[C2-001] 기초수학

Lecture 05: Affine Transformation

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Topics

- More Determinant

- Affine Transformation

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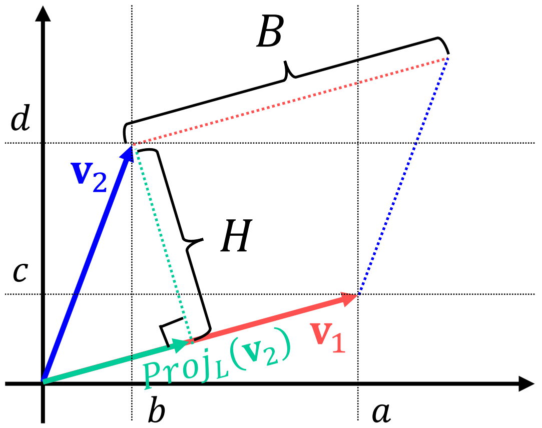
More Determinant: Area of a Parallelogram

- Determinant and Area of a Parallelogram
  - $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2]$, $\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$



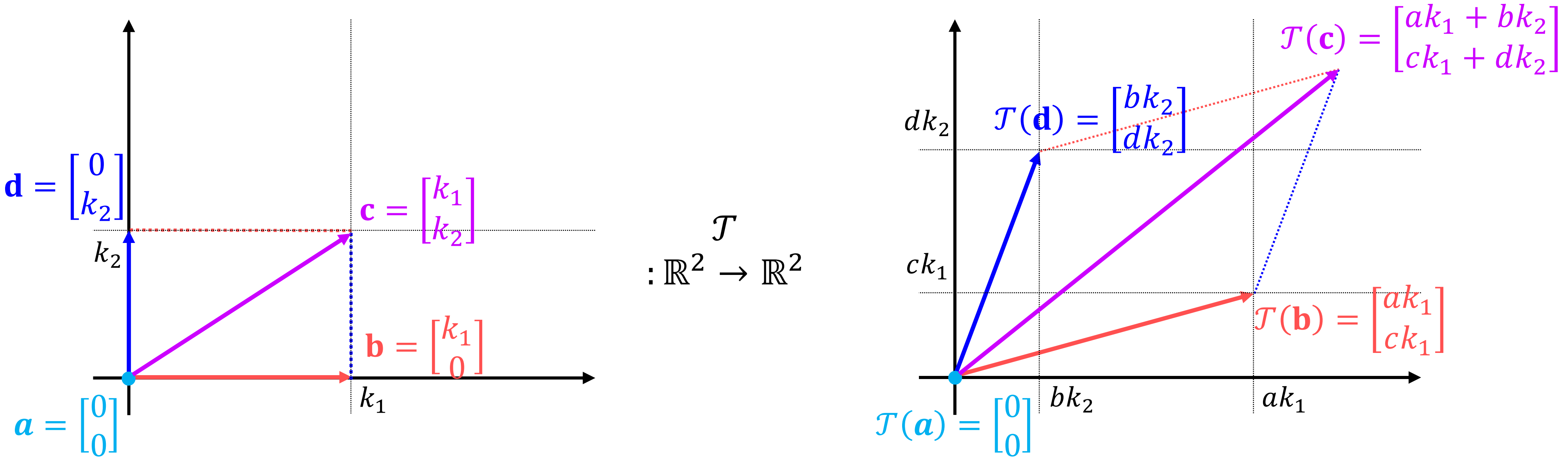
  - Area of a parallelogram
    - $B = \|\mathbf{v}_1\| \longrightarrow B^2 = \|\mathbf{v}_1\|^2 = \mathbf{v}_1 \cdot \mathbf{v}_1$
    - $H^2 = \|\mathbf{v}_2\|^2 - \|Proj_L(\mathbf{v}_2)\|^2 = \mathbf{v}_2 \cdot \mathbf{v}_2 - \left\|\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\right\|^2 = \mathbf{v}_2 \cdot \mathbf{v}_2 - \left(\frac{(\mathbf{v}_2 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_1)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{v}_1)}\mathbf{v}_1 \cdot \mathbf{v}_1\right)$
    - $S = BH \longrightarrow S^2 = B^2H^2 = (\mathbf{v}_1 \cdot \mathbf{v}_1)\left(\mathbf{v}_2 \cdot \mathbf{v}_2 - \left(\frac{(\mathbf{v}_2 \cdot \mathbf{v}_1)^2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\right) = (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1)^2$

      $= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 = a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - (a^2b^2 + 2abcd + c^2d^2)$

      $= a^2d^2 - 2abcd + c^2d^2 = (ad - bc)^2 = \left(det(\mathbf{A})\right)^2$
    - $\therefore S = |det(\mathbf{A})|$

More Determinant: Scaling

- Determinant as Scaling Factor
  - $\mathcal{T}:\ \mathbb{R}^2 \rightarrow \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}$
  - $\mathbf{R} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \xrightarrow{\mathcal{T}} \mathcal{T}(\mathbf{R}) = \begin{bmatrix} ak_1 & bk_2 \\ ck_1 & dk_2 \end{bmatrix} = \mathbf{P}$
  - $det(\mathbf{P}) = |k_1 k_2 ad - k_1 k_2 bc| = |k_1 k_2 (ad - bc)| = |k_1 k_2 \cdot det(\mathbf{A})|$



Topics

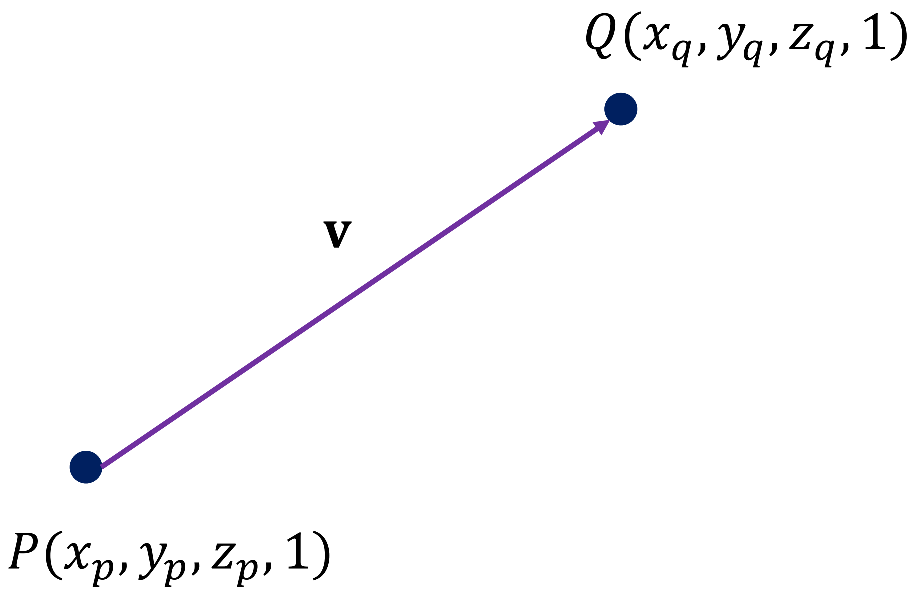
- More Determinant
- Affine Transformation

Affine Space

- What Affine Space is

— Points can be related to vectors by means of an affine space

— An affine space consists of a set of points W and a vector space V



Affine Transformation

- Matrix Definition for Affine Transformation
  - $\mathbf{Ax} + \mathbf{y}$
    - $\mathbf{A}$: $m \times n$ matrix
    - $\mathbf{x}$: the point coordinates $(x_1, x_1, \cdots, x_n)$
    - $\mathbf{y}$: m - dimensional vector

$$\begin{bmatrix} \mathbf{A} & \mathbf{y} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Ax} + \mathbf{y} \\ 1 \end{bmatrix}$$

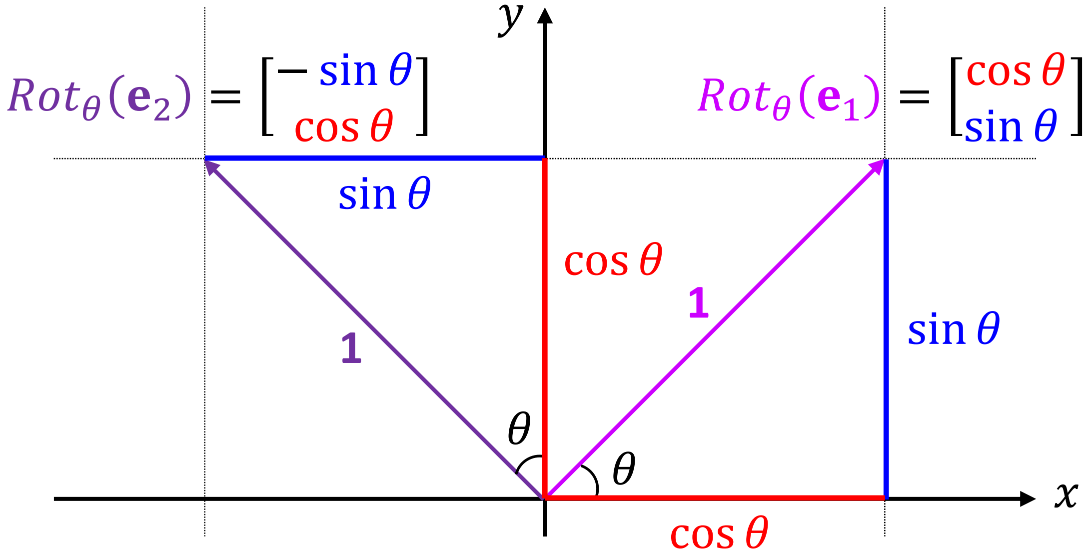
Affine Transformation: **Translation**

- Translation
  - For a single point, it is the same as adding a vector $\mathbf{t}$ to it.
  - All points are shifted equally in space, the size and shape of the object will not change.
  - $\mathcal{T}(O) = \mathbf{t} + O = t_x\hat{\mathbf{i}} + t_y\hat{\mathbf{j}} + t_z\hat{\mathbf{k}} + O$
  - $\mathcal{T}(\hat{\mathbf{i}}) = \mathcal{T}(P - Q) = \mathcal{T}(P) - \mathcal{T}(Q) = (\mathbf{t} + P) - (\mathbf{t} + Q) = P - Q = \hat{\mathbf{i}}$

- Generalized Translation Matrix
  - $\mathbf{T_t} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$

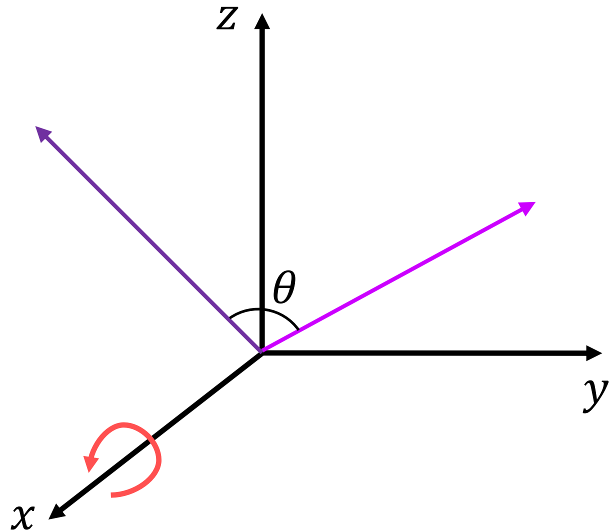
Affine Transformation: **2D Rotation**

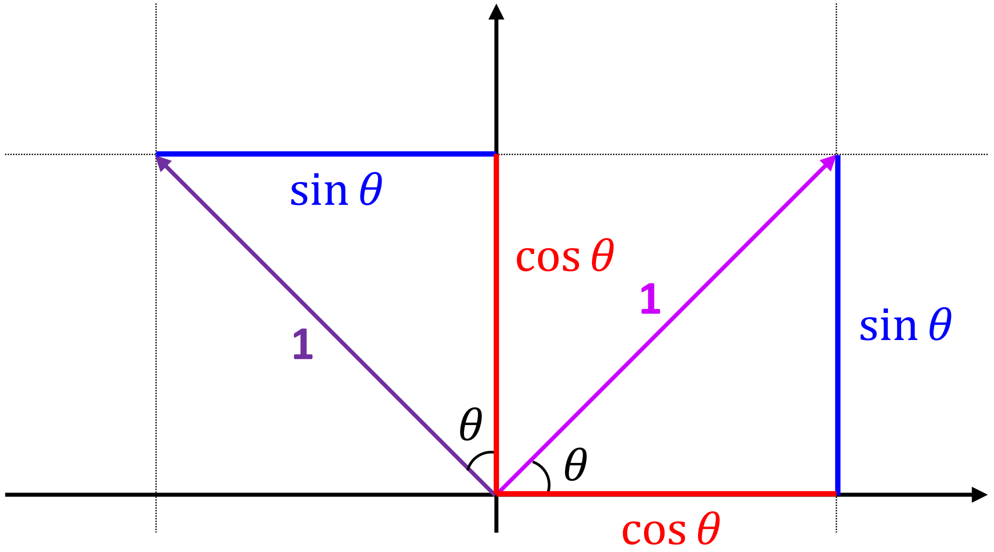
- Rotation in $\mathbb{R}^2$
  - Considering the rotation of a vector, its direction is rigidly changed around an axis without changing its length.
  - $Rot_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2,\ Rot_\theta(\mathbf{x}) = \mathbf{Ax},\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2]$
  - $Rot_\theta(\mathbf{x}) = \mathbf{Ax} = [Rot_\theta(\mathbf{e}_1) \quad Rot_\theta(\mathbf{e}_2)]\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\mathbf{x}$



Affine Transformation: **3D Rotation**

- Rotation in $\mathbb{R}^3$ Under the $\boldsymbol{x}$-axis
  - $Rot_\theta : \mathbb{R}^3 \to \mathbb{R}^3,\ Rot_\theta(\mathbf{x}) = \mathbf{R}_x\mathbf{x},\ \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$
  - $\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$





Affine Transformation: **3D Rotation**

- Rotation in $\mathbb{R}^3$ Under Each axis
  - Their determinants are equal to 1, and these are all orthogonal.
  - $\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $\mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$, $\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Generalized Rotation Matrix
  - $\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z = \mathbf{R} = \begin{bmatrix} C_yC_z & -C_yS_z & S_y \\ S_xS_yC_z + C_xS_z & -S_xS_yS_z + C_xC_z & -S_xC_y \\ -C_xS_yC_z + S_xS_z & C_xS_yS_z + S_xC_z & C_xC_y \end{bmatrix} \longrightarrow \mathbf{R}_{xyz} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$

Affine Transformation: **Scaling**

- Scaling
  - It is like a scalar multiplication but not quite the same. In scaling in affine transformation, we consider the positive factor as a scale factor.
  - $\mathcal{T}(\hat{\mathbf{i}}) = s_x\hat{\mathbf{i}},\ \mathcal{T}(\hat{\mathbf{j}}) = s_y\hat{\mathbf{j}},\ \mathcal{T}(\hat{\mathbf{k}}) = s_z\hat{\mathbf{k}}$
    - Uniform scaling: all scale factors are equal
    - Non-uniform scaling: different scale factors in each axis

- Generalized Scaling Matrix
  - $\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \longrightarrow \mathbf{S}_{xyz} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$

Affine Transformation: **Reflection**

- Reflection
  - The reflection transformation symmetrically maps an object across a plane or though a point.
  - Examples of possible reflection
    - $\begin{matrix} x' = -x \\ y' = y \\ z' = z \end{matrix}, \quad \begin{matrix} x' = x \\ y' = -y \\ z' = z \end{matrix}, \quad \begin{matrix} x' = -x \\ y' = y \\ z' = -z \end{matrix}$

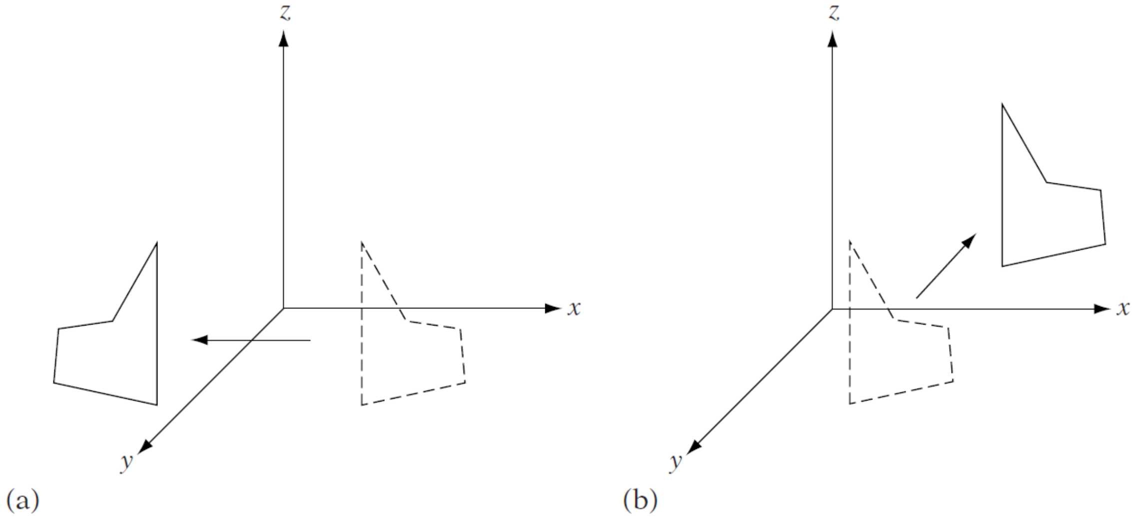


FIGURE 4.9 (a) yz reflection, and (b) xz reflection.

- Reflection Matrix through the Origin
  - $\mathbf{F}_O = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \mathbf{F} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$

Using Affine Transformation

- Matrix Composition
  - For the final world transformation, we will concatenate a sequence of these translation, rotation, and scaling transformations together.
  - Note: The concatenation of transformations is not commutative.

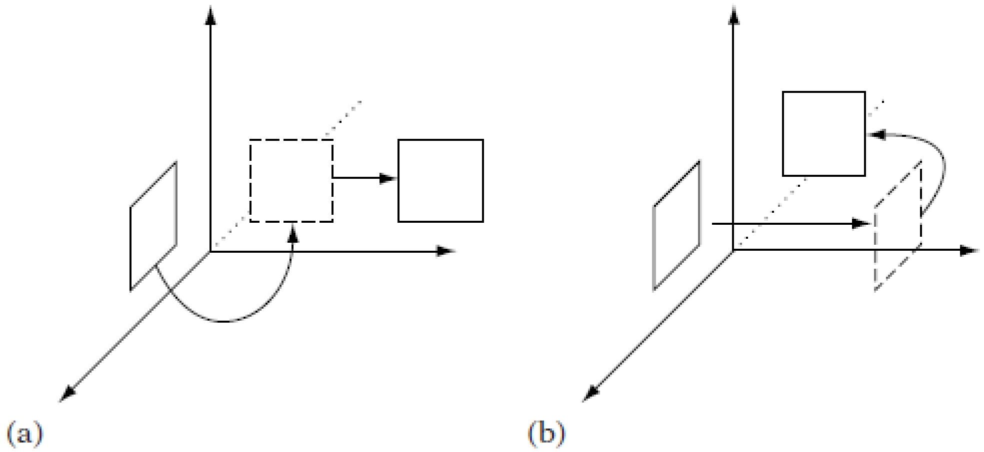


FIGURE 4.16 (a) Rotation, then translation and (b) translation, then rotation.

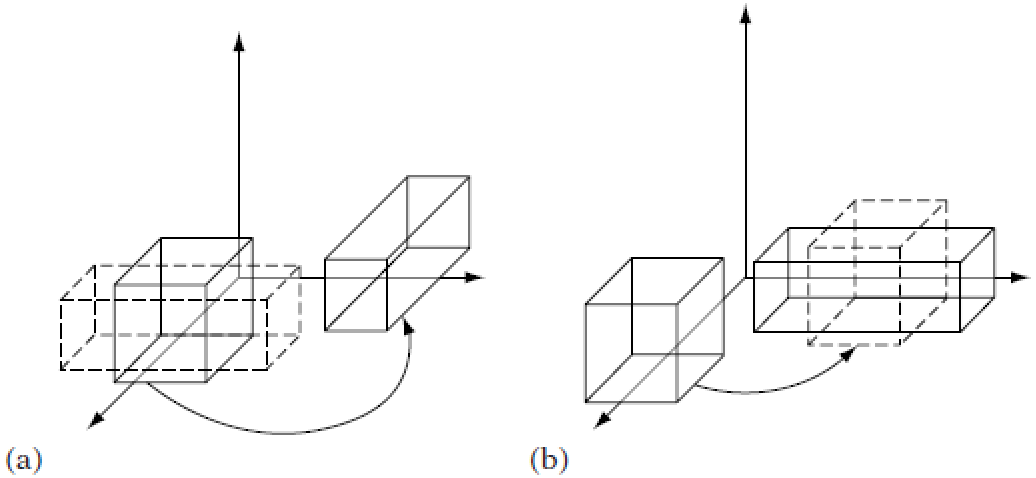
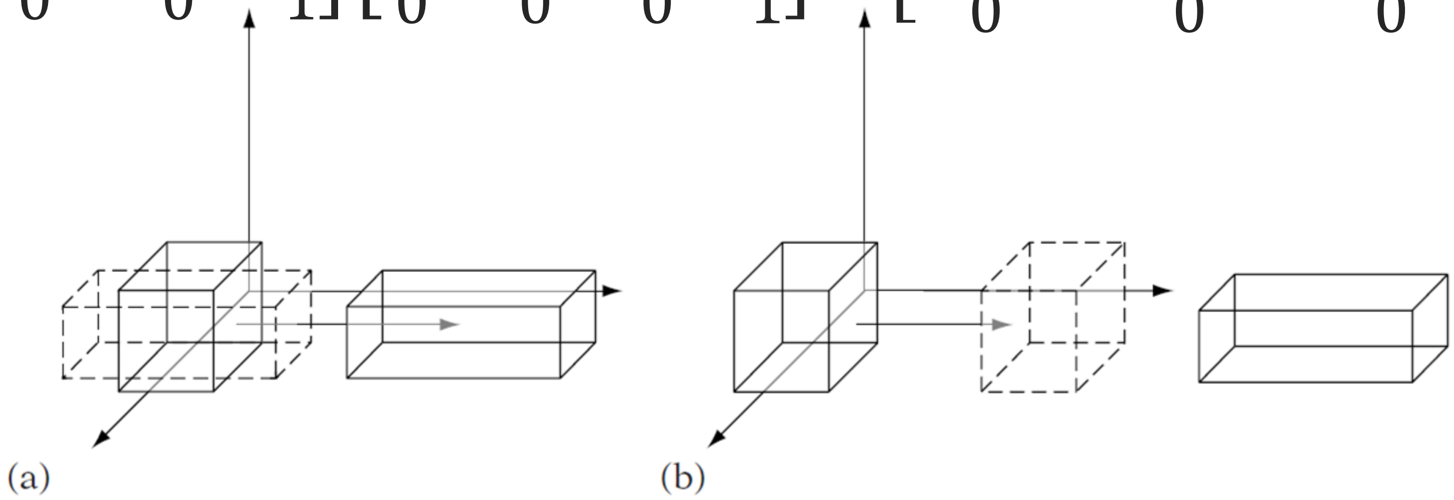


FIGURE 4.17 (a) Scale, then rotation and (b) rotation, then scale.

Using Affine Transformation

- Matrix Decomposition
  - It is sometimes useful to break an affine transformation matrix into its component basic affine transformations.
  - This is called matrix decomposition.
  - $\mathbf{RS} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x r_{11} & s_y r_{12} & s_z r_{13} & 0 \\ s_x r_{21} & s_y r_{22} & s_z r_{23} & 0 \\ s_x r_{31} & s_y r_{32} & s_z r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



Linear Transformation: **Projection**

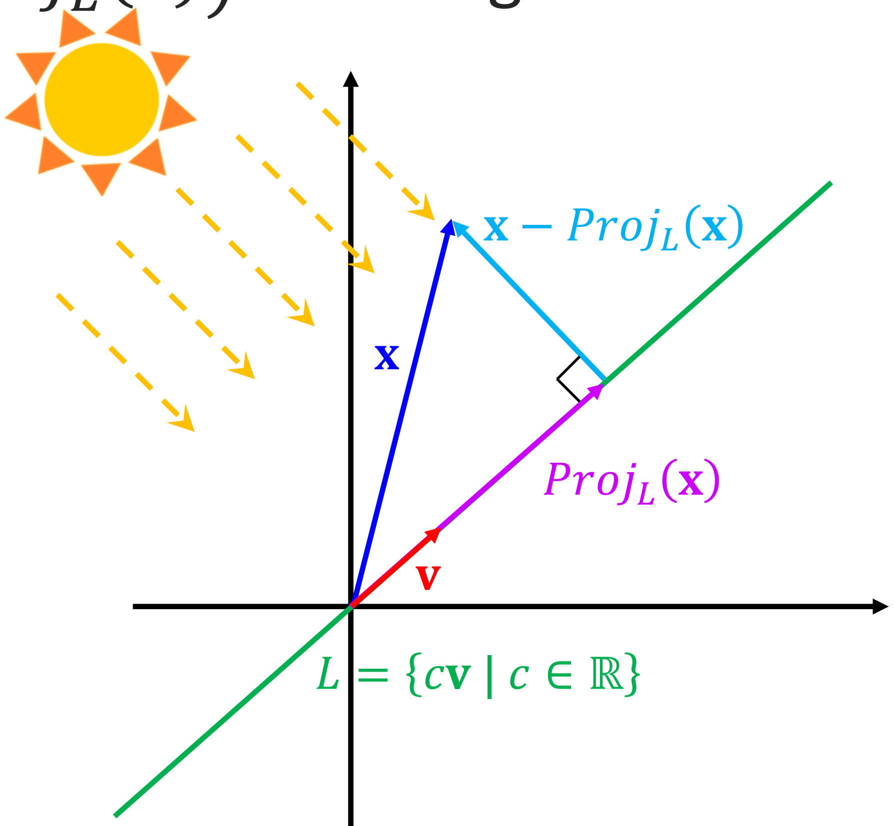
- Introduction To Projection

① $\mathrm{Proj}_{\mathrm{L}}(\mathbf{x})$: Shadow of $\mathbf{x}$ on L

② $Proj_L(\mathbf{x})$: Same vector in L where $\left(\mathbf{x} - Proj_L(\mathbf{x})\right)$ is orthogonal to $L = c\mathbf{v}$

- $(\mathbf{x} - c\mathbf{v}) \cdot \mathbf{v} = 0 \longrightarrow \mathbf{x} \cdot \mathbf{v} - c\mathbf{v} \cdot \mathbf{v} = 0$
  $\longrightarrow \mathbf{x} \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} \longrightarrow c = \frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$

- $Proj_L(\mathbf{x}) = c\mathbf{v} = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$



Linear Transformation: **Projection**

- Projection As Matrix-Vector Product
  - $Proj_L: \mathbb{R}^n \to \mathbb{R}^n,\ Proj_L(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{x}\cdot\mathbf{v})\mathbf{v}\ \ (\mathbf{v}\text{: unit vector}, \hat{\mathbf{u}})$
  - $Proj_L(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{x}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$

- Linear Transform of Projection

① $Proj_L(\mathbf{a}+\mathbf{b}) = \big((\mathbf{a}+\mathbf{b})\cdot\hat{\mathbf{u}}\big)\hat{\mathbf{u}} = (\mathbf{a}\cdot\hat{\mathbf{u}}+\mathbf{b}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = (\mathbf{a}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{b}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ $= Proj_L(\mathbf{a}) + Proj_L(\mathbf{b})$

② $Proj_L(c\mathbf{a}) = \big((c\mathbf{a})\cdot\hat{\mathbf{u}}\big)\hat{\mathbf{u}} = c(\mathbf{a}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = cProj_L(\mathbf{a})$

- $Proj_L(\mathbf{x}) = (\mathbf{x}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{A}\mathbf{x} = \left[\left(\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}\quad\left(\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}\right]\mathbf{x}$

$$= \left[u_1\begin{bmatrix}u_1\\u_2\end{bmatrix}\quad u_2\begin{bmatrix}u_1\\u_2\end{bmatrix}\right]\mathbf{x} = \begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}\mathbf{x}$$

Next Lecture

- Eigenvalues and Eigenvectors