1) why measure errors → to determine The accuracy of results
→ to develop stopping criteria for interative algorithms

Absolute error (E_t)

→ The difference between The exact value in a calculation and The approximate value found using numerical method

$$E_t = \text{exact} - \text{Approximate}$$

Relative Absolute error (ε_t)

→ The ratio between The absolute error and exact value

$$\varepsilon_t = \frac{\text{Absolute Error}}{\text{exact value}}$$

Approximate error (E_a)

→ The difference between present approximate and Previous approximate.

$$E_a = \text{Present approximate} - \text{Previous approximate}$$

Relative Approximate error

→ The ratio between The approximate error and present approximate

$$\varepsilon_a = \frac{\text{Approximate error}}{\text{present Approximate}}$$

★ Source of numerical error → (Round off - Truncation)

- Round off : caused by representing a number approximately
- Truncation : caused by truncation or approximating a math Procedure

★ taylor series :

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \text{------}$$

★ MaClauren series : $(X=0)$

$$f(x+h) := f(0) + h f'(0) + \frac{h^2}{2!} f''(0) + \text{----}$$

★ Absolute error $\longrightarrow$ $R_n = \dfrac{(x-h)^{n+1} \times f^{n+1}(c)}{(n+1)!}$ $\quad c \in [x, x+h]$

Gauss elemination

① forward elemination ② BacK substitution

③ no. of steps of forward elemination $(n-1)$

★ Pit falls for gauss elimination :

① Division by zero ② large round off errors

★ Steps of Gauss elemination

① Column ① → $|a_{pk}|$ اكبر قيمة $\quad a_{21} \leftarrow \begin{cases} P=2 \\ K=1 \end{cases}$

$K \leq P \leq n$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -6 & 1 & -1 & 3 \\ 3 & 1 & 1 & 4 \end{array}\right]_{n \times n}$$

② swap R_2 with R_1, $\frac{R_1}{a_{11}} \times a_{21} - R_2 \rightarrow R_2$

$\frac{R_1}{a_{11}} \times a_{31} - R_3 \rightarrow R_3$

★ det(A) =

$$\left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{5}{6} & -\frac{5}{6} & -\frac{5}{2} \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{11}{2} \end{array}\right]$$

③ Column ② → $|a_{pk}|$ → a_{32} → اكبر قيمة

④ swap R_3 with R_2 → $\frac{R_2}{a_{22}} \times a_{32} - R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{11}{2} \\ 0 & 0 & \frac{10}{9} & \frac{50}{9} \end{array}\right]$$

$x_1 = ?$, $x_2 = ?$, $x_3 = ?$

★ Jacobi ---- steps

① $x =$ ———— , $y =$ ———— , $z =$ ———— $X_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

② start by → $x_0 = 1$, $y_0 = 1$, $z_0 = 1$
given

③ Then → use The x, y, z results sub. values

④ $\epsilon = \frac{x^{new} - x^{old}}{x^{new}} \times 100 =$ — %.

⑤ max. $\epsilon =$ ————

n	x	ϵ_1	y	ϵ_2	z	ϵ_3
0	1	—	1	—	1	—
1	—	50%	—	—	—	—
2						
↓	↓	↓	↓	↓	↓	↓

★ Gauss seidel ———— steps

① $x =$ ———— , $y =$ ———— , $z =$ ———— $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

② Start by → $x = 0$, $y = 0$, $z = 0$ (given)

③ Then → use x in y and use x, y in z

n	x	y	z
0	0	0	0
1	—	—	—
2	—	—	—

← at y يستخدم ال x^{new}
at z ———— يستخدم ال x^{new} , y^{new}

★ Bisection Method

① $X_L = \_\_\_$, $X_u = \_\_\_$ ← given

② $X_m = \frac{X_L + X_u}{2}$

n	X_L	X_u	X_m	ϵ	$\pm$

③ $F(X_L) \cdot f(X_m) < 0 \rightarrow X_L = X_L , X_u = X_m$

$F(X_L) \cdot F(X_m) > 0 \rightarrow X_L = X_m , X_u = X_u$

$F(X_L) \cdot F(X_m) = 0 \rightarrow$ Algorithm Stop root = X_m

④ root = last X_m , $\mathcal{E}_s$ = last(ϵ)

★ Newton Raphson

① $f'(x_i)$ ($x_0 \rightarrow$ given)

② $X_{i+1} = X_i - \frac{F(X_i)}{F'(X_i)}$

③ root = last X_n

n	X_n	$\lvert \epsilon_a \rvert$

★ Bisection method

Advantages	Drawbacks
① Always Convergent	① slow Convergence
② The root bracket gets halved with each iteration - guranteed	② If one of the initial guesses is Close to the root, The convergence is slower

★ Newton Raphson

Advantages	Drawbacks
① Converges fast (quadratic convergence)	① Divergence at inflection points
② Require only one guess	② Division by Zero
	③ Root Jumping

* Secant Method

① $x_{i+1} = x_i - \dfrac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$

② $|\epsilon_a| = \left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100 = \text{—} \%$

③

n	x_{i-1}	x_i	x_{i+1}	ϵ
↓	↓	↓	↓	↓

$x_{i-1} = \text{—}$? (given)

$x_1 = \text{—}$? (given)

← بتغير ال x_i, x_{i-1}
في كل iteration

* Interpolation

① Direct Method: $y = a_0 + a_1 x + \text{———} + a_n x^n$

* "setup $(n+1)$ eq. to find $(n+1)$ constants"

② Linear Interpolation: $f(x) = a_0 + a_1 x$ $\quad a_0, a_1 \rightarrow$ given

③ Quadratic Interpolation: $f(x) = a_0 + a_1 x + a_2 x^2$ $\quad a_0, a_1, a_2 \rightarrow$ given

④ Cubic Interpolation: $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\quad a_0, a_1, a_2, a_3 \rightarrow$ given

* Newton's Divided difference Method

① Linear interpolation:

$F_1(x) = b_0 + b_1(x - x_0)$ $\quad b_0 = F(x_0)$, $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

② Quadratic Interpolation:

$F_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$ $\quad b_0 = f(x_0)$, $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$F_2(x) = f(x_0) + f(x_1, x_2)(x - x_0) + f(x_2, x_1, x_0)(x - x_0)(x - x_1)$ $\quad b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$

③ Cubic Interpolation

$$F_3(X) = F(X_0) + f(X_1, X_0)(X - X_0) + f(X_2, X_1, X_0)(X - X_0)(X - X_1) + f(X_3, X_2, X_1, X_0)(X - X_0)(X - X_1)(X - X_2)$$

	b_0	b_1	b_2	b_3
X_0	$F(X_0)$			
		$F(X_1, X_0)$		
X_1	$f(X_1)$		$F(X_2, X_1, X_0)$	
		$F(X_2, X_1)$		$F(X_3, X_2, X_1, X_0)$
X_2	$f(X_2)$		$F(X_3, X_2, X_1)$	
		$F(X_3, X_2)$		
X_3	$f(X_3)$			

★ Lagrangian Interpolation

$$\rightarrow F_n(X) = \sum_{i=0}^{n} L_i(X)\, F(X_i) \quad , \quad L_i(X) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{X - X_j}{X_i - X_j}$$

$$\rightarrow P_1(X) = \frac{X - X_1}{X_0 - X_1} y_0 + \frac{X - X_0}{X_1 - X_0} y_1 = \frac{(X_1 - X) y_0 + (X - X_0) y_1}{X_1 - X_0}$$

$$\rightarrow P_2(X) = \frac{(X - X_1)(X - X_2)}{(X_0 - X_1)(X_0 - X_2)} y_0 + \frac{(X - X_0)(X - X_2)}{(X_1 - X_0)(X_1 - X_2)} y_1 + \frac{(X - X_0)(X - X_1)}{(X_2 - X_0)(X_2 - X_1)} y_2$$

★ linear Interpolation

$V(t_0), V(t_1)$ given

$$V(t) = \sum_{i=0}^{1} L_i(t)\, V(t_i) = L_0(t)\, V(t_0) + L_1(t)\, V(t_1)$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1} \quad , \quad L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

★ Quadratic Interpolation

$$V(t) = \sum_{i=0}^{2} l_i(t)\, V(t_i) = L_0(t)\, V(t_0) + L_1(t)\, V(t_1) + L_2(t)\, V(t_2)$$

$$L_0(t) = \prod_{\substack{j=0\\ j\neq 0}}^{2} \frac{t-t_j}{t_0-t_j} = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right), \quad L_1(t) = \prod_{\substack{j=0\\ j\neq 1}}^{2} \frac{t-t_j}{t_1-t_j} = \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0\\ j\neq 2}}^{2} \frac{t-t_j}{t_2-t_j} = \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)$$

★ Cubic Interpolation

$$V(t) = \sum_{i=0}^{3} l_i(t)\, V(t_i) = l_0(t)\, V(t_0) + l_1(t)\, V(t_1) + L_2(t)\, V(t_2) + L_3(t)\, V(t_3)$$

$$L_0(t) = \prod_{\substack{j=0\\ j\neq 0}}^{3} \frac{t-t_j}{t_0-t_j} = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

$$L_1(t) = \prod_{\substack{j=0\\ j\neq 1}}^{3} \frac{t-t_j}{t_1-t_j} = \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)$$

$$L_2(t) = \prod_{\substack{j=0\\ j\neq 2}}^{3} \frac{t-t_j}{t_2-t_j} = \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)$$

$$L_3(t) = \prod_{\substack{j=0\\ j\neq 3}}^{3} \frac{t-t_j}{t_3-t_j} = \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)$$

★ After Quadratic Interpolation:

$$|\epsilon_a| = \left|\frac{\text{Quadratic} - \text{Linear}}{\text{Quadratic}}\right| \times 100 = \text{—}\ \%$$

★ After Cubic Interpolation:

$$|\epsilon_a| = \left|\frac{\text{Cubic} - \text{Quadratic}}{\text{Cubic}}\right| \times 100 = \text{—}\ \%$$