ベクトル空間 V 上の線形変換全体の集合を $\mathrm{End}(V)$ と表す。$\mathrm{End}(V) = \mathrm{Hom}(V, V)$ である。

$$f \in \mathrm{End}(V) \overset{\text{def}}{\iff} f : V \to V \tag{1}$$

....

遷移行列

$$T^{(N)}(u) := \mathrm{tr}_{V^{(0)}} \left(R^{0N}(u) R^{0N-1}(u) \cdots R^{02}(u) R^{01}(u) \right) \in \mathrm{End}\left(\bigotimes_{i=1}^{N} V^{(i)} \right) \tag{2}$$

$$T^{(N)}(u) = \left(T^{(N)}(u)^{j_1 \cdots j_N}_{j'_1 \cdots j'_N} \right)^{j_1 \cdots j_N}_{j'_1 \cdots j'_N} \tag{3}$$

$$T^{(N)}(u)^{j_1 \cdots j_N}_{j'_1 \cdots j'_N} := \sum_{i_0, \ldots, i_{N-1}} R^{i_0 j_N}_{i_{N-1} j'_N}(u) R^{i_{N-1} j_{N-1}}_{i_{N-2} j'_{N-1}}(u) \ldots R^{i_2 j_2}_{i_1 j'_2}(u) R^{i_1 j_1}_{i_0 j'_1}(u) \tag{4}$$

....

YBE Yang-Baxter 方程式

$R(u) \in \mathrm{End}(V \otimes V)$ とする。

$R(u) e_i \otimes e_j = \sum_{i'j'} e_{i_{i'}} \otimes e_{j'} R^{i'j'}_{ij}(u)$ として $R(u) = \left(R^{i'j'}_{ij}(u) \right)_{\substack{i,j=0,\ldots,N-1 \\ i',j'=0,\ldots,N-1}}$

YBE は $\mathrm{End}(V \otimes V \otimes V)$ 上の

$$R^{01}(u) R^{02}(u+v) R^{12}(v) = R^{12}(v) R^{02}(u+v) R^{01}(u) \in \mathrm{End}(\overset{0}{V} \otimes \overset{1}{V} \otimes \overset{2}{V}) \tag{5}$$

R^{01} は $\overset{0}{V} \otimes \overset{1}{V} \otimes \overset{2}{V}$ の $\overset{0}{V} \otimes \overset{1}{V}$ 上に

$$R^{01}(u) e_i \otimes e_j \otimes e_k = \sum_{i,j} e_{i'} \otimes e_{j'} \otimes e_k R^{i'j'}_{ij}(u) \tag{6}$$

....

$R(u) = u\mathrm{Id} + P$ とする。YBE $(R^{01}(u)R^{02}(u+v)R^{12}(v) = R^{12}(v)R^{02}(u+v)R^{01}(u))$ は次のように表される。

$$\begin{aligned} &(u\mathrm{Id}^{01} + P^{01})((u+v)\mathrm{Id}^{02} + P^{02})(v\mathrm{Id}^{12} + P^{12}) \\ &\qquad = (v\mathrm{Id}^{12} + P^{12})((u+v)\mathrm{Id}^{02} + P^{02})(u\mathrm{Id}^{01} + P^{01}) \end{aligned} \tag{7}$$

$$R^{01}(u) = u \cdot \mathrm{Id}^{01} + P^{01} \tag{8}$$

$$R^{02}(u+v) = (u+v) \cdot \mathrm{Id}^{02} + P^{02} \tag{9}$$

$$R^{12}(v) = v \cdot \mathrm{Id}^{12} + P^{12} \tag{10}$$

....

$\dim V = 2$ の時、Yang-Baxter 方程式 の McGuire-Yang の解

$$R(u) = u \cdot \mathrm{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2} + P = (R_{ab,cd}(u))_{ab,cd} = \left(R^{ab}_{cd}(u) \right)^{ab}_{cd} \tag{11}$$

$$= \begin{pmatrix} R^{00}_{00}(u) & R^{00}_{01}(u) & \cdots & R^{00}_{11}(u) \\ R^{01}_{00}(u) & R^{01}_{01}(u) & \cdots & R^{01}_{11}(u) \\ \vdots & \vdots & \ddots & \vdots \\ R^{11}_{00}(u) & R^{11}_{01}(u) & \cdots & R^{11}_{11}(u) \end{pmatrix} = \begin{pmatrix} u+1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & u+1 \end{pmatrix} \tag{12}$$

$$R_{00}^{00}(u) = R_{11}^{11}(u) = u + 1 \tag{13}$$
$$R_{01}^{01}(u) = R_{10}^{10}(u) = u \tag{14}$$
$$R_{10}^{01}(u) = R_{01}^{10}(u) = 1 \tag{15}$$
$$R_{cd}^{ab}(u) = 0 \ (a + b \neq c + d) \tag{16}$$

$V = \mathbb{C}^2 = \langle e_1, e_2 \rangle$
以下を行列表示せよ。

1. $T^{(1)}(u) \in \mathrm{End}\left(V^{(1)}\right)$

. . . .

$$T^{(1)}(u) = \left(T^{(1)}(u)_{j_1'}^{j_1}\right)_{j_1'}^{j_1} \tag{17}$$
$$T^{(1)}(u)_{j_1'}^{j_1} = \sum_{i_0} R_{i_0 j_1'}^{i_0 j_1}(u) \tag{18}$$
$$T^{(1)}(u)_1^1 = R_{11}^{11}(u) + R_{21}^{21}(u) = u + 1 + u = 2u + 1 \tag{19}$$
$$T^{(1)}(u)_2^1 = R_{12}^{11}(u) + R_{22}^{21}(u) = 0 + 0 = 0 \tag{20}$$
$$T^{(1)}(u)_1^2 = R_{11}^{12}(u) + R_{21}^{22}(u) = 0 + 0 = 0 \tag{21}$$
$$T^{(1)}(u)_2^2 = R_{12}^{12}(u) + R_{22}^{22}(u) = u + u + 1 = 2u + 1 \tag{22}$$

$$T^{(1)}(u) = \begin{pmatrix} 2u+1 & 0 \\ 0 & 2u+1 \end{pmatrix} \tag{23}$$

---

2. $T^{(2)}(u) \in \mathrm{End}\left(V^{(1)} \otimes V^{(2)}\right)$

. . . .

$$T^{(2)}(u) = \left(T^{(2)}(u)_{j_1' j_2'}^{j_1 j_2}\right)_{j_1' j_2'}^{j_1 j_2}, \quad T^{(2)}(u)_{j_1' j_2'}^{j_1 j_2} = \sum_{i_0, i_1} R_{i_1 j_2'}^{i_0 j_2}(u) R_{i_0 j_1'}^{i_1 j_1}(u) \tag{24}$$

$$\begin{aligned}
T^{(2)}(u)_{11}^{11} &= \sum_{i_0 i_1} R_{i_1 1}^{i_0 1}(u) R_{i_0 1}^{i_1 1}(u) && (25) \\
&= R_{11}^{11}(u) R_{11}^{11}(u) + R_{11}^{21}(u) R_{21}^{11}(u) + R_{21}^{11}(u) R_{11}^{21}(u) + R_{21}^{21}(u) R_{21}^{21}(u) \\
&= (u+1)^2 + 0 + 0 + u^2 = 2u^2 + 2u + 1 && (26) \\
T^{(2)}(u)_{11}^{12} &= \sum_{i_0 i_1} R_{i_1 1}^{i_0 2}(u) R_{i_0 1}^{i_1 1}(u) && (27) \\
&= R_{11}^{12}(u) R_{11}^{11}(u) + R_{11}^{22}(u) R_{21}^{11}(u) + R_{21}^{12}(u) R_{11}^{21}(u) + R_{21}^{22}(u) R_{21}^{21}(u) \\
&= 0 + 0 + 0 + 0 = 0 && (28) \\
T^{(2)}(u)_{11}^{21} &= T^{(2)}(u)_{12}^{11} = T^{(2)}(u)_{21}^{11} = 0 && (29)
\end{aligned}$$

$$T^{(2)}(u)^{22}_{11} = \sum_{i_0 i_1} R^{i_0 2}_{i_1 1}(u) R^{i_1 2}_{i_0 1}(u) \tag{30}$$
$$= R^{12}_{11}(u)R^{12}_{11}(u) + R^{22}_{11}(u)R^{12}_{21}(u) + R^{12}_{21}(u)R^{22}_{11}(u) + R^{22}_{21}(u)R^{22}_{21}(u)$$
$$= 0 + 0 + 0 + 0 = 0 \tag{31}$$
$$T^{(2)}(u)^{11}_{22} = 0 \tag{32}$$
$$T^{(2)}(u)^{12}_{12} = \sum_{i_0 i_1} R^{i_0 2}_{i_1 2}(u) R^{i_1 1}_{i_0 1}(u) \tag{33}$$
$$= R^{12}_{12}(u)R^{11}_{11}(u) + R^{22}_{12}(u)R^{11}_{21}(u) + R^{12}_{22}(u)R^{21}_{11}(u) + R^{22}_{22}(u)R^{21}_{21}(u)$$
$$= u(u+1) + 0 + 0 + (u+1)u = 2u^2 + 2u \tag{34}$$
$$T^{(2)}(u)^{21}_{21} = 2u^2 + 2u \tag{35}$$
$$T^{(2)}(u)^{12}_{21} = \sum_{i_0 i_1} R^{i_0 2}_{i_1 1}(u) R^{i_1 1}_{i_0 2}(u) \tag{36}$$
$$= R^{12}_{11}(u)R^{11}_{12}(u) + R^{22}_{11}(u)R^{11}_{22}(u) + R^{12}_{21}(u)R^{21}_{12}(u) + R^{22}_{21}(u)R^{21}_{22}(u)$$
$$= 0 + 0 + 1 \cdot 1 + 0 = 1 \tag{37}$$
$$T^{(2)}(u)^{21}_{12} = 1 \tag{38}$$

$$T^{(2)}(u)^{21}_{22} = T^{(2)}(u)^{12}_{22} = T^{(2)}(u)^{22}_{21} = T^{(2)}(u)^{22}_{12} = 0 \tag{39}$$
$$T^{(2)}(u)^{22}_{22} = \sum_{i_0 i_1} R^{i_0 2}_{i_1 2}(u) R^{i_1 2}_{i_0 2}(u) \tag{40}$$
$$= R^{12}_{12}(u)R^{12}_{12}(u) + R^{22}_{12}(u)R^{12}_{22}(u) + R^{12}_{22}(u)R^{22}_{12}(u) + R^{22}_{22}(u)R^{22}_{22}(u)$$
$$= u^2 + 0 + 0 + (u+1)^2 = 2u^2 + 2u + 1 \tag{41}$$

$$T^{(2)}(u) = \begin{pmatrix} 2u^2+2u+1 & 0 & 0 & 0 \\ 0 & 2u^2+2u & 1 & 0 \\ 0 & 1 & 2u^2+2u & 0 \\ 0 & 0 & 0 & 2u^2+2u+1 \end{pmatrix} \tag{42}$$

---

3. $T^{(3)}(u) \in \mathrm{End}\left(V^{(1)} \otimes V^{(2)} \otimes V^{(3)}\right)$

. . . .

$$T^{(3)}(u) = \left(T^{(3)}(u)^{j_1 j_2 j_3}_{j'_1 j'_2 j'_3}\right)^{j_1 j_2 j_3}_{j'_1 j'_2 j'_3} \tag{43}$$

$$T^{(3)}(u)^{j_1 j_2 j_3}_{j'_1 j'_2 j'_3} = \sum_{i_0, i_1, i_2} R^{i_0 j_3}_{i_2 j'_3}(u) R^{i_2 j_2}_{i_1 j'_2}(u) R^{i_1 j_1}_{i_0 j'_1}(u) \tag{44}$$
$$= R^{1 j_3}_{1 j'_3}(u) R^{1 j_2}_{1 j'_2}(u) R^{1 j_1}_{1 j'_1}(u) + R^{1 j_3}_{2 j'_3}(u) R^{2 j_2}_{1 j'_2}(u) R^{1 j_1}_{1 j'_1}(u) \tag{45}$$
$$+ R^{1 j_3}_{1 j'_3}(u) R^{1 j_2}_{2 j'_2}(u) R^{2 j_1}_{1 j'_1}(u) + R^{1 j_3}_{2 j'_3}(u) R^{2 j_2}_{2 j'_2}(u) R^{2 j_1}_{1 j'_1}(u) \tag{46}$$
$$+ R^{2 j_3}_{1 j'_3}(u) R^{1 j_2}_{1 j'_2}(u) R^{1 j_1}_{2 j'_1}(u) + R^{2 j_3}_{2 j'_3}(u) R^{2 j_2}_{1 j'_2}(u) R^{1 j_1}_{2 j'_1}(u) \tag{47}$$
$$+ R^{2 j_3}_{1 j'_3}(u) R^{1 j_2}_{2 j'_2}(u) R^{2 j_1}_{2 j'_1}(u) + R^{2 j_3}_{2 j'_3}(u) R^{2 j_2}_{2 j'_2}(u) R^{2 j_1}_{2 j'_1}(u) \tag{48}$$

$$
\begin{align}
T^{(3)}(u)^{111}_{111} &= T^{(3)}(u)^{222}_{222} = (u+1)^3 + u^3 \tag{49}\\
T^{(3)}(u)^{112}_{112} &= T^{(3)}(u)^{121}_{121} = T^{(3)}(u)^{211}_{211} = u(u+1)(2u+1) \tag{50}\\
T^{(3)}(u)^{122}_{122} &= T^{(3)}(u)^{212}_{212} = T^{(3)}(u)^{221}_{221} = u(u+1)(2u+1) \tag{51}\\
T^{(3)}(u)^{112}_{121} &= T^{(3)}(u)^{121}_{211} = T^{(3)}(u)^{122}_{221} = T^{(3)}(u)^{211}_{112} \tag{52}\\
&= T^{(3)}(u)^{212}_{122} = T^{(3)}(u)^{221}_{212} = u+1 \tag{53}\\
T^{(3)}(u)^{212}_{221} &= T^{(3)}(u)^{122}_{212} = T^{(3)}(u)^{112}_{211} = T^{(3)}(u)^{221}_{122} \tag{54}\\
&= T^{(3)}(u)^{211}_{121} = T^{(3)}(u)^{121}_{112} = u \tag{55}
\end{align}
$$

$$
\begin{align}
T^{(3)}(u) &= \begin{pmatrix}
T^{111}_{111} & T^{111}_{112} & T^{111}_{121} & T^{111}_{122} & T^{111}_{211} & T^{111}_{212} & T^{111}_{221} & T^{111}_{222}\\
T^{112}_{111} & T^{112}_{112} & & & & & & \vdots\\
\vdots & & \ddots & & & & & \vdots\\
T^{222}_{111} & \cdots & & & & & & T^{222}_{222}
\end{pmatrix} \tag{56}\\
&= \begin{pmatrix}
T^{111}_{111} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & T^{112}_{112} & T^{112}_{121} & 0 & T^{112}_{211} & 0 & 0 & 0\\
0 & T^{121}_{112} & T^{121}_{121} & 0 & T^{121}_{211} & 0 & 0 & 0\\
0 & 0 & 0 & T^{122}_{122} & 0 & T^{122}_{212} & T^{122}_{221} & 0\\
0 & T^{211}_{112} & T^{211}_{121} & 0 & T^{211}_{211} & 0 & 0 & 0\\
0 & 0 & 0 & T^{212}_{122} & 0 & T^{212}_{212} & T^{212}_{221} & 0\\
0 & 0 & 0 & T^{221}_{122} & 0 & T^{221}_{212} & T^{221}_{221} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & T^{222}_{222}
\end{pmatrix} \tag{57}\\
&= \begin{pmatrix}
(u+1)^3+u^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha & u+1 & 0 & u & 0 & 0 & 0\\
0 & u & \alpha & 0 & u+1 & 0 & 0 & 0\\
0 & 0 & 0 & \alpha & 0 & u & u+1 & 0\\
0 & u+1 & u & 0 & \alpha & 0 & 0 & 0\\
0 & 0 & 0 & u+1 & 0 & \alpha & u & 0\\
0 & 0 & 0 & u & 0 & u+1 & \alpha & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (u+1)^3+u^3
\end{pmatrix} \tag{58}
\end{align}
$$

スペースの関係で $\alpha = u(u+1)(2u+1)$ と置いている。
