微分幾何

曲率と捩率

任意の媒介変数 t による空間曲線 $\boldsymbol{x}(t)=(e^t,e^{-t},\sqrt{2}t)$ の曲率と捩率を定義に従って求める。

まず、$\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=\boxed{1}$ である。よって、$\left|\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)\right|=e^t+\boxed{2}$ となるので、s を弧長パラメーターとすると、$\dfrac{\mathrm{d}s}{\mathrm{d}t}=e^t+\boxed{2}$ である。したがって、$\boldsymbol{e}_1=\dfrac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(t)=\dfrac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}\Big/\dfrac{\mathrm{d}s}{\mathrm{d}t}=\boxed{3}$ となる。次に、$\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_1=\dfrac{(2,2,\boxed{4})}{(e^t+e^{-t})^2}$ である。これにより、$\dfrac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1=\dfrac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\Big/\dfrac{\mathrm{d}s}{\mathrm{d}t}=\boxed{5}$ を得る。

よって、曲率 $\kappa=\left|\dfrac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1\right|=\dfrac{\boxed{6}}{(e^t+e^{-t})^2}$ となる。ここで、$\boldsymbol{e}_2=\dfrac{\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1}{\kappa}$ 、および $\boldsymbol{e}_3=\boldsymbol{e}_1\times\boldsymbol{e}_2$ とおくと $\boldsymbol{e}_3=\dfrac{(-e^{-t},\boxed{7},\sqrt{2})}{e^t+e^{-t}}$ である。さらに、$\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_3=\dfrac{\boxed{8}}{(e^t+e^{-t})^2}$ である。$\dfrac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_3=\dfrac{\mathrm{d}\boldsymbol{e}_3}{\mathrm{d}t}\Big/\dfrac{\mathrm{d}s}{\mathrm{d}t}=-\tau\boldsymbol{e}_2$ なる τ が捩率なので、$\tau=\boxed{9}$ となる。

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1. $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=\boxed{1}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=(e^t,-e^{-t},\sqrt{2}) \tag{1}$$

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2. $\left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)\right|=e^t+\boxed{2}$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)\right|=\sqrt{e^{2t}+e^{-2t}+2}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t} \tag{2}$$

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3. $\boldsymbol{e}_1=\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(t)=\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}\Big/\frac{\mathrm{d}s}{\mathrm{d}t}=\boxed{3}$

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}\Big/\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{e^t+e^{-t}}(e^t,-e^{-t},\sqrt{2}) \tag{3}$$

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4. $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_1=\frac{(2,2,\boxed{4})}{(e^t+e^{-t})^2}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_1=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{e^t+e^{-t}}(e^t,-e^{-t},\sqrt{2})\right)=\frac{1}{(e^t+e^{-t})^2}(2,2,\sqrt{2}(-e^t+e^{-t})) \tag{4}$$

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5. $\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1 = \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\Big/\frac{\mathrm{d}s}{\mathrm{d}t} = \boxed{5}$

$$\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\Big/\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{e^t+e^{-t}}\frac{1}{(e^t+e^{-t})^2}(2,2,\sqrt{2}(-e^t+e^{-t})) \tag{5}$$

$$= \frac{1}{(e^t+e^{-t})^3}(2,2,\sqrt{2}(-e^t+e^{-t})) \tag{6}$$

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6. $\kappa = \left|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1\right| = \frac{\boxed{6}}{(e^t+e^{-t})^2}$

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1\right| = \frac{1}{(e^t+e^{-t})^3}\sqrt{2^2+2^2+\sqrt{2}^2(-e^t+e^{-t})^2} = \frac{\sqrt{2}}{(e^t+e^{-t})^2} \tag{7}$$

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7. $\boldsymbol{e}_3 = \frac{(-e^{-t},\boxed{7},\sqrt{2})}{e^t+e^{-t}}$

$$\boldsymbol{e}_2 = \frac{\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{e}_1}{\kappa} = \frac{1}{e^t+e^{-t}}(\sqrt{2},\sqrt{2},(-e^t+e^{-t})) \tag{8}$$

$$(e^t,-e^{-t},\sqrt{2}) \times (\sqrt{2},\sqrt{2},(-e^t+e^{-t})) \tag{9}$$

$$= \left(\begin{vmatrix} -e^{-t} & \sqrt{2} \\ \sqrt{2} & -e^t+e^{-t} \end{vmatrix}, \begin{vmatrix} \sqrt{2} & e^t \\ -e^t+e^{-t} & \sqrt{2} \end{vmatrix}, \begin{vmatrix} e^t & -e^{-t} \\ \sqrt{2} & \sqrt{2} \end{vmatrix}\right) \tag{10}$$

$$= \left(-e^{-t}(e^t+e^{-t}),\ e^t(e^t+e^{-t}),\ \sqrt{2}(e^t+e^{-t})\right) \tag{11}$$

$$= (e^t+e^{-t})\left(-e^{-t},e^t,\sqrt{2}\right) \tag{12}$$

$$\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2 \tag{13}$$

$$= \frac{1}{e^t+e^{-t}}(e^t,-e^{-t},\sqrt{2}) \times \frac{1}{e^t+e^{-t}}(\sqrt{2},\sqrt{2},(-e^t+e^{-t})) \tag{14}$$

$$\frac{1}{e^t+e^{-t}}(-e^{-t},e^t,\sqrt{2}) \tag{15}$$

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8. $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_3 = \frac{\boxed{8}}{(e^t+e^{-t})^2}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_3 = \frac{1}{(e^t+e^{-t})^2}\left(1+e^{-2t}+1-e^{-2t},\ e^{2t}+1-e^{2t}+1,\ -\sqrt{2}(e^t-e^{-t})\right) \tag{16}$$

$$= \frac{1}{(e^t+e^{-t})^2}\left(2,\ 2,\ -\sqrt{2}(e^t-e^{-t})\right) \tag{17}$$

. . . .

9. $\tau = \boxed{9}$

$$\tau = -\left|\frac{\mathrm{d}\boldsymbol{e}_3}{\mathrm{d}t}\Big/\frac{\mathrm{d}s}{\mathrm{d}t}\right| \tag{18}$$

$$= -\left|\frac{1}{(e^t+e^{-t})^2}\left(2,\ 2,\ -\sqrt{2}(e^t-e^{-t})\right)\frac{1}{e^t+e^{-t}}\right| \tag{19}$$

$$= -\frac{1}{(e^t+e^{-t})^3}(\sqrt{2(e^t+e^{-t})^2}) \tag{20}$$

$$= -\frac{\sqrt{2}}{(e^t+e^{-t})^2} \tag{21}$$
