1. 不定積分

(a) $\displaystyle\int \cos(5x-7)dx = \frac{1}{5}\sin(5x-7)+C$

(b) $\displaystyle\int \frac{dx}{6+x^2}$

$x=\sqrt{6}\tan t$ と置くと積分する式は

$$\frac{1}{6+x^2} = \frac{1}{6(1+\tan^2 t)} = \frac{\cos^2 t}{6} \tag{1}$$

である。また、$\frac{dx}{dt} = \frac{\sqrt{6}}{\cos^2 t}$ より $dx = \frac{\sqrt{6}}{\cos^2 t}dt$ であるので、

$$\int \frac{dx}{6+x^2} = \int \frac{\cos^2 t}{6}\cdot\frac{\sqrt{6}}{\cos^2 t}dt \tag{2}$$

$$= \int \frac{\sqrt{6}}{6}dt = \frac{1}{\sqrt{6}}t + C \tag{3}$$

$x=\sqrt{6}\tan t$ から $t=\arctan\frac{x}{\sqrt{6}}$ であるので

$$\int \frac{dx}{6+x^2} = \frac{1}{\sqrt{6}}\arctan\frac{x}{\sqrt{6}} + C \tag{4}$$

(c) $\displaystyle\int \frac{3\log x+2}{x}dx$

$t=3\log x+2$ と置くと $dt=\frac{3}{x}dx$ であるので、

$$\int \frac{3\log x+2}{x}dx = \int \frac{t}{3}dt = \frac{t^2}{6}+C = \frac{1}{6}(3\log x+2)^2+C \tag{5}$$

(d) $\displaystyle\int (x^2+2)e^x dx$

部分積分を利用し、

$$\int (x^2+2)e^x dx = (x^2+2)e^x - \int 2xe^x dx \tag{6}$$

$$= (x^2+2)e^x - \left(2xe^x - \int 2e^x dx\right) \tag{7}$$

$$= (x^2+2)e^x - (2xe^x - 2e^x) + C \tag{8}$$

$$= e^x(x^2-2x+4)+C \tag{9}$$

(e) $\displaystyle\int \frac{2-9x}{(x+4)(x^2+3)}dx$

被積分関数を 3 つの分数に分ける。

$$\frac{2-9x}{(x+4)(x^2+3)} = \frac{2}{x+4} - \frac{2x}{x^2+3} - \frac{1}{x^2+3} \tag{10}$$

それぞれ別々に不定積分を行う。

$$\int \frac{2}{x+4}dx = 2\log(x+4) + C \tag{11}$$

$$\int \frac{2x}{x^2+3}dx = \log(x^2+3) + C \tag{12}$$

$\int \frac{1}{x^2+3}dx$ は $x = \sqrt{3}\tan t$ と置き積分を行う。

$$\int \frac{1}{x^2+3}dx = \int \frac{\cos^2 t}{3} \cdot \frac{\sqrt{3}}{\cos^2 t}dt \tag{13}$$

$$= \frac{1}{\sqrt{3}}t + C = \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C \tag{14}$$

これらを合わせると

$$\int \frac{2-9x}{(x+4)(x^2+3)}dx = \log\frac{(x+4)^2}{x^2+3} - \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} \tag{15}$$

2. $\displaystyle\int_{-\infty}^{-2} \frac{dx}{\sqrt[5]{x^6}}$

$x = -\frac{1}{t}$ と置くと

$x:$	$-\infty$	$\to$	-2
$t:$	0	$\to$	$1/2$

となるので、

$$\int_{-\infty}^{-2} \frac{dx}{\sqrt[5]{x^6}} = \int_0^{\frac{1}{2}} t^{\frac{6}{5}} \cdot t^{-2}dt = \left[5t^{\frac{1}{5}}\right]_0^{\frac{1}{2}} = \frac{5}{\sqrt[5]{2}} \tag{16}$$

3. $\displaystyle\lim_{x\to 1}\left(\sin\frac{\pi}{2}x\right)^{\frac{1}{x-1}}$

$$t = x-1\text{の時,}\quad \left(\sin\frac{\pi}{2}x\right)^{\frac{1}{x-1}} = \left(\sin\frac{\pi}{2}(t+1)\right)^{\frac{1}{t}} = \left(\cos\frac{\pi}{2}t\right)^{\frac{1}{t}} \tag{17}$$

$-1 \leq t \leq 1$ において

$$0 \leq \cos\frac{\pi}{2}t \leq 1 \tag{18}$$

であるから、

$$\lim_{t\to 0} 0^{\frac{1}{t}} \leq \lim_{t\to 0}\left(\cos\frac{\pi}{2}t\right)^{\frac{1}{t}} \leq \lim_{t\to 0} 1^{\frac{1}{t}} \tag{19}$$

$\lim_{t\to 0} 0^{\frac{1}{t}} = 0,\ \lim_{t\to 0} 1^{\frac{1}{t}} = 1$ より

$$0 \le \lim_{t\to 0}\left(\cos\frac{\pi}{2}t\right)^{\frac{1}{t}} \le 1 \tag{20}$$

4. $z = \log(3x+4y)$ の第 2 次偏導関数を求めよ。
   1 階偏導関数は次の 2 つ。

$$\frac{\partial z}{\partial x} = \frac{3}{3x+4y}, \quad \frac{\partial z}{\partial y} = \frac{4}{3x+4y} \tag{21}$$

   これより 2 階偏導関数は次の 3 つ。

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}z = -\frac{9}{(3x+4y)^2}, \quad \frac{\partial}{\partial x}\frac{\partial}{\partial y}z = -\frac{12}{(3x+4y)^2}, \quad \frac{\partial}{\partial y}\frac{\partial}{\partial y}z = -\frac{16}{(3x+4y)^2} \tag{22}$$

5. $z = x^2y^3, x = \sin uv, y = \cos(u+v)$ の時、$\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$ を求めよ。
   媒介変数の偏微分は次の式を満たす。

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \tag{23}$$

   そこで、それぞれを求めると

$$\frac{\partial z}{\partial x} = 2xy^3 \qquad \frac{\partial z}{\partial y} = 3x^2y^2 \qquad \frac{\partial x}{\partial u} = v\cos uv \tag{24}$$

$$\frac{\partial x}{\partial v} = u\cos uv \qquad \frac{\partial y}{\partial u} = -\sin(u+v) \qquad \frac{\partial y}{\partial v} = -\sin(u+v) \tag{25}$$

   これらを用いると

$$\frac{\partial z}{\partial u} = 2xy^3 \cdot v\cos uv - 3x^2y^2 \cdot \sin(u+v) \tag{26}$$

$$\frac{\partial z}{\partial v} = 2xy^3 \cdot u\cos uv - 3x^2y^2 \cdot \sin(u+v) \tag{27}$$

6.

$$\iint_D (4xy - y^3)dxdy, \quad D: 0 \le x \le 1, 0 \le y \le 2x \tag{28}$$

$$\iint_D (4xy - y^3)dxdy = \int_0^1 \int_0^{2x} (4xy - y^2)dydx \tag{29}$$

$$= \int_0^1 \left[2xy^2 - \frac{1}{3}y^3\right]_{y=0}^{y=2x} dx \tag{30}$$

$$= \int_0^1 \left(8x^3 - \frac{8}{3}x^3\right) dx \tag{31}$$

$$= \frac{4}{3}\left[x^4\right]_{x=0}^{x=1} = \frac{4}{3} \tag{32}$$