[C2-001] 기초수학

Lecture 01: Vectors

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Immersive Reality & Intelligent Systems Lab (IRIS LAB)

Graduate School of Advanced Imaging Science, Multimedia & Film (GSAIM)

Chung-Ang University (CAU)

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Introduction: **IRIS LAB**

Immersive Reality & Intelligent Systems Lab (IRIS LAB)

Graduate School of Advanced Imaging Science, Multimedia & Film (GSAIM), Chung-Ang Univ.

Advisor

Prof. Hak Gu Kim

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Introduction to IRIS LAB@CAU

CAU IRIS LAB

- **IRIS@CAU: Immersive Reality & Intelligent Systems (IRIS)**
  : Convergence of AI & VR/Metaverse
- **Major – VR/Game/Metaverse**
  : AI/ML-based 3D VR & Virtual human (Digital twin), AI in Metaverse

Main Research

- **Immersive Content Analysis**
  : Convergence of AI & VR/Metaverse
- **Attention-aware Processing**
  : Human vision-based AI modeling
- **Domain Knowledge Learning**
  : Interaction between human and AI

Recent Publications

Journals (2020, 2021)

- **1 IEEE TIP** (JCR Top 5.7%, IF: 10.856)
- **4 IEEE TCSVT** (JCR Top 15.7%, IF: 4.685)

Conferences (2020, 2021)

- **2 CVPR** (*Top-tier AI & CV conf.*)
- **2 AAAI** (*Top-tier AI & CV conf.*)
- **1 ECCV** (*Top-tier AI & CV conf.*)

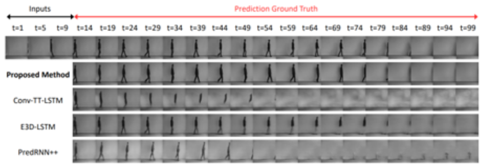
Immersive Content Analysis

Stereoscopic 3D (S3D) depth editing

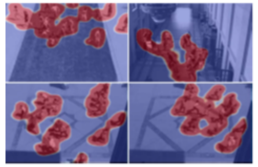
360° image quality assessment & VR Sickness assessment

- ❖ AI-based S3D Content Editing
- ❖ AI-based 360° Image & Video Analysis for VR/Metaverse Content Creation

Attention-Aware Processing



Long-term video prediction

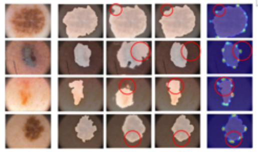


Anomaly detection

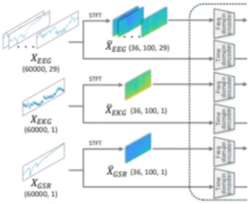
Video interpolation

- ❖ Human Visual Perception-based Video Understanding and Analysis
- ❖ AI-based Expression & Action Analysis

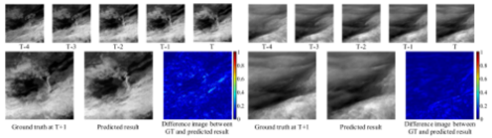
Domain Knowledge Learning



Medical image analysis



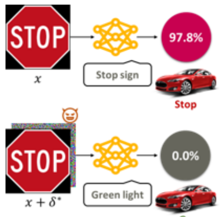
Physiological signal encoding



Weather forecasting

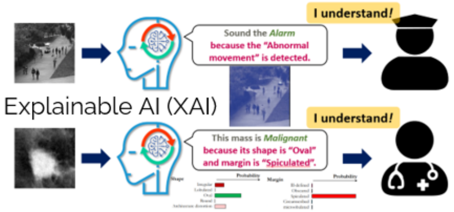
- ❖ Multi-Modal Learning (video & audio)
- ❖ Interactive Learning between Human Expert and AI Agent

Adversarial Attack & Defense



Adversarial attack

Natural-looking adversarial examples



Explainable AI (XAI)

- ❖ Robustness of Deep Neural Network for Safe AI
- ❖ Explainable AI for reliable AI

Basic Course Information

- **Time & Location**
  - Time: Mon. 09:00 – 11:00
  - Location: Rm #104, 309 Bldg. in CAU, Seoul

- **Instructor:** Hak Gu Kim
  - E-mail: hakgukim@cau.ac.kr
  - Webpage: www.irislab.cau.ac.kr
  - Office: Rm #818, 305 Building, CAU, Seoul

Learning Objectives

- Study Basic Mathematics for Artificial Intelligence (AI)
  - Understand the basics mathematics for machine learning (ML) and AI
  - Learn the link between mathematical theory and AI fields
  - Learn how mathematics is applied to AI-based applications

Organization

- **W1 (01 Aug.):** Vectors
- **W2 (08 Aug.):** Matrix & Linear Transformation I
- **W3 (15 Aug.):** Linear Transformation II
- **W4 (22 Aug.):** Matrix Inverse
- **W5 (29 Aug.):** Determinant & Affine Transformation
- **W6 (05 Sept.):** Eigenvalue & Eigenvector

Topics

- Vectors
- Lines
- Planes

Topics

- Vectors
- Lines
- Planes

Systems of Linear Equations

- For unknown variables $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$,

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

- Three cases of solutions:

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 \phantom{{}+x_2} + 3x_3 &= 1 \end{aligned}$$

No solution

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ x_2 + 3x_3 &= 1 \end{aligned}$$

Unique solution

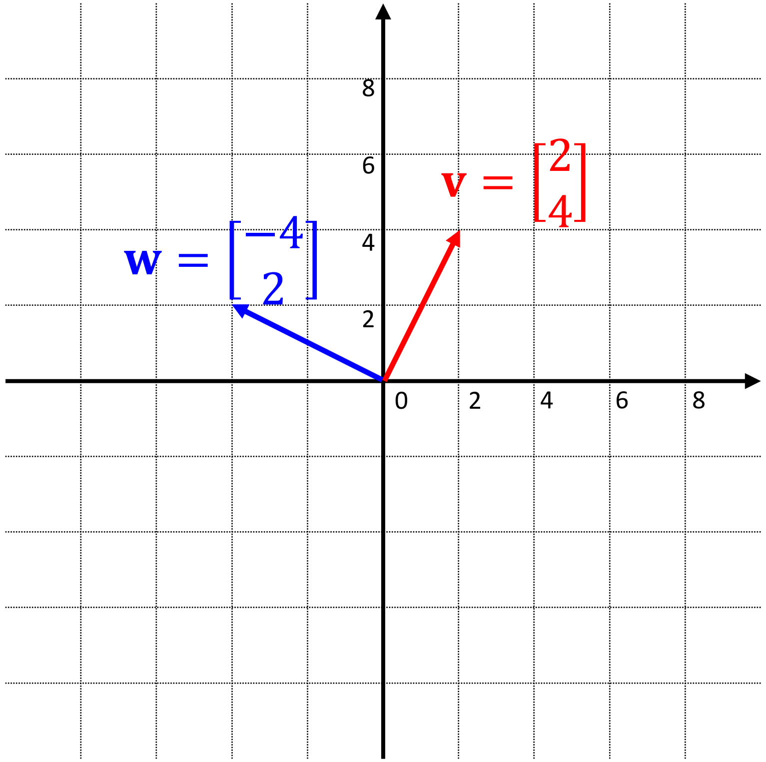
$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 \phantom{{}+x_2} + 3x_3 &= 5 \end{aligned}$$

Infinitely many solutions

Vectors

- A geometric vector $\mathbf{v}$ is an entity with magnitude and direction
  — Length: magnitude of the vector
  — Arrowhead: direction of the vector

- A vector does not have a location
  — The vectors with the same magnitude and direction are equal



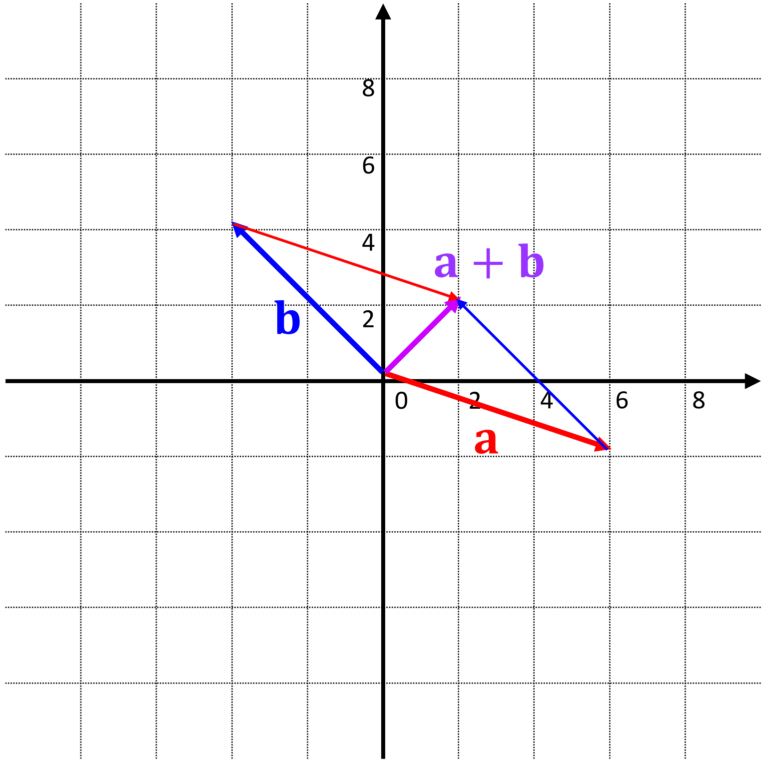
Vector **Basic Operation**

- Vector Addition

— $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$: 2-dimensional real coordinate space

  - $\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

- Example of Vector Addition
  - $\mathbf{a} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$
  - $\mathbf{a} + \mathbf{b} = \begin{bmatrix} 6 + (-4) \\ (-2) + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
  - $\mathbf{b} + \mathbf{a} = \begin{bmatrix} (-4) + 6 \\ 4 + (-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



Vector **Basic Operation**

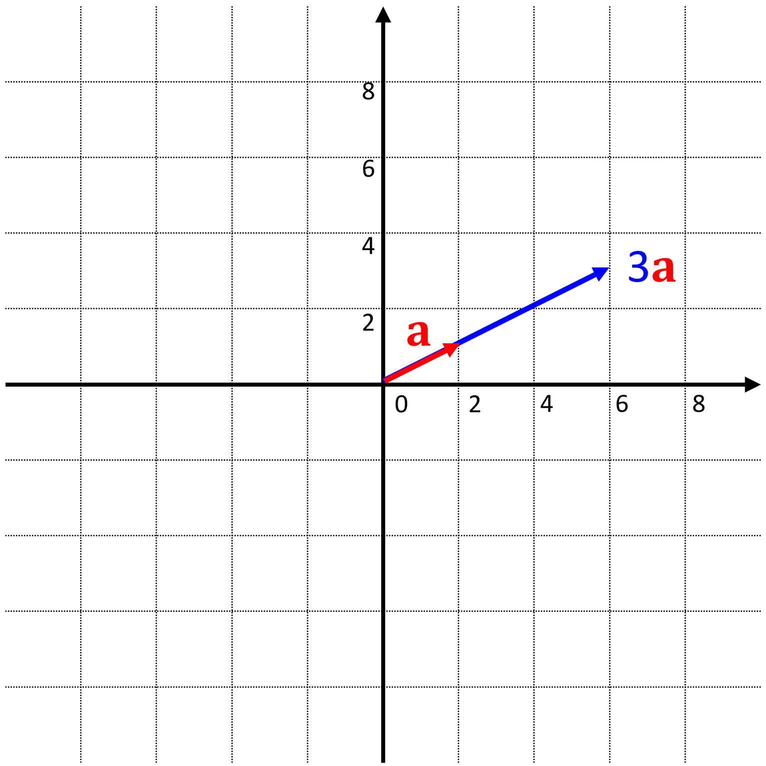
- Scalar Multiplication

— Changes the length of a vector multiplying it by a single real value (scalar)

  - $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$
  - $c\mathbf{v} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$

- Example of Scalar Multiplication

  - $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
  - $3\mathbf{a} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$



Vector **Basic Operation**

- Algebraic Rules for Vector Addition
  - Commutative property: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
  - Associative property: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  - Additive identity: $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  - Additive inverse: For every $\mathbf{v}$, there is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

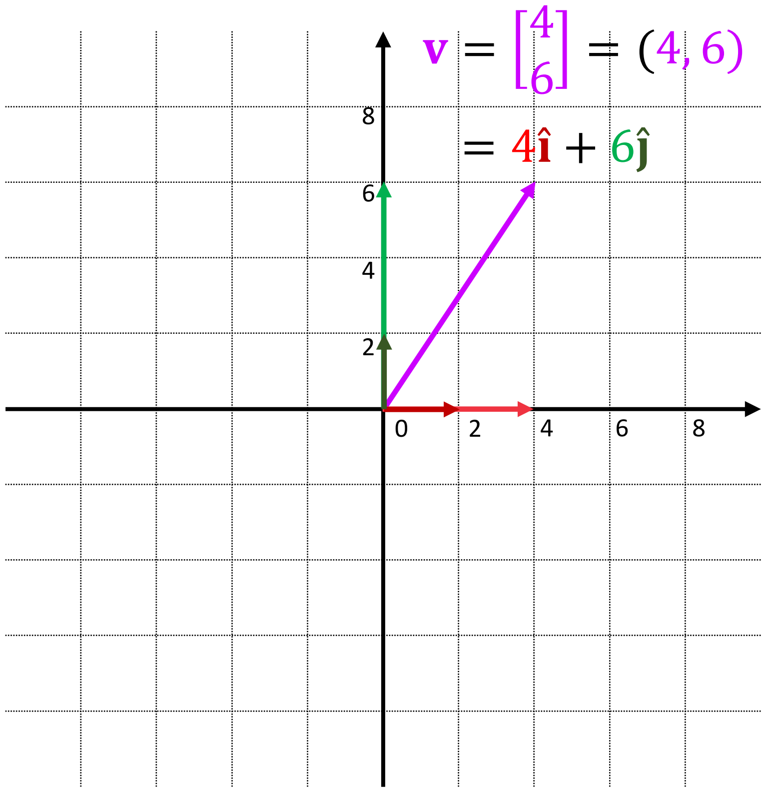
- Algebraic Rules for Scalar Multiplication
  - Associative property: $(ab)\mathbf{v} = a(b\mathbf{v})$
  - Distributive property: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ and $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$
  - Multiplicative identity: $1 \cdot \mathbf{v} = \mathbf{v}$

Special Vectors

- Unit Vector (Normalized Vector): $\hat{\mathbf{v}}$

— A vector whose magnitude is 1

  - In $\mathbb{R}^2$: $\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - In $\mathbb{R}^3$: $\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



- Zero Vector: $\mathbf{0}$

— A vector has a magnitude of zero but no direction

  - In $\mathbb{R}^2$: $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

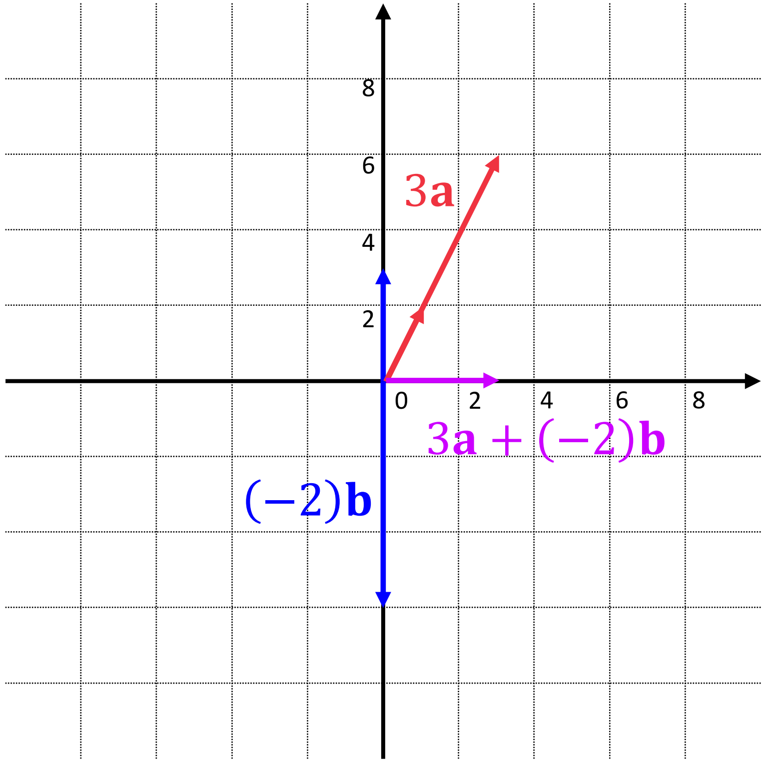
Linear Combination

- Linear Combination

— Given $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, we can create a new vector $\mathbf{v}$ like this:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad where\ c_1, \cdots, c_n \in \mathbb{R}$$

- Example of Linear Combination
  - $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
  - $3\mathbf{a} + (-2)\mathbf{b} = \begin{bmatrix} 3 \cdot 1 + (-2) \cdot 0 \\ 3 \cdot 2 + (-2) \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$



Span

- If we take all the possible linear combinations of all vectors in $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, the set T of vectors thus created is the Span of S

$$Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ \cdots + c_n\mathbf{v}_n \mid c_i \in \mathbb{R} \text{ for } 1 \le i \le n\}$$

- Example of Span
  - Can we represent a point, $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ using two vectors, $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$?

$$c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{x} \longrightarrow c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \mathbf{x} \longrightarrow \begin{aligned} 1 \cdot c_1 + 0 \cdot c_2 &= x_1 \\ 2 \cdot c_1 + 3 \cdot c_2 &= x_2 \end{aligned}$$

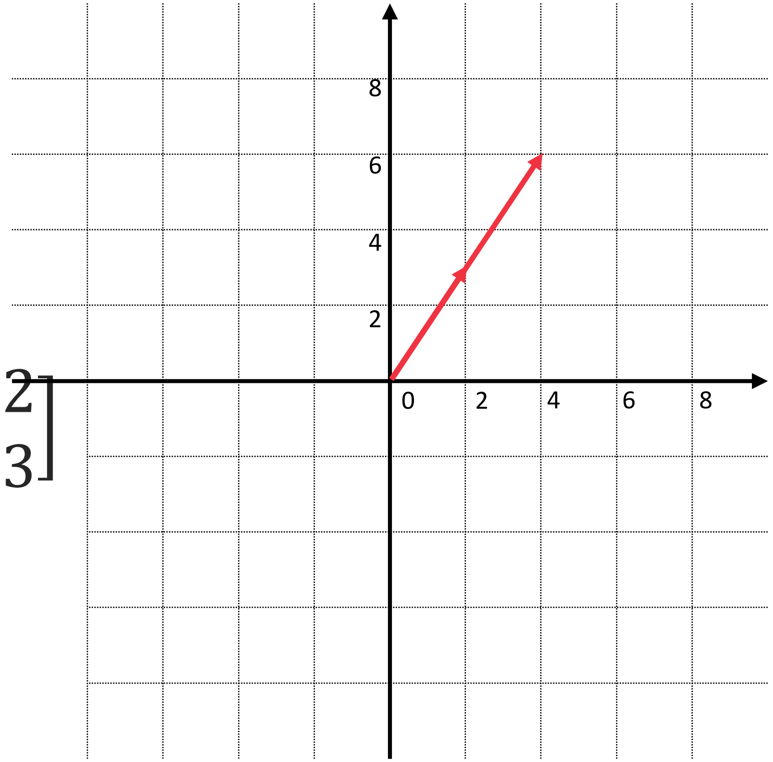
$$\longrightarrow \begin{aligned} 2 \cdot c_1 + 0 \cdot c_2 &= 2x_1 \\ 2 \cdot c_1 + 3 \cdot c_2 &= \ \ x_2 \end{aligned} \longrightarrow \begin{aligned} c_1 &= x_1 \\ c_2 &= \frac{1}{3}(x_1 - 2x_2) \end{aligned}$$

Linearly Dependent

- For $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \ \ where\ c_1, \cdots, c_n \in \mathbb{R}$,
  $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly dependent $\Leftrightarrow c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$
  for c_i, not all are zero (at least 1 is non-zero)

- Example of Linearly Dependent Set
  - Set of vectors: $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$
  - Linearly dependent? Yes, $Span(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^1$

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \cdot 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (c_1 + 2c_2) \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

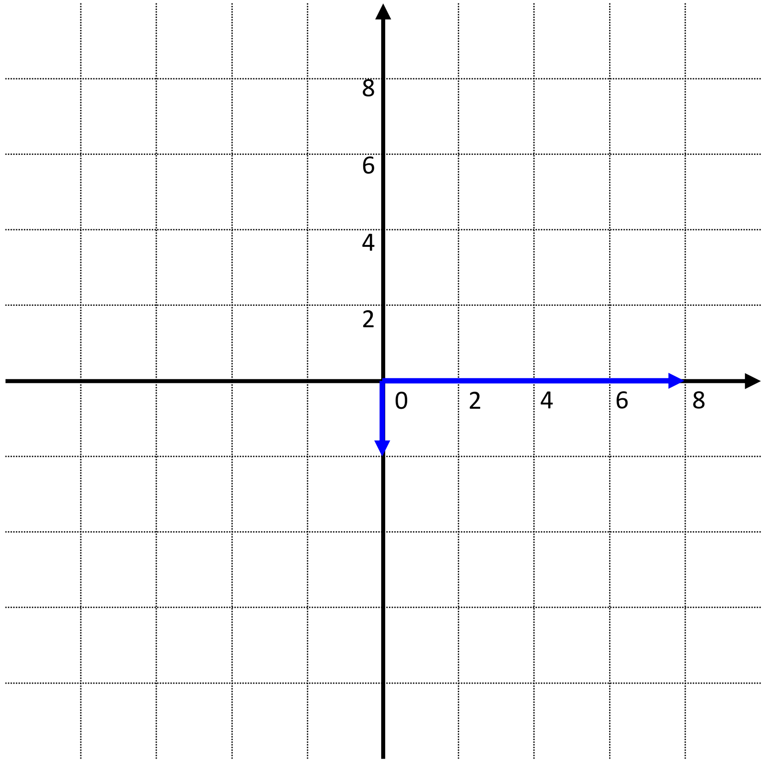


Linearly Independent

- For $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \ \ where\ c_1, \cdots, c_n \in \mathbb{R}$,
  $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent $\Leftrightarrow c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$
  for c_i, only solution is $c_i = 0$ for $1 \le i \le n$

- Example of Linearly Independent Set
  - Set of vectors: $\left\{ \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$
  - Linearly independent? Yes, $Span(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$

  $c_1 \begin{bmatrix} 8 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ⇢ only solution: $c_1 = c_2 = 0$



Dot Product

- Given two vectors $\mathbf{v}$ and $\mathbf{w}$ with an angle θ between them, the dot product $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

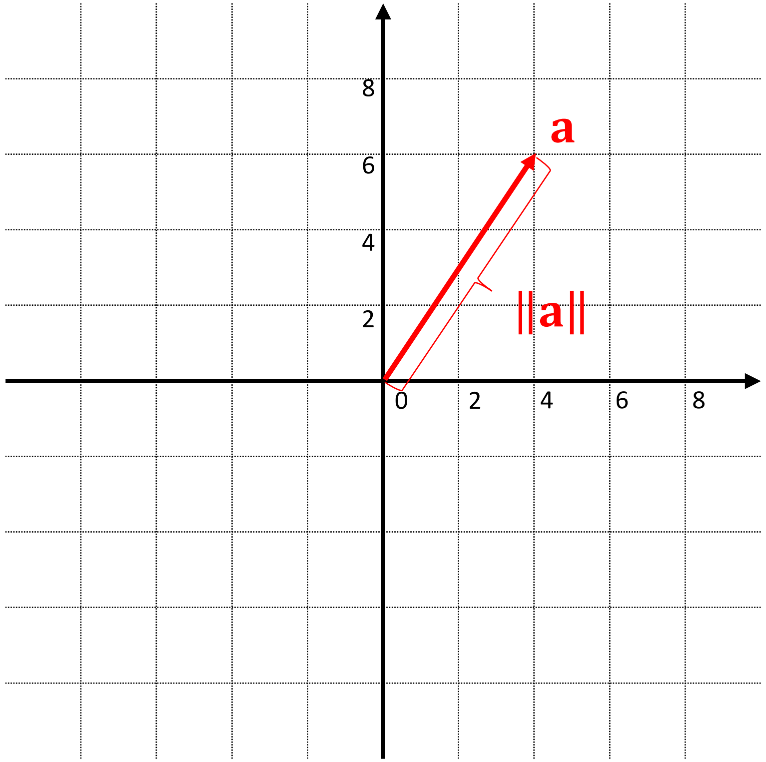
- Algebraic Rules for Dot Product
  - Symmetry: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
  - Additivity: $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
  - Homogeneity: $c(\mathbf{v} \cdot \mathbf{w}) = c(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$
  - Positivity: $\mathbf{v} \cdot \mathbf{v} \geq 0$
  - Definiteness: $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = 0$

Length of Vector

- A norm $\|\mathbf{v}\|$ is defined as a real-valued size measuring function on a vector $\mathbf{v}$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

- Example of Length
  - $\mathbf{a} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \dashrightarrow \|\mathbf{a}\| = \sqrt{4^2 + 6^2} = \sqrt{52}$
  - $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \dashrightarrow \|\mathbf{b}\| = \sqrt{1^2 + 5^2 + 3^2} = \sqrt{35}$

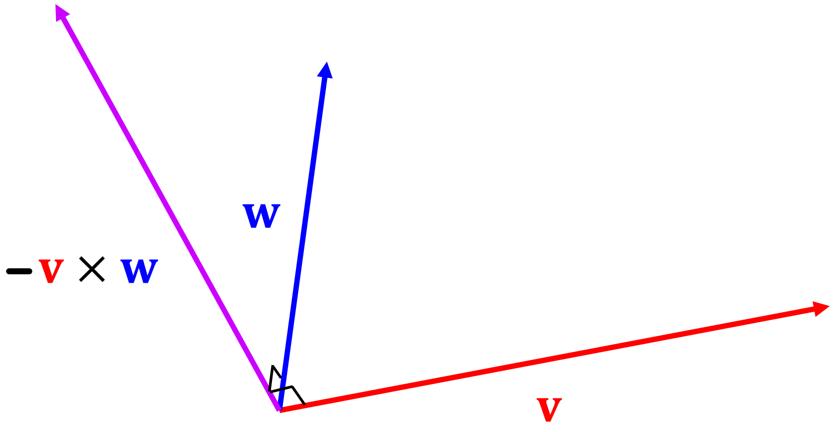


Cross Product

- Cross Product is the way to find a new vector $\mathbf{u}$ orthogonal to both two vectors $\mathbf{v}$ and $\mathbf{w}$:

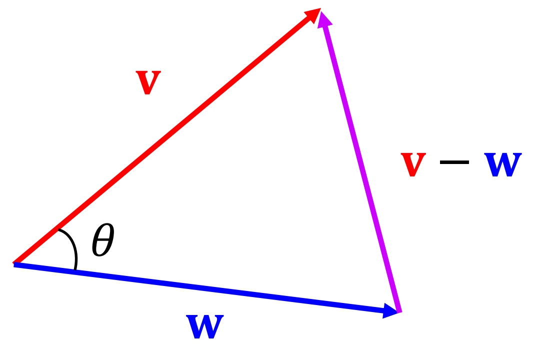
$$\mathbf{v} \times \mathbf{w} = \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

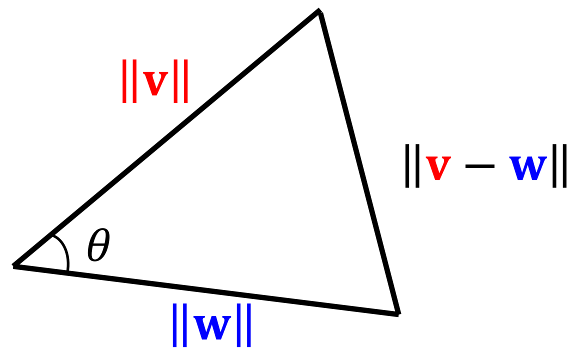
- Algebraic Rules for Cross Product
  - $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
  - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
  - $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
  - $a(\mathbf{v} \times \mathbf{w}) = (a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w})$
  - $\mathbf{v} \times \mathbf{0} = \mathbf{0} \times \mathbf{v} = \mathbf{0}$
  - $\mathbf{v} \times \mathbf{v} = \mathbf{0}$



Angle Between Two Vectors

- Relation Between Dot Product & Cosine θ
  - Given non-zero two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,
  - $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \;\cdots$ ①: *Law of Cosine*
  - $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \;\cdots$ ②
  - $\|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \;\cdots$ ① = ②: $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$
  - $\therefore\ \mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$





Angle Between Two Vectors

- Relation Between Cross Product & Sine θ
  - $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \Rightarrow \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$
  - $\|\mathbf{v} \times \mathbf{w}\|^2 = (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2$

    $= v_2^2 w_3^2 - 2 v_2 v_3 w_2 w_3 + v_3^2 w_2^2 + v_3^2 w_1^2 - 2 v_1 v_3 w_1 w_3 + v_1^2 w_3^2$

    $+ v_1^2 w_2^2 - 2 v_1 v_2 w_1 w_2 + v_2^2 w_1^2$

    $= v_1^2 (w_2^2 + w_3^2) + v_2^2 (w_1^2 + w_3^2) + v_3^2 (w_1^2 + w_2^2)$

    $-2(v_2 v_3 w_2 w_3 + v_1 v_3 w_1 w_3 + v_1 v_2 w_1 w_2)$
  - $\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \cos^2\theta = (\mathbf{v} \cdot \mathbf{w})^2 = (v_1 w_1 + v_2 w_2 + v_3 w_3)(v_1 w_1 + v_2 w_2 + v_3 w_3)$

    $= v_1^2 w_1^2 + v_2^2 w_2^2 + v_3^2 w_3^2 + 2(v_1 v_2 w_1 w_2 + v_1 v_3 w_1 w_3 + v_2 v_3 w_2 w_3)$
  - $\|\mathbf{v} \times \mathbf{w}\|^2 + \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \cos^2\theta = (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$
  - $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (1 - \cos^2\theta) = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin^2\theta$. $\therefore \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$

Summary: Dot Product & Cross Product

- Dot Product
  - In any $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$
  - $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$
  - How much two vectors move in the same direction

- Cross Product
  - Only defined in $\mathbb{R}^3$
  - $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$
  - How perpendicular two vectors are

Topics

- Vectors
- Lines
- Planes

Line

- Parametric Representation of Straight Line

— A line has been referred to as the shortest distance between two points

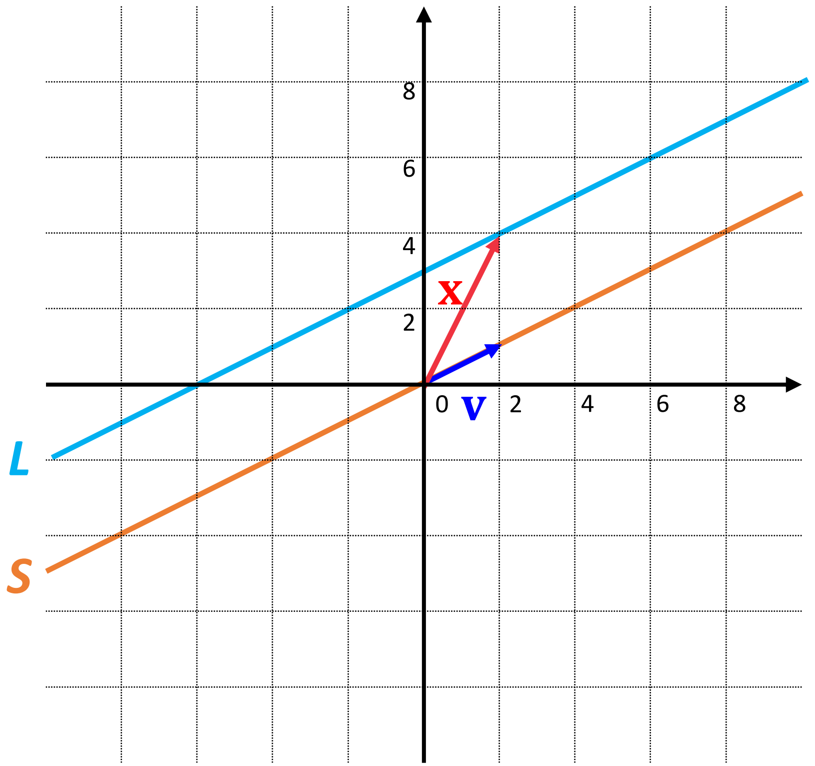
— Parametric equation is one possible representation of a generalized line across all dimensions

- Set of collinear vectors: $S = \{c\mathbf{v} \mid c \in \mathbb{R}\}$

— Slope: $\mathbf{v}$

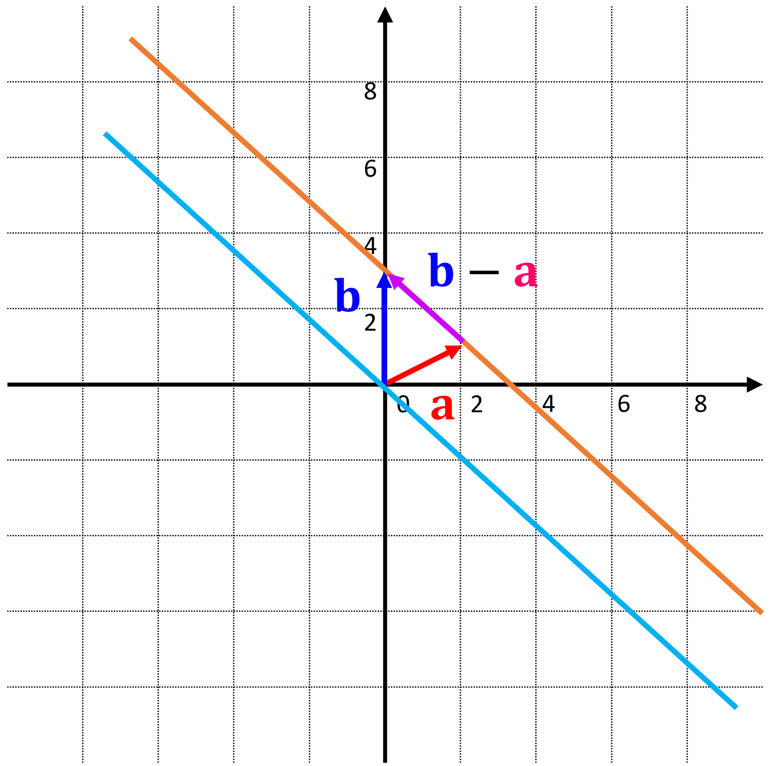
- Line: $L = \{\mathbf{x} + t\mathbf{v} \mid t \in \mathbb{R}\}$

— $\mathbf{x}$: Position vector



Line

- Examples of Parametric Representation of Line
  - $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
    - Slope: $\mathbf{b} - \mathbf{a}$
  - $\therefore L = \{\mathbf{b} + t(\mathbf{b} - \mathbf{a}) \mid t \in \mathbb{R}\} = \left\{\begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \mid t \in \mathbb{R}\right\}$
  - $\mathbf{P}_1 = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
  - $L = \{\mathbf{P}_1 + t(\mathbf{P}_1 - \mathbf{P}_2) \mid t \in \mathbb{R}\} = \left\{\begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \mid t \in \mathbb{R}\right\}$
  - $\therefore \ x = -1 - t, \ y = 2 - t, \ z = 7 + 3t$



Topics

- Vectors
- Lines
- Planes

Planes

- Generalized Equation of A Plane in $\mathbb{R}^3$

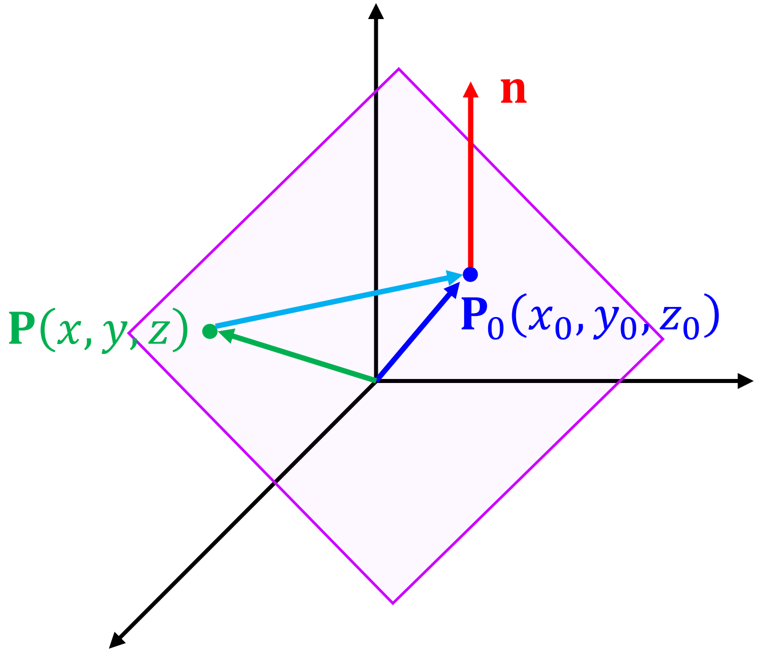
— A surface which lies evenly with the straight lines on itself

- Normal vector $\mathbf{n}$: It is perpendicular to everything on the plane: $\mathbf{n} \cdot \mathbf{v} = 0$

— $\mathbf{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{P}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

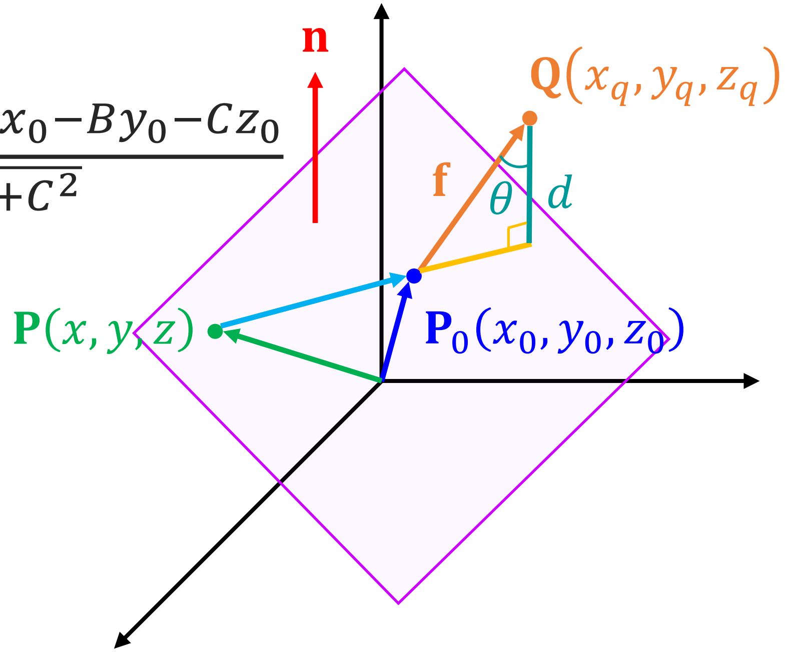
— $\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_0) = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$

— $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$



Planes

- Distance From Point To Plane
  - $\mathbf{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \mathbf{P}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix}$
  - $\mathbf{f} = (x_0 - x_p)\hat{\mathbf{i}} + (y_0 - y_p)\hat{\mathbf{j}} + (z_0 - z_p)\hat{\mathbf{k}}$
  - $d = \|\mathbf{f}\| \cos\theta \quad \rightarrow \qquad d = \frac{\|\mathbf{n}\|\|\mathbf{f}\| \cos\theta}{\|\mathbf{n}\|} = \frac{\mathbf{n}\cdot\mathbf{f}}{\|\mathbf{n}\|}$
  - $\frac{\mathbf{n}\cdot\mathbf{f}}{\|\mathbf{n}\|} = \frac{A(x_0-x_p)+B(y_0-y_p)+C(z_0-z_p)}{\sqrt{A^2+B^2+C^2}} = \frac{Ax_q+By_q+Cz_q-Ax_0-By_0-Cz_0}{\sqrt{A^2+B^2+C^2}}$



- Example of Distance From Point To Plane
  - Distance from $\mathbf{Q}(2, 3, 1)$ to plane $x - 2y + 3z = 5$
  - $d = \frac{1\cdot2-2\cdot3+3\cdot1-5}{\sqrt{1+4+9}} = \frac{-6}{\sqrt{14}}$

Planes

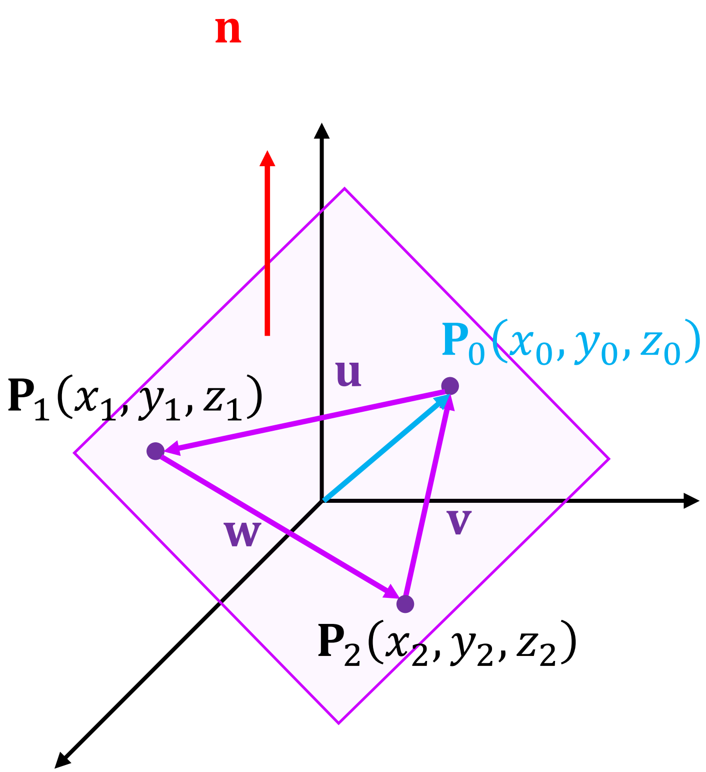
- Parametric Representation of A Plane

— Parametric equation is one possible representation of a generalized plane across all dimensions

- Set of coplanar vectors: $S = \{s\mathbf{u}, t\mathbf{v} \mid s, t \in \mathbb{R}\}$

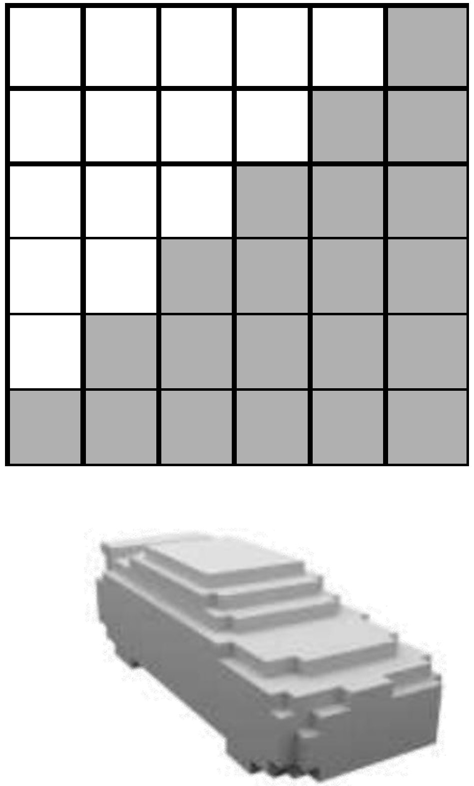
- Plane: $P = \{\mathbf{P}_0 + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$

— $\mathbf{P}_0$: Position vector

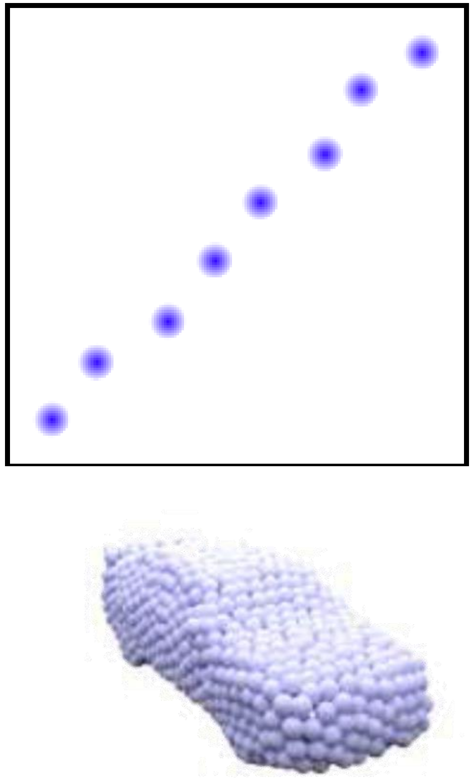


Examples of Explicit **3D Representations**

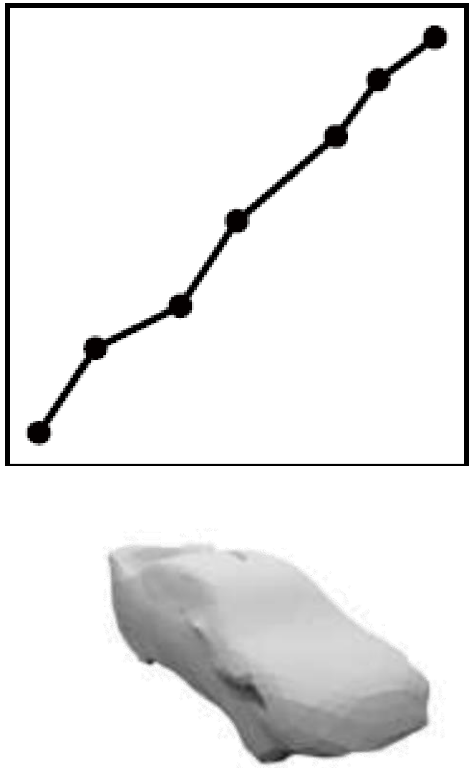
- Most of 3D representations *discretize* the output space differently:



Voxel-based
representation



Point-based
representation



Mesh-based
representation

Next Lecture

- Matrix