[C2-001] 기초수학

Lecture 02: Matrix & Linear Transformation I

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Recap: Vectors

- Linear Combination

— Given $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, we can create a new vector $\mathbf{v}$ like this:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \ \ where\ c_1, \cdots, c_n \in \mathbb{R}$$

- Span

— If we take all the possible linear combinations of all vectors in $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, the set T of vectors thus created is the Span of S

$$Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

Recap: Vectors

- Linear Dependent

— For $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \; where \; c_1, \cdots, c_n \in \mathbb{R}$,
$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly dependent $\Leftrightarrow c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$
for c_i, not all are zero (at least 1 is non-zero)

- Linear Independent

— For $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \; where \; c_1, \cdots, c_n \in \mathbb{R}$,
$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent $\Leftrightarrow c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ for c_i, only solution is $c_i = 0$ for $1 \leq i \leq n$

Recap: Vectors

- Dot Product

— In any $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$

— $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$

— How much two vectors move in the same direction

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

- Cross Product

— Only defined in $\mathbb{R}^3$

— $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$

— How perpendicular two vectors are

$$\mathbf{v} \times \mathbf{w} = \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

Topics

- Planes
- Matrix
- Linear Transformation I

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Planes

- Generalized Equation of A Plane in $\mathbb{R}^3$

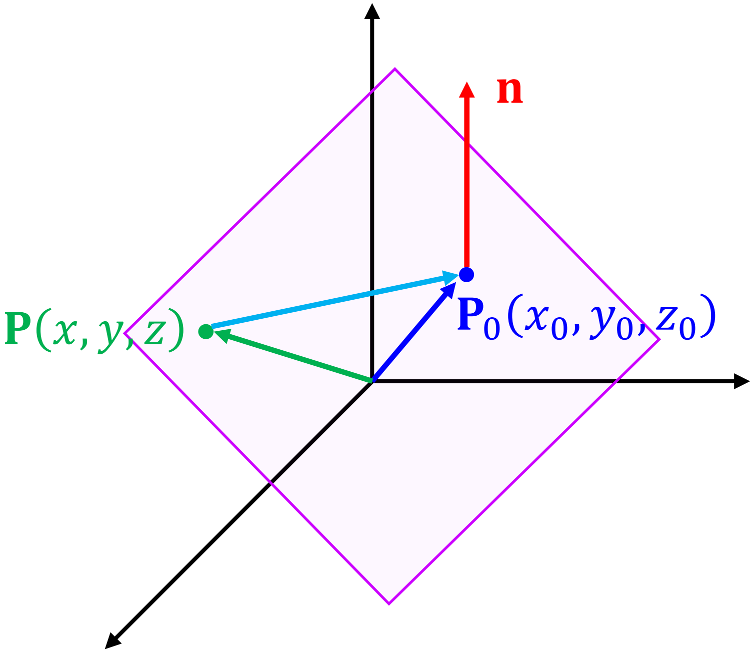
— A surface which lies evenly with the straight lines on itself

- Normal vector $\mathbf{n}$: It is perpendicular to everything on the plane: $\mathbf{n} \cdot \mathbf{v} = 0$

— $\mathbf{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{P}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

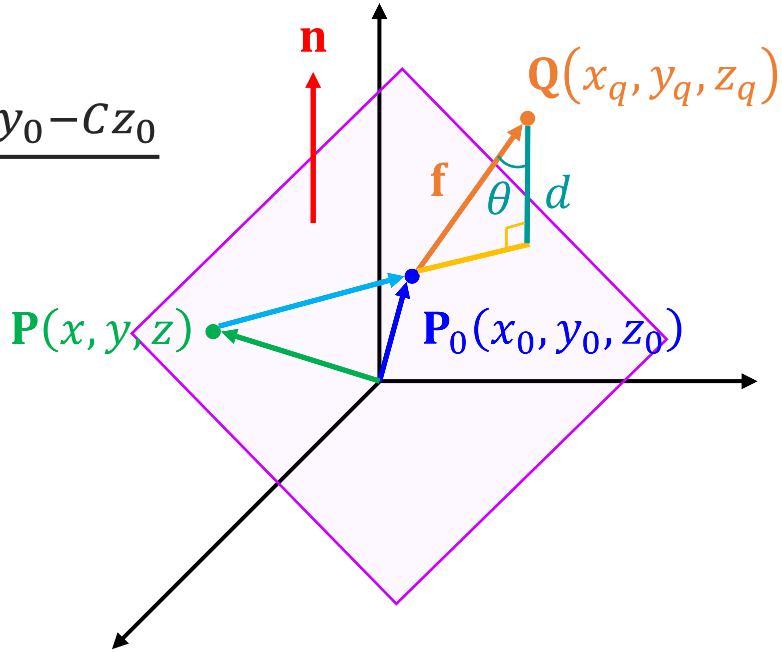
— $\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_0) = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$

— $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$



Planes

- Distance From Point To Plane
  - $\mathbf{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \mathbf{P}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix}$
  - $\mathbf{f} = (x_0 - x_q)\hat{\mathbf{i}} + (y_0 - y_q)\hat{\mathbf{j}} + (z_0 - z_q)\hat{\mathbf{k}}$
  - $d = \|\mathbf{f}\| \cos\theta \quad \rightarrow \qquad d = \dfrac{\|\mathbf{n}\|\|\mathbf{f}\| \cos\theta}{\|\mathbf{n}\|} = \dfrac{|\mathbf{n}\cdot\mathbf{f}|}{\|\mathbf{n}\|}$
  - $\dfrac{|\mathbf{n}\cdot\mathbf{f}|}{\|\mathbf{n}\|} = \dfrac{A(x_0-x_q)+B(y_0-y_q)+C(z_0-z_q)}{\sqrt{A^2+B^2+C^2}} = \dfrac{Ax_q+By_q+Cz_q-Ax_0-By_0-Cz_0}{\sqrt{A^2+B^2+C^2}}$



- Example of Distance From Point To Plane
  - Distance from $\mathbf{Q}(2, 3, 1)$ to plane $x - 2y + 3z = 5$
  - $d = \dfrac{1\cdot2-2\cdot3+3\cdot1-5}{\sqrt{1+4+9}} = \dfrac{-6}{\sqrt{14}}$

Planes

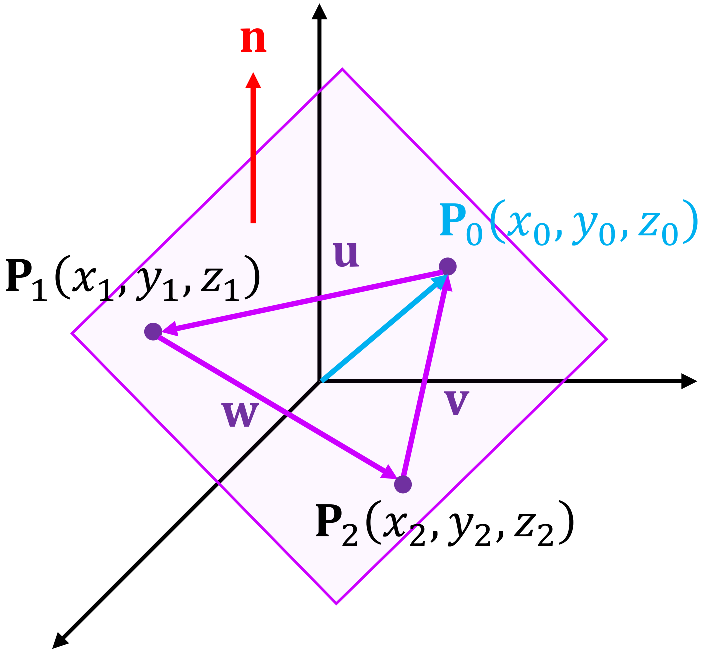
- Parametric Representation of A Plane

— Parametric equation is one possible representation of a generalized plane across all dimensions

- Set of coplanar vectors: $S = \{s\mathbf{u}, t\mathbf{v} \mid s, t \in \mathbb{R}\}$

- Plane: $P = \{\mathbf{P}_0 + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$

— $\mathbf{P}_0$: Position vector



Topics

- Planes
- Matrix
- Linear Transformation I

Introduction: **Matrix**

- A matrix is a rectangular, 2D array of values
— Each individual value in a matrix is called an element

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

- A matrix is described as having m rows by n columns, or being a $m \times n$ matrix
— A row is a horizontal group of elements from left to right
— A column is a vertical, top-to-bottom group

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 35 & -15 \\ 2 & 52 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 6 & 3 \end{bmatrix}$$

Introduction: **Matrix**

- Trace of A Matrix

— The trace of a matrix is the sum of the main diagonal elements

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 8 \\ -4 & 9 & 5 \end{bmatrix}$$

- Triangular & Diagonal Matrix
  - In matrix $\mathbf{U}$, if all elements below the diagonal are 0, it is an upper triangular matrix
  - In matrix $\mathbf{L}$, if all elements above the diagonal are 0, it is a lower triangular matrix
  - A diagonal matrix, $\mathbf{D}$, is a square matrix that has non-diagonal elements of zero

$$\mathbf{U} = \begin{bmatrix} 3 & -5 & 0 & 1 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -6 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Simple Operations of Matrix

- Matrix Addition
  - $\mathbf{S} = \mathbf{A} + \mathbf{B}$
  - $\begin{bmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & \ddots & \vdots \\ s_{m,1} & \cdots & s_{m,n} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$

- Scalar Multiplication
  - $\mathbf{P} = c\mathbf{A}$
  - $\begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{m,1} & \cdots & p_{m,n} \end{bmatrix} = \begin{bmatrix} c \cdot a_{1,1} & \cdots & c \cdot a_{1,n} \\ \vdots & \ddots & \vdots \\ c \cdot a_{m,1} & \cdots & c \cdot a_{m,n} \end{bmatrix}$

Simple Operations of Matrix

- Algebraic Rules of Matrix Addition and Scalar Multiplication
  - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
  - $\mathbf{A} + \mathbf{0} = \mathbf{A}$
  - $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
  - $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$
  - $a(b\mathbf{A}) = (ab)\mathbf{A}$
  - $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$
  - $1\mathbf{A} = \mathbf{A}$

Simple Operations of Matrix

- Transpose

— The transpose of a matrix $\mathbf{A}$, $\mathbf{A}^T$, interchanges the rows and columns of $\boldsymbol{A}$

— It does this by exchanging elements across the matrix's main diagonal

  - $(\mathbf{A}^T)_{i,j} = (\mathbf{A})_{j,i}$

— The main diagonal doesn't change, or is invariant

  - $(\mathbf{A}^T)_{i,i} = (\mathbf{A})_{i,i}$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 6 & 3 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 2 & 3 \end{bmatrix}$$

Simple Operations of Matrix

- Symmetric Matrix

— A symmetric matrix is a matrix where cross-diagonal entries are equal

- $(\mathbf{A})_{i,j} = (\mathbf{A})_{j,i}$
- $(\mathbf{A}^T)_{j,i} = (\mathbf{A})_{i,j} = (\mathbf{A})_{j,i}$

— A skew symmetric matrix is a matrix where cross-diagonal entries are negated and the diagonal is 0

- $(\mathbf{A})_{i,j} = -(\mathbf{A})_{j,i}$
- $(\mathbf{A}^T)_{j,i} = (\mathbf{A})_{i,j} = -(\mathbf{A})_{j,i}$

$$\mathbf{S} = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 1 & 2 & -5 & 0 \\ 2 & -5 & 1 & 9 \\ 3 & 0 & 9 & 1 \end{bmatrix}$$

Symmetry matrix

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & -5 & 0 \\ -2 & 5 & 0 & 9 \\ -3 & 0 & -9 & 0 \end{bmatrix}$$

Skew symmetry matrix

Simple Operations of Matrix

- Algebraic Rules Involving The Transpose
  - $(\mathbf{A}^T)^T = \mathbf{A}$
  - $(a\mathbf{A}^T) = a\mathbf{A}^T$
  - $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

Vector Representation

- Block Matrix
  - A matrix can be represented by submatrices, rather than by individual elements.
  - This is know as a *block matrix*.

$$\mathbf{B} = \begin{bmatrix} 2 & 3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

  - A matrix can be represented by a set of row or column matrices.

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

Vector Representation

- Matrix Product
  - Multiplying a matrix by a compatible vector will *transform* the vector.
  - Multiplying matrices together will create a single matrix that performs their *combined transformations*.
  - $\mathbf{C} = \mathbf{AB}$

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{bmatrix}$$

Vector Representation

- Representation of Matrix Multiplication

— By a collection of rows and a collection of columns:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \end{bmatrix} [\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{bmatrix}$$

— By using block matrices:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{AE} + \mathbf{BG} & \mathbf{AF} + \mathbf{BH} \\ \mathbf{CE} + \mathbf{DG} & \mathbf{CD} + \mathbf{DH} \end{bmatrix}$$

Vector Representation

- Algebraic Rules for Matrix Multiplication
  - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
  - $a(\mathbf{BC}) = (a\mathbf{B})\mathbf{C}$
  - $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
  - $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
  - $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Identity Matrix

- The identity matrix $\mathbf{I}$ is like a scalar or vector by 1 in matrix multiplication

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} = \mathbf{A}$$

— A particular identity matrix is a diagonal square matrix (diagonal is all 1s)

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

— If a particular identity matrix is needed, it is sometimes referred to as $\mathbf{I}_n$

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topics

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Matrix-Vector Multiplication

- $\mathbf{Ax} = \mathbf{b}$
  - $\mathbf{A}$: $m \times n$
  - $\mathbf{x}$: $n \times 1$
  - $\mathbf{b}$: $m \times 1$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_m \cdot \mathbf{x} \end{bmatrix}$$

Matrix-Vector Multiplication

- Linear Combination of column vectors of a matrix
  - $\mathbf{Ax} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$

- Example of Matrix-Vector Multiplication
  - $\begin{bmatrix} -3 & 0 & 3 & 2 \\ 1 & 7 & -1 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} (-3)\cdot 2 + 0 \cdot (-3) + 3 \cdot 4 + 2 \cdot (-1) \\ 1 \cdot 2 + 7 \cdot (-3) + (-1) \cdot 4 + 9 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \end{bmatrix}$

Linear System

- Solving Linear Systems using Reduced Row Echelon Form (RREF)
  - $$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 7 \\ x_1 + 2x_2 + 2x_3 - x_4 &= 12 \\ 2x_1 + 4x_2 + \quad +6x_4 &= 4 \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 1 & 1 & 7 \\ 0 & 0 & -1 & 2 & -5 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right] \rightarrow$$
  - $$\left[\begin{array}{cccc|c} \mathbf{1} & 2 & 1 & 1 & 7 \\ 0 & 0 & \mathbf{1} & -2 & 5 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 1 & 1 & 7 \\ 0 & 0 & \mathbf{1} & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 3 & 2 \\ 0 & 0 & \mathbf{1} & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] : rref(\mathbf{A})$$
  - $$\begin{aligned} x_1 &= 2 - 2x_2 - 3x_4 \\ x_3 &= 5 + 2x_4 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Linear System

- Solving Linear Systems using Reduced Row Echelon Form (RREF)
  - $\begin{matrix} x+y+z=3 \\ x+2y+3z=0 \\ x+3y+4z=-2 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 3 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow$
  - $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & -1 & 6 \\ 0 & \mathbf{1} & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 5 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & -1 \end{array}\right] : rref(\mathbf{A})$
  - $\begin{matrix} x=5 \\ y=-1 \\ z=-1 \end{matrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$

Linear System

- Solving Linear Systems using Reduced Row Echelon Form (RREF)

- $\begin{matrix} x_1 + 2x_2 + x_3 + x_4 = 8 \\ x_1 + 2x_2 + 2x_3 - x_4 = 12 \\ 2x_1 + 4x_2 + \quad +6x_4 = 4 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & |\ 8 \\ 1 & 2 & 2 & -1 & |12 \\ 2 & 4 & 0 & 6 & |\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |8 \\ 0 & 0 & \mathbf{1} & -2 & |4 \\ 2 & 4 & 0 & 6 & |4 \end{bmatrix} \rightarrow$

- $\begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |\ 8 \\ 0 & 0 & \mathbf{1} & -2 & |\ 4 \\ 0 & 0 & -2 & 4 & |-12 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |\ 8 \\ 0 & 0 & \mathbf{1} & -2 & |\ 4 \\ 0 & 0 & 0 & 0 & |-4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 & |\ 4 \\ 0 & 0 & \mathbf{1} & -2 & |\ 4 \\ 0 & 0 & 0 & 0 & |-4 \end{bmatrix}$

$: rref(\mathbf{A})$

- $\begin{matrix} x_1 + 2x_2 + x_4 = 4 \\ x_3 - 2x_4 = 4 \\ 0 = -4 \end{matrix}$ **: No solution**

Linear Transformation I: **Null Space**

- Null Space

— The null space is the set all vectors in V that map to **0**

— $N = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$: Null space of $\mathbf{A}$, $N(\mathbf{A})$

- $\mathbf{Ax} = \mathbf{0}$: Homogeneous eq., $N = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$ is the subspace of **A**?
  - $\mathbf{A0} = \mathbf{0}$
  - $\mathbf{v}_1,\ \mathbf{v}_2 \in N,\ \mathbf{Av}_1 = \mathbf{0},\ \mathbf{Av}_2 = \mathbf{0}\ \rightarrow \mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{Av}_1 + \mathbf{Av}_2 = \mathbf{0}$
  - $\mathbf{v}_1 \in N,\ c \in \mathbb{R},\ \mathbf{A}(c\mathbf{v}_1) = c\mathbf{Av}_1 = \mathbf{0}$

Linear Transformation I: **Null Space**

- Example of Null Space
  - $\mathbf{Ax} = \mathbf{0} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} : N(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^4 \mid \mathbf{Ax} = \mathbf{0}\}$
  - $\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & | & 0 \\ 1 & 2 & 3 & 4 & | & 0 \\ 4 & 3 & 2 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 0 & -1 & -2 & | & 0 \\ 0 & \mathbf{1} & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} : rref(\mathbf{A})$$

Linear Transformation I: **Null Space**

- Example of Null Space
  - $\begin{bmatrix} \color{red}{\mathbf{1}} & 0 & -1 & -2 & | & 0 \\ 0 & \color{red}{\mathbf{1}} & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} : rref(\mathbf{A})$
  - $\begin{matrix} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{matrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$
  - $N(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) = N(rref(\mathbf{A}))$

Linear Transformation I: **Null Space**

- Relation between Null Space and Column Vector of Matrix $\mathbf{A}$
  - $\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$
  - $\mathbf{Ax} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$
  - $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent

    $\Leftrightarrow x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ for x_i, only solution is $x_i = 0\ for\ 1 \leq i \leq n$

    $\Leftrightarrow N(\mathbf{A}) = \{\mathbf{0}\} \;\leftarrow x_1, x_2, \cdots, x_n = 0$

Linear Transformation I: **Column Space**

- The column space is the vector space spanned by the matrix's column vectors
  - $\mathbf{A} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n],\ \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^m \ \rightarrow C(\mathbf{A}) = Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$
  - $\{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \mid x_1, x_2, \cdots, x_n \in \mathbb{R}\} = Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) = C(\mathbf{A})$
  - $\mathbf{Ax} = \mathbf{b}_1,\ if\ \mathbf{b}_1 \notin C(\mathbf{A}) \ \Longrightarrow \mathbf{Ax} = \mathbf{b}_1$ has no solution
  - $\mathbf{Ax} = \mathbf{b}_2,\ if\ \mathbf{b}_2 \in C(\mathbf{A}) \ \Longrightarrow \mathbf{Ax} = \mathbf{b}_2$ has at least one solution

Linear Transformation I: **Column Space**

- Basis for Column Space & Null Space
  - $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \rightarrow C(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$

- These vectors are basis of $C(\mathbf{A})$?

— If these vectors are *linearly independent*, they would be the basis of $C(\mathbf{A})$

— Linearly independent $\Leftrightarrow N(\mathbf{A}) = \{\mathbf{0}\} \Leftrightarrow N\big(rref(\mathbf{A})\big) = \{\mathbf{0}\}$

Linear Transformation I: **Column Space**

- Basis for Column Space & Null Space
  - $\mathbf{A}=\begin{bmatrix}1&1&1&1\\2&1&4&3\\3&4&1&2\end{bmatrix}\rightarrow\left[\begin{array}{cccc|c}1&1&1&1&0\\2&1&4&3&0\\3&4&1&2&0\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}\color{red}{\mathbf{1}}&1&1&1&0\\0&1&-2&-1&0\\0&1&-2&-1&0\end{array}\right]\rightarrow$

$$\left[\begin{array}{cccc|c}\color{red}{\mathbf{1}}&1&1&1&0\\0&1&-2&-1&0\\0&0&0&0&0\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}\color{red}{\mathbf{1}}&0&3&2&0\\0&\color{red}{\mathbf{1}}&-2&-1&0\\0&0&0&0&0\end{array}\right]:rref(\mathbf{A})$$

  - $\begin{matrix}x_1=-3x_3-2x_4\\x_2=+2x_3+x_4\end{matrix}\rightarrow N(\mathbf{A})=N\big(rref(\mathbf{A})\big)=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-3\\2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

Linear Transformation I: **Column Space**

- Find the Basis for Column Space
  - $\mathbf{Ax} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}1\\1\\4\end{bmatrix} + x_3\begin{bmatrix}1\\4\\1\end{bmatrix} + x_4\begin{bmatrix}1\\3\\2\end{bmatrix} = \mathbf{0}$
  - $N(\mathbf{A}) = \begin{cases} x_1 = -3x_3 - 2x_4 \\ x_2 = +2x_3 + x_4 \end{cases}$
  - If $x_3 = 0,\ x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}1\\1\\4\end{bmatrix} = -x_4\begin{bmatrix}1\\3\\2\end{bmatrix} \rightarrow x_4 = -1: \begin{cases} x_1 = -3\cdot 0 - 2\cdot(-1) = 2 \\ x_2 = +2\cdot 0 + 1\cdot(-1) = -1 \end{cases}$
  - $2\begin{bmatrix}1\\2\\3\end{bmatrix} + (-1)\begin{bmatrix}1\\1\\4\end{bmatrix} = \begin{bmatrix}1\\3\\2\end{bmatrix}$: Linearly dependent

Linear Transformation I: **Column Space**

- Find the Basis for Column Space
  - If $x_4 = 0,\ x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = -x_3 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \rightarrow x_3 = -1: \begin{cases} x_1 = -3 \cdot (-1) - 2 \cdot 0 = 3 \\ x_2 = +2 \cdot (-1) + 1 \cdot 0 = -2 \end{cases}$
  - $3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$: Linearly dependent
  - $C(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right)$: Linearly independent $\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$: a basis for $C(\mathbf{A})$

Linear Transformation I: **Dimension**

- Nullity is the dimension of null space (= # of free variables in $rref(\mathbf{A})$)
  - $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix}$, $N(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^5 \mid \mathbf{Ax} = \mathbf{0}\} = N(rref(\mathbf{A}))$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 2 & 0 \\ 1 & 1 & 3 & 1 & 4 & 0 \end{array}\right] \rightarrow rref(\mathbf{A}) = \begin{bmatrix} \mathbf{1} & 1 & 0 & 7 & -2 \\ 0 & 0 & \mathbf{1} & -2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \mathbf{1} & 1 & 0 & 7 & -2 \\ 0 & 0 & \mathbf{1} & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\begin{cases} x_1 + x_2 + 7x_4 - 2x_5 = 0 \\ x_3 - 2x_4 + 2x_5 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -x_2 - 7x_4 + 2x_5 \\ x_3 = 2x_4 - 2x_5 \end{cases}$

Linear Transformation I: **Dimension**

- Nullity is the dimension of null space (= # of free variables in $rref(\mathbf{A})$)

— Dimension of a subspace: The number of vectors in a basis for the subspace

  - $dim(\mathbf{A}) = nullity(\mathbf{A}) = 3$

  - $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad N(\mathbf{A}) = N(rref(\mathbf{A})) = Span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$

Linear Transformation I: **Dimension**

- Rank is a dimension of column space

— The dimension of the vector space generated (or spanned) by its columns

- $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix} \rightarrow C(\mathbf{A}) = Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$: Basis for $C(\mathbf{A})$?

- $\rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & -1 & -2 & -2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 & 4 \\ 0 & \mathbf{1} & 2 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- Basis for $C(\mathbf{A}) = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$

- $dim\big(C(\mathbf{A})\big) = rank(\mathbf{A}) = 3$

Next Lecture

- Linear Transformation II