1. 加法定理

$$
\begin{aligned}
f(x-u)f(x+u)f(y-v)f(y+v)-f(x-v)f(x+v)f(y-u)f(y+u)& \\
=f(x-y)f(x+y)f(u-v)f(u+v)&
\end{aligned}
\tag{1}
$$

$f(x)$ が次の式である時、上の加法定理 (1) が成立することを示せ。

(a) $f(x)=x$

(b) $f(x)=\sin x$

(c) $f(x)=\theta_1(x \mid \tau)$

(a) $f(x)=x$

$$
\begin{aligned}
(\text{左辺}) &= (x-u)(x+u)(y-v)(y+v)-(x-v)(x+v)(y-u)(y+u) && (2)\\
&= \left(x^2-u^2\right)\left(y^2-v^2\right)-\left(x^2-v^2\right)\left(y^2-u^2\right) && (3)\\
&= -x^2v^2-y^2u^2+x^2u^2+y^2v^2 && (4)\\
&= x^2(u^2-v^2)-y^2(u^2-v^2) && (5)\\
&= (x^2-y^2)(u^2-v^2) && (6)\\
&= (x-y)(x+y)(u-v)(u+v)=(\text{右辺}) && (7)
\end{aligned}
$$

(b) $f(x)=\sin x$

$\sin\alpha$ と $\sin\beta$ の積の公式

$$\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right) \tag{8}$$

$$\sin(x-u)\sin(x+u)=\frac{1}{2}\left(\cos 2u-\cos 2x\right) \tag{9}$$

$$\sin(y-v)\sin(y+v)=\frac{1}{2}\left(\cos 2v-\cos 2y\right) \tag{10}$$

$$\sin(x-v)\sin(x+v)=\frac{1}{2}\left(\cos 2v-\cos 2x\right) \tag{11}$$

$$\sin(y-u)\sin(y+u)=\frac{1}{2}\left(\cos 2u-\cos 2y\right) \tag{12}$$

$$\sin(x-y)\sin(x+y)=\frac{1}{2}\left(\cos 2y-\cos 2x\right) \tag{13}$$

$$\sin(u-v)\sin(u+v)=\frac{1}{2}\left(\cos 2v-\cos 2u\right) \tag{14}$$

$$
\begin{aligned}
(\text{左辺}) &= \frac{1}{4}(\cos 2u - \cos 2x)(\cos 2v - \cos 2y) && (15)\\
&\quad - \frac{1}{4}(\cos 2v - \cos 2x)(\cos 2u - \cos 2y) && (16)\\
&= \frac{1}{4}(-\cos 2x \cos 2v - \cos 2y \cos 2u + \cos 2x \cos 2u + \cos 2y \cos 2v) && (17)\\
&= \frac{1}{4}(\cos 2x - \cos 2y)(\cos 2u - \cos 2v) && (18)\\
&= \sin(x-y)\sin(x+y)\sin(u-v)\sin(u+v) = (\text{右辺}) && (19)
\end{aligned}
$$

(c) $f(x) = \theta_1(x \mid \tau)$

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2. テータ関数

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u \mid \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}\left[(n+a)^2\tau + 2(n+a)(u+b)\right] \tag{20}$$

$$\left(a, b \in \mathbb{R},\ u \in \mathbb{C},\ \tau \in \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\},\ \mathbf{e}[x] \overset{\text{def}}{=} \exp \sqrt{-1}\pi x\right) \tag{21}$$

の変換性 (準周期性) は

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u+1 \mid \tau) = \mathbf{e}[2a]\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u \mid \tau) \tag{22}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u+\tau \mid \tau) = \mathbf{e}[-\tau - 2(u+b)]\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u \mid \tau) \tag{23}$$

であり、$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u \mid \tau) = 0$ の $\mathrm{mod}(\mathbb{Z} \oplus \tau\mathbb{Z})$ での zero 点の個数は 1 個で、それは $\frac{1}{2} - \frac{a}{2} + \left(\frac{1}{2} - \frac{b}{2}\right)\tau \mod (\mathbb{Z} \oplus \tau\mathbb{Z})$ であった。
テータ関数の記号は次のものを用いる。

$$\theta_1(u \mid \tau) = -\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (u+\tau \mid \tau), \qquad \theta_2(u \mid \tau) = \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (u+\tau \mid \tau) \tag{24}$$

$$\theta_3(u \mid \tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u+\tau \mid \tau), \qquad \theta_4(u \mid \tau) = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (u+\tau \mid \tau) \tag{25}$$

(a) テータ関数の定義を用いて次を表わせ。

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$$\alpha \in \mathbb{R}\text{に対して}\quad \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(u+\alpha \mid \tau) = \vartheta\begin{bmatrix} a \\ b+\alpha \end{bmatrix}(u \mid \tau) \tag{26}$$

i. $\theta_1\left(u+\frac{1}{2}\middle|\tau\right)$ を $\theta_2\left(u \mid \tau\right)$ を用いて表わせ。

$$\begin{aligned} \theta_1\left(u+\frac{1}{2}\middle|\tau\right) &= -\vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(u+\frac{1}{2}+\tau \mid \tau) && (27) \\ &= -\vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(u+1+\tau \mid \tau) && (28) \\ &= -\mathbf{e}[1]\vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(u+\tau \mid \tau) && (29) \\ &= -\mathbf{e}[1]\theta_2(u \mid \tau) && (30) \end{aligned}$$

ii. $\theta_2\left(u+\frac{1}{2}\middle|\tau\right)$ を $\theta_1\left(u|\tau\right)$ を用いて表わせ。

$$\begin{aligned} \theta_2\left(u+\frac{1}{2}\middle|\tau\right) &= \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(u+\frac{1}{2}+\tau \mid \tau) && (31) \\ &= \vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(u+\tau \mid \tau) && (32) \\ &= -\theta_1(u \mid \tau) && (33) \end{aligned}$$

iii. $\theta_3\left(u+\frac{1}{2}\middle|\tau\right)$ を $\theta_4\left(u|\tau\right)$ を用いて表わせ。

$$\begin{aligned} \theta_3(u+\frac{1}{2} \mid \tau) &= \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u+\frac{1}{2}+\tau \mid \tau) && (34) \\ &= \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(u+\tau \mid \tau) && (35) \\ &= \theta_4(u \mid \tau) && (36) \end{aligned}$$

iv. $\theta_4\left(u+\frac{1}{2}\middle|\tau\right)$ を $\theta_3\left(u|\tau\right)$ を用いて表わせ。

$$\theta_4(u+\frac{1}{2}\mid\tau)=\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+\frac{1}{2}+\tau\mid\tau) \tag{37}$$
$$=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+1+\tau\mid\tau) \tag{38}$$
$$=\mathbf{e}[0]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+\tau\mid\tau) \tag{39}$$
$$=\theta_3(u\mid\tau) \tag{40}$$

（b）テータ関数の定義を用いて次を表わせ。

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(u\mid\tau)=\sum_{n\in\mathbb{Z}}\mathbf{e}\left[(n+a)^2\tau+2(n+a)(u+b)\right] \tag{41}$$
$$=\sum_{n\in\mathbb{Z}}\mathbf{e}\left[n^2\tau+2n(u+a\tau+b)+a^2\tau+2a(u+b)\right] \tag{42}$$
$$=\mathbf{e}[a^2\tau+2a(u+b)]\sum_{n\in\mathbb{Z}}\mathbf{e}[n^2\tau+2n(u+a\tau+b)] \tag{43}$$
$$=\mathbf{e}[a^2\tau+2a(u+b)]\vartheta\begin{bmatrix}0\\b\end{bmatrix}(u+a\tau\mid\tau) \tag{44}$$

i. $\theta_1\left(u+\frac{\tau}{2}\middle|\tau\right)$ を $\theta_4\left(u|\tau\right)$ を用いて表わせ。

$$\theta_1(u+\frac{\tau}{2}\mid\tau) \tag{45}$$
$$=-\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(u+\frac{3}{2}\tau\mid\tau) \tag{46}$$
$$=-\mathbf{e}[\frac{\tau}{4}+u+\frac{3}{2}\tau+\frac{1}{2}]\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+\frac{1}{2}\tau+\frac{3}{2}\tau\mid\tau) \tag{47}$$
$$=-\mathbf{e}[\frac{\tau}{4}+u+\frac{3}{2}\tau+\frac{1}{2}]\mathbf{e}[-\tau-2(u+\tau+\frac{1}{2})]\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+\tau\mid\tau) \tag{48}$$
$$=-\mathbf{e}[-\frac{5}{4}\tau-u-\frac{1}{2}]\theta_4(u\mid\tau) \tag{49}$$

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ii. $\theta_2\left(u+\frac{\tau}{2}\middle|\tau\right)$ を $\theta_3\left(u|\tau\right)$ を用いて表わせ。

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$$\theta_2(u+\frac{\tau}{2}\mid\tau) = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(u+\frac{3}{2}\tau\mid\tau) \tag{50}$$

$$= \mathbf{e}[\frac{\tau}{4}+u+\frac{3}{2}\tau]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+2\tau\mid\tau) \tag{51}$$

$$= \mathbf{e}[\frac{\tau}{4}+u+\frac{3}{2}\tau]\mathbf{e}[-\tau-2(u+\tau)]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+\tau\mid\tau) \tag{52}$$

$$= \mathbf{e}[-\frac{5}{4}\tau-u]\theta_3(u\mid\tau) \tag{53}$$

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iii. $\theta_3\left(u+\frac{\tau}{2}\middle|\tau\right)$ を $\theta_2\left(u|\tau\right)$ を用いて表わせ。

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$$\theta_3\left(u+\frac{\tau}{2}\middle|\tau\right) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{\tau}{2}+\tau\mid\tau) \tag{54}$$

$$= \mathbf{e}[-\frac{\tau}{4}-u-\tau]\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(u+\tau\mid\tau) \tag{55}$$

$$= \mathbf{e}[-\frac{5}{4}\tau-u]\theta_2\left(u|\tau\right) \tag{56}$$

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iv. $\theta_4\left(u+\frac{\tau}{2}\middle|\tau\right)$ を $\theta_1\left(u|\tau\right)$ を用いて表わせ。

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$$\theta_4\left(u+\frac{\tau}{2}\middle|\tau\right) = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+\frac{\tau}{2}+\tau\mid\tau) \tag{57}$$

$$= \mathbf{e}[-\frac{\tau}{4}-(u+\tau+\frac{1}{2})]\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(u+\tau\mid\tau) \tag{58}$$

$$= \mathbf{e}[-\frac{5}{4}\tau-u-\frac{1}{2}]\theta_1\left(u|\tau\right) \tag{59}$$

(c)　i. $\theta_1\left(u+\frac{1}{2}\middle|2\tau\right)$ を $\theta_2\left(u|2\tau\right)$ を用いて表わせ。

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$$\theta_1\left(u+\frac{1}{2}\middle|2\tau\right) = -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\left(u+\frac{1}{2}+2\tau \mid 2\tau\right) \tag{60}$$

$$= -\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\left(u+1+2\tau \mid 2\tau\right) \tag{61}$$

$$= -\mathbf{e}[1]\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\left(u+2\tau \mid 2\tau\right) \tag{62}$$

$$= -\mathbf{e}[1]\theta_2\left(u|2\tau\right) \tag{63}$$

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ii. $\theta_2\left(u+\frac{1}{2}\middle|2\tau\right)$ を $\theta_1\left(u|2\tau\right)$ を用いて表わせ。

$$\theta_2\left(u+\frac{1}{2}\middle|2\tau\right) = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\left(u+\frac{1}{2}+2\tau \mid 2\tau\right) \tag{64}$$

$$= \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\left(u+2\tau \mid 2\tau\right) \tag{65}$$

$$= -\theta_1\left(u|2\tau\right) \tag{66}$$

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iii. $\theta_3\left(u+\frac{1}{2}\middle|2\tau\right)$ を $\theta_4\left(u|2\tau\right)$ を用いて表わせ。

$$\theta_3\left(u+\frac{1}{2}\middle|2\tau\right) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(u+\frac{1}{2}+2\tau \mid 2\tau\right) \tag{67}$$

$$= \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\left(u+2\tau \mid 2\tau\right) \tag{68}$$

$$= \theta_4\left(u|2\tau\right) \tag{69}$$

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iv. $\theta_4\left(u+\frac{1}{2}\middle|2\tau\right)$ を $\theta_3\left(u|2\tau\right)$ を用いて表わせ。

$$\theta_4\left(u+\frac{1}{2}\middle|2\tau\right)=\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+\frac{1}{2}+2\tau\mid 2\tau) \tag{70}$$

$$=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+1+2\tau\mid 2\tau) \tag{71}$$

$$=\mathbf{e}[0]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+2\tau\mid 2\tau) \tag{72}$$

$$=\theta_3(u|2\tau) \tag{73}$$

(d) i. $\theta_1(u+\tau|2\tau)$ を $\theta_4(u|2\tau)$ を用いて表わせ。

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$$\theta_1(u+\tau|2\tau) \tag{74}$$

$$=-\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(u+\tau+2\tau\mid 2\tau) \tag{75}$$

$$=-\mathbf{e}[\frac{2\tau}{4}+u+3\tau+\frac{1}{2}]\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+2\tau+2\tau\mid 2\tau) \tag{76}$$

$$=-\mathbf{e}[\frac{7}{2}\tau+u+\frac{1}{2}]\mathbf{e}[-2\tau-2(u+2\tau+\frac{1}{2})]\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(u+2\tau\mid 2\tau) \tag{77}$$

$$=-\mathbf{e}[-\frac{5}{2}\tau-u-\frac{1}{2}]\theta_4(u|2\tau) \tag{78}$$

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ii. $\theta_2(u+\tau|2\tau)$ を $\theta_3(u|2\tau)$ を用いて表わせ。

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$$\theta_2(u+\tau|2\tau)=\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(u+\tau+2\tau\mid 2\tau) \tag{79}$$

$$=\mathbf{e}[\frac{2\tau}{4}+u+3\tau]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+2\tau+2\tau\mid 2\tau) \tag{80}$$

$$=\mathbf{e}[\frac{7}{2}\tau+u]\mathbf{e}[-2\tau-2(u+2\tau)]\vartheta\begin{bmatrix}0\\0\end{bmatrix}(u+2\tau\mid 2\tau) \tag{81}$$

$$=\mathbf{e}[-\frac{5}{2}\tau-u]\theta_3(u\mid 2\tau) \tag{82}$$

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iii. $\theta_3(u+\tau|2\tau)$ を $\theta_2(u|2\tau)$ を用いて表わせ。

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$$\theta_3\left(u+\tau|2\tau\right) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(u+\tau+2\tau \mid 2\tau\right) \tag{83}$$

$$= \mathbf{e}[-\frac{2\tau}{4} - u - 2\tau]\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\left(u+2\tau \mid 2\tau\right) \tag{84}$$

$$= \mathbf{e}[-\frac{5}{2}\tau - u]\theta_2\left(u \mid 2\tau\right) \tag{85}$$

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iv. $\theta_4\left(u+\tau|2\tau\right)$ を $\theta_1\left(u|2\tau\right)$ を用いて表わせ。

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$$\theta_4\left(u+\tau \mid 2\tau\right) = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\left(u+\tau+2\tau \mid 2\tau\right) \tag{86}$$

$$= \mathbf{e}[-\frac{2\tau}{4} - u - 2\tau - \frac{1}{2}]\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\left(u+2\tau \mid 2\tau\right) \tag{87}$$

$$= -\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_1\left(u \mid 2\tau\right) \tag{88}$$

(e) $\theta_1(u \mid 2\tau)$, $\theta_2(u \mid 2\tau)$, $\theta_3(u \mid 2\tau)$, $\theta_4(u \mid 2\tau)$ の $u \mapsto u+1$ としたときの変換性 (準周期性) を求めよ。

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$$\theta_1(u+\frac{1}{2} \mid 2\tau) = -\mathbf{e}[1]\theta_2\left(u \mid 2\tau\right) \tag{89}$$

$$\theta_2(u+\frac{1}{2} \mid 2\tau) = -\theta_1\left(u \mid 2\tau\right) \tag{90}$$

$$\theta_3(u+\frac{1}{2} \mid 2\tau) = \theta_4(u \mid 2\tau) \tag{91}$$

$$\theta_4(u+\frac{1}{2} \mid 2\tau) = \theta_3(u \mid 2\tau) \tag{92}$$

$$\theta_1(u+1 \mid 2\tau) = -\mathbf{e}[1]\theta_2(u+\frac{1}{2} \mid 2\tau) = \mathbf{e}[1]\theta_1\left(u \mid 2\tau\right) \tag{93}$$

$$\theta_2(u+1 \mid 2\tau) = -\theta_1(u+\frac{1}{2} \mid 2\tau) = \mathbf{e}[1]\theta_2\left(u \mid 2\tau\right) \tag{94}$$

$$\theta_3(u+1 \mid 2\tau) = \theta_4(u+\frac{1}{2} \mid 2\tau) = \theta_3(u \mid 2\tau) \tag{95}$$

$$\theta_4(u+1 \mid 2\tau) = \theta_4(u+\frac{1}{2} \mid 2\tau) = \theta_3(u \mid 2\tau) \tag{96}$$

(f) $\theta_1(u \mid 2\tau)$, $\theta_2(u \mid 2\tau)$, $\theta_3(u \mid 2\tau)$, $\theta_4(u \mid 2\tau)$ の $u \mapsto u+2\tau$ としたときの変換性 (準周期性) を求めよ。

. . . .

$$\theta_1(u+\tau \mid 2\tau) = -\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_4(u \mid 2\tau) \tag{97}$$

$$\theta_2(u+\tau \mid 2\tau) = \mathbf{e}[-\frac{5}{2}\tau - u]\theta_3(u \mid 2\tau) \tag{98}$$

$$\theta_3(u+\tau \mid 2\tau) = \mathbf{e}[-\frac{5}{2}\tau - u]\theta_2(u \mid 2\tau) \tag{99}$$

$$\theta_4(u+\tau \mid 2\tau) = -\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_1(u \mid 2\tau) \tag{100}$$

$$\theta_1(u+2\tau \mid 2\tau) = -\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_4(u+\tau \mid 2\tau) \tag{101}$$

$$= \mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_1(u \mid 2\tau) \tag{102}$$

$$= \mathbf{e}[-5\tau - 2u - 1]\theta_1(u \mid 2\tau) \tag{103}$$

$$\theta_2(u+2\tau \mid 2\tau) = \mathbf{e}[-\frac{5}{2}\tau - u]\theta_3(u+\tau \mid 2\tau) \tag{104}$$

$$= \mathbf{e}[-\frac{5}{2}\tau - u]\mathbf{e}[-\frac{5}{2}\tau - u]\theta_2(u \mid 2\tau) \tag{105}$$

$$= \mathbf{e}[-5\tau - 2u]\theta_2(u \mid 2\tau) \tag{106}$$

$$\theta_3(u+2\tau \mid 2\tau) = \mathbf{e}[-\frac{5}{2}\tau - u]\theta_2(u+\tau \mid 2\tau) \tag{107}$$

$$= \mathbf{e}[-\frac{5}{2}\tau - u]\mathbf{e}[-\frac{5}{2}\tau - u]\theta_3(u \mid 2\tau) \tag{108}$$

$$= \mathbf{e}[-5\tau - 2u]\theta_3(u \mid 2\tau) \tag{109}$$

$$\theta_4(u+2\tau \mid 2\tau) = -\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_1(u+\tau \mid 2\tau) \tag{110}$$

$$= \mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\mathbf{e}[-\frac{5}{2}\tau - u - \frac{1}{2}]\theta_4(u \mid 2\tau) \tag{111}$$

$$= \mathbf{e}[-5\tau - 2u - 1]\theta_4(u \mid 2\tau) \tag{112}$$