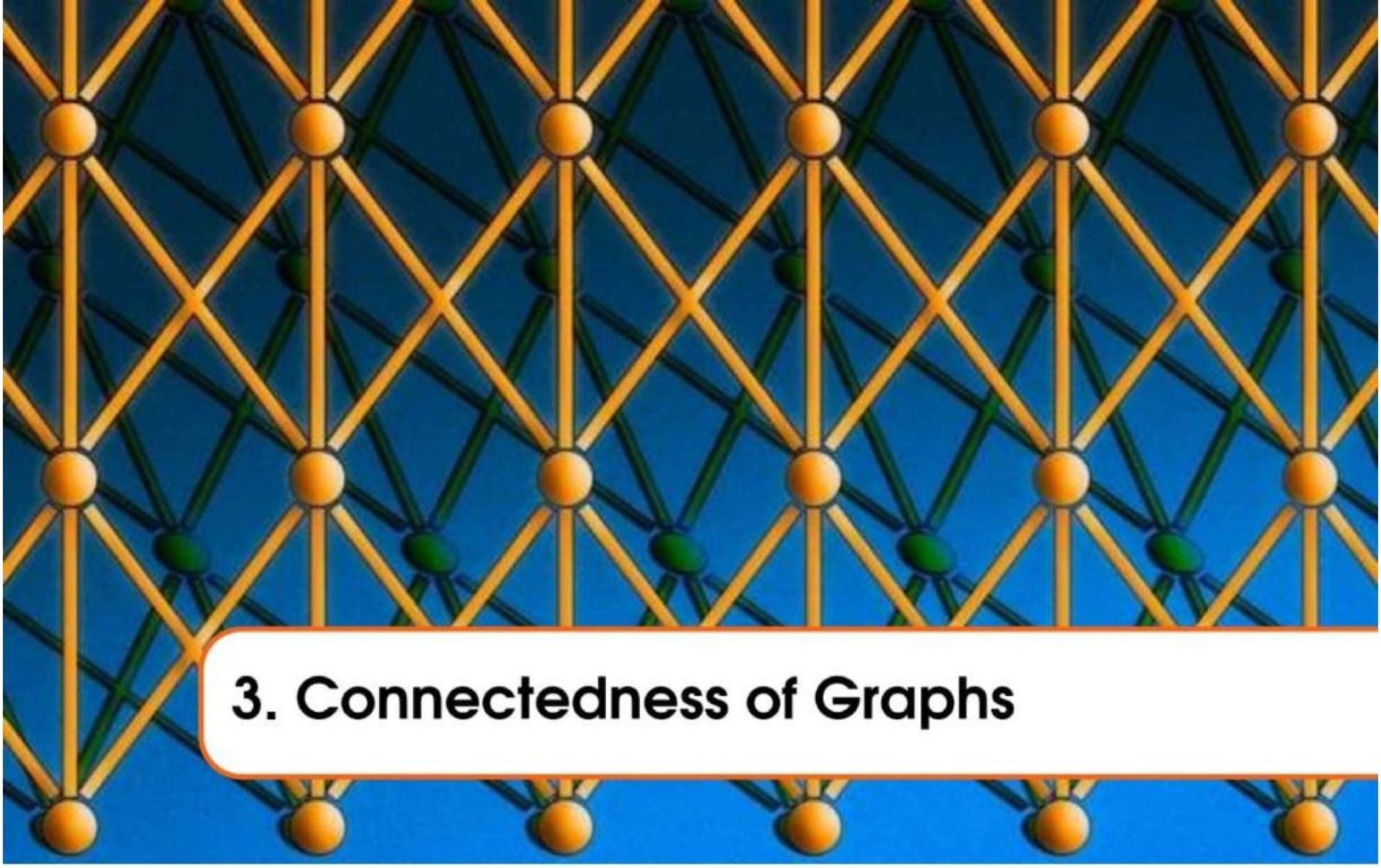

3. Connectedness of Graphs

3.1 Paths, Cycles and Distances in Graphs

Definition 3.1.1 — Walks. A *walk* in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a *walk* is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a *closed walk*.

A walk can end on the same vertex on which it began or on a different vertex. A walk can travel over any edge and any vertex any number of times.

Definition 3.1.2 — Trails and Tours. A *trail* is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A *tour* is a trail that begins and ends on the same vertex.

Definition 3.1.3 — Paths and Cycles. A *path* is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A *cycle* or a *circuit* is a path that begins and ends on the same vertex.

Definition 3.1.4 — Length of Paths and Cycles. The *length* of a walk or circuit or path or cycle is the number of edges in it.

A path of order n is denoted by P_n and a cycle of order n is denoted by C_n. Every edge of G can be considered as a path of length 1. Note that the length of a path on n vertices is $n-1$.

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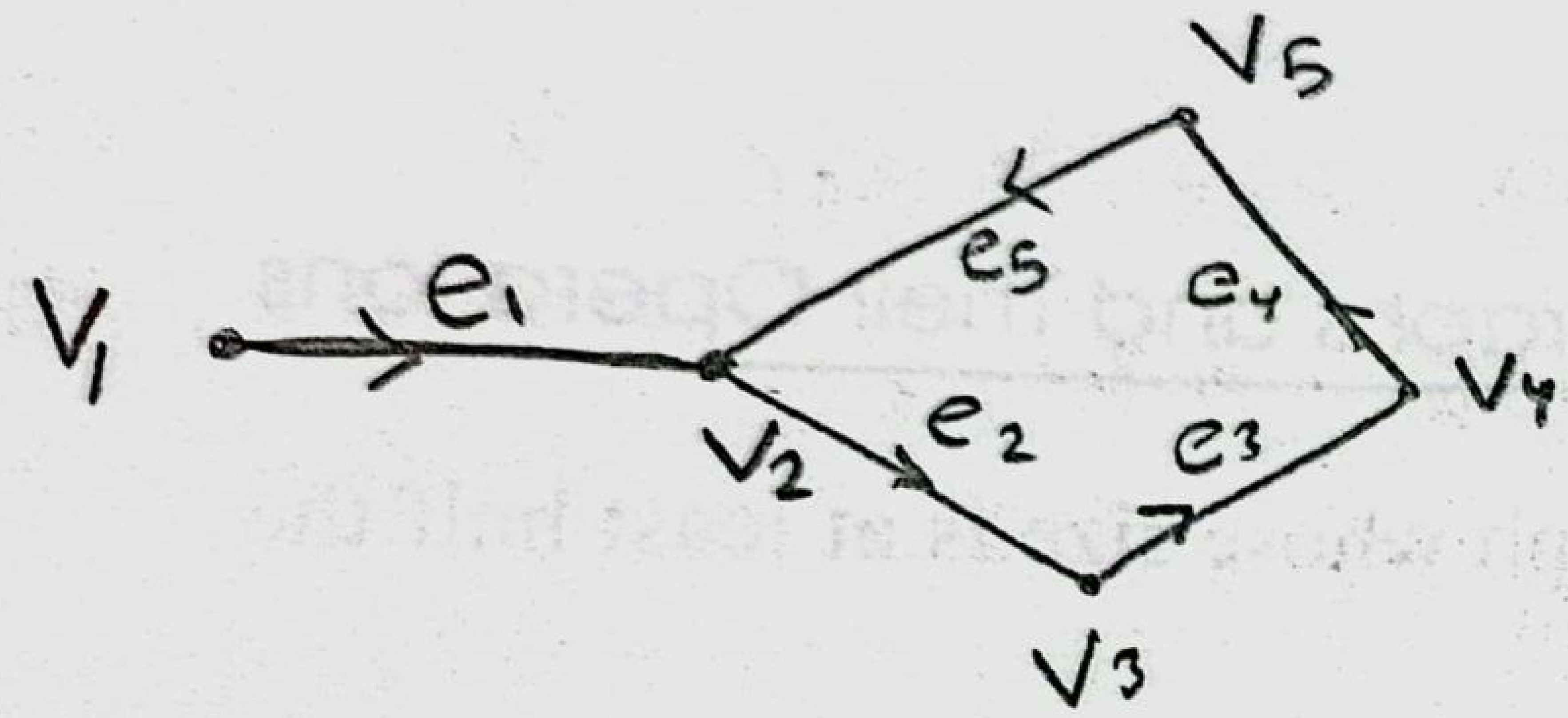
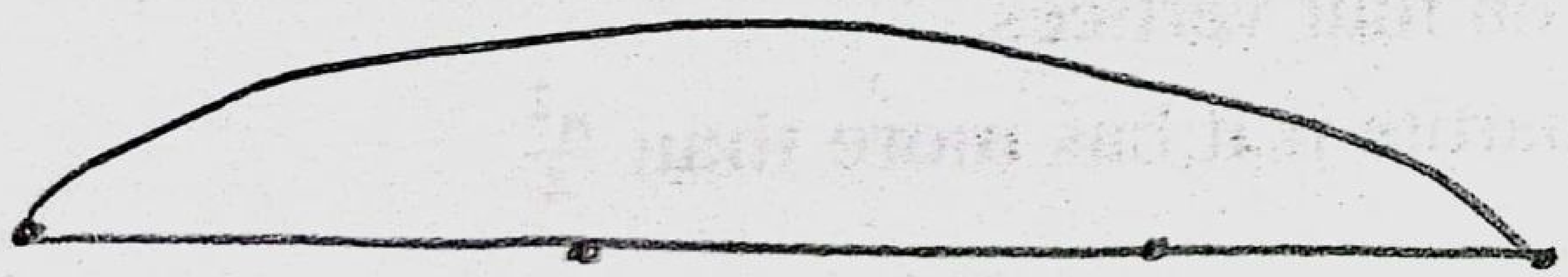


$(V_1\, e_1, V_2\, e_2, V_3\, e_3, V_4, e_4, V_5, e_5)$

walk

Path



cycle

A cycle having odd length is usually called an *odd cycle* and a cycle having even length is called an *even cycle*.

Definition 3.1.5 — Distance between two vertices. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$ or simply $d(u,v)$, is the length (number of edges) of a *shortest path* (also called a *graph geodesic*) connecting them. This distance is also known as the *geodesic distance*.

Definition 3.1.6 — Eccentricity of a Vertex. The *eccentricity* of a vertex v, denoted by $\varepsilon(v)$, is the greatest geodesic distance between v and any other vertex. It can be thought of as how far a vertex is from the vertex most distant from it in the graph.

Definition 3.1.7 — Radius of a Graph. The *radius* r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = \min\limits_{v \in V(G)} \varepsilon(v)$.

Definition 3.1.8 — Diameter of a Graph. The *diameter* of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = \max\limits_{v \in V(G)} \varepsilon(v)$.

Here, note that the diameter of a graph need not be twice its radius unlike in geometry. We can even see many graphs having same radius and diameter. Complete graphs are examples of the graphs with radius equals to diameter.

Definition 3.1.9 — Center of a Graph. A *center* of a graph G is a vertex of G whose eccentricity equal to the radius of G.

Definition 3.1.10 — Peripheral Vertex of a Graph. A *peripheral vertex* in a graph of diameter d is one that is distance d from some other vertex. That is, a peripheral vertex is a a vertex that achieves the diameter. More formally, a vertex v of G is peripheral vertex of a graph G, if $\varepsilon(v) = d$.

For a general graph, there may be several centers and a center is not necessarily on a diameter.

The distances between vertices in the above graph are given in Table 3.1. Note that a vertex v_i is represented by i in the table (to save the space).

Note that the radius of G is given by $r(G) = \min\{\varepsilon(v)\} = 4$ and the diameter of G is given by $diam(G) = \max\{\varepsilon(v)\} = 6$ and all eight central vertices are represented by white vertices in Figure 3.1.

Definition 3.1.11 — Geodetic Graph. A graph in which any two vertices are connected by a unique shortest path is called a *geodetic graph*.

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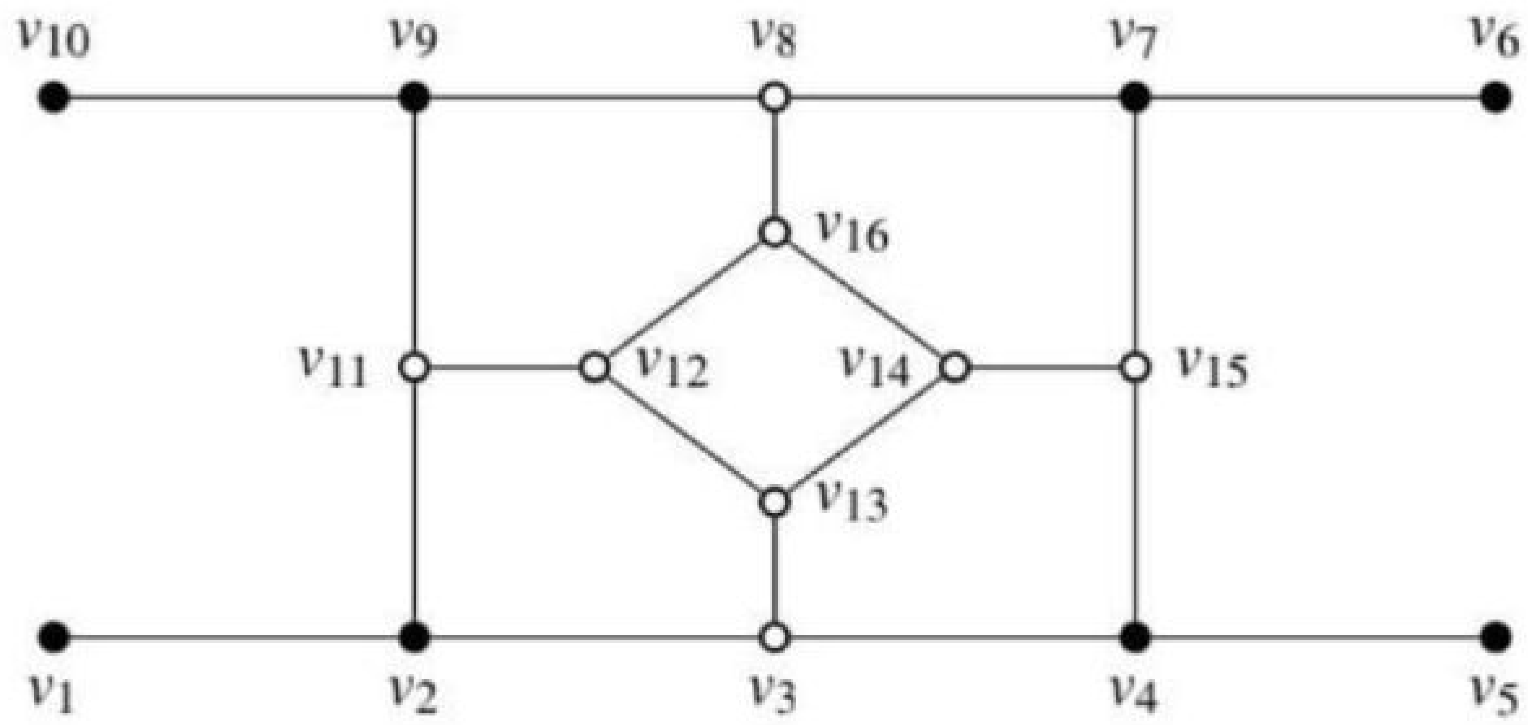


Figure 3.1: A graph with eight centers.

v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ε
1	0	1	2	3	4	6	5	4	3	4	2	3	3	4	4	4	6
2	1	0	1	2	3	5	4	3	2	3	1	2	2	3	3	3	5
3	2	1	0	1	2	4	3	4	3	4	2	2	1	1	2	3	4
4	3	2	1	0	1	3	2	3	4	5	5	3	2	2	1	3	5
5	4	3	2	1	0	4	3	4	5	6	4	4	3	3	2	4	6
6	6	5	4	3	4	0	1	2	3	4	4	4	4	3	2	3	6
7	5	4	3	2	3	1	0	1	2	3	3	3	3	2	1	2	5
8	4	3	4	3	4	2	1	0	1	2	2	2	3	2	2	1	4
9	3	2	3	4	5	3	2	1	0	1	1	2	3	3	3	2	5
10	4	3	4	5	6	4	3	2	1	0	2	3	4	4	4	3	6
11	2	1	2	3	4	4	3	2	1	1	0	1	2	3	4	2	4
12	3	2	2	3	4	4	2	1	2	3	1	0	1	2	3	1	4
13	3	2	1	2	3	4	3	3	3	4	2	1	0	1	2	2	4
14	4	3	2	2	3	3	2	2	3	4	3	2	1	0	1	1	4
15	4	3	2	1	2	2	1	2	3	4	4	3	2	1	0	2	4
16	4	3	3	3	4	3	2	1	2	3	2	3	2	1	2	0	4

Table 3.1: Eccentricities of vertices of the graph in Figure 3.1.

Theorem 3.1.1 If G is a simple graph with $diam(G) \geq 3$, then $diam(\bar{G}) \leq 3$.

Proof. If $diam(G) \geq 3$, then there exist at least two non-adjacent vertices u and v in G such that u and v have no common neighbours in G. Hence, every vertex x in $G - \{u, v\}$ is non-adjacent to u or v or both in G. This makes x adjacent to u or v or both in $\bar{G}$. Moreover, $uv \in E(\bar{G})$. So, for every pair of vertices x, y, there is an x, y path of length at most 3 in $\bar{G}$ through the edge uv. Hence, $diam(\bar{G}) \leq 3$. ■

3.2 Connected Graphs

Definition 3.2.1 — Connectedness in a Graph. Two vertices u and v are said to be *connected* if there exists a path between them. If there is a path between two vertices u and v, then u is said to be *reachable* from v and vice versa. A graph G is said to be *connected* every path → tour → walk

Walk : $v_1 e_{18} v_{12}, e_{15} v_{13} e_{12} v_{14}, e_{13}, v_{16} e_{14}, v_{12} e_{18}, v_{11} e_{17}, v_9$

Cycle : $v_9 e_{17}, v_{11} e_{18}, v_{12} e_{14}, v_{16} e_{19}, v_8 e_8, v_9$

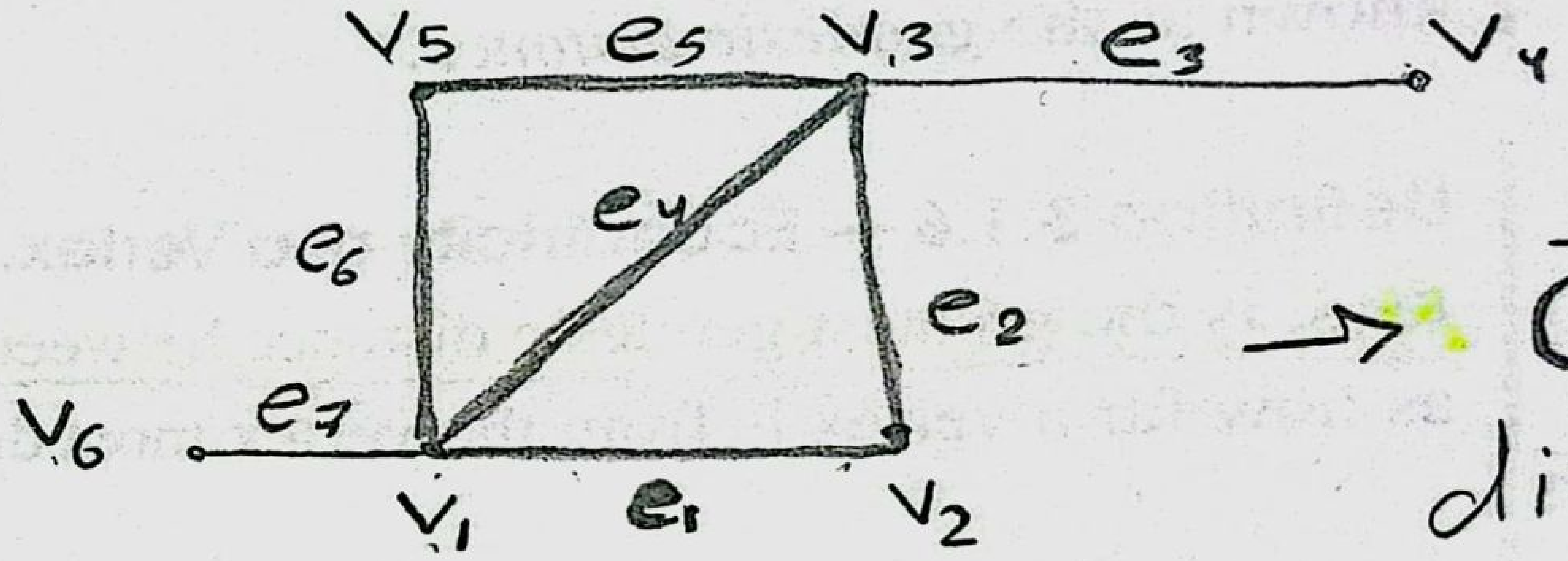
Path : $v_9 e_{17}, v_{11} e_{18}, v_{12} e_{14}, v_{16} e_{19}, v_8$

V_i	1	2	3	4	5	6	ε
1	0	1	1	2	1	1	2
2	1	0	1	2	2	2	2
3	1	1	0	1	1	2	2
4	2	2	1	0	2	3	3
5	1	2	1	2	0	2	2
6	1	2	2	3	2	0	3

$d(G) = 3$

$r(G) = 2$

center(G) = v_1, v_2, v_3, v_5



→ $\bar{G}$

$\dim(\bar{G}) \leq 3$

if there exist paths between any two vertices in G.

Definition 3.2.2 — Component of a Graph. A connected *component* or simply, a *component* of a graph G is a maximal connected subgraph of G.

Each vertex belongs to exactly one connected component, as does each edge. A connected graph has only one component.

A graph having more than one component is a *disconnected graph* (In other words, a disconnected graph is a graph which is not connected). The number of components of a graph G is denoted by $\omega(G)$.

In view of the above notions, the following theorem characterises bipartite graphs.

Theorem 3.2.1 A connected graph G is bipartite if and only if G has no odd cycles.

Proof. Suppose that G is a bipartite graph with bipartition (X,Y). Assume for contradiction that there exists a cycle $v_1, v_2, v_3, \ldots\ldots, v_k, v_1$ in G with k odd. Without loss of generality, we may additionally assume that $v_1 \in X$. Since G is bipartite, $v_2 \in Y$, $v_3 \in X$, $v_4 \in Y$ and so on. That is, $v_i \in X$ for odd values of i and $v_i \in Y$ for even values of i. Therefore, $v_k \in X$. But, then the edge $v_k, v_1 \in E$ is an edge with both endpoints in X, which contradicts the fact that G is bipartite. Hence, a bipartite graph G has no odd cycles.

Conversely, assume that G is a graph with no odd cycles. Let $d(u,v)$ denote the distance between two vertices u and v in G. Pick an arbitrary vertex $u \in V$ and define $X = \{x \in V(G) : d(x,u) \text{ is even}\}$. Clearly, $u \in X$ as $d(u,u) = 0$. Now, define another $Y = \{y \in V(G) : d(u,y) \text{ is odd}\}$. That is, $Y = V - X$. If possible, assume that there exists an edge $vw \in E(G)$ such that $v,w \in X$ (or $v,w \in Y$). Then, by construction $d(u,v)$ and $d(u,w)$ are both even (or odd). Let $P(u,w)$ and $P(u,v)$ be the shortest paths connecting u to w, and u to v respectively. Then, the cycle given by $P(u,w) \cup \{wv\} \cup P(v,u)$ has odd length $1 + d(u,w) + d(u,v)$, which is a contradiction. Therefore, no such edge wv may exist and G is bipartite. ■

Theorem 3.2.2 A graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in the subset V_2.

Proof. Suppose that such a partitioning exists. Consider two arbitrary vertices u and v of G, such that $u \in V_1$ and $v \in V_2$. No path can exist between vertices u and v; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, assume that G is a disconnected graph. Consider a vertex u in G. Let V_1 be the set of all vertices that are joined by paths to u. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any vertex in V_2 by an edge. Hence, we get the required partition. ■

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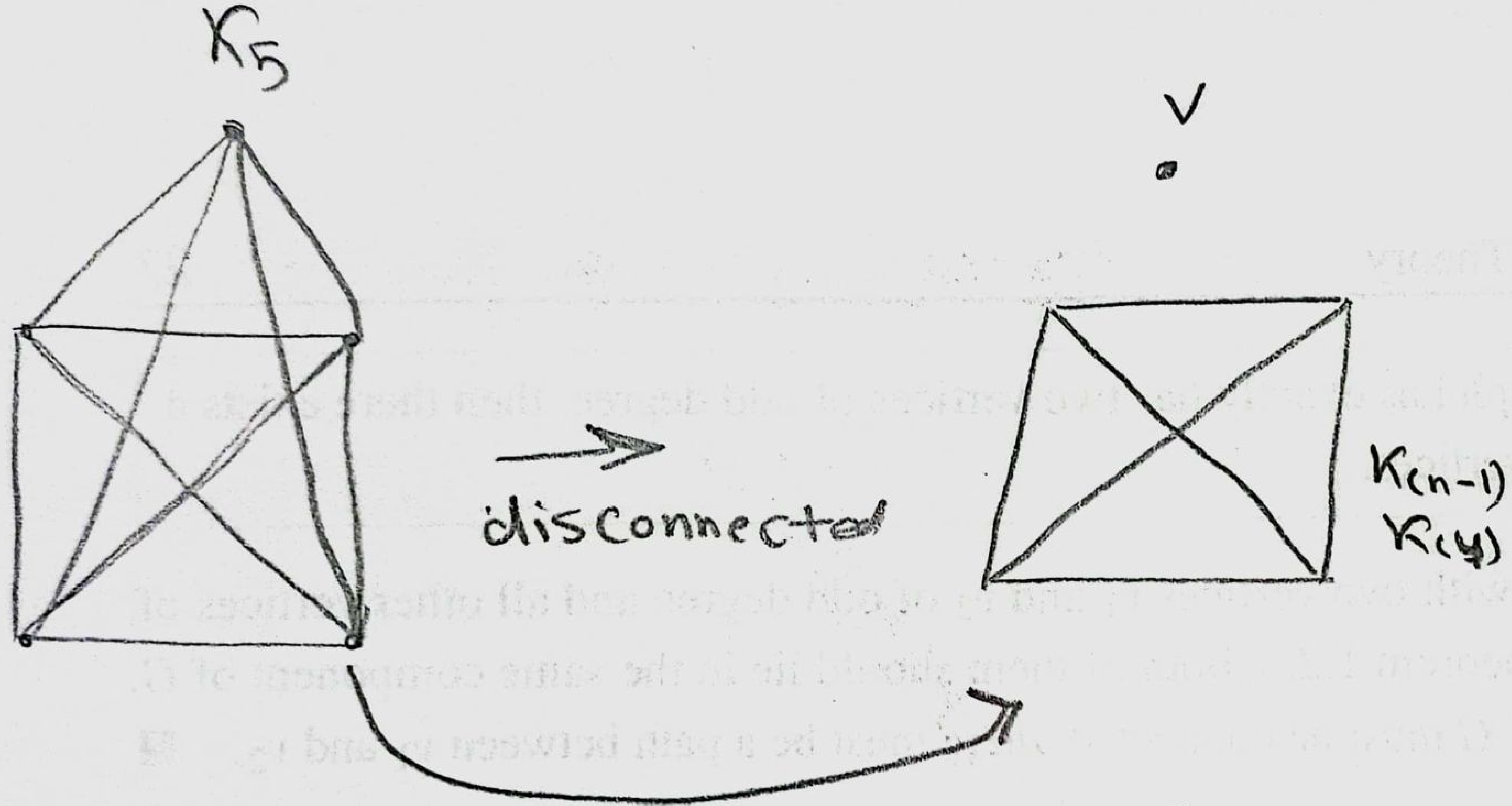
Problem 3.1 Show that an acyclic graph on n vertices and k components has $n-k$ edges.

Solution. The solution follows directly from the first part of the above theorem.

Problem 3.2 Show that every graph on n vertices having more than $\frac{(n-1)(n-2)}{2}$ edges is connected.

Solution. Consider the complete graph K_n and v be an arbitrary vertex of K_n. Now remove all $n-1$ edges of K_n incident on v so that it becomes disconnected with K_{n-1} as one component and the isolated vertex v as the second component. Clearly, this disconnected graph has $\frac{(n-1)(n-2)}{2}$ edges (all of which belong to the first component). Since all pairs of vertices in the first component K_{n-1} are any adjacent to each other, any new edge drawn must be joining a vertex in K_{n-1} and the isolated vertex v, making the revised graph connected. ■



$$\frac{(5-1)(5-2)}{2} = \frac{12}{2} = 6$$

In a graph (G) If order = 7 and edges = 20 Then The graph is Connected

$$\frac{(7-1)(7-2)}{2} = 15$$

$20 > 15$