* Mathematical Induction

Mathematical Induction is a method of mathematical proof typically used to establish a given statement for all natural numbers.

It is a form of direct proof, and it is done in two steps.

① Base Case - to prove the given statement for the first natural ~~step~~. number.

② Inductive Step - prove that the given statement for any one natural no. implies the give statement for the next natural no.

Ex. Dominoes.

Steps: ① Show that $P(1)$ is true
let $n=1$ and work it out

② Assume $P(k)$ is true
put $n=k$ for all values of n

③ Show that $P(k) \rightarrow P(k+1)$
use $P(k)$ to show that $P(k+1)$ is true

④ End of proof.
" Thus $P(n)$ is true "

Ex. ① prove $P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$\Rightarrow$

Step ① Show that $P(1)$ is true
$\therefore$ put $n=1$

$$P(1) \quad 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

So $P(1)$ is true

Step ② Assume $P(k)$ is true

// put $n = k$

$P(k): 1+2+3+\cdots+k = \frac{k(k+1)}{2}$ is true

Step ③ Show $P(k) \rightarrow P(k+1)$

put $n = k+1$

Goal

$P(k+1): 1+2+3+\cdots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2}$

Now

$\therefore P(k+1): 1+2+3+\cdots+k+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)[(k+1)+1]}{2} = RHS$

So $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true

Ex ② Prove

$P(n): 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$\Rightarrow$

① Show $P(1) =$ true

put $n = 1$

$P(1): 1^2 = \frac{1(1+1)(2(1)+1)}{6}$

$= \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6}$

$= 1$

So $P(1)$ is true

② Assume $P(k)$ is true

put $n=k$

$\therefore P(k): 1^2+2^2+3^2+\cdots+k^2 = \dfrac{k(k+1)(2k+1)}{6}$

$P(k)$ is true

③ Show $P(k) \rightarrow P(k+1)$ is true

put $n=k+1$

Goal:

$$P(k+1): 1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Now,

$$P(k+1): 1^2+2^2+3^2+\cdots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = RHS$$

So $P(k) \rightarrow P(k+1)$

Thus $P(n)$ is true

Ex ③ Prove

$$P(n) : \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

① Show that $P(1)$ is true

put $n=1$

$$P(1) : \left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$$

So $P(1)$ is true

② Assume $P(k)$ is true

put $n=k$

$\therefore P(k) : \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$ is true

③ Show $P(k) \rightarrow P(k+1)$ is true

put $n=k+1$

Goal

$\therefore P(k+1) : \left(\frac{a}{b}\right)^{k+1} = \frac{a^{k+1}}{b^{k+1}}$

Now

$$P(k+1) : \left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k \left(\frac{a}{b}\right)^1$$

$$= \frac{a^k}{b^k} \cdot \frac{a^1}{b^1}$$

$$= \frac{a^k \cdot a^1}{b^k \cdot b^1}$$

$$= \frac{a^{k+1}}{b^{k+1}} = RHS$$

So, $P(k) \rightarrow P(k+1)$

Thus $P(n)$ is true

Ex. 4 Prove that

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

$\Rightarrow$

① Show $P(1)$ is true

put $n=1$

$$\therefore P(1): \frac{1}{[3(1)-2][3(1)+1]}=\frac{1}{3(1)+1}$$

$$\frac{1}{(1)(4)}=\frac{1}{4}$$

$$=\frac{1}{4}$$

So $P(1)$ is true

② Show $P(k)$ is true

put $n=k$

$$P(k): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$

So $P(k)$ is true

③ Show that $P(k) \rightarrow P(k+1)$ is true

put $n=k+1$

Goal:

$$P(k+1): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Now

$$P(k+1): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$= \frac{k}{3k+1} + \frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)}{(3k+1)(3k+4)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)}{(3k+4)}$$

$$= \frac{(k+1)}{3(k+1)+1}$$

$= RHS$

So $P(k+1)$ is true

Thus $P(n)$ is true

Ex. 5 Prove by mathematical induction $8^n - 3^n$ is multiple of 5 for $n \geq 1$

$\Rightarrow$

$P(n) : 8^n - 3^n$

1. Show that $P(1)$ is true

put $n = 1$

$\therefore P(1) : 8^1 - 3^1 = 8 - 3 = 5$

$\therefore$ so $P(1)$ is true.

2. show that $P(k)$ is true

put $n = k$

$\therefore P(k) : 8^k - 3^k$ is true

3. Show that $P(k) \rightarrow P(k+1)$ is true

put $n = (k+1)$.

Goal

$P(k+1) : 8^{k+1} - 3^{k+1}$ is true

$$\therefore P(k+1) = 8^{k+1} - 3^{k+1}$$
$$= 8^k \cdot 8^1 - 3^k \cdot 3$$
$$= 8^k (5+3) - 3^k \cdot 3$$
$$= 8^k \cdot 5 + 8^k \cdot 3 - 3^k \cdot 3$$
$$= 8^k \cdot 5 + 3(8^k - 3^k)$$

$\therefore P(k+1) = 5 \cdot 8^k + 3 \cdot (8^k - 3^k)$ is true

thus $P(n)$ is true

—x—

Ex. 6 Prove that $5^n - 1$ is divisible by 4 for $n \geq 1$

$\Rightarrow$

$P(n) : 5^n - 1$

1. show that $P(1)$ is true

put $n = 1$

$P(1) : 5^1 - 1 = 5 - 1 = 4$ divisible by 4

Thus $P(1)$ is true

② Show that $P(k)$ is true
put $n = k$

$\therefore P(k) : 5^k - 1$ is true i.e. divisible by 4

③ Show that $P(k) \to P(k+1)$
put $n = k+1$

$\therefore P(k+1) : 5^{k+1} - 1$

$$= 5^{k+1} - 1$$
$$= 5^k \cdot 5^1 - 1$$
$$= 5^k \cdot 5^1 - \underline{5 + 4}$$
$$= 5(5^k - 1) + 4$$ is true

i.e. divisible by 4

Thus $P(n)$ i.e. $5^n - 1$ is true

Ex. ⑦ Show that $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

$\Rightarrow$ For $n = 1$

$\therefore 1 + 2^1 = 3 = 2^{1+1} - 1 =$

For $n = k$

$$1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1$$

Now

For $n = k+1$

$P(k+1) : 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$\therefore$

$$P(k+1) = 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1}$$
$$= (2^{k+1} - 1) + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

prove

Ex. ⑧ $\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Ex. ⑧ $\frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

$\longrightarrow$

put $n=1$ $\quad \frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{(1)(3)}=\frac{1}{2(1)+1}$

put $n=k$

$P(k): \frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

put $n=k+1$

Goal

$P(k+1): \frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2k-1)(2k+1)}+$

$\cdots+\frac{1}{[2(k+1)-1][2(k+1)+1]}=\frac{(k+1)}{2(k+1)+1}$

Now,

$P(k+1): \frac{1}{1(3)}+\frac{1}{3(5)}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{[2(k+1)-1][2(k+1)+1]}$

$=\frac{k}{2k+1}+\frac{1}{[2(k+1)-1][2(k+1)+1]}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{1}{2k+1}\left[k+\frac{1}{2k+3}\right]$

$=\frac{1}{2k+1}\left[\frac{2k^2+3k+1}{2k+3}\right]=\frac{1}{\cancel{2k+1}}\Big[\quad = \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$

$$= \frac{(k+1)}{2(k+1)+1}$$

Ex. 51 $P(n) = 1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$

$\rightarrow$ put $n=1$

$$3(1)-2 = 1 = \frac{1(3(1)-1)}{2}$$

put $n=k$

$$P(k) = 1+4+7+\cdots+(3k-2) = \frac{k(3k-1)}{2}$$

put $n=k+1$

$$P(k+1) = 1+4+7+\cdots+(3k-2)+(3(k+1)-2)$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Now,

$$P(k+1) = 1+4+7+\cdots+(3k-2)+(3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3k+1)$$

$$= \frac{k(3k-1)+2(3k+1)}{2}$$

$$= \frac{3k^2-k+6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Ex (10) For $n \geq 1$

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

$\Rightarrow$ For $n = 1$

$P(1) : 2 + 2^2 = 6 = 2^{1+1} - 2$ is true

for $n = k$

$P(k) : 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^k = 2^{k+1} - 2$

is true

Now for $n = k+1$

$$P(k+1) : \underbrace{2 + 2^2 + 2^3 + 2^4 + \cdots + 2^k}_{P(k)} + 2^{k+1} = 2^{(k+1)+1} - 2$$

$$2^{k+1} - 2 + 2^{k+1} = 2^{k+2} - 2$$

$$2^1 \cdot 2^{k+1} - 2 = 2^{k+2} - 2$$

$$2^{k+2} - 2 = 2^{k+2} - 2$$

$\therefore$ $P(k+1)$ is true

Ex (11) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

$\rightarrow$ For $n = 1$ $1 = 1^2$ is true

for $n = k$

$P(k) : 1 + 3 + 5 + \cdots + (2k-1) = k^2$

is true

For $n = k+1$

$$P(k+1) : \underbrace{1 + 3 + 5 + \cdots + (2k-1)}_{P(k)} + (2(k+1)-1) = (k+1)^2$$

$=$

$= k^2 + (2(k+1) - 1)$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= (k+1)^2$ is true

① prove that 2 divides n^2+n whenever n is +ve int.

→ $P(n): n^2+n$

$P(1): 1^2+1=2$ divisible by 2 ∴ P(1) is True

$P(k): k^2+k$ is true

$P(k+1): (k+1)^2+(k+1)$

$= k^2+2k+1+k+1$

$= (k^2+k)+(2k+2)$

$= \underbrace{(k^2+k)}_{P(k)}+2(k+1)$

∴ P(k+1) is true

② prove that n^3+2n is divisible by 3 whenever n is +ve int

→

$P(n): n^3+2n$

$P(1): 1^3+2(1) = 1+2=3$ is divisible by 3

∴ P(1) is true

$P(k): k^3+2k$ ∴ P(k) is true)

$P(k+1): (k+1)^3+2(k+1)$

$= (k+1)(k+1)^2+2(k+1)$

$= (k+1)(k^2+2k+1)+2k+2$

$= k^3+2k^2+k+k^2+2k+1+2k+2$

$= (k^3+2k)+3k^2+3k+3$

$= (k^3+2k)+3(k^2+k+1)$

∴ P(k+1) is true

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③ Use mathematical induction to show $1+5+9+\cdots+(4n-3) = n(2n-1)$, $\forall n \geq 1$, $n \in Z$

→ $P(n): 1+5+9+\cdots+(4n-3) = n(2n-1)$

① Show P(1) is True, put n=1

$P(1): 1 = 1\cdot(2\cdot1-1) = 1$

∴ P(1) is True

② Show P(k) is True, put n=k

$P(k): 1+5+9+\cdots+(4k-3) = k\cdot(2k-1)$

∴ P(k) is True

③ Show P(k) → P(k+1) is True

put n=k+1

∴ $P(k+1): 1+5+9+\cdots+(4k-3)+(4(k+1)-3) = (k+1)(2(k+1)-1)$

Now

$P(k+1): P(k)+(4(k+1)-3)$

$= P(k)+(4k+1)$

$= k(2k-1)+(4k+1)$

$= 2k^2-k+4k+1$

$= 2k^2+3k+1$

$= (k+1)(2k+1)$

$= RHS$

∴ P(k+1) is True