System of Linear Equations

A System of Linear Equations can be written as set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

1-Gaussian Elimination Method

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps
1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$

$$. \qquad .$$

$$. \qquad .$$

$$. \qquad .$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$- \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\overline{\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1}$$

$$\text{or} \quad a_{22}'x_2 + \ldots + a_{2n}'x_n = b_2'$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$\begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix}$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{22}'x_2 + a_{23}'x_3 + \ldots + a_{2n}'x_n = b_2'$$

$$a_{33}''x_3 + \ldots + a_{3n}''x_n = b_3''$$

$$\vdots \qquad \vdots$$

$$a_{n3}''x_3 + \ldots + a_{nn}''x_n = b_n''$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + ... + a''_n x_n = b''_3$$

$$\begin{matrix} . & . \\ . & . \\ . & . \end{matrix}$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)}x_{i+1} - a_{i,i+2}^{(i-1)}x_{i+2} - \ldots - a_{i,n}^{(i-1)}x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

Example 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Results in a matrix template of the form:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination

Number of steps of forward elimination is
$(n-1)=(3-1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\dfrac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ - \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\dfrac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by -4.8 and multiply it by -16.8,

$$\frac{-16.8}{-4.8} = 3.5.$$

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{c} \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & 335.968 \end{bmatrix} \\ - \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Gauss Elimination Pitfalls

Pitfall #1. Division by zero

$$10x_2 - 7x_3 = 3$$

$$6x_1 + 2x_2 + 3x_3 = 11$$

$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$12x_1 + 10x_2 - 7x_3 = 15$$

$$6x_1 + 5x_2 + 3x_3 = 14$$

$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?
YES

$$12x_1 + 10x_2 - 7x_3 = 15$$

$$6x_1 + 5x_2 + 3x_3 = 14$$

$$24x_1 - x_2 + 5x_3 = 28$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix} \longrightarrow \begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 12 & -21 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}$$

Division by zero is a possibility at any step
of forward elimination

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error

- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero

- Reduces round off error

Gauss Elimination with Partial Pivoting

What is Different About Partial Pivoting?

At the beginning of the kth step of forward elimination, find the maximum of

$$\left|a_{kk}\right|, \left|a_{k+1,k}\right|, \ldots\ldots\ldots\ldots, \left|a_{nk}\right|$$

If the maximum of the values is $\left|a_{pk}\right|$

in the pth row, $k \leq p \leq n,$ then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}' & a_{23}' & \cdots & a_{2n}' \\ 0 & a_{32}' & a_{33}' & \cdots & a_{3n}' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}' & a_{n3}' & a_{n4}' & a_{nn}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

Example (2nd step of FE)

$$
\begin{bmatrix}
6 & 14 & 5.1 & 3.7 & 6 \\
0 & -7 & 6 & 1 & 2 \\
0 & 4 & 12 & 1 & 11 \\
0 & 9 & 23 & 6 & 8 \\
0 & -17 & 12 & 11 & 43
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
=
\begin{bmatrix}
5 \\
-6 \\
8 \\
9 \\
3
\end{bmatrix}
$$

Which two rows would you switch?

Example (2nd step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps
1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22}' & a_{23}' & \cdots & a_{2n}' \\
0 & a_{32}' & a_{33}' & \cdots & a_{3n}' \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & a_{n2}' & a_{n3}' & a_{n4}' & a_{nn}'
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2' \\
b_3' \\
\vdots \\
b_n'
\end{bmatrix}
$$

Matrix Form at End of Forward Elimination

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\
0 & 0 & a''_{33} & \cdots & a''_{3n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & a_{nn}^{(n-1)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)}
\end{bmatrix}
$$

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_n x_n = b''_3$$

$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum\limits_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

Gauss Elimination with Partial Pivoting Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

$|10|$, $|-3|$, and $|5|$ or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \implies \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \implies \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Determinant of a Square Matrix
Using Gauss Elimination

Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then det(A)=det(B)

Theorem of Determinants

The determinant of an upper triangular matrix

$[A]_{nxn}$ is given by

$$\det\left(A\right) = a_{11} \times a_{22} \times ... \times a_{ii} \times ... \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.

$$[A]_{n \times n} \to [U]_{n \times n}$$

$$\det(A) = \det(U)$$

Example

Using Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\dfrac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\dfrac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by -4.8 and multiply it by -16.8,

$$\frac{-16.8}{-4.8} = 3.5.$$

$$\left(\begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix}\right) \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\det(A) = u_{11} \times u_{22} \times u_{33}$$
$$= 25 \times (-4.8) \times 0.7$$
$$= -84.00$$