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DSGT - 1

* **Propositions (statements)**
  - A declarative sentence which is either true or false, but not both.

Ex. ① There are 7 days in week. → TRUE
② $2+2=5$. → FALSE
③ The earth is flat. → FALSE
④ It will rain tomorrow. → ?

∴ sentences which are not propositions -

Ex. ① $x+3=5$
② Bring that book!
③ When is your interview?

* **Notations**
  - Primary, primitive or atomic statements
    - sentences which cannot be further split or broken down into simpler sentence
    - primary statements denoted by lower-case letters ($p, q \ldots$)

* **Logical Connectives**
  - compound statements
    - Complicated statements can be obtained from the primary statements by using certain connecting words
    - Ex. Not, And, Or, but, while etc.

① **Negation**
- The negation of a statement is formed by either introducing the word "not" at proper place or by prefixing the statement with the phrase "It is not the case that".
- If "p" denotes a statement, the negation of "p" is denoted by "$\sim p$" or ($\bar{p}$).
- i.e. if p is true then $\sim p$ is false & vice-versa.

Ex. ① $p \rightarrow$ " I am going for walk "
$\sim p \rightarrow$ " I am not going for walk "
OR " It is not the case that I am going for a walk ".

Ex. ② $q \rightarrow$ " 3 is not a prime number "
$\sim q \rightarrow$ " 3 is a prime number ".

② Conjunction (" And ")

If p and q are ~~compound~~ statements, then " $p \wedge q$ " or " p and q " is called as conjunction of p & q.

Ex. ① p : The sun is shining.
q : The birds are singing.
Then $p \wedge q \rightarrow$ " The sun is shining and the birds are singing ".

Ex. ② p : 2 is prime number
q : Ram is an intelligent boy.
Then $p \wedge q \rightarrow$ " 2 is prime number and Ram is an intelligent boy ".

- "but" & " while " are treated as equivalent to "and"

Translate the statement into symbolic form

Ex. ① Amar is poor but happy.
→ p : Amar is poor
q : Amar is happy.
∴ $p \wedge q$

Ex. ② We watch television while we have dinner
→ p : we watch television
q : we have dinner
∴ $p \wedge q$

③ Disjunction ("or")

If p & q are statements, then the compound statement "p or q" is called as the disjunction of p & q and is denoted by "$p \vee q$".

Ex. ① There is an error in the program or the data is wrong.

→ p : There is an error in the program
q : The data is wrong.
∴ $p \vee q$.

Ex. ② Either I will read a book or goto sleep.

→ p : I will read a book
q : I will goto sleep
∴ $p \vee q$

In Ex. ① "or" is inclusive - i.e at least one possibility exists or even both

In Ex. ② "or" is exclusive - i.e only one possibility can exists but not the both.

Notations = $\vee$: inclusive or

$\oplus$ $\underline{\vee}$: exclusive or

④ Conditional ("If... then")

If p & q are statements, the compound statement "If p then q", denoted by $p \rightarrow q$ is called as conditional statement or implication.

$p \rightarrow q$

- where p is antecedent & q is consequent
- converse of "$p \rightarrow q$" is the conditional "$q \rightarrow p$"
- contrapositive of "$p \rightarrow q$" is the conditional "$\sim q \rightarrow \sim p$"
- inverse of "$p \rightarrow q$" is $\sim p \rightarrow \sim q$

p - Hypothesis

q - Conclusion

Ex. ① p : Hari works hard

q : Hari will pass the exam.

$\therefore$ $p \to q$: If Hari works hard, then he will pass the exam.

Ex. ② Give the converse & contrapositive of the conditional statement

" If it rains, then I carry an umbrella"

→ let p : It rains

q : I carry an umbrella

Converse of $p \to q$ is

$q \to p$: If I carry an umbrella, then it rains

Contrapositive of $p \to q$ is

$\sim q \to \sim p$: If I do not carry an umbrella, then it does not rains.

Ex. ③ Write in symbolic form

" farmers will face hardship if the dry spell continues"

→ let p : Farmers will face harship

q : The dry spell continues

$\therefore$ $q \to p$

⑤ Biconditional (" If and only if") "iff"

If p & q are statements, the compound statement " p if and only if q", denoted by $p \leftrightarrow q$, is a biconditional statement

" If p then q, and conversely"

Ex. ① An integer is even if and only if it is divisible by 2.

② Two lines are parallel if and only if they have the same slope.

* Propositional or Statement Form

Ex. ① $\sim(p \vee q) \rightarrow p$

② $(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$ etc.

- A statement form has no fixed truth value
- It value depends on value assigned to its variables

Notations : Truth → "T" (1)

False → "F" (0)

—x—x—x—

Ex. ① using the following statements

p: Mohan is rich

q: Mohan is happy

write the following statements in symbolic form

i> Mohan is rich but unhappy $(p \wedge \sim q)$

ii> Mohan is poor but happy $(\sim p \wedge q)$

iii> Mohan is neither rich nor happy $\sim(p \vee q)$ or $\sim p \wedge \sim q$

iv> Mohan is poor or he is both rich and unhappy

$\sim p \vee (p \wedge \sim q)$

Ex. ② p : Rajani is tall

q : Rajanis beautiful

write the following statements in symbolic form

i> Rajani is tall and beautiful $(p \wedge q)$

ii> Rajani is tall but not beautiful $(p \wedge \sim q)$

iii> It is false that Rajani is short or beautiful

$\sim(\sim p \vee q)$

iv> Rajani is tall or Rajani is short and beautiful

$p \vee (\sim p \wedge q)$

Ex. ③ p: I will study discrete mathematics
q: I will goto movie
r: I am in a good mood
write the following statements in symbolic form
i> If I am not in a good mood, then I will go to a movie $(\sim r \rightarrow q)$
ii> I will not goto a movie & I will study discrete mathematics $(\sim q \wedge p)$
iii> I will goto a movie only if I will not study discrete mathematics $(q \rightarrow \sim p)$
iv> If I will not study discrete mathematics, then I am not in a good mood.
$(\sim p \rightarrow \sim r)$

Ex. ④ Put the following statements into symbolic form
i> whenever weather is nice, then only we will have a picnic
→
p : weather is nice
q : we will have a picnic
The statement is equivalent to " we will have a picnic only if the weather is nice"
$\therefore q \rightarrow p$

ii> Program is readable only if it is well structured
→
p: program is readable
q: Program is well structured
$\therefore p \rightarrow q$

iii> Unless he studies, he will fail in the exam.
→ p: He studies & q: He will fail in exam
" If he does not study, then he will fail in exam"
$\therefore \sim p \rightarrow q$

Ex. 5 p: I am bored

q: I am waiting for one hour

r: there is no bus

translate the following into English

i> $(q \vee r) \rightarrow p$

"If I am waiting for one hour or there is no bus, then I get bored".

ii> $\sim q \rightarrow \sim p$

"If I am not waiting for one hour, then I am not bored".

iii> $(q \rightarrow p) \vee (r \rightarrow p)$

"If I am waiting for one hour then I am bored, or if there is no bus, then I am bored".

Ex. 6 Write the following statements in symbolic form

i> Gopal is intelligent and rich.

ii> Gopal is intelligent but not rich.

iii> Gopal is either intelligent or rich

→ let, p: Gopal is intelligent

q: Gopal is rich

i> $p \wedge q$

ii> $p \wedge \sim q$

iii> $p \vee q$

Write logical negations of above statements

i> $\sim (p \wedge q)$ OR $\sim p \vee \sim q$

ii> $\sim (p \wedge \sim q)$ OR $\sim p \vee q$

iii> $\sim (p \vee q)$ OR $\sim p \wedge \sim q$

* Truth Tables

A table giving all possible truth values of a statement form, corresponding to the truth values assigned to its variables, is called truth table.

- If a statement form consists of n-distinct variables, then the table will contains 2^n values
- Ex. $2^2 = 4$, $2^3 = 8$

p	$\sim p$
T	F
F	T

① Negation

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

② And

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

③ OR

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

④ conditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

⑤ Biconditional

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

⑥ Exclusive-OR

p	q	$p \times q$
T	T	T
T	F	F
F	T	F
F	F	T

priority of operators

operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Ex. 1. construct the truth tables for the following statement form

i) $(\sim p \vee q) \rightarrow q$

ii) $\sim(p \wedge q) \vee (p \leftrightarrow q)$

iii) $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

→ i)

p	q	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \rightarrow q$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

ii)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \leftrightarrow q$	$\sim(p \wedge q) \vee (p \leftrightarrow q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	T	T	T

iii)

p	q	r	$\sim p$	$\sim p \rightarrow r$	$p \leftrightarrow q$	$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Ex. ② If $p \to q$ is false, determine the truth value of $(\sim(p \wedge q)) \to q$

→

$p \to q$ is false when $p = T$ & $q = F$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$(\sim(p \wedge q)) \to q$
T	F	F	T	F

Ex. ③ If p and q are false propositions, find the truth value of $(p \vee q) \wedge (\sim p \vee \sim q)$

→

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(p \vee q) \wedge (\sim p \vee \sim q)$
F	F	F	T	T	T	F

Ex. ④ If $p \to q$ is true, can we determine the truth value of $\sim p \vee (p \to q)$? Explain your answer.

→

p	q	$\sim p$	$p \to q$	$\sim p \vee (p \to q)$
T	T	F	T	T
F	T	T	T	T
F	F	T	T	T

Yes, it is possible to determine the truth value & its is "T".

Ex. ⑤ Let, p and q = T & r and s = F, find the truth values of following.

i) $p \vee (q \wedge r)$ ii) $p \to (r \wedge s)$

→

i)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	F	F	T

→

ii)

p	r	s	$r \wedge s$	$p \to (r \wedge s)$
T	F	F	F	F

Ex. ⑥ $p \to q$ is false, determine the truth value of: $(\sim p \vee \sim q) \to q$

→

p	q	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	$(\sim p \vee \sim q) \to q$
T	F	F	T	T	F

Ex. ⑦ $p \to q$ is true, can you determine the value of $\sim p \vee (p \leftrightarrow q)$

→

p	q	$\sim p$	$p \leftrightarrow q$	$\sim p \vee (p \leftrightarrow q)$
T	T	F	T	T
F	T	T	F	T
F	F	T	T	T

* Tautology

- A statement form is called Tautology if it always assums the truth value 'T' irrespective of truth values assigned to its variables.

Contradiction

- A statement form is called a contradiction if it always assums the truth value 'F' irrespective of the truth values assigned to its variables.

Contingency

- A statement form which is neither a tautology nor a contradiction is called a contingency.

Ex. ① $p \vee \sim p$ is tautology & $p \wedge \sim p$ is contradiction

→

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Ex. ② $p \rightarrow p$ is tautology

p	p	$p \rightarrow p$
T	T	T
F	F	T

Ex. ③ construct truth tables to determine whether each of the following is a tautology, contradiction or contingency.

i> $p \rightarrow (q \rightarrow p)$

ii> $(p \wedge q) \rightarrow p$

iii> $(p \wedge q) \wedge \sim (p \vee q)$

iv> $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

⇒ i>

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Hence $p \rightarrow (q \rightarrow p)$ is tautology

ii>

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Hence, $(p \wedge q) \rightarrow p$ is tautology

iii>

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence, $(p \wedge q) \wedge \sim (p \vee q)$ is contradiction

Ex. Show that $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology, without using truth table.

→ we have to only show that,
$p \wedge (p \rightarrow q)$ is true, q is true since in the other cases $(p \wedge (p \rightarrow q)) \rightarrow q$ is anyway true.
Now, $p \wedge (p \rightarrow q)$ is T implies p is T and $p \rightarrow q$ is T. These together means that q is T.
Hence, the required form is a tautology.

* **Equivalence of statement forms**

Two statement forms are logically equivalent if both have the same truth values, whatever may be the truth values assigned to the statements variable, occuring in both forms.

① **Idempotence**

p and $p \wedge p$ are logically equivalent.

p	$p \wedge p$
T	T
F	F

p and $p \rightarrow p$ are logically equivalent

② i> $p \wedge q$ and $q \wedge p$ are logically equivalent
ii> $p \vee q$ and $q \vee p$ are logically equivalent
iii> p and $\sim(\sim p)$ are logically equivalent

③ **Contrapositive** : $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are equivalent

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

④ Distributivity

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ equivalent

⑤ $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ equivalent

⑥ De Morgan's Law : $\sim(p \wedge q)$ and $\sim p \vee \sim q$

⑦ $\sim(p \vee q)$ and $\sim p \wedge \sim q$ equivalent

⑧ Elimination of conditional : $p \rightarrow q$ and $\sim p \vee q$

⑨ $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ which is logically equivalent to $(\sim p \vee q) \wedge (\sim q \vee p)$

* Logical Identities

1. $p \equiv p \vee p$
2. $p \equiv p \wedge p$
3. $p \vee q \equiv q \vee p$ } Commutative
4. $p \wedge q \equiv q \wedge p$
5. $p \vee (q \vee r) \equiv (p \vee q) \vee r$ } Associative
6. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } Distributivity
8. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
9. $p \equiv \sim(\sim p)$
10. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ } De Morgans Law
11. $\sim(p \wedge q) \equiv \sim p \vee \sim q$
12. $p \vee \sim p \equiv$ Tautology
13. $p \wedge \sim p \equiv$ Contradiction
14. $p \vee (p \wedge q) \equiv p$
15. $p \wedge (p \vee q) \equiv p$

* Predicates -

① " x is tall and handsome".
② " $x+3=5$ "
③ " $x+y \geq 10$ "

These sentences are not propositions, since they do not have truth value. However, if values are assigned to the variables, each of them becomes a proposition, which is either true or false.

- An assertion (sentence) that contains one or more variables is called a predicate; its truth value is predicted after assigning truth values to its variables.

Notations - $P(x_1, x_2, \ldots x_n)$
where P is predicate containing n-variables $x_1, x_2 \ldots x_n$ is called an n-place predicate.

Ex. ① " x is a city in India " $\longrightarrow$ $P(x)$
② " x is the father of y " $\longrightarrow$ $P(x,y)$
③ " $x+y \geq z$ " $\longrightarrow$ $P(x,y,z)$

- Here x, y, and z are variables or arguments.
- The value which the variables may assume constitute a collection / set called universe of disclosure.
- Value is then ~~bind~~ bound to the variable appearing in a predicate.
- A predicate becomes a proposition only when all its variables are bound.

Quantification

- method of binding individual variables in a predicate.
- The most common forms of quantification are universal and existential.

① Universal Quantifier

If $P(x)$ is a predicate with the individual variable x as argument, the the assertion "For all x, $P(x)$", which is interpreted as "for all values of x, the assertion $P(x)$ is true" is a statement in which the variable x is said to be universally quantified.

Notation - $\forall$ "For all" / "For every" / "For each"

If $P(x)$ is true for every possible value of x, then $\forall x\, P(x)$ is true, otherwise false.

② Existential Quantifier

Suppose for the predicate $P(x)$, $\forall x\, P(x)$ is false, but there exists atleast one value of x for which $P(x)$ is true, then in this proposition, x is bound by existential quantification.

Notation - $\exists x\, P(x)$ means "there exist a value of x for which $P(x)$ is true".

Ex. ① Let $P(x)$: "$x+3=5$" predicate.

The proposition

$\exists x\, P(x)$ is true (by setting $x=2$)

but $\forall x\, P(x)$ is false.

Let, $P(x,y)$ be two-place predicate

i> $\exists x \forall y \; P(x,y)$ – " There exists a value of x such that for all values of y, $P(x,y)$ is true"

ii> $\forall y \exists x \; P(x,y)$ – " For each value of y, there exist an x, such that $P(x,y)$ is true

iii> $\exists x \exists y \; P(x,y)$ – " There exist a value of x & value of y such that $P(x,y)$ is true"

iv> $\forall x \forall y \; P(x,y)$ – " For all values of x & y, $P(x)$ is true"

* <u>Negation of Quantified Statements</u>

Ex. ① " All invited guests were present for dinner"

Negation – " All invited guests were not present for the dinner "

Let, x : x is guest

$P(x)$: x was present for dinner

$\therefore$ $\forall x \; P(x)$ & negation is $\exists x \; (\sim P(x))$

Statement	Negation
$\forall x \; (P(x))$	$\exists x \; (\sim P(x))$
$\exists x \; (\sim P(x))$	$\forall x \; (P(x))$
$\forall x \; (\sim P(x))$	$\exists x \; (P(x))$
$\exists x \; P(x)$	$\forall x \; (\sim P(x))$

Ex. ① Negate the following in such a way that the symbol $\sim$ does not appear outside the square brackets

i> $\forall x \; [x^2 \geq 0]$

ii> $\exists x \; [x \cdot 2 = 1]$

iii> $\forall x \; \exists y \; [x + y = 1]$

Ans: i> $\exists x \; [x^2 < 0]$

ii> $\forall x \; [x \cdot 2 \neq 1]$

iii> $\exists x \; \forall y \; [x + y \neq 1]$

Ex. ② Transcribe the following into logical notation. Let the universe of disclosure be the real no.

i> For any value of x, x^2 is non-negative.

ii> For every value of x, there is some value of y such that $x \cdot y = 1$

iii> For every values of x, there is some value of y such that $x - y = 1$.

iv> There are positive values of x & y, such that $x \cdot y > 0$

v> There is value of x such that if y is positive, then $x+y$ is negative

⇒

i> $\forall x\ [x^2 \geq 0]$

ii> $\forall x\ \exists y\ [x \cdot y = 1]$

iii> $\forall x\ \exists y\ [x - y = 1]$

iv> $\exists x\ \exists y\ [(x > 0) \wedge (y > 0) \wedge (x \cdot y > 0)]$

v> $\exists x\ \forall y\ [(y > 0) \rightarrow (x + y < 0)]$

Ex. ③ For the universe of all integers, let $P(x)$, $Q(x)$, $R(x)$, $S(x)$ and $T(x)$ be the following statements:

$P(x)$: $x \geq 0$

$Q(x)$: x is even

$R(x)$: x is a perfect square

$S(x)$: x is divisible by 4

$T(x)$: x is divisible by 5

Write the following statements in symbolic forms:

- i> Atleast one integer is even
- ii> There exists a positive integer that is even
- iii> If x is even, then x is not divisible by 5.
- iv> No even integer is divisible by 5
- v> There exists an even integer divisible by 5
- vi> If x is even & x is a perfect square, then x is divisible by 4

$\Rightarrow$ i> $\exists x \; Q(x)$

ii> $\exists x \; [P(x) \wedge Q(x)]$

iii> $\forall x \; [Q(x) \rightarrow \sim T(x)]$

iv> $\forall x \; [Q(x) \rightarrow \sim T(x)]$

v> $\exists x \; [Q(x) \wedge T(x)]$

vi> $\forall x \; [Q(x) \wedge R(x) \rightarrow S(x)]$

Ex. 4 Write the following statements in symbolic form, using quantifiers

i> All students have taken a course in communication skills.

ii> There is a girl student in the class, who is also a sport person.

iii> Some students are intelligent, but not hardworking

$\Rightarrow$

Soln. i> let, $P(x)$: student x has taken a course in communication skills

$\therefore \quad \forall x \; P(x)$

ii> let, $P(x)$: x is a student

$Q(x)$: x is a girl

$R(x)$: x is sport person

$\therefore \quad \exists x \; [P(x) \wedge Q(x) \wedge R(x)]$

iii> Let, $P(x)$: x is intelligent

$Q(x)$: x is hardworking

$\therefore \quad \exists x \; [P(x) \wedge \sim Q(x)]$

Ex. 5 Negate each of the following

i> $\forall x, \; |x| = x$

ii> $\exists x, \; x^2 = x$

$\Rightarrow$

i> $\exists x, \; |x| \neq x$

ii> $\forall x, \; x^2 \neq x$