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High Caliber Technical Education in an Environment that Promotes Excellence

TEST (I / II) / PRELIMINARY EXAMINATION

Name of Candidate : ____________________

(IN BLOCK LETTERS) (Surname) (First Name) (Middle Name)

Year : FE / SE / TE / BE Branch : ________ Division : ________ Roll No. : ________

Semester : I / II Name of Subject : ____________________

Total Supplements : 1 + ______ = ______

Signature of Student Signature of Supervisor

Question Number	1	2	3	4	5	6	7	8	9	10	Total
Marks Obtained											
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Signature of Examiner : Signature of Moderator :

(Start From here only)

* Recurrence Relations (or Difference Equation)

In many discrete computation problems, it is easier to obtain the numeric function in the form of a relation betn its terms.

The recursive formula for defining the numeric function (or sequence) is called a recurrence relation.

If $a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$ is a numeric function then the recurrence relation for 'a' is an equation relating a_r, for any r, to one or more a_i's $(i<r)$.

In other words, a recurrence relation on the numeric function 'a' is a formula that relates all the terms of 'a' to previous terms of 'a'.

A recurrence relation is also called difference equation.

* Numeric function $a: N \to R$

Function whose domain set is of natural no. & range set is of real no.

Ex. (i) Consider recurrence relation

$$a_r = a_{r-1} + 3 \quad , \quad r \geq 1 \text{ with } a_0 = 2$$

Here

$$a_1 = a_0 + 3 = 2+3 = 5$$
$$a_2 = a_1 + 3 = 5+3 = 8$$
$$a_3 = a_2 + 3 = 8+3 = 11$$
$$\ldots$$

$\therefore$ Numeric function $a = \{2, 5, 8, 11 \ldots\}$

The condition $a_0 = 2$ is initial condition

Ex. (2) Fibonacci Sequence of numbers

$$a_r = a_{r-2} + a_{r-1} \quad , \quad r \geq 2$$

with $a_0 = 1$ and
$a_1 = 1$

Here

$$a_2 = a_0 + a_1 = 1+1 = 2$$
$$a_3 = a_1 + a_2 = 1+2 = 3$$
$$a_4 = a_2 + a_3 = 2+3 = 5$$
$$\ldots$$

Thus, Fibonacci Sequence is $1, 1, 2, 3, 5, 8 \ldots$

The numeric function which is computed using recurrence relation is known as the solution of the recurrence relation.

In Ex. (1) — $a = \{2, 5, 8, 11 \ldots\}$ } Solutions of
In Ex. (2) — $a = \{1, 1, 2, 3, 5, 8, \ldots\}$ } recurrence relⁿ.

(i) <u>Linear Recurrence Relations with Constant Coefficients</u>

Suppose, r & k are non-negative integers

A recurrence relation of the form —

$$\boxed{C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = F(r)}$$

for $r \geq k$

where $C_0, C_1, C_2 \ldots C_k$ are constants,

is called a linear recurrence relation with constant coefficient of order k, provided C_0 & C_k are non-zero.

- The relation $a_r - 2a_{r-1} = 2r$ is a first order linear recurrence relation with const.

- Similarly, $a_r + 2a_{r-3} = r^2$ is third order linear recurrence relation.

- But $a_r^2 + a_{r-1} = 5$ is not linear recurr. relⁿ

* To solve kth order linear recurrence relⁿ with constant coeff., we require k initial conditions to determine the numeric function uniquely.

With fewer than k^{th} initial conditions, the numeric function computed is not unique.

Ex. Second order relⁿ with $a_0 = 2$

$$\boxed{a_r + a_{r-1} + a_{r-2} = 4}$$

∴ Numeric functions which satisfys given recurrence relⁿ & initial condⁿ. are

① 2, 0, 2, 2, 0, 2, 2, 0, 2, ...

② 2, 2, 0, 2, 2, 0, 2, 2, 0, ...

③ 2, 5, −3, 2, 5, −3, 2, 5, −3, ...

It is not unique.

② Homogeneous solutions

Each linear recurrence relation is related with its homogenous equations & solution of homogeneous equation is called homogeneous solution.

Consider kth ordere linear recurrence relⁿ

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Homogeneous recurr. relⁿ is given by

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = 0$$

∴ This means that for any linear recurrence relation

$$F(r) = 0$$

then given eqⁿ is homogeneous recu. relⁿ

Ex ① linear recurrence relⁿ.

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 2r$$

Its homogenous recur. relⁿ is

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 0$$

Homogeneous solution –

To find homogeneous solution, define the term characteristic equation.

– Homogeneous k^{th} order linear recu. relⁿ.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = 0$$

It's characteristic equation is

$$C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k = 0$$

where, α is degree of k.

Therefore, it has 'k' roots, called characteristic roots.

– Suppose, $(\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_k)$ all roots are distinct, then solution of the homogeneous recurrence relⁿ is given by –

$$a_r = A_1 \cdot \alpha_1^r + A_2 \cdot \alpha_2^r + \cdots + A_k \cdot \alpha_k^r$$

where

$(A_1, A_2, \cdots A_k)$ constants determined by initial condition.

– If any root is repeated (eg. α_1) m-times then term $A_1 \alpha_1^r$ is replaced by

$$(A_1 r^{m-1} + A_2 r^{m-2} + \cdots + A_{m-1} r + A_m)\alpha_1^r$$

where A_i – calculated using initial condition.

Ex. 1 $a_r - 10 \cdot a_{r-1} + 9a_{r-2} = 0$

with

$a_0 = 3$ and $a_1 = 11$

Find homogeneous solution?

$\Rightarrow$

$a_r - 10 \cdot a_{r-1} + 9 a_{r-2} = 0$

$\therefore$ its characteristic equation is

$$\boxed{\alpha^2 - 10\alpha + 9 = 0}$$

Solve, $(\alpha - 1)(\alpha - 9) = 0$

characteristic roots $\Rightarrow \alpha = 1, 9$ (i.e $\alpha_1 = 1, \alpha_2 = 9$)

Now, all roots $(1, 9)$ are distinct,

so solution is given as follows

$a_r = A_1 \alpha_1^r + A_2 \alpha_2^r$

$a_r = A_1 (1)^r + A_2 (9)^r$ — (1) ($\because A_1, A_2 =$ constants)

Now,

To find A_1 & A_2 put $r = 0$ in eq (1)

$a_0 = A_1 (1)^0 + A_2 (9)^0$

$a_0 = A_1 + A_2$ we have $a_0 = 3$

So,

$$\boxed{A_1 + A_2 = 3} \quad — (2)$$

Now

put $r = 1$ in eq^n (1)

$a_1 = A_1 (1)^1 + A_2 (9)^1$

$a_1 = A_1 + 9A_2$ we have $a_1 = 11$

So,

$$\boxed{A_1 + 9A_2 = 11} \quad — (3)$$

by solving eq^n (2) & (3)

$A_1 = 2$ & $A_2 = 1$

also we have $\alpha_1 = 1$ & $\alpha_2 = 9$

$\therefore$ Solution to homo. recu. rel^n is

from eq^n (1)

$a_r = 2(1)^r + (9)^r$

$\therefore$ $\boxed{a_r = 2 + 9^r}$

$A_1 = (\cancel{A_2 - 3}) = 3 - A_2$

$\therefore A_1 + 9A_2 = 11$

$\therefore (A_2 - 3) + 9A_2 = 11$

$\therefore 8A_2 = 8$ $\boxed{A_2 = 1}$

Now, $A_1 = \cancel{A_2}\ 3 - A_2$

$= 3 - 1$

$\boxed{A_1 = 2}$

Ex. 2 $a_r - 8 \cdot a_{r-1} + 16 \cdot a_{r-2} = 0$

where

$a_2 = 16$ & $a_3 = 80$

Find homogeneous solutions?

$\Rightarrow$

Given, $a_r - 8a_{r-1} + 16a_{r-2} = 0$

Then, characteristic eqn. is

$\alpha^2 - 8\alpha + 16 = 0$

Solve

$(\alpha - 4)^2 = 0$

$\therefore$ $\alpha = 4, 4$ i.e $\alpha_1 = 4$, $\alpha_2 = 4$ $(m=2)$

Same characteristic roots. Therefore, the homogeneous solution is given as

$a_r = (A_1 \cdot r + A_2)\alpha_1^r$ — ①

Now,

given $a_2 = 16$ & $a_3 = 8$ put it into eqn ①

$a_2 = (A_1(2) + A_2)4^2$

$16 = (2A_1 + A_2)4^2$

$\therefore$ $\boxed{2A_1 + A_2 = 1}$ — ②

and

$a_3 = (A_1(3) + A_2)4^3$

$80 = (3A_1 + A_2)64$

$\therefore$ $3A_1 + A_2 = 80/64$

i.e

$\boxed{3A_1 + A_2 = 5/4}$ — ③ $\boxed{3A_1 + A_2 = 5/4}$

From eqn ② & ③

$A_1 = 1/4$

$A_2 = 1/2$

& $\alpha_1 = 4$

The homogeneous solution is

$a_r = (A_1 \cdot r + A_2)\alpha_1^r$

$\boxed{a_r = \left(\frac{1}{4}r + \frac{1}{2}\right)4^r}$

(3) Total Solutions

The homogeneous recur. relⁿ. is obtained by putting $F(r)=0$ in the given k^{th} order linear recur. relⁿ.

The solution which satisfies the $F(r)$ is called particular solution. There is no general procedure for determining the particular solution.

$a_r^{(h)}$ – Homogeneous solution

$a_r^{(p)}$ – particular solution

∴ Total solution

$$a_r = a_r^{(h)} + a_r^{(p)}$$

As there is no general procedure for determining particular solution. But in some cases, it can be obtained by the method of inspection of $F(r)$.

$F(r)$	Form of particular solⁿ.
(1) A const. d	(1) A constant P
(2) Linear Function $d_0 + d_1 r$	(2) Linear function $P_0 + P_1 \cdot r$
(3) nth degree polynomial $d_0 + d_1 r + d_2 r^2 + \cdots + d_n r^n$	(3) nth degree polynomial $P_0 + P_1 \cdot r + P_2 \cdot r^2 + \cdots + P_n r^n$
(4) Exponential function db^r – provided 'b' is not characteristic root	(4) Exponential function Pb^r.
(5) Exponential Function – 'b' is characteristic root of the equation with multiplicity $(m-1)$	(5) Exponential function $P r^{m} \cdot b^r$

Ex. ① Solve $a_r - a_{(r-1)} - 6 \cdot a_{(r-2)} = -30$

given $a_0 = 20$ & $a_1 = -5$

$\Rightarrow$

$$a_r - a_{(r-1)} - 6a_{(r-2)} = -30$$

$\therefore$ characteristic equation (put $F(r)=0$ i.e -3

$$\alpha^2 - \alpha - 6 = 0$$

$\alpha = -2, 3$ i.e $\alpha_1 = -2$ & $\alpha_2 = 3$

$\therefore$ Homogeneous solution is

$$a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r$$

$$a_r^{(h)} = A_1(-2)^r + A_2(3)^r \quad \text{--- ①}$$

Now,

particular solution

$F(r) = -30$ i.e constant value

so $a_r = P$ for all r

i.e $a_r = a_{r-1} = a_{r-2} = P$ (constant).

$\therefore$ $a_r - a_{(r-1)} - 6a_{(r-2)} = -30$

becomes

$$P - P - 6P = -30$$

$\therefore$ $P = 5$

$$a_r^{(P)} = 5$$

$\therefore$ Total solution $a_r = a_r^{(h)} + a_r^{(P)}$

$$a_r = A_1(-2)^r + A_2(3)^r + 5$$

we have $a_0 = 20$ & $a_1 = -5$

so to findout A_1 & A_2

put $r=0$ & $r=1$

$\therefore$ For $r=0$ eqⁿ ② becomes

$$a_0 = A_1(-2)^0 + A_2(3)^0 + 5$$

$$20 = A_1 + A_2 + 5$$

$\therefore$ $\boxed{A_1 + A_2 = 15}$ --- ③

For $r=1$ eqⁿ ② becomes.

$$a_1 = A_1(-2)^1 + A_2(3)^1 + 5$$

$$-5 = -2A_1 + 3A_2 + 5$$

$\therefore$ $\boxed{-2A_1 + 3A_2 = -10}$ --- ④

DSGT-3 H. E. Suryavanshi 2

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From eqⁿ 3 & 4 $A_1 = 11$ & $A_2 = 4$

Put $A_1 = 11$, $A_2 = 4$, into eqⁿ 2

Total / complete solution is

$$\boxed{a_r = 11(-2)^r + 4(3)^r + 5}$$

Ex 2 Solve $a_r - 7a_{(r-1)} + 10 \cdot a_{(r-2)} = 6 + 8r$

with $a_0 = 1$ and $a_1 = 2$

$\Rightarrow$

Given $a_r - 7a_{r-1} + 10a_{r-2} = 6 + 8r$

$\therefore a_r - 7a_{r-1} + 10a_{r-2} = 0$

characteristic equation is

$\alpha^2 - 7\alpha + 10 = 0$

$\alpha = 2, 5$

i.e. $\alpha_1 = 2$, $\alpha_2 = 5$

Homogeneous solution $a_r^{(h)}$ is

$$a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r$$

$$a_r^{(h)} = A_1 (2)^r + A_2 (5)^r$$

Now,

For particular solution $a_r^{(p)}$

$F(r)$ – linear polynomial

$$\therefore \quad a_r = P_0 + P_1 r$$

$$a_{(r-1)} = P_0 + P_1 (r-1)$$

$$a_{(r-2)} = P_0 + P_1 (r-2)$$

by putting these values, we have

$$(P_0+P_1 r) - 7\left(P_0+P_1(r-1)\right) + 10\left(P_0+P_1(r-2)\right) = 6+8r$$

$$(4P_0 - 13P_1) + 4P_1 r = 6+8r$$

$$\therefore \quad 4P_0 - 13P_1 = 6$$

$$\& \quad 4P_1 r = 8r$$

So

$$P_0 = 8 \quad \& \quad P_1 = 2$$

$\therefore$ particular solution $a_r^p = P_0 + P_1 r$ becomes

$$a_r^p = 8 + 2r$$

$\therefore$ Total / complete solution is

$$a_r = a_r^h + a_r^p$$

$$a_r = A_1 2^r + A_2 5^r + 8 + 2r \quad \text{—} (1)$$

We have

$$a_0 = 1 \quad \& \quad a_1 = 2$$

So put $r=0$ & $r=1$ into eq^n (1)

For $r=0$, $\quad a_0 = A_1 2^0 + A_2 5^0 + 8 + 2(0)$

$$a_0 = A_1 + A_2 + 8$$

Since $a_0 = 1$, $\quad A_1 + A_2 + 8 = 1$

$$\boxed{A_1 + A_2 = -7}$$

For $r=1$, $\quad a_1 = A_1 2^1 + A_2 5^1 + 8 + 2(1)$

$$a_1 = 2A_1 + 5A_2 + 10$$

Since $a_1 = 2$ $\quad 2A_1 + 5A_2 + 10 = 2 \Rightarrow \boxed{2A_1 + 5A_2 = -8}$