


\* Generating Functions \*

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<p>Name of Subject : _____</p>								
<p>SEM - <input type="checkbox"/> I / <input type="checkbox"/> II TEST NUMBER - <input type="checkbox"/> I / <input type="checkbox"/> II / <input type="checkbox"/> III / <input type="checkbox"/> IV / <input type="checkbox"/> V (Tick appropriate)</p>								
Question Number	1	2	3	4	5	6	Total	<p>Signature of Student</p>  <p>Signature of Supervisor</p>
Marks Obtained								
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(Start From here only)

Definition: Let  $a_0, a_1, a_2, \dots, \infty$  be a series of real no. denoted as  $\{a_n\}$

Then a series in power of  $x$ , such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \infty$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \text{ is called generating functions}$$

Application: ① To solve counting problems

② To solve recurrence relations.

Ex. ① Find the generating function for  $1, -1, 1, -1, \dots, \infty$

→

let  $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \infty$  — ①

∴ put  $a_0 = 1, a_1 = -1, a_2 = 1, a_3 = -1$  and so on into ①

$$g(x) = 1 - x + x^2 - x^3 + x^4 \dots \infty$$
 — ②

∴  $g(x) = \frac{1}{1+x}$  is generating function

proof:

$$\begin{array}{r}
 1-x+x^2-x^3 \dots \dots \leftarrow \text{eqn (2) series} \\
 1+x \sqrt{\begin{array}{r} 1 \\ -(1+x) \\ \hline -x \\ -(-x-x^2) \\ \hline x^2 \\ -(x^2+x^3) \\ \hline -x^3 \\ -(-x^3-x^4) \\ \hline x^4 \\ \dots \dots \end{array}}
 \end{array}$$

Ex. (2) Find generating function for  
1, 0, 0, 1, 0, 0, 1, 0, 0, \dots

→

let

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \quad (1)$$

put

$$a_0 = 1, a_1 = 0, a_2 = 0, a_3 = 1 \text{ \& so on in (1)}$$

$$g(x) = 1 + x^3 + x^6 + x^9 + \dots$$

$$\therefore \boxed{g(x) = \frac{1}{1-x^3}} \text{ is generating function}$$

proof:

$$\begin{array}{r}
 1+x^3+x^6 \leftarrow \text{eqn (1)} \\
 1-x^3 \sqrt{\begin{array}{r} 1 \\ -(1-x^3) \\ \hline x^3 \\ -(x^3-x^6) \\ \hline x^6 \\ -(x^6-x^9) \\ \hline x^9 \\ \dots \dots \end{array}}
 \end{array}$$

Ex-3 Find generating function for  $1, 2, 2^2, 2^3, 2^4, \dots$   
or  $1, 2, 4, 8, 16, \dots$

→ let

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \quad \text{--- (1)}$$

Put

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2^2, \quad a_3 = 2^3 \text{ and so on}$$

int (1)

$$\therefore g(x) = 1 + 2x + 2^2x^2 + 2^3x^3 + \dots \quad \text{--- (2)}$$

Now put  $y = 2x$

$$g(y) = 1 + y + y^2 + y^3 + y^4 + \dots$$

$$g(y) = \frac{1}{1-y}$$

$$\therefore \boxed{g(x) = \frac{1}{1-2x}} \quad (\text{because } y = 2x)$$

Numeric fun.

Generating fun.

(1)  $a_r = k \cdot a^r$

$$A(z) = \frac{k}{1-az}$$

(2)  $a_r = r$

$$A(z) = \frac{z}{(1-z)^2}$$

(3)  $a_r = b_r \cdot a^r$

$$A(z) = \frac{abz}{(1-az)^2}$$

(4)  $a_r = \frac{1}{r!}$

$$A(z) = e^z$$

(5)  $a_r = \begin{cases} n C_r, & 0 \leq r \leq n \\ 0, & r > n \end{cases}$

$$A(z) = (1+z)^n$$

Ex. ①  $a = \{4^0, 4^1, 4^2, 4^3, 4^4, \dots\}$

$$g(x) = 4^0 + 4^1x + 4^2x^2 + 4^3x^3 + \dots$$

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$\boxed{g(x) = \frac{1}{1-4x}}$$

Ex. ②  $b = 8 \cdot 9^r \quad r \geq 0$

$$\begin{aligned} b &= 8 \cdot 9^r \\ &= 8 \cdot g(x) \end{aligned}$$

$$g(x) = 9^0 + 9^1x + 9^2x^2 + 9^3x^3 + \dots$$

$$= \frac{1}{1-9x}$$

$$\therefore \boxed{B(x) = 8 \cdot \left( \frac{1}{1-9x} \right)} \Rightarrow \boxed{B(x) = \frac{8}{1-9x}}$$

Ex. ③  $c = 3^r + 4^r$

$$C(z) = A(z) + B(z)$$

$$\therefore C(z) = \frac{1}{1-3z} + \frac{1}{1-4z}$$

Ex. ④  $c = 3^r \cdot 5^r$

$$C(z) = A(z) \cdot B(z)$$

$$C(z) = \left[ \frac{1}{1-3z} \right] \left[ \frac{1}{1-5z} \right]$$

Ex. ⑤ Determine generating function of

i)  $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

ii)  $a_r = 5, \quad r \geq 0$



$$i) \quad c(z) = A(z) + B(z)$$

$$A(z) = 3^r = \frac{1}{1-3z}$$

$$B(z) = 4^{r+1} = 4^r \cdot 4 = 4 \cdot 4^r = 4 \left( \frac{1}{1-4z} \right) = \left[ \frac{4}{1-4z} \right]$$

$$\therefore c(z) = \left[ \frac{1}{1-3z} \right] + \left[ \frac{4}{1-4z} \right]$$


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$$ii) \quad q_r = 5, \quad r \geq 0$$

$$a = \{5, 5, 5, 5, \dots\}$$

$$\begin{aligned} g(x) &= 5 + 5x + 5x^2 + 5x^3 + \dots \\ &= 5(1 + x + x^2 + x^3 + \dots) \\ &= 5 \left[ \frac{1}{1-x} \right] \end{aligned}$$

$$\therefore \boxed{g(x) = \frac{5}{1-x}}$$

Ex. ⑥ Determine numeric fun. corresponding to following generating function

$$i) \quad \frac{1}{(1+z)}$$

$$ii) \quad \frac{3-5z}{(1-2z-3z^2)}$$

— i)

$$A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$$

$$\begin{aligned} \therefore A(z) &= 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + \dots \\ &= 1 - z + z^2 - z^3 + z^4 + \dots \end{aligned}$$

Numeric function

$$q_r = (-1)^r$$

$$\text{ii)} \quad A(z) = \frac{3-5z}{(1-3z-3z^2)}$$

$$= \frac{3-5z}{(1-3z)(1+z)}$$

$$\therefore \frac{3-5z}{(1-3z)(1+z)} = \frac{A}{(1-3z)} + \frac{B}{(1+z)}$$

on simplifying

$$(3-5z) = (1-3z)(1+z) \left[ \frac{A}{1-3z} + \frac{B}{1+z} \right]$$

$$3-5z = A(1+z) + B(1-3z)$$

$$= A + Az + B - 3zB$$

$$3-5z = (A+B) + z(A-3B)$$

on equating

$$A+B=3 \quad \text{--- (i)}$$

$$A-3B=-5 \quad \text{--- (ii)}$$

put  $A=3-B$  into (ii)

$$A-3B=-5$$

$$(3-B)-3B=-5$$

$$3-4B=-5$$

$$3+5=4B$$

$$4B=8$$

$$\boxed{B=2}$$

put  $B=2$  into (i)

$$A+B=3$$

$$A+2=3$$

$$\boxed{A=1}$$

$$\therefore A(z) = \frac{A}{1-3z} + \frac{B}{1+z} = \frac{1}{1-3z} + \frac{2}{1+z}$$

Now  $B(z) = \frac{1}{1-3z} \Rightarrow 3^r$

$$C(z) = \frac{2}{1+z} \Rightarrow 2(-1)^r$$

Numeric function

$$\therefore \boxed{A(z) = 3^r + 2(-1)^r}$$

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