{1000}$ the original amount of C-14. Determine the age of the fossil.

Solution

Similar to example 2, the IVP that represents this problem is

$$\frac{dS}{dt} = kS \quad , \quad S(t_0) = S_0 \qquad (4)$$

And has the solution

$$S(t) = S_0\, e^{kt}$$

To determine the value of k we use the fact that

$$\frac{S_0}{2} = S(5600) \quad or \quad \frac{S_0}{2} = S_0\, e^{5600\, t}$$

$$5600\, k = \ln\left(\frac{1}{2}\right) = -\ln 2\,. \quad Thus,$$

$k = -\frac{\ln 2}{5600} = -0.00012378$ and the amount at any time is:

$$S(t) = S_0\, e^{-0.00012378}$$

When $S(t) = \frac{1}{1000}\, S_0 \;\; then, \; \frac{S_0}{1000} = S_0\, e^{-0.00012378\, t}$

Thus, $-0.00012378\,t = \ln\frac{1}{1000} = -\ln 1000$ and required age of the fossilized bone is

$$t = \frac{\ln 1000}{0.00012378} \cong 55.800\,years$$

8-4 Series Circuits

In a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor $\left(L\left(\frac{di}{dt}\right)\right)$ and the voltage drp across the resistor (iR) is the same as the impressed voltage $\left(E(t)\right)$ on the circuit. Thus we obtain the linear differential equation for the $i(t)$ is

$$L\frac{di}{dt} + Ri = E(t) \qquad (5)$$

Where the constants L and R denotes to the inductance and resistance (respectively).

The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{c}$ where q is the charge on the capacitor. Hence for the series circuit Kirchhoff's second law gives

$$Ri + \frac{1}{c}q = E(t) \qquad (6)$$

But current i and charge q are related by $i = \frac{dq}{dt}$. So equation (6) becomes the linear differential equation.

$$R\frac{dq}{dt} + \frac{1}{c}q = E(t) \qquad (7)$$

Example 4

A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and resistance 10 ohms. Determine the current i if the initial current is zero.

Solution

From equation (5) we solve the equation

$$\frac{1}{2}\frac{di}{dt} + 10i = 12 \quad subject\ to\ \ i(0) = 0$$

Since it is linear differential equation of the first order and may take the form.

$$\frac{di}{dt} + 20i = 24\,, \quad then$$

$$\mu(t) = e^{\int 20\ dt} = e^{20\,t} + c\,,\ \ c\ is\ constant$$

And the solution is given by using the formula

$$\mu(t)\, i(t) \;=\; 24\int e^{20\,t}\ dt\ + c\ \ ,(\ c\ \ is\ constant\ )\quad then$$

$$e^{20t}\, i(t) \;=\; 24\int e^{20\,t}\ dt\ + c\ \ ,(\ c\ \ is\ constant\ ),\quad thus$$

$$(t) = \frac{24}{20} + c\ e^{-\,20\,t} = \frac{6}{5} + c\ e^{-\,20\,t}$$

The general solution of equation (6) can be given similarly and has the form

$$i(t) = \frac{e^{-\frac{R}{L}t}}{L}\ \int e^{-\frac{R}{L}t}\, E(t)dt\ + c\ e^{-\frac{R}{L}t}$$

In particular if $E(t) = E_0$

$$i(t) = \frac{E_0}{R} + ce^{-\frac{R}{L}t}$$

8-5 Orthogonal Trajectories

Definition 1

Two lines L_1 (with slope m_1) and L_2 (with slope m_2) are **perpendicular** if and only if $\boldsymbol{m_1 m_2 = -1}$

Definition 2

Two curves L_1 (with tangent line T_1) and L_2 (with tangent line T_2) are **orthogonal** are a point if and only if their tangent lines $T_1\ and\ T_2$ are **perpendicular** at the point of intersection.

Example 5

Show that the two curves defined by:

$$y = x^3\ and\ \ x^2 + 3y^2 = 4$$

Are orthogonal

Solution

By solving the equations of the curves together, it is clear that the points of intersection are (1,1) and (-1,-1). Now the slope of the tangent line for the curve $y = x^3\ is\ \frac{dy}{dx} = 3x^2$ and hence

$$\left(\frac{dy}{dx}\right)_{x=1} = \left(\frac{dy}{dx}\right)_{x=-1} = 3 \qquad (i)$$

Now, for the second curve $x^2 + 3y^2 = 4$

$$2x + 6y\frac{dy}{dx} = 0 \quad , i.e. \ \frac{dy}{dx} = -\frac{x}{3y}$$

At the points of intersection we have

$$\left(\frac{dy}{dx}\right)_{x=1} = \left(\frac{dy}{dx}\right)_{x=-1} = -\frac{1}{3} \qquad (ii)$$

From (i) and (ii), the two curves are orthogonal.

Definition 1 (Orthogonal trajectories)

When all curves of one family of curves $G(x, y, c_1) = 0$ intersect orthogonally all the curves of another family $H(x, y, c_2) = 0$, then the families are said to be orthogonal trajectories of each other.

To find orthogonal trajectories of a given family of curves we first find the differential equation (by using the technique of the formation of the differential equation, given in section 1-2). The replacing y' in this differential equation $-\frac{1}{y'}\left(where\ y' = \frac{dy}{dx}\right)$ the resulting differential equation is the equation of the required orthogonal trajectories. Solving it the orthogonal trajectories is obtained.

Example 6

Find orthogonal trajectories of the family of the rectangular hyperbolas

$$y = \frac{k}{x} \;, \quad k\ constant$$

Solution

The equation can put in the form $xy = k$ and by differentiating both side of the equation, we have

$$xy' + y = 0 \quad implies \quad y' = -\frac{y}{x}$$

Now, replacing $y'\ by\ -\frac{1}{y}\ we\ obtain$

$$\frac{dy}{dx} = \frac{x}{y} \quad implies\ \ y\,dy = x\,dx$$

Integrating both sides, we have

$$\int y\,dy = \int x\,dx \qquad thus, the\ trajectories\ are$$

Which represents another family of hyperbolas.

Example 7

Find orthogonal trajectories of the family of the circle centered at the origin:

$$x^2 + y^2 = R^2 \,, \quad (R\ is\ the\ radies)$$

Solution

Differentiating both sides we have the differential equation

$$x + y\,y' = 0$$

Replacing y' by $-\frac{1}{y'}$ yields $\frac{dy}{dx} = \frac{y}{x}$. Thus, have

$$\int \frac{dy}{y} = \int \frac{dx}{x} + \ln c, \quad (c\ constant). \quad this\ implies\ that$$

$$\ln y = -\ln c\,x \quad i.e. \quad y = cx$$

Which a family of straight lines passes the origin.

Exercises

1- The population of a certain community is known to increase at a rate proportional to the number of people present at any time .If the population has doubled after 10 years, how long will it take to triple?

2- If the rate of the population growth of Suez proportional with its population at any time. If the rate of increase is 2% yearly, and the population is now 200000, haw long it take to double? What is the population after 10 years.

3- A person invests 10000 Egyptian pounds in a bank gives 10% interest, how long will it take to be one million pounds? And how long will it take to triple?

4- The population of bacteria in a culture growth at a rate proportional to the number of bacteria at any time. After 4 hours it is observed that 500 bacteria present. After 10 hours there are 3000 bacteria present. What is the initial number of bacteria?

5- If there are 100 milligrams of a radioactive substance present. After 6 hours the mass decreased by 3%. If the rate of decay is proportional to the amount of the substance present at any time. Find the amount remaining after 12 hours and after 24 hours.

6- Determine the half-life of the radioactive substance described in problem 5.

7- In a piece of burned wood, it was found that 85.5% of the C-4 has decayed. Use the information discussed before to determine the approximate age of the wood.

In the problems 8- 14 find the orthogonal trajectories of the given family of curves

8- $2x + 3y = k$

9- $y = k\,x^2$

10- $y = x\frac{x}{1-kx}$

11- $x^2 + y^2 = 2k\,x$

12- $y = \frac{1}{\ln kx}$

13- $y = (x-k)^2$

14- Find the member of orthogonal trajectories for $x + y = k\,e^y$ that passes through (0,5)

15- Find the member of orthogonal trajectories for $3xy^2 = 2 + 3k\,x$ that passes through (1,10)

References

[1] Blanchard,P.,David,R.L.and Hall,G.R.,Differential Equations (4th end),Brooks/Cole Publishing Co,2011

[2] Dennis G. Zill, A first course in differential equation, PWS –Kent publishing Company Boston, (1993).

[3] Edward,C.H and David,E.Penny, Elementary differential equations with applications(3rd end), Printic-hall-England Cliffs,1996

[4] Finizio N.and Ladas,G. Ordinary differential equations with modern application,(3rd edition).