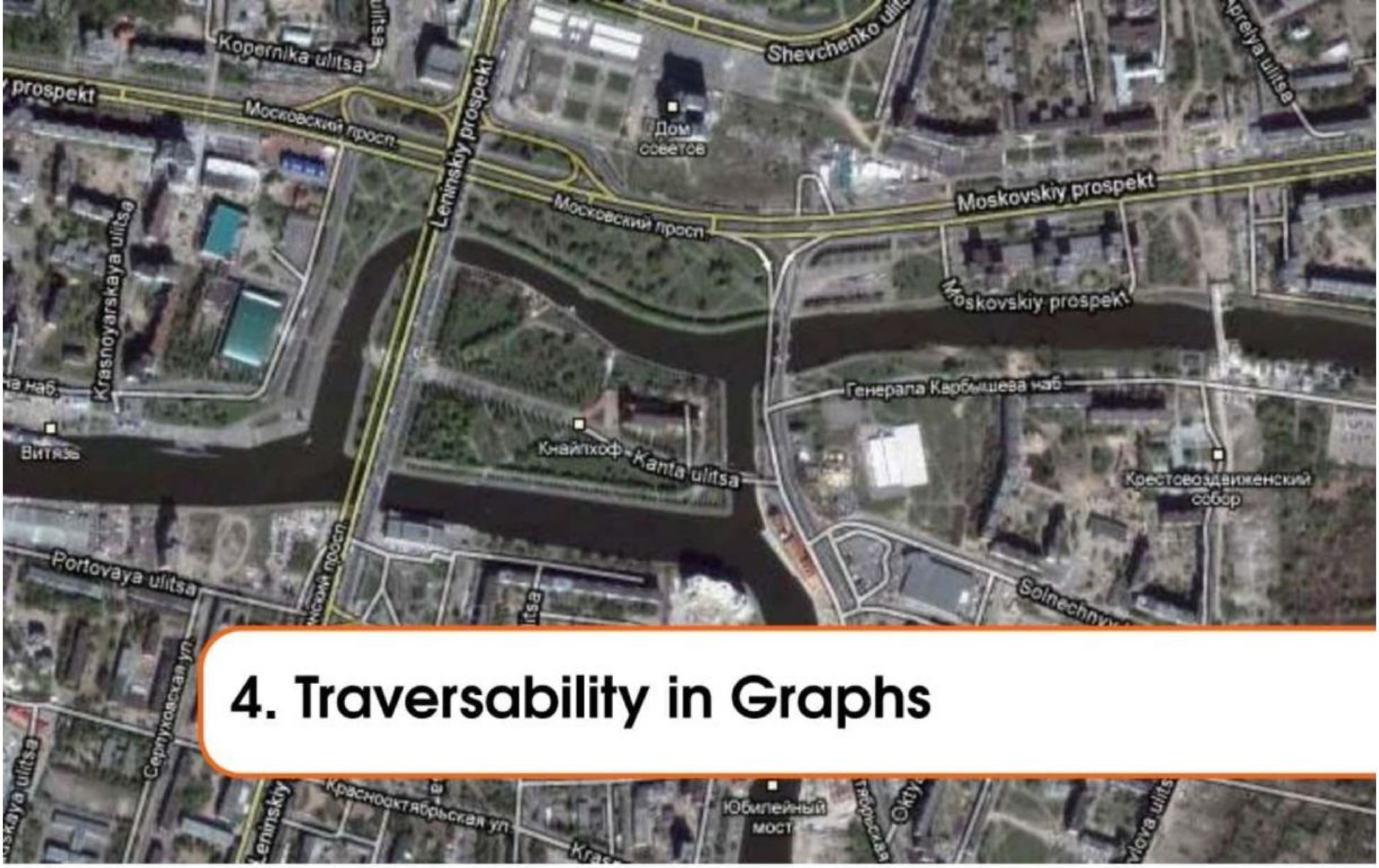

4. Traversability in Graphs

4.1 Königsberg Seven Bridge Problem

The city of *Königsberg* in Prussia (now Kaliningrad, Russia) was situated on either sides of the *Pregel River* and included two large islands which were connected to each other and the mainlands by *seven bridges* (see the below picture).

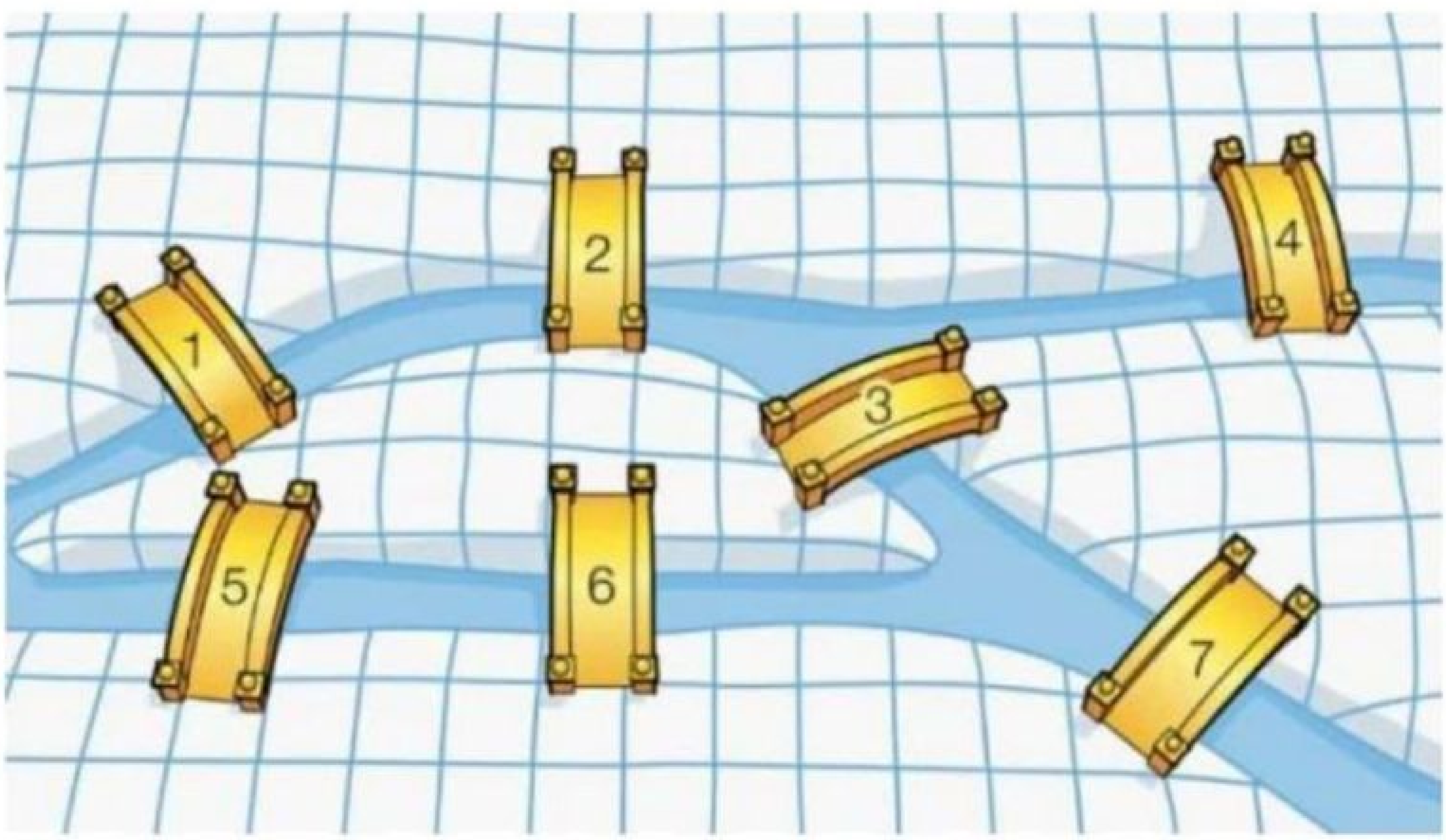

Figure 4.1: Königsberg's seven bridge problem.

The problem was to devise a walk through the city that would cross each bridge once and only once, subject to the following conditions:

(i) The islands could only be reached by the bridges;
(ii) Every bridge once accessed must be crossed to its other end;
(iii) The starting and ending points of the walk are the same.

The Königsberg seven bridge problem was instrumental to the origination of Graph Theory as a branch of modern Mathematics.

In 1736, a Swiss Mathematician *Leonard Euler* introduced a graphical model to this problem by representing each land area by a vertex and each bridge by an edge connecting corresponding vertices (see the following figure).

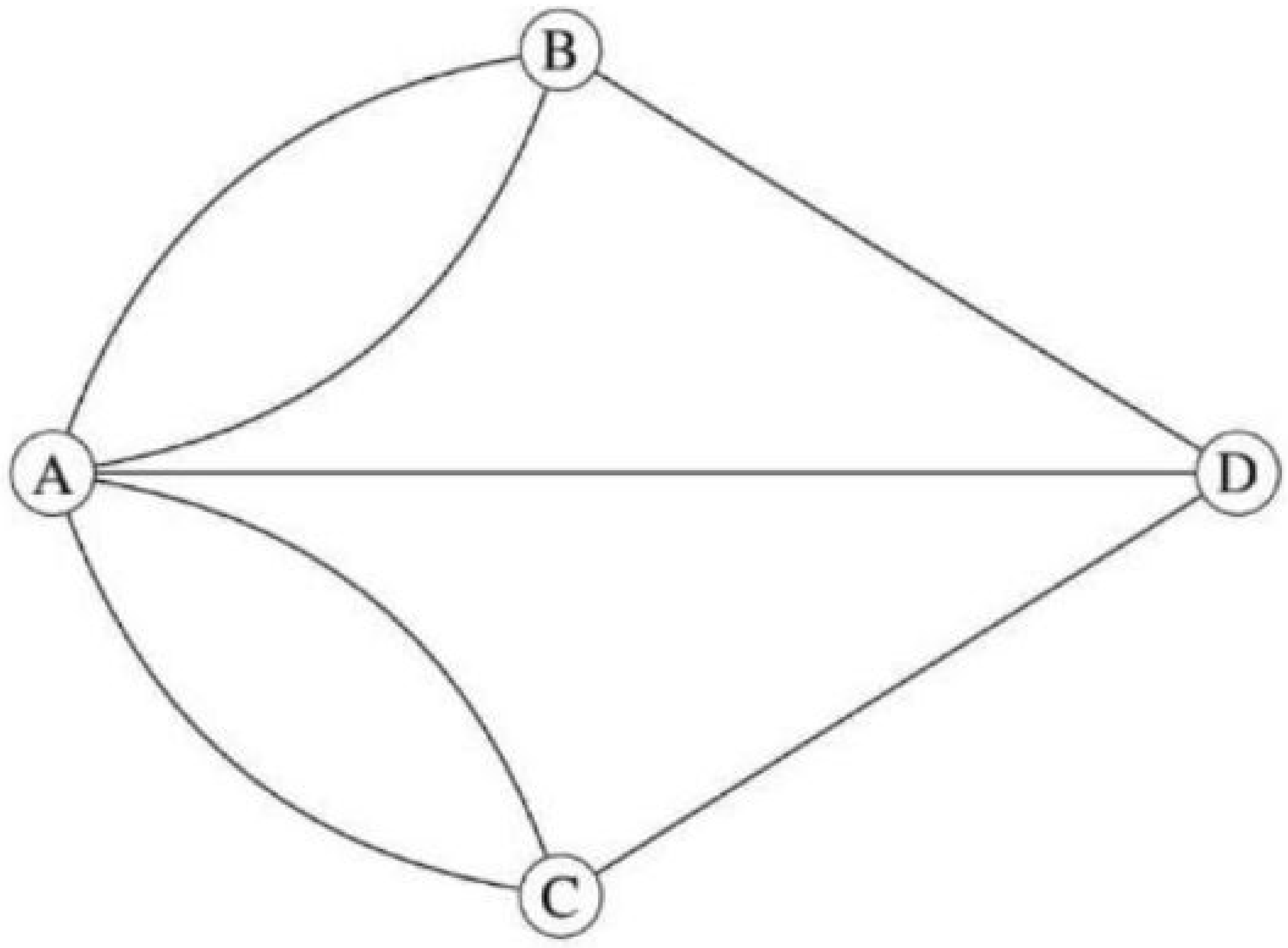

Figure 4.2: Graphical representation of seven bridge problem

Using this graphical model, Euler proved that no such walk (trail) exists.

4.2 Eulerian Graphs

Definition 4.2.1 — Traversable Graph. An *Eulerian trail* or *Euler walk* in an undirected graph is a walk that uses each edge exactly once. If an Euler trail exists in a given graph G, then G is called a *traversable graph* or a *semi-Eulerian graph*.

Definition 4.2.2 — Eulerian Graph. An *Eulerian cycle* or *Eulerian circuit* or *Euler tour* in an undirected graph is a cycle that uses each edge exactly once. If such an Euler cycle exists in the graph concerned, then the graph is called an *Eulerian graph* or a *unicursal graph*.

The following theorem characterises the class of Eulerian graphs:

Theorem 4.2.1 — Euler Theorem. A connected graph G is Eulerian if and only if every vertex in G is of even degree.

Proof. If G is Eulerian, then there is an Euler circuit, say P, in G. Every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the

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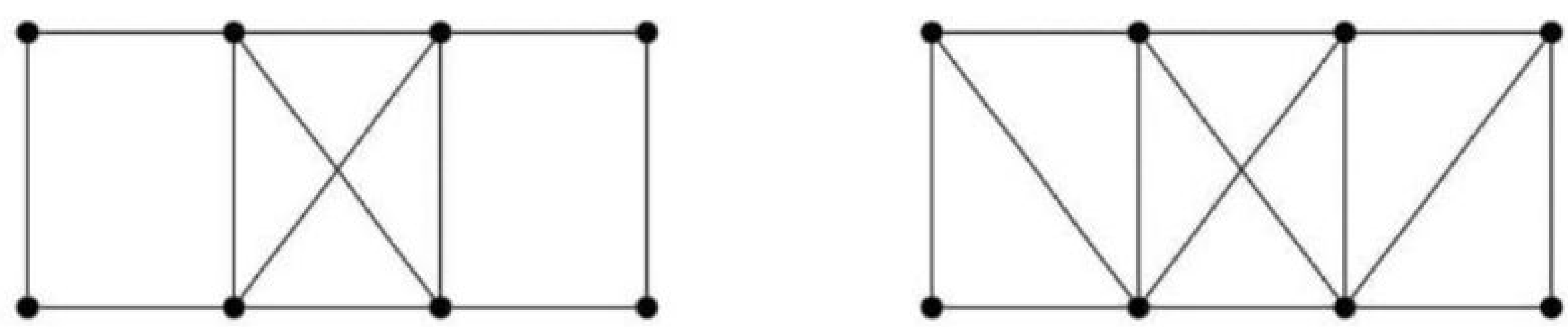
one after it in the list. This circuit uses every edge exactly once. So, every edge is accounted for and there are no repeats. Thus every degree must be even.

Conversely, let us assume that each vertex of G has even degree. We need to show that G is Eulerian. We prove the result by induction on the number of edges of G. Let us start with a vertex $v_0 \in V(G)$. As G is connected, there exists a vertex $v_1 \in V(G)$ that is adjacent to v_0. Since G is a simple graph and $d(v_i) \geq 2$, for each vertex $v_i \in V(G)$, there exists a vertex $v2 \in V(G)$, that is adjacent to v_1 with $v_2 \neq v_0$. Similarly, there exists a vertex $v_3 \in V(G)$, that is adjacent to v_2 with $v_3 \neq v_1$. Note that either $v_3 = v_0$, in which case, we have a circuit $v_0v_1v_2v_0$ or else one can proceed as above to get a vertex $v_4 \in V(G)$ and so on. As the number of vertices is finite, the process of getting a new vertex will finally end with a vertex v_i being adjacent to a vertex v_k, for some i, $0 \leq i \leq k-2$. Hence, $v_i - v_{i+1} - v_{i+2} - \ldots - v_k - v_i$ forms a circuit, say C, in G.

If C contains every edge of G, then C gives rise to a closed Eulerian trail and we are done. So, let us assume that $E(C)$ is a proper subset of $E(G)$. Now, consider the graph G_1 that is obtained by removing all the edges in C from G. Then, G_1 may be a disconnected graph but each vertex of G_1 still has even degree. Hence, we can do the same process explained above to G_1 also to get a closed Eulerian trail, say C_1. As each component of G_1 has at least one vertex in common with C, if C_1 contains all edges of G_1, then $C \cup C_1$ is a closed Euler trail in G. If not, let G_2 be the graph obtained by removing the edges of C_1 from G_1. (That is, $G_2 = G_1 - E(C \cup C_1)$).

Since G is a finite graph, we can proceed to find out a finite number of cycles only. Let the process of finding cycles, as explained above, ends after a finite number of steps, say r. Then, the reduced graph $G_r = G_{r-1} - E(C_{r-1}) = G - E(C \cup C_1 \cup C_{r-1})$ will be an empty graph (null graph). Then, $C \cup C_1 \cup C_2 \ldots \cup C_{r-1}$ is a closed Euler trail in G. Therefore, G is Eulerian. This completes the proof. ■

Illustrations to an Eulerian graph and a non-Eulerian graph are given in Figure 4.3.



(a) An Eulerian graph.

(b) A non-Eulerian graph

Figure 4.3: Examples of Eulerian and non-Eulerian graphs

In the first graph in Figure 4.3, every vertex has even degree and hence by Theorem 4.2.1, it is Eulerian. In the second graph, some vertices have odd degree and hence it is not Eulerian.

Note: In an Euler graph, it can be noted that every edge of G is contained in exactly one cycle of G. Hence, we have the following Theorem.

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Theorem 4.2.2 A connected graph G is Eulerian if and only if it can be decomposed into edge-disjoint cycles.

Proof. Assume that G can be decomposed into edge-disjoint cycles. Since the degree of every vertex in a cycle is 2, the degree of every vertex in G is two or multiples of 2. That is, all vertices in G are even degree vertices. Then, by Theorem 4.2.1, G is Eulerian.

Converse part is exactly the same as that of Theorem 4.2.1. ■

Theorem 4.2.3 A connected graph G is traversable if and only if it has exactly two odd degree vertices.

Proof. In a traversable graph, there must be an Euler trail. The starting vertex and terminal vertex need not be the same. Therefore, these two vertices can have odd degrees. Remaining part of the theorem is exactly as in the proof of Theorem 4.2.1. ■

4.3 Chinese Postman Problem

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In his job, a postman picks up mail at the post office, delivers it, and then returns to the post office. He must, of course, cover each street in his area at least once. Subject to this condition, he wishes to choose his route in such a way that walks as little as possible. This problem is known as the Chinese postman problem, since it was first considered by a Chinese mathematician, Guan in 1960.

We refer to the street system as a weighted graph (G, w) whose vertices represent the intersections of the streets, whose edges represent the streets (one-way or two-way) and the weight represents the distance between two intersections, of course, a positive real number. A closed walk that covers each edge at least once in G is called a *postman tour*. Clearly, the Chinese postman problem is just that of finding a minimum-weight postman tour. We will refer to such a postman tour as an optimal tour.

An algorithm for finding an optimal Chinese postman route is as follows:

- S-1 : List all odd vertices.
- S-2 : List all possible pairings of odd vertices.
- S-3 : For each pairing find the edges that connect the vertices with the minimum weight.
- S-4 : Find the pairings such that the sum of the weights is minimised.
- S-5 : On the original graph add the edges that have been found in Step 4.
- S-6 : The length of an optimal Chinese postman route is the sum of all the edges added to the total found in Step 4.
- S-7 : A route corresponding to this minimum weight can then be easily found.

■ **Example 4.1** Consider the following weighted graph:

1. The odd vertices are A and H; There is only one way of pairing these odd vertices, namely AH;
2. The shortest way of joining A to H is using the path $\{AB, BF, FH\}$, a total length of 160;

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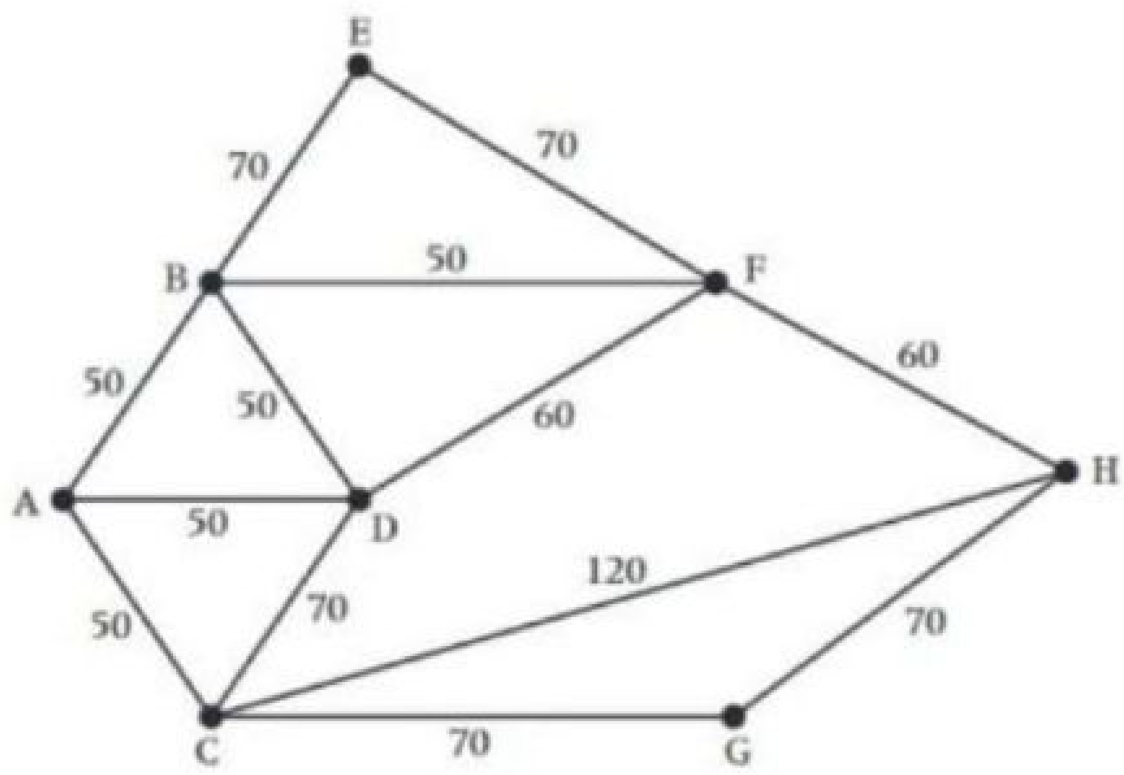

Figure 4.4: An example for Chinese Postman Problem

3. Draw these edges onto the original network.
4. The length of the optimal Chinese postman route is the sum of all the edges in the original network, which is $840m$, plus the answer found in Step 4, which is $160m$. Hence the length of the optimal Chinese postman route is $1000m$.
5. One possible route corresponding to this length is $ADCGHCABDFBEFHFBA$, but many other possible routes of the same minimum length can be found.

■

4.4 Hamiltonian Graphs

Definition 4.4.1 — Traceable Graphs. A *Hamiltonian path* (or *traceable path*) is a path in an undirected (or directed graph) that visits each vertex exactly once. A graph that contains a Hamiltonian path is called a *traceable graph*.

Definition 4.4.2 — Hamiltonian Graphs. A *Hamiltonian cycle*, or a *Hamiltonian circuit*, or a *vertex tour* or a *graph cycle* is a cycle that visits each vertex exactly once (except for the vertex that is both the start and end, which is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Hamiltonian graphs are named after the famous mathematician *William Rowan Hamilton* who invented the Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron.

A necessary and sufficient condition for a graph to be a Hamiltonian is still to be determined. But there are a few sufficient conditions for certain graphs to be Hamiltonian. The following theorem is one of those results.

Theorem 4.4.1 — Dirac's Theorem. Every graph G with $n \geq 3$ vertices and minimum degree $\delta(G) \geq \frac{n}{2}$ has a Hamilton cycle.

Proof. Suppose that $G = (V, E)$ satisfies the hypotheses of the theorem. Then G is connected, since otherwise the degree of any vertex in a smallest component C of G would be at most

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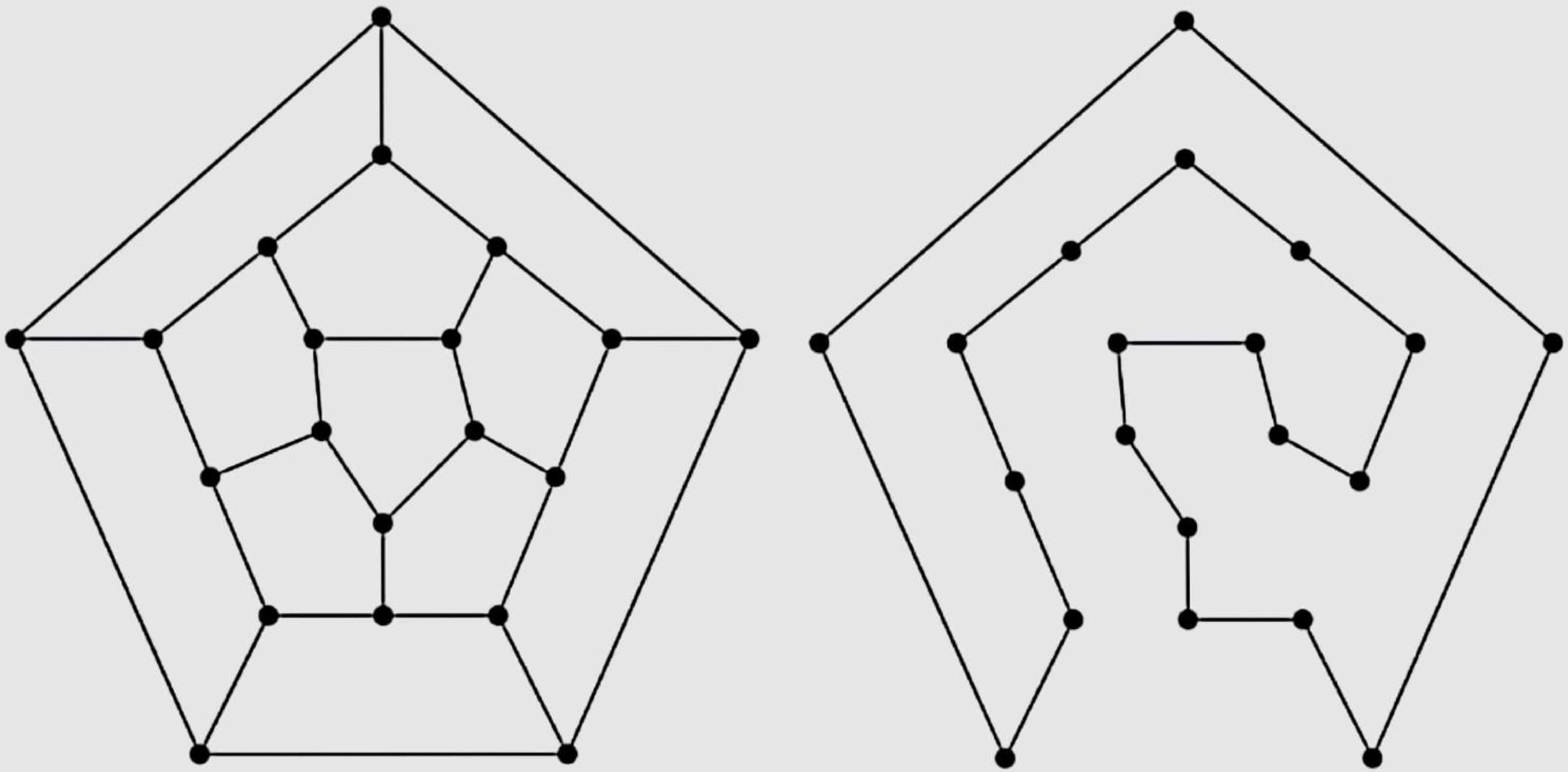


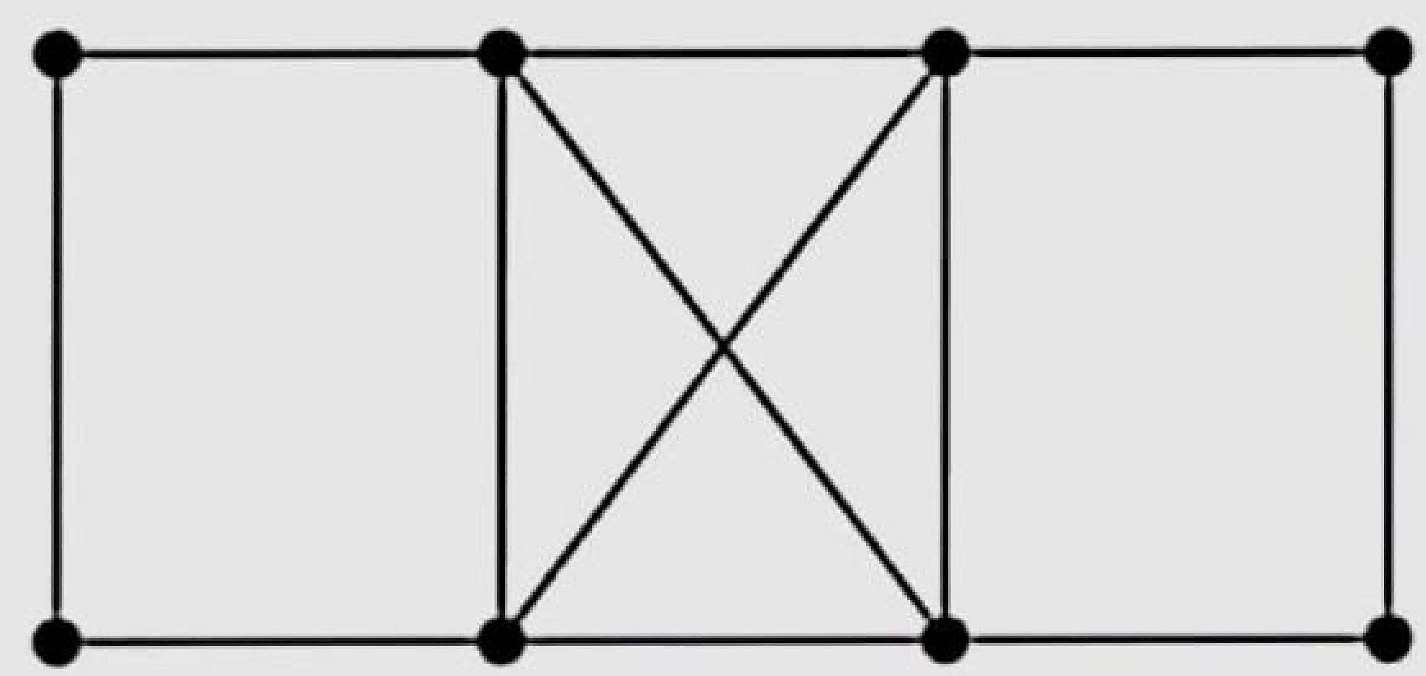
(a) (b)

Figure 4.5: Dodecahedron and a Hamilton cycle in it.

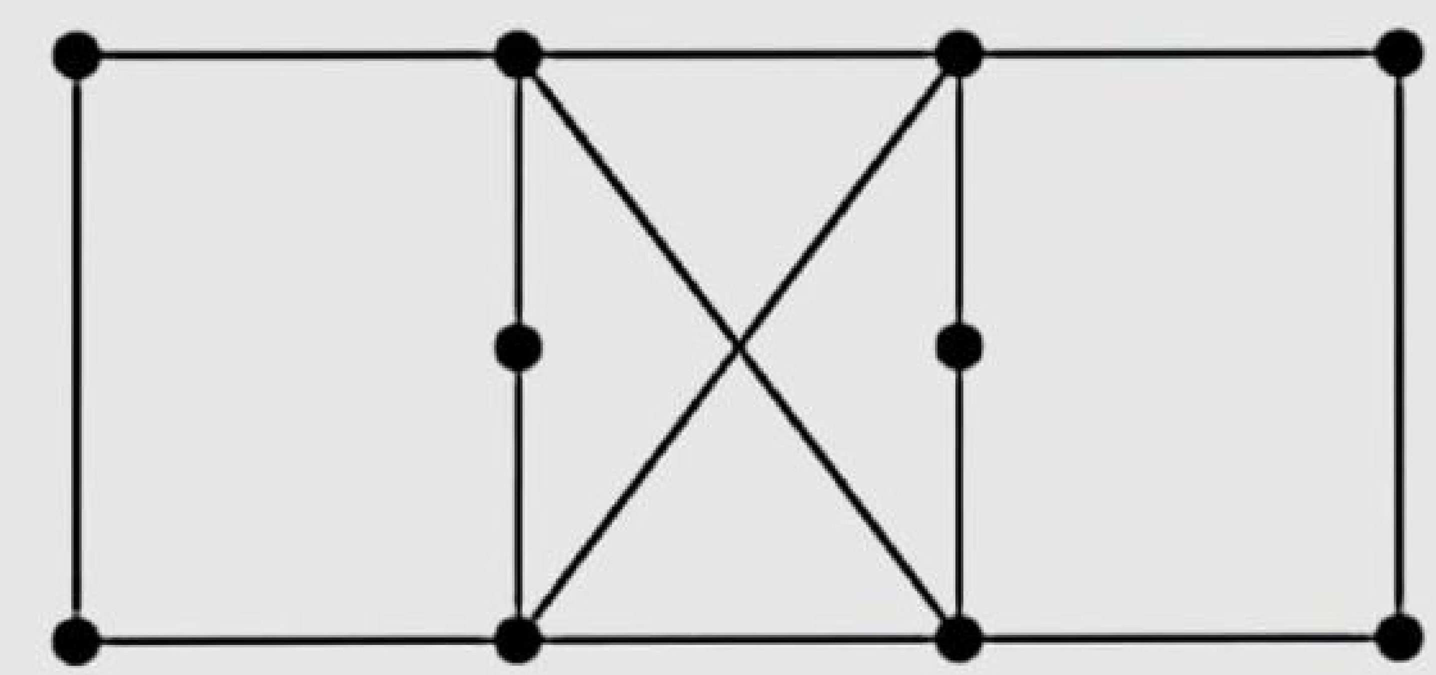
4.5 Some Illustrations

We can find out graphs, which are either Eulerian or Hamiltonian or simultaneously both, whereas some graph are neither Eulerian nor Hamiltonian. We note that the dodecahedron is an example for a Hamiltonian graph which is not Eulerian (see Figure 4.5a).

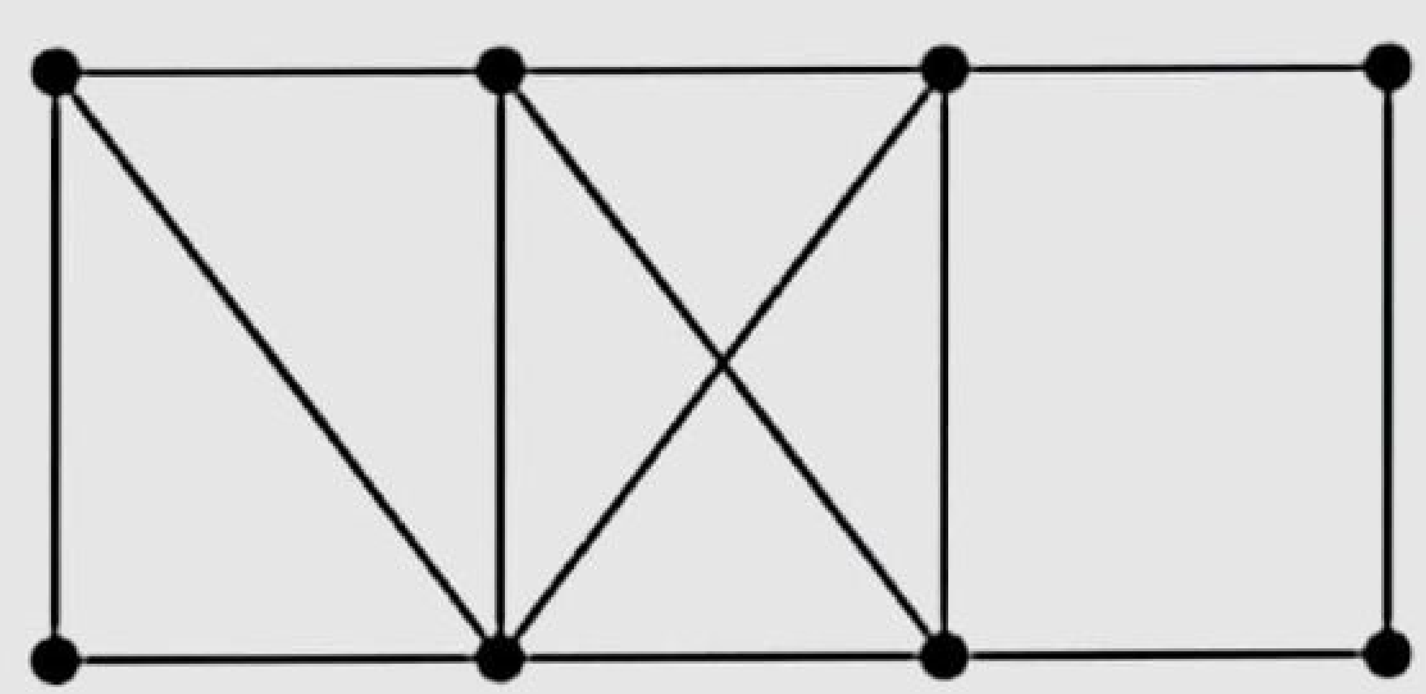
Let us now examine some examples all possible types of graphs in this category.



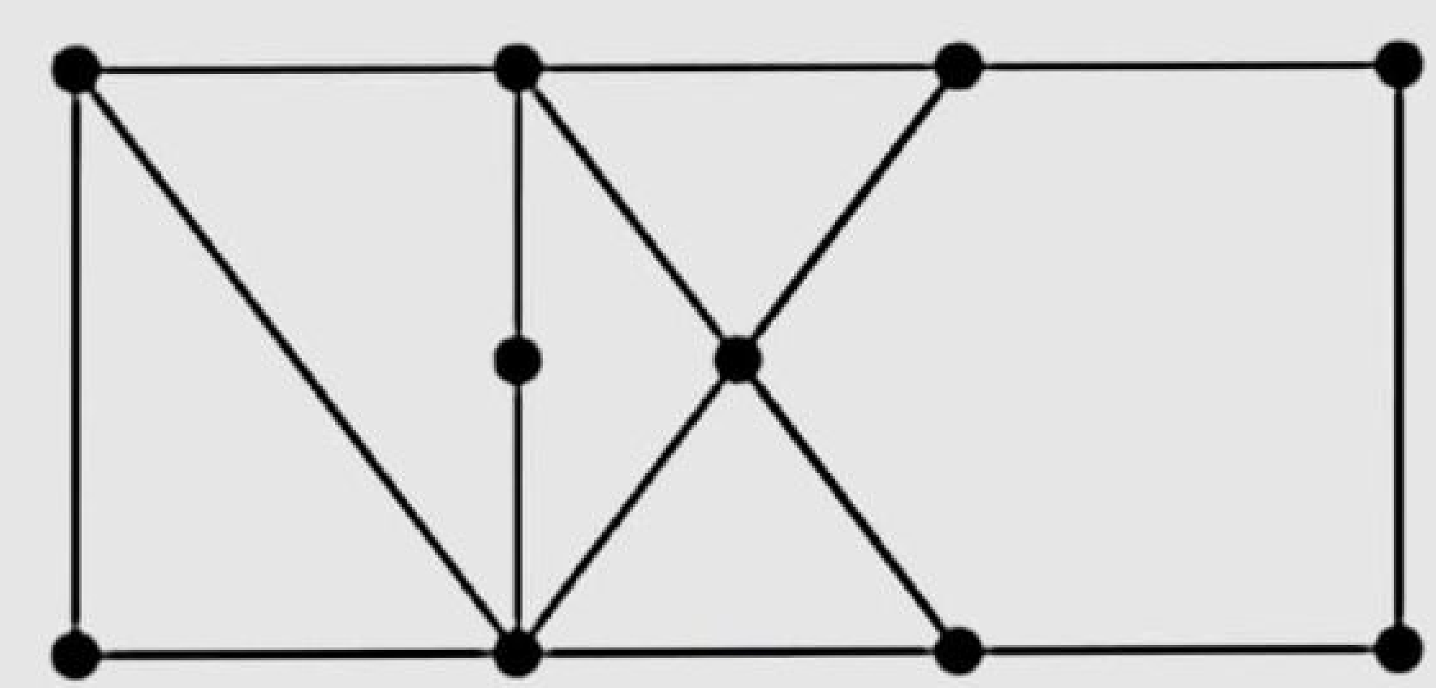
(a) A graph which is both Eulerian and Hamiltonian



(b) A graph which is Eulerian, but not Hamiltonian.



(c) A graph which is Hamiltonian, but not Eulerian.



(d) A graph which is neither Eulerian nor Hamiltonian.

Figure 4.9: Traversability in graphs