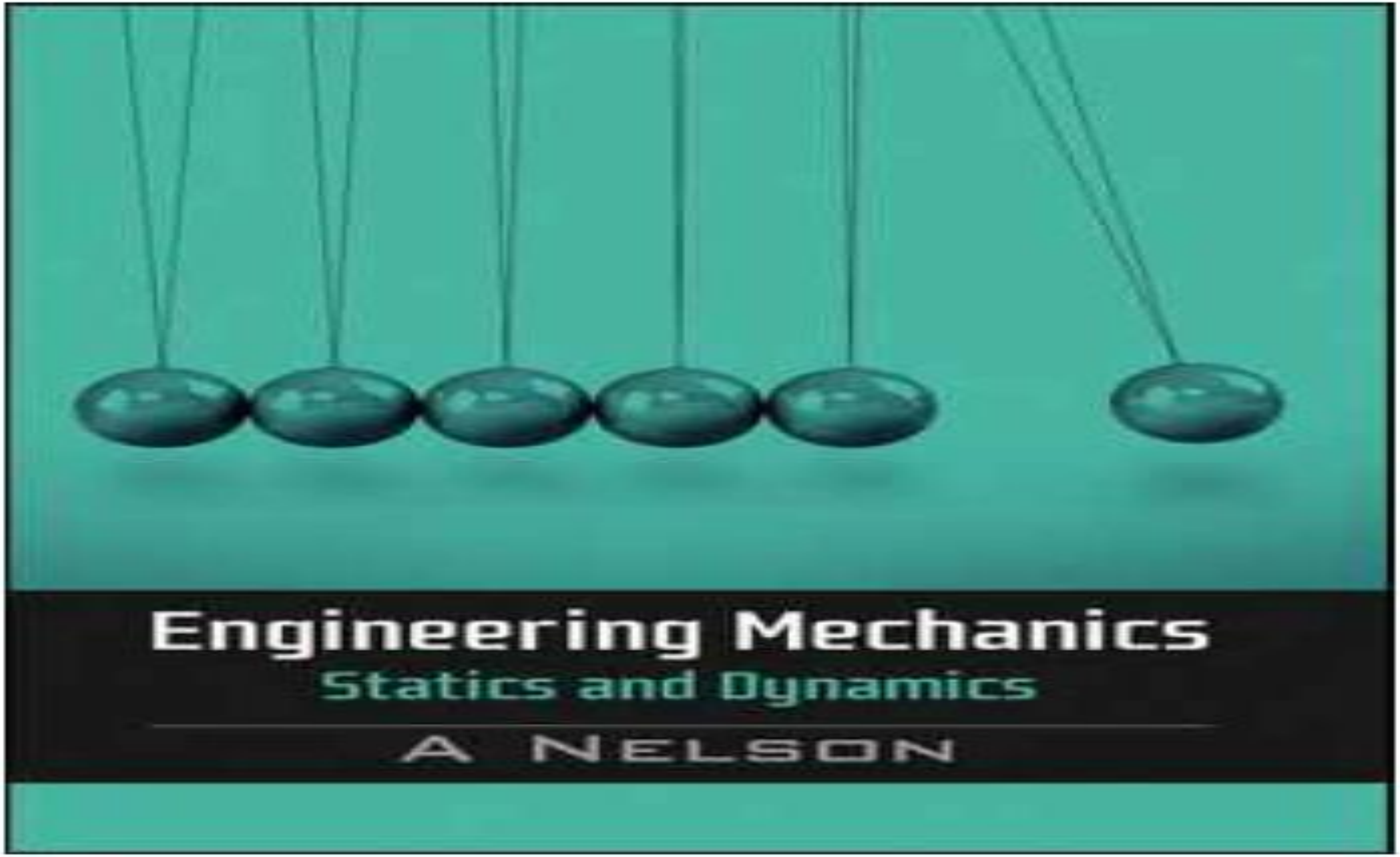
Lecture (1)
Mechanics

DR./ IBRAHIM ABADY


Engineering Mechanics
Statics and Dynamics
A Nelson

1 Introduction

1.1 INTRODUCTION TO MECHANICS

Mechanics is the oldest of *physical sciences*, which deals with the state of rest or motion of bodies under the action of forces. As we know that matter can exist in three different states—solids, liquids and gases—and that their behaviours under the action of forces vary, we study them separately under different headings. Hence, mechanics can be classified into **mechanics of solids** and **mechanics of fluids**. Mechanics of fluids or fluid mechanics deals with the study of liquids and gases (together called fluids) at rest or in motion. This will be covered in books on fluid mechanics and will not be covered in this study. In this book, we shall study mechanics of solids only.

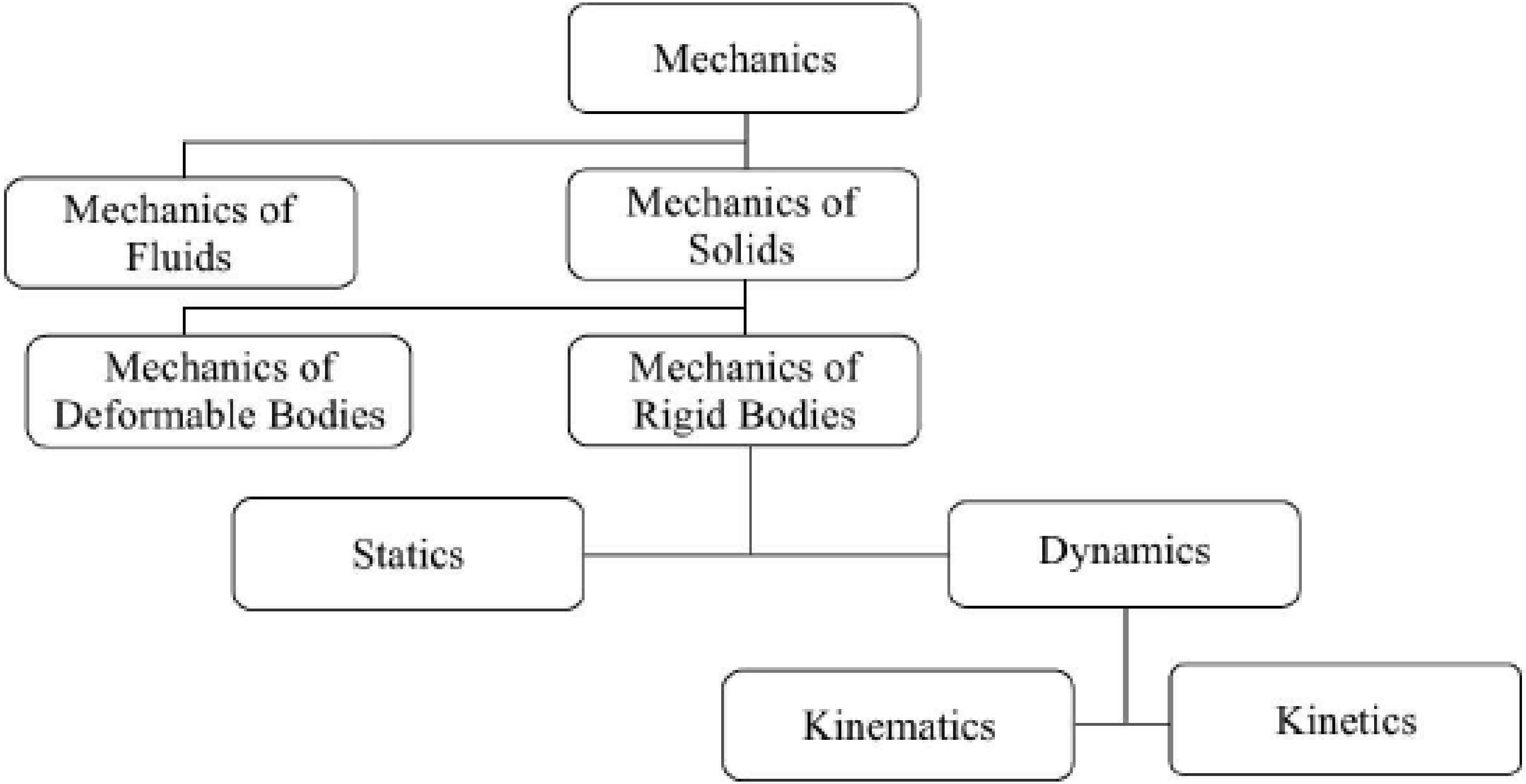


Chart 1.1 Branches of mechanics

Table 1.1 Fundamental and derived quantities in mechanics

S.No.	*Quantity*	*Dimension*	*Unit*
1	Length	L	m
2	Mass	M	kg
3	Time	T	s
4	Area	L^2	m^2
5	Volume	L^3	m^3
6	Velocity	LT^{-1}	m/s
7	Acceleration	LT^{-2}	m/s^2
8	Momentum	$M\,LT^{-1}$	N.s
9	Force	$M\,LT^{-2}$	N
10	Work done	$M\,L^2T^{-2}$	N.m (or) J
11	Energy	$M\,L^2T^{-2}$	N.m (or) J
12	Moment of force	$M\,L^2T^{-2}$	N.m
13	Torque	$M\,L^2T^{-2}$	N.m
14	Power	$M\,L^2T^{-3}$	N.m/s (or) W
15	Impulse	$M\,LT^{-1}$	N.s

1.5.5 Dimensional Homogeneity

While expressing physical quantities in terms of other quantities using mathematical equations, the following rule must be checked. That is, the dimensions of the terms on both sides of the equation must be equal. This is known as **dimensional homogeneity**. For instance, the velocity of a body in rectilinear motion (which we will learn later) is given by the equation,

$$v = v_o + at$$

where v_o is the initial velocity, a is the acceleration and t is the time. The dimensions of the terms on both sides of the equation can be written as

$$LT^{-1} = LT^{-1} + LT^{-2}T$$

or $$LT^{-1} = LT^{-1} + LT^{-1}$$

We see that the dimensions of the terms on both sides of the equation are equal. Hence, this equation is said to be dimensionally *homogeneous*. Consider now the following equation, We see that the dimensions of the terms on both sides of the equation are *not* equal. Hence, this equation is said to be dimensionally *non-homogeneous* and it can be seen that the correct equation must be

$$s = v_o t + (1/2)at^2$$

for it to be dimensionally homogeneous.

1.7 LAWS OF MECHANICS

The mechanics of bodies is governed by two basic laws: the laws of motion and the force laws. Laws of motion were stated by Isaac Newton.

First Law *Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces acting on it.*

Stated in an alternate way, to cause a change in velocity or in other words, to accelerate a body, an external force must act on it.

Second Law *If the resultant force acting on a particle is not zero, the particle will have acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.*

The mathematical statement of this law is given as

$$\vec{F} = m\vec{a}$$

where $\vec{F}$ represents the resultant of a system of forces acting on a particle, m mass of the particle and $\vec{a}$ its acceleration.

Third Law *To every action there is always an equal and opposite reaction.*

Forces always occur in pairs of active and reactive forces. A single isolated force is an impossibility. When one body exerts a force on a second body, the second body always exerts a force on the first.

Newton's Law of Universal Gravitation *Any two particles of mass, m_1 and m_2, are mutually attracted along the line connecting them with equal and opposite forces of magnitude F proportional to the product of their masses and inversely proportional to the square of the distance r between them.*

Mathematically, $$F = \frac{Gm_1m_2}{r^2},$$

where G is the universal gravitational constant and r is the distance between the two particles.

$$G = 6.673 \times 10^{-11}\ \text{N.m}^2/\text{kg}^2$$

Consider a body of mass m located on the surface of the earth, whose mass is M and radius R. Then the force of attraction between the two bodies is given as

$$F = \frac{GMm}{R^2} = m\left[\frac{GM}{R^2}\right]$$

We can see that the terms within the bracket are constants, hence we can write

$$F = mg$$

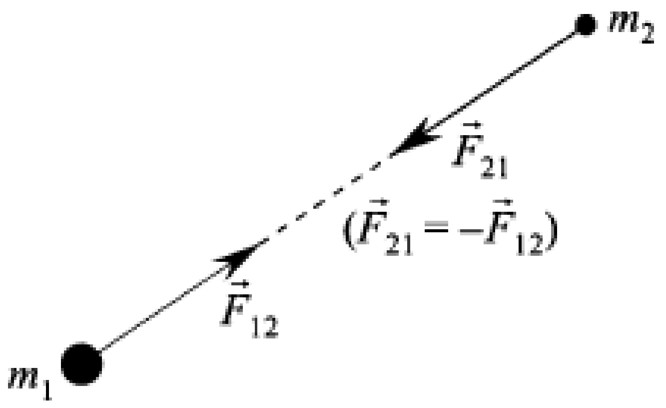


Fig. 1.1 Gravitational force of attraction

where g is a constant called the **acceleration due to gravity**. Here, the force F is also called the **force of gravity**. As the earth is not perfectly spherical, the value of R varies for different latitudes and for different altitudes. However, if we consider smaller altitudes, the value of g can be assumed constant. Hence, for all calculation purposes, a value of g equal to 9.81 m/s^2 can be used.

Mass vs. Weight Mass is an intrinsic property of a body. It is a scalar quantity. Weight of a body is defined as the gravitational force exerted by the earth on the body. As it is a force, weight is a vector quantity. Its magnitude is given as a product of mass and acceleration due to gravity.

② Vector Algebra

2.1 INTRODUCTION

Certain physical quantities can be specified *completely* by stating their magnitude (a number and unit) only. Such physical quantities are termed **scalars**. The examples are mass, length, time, density, energy, power, etc.

10 kg of mass; 2.3 m length; 35 s duration; 1000 kg/m^3 mass density

50 J of energy; 10 kW of power

As these are merely numbers with units, they follow the normal rules of arithmetic and algebra. For instance, if the times taken for two related activities are respectively 3 s and 5 s, then the total time taken for the two activities is given by normal arithmetic as $3 + 5 = 8$ s.

There are certain other physical quantities, which cannot be specified completely by stating their magnitude alone. Hence, in addition to the magnitude, we need to specify the direction in which they act. Such quantities, which are specified completely by their magnitude and direction and in addition, which obey the parallelogram law of addition (which we will learn later in the chapter) are termed

vectors. The examples are displacement, velocity, acceleration, momentum, force, etc.

a displacement of *10 m towards south*

a car moving with a velocity of *20 m/s along the eastern direction*

gravitational acceleration of *9.81* m/s^2 *acting towards the centre of the earth*

50 N weight acting vertically downwards

As these quantities are specified with direction as well as magnitude, the normal rules of mathematical operations cannot be performed on them. Hence, we must establish new rules for operation of vectors. Thus, for this reason we discuss the basics of vector algebra in this chapter, which will form the foundation for the succeeding chapters, as vectors are widely used in the solution of statics and dynamics problems.

2.2 REPRESENTATION OF VECTORS

Graphically, a vector is represented by a line segment drawn from a point called the **point of application** of the vector such that its length is proportional to the magnitude of the vector. Its inclination with

a reference line represents the **direction** of the vector. The arrowhead indicates **sense** of the vector, i.e., whether it is acting away or towards the point of application.

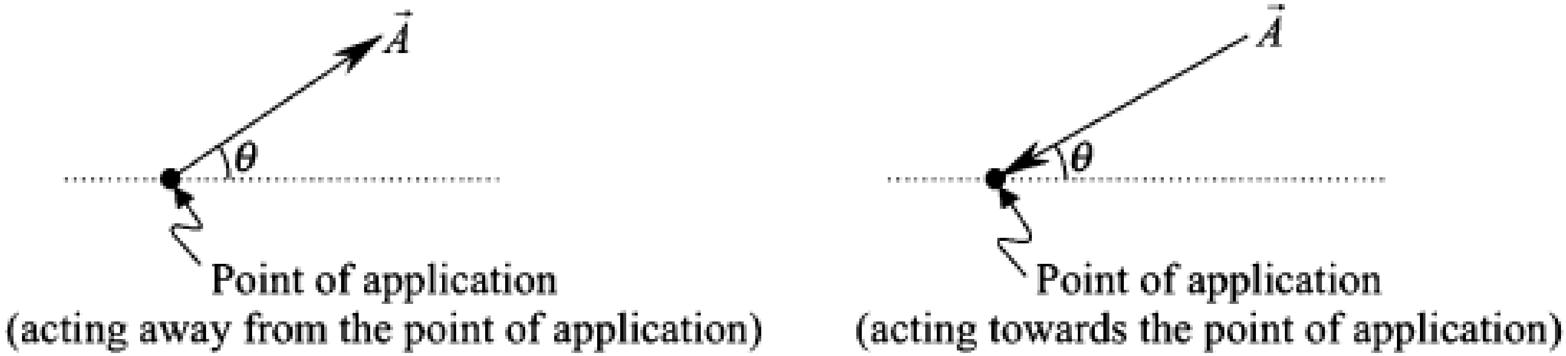


Fig. 2.1

For instance, a body moving with a velocity of 10 m/s along NE direction is shown graphically in Fig. 2.2.

Mathematically, a vector is represented by an **arrow** placed over the symbol used to denote the vector like $\vec{a}$. The **magnitude** of a vector is a positive quantity corresponding to the length of the vector. It is denoted by $|\vec{a}|$ or simply 'a.'

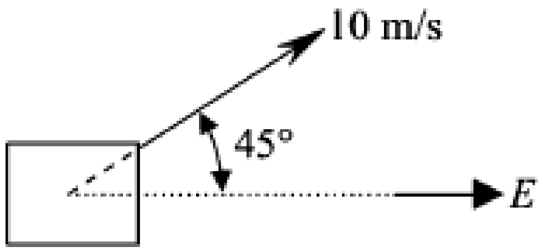


Fig. 2.2 Graphical representation

Unit Vector A vector whose magnitude is *unity* is called a **unit vector**. It is normally represented by a circumflex placed over the letter. For example: $\hat{n}$. Any vector can be represented as a product of its *magnitude* and *unit vector* along its direction, i.e.,

$$\vec{a} = |\vec{a}|\hat{n} = a\hat{n} \tag{2.1}$$

Here the length of the vector is specified by its magnitude and the direction by the unit vector along its direction. The unit vector along the direction of any vector can be obtained as

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} \tag{2.2}$$

The unit vectors that are of importance to us are the unit vectors along the rectangular coordinate axes. For a rectangular coordinate system, the unit vectors along X, Y and Z axes are normally represented respectively as $\hat{i}$, $\hat{j}$, and $\hat{k}$, or simply $\vec{i}$, $\vec{j}$, and $\vec{k}$.

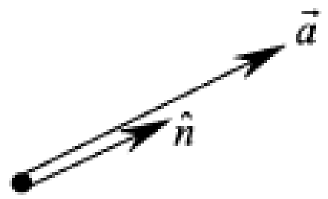


Fig. 2.7 Unit vector along the direction of any vector

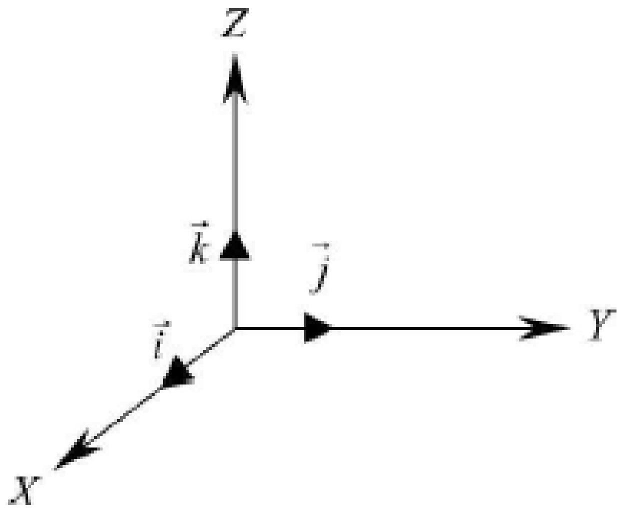


Fig. 2.8 Unit vectors along rectangular coordinate axes

There are other unit vectors too pointing along the path of the particle, $\hat{e}_t$, and normal to the path, $\hat{e}_n$. Though their magnitudes are unity, their directions, unlike the above rectangular unit vectors, keep on changing with respect to time.

Null Vector A vector whose magnitude is *zero* is called a **null vector**. It is similar to zero in scalars, but it is differentiated by a vector sign as $\vec{O}$.

Negative of a Vector A vector having the same magnitude, but direction opposite to that of a given vector is called the **negative** of the vector. Suppose the given vector is $\vec{a}$, then its negative vector is denoted as $-\vec{a}$. Graphically, it is represented as shown in Fig. 2.9.

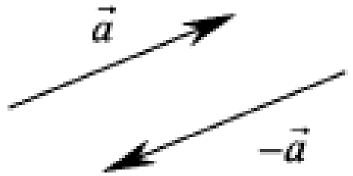


Fig. 2.9 Negative of a vector

2.4 MATHEMATICAL OPERATIONS OF VECTORS

2.4.1 Addition of Vectors

The addition of vectors can be determined graphically using the **parallelogram law**. It states that, when two concurrent vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram, then the diagonal passing through their point of concurrency represents the sum of vectors $\vec{a} + \vec{b}$ in magnitude and direction.

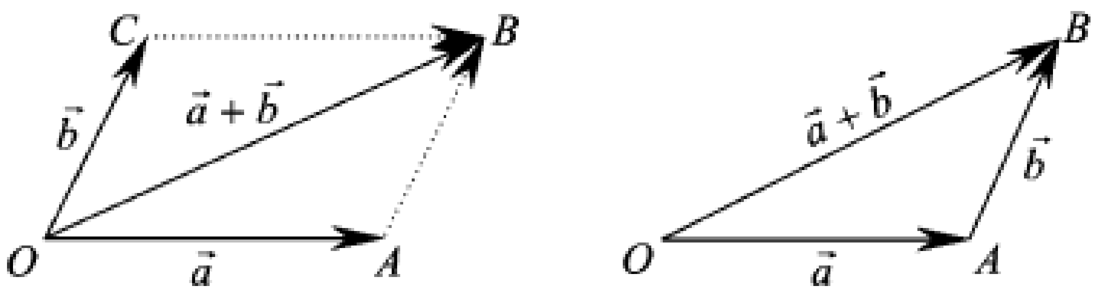
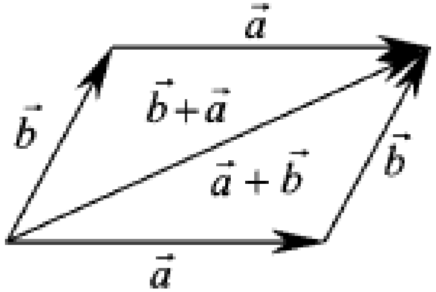


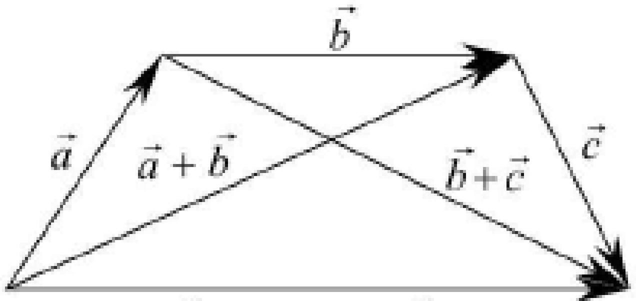
Fig. 2.10 Addition of vectors

Alternatively, treating $\vec{b}$ as a free vector, we can construct the triangle OAB and the summation can be determined using **triangle law**. It states that if two vectors $\vec{a}$ and $\vec{b}$ can be represented by the two sides of a triangle (in magnitude and direction) taken in order, then the third side (closing side) represents the sum of the two vectors in the opposite order.

The addition of vectors is independent of the order in which the vectors are selected. Hence, we say that vectors follow **commutative law**. Also, the vectors can be grouped without any restriction. Hence, vectors also follow **associative law** as shown in Fig. 2.11.



Commutative law



$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

Associative law

Fig. 2.11

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(Commutative law)} \qquad (2.3)$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c} \qquad \text{(Associative law)} \qquad (2.4)$$

Thus, we should note that vectors, because of their directions do not add up algebraically, but follow the parallelogram law of addition.

Corollary It is worthwhile mentioning at this point that some physical quantities like *angular rotation*, though having magnitude and direction do not obey the parallelogram law. Hence, it is for this reason we included in the definition of vectors at the beginning that vectors not only must have magnitude and direction but also that they should obey the parallelogram law of addition.

2.4.2 Subtraction of Vectors

The difference of two vectors is obtained by adding the first vector with the negative of the second vector, i.e., $\vec{a} + (-\vec{b})$. The negative of $\vec{b}$ is given by $\overrightarrow{AC}$. Hence, $\overrightarrow{OC}$ gives the difference of $\vec{a}$ and $\vec{b}$.

The graphical methods so far discussed become tedious to work with when the number of vectors are more and when the vectors are three-dimensional or in space. Hence, we introduce the analytical method, called the *resolution of vectors*, to simplify such problems.

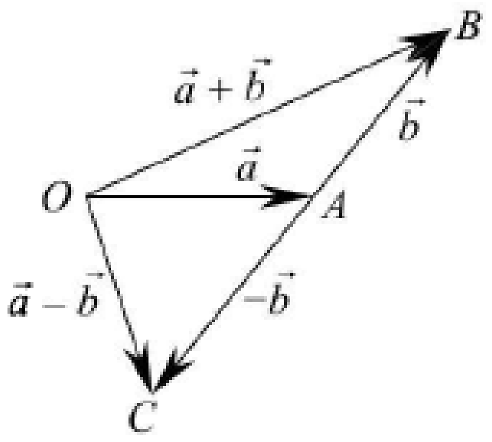


Fig. 2.12 Subtraction of vectors

2.4.3 Resolution of a Vector

This method is the inverse of the parallelogram law. That is, just as two vectors can be added to give a single vector, any vector can be resolved into two components along a pair of given axes. As there could be infinite number of such pairs of axes, we will have infinite number of components to a given vector.

However, we are interested only in a particular case, where the components are taken about a pair of perpendicular axes. Such axes are called *orthogonal* or *rectangular* axes and the components are called *rectangular components*.

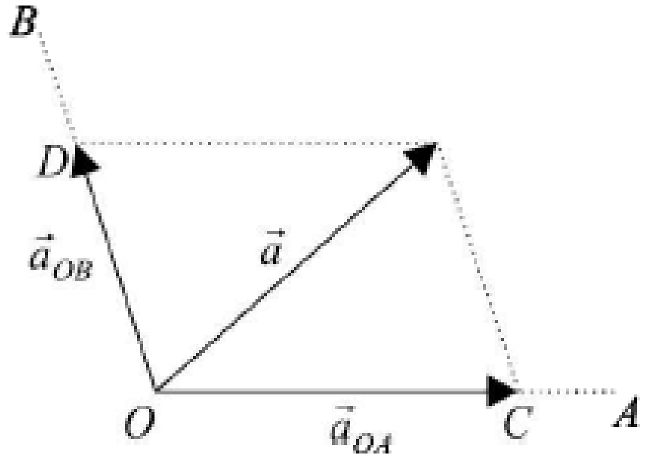


Fig. 2.13(a) Resolution of a vector

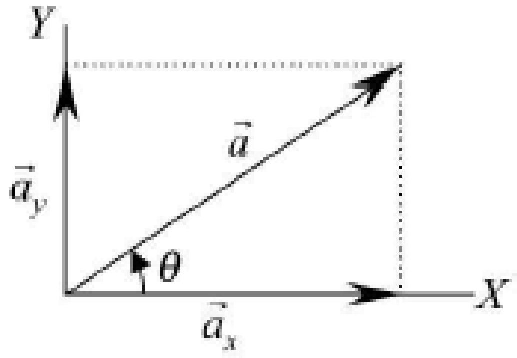


Fig. 2.13(b) Rectangular components of a vector

The rectangular components of a vector are obtained by projecting the vector onto X and Y axes as shown in Fig. 2.13(b). Hence, the vector can be written as

$$\vec{a} = \vec{a}_x + \vec{a}_y \tag{2.5}$$

where $\vec{a}_x$ and $\vec{a}_y$ are called **vector components** of $\vec{a}$.

As we already saw that any vector could be represented as a product of its magnitude and the unit vector along its direction, we can also express the vector $\vec{a}$ in Eq. 2.5 as

$$\vec{a} = a_x\vec{i} + a_y\vec{j} \tag{2.6}$$

where a_x and a_y are called **scalar components** or simply **components** of the vector, and $\vec{i}$ and $\vec{j}$ are the unit vectors along X and Y axes respectively. If θ is the inclination of $\vec{a}$ with respect to the X-axis, then

$$\vec{a} = a\cos\theta\vec{i} + a\sin\theta\vec{j} \tag{2.7}$$

The magnitude and direction of the vector can be expressed in terms of its components as

$$|\vec{a}| = a = \sqrt{a_x^2 + a_y^2} \tag{2.8}$$

$$\theta = \tan^{-1}\left[\frac{a_y}{a_x}\right] \tag{2.9}$$

The same procedure can be extended to a vector in three-dimensional space with the addition of a component along the Z-direction. Hence, a vector in space can be expressed in terms of its components as

$$\begin{aligned}\vec{a} &= \vec{a}_x + \vec{a}_y + \vec{a}_z \\ &= a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \\ &= a\cos\theta_x\vec{i} + a\cos\theta_y\vec{j} + a\cos\theta_z\vec{k} \\ &= a\,[\cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}] \qquad (2.10)\end{aligned}$$

where θ_x, θ_y and θ_z are the angles made by the vector with respect to the X, Y and Z axes respectively.

Its magnitude and direction are given in terms of the components as

$$|\vec{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{2.11}$$

$$\cos\theta_x = \frac{a_x}{a}, \cos\theta_y = \frac{a_y}{a} \text{ and } \cos\theta_z = \frac{a_z}{a} \tag{2.12}$$

where $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ are called **direction cosines**. As any vector can be represented as a product of its magnitude and the unit vector along its direction, from Eq. (2.10), we readily see that the unit vector along the direction of the vector is

$$\hat{n} = \cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k} \tag{2.13}$$

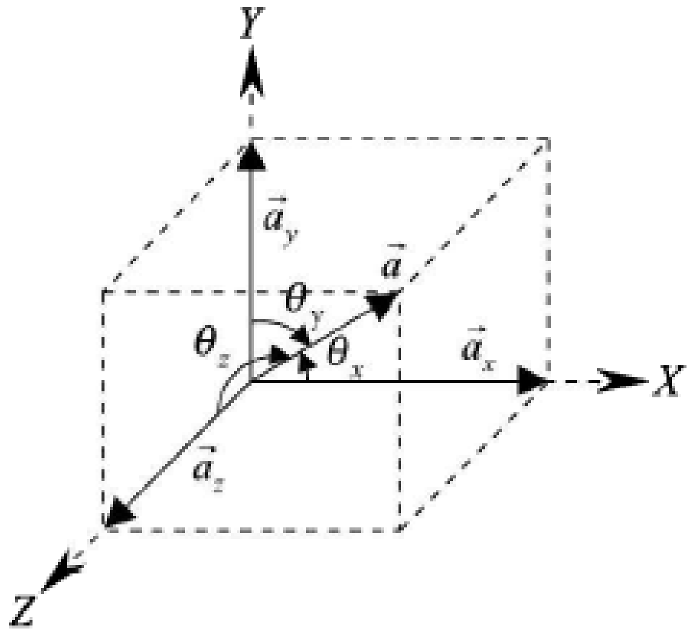


Fig. 2.14 Components of a vector in space

Since the magnitude of the unit vector is unity,

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1 \tag{2.14}$$

If the inclinations of the vector with respect to any two orthogonal axes are known, then its inclination with the third axis can be determined from the above expression.

Addition of Vectors Analytically If two vectors can be represented in terms of their components as

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

and

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

then their summation is obtained by adding the corresponding components along the *X*, *Y* and *Z* directions, i.e.,

$$\vec{a} + \vec{b} = (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k} \tag{2.15}$$

and their difference is given as

$$\vec{a} - \vec{b} = (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} + (a_z - b_z)\vec{k} \tag{2.16}$$

The physical application of summation and difference of two vectors is to determine the **displacement vector**. Suppose a body is at point A at an instant of time, then its **position vector** with respect to a coordinate system is obtained by drawing a line segment from the origin to the point A. If (x_1, y_1, z_1) be the coordinates of point A, then its position vector is given as

$$\overrightarrow{OA} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

At a later instant of time, if it is at point B, whose coordinates are (x_2, y_2, z_2), then its position vector is given as

$$\overrightarrow{OB} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

Hence, in this time interval, the body has displaced from A to B and the displacement is obtained by drawing a line segment from A to B. Note that the actual path traveled may not be a straight line. Using the addition of vectors, we know that,

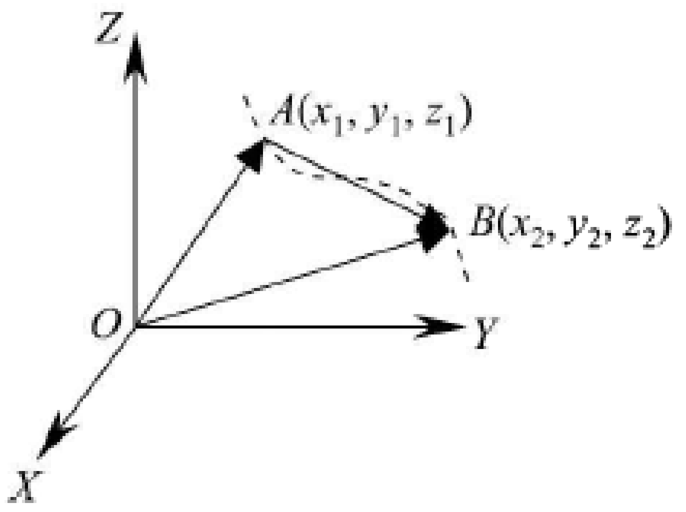


Fig. 2.15 Displacement vector

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\therefore \qquad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

which is the displacement vector.

Then unit vector along AB is given as

$$\hat{n}_{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

Example 2.1 Find the magnitude of the vector $5\vec{i} + 6\vec{j} - 2\vec{k}$ and unit vector along its direction.

Solution Let $\vec{a} = 5\vec{i} + 6\vec{j} - 2\vec{k}$. Its magnitude is given as

$$|\vec{a}| = a = \sqrt{5^2 + 6^2 + (-2)^2} = \sqrt{65} \text{ units}$$

Therefore, unit vector along its direction is obtained as

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\vec{i} + 6\vec{j} - 2\vec{k}}{\sqrt{65}}$$

Example 2.2 Determine the unit vector parallel to the sum of vectors $(2\vec{i} + 4\vec{j} - 5\vec{k})$ and $(\vec{i} + 2\vec{j} + 3\vec{k})$.

Solution Let $\vec{a} = 2\vec{i} + 4\vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$, then

$$\vec{a} + \vec{b} = (2\vec{i} + 4\vec{j} - 5\vec{k}) + (\vec{i} + 2\vec{j} + 3\vec{k})$$

$$= 3\vec{i} + 6\vec{j} - 2\vec{k}$$

Therefore, unit vector parallel to $\vec{a} + \vec{b}$ is

$$\hat{n} = \frac{3\vec{i} + 6\vec{j} - 2\vec{k}}{\sqrt{3^2 + 6^2 + (-2)^2}}$$

$$= \frac{3\vec{i} + 6\vec{j} - 2\vec{k}}{7}$$

Example 2.3 Find the direction cosines of the vector $2\vec{i} + 3\vec{j} + 4\vec{k}$.

Solution Let $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, then

$$|\vec{a}| = a = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \text{ units}$$

Therefore, direction cosines are

$$\cos\theta_x = \frac{a_x}{a} = \frac{2}{\sqrt{29}},$$

$$\cos\theta_y = \frac{a_y}{a} = \frac{3}{\sqrt{29}}$$

and

$$\cos\theta_z = \frac{a_z}{a} = \frac{4}{\sqrt{29}}$$

Example 2.4 If the position vectors of two points A and B are $2\vec{i}-9\vec{j}-4\vec{k}$ and $6\vec{i}-3\vec{j}-8\vec{k}$ respectively, then find $\overrightarrow{AB}$ and its magnitude.

Solution Given that $\overrightarrow{OA} = 2\vec{i} - 9\vec{j} - 4\vec{k}$ and $\overrightarrow{OB} = 6\vec{i} - 3\vec{j} - 8\vec{k}$,

then
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (6\vec{i} - 3\vec{j} - 8\vec{k}) - (2\vec{i} - 9\vec{j} - 4\vec{k}) \\ &= 4\vec{i} + 6\vec{j} - 4\vec{k}\end{aligned}$$

Therefore, its magnitude is

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{4^2 + 6^2 + (-4)^2} \\ &= \sqrt{68} \text{ units}\end{aligned}$$

2.4.4 Multiplication of Vectors

Multiplication of a Vector by a Scalar If m is a scalar, then the product of a vector $\vec{a}$ by the scalar m is a vector whose magnitude is $m|\vec{a}|$. Its direction is *same* as $\vec{a}$ if m is *positive* and *opposite* to that of $\vec{a}$ if m is *negative*.

Hence, if $m = 1 \;\Rightarrow\; 1(\vec{a}) = \vec{a}$

if $m = -1 \;\Rightarrow\; -1(\vec{a}) = -\vec{a}$

In general, if $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, then

$$\begin{aligned} m\vec{a} &= m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \\ &= ma_x\vec{i} + ma_y\vec{j} + ma_z\vec{k} \end{aligned} \tag{2.17}$$

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In general, if $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, then

$$m\vec{a} = m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

$$= ma_x\vec{i} + ma_y\vec{j} + ma_z\vec{k} \tag{2.17}$$

Scalar (or) Dot Product Scalar product of two vectors is defined as *the product of their magnitudes and the cosine of the included angle between them*. If $\vec{a}$ and $\vec{b}$ are two vectors, then the scalar product is expressed as

$$\vec{a}.\vec{b} = ab\cos\theta \tag{2.18}$$

Since a and b are scalars, and $\cos\theta$ is a pure number, the scalar product of two vectors is a **scalar**. Because of the notation used, scalar product is also called **dot product** and it is read as $\vec{a}$ dot $\vec{b}$. Given two vectors, then the angle between them can be obtained as

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} \tag{2.19}$$

Geometrical Interpretation of Scalar Product From Fig. 2.16, we can see that the dot product of two vectors can also be written as

$$\vec{a}.\vec{b} = ab\cos\theta$$

$$= a\,[\text{projection of } \vec{b} \text{ on } \vec{a}] \qquad (2.20)$$

Also, $$\vec{a}.\vec{b} = b\,[\text{projection of } \vec{a} \text{ on } \vec{b}] \qquad (2.21)$$

Hence, the dot product of two vectors is the product of magnitude of one of the vectors and the projection of the other vector on the first vector.

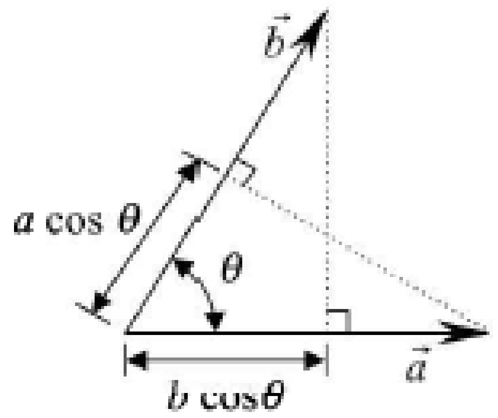

Fig. 2.16 Geometrical interpretation of dot product

Therefore, projection of $\vec{b}$ on $\vec{a}$ $= b\cos\theta = \dfrac{\vec{a}.\vec{b}}{a}$ (2.22)

Similarly, projection of $\vec{a}$ on $\vec{b}$ $= a\cos\theta = \dfrac{\vec{a}.\vec{b}}{b}$ (2.23)

Dot Product of Unit Vectors The dot product of two *collinear* or *parallel* vectors is equal to the product of their magnitudes as the included angle θ is zero or $\cos\theta = 1$. In the same way, the dot product of two *perpendicular* vectors is zero as the included angle θ is 90° or $\cos\theta = 0$. Based on this, we can determine the dot product of unit vectors:

$$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1 \tag{2.24}$$

$$\vec{i}.\vec{j} = \vec{j}.\vec{i} = \vec{j}.\vec{k} = \vec{k}.\vec{j} = \vec{k}.\vec{i} = \vec{i}.\vec{k} = 0 \tag{2.25}$$

In general, if $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

then,

$$\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z \tag{2.26}$$

If $\vec{a} = \vec{b}$, then

$$\begin{aligned}\vec{a}.\vec{a} &= a_x a_x + a_y a_y + a_z a_z \\ &= a_x^2 + a_y^2 + a_z^2 \\ &= \left[\sqrt{a_x^2 + a_y^2 + a_z^2}\right]^2 \\ &= a^2 \end{aligned} \tag{2.27}$$

Dot product is a product of two scalars. As scalars follow the commutative law, the dot product of vectors also follows **commutative law**, i.e.,

$$\vec{a}.\vec{b} = ab\cos\theta = ba\cos\theta = \vec{b}.\vec{a} \qquad (2.28)$$

Associative property, i.e.,

$$(\vec{a}.\vec{b}).\vec{c} \qquad (2.29)$$

cannot apply to vectors as $\vec{a}.\vec{b}$ is a scalar and dot product of a scalar and a vector is *meaningless*.

Scalar product follows **distributive law**, i.e.,

$$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c} \qquad (2.30)$$

A number of important physical quantities can be described as the scalar product of two vectors. For instance, **work done** is defined as a product of force component in the direction of displacement and the displacement. Hence, work done can be expressed as

$$\text{Work done} = \vec{F}.d\vec{r} \qquad (2.31)$$

Example 2.5 If $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 7\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$, find (i) $\vec{a} + 2\vec{b} + 3\vec{c}$, (ii) $\vec{a} - \vec{b} - \vec{c}$, iii) $3\vec{a} - 2\vec{b} + 4\vec{c}$. Also, find the magnitude of the vector in each case.

Solution

(i)
$$\vec{a} + 2\vec{b} + 3\vec{c} = (3\vec{i} - \vec{j} - 4\vec{k}) + 2(-2\vec{i} + 4\vec{j} - 7\vec{k}) + 3(\vec{i} + 2\vec{j} + \vec{k})$$
$$= 2\vec{i} + 13\vec{j} - 15\vec{k}$$
$$|\vec{a} + 2\vec{b} + 3\vec{c}| = \sqrt{2^2 + 13^2 + (-15)^2} = \sqrt{398} \text{ units}$$

(ii)
$$\vec{a} - \vec{b} - \vec{c} = (3\vec{i} - \vec{j} - 4\vec{k}) - (-2\vec{i} + 4\vec{j} - 7\vec{k}) - (\vec{i} + 2\vec{j} + \vec{k})$$
$$= 4\vec{i} - 7\vec{j} + 2\vec{k}$$
$$|\vec{a} - \vec{b} - \vec{c}| = \sqrt{4^2 + (-7)^2 + 2^2} = \sqrt{69} \text{ units}$$

(iii)
$$3\vec{a} - 2\vec{b} + 4\vec{c} = 3(3\vec{i} - \vec{j} - 4\vec{k}) - 2(-2\vec{i} + 4\vec{j} - 7\vec{k}) + 4(\vec{i} + 2\vec{j} + \vec{k})$$
$$= 17\vec{i} - 3\vec{j} + 6\vec{k}$$
$$|3\vec{a} - 2\vec{b} + 4\vec{c}| = \sqrt{17^2 + (-3)^2 + (6)^2} = \sqrt{334} \text{ units}$$

Example 2.6 If $\vec{a} = 2\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} - 3\vec{j} + \vec{k}$, then find $\vec{a}.\vec{b}$ and the angle between them and the projection of $\vec{b}$ on $\vec{a}$.

Solution The dot product of two vectors is given as

$$\vec{a}.\vec{b} = (2\vec{i} + \vec{j} + 2\vec{k}).(\vec{i} - 3\vec{j} + \vec{k})$$

$$= 2 - 3 + 2 = 1$$

Also, $$|\vec{a}| = a = \sqrt{2^2 + 1^2 + 2^2} = 3$$

and $$|\vec{b}| = b = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11}$$

Therefore, the angle between the two vectors is given as

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \frac{1}{3\sqrt{11}} = 0.101$$

$\Rightarrow$ $$\theta = 84.23°$$

Projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}.\vec{b}}{a} = \dfrac{1}{3}$

Example 2.7 If $\vec{a} = 2\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$, then find the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$.

Solution $\vec{a} + \vec{b} = (2\vec{i} + 2\vec{j} + 2\vec{k}) + (\vec{i} + 2\vec{j} + \vec{k}) = 3\vec{i} + 4\vec{j} + 3\vec{k}$

Hence, its magnitude is

$$|\vec{a} + \vec{b}| = \sqrt{3^2 + 4^2 + 3^2} = \sqrt{34} \text{ units}$$

Similarly, $\vec{a} - \vec{b} = (2\vec{i} + 2\vec{j} + 2\vec{k}) - (\vec{i} + 2\vec{j} + \vec{k}) = \vec{i} + \vec{k}$

and its magnitude is

$$|\vec{a} - \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ units}$$

Therefore, the angle between the two vectors is given as

$$\cos\theta = \frac{(\vec{a} + \vec{b}).(\vec{a} - \vec{b})}{|\vec{a} + \vec{b}||\vec{a} - \vec{b}|} = \frac{(3\vec{i} + 4\vec{j} + 3\vec{k}).(\vec{i} + \vec{k})}{\sqrt{34}\sqrt{2}} = \frac{3+3}{\sqrt{68}}$$

$$\Rightarrow \qquad \theta = 43.31°$$

Example 2.8 Find the angle, which $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ makes with the Z-axis.

Solution The magnitude of $\vec{a}$ is given as

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \text{ units}$$

Since the unit vector along the Z-axis is $\vec{k}$, we take the dot product of $\vec{a}$ with $\vec{k}$

$$\vec{a}.\vec{k} = (\vec{i} + 2\vec{j} + 3\vec{k}).(\vec{k}) = 3$$

Since dot product of two vectors is also given as $\vec{a}.\vec{k} = (a)(1)\cos\theta$

$$(a)(1)\cos\theta = 3$$

$$\sqrt{14}\,.1\cos\theta = 3$$

$$\Rightarrow \qquad \cos\theta = 3/\sqrt{14}$$

$$\therefore \qquad \theta = 36.7°$$

Example 2.9 The dot product of a vector with vectors $(3\vec{i} - 5\vec{k})$, $(2\vec{i} + 7\vec{j})$ and $(\vec{i} + \vec{j} + \vec{k})$ are respectively -1, 6 and 5. Find the vector.

Solution Assume the unknown vector to be $x\vec{i} + y\vec{j} + z\vec{k}$. Taking dot product of this vector with each of the given vectors,

$$3x - 5z = -1 \qquad \text{(a)}$$

$$2x + 7y = 6 \qquad \text{(b)}$$

$$x + y + z = 5 \qquad \text{(c)}$$

From equations (a) and (b), we get the values of y and z in terms of x

$$z = \left[\frac{1 + 3x}{5}\right] \qquad \text{(d)}$$

$$y = \left[\frac{6 - 2x}{7}\right] \qquad \text{(e)}$$

Substituting these values in equation(c)

$$x + y + z = 5$$

$$x + \left[\frac{6 - 2x}{7}\right] + \left[\frac{1 + 3x}{5}\right] = 5$$

$$\frac{35x + 30 - 10x + 7 + 21x}{35} = 5$$

$$46x + 37 = 175$$

$$46x = 138$$

$$\Rightarrow \quad x = \frac{138}{46} = 3$$

From equation (e),

$$y = \left[\frac{6-2x}{7}\right] = \left[\frac{6-2(3)}{7}\right] = 0$$

From equation (d),

$$z = \left[\frac{1+3x}{5}\right] = \left[\frac{1+3(3)}{5}\right] = \frac{10}{5} = 2$$

Therefore, the vector is

$$x\vec{i} + y\vec{j} + z\vec{k} = 3\vec{i} + 2\vec{k}$$

Vector (or) Cross Product Vector product of two vectors is a **vector**, whose magnitude is the product of magnitudes of the two vectors and sine of their included angle; its direction is determined by the right-hand screw rule. If $\vec{a}$ and $\vec{b}$ are two vectors, then their vector product is expressed as

$$\vec{a} \times \vec{b} = ab \sin\theta \hat{n} \tag{2.32}$$

where θ is the smaller of the angles between the vectors, i.e., $\theta \leq 180°$. Hence, $\sin\theta$ is always positive.

Because of the notation used, vector product is also called **cross product** and it is read as $\vec{a}$ cross $\vec{b}$. Its magnitude is given as

$$|\vec{a} \times \vec{b}| = ab \sin\theta \tag{2.33}$$

and its direction is specified by right-hand screw rule. Suppose a right-hand screw with its axis perpendicular to the plane formed by the two vectors and passing through their point of concurrency is rotated from $\vec{a}$ to $\vec{b}$ through the smaller angle between them, then the direction of advance of the screw gives the direction of the vector product. Alternatively, if we curl the fingers of our right hand around an axis perpendicular to the plane formed by the two vectors and passing through the point of concurrency while moving from $\vec{a}$ to $\vec{b}$, then the extended thumb specifies the sense of the cross product.

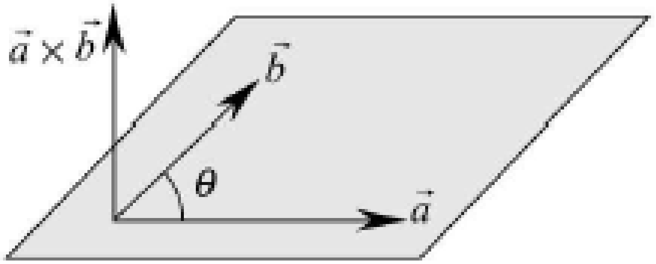

Fig. 2.17 Graphical representation of cross product

Note: Once we define the direction of the X and Y axes, to define the Z-direction in a right-hand triad, we must follow the right-hand screw rule as explained above. When we curl the fingers of our right hand from the X-axis to Y-axis through the smaller angle, then the direction of the extended thumb indicates the direction of the Z-axis. The following figures will clarify this.

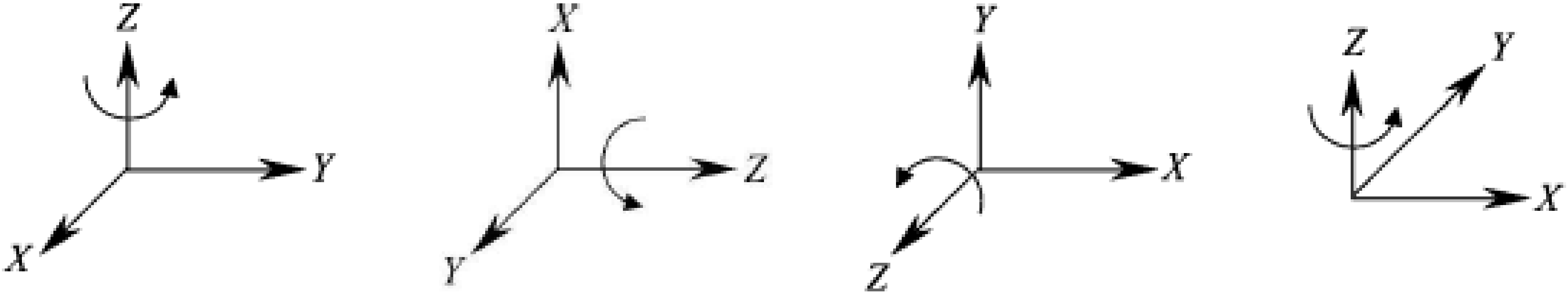

Fig. 2.18 Right-hand triad

Geometrical Interpretation of Cross Product Construct a parallelogram $OABC$ with the two vectors $\vec{a}$ and $\vec{b}$ as its adjacent sides. From Fig. 2.19, we see that $b \sin\theta$ is the perpendicular height DC of the parallelogram.

Then magnitude of the cross product of two vectors can be written as

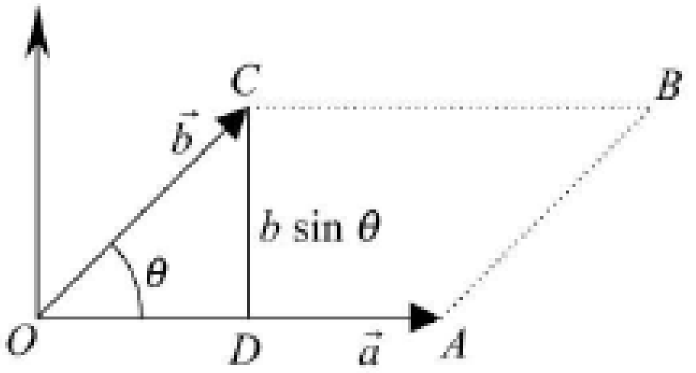

Fig. 2.19 Geometrical interpretation of cross product

$$
\begin{aligned}
|\vec{a} \times \vec{b}| &= ab \sin\theta \\
&= [a][b \sin\theta] \\
&= \text{base} \times \text{perpendicular height of parallelogram} \\
&= \text{area of parallelogram } OABC \qquad (2.34)
\end{aligned}
$$

Hence, the magnitude of the cross product of two vectors is the *area of the parallelogram formed by the two vectors as adjacent sides*.

Cross Product of Rectangular Unit Vectors When two vectors are *parallel*, then the angle between them is zero or $\sin\theta = 0$. Then

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{O} \qquad (2.35)$$

When two vectors are perpendicular to each other, then the angle between them is 90° or $\sin\theta = 1$. Then

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} & \vec{j} \times \vec{i} &= -\vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} & \vec{k} \times \vec{j} &= -\vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} & \vec{i} \times \vec{k} &= -\vec{j} \end{aligned} \tag{2.36}$$

To remember the above cross product of two rectangular unit vectors, we follow the sign convention as explained in Fig. 2.20.

The unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are arranged in an anti-clockwise sequence around the circumference of a circle. While taking cross product of two unit vectors, if we move in the direction of the sequence, then it is taken as **positive**, otherwise, it is **negative**.

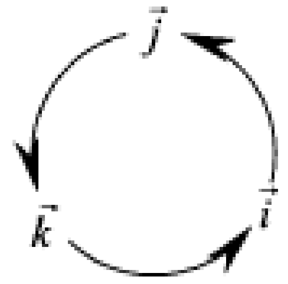

Fig. 2.20 Positive sign convention sequence

In general, if $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

then

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \times (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) \\ &= (a_xb_y - a_yb_x)\vec{k} + (a_zb_x - a_xb_z)\vec{j} + (a_yb_z - a_zb_y)\vec{i} \\ &= (a_yb_z - a_zb_y)\vec{i} - (a_xb_z - a_zb_x)\vec{j} + (a_xb_y - a_yb_x)\vec{k} \end{aligned} \tag{2.37}$$

which can also be written in determinant form as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{2.38}$$

Vector product is **not commutative**, i.e.,

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \tag{2.39}$$

Instead,

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a}) \tag{2.40}$$

Vector product is **not associative**, i.e.,

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \tag{2.41}$$

Vector product is **distributive** with respect to vector addition, i.e.,

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \tag{2.42}$$

A number of important physical quantities can be described as vector product of two vectors. For example, **moment of a force** about the origin is defined as the cross product of the position vector of the point of application of the force and the force vector.

Lecture (3)

Mechanics

DR./ IBRAHIM ABADY

Example 2.10 If $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$, compute $\vec{a} \times \vec{b}$.

Solution The cross product of the two vectors is given as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(1 - 2) - \vec{j}(-3 - 4) + \vec{k}(3 + 2)$$

$$= -\vec{i} + 7\vec{j} + 5\vec{k}$$

Example 2.11 Find the area of the parallelogram with adjacent sides $2\vec{i} - 3\vec{j}$ and $3\vec{i} - \vec{k}$.

Solution The area of the parallelogram with the given vectors as adjacent sides is given by the magnitude of their cross product.

Let $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = 3\vec{i} - \vec{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= \vec{i}(3 - 0) - \vec{j}(-2 - 0) + \vec{k}(0 + 9)$$

$$= 3\vec{i} + 2\vec{j} + 9\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 9^2}$$

$$= 9.7 \text{ units}$$

Example 2.13 Find 't' if $4\vec{i} + (2t/3)\vec{j} + t\vec{k}$ is parallel to $\vec{i} + 2\vec{j} + 3\vec{k}$.

Solution When two vectors are parallel, then their cross product is zero.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2t/3 & t \\ 1 & 2 & 3 \end{vmatrix}$$

$$\vec{O} = \vec{i}(2t - 2t) - \vec{j}(12 - t) + \vec{k}(8 - 2t/3)$$

$$= (t - 12)\vec{j} + (8 - 2t/3)\vec{k}$$

$$\Rightarrow \quad t = 12$$

Scalar Triple Product For a set of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, the scalar triple product is defined as

$$(\vec{a} \times \vec{b}).\vec{c} \tag{2.43}$$

We know,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - a_z b_y)\vec{i} - (a_x b_z - a_z b_x)\vec{j} + (a_x b_y - a_y b_x)\vec{k}$$

Therefore, $$(\vec{a} \times \vec{b}).\vec{c} = (a_y b_z - a_z b_y)(c_x) - (a_x b_z - a_z b_x)(c_y) + (a_x b_y - a_y b_x)(c_z)$$

On rearranging,

$$(\vec{a} \times \vec{b}).\vec{c} = a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) + a_z(b_x c_y - b_y c_x)$$

which can also be represented in determinant form as

$$(\vec{a} \times \vec{b}).\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{2.44}$$

Geometrical Interpretation of Scalar Triple Product If we assume $\vec{a}$ and $\vec{b}$ to lie on the X–Y plane, then $\vec{a} \times \vec{b}$ is a vector, whose magnitude, i.e., $|\vec{a} \times \vec{b}|$ is the area of the parallelogram $OABC$ and its direction is perpendicular to X–Y plane or along Z direction. Then

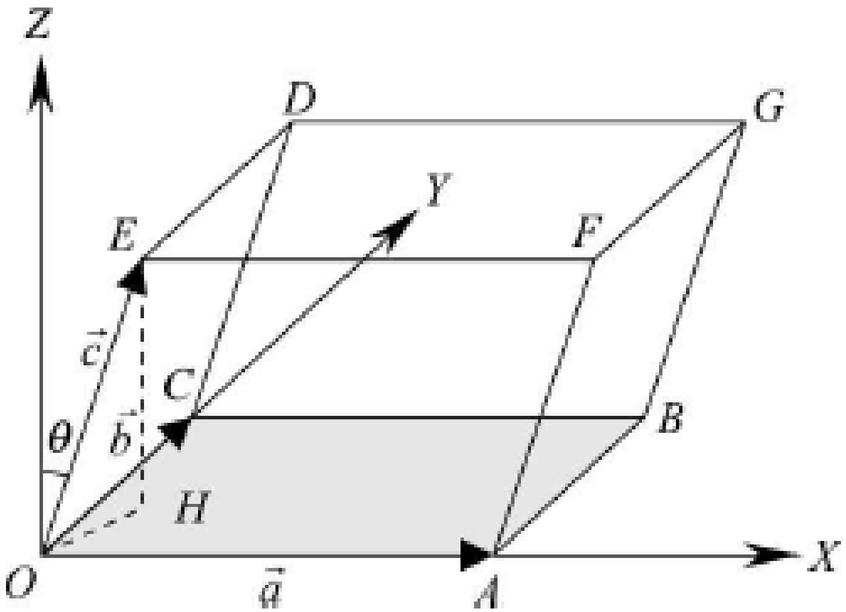

Fig. 2.21

$$\begin{aligned}(\vec{a} \times \vec{b}).\vec{c} &= |\vec{a} \times \vec{b}||\vec{c}| \cos\theta \\ &= [\text{magnitude of } \vec{a} \times \vec{b}][\text{magnitude of } \vec{c}][\cos\theta] \\ &= [\text{area of parallelogram } OABC]\ [\text{height } EH] \\ &= \text{volume of parallelepiped with sides } \vec{a}, \vec{b} \text{ and } \vec{c}\end{aligned} \tag{2.45}$$

Note: The scalar triple product is *zero*, if all the three vectors are *coplanar*.

Scalar triple product is commutative

$$(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c}) \tag{2.46}$$

Also,

$$\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b}), \tag{2.47}$$

Vector Triple Product For a set of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, the vector triple product is defined as

$$\vec{a} \times (\vec{b} \times \vec{c}) \tag{2.48}$$

Quiz [1]:

The student can easily verify the proof of this expression

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$$

Example 2.14 If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{c} = 2\vec{i} - 4\vec{j} + \vec{k}$, find $\vec{a}.(\vec{b} \times \vec{c})$ and $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution The scalar triple product is given as

$$\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & -2 & -3 \\ 1 & 3 & 2 \\ 2 & -4 & 1 \end{vmatrix}$$

$$= 1(3 + 8) - (-2)(1 - 4) - 3(-4 - 6)$$

$$= 11 - 6 + 30 = 35 \text{ units}$$

The vector triple product is given as

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$$

$$\vec{a}.\vec{c} = (\vec{i} - 2\vec{j} - 3\vec{k}).(2\vec{i} - 4\vec{j} + \vec{k})$$

$$= 2 + 8 - 3 = 7 \text{ units}$$

$$\vec{a}.\vec{b} = (\vec{i} - 2\vec{j} - 3\vec{k}).(\vec{i} + 3\vec{j} + 2\vec{k})$$

$$= 1 - 6 - 6 = -11 \text{ units}$$

Therefore,

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$$

$$= (\vec{i} + 3\vec{j} + 2\vec{k})(7) - (2\vec{i} - 4\vec{j} + \vec{k})(-11)$$

$$= 29\vec{i} - 23\vec{j} + 25\vec{k}$$

Example 2.15 Given the vectors $\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = 4\vec{i} + b_y\vec{j} - 2\vec{k}$ and $\vec{c} = 2\vec{i} - 2\vec{j} + 2\vec{k}$, determine the value of b_y for which the three vectors are coplanar.

Solution Considering the scalar triple product,

$$(\vec{a} \times \vec{b}).\vec{c} = \begin{vmatrix} 3 & -1 & 1 \\ 4 & b_y & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

$$= 3(2b_y - 4) - (-1)(8 + 4) + 1(-8 - 2b_y)$$

$$= 6b_y - 12 + 12 - 8 - 2b_y = 4b_y - 8$$

The three vectors are coplanar if their scalar triple product is zero.

Therefore,

$$0 = 4b_y - 8 \quad \Rightarrow \quad b_y = 2$$