DM unit-3 H.E. Suryavanshi

* Generating Functions *

Definition: Let $a_0, a_1, a_2 \ldots\ldots \infty$ be a series of real no. denoted as $\{a_n\}$

Then a series in power of x, such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots \infty$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$ is called generating functions

Application: ① To solve counting problems
② To solve recurrence relations.

Ex. ① Find the generating function for $1, -1, 1, -1, \ldots\ldots \infty$

→

let $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots \infty$ — ①

∴ put $a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$ and so on into ①

$$g(x) = 1 - x + x^2 - x^3 + x^4 \ldots\ldots \infty \quad — ②$$

∴ $\boxed{g(x) = \frac{1}{1+x}}$ is generating function proof:

eqⁿ ② series

$$1 - x + x^2 - x^3 \ldots\ldots$$

$$1+x \;\Big)\; 1$$

$$-(1+x)$$

$$-x$$

$$-(-x-x^2)$$

$$x^2$$

$$-(x^2+x^3)$$

$$-x^3$$

$$-(-x^3-x^4)$$

$$x^4$$

$$\ldots\ldots$$

Ex. ② Find generating function for

1, 0, 0, 1, 0, 0, 1, 0, 0 .....

$\longrightarrow$

let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad ①$$

put

$a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ & so on in ①

$$g(x) = 1 + x^3 + x^6 + x^9 + \ldots\ldots$$

$\therefore$ $\boxed{g(x) = \dfrac{1}{1-x^3}}$ is generating function

proof: eqⁿ ①

$$1 + x^3 + x^6$$

$$1-x^3 \;\Big)\; 1$$

$$-(1-x^3)$$

$$x^3$$

$$-(x^3-x^6)$$

$$x^6$$

$$-(x^6-x^9)$$

$$x^9$$

$$\ldots\ldots$$

Ex. (3) Find generating function for $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$

or $1, 2, 4, 8, 16, \ldots$

→ Let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{--- (1)}$$

Put

$a_0 = 1$, $a_1 = 2$, $a_2 = 2^2$, $a_3 = 2^3$ and so on int (1)

$$\therefore g(x) = 1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots \quad \text{--- (2)}$$

Now put $y = 2x$

$$g(y) = 1 + y + y^2 + y^3 + y^4 + \cdots$$

$$g(y) = \frac{1}{1-y}$$

$$\therefore \boxed{g(x) = \frac{1}{1-2x}} \quad (\text{becoz } y = 2x)$$

Numeric fun.	Generating fun.
(1) $a_r = k \cdot a^r$	$A(z) = \frac{k}{1-az}$
(2) $a_r = r$	$A(z) = \frac{z}{(1-z)^2}$
(3) $a_r = b_r \cdot a^r$	$A(z) = \frac{abz}{(1-az)^2}$
(4) $a_r = \frac{1}{r!}$	$A(z) = e^z$
(5) $a_r = \begin{cases} {}^nC_r, & 0 \le r \le n \\ 0, & r > n \end{cases}$	$A(z) = (1+z)^n$

Ex. 1 $a = \{4^0, 4^1, 4^2, 4^3, 4^4 \ldots\}$

$$g(x) = 4^0 + 4^1 x + 4^2 x + 4^3 x + \cdots$$

$$g(x) = a_0 + a_1 x + a_2 x + a_3 x + \cdots$$

$$\boxed{g(x) = \frac{1}{1-4x}}$$

Ex. 2 $b = 8 \cdot 9^r \qquad r \geq 0$

$$b = 8 \cdot 9^r$$
$$= 8 \cdot g(x)$$

$$g(x) = 9^0 + 9^1 x + 9^2 x^2 + 9^3 x^3 + \cdots$$
$$= \frac{1}{1-9x}$$

$$\therefore \boxed{B(x) = 8 \cdot \left(\frac{1}{1-9x}\right)} \Rightarrow \boxed{B(x) = \frac{8}{1-9x}}$$

Ex. 3 $C_r = 3^r + 4^r$

$$C(z) = A(z) + B(z)$$

$$\therefore C(z) = \frac{1}{1-3z} + \frac{1}{1-4z}$$

Ex. 4 $C_r = 3^r \cdot 5^r$

$$C(z) = A(z) \cdot B(z)$$

$$C(z) = \left[\frac{1}{1-3z}\right]\left[\frac{1}{1-5z}\right].$$

Ex. 5 Determine generating function of

i) $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

ii) $a_r = 5, \quad r \geq 0$

i) $c(z) = A(z) + B(z)$

$$A(z) = 3^r = \frac{1}{1-3z}$$

$$B(z) = 4^{r+1} = 4^r \cdot 4^1 = 4 \cdot 4^r = 4\left(\frac{1}{1-4z}\right)$$

$$= \left[\frac{4}{1-4z}\right]$$

$$\therefore c(z) = \left[\frac{1}{1-3z}\right] + \left[\frac{4}{1-4z}\right]$$

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ii) $a_r = 5, \; r \geq 0$

$$a = \{5, 5, 5, 5, \ldots\}$$

$$g(x) = 5 + 5x + 5x^2 + 5x^3 + \ldots$$

$$= 5(1 + x + x^2 + x^3 + \ldots)$$

$$= 5\left[\frac{1}{1-x}\right]$$

$$\therefore \boxed{g(x) = \frac{5}{1-x}}$$

Ex. 6 Determine numeric fun. corresponding to following generating function

i) $\frac{1}{(1+z)}$ ii) $\frac{3-5z}{(1-2z-3z^2)}$

— i)

$$A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$$

$$\therefore A(z) = 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + \ldots$$

$$= 1 - z + z^2 - z^3 + z^4 + \ldots$$

Numeric function

$$a_r = (-1)^r$$

ii>

$$A(z) = \frac{3-5z}{(1-2z-3z^2)}$$

$$= \frac{3-5z}{(1-3z)(1+z)}$$

$$\therefore \frac{3-5z}{(1-3z)(1+z)} = \frac{A}{(1-3z)} + \frac{B}{(1+z)}$$

on simplifying

$$(3-5z) = (1-3z)(1+z)\left[\frac{A}{1-3z} + \frac{B}{1+z}\right]$$

$$3-5z = A(1+z) + B(1-3z)$$

$$= A + Az + B - 3zB$$

$$3-5z = (A+B) + z(A-3B)$$

on equating

$$A+B = 3 \quad \text{——} \quad (i)$$

$$A-3B = -5 \quad \text{——} \quad (ii)$$

put $A = 3-B$ into (ii)

$$A-3B = -5$$

$$(3-B) - 3B = -5$$

$$3-4B = -5$$

$$3+5 = 4B$$

$$4B = 8$$

$$\boxed{B = 2}$$

put $B=2$ into (i)

$$A+B = 3$$

$$A+2 = 3$$

$$\boxed{A = 1}$$

$$\therefore A(z) = \frac{A}{1-3z} + \frac{B}{1+z} = \frac{1}{1-3z} + \frac{2}{1+z}$$

Now $B(z) = \frac{1}{1-3z} \Rightarrow 3^r$

$C(z) = \frac{2}{1+z} \Rightarrow 2(-1)^r$

Numeric function

$\therefore \boxed{A(z) = 3^r + 2(-1)^r}$