
$N = 2$ のとき、 $g, h \in \text{End}(\mathbb{C}^2)$ は $g^2 = h^2 = \text{Id}_{\mathbb{C}^2}$ ($g^{-1} = g, h^{-1} = h$)

1. $R_{8v}(0) = P$ 、 $P(x \otimes y) = y \otimes x$ for any $x, y \in \mathbb{C}^N$
2. $(g \otimes g)^{-1} R_{8v}(u) (g \otimes g) = (h \otimes h)^{-1} R_{8v}(u) (h \otimes h) = R_{8v}(u)$
3. $R_{8v}(u + 1) = e[-1](g^{-1} \otimes 1) R_{8v}(u) (g \otimes 1)$
4. $R_{8v}(u + \tau) = e[-\tau - 2(u + \frac{\eta}{2} + \frac{1}{2})](h \otimes 1) R_{8v}(u) (h^{-1} \otimes 1)$
5. $R_{8v}(u)$ の各成分は u の正則関数である。

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$$R_{8v}(u) = \frac{1}{2} \sum_{\alpha, \beta=0}^1 w_{\alpha, \beta}(u \mid \tau) h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \quad (1)$$

$$w_{\alpha, \beta} = \frac{\vartheta \left[\begin{smallmatrix} 1/2 + \alpha/2 \\ 1/2 + \beta/2 \end{smallmatrix} \right] (u + \frac{\eta}{2} \mid \tau)}{\vartheta \left[\begin{smallmatrix} 1/2 + \alpha/2 \\ 1/2 + \beta/2 \end{smallmatrix} \right] (\frac{\eta}{2} \mid \tau)} \quad (2)$$

$R_{8v}(u)$ が unitary 関係式

$$P R_{8v}(u) P R_{8v}(-u) = \rho(u) \text{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2} \quad (3)$$

を満たすことを示せ。

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$P, R_{8v} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ は次の様に表せる。

$$P = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \quad (4)$$

$$R_{8v}(u) = \begin{pmatrix} a_u & & & d_u \\ & b_u & c_u & \\ & c_u & b_u & \\ d_u & & & a_u \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \quad (5)$$

ただし、 a_u, b_u, c_u, d_u は次を表している。

$$a_u = a(u | \tau) = \frac{\theta_4(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} \quad (6)$$

$$b_u = b(u | \tau) = \frac{\theta_1(u | 2\tau)\theta_4(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} \quad (7)$$

$$c_u = c(u | \tau) = \frac{\theta_4(u | 2\tau)\theta_4(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (8)$$

$$d_u = d(u | \tau) = \frac{\theta_1(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (9)$$

$PR_{8v}(u)PR_{8v}(-u)$ を計算する。

$$PR_{8v}(u)PR_{8v}(-u) \quad (10)$$

$$= \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_u & & & d_u \\ & b_u & c_u & \\ & c_u & b_u & \\ d_u & & & a_u \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_{-u} & & & d_{-u} \\ & b_{-u} & c_{-u} & \\ & c_{-u} & b_{-u} & \\ d_{-u} & & & a_{-u} \end{pmatrix} \quad (11)$$

$$= \begin{pmatrix} a_u a_{-u} + d_u d_{-u} & & & a_u d_{-u} + a_{-u} d_u \\ & b_u b_{-u} + c_u c_{-u} & b_u c_{-u} + b_{-u} c_u & \\ & b_u c_{-u} + b_{-u} c_u & b_u b_{-u} + c_u c_{-u} & \\ a_u d_{-u} + a_{-u} d_u & & & a_u a_{-u} + d_u d_{-u} \end{pmatrix} \quad (12)$$

θ_1 は奇関数で、 $\theta_2, \theta_3, \theta_4$ は偶関数である。これを用いて $a_u d_{-u} + a_{-u} d_u$ は次の様に求まる。

$$a_u d_{-u} = \frac{\theta_4(u | 2\tau)\theta_1(u + \eta | 2\tau)\theta_1(-u | 2\tau)\theta_1(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (13)$$

$$= -\frac{\theta_4(u | 2\tau)\theta_1(u + \eta | 2\tau)\theta_1(u | 2\tau)\theta_1(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (14)$$

$$(15)$$

$$a_{-u} d_u = \frac{\theta_4(-u | 2\tau)\theta_1(-u + \eta | 2\tau)\theta_1(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (16)$$

$$= \frac{\theta_4(u | 2\tau)\theta_1(-u + \eta | 2\tau)\theta_1(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (17)$$

$$a_u d_{-u} + a_{-u} d_u = 0 \quad (18)$$

同様に $b_u c_{-u} + b_{-u} c_u = 0$ であることが求まる。

よって、次のような行列となる。

$$PR_{8v}(u)PR_{8v}(-u) \quad (19)$$

$$= \begin{pmatrix} a_u a_{-u} + d_u d_{-u} & & & \\ & b_u b_{-u} + c_u c_{-u} & & \\ & & b_u b_{-u} + c_u c_{-u} & \\ & & & a_u a_{-u} + d_u d_{-u} \end{pmatrix} \quad (20)$$

$a_u a_{-u} + d_u d_{-u}$ と $b_u b_{-u} + c_u c_{-u}$ は u の関数として偶関数である。その為、 $PR_{8v}(u)PR_{8v}(-u)$ は偶関数で表すことが出来る。

$$PR_{8v}(u)PR_{8v}(-u) = \rho(u)\text{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2} \quad (21)$$

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$$a_u a_{-u} + d_u d_{-u} \quad (22)$$

$$= \frac{\theta_4(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} \frac{\theta_4(-u | 2\tau)\theta_1(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} + \frac{\theta_1(u | 2\tau)\theta_1(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \frac{\theta_1(-u | 2\tau)\theta_1(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (23)$$

$$= \frac{\theta_1(u + \eta | 2\tau)\theta_1(-u + \eta | 2\tau)}{(\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(\eta | 2\tau))^2} ((\theta_4(u | 2\tau)\theta_4(\eta | 2\tau))^2 - (\theta_1(u | 2\tau)\theta_1(\eta | 2\tau))^2) \quad (24)$$

$$b_u b_{-u} + c_u c_{-u} \quad (25)$$

$$= \frac{\theta_1(u | 2\tau)\theta_4(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} \frac{\theta_1(-u | 2\tau)\theta_4(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)} + \frac{\theta_4(u | 2\tau)\theta_4(u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \frac{\theta_4(-u | 2\tau)\theta_4(-u + \eta | 2\tau)}{\theta_4(0 | 2\tau)\theta_4(\eta | 2\tau)} \quad (26)$$

$$= \frac{\theta_4(u + \eta | 2\tau)\theta_4(-u + \eta | 2\tau)}{(\theta_4(0 | 2\tau)\theta_1(\eta | 2\tau)\theta_4(\eta | 2\tau))^2} (-(\theta_1(u | 2\tau)\theta_4(\eta | 2\tau))^2 + (\theta_4(u | 2\tau)\theta_1(\eta | 2\tau))^2) \quad (27)$$

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$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u | \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}[(n + a)^2 + 2(n + a)(u + b)] \quad (28)$$

$$a, b \in \mathbb{R}, \quad u \in \mathbb{C}, \quad \tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im} z \geq 0\}, \quad \mathbf{e}[x] \stackrel{\text{def}}{=} \exp(i\pi x) \quad (29)$$

$$\theta_1(u \mid \tau) = -\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (u + \tau \mid \tau) \quad (30)$$

$$= -\mathbf{e}[-\tau - 2(u + 1/2)] \sum_{n \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + 2(n + 1/2)(u + 1/2)] \quad (31)$$

$$\theta_4(u \mid \tau) = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (u + \tau \mid \tau) \quad (32)$$

$$= \mathbf{e}[-\tau - 2(u + 1/2)] \sum_{n \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(u + \tau + 1/2)] \quad (33)$$

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$$\theta_1(\alpha \mid \tau)\theta_1(\beta \mid \tau) \quad (34)$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + 2(n + 1/2)(\alpha + 1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[(m + 1/2)^2 + 2(m + 1/2)(\beta + 1/2)] \quad (35)$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n, m \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + 2(n + 1/2)(\alpha + 1/2) + (m + 1/2)^2 + 2(m + 1/2)(\beta + 1/2)] \quad (36)$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n, m \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + (m + 1/2)^2 + 2(n + 1/2)(\alpha + 1/2) + 2(m + 1/2)(\beta + 1/2)] \quad (37)$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n, m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n + m^2 + 2m + 2n\alpha + 2m\beta + \alpha + \beta + 3/2] \quad (38)$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2) + \alpha + \beta + 3/2] \sum_{n, m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n + m^2 + 2m + 2n\alpha + 2m\beta] \quad (39)$$

$$\theta_1(\alpha \mid \tau)\theta_4(\beta \mid \tau) \quad (40)$$

$$= -\mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + 2(n + 1/2)(\alpha + 1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[m^2 + 2m(\beta + 1/2)] \quad (41)$$

$$= \sum_{n, m \in \mathbb{Z}} \mathbf{e}[(n + 1/2)^2 + 2(n + 1/2)(\alpha + 1/2) + m^2 + 2m(\beta + 1/2)] \quad (42)$$

$$= \sum_{n, m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n + m^2 + m + 2n\alpha + 2m\beta + \alpha + 3/4] \quad (43)$$

$$\theta_4(\alpha \mid \tau) \theta_4(\beta \mid \tau) \tag{44}$$

$$= \mathbf{e}[-2\tau - 4(u + 1/2)] \sum_{n \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(\alpha + 1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[m^2 + 2m(\beta + 1/2)] \tag{45}$$

$$= \sum_{n, m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(\alpha + 1/2) + m^2 + 2m(\beta + 1/2)] \tag{46}$$

$$= \sum_{n, m \in \mathbb{Z}} \mathbf{e}[n^2 + n + m^2 + m + 2n\alpha + 2m\beta] \tag{47}$$
