$N=2$ のとき、$g,h \in \mathrm{End}(\mathbb{C}^2)$ は $g^2=h^2=\mathrm{Id}_{\mathbb{C}^2}$ $(g^{-1}=g, h^{-1}=h)$

1. $R_{8v}(0)=P$、$P(x\otimes y)=y\otimes x$ for any $x,y\in\mathbb{C}^N$
2. $(g\otimes g)^{-1}R_{8v}(u)(g\otimes g)=(h\otimes h)^{-1}R_{8v}(u)(h\otimes h)=R_{8v}(u)$
3. $R_{8v}(u+1)=\mathrm{e}[-1](g^{-1}\otimes 1)R_{8v}(u)(g\otimes 1)$
4. $R_{8v}(u+\tau)=\mathrm{e}[-\tau-2(u+\frac{\eta}{2}+\frac{1}{2})](h\otimes 1)R_{8v}(u)(h^{-1}\otimes 1)$
5. $R_{8v}(u)$ の各成分は u の正則関数である。

$$R_{8v}(u)=\frac{1}{2}\sum_{\alpha,\beta=0}^{1} w_{\alpha,\beta}(u\mid\tau)h^{-\alpha}g^{\beta}\otimes g^{-\beta}h^{\alpha}\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2) \tag{1}$$

$$w_{\alpha,\beta}=\frac{\vartheta\begin{bmatrix}1/2+\alpha/2\\1/2+\beta/2\end{bmatrix}(u+\frac{\eta}{2}\mid\tau)}{\vartheta\begin{bmatrix}1/2+\alpha/2\\1/2+\beta/2\end{bmatrix}(\frac{\eta}{2}\mid\tau)} \tag{2}$$

$R_{8v}(u)$ が unitary 関係式

$$PR_{8v}(u)PR_{8v}(-u)=\rho(u)\mathrm{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2} \tag{3}$$

を満たすことを示せ。

$P, R_{8v}\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$ は次の様に表せる。

$$P=\begin{pmatrix}1&&&\\&0&1&\\&1&0&\\&&&1\end{pmatrix}\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2) \tag{4}$$

$$R_{8v}(u)=\begin{pmatrix}a_u&&&d_u\\&b_u&c_u&\\&c_u&b_u&\\d_u&&&a_u\end{pmatrix}\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2) \tag{5}$$

ただし、a_u, b_u, c_u, d_u は次を表している。

$$a_u = a(u \mid \tau) = \frac{\theta_4(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)} \tag{6}$$
$$b_u = b(u \mid \tau) = \frac{\theta_1(u \mid 2\tau)\theta_4(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)} \tag{7}$$
$$c_u = c(u \mid \tau) = \frac{\theta_4(u \mid 2\tau)\theta_4(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{8}$$
$$d_u = d(u \mid \tau) = \frac{\theta_1(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{9}$$

$PR_{8v}(u)PR_{8v}(-u)$ を計算する。

$$PR_{8v}(u)PR_{8v}(-u) \tag{10}$$
$$= \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_u & & & d_u \\ & b_u & c_u & \\ & c_u & b_u & \\ d_u & & & a_u \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_{-u} & & & d_{-u} \\ & b_{-u} & c_{-u} & \\ & c_{-u} & b_{-u} & \\ d_{-u} & & & a_{-u} \end{pmatrix} \tag{11}$$
$$= \begin{pmatrix} a_u a_{-u} + d_u d_{-u} & & & a_u d_{-u} + a_{-u} d_u \\ & b_u b_{-u} + c_u c_{-u} & b_u c_{-u} + b_{-u} c_u & \\ & b_u c_{-u} + b_{-u} c_u & b_u b_{-u} + c_u c_{-u} & \\ a_u d_{-u} + a_{-u} d_u & & & a_u a_{-u} + d_u d_{-u} \end{pmatrix} \tag{12}$$

θ_1 は奇関数で、$\theta_2, \theta_3, \theta_4$ は偶関数である。これを用いて $a_u d_{-u} + a_{-u} d_u$ は次の様に求まる。

$$a_u d_{-u} = \frac{\theta_4(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)\theta_1(-u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{13}$$
$$= -\frac{\theta_4(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)\theta_1(u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{14}$$
$$\tag{15}$$
$$a_{-u} d_u = \frac{\theta_4(-u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)\theta_1(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{16}$$
$$= \frac{\theta_4(u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)\theta_1(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{17}$$

$$a_u d_{-u} + a_{-u} d_u = 0 \tag{18}$$

同様に $b_u c_{-u} + b_{-u} c_u = 0$ であることが求まる。

よって、次のような行列となる。

$$PR_{8v}(u)PR_{8v}(-u) \tag{19}$$

$$= \begin{pmatrix} a_u a_{-u} + d_u d_{-u} & & & \\ & b_u b_{-u} + c_u c_{-u} & & \\ & & b_u b_{-u} + c_u c_{-u} & \\ & & & a_u a_{-u} + d_u d_{-u} \end{pmatrix} \tag{20}$$

$a_u a_{-u} + d_u d_{-u}$ と $b_u b_{-u} + c_u c_{-u}$ は u の関数として偶関数である。その為、$PR_{8v}(u)PR_{8v}(-u)$ は偶関数で表すことが出来る。

$$PR_{8v}(u)PR_{8v}(-u) = \rho(u)\mathrm{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2} \tag{21}$$

. . . .

$$a_u a_{-u} + d_u d_{-u} \tag{22}$$

$$= \frac{\theta_4(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)} \frac{\theta_4(-u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)}$$

$$+ \frac{\theta_1(u \mid 2\tau)\theta_1(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \frac{\theta_1(-u \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{23}$$

$$= \frac{\theta_1(u+\eta \mid 2\tau)\theta_1(-u+\eta \mid 2\tau)}{(\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(\eta \mid 2\tau))^2}((\theta_4(u \mid 2\tau)\theta_4(\eta \mid 2\tau))^2 - (\theta_1(u \mid 2\tau)\theta_1(\eta \mid 2\tau))^2) \tag{24}$$

$$b_u b_{-u} + c_u c_{-u} \tag{25}$$

$$= \frac{\theta_1(u \mid 2\tau)\theta_4(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)} \frac{\theta_1(-u \mid 2\tau)\theta_4(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)}$$

$$+ \frac{\theta_4(u \mid 2\tau)\theta_4(u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \frac{\theta_4(-u \mid 2\tau)\theta_4(-u+\eta \mid 2\tau)}{\theta_4(0 \mid 2\tau)\theta_4(\eta \mid 2\tau)} \tag{26}$$

$$= \frac{\theta_4(u+\eta \mid 2\tau)\theta_4(-u+\eta \mid 2\tau)}{(\theta_4(0 \mid 2\tau)\theta_1(\eta \mid 2\tau)\theta_4(\eta \mid 2\tau))^2}(-(\theta_1(u \mid 2\tau)\theta_4(\eta \mid 2\tau))^2 + (\theta_4(u \mid 2\tau)\theta_1(\eta \mid 2\tau))^2) \tag{27}$$

. . . .

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u \mid \tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}[(n+a)^2 + 2(n+a)(u+b)] \tag{28}$$

$$a, b \in \mathbb{R}, \quad u \in \mathbb{C}, \quad \tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}, \quad \mathbf{e}[x] \stackrel{\mathrm{def}}{=} \exp(i\pi x) \tag{29}$$

$$\theta_1(u \mid \tau) = -\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (u + \tau \mid \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + 2(n+1/2)(u+1/2)] \tag{30}$$

$$\theta_4(u \mid \tau) = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (u + \tau \mid \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(u+1/2)] \tag{31}$$

....

$$\theta_1(\alpha \mid \tau)\theta_1(\beta \mid \tau) \tag{32}$$

$$= \sum_{n \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + 2(n+1/2)(\alpha+1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[(m+1/2)^2 + 2(m+1/2)(\beta+1/2)] \tag{33}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + 2(n+1/2)(\alpha+1/2) + (m+1/2)^2 + 2(m+1/2)(\beta+1/2)] \tag{34}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + (m+1/2)^2 + 2(n+1/2)(\alpha+1/2) + 2(m+1/2)(\beta+1/2)] \tag{35}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n + m^2 + 2m + 2n\alpha + 2m\beta + \alpha + \beta + 3/2] \tag{36}$$

$$\theta_1(\alpha \mid \tau)\theta_4(\beta \mid \tau) \tag{37}$$

$$= -\sum_{n \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + 2(n+1/2)(\alpha+1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[m^2 + 2m(\beta+1/2)] \tag{38}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[(n+1/2)^2 + 2(n+1/2)(\alpha+1/2) + m^2 + 2m(\beta+1/2)] \tag{39}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n + m^2 + m + 2n\alpha + 2m\beta + \alpha + 3/4] \tag{40}$$

$$\theta_4(\alpha \mid \tau)\theta_4(\beta \mid \tau) \tag{41}$$

$$= -\sum_{n \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(\alpha+1/2)] \sum_{m \in \mathbb{Z}} \mathbf{e}[m^2 + 2m(\beta+1/2)] \tag{42}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[n^2 + 2n(\alpha+1/2) + m^2 + 2m(\beta+1/2)] \tag{43}$$

$$= \sum_{n,m \in \mathbb{Z}} \mathbf{e}[n^2 + n + m^2 + m + 2n\alpha + 2m\beta] \tag{44}$$