[C2-001] 기초수학

Lecture 06: Eigenvalue & Eigenvector

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Topics

- Eigenvalue and eigenvectors

- Application of eigenvalues

Topics

- Eigenvalue and eigenvectors

- Application of eigenvalues

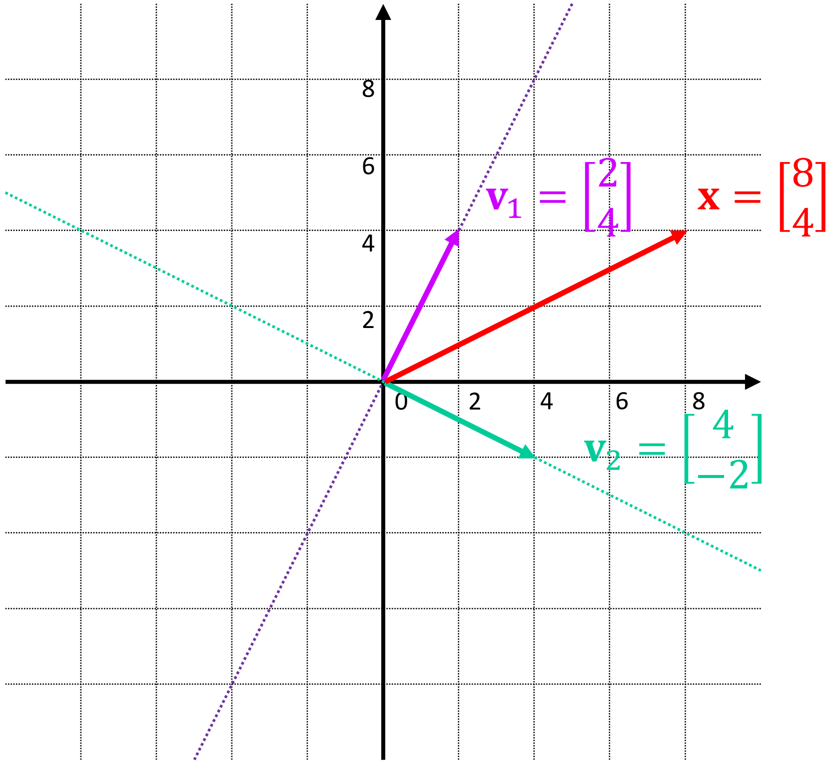
Eigenvalue, Eigenvector, Eigenspace

- Eigenvector
  - If we have an $n \times n$ matrix $\mathbf{A}$, then a nonzero vector $\mathbf{x}$ is called an *eigenvector* if there is some scalar value λ such that $\mathbf{Ax} = \lambda\mathbf{x}$.

- Eigenvalue
  - The scalar value λ is the *eigenvalue* associated with that eigenvector.

- Eigenspace
  - The collection of eigenvectors associated with each λ for the linear transformation applied to the eigenvector.

$\mathcal{T}: \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathcal{T}(\mathbf{v}) = \lambda\mathbf{v}$



Characteristic Equation

- Characteristic Equation of Matrix A
  - $\mathbf{Ax} = \lambda\mathbf{x} \rightarrow \mathbf{Ax} = \lambda\mathbf{Ix}$ for *nonzero* $\mathbf{x}$

    $\lambda\mathbf{Ix} - \mathbf{Ax} = \mathbf{0} \ \rightarrow (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$: It means that $\mathbf{x} \in N(\lambda\mathbf{I} - \mathbf{A})$.

    $\mathbf{Bx} = \mathbf{0},\ N(\mathbf{B}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Bx} = \mathbf{0}\}$

  - The column vectors of matrix $\mathbf{A}$ are linearly independent $\Leftrightarrow N(\mathbf{A}) = \{\mathbf{0}\}$

    $\Rightarrow$ The column vectors of $\lambda\mathbf{I} - \mathbf{A}$ must be *linearly dependent*.

    $\Rightarrow$ *Non-invertible*

    $\Rightarrow det(\lambda\mathbf{I} - \mathbf{A}) = 0$: Characteristic equation of $\mathbf{A}$

Example of **Eigenvalue of** 2×2 **Matrix**

- λ is eigenvalue of $\mathbf{A}$ iff $det(\lambda\mathbf{I} - \mathbf{A}) = 0$
- Find the eigenvalue of a given matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\to det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right) = det\left(\begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix}\right) = 0$$

$$\to \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\to \therefore \lambda = 5, -1$$

Example of **Eigenvector & Eigenspace of** 2×2 **Matrix**

- Eigenspace, $E_\lambda = N(\lambda\mathbf{I} - \mathbf{A})$
- Find the eigenvector and Eigenspace of a given matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\rightarrow \begin{cases} \lambda = 5, \quad E_{\lambda=5} = N\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right) = N\left(\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\right) \\ \lambda = -1, \quad E_{\lambda=-1} = N\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right) = N\left(\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\right) \end{cases}$$

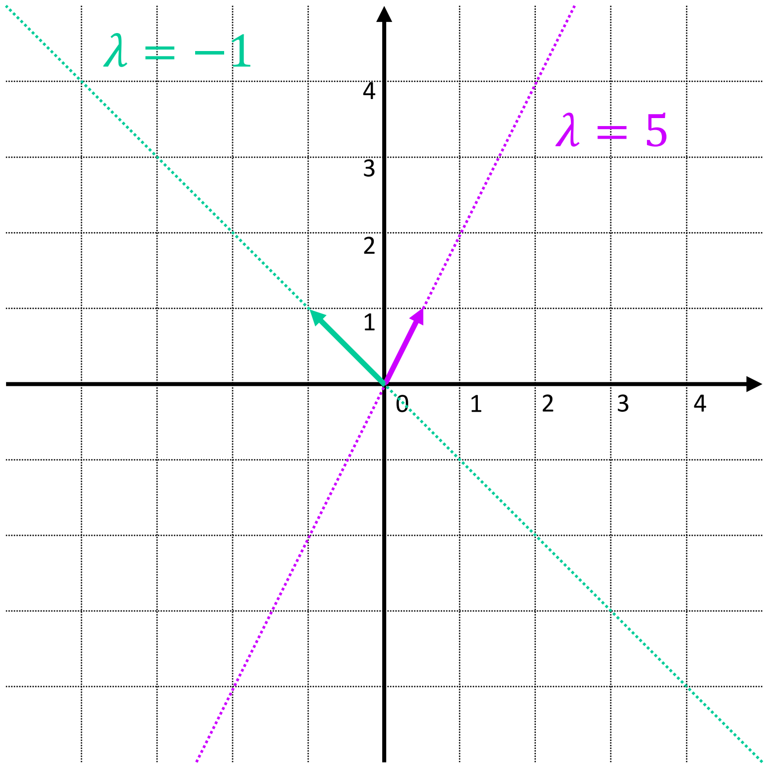
$$\rightarrow \begin{cases} \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 - \frac{1}{2}x_2 = 0 \\ \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 + x_2 = 0 \end{cases}$$

Example of **Eigenvector & Eigenspace of** 2×2 **Matrix**

- Eigenspace, $E_\lambda = N(\lambda \mathbf{I} - \mathbf{A})$
- Find the eigenvector and Eigenspace of a given matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\therefore\ E_{\lambda=5} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \middle|\, t \in \mathbb{R} \right\} = Span\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

$$E_{\lambda=-1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \middle|\, t \in \mathbb{R} \right\} = Span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$



Example of **Eigenvalue of** 3×3 **Matrix**

- Find the eigenvalue of a given matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

$$\rightarrow det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\right) = det\left(\begin{bmatrix} \lambda+1 & -2 & -2 \\ -2 & \lambda-2 & 1 \\ -2 & 1 & \lambda-2 \end{bmatrix}\right) = 0$$

$$\rightarrow \begin{vmatrix} \lambda+1 & -2 & -2 \\ -2 & \lambda-2 & 1 \\ -2 & 1 & \lambda-2 \end{vmatrix} \begin{matrix} \lambda+1 & -2 \\ -2 & \lambda-2 \\ -2 & 1 \end{matrix}$$

$$= (\lambda+1)(\lambda-2)(\lambda-2) + 4 + 4 - 4(\lambda-2) - (\lambda+1) - 4(\lambda-2)$$

$$= (\lambda+1)(\lambda^2 - 4\lambda + 4 + 8 - 9\lambda + 15) = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = (\lambda-3)(\lambda^2-9) = 0$$

$$\therefore \lambda = 3, -3$$

Example of **Eigenvalue of** 3×3 **Matrix**

- Find the eigenvalue of a given matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

$$\rightarrow \lambda = 3: \begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0 \rightarrow x_1 = \frac{1}{2}x_2 + \frac{1}{2}x_3$$

$$\therefore E_{\lambda=3} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\} = Span\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Example of **Eigenvalue of** 3×3 **Matrix**

- Find the eigenvalue of a given matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

$$\rightarrow \lambda = -3: \begin{bmatrix} -2 & -2 & -2 \\ -2 & -5 & 1 \\ -2 & 1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_2 \\ \qquad\qquad x_2 - x_3 = 0 \end{cases}$$

$$\therefore E_{\lambda=-3} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} = Span\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)$$

Topics

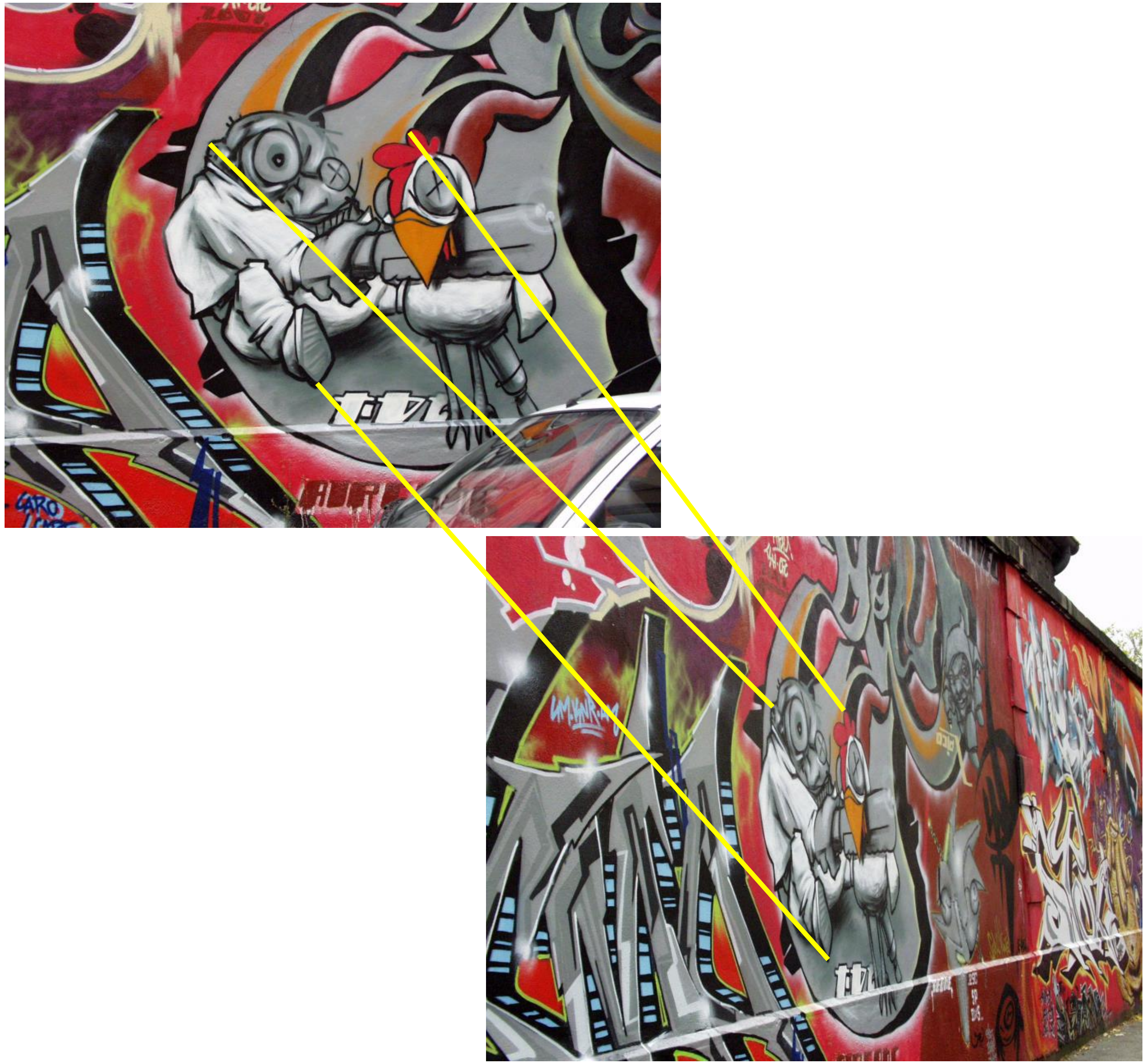
- Eigenvalue and eigenvectors
- Application of eigenvalues

Applications of Features

Object recognition

Localization in robots

Applications of Features



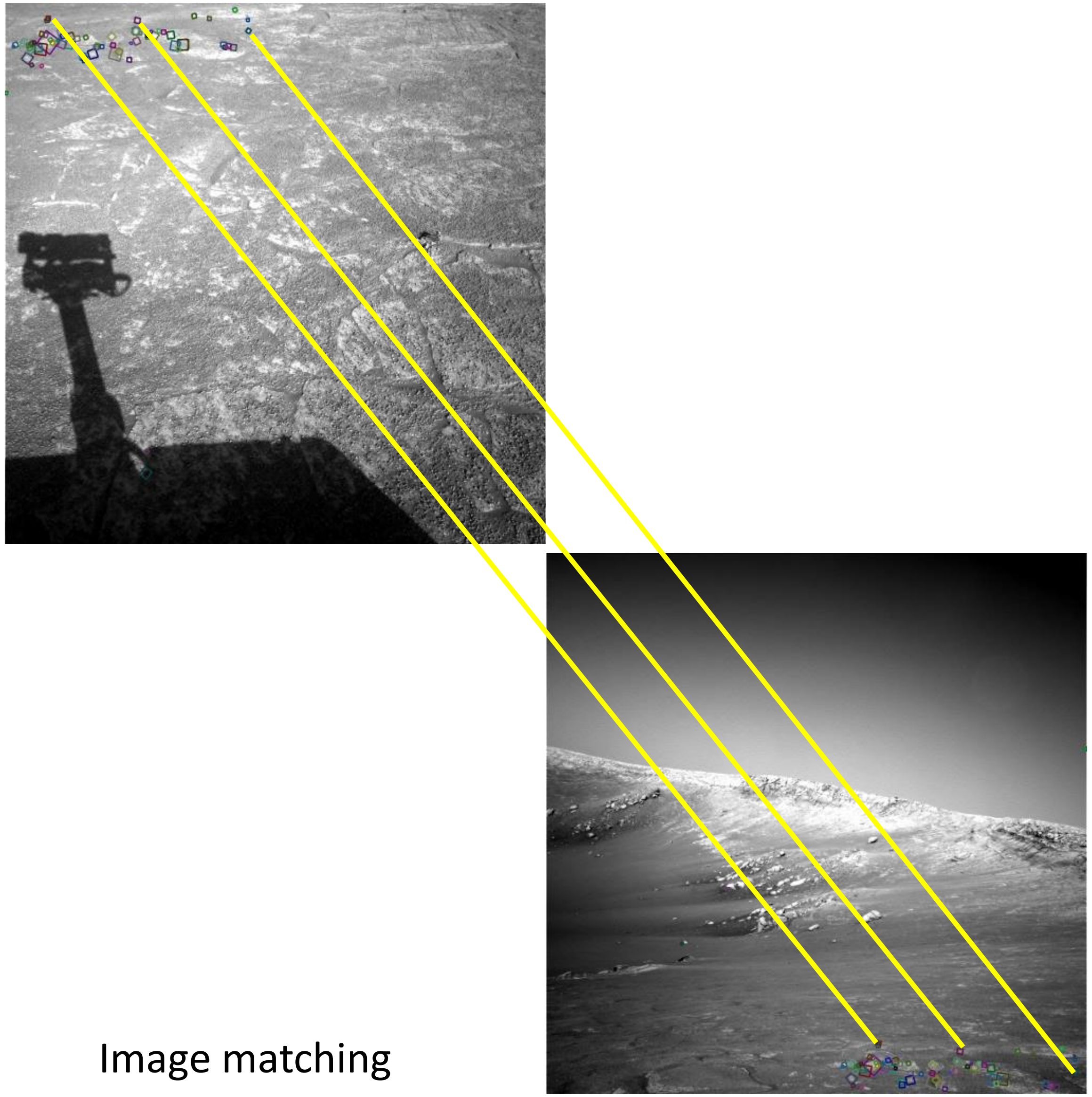


Image matching

Characteristics of **Good Features**

— **Locality**
  - Features are local, robust to occlusion and clutter

— **Accurate**
  - Precise localization

— **Robustness**
  - Noise, blur, compression, etc. do not have a big impact on the feature

— **Distinctiveness**
  - Individual features can be easily matched

— **Efficiency**
  - Close to real-time performance

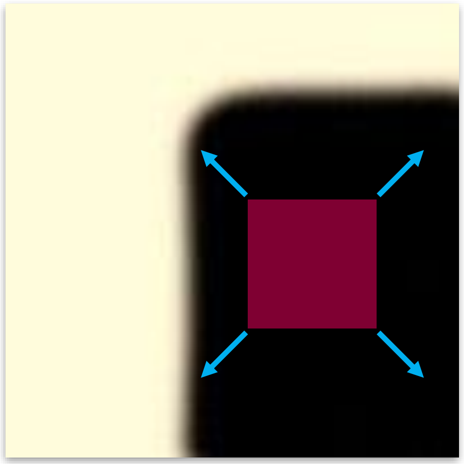
Corner

- **Key Property:**

— In the region around a corner, image gradient has two or more dominant directions

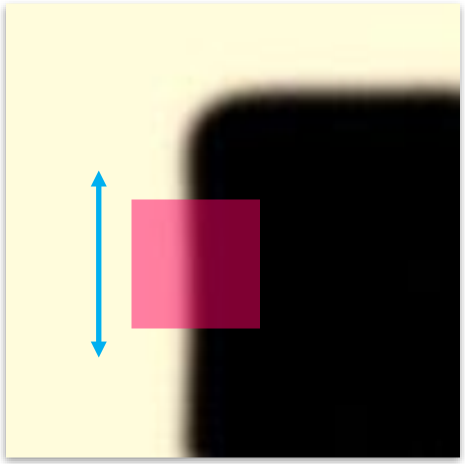
— Corners are robust and distinctive

Flat



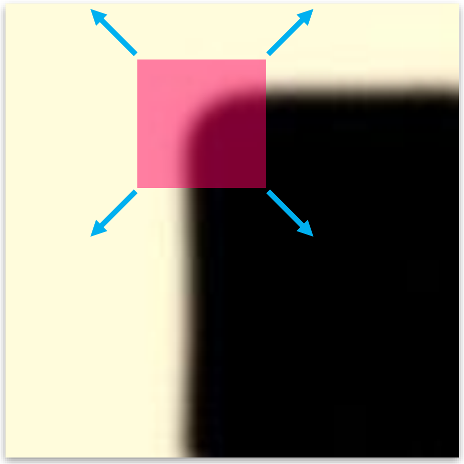
No changes
in all directions

Edge



No changes
along the edge direction

Corner



Significant changes
in all directions

C. Harris and M. Stephens, A combined Corner and Edge Detector, **Alvey Vision Conference**, 1988

Corner Detection

- Edge detectors perform poorly at corners

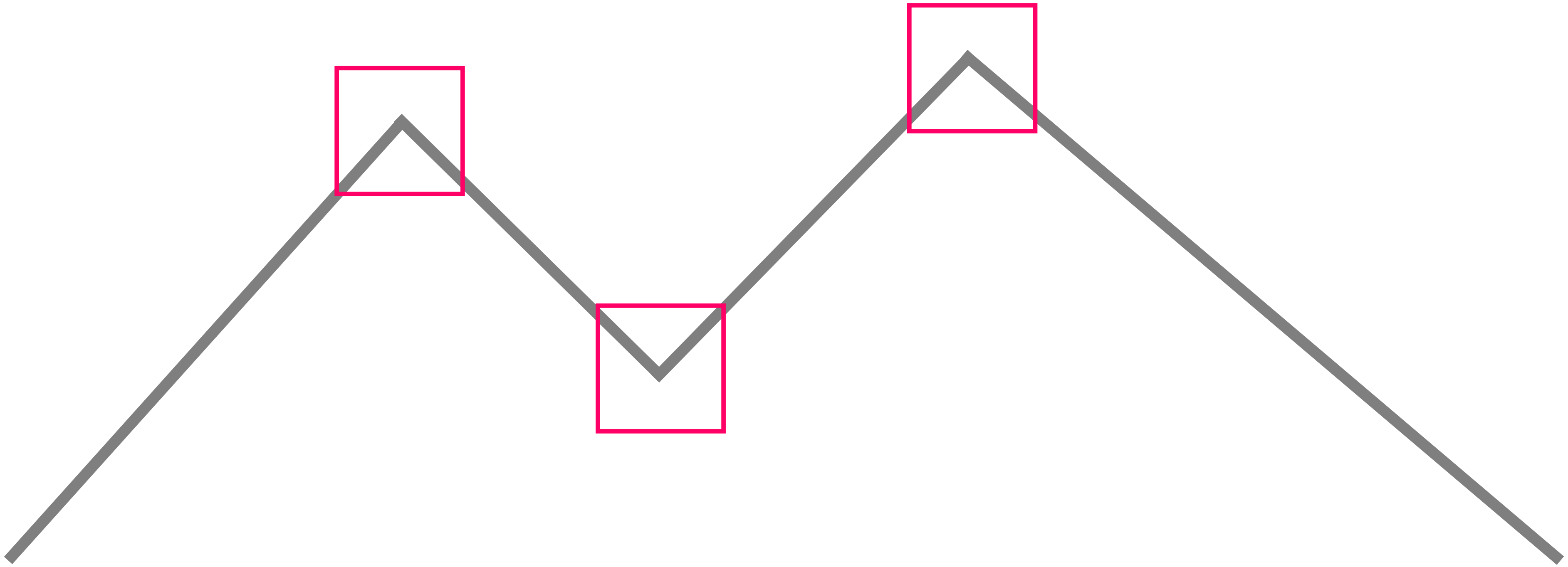
- **Observations:**

— The gradient is ill-defined exactly at a corner

— Near a corner, the gradient has two or more distinct values

How to Find a Corner

- Easily recognized by looking through a small window
- Shifting the window should give large change in intensity



Harris Corner Detection

① Compute **image gradient** over small region

② Compute the **covariance matrix**

③ Compute **eigenvectors** and **eigenvalues**

④ Use **threshold on eigenvalues** to detect corners

Harris Corner Detection: ① **Image Gradient**

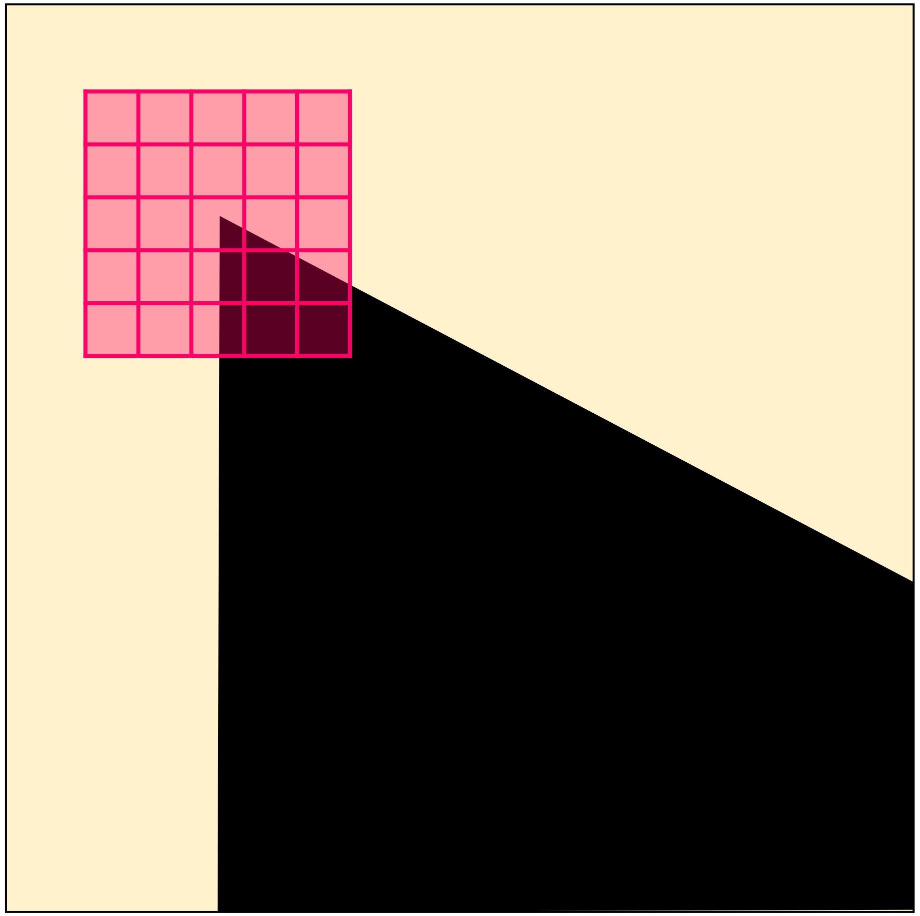
- Compute **image gradient** over small region

— Gradient in x direction:

$$I_y = \frac{\partial I}{\partial y}$$

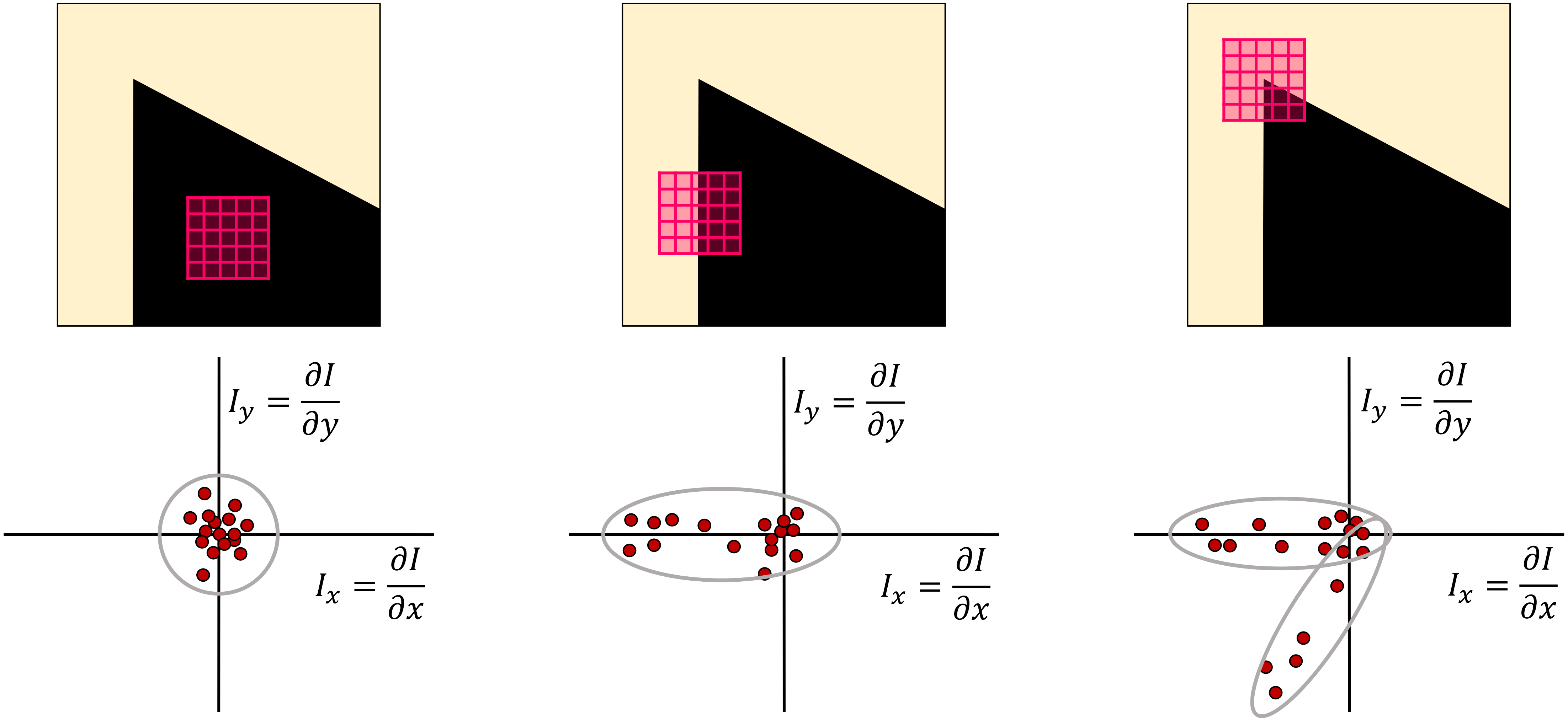
— Gradient in y direction:

$$I_x = \frac{\partial I}{\partial x}$$



Harris Corner Detection: ① **Image Gradient**

- Distribution reveals **edge orientation** and **magnitude**



Harris Corner Detection: ② **Covariance Matrix**

- Compute the **covariance matrix**

$$\mathbf{M} = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detection: ② Covariance Matrix

- Compute the **covariance matrix**

Sum over local window region around corner

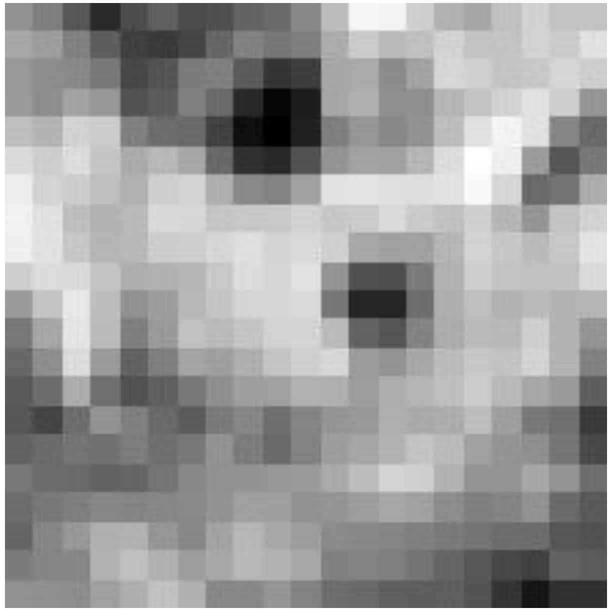
Gradient with respect to each direction (x or y)

$$\mathbf{M} = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detection: ② **Covariance Matrix**

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} w(x,y)[I(x+u,y+v) - I(x,y)]^2$$

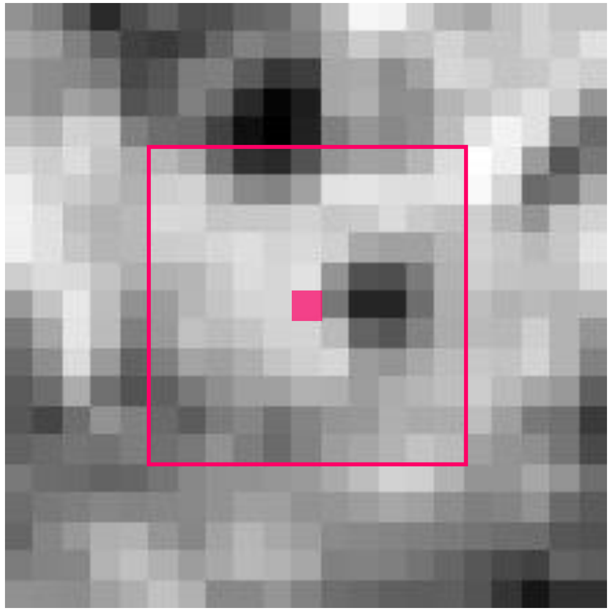


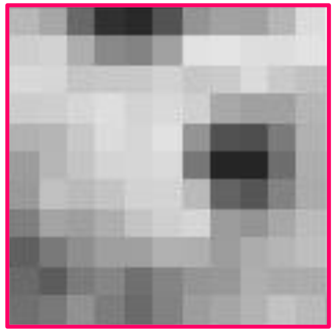
Harris Corner Detection: ② **Covariance Matrix**

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u, v) = \sum_{(x,y)\in P} w(x, y)[I(x + u, y + v) - \boxed{I(x, y)}]^2$$

Center pixel





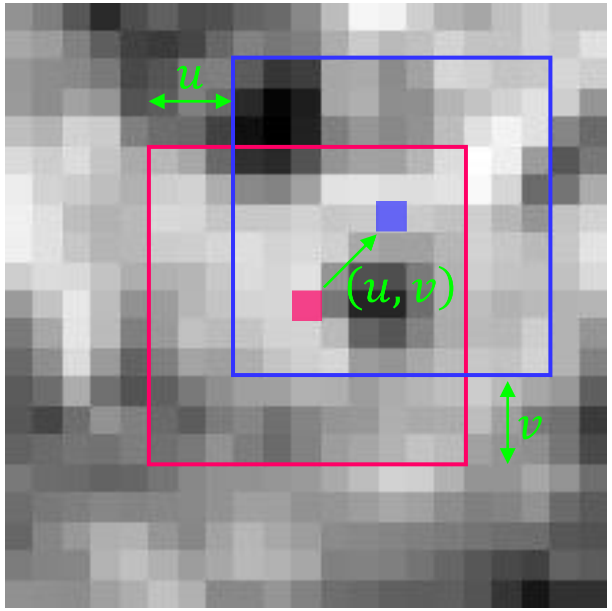
$I(x, y)$

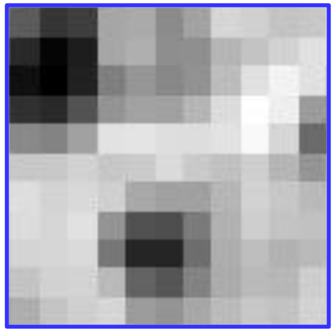
Harris Corner Detection: ② Covariance Matrix

- Change of intensity for the shift value $[u, v]$, error function:

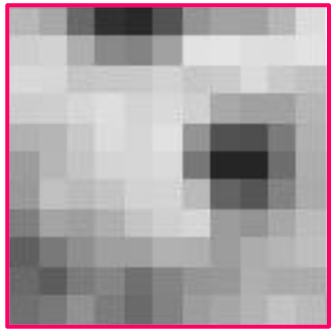
$$E(u,v) = \sum_{(x,y)\in P} w(x,y)[\boxed{I(x+u,y+v)} - \boxed{I(x,y)}]^2$$

Neighboring pixels   Center pixel





$I(x+u, y+v)$

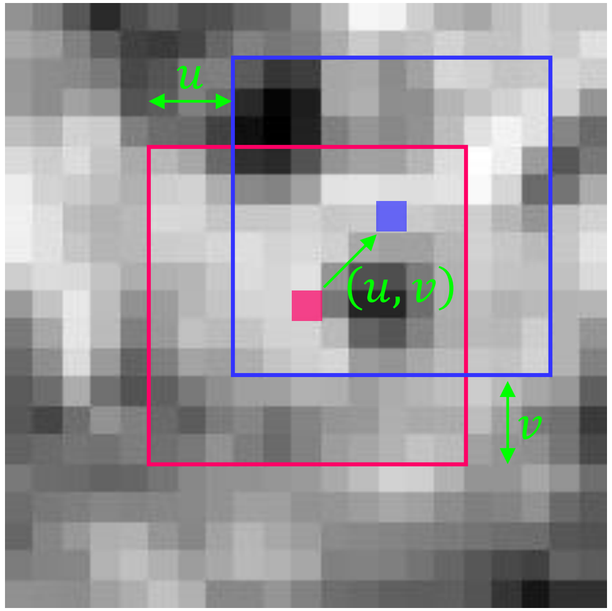


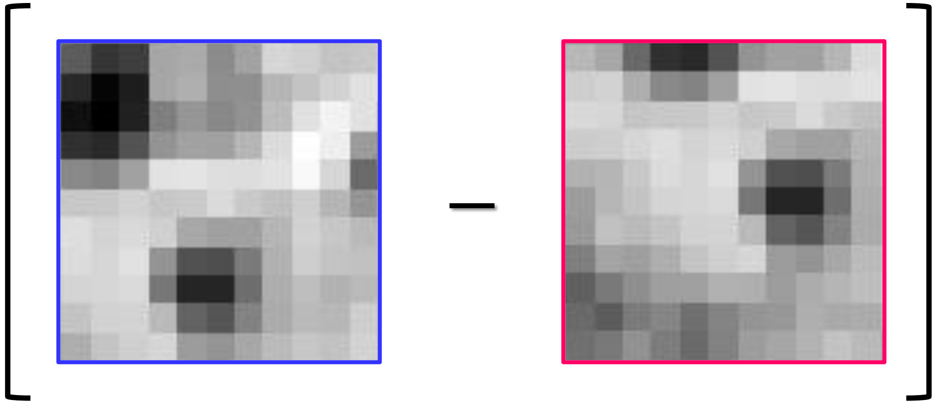
$I(x,y)$

Harris Corner Detection: ② Covariance Matrix

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} w(x,y)\big[\underbrace{\boxed{I(x+u,y+v)}}_{\text{Neighboring pixels}} - \underbrace{\boxed{I(x,y)}}_{\text{Center pixel}}\big]^2$$



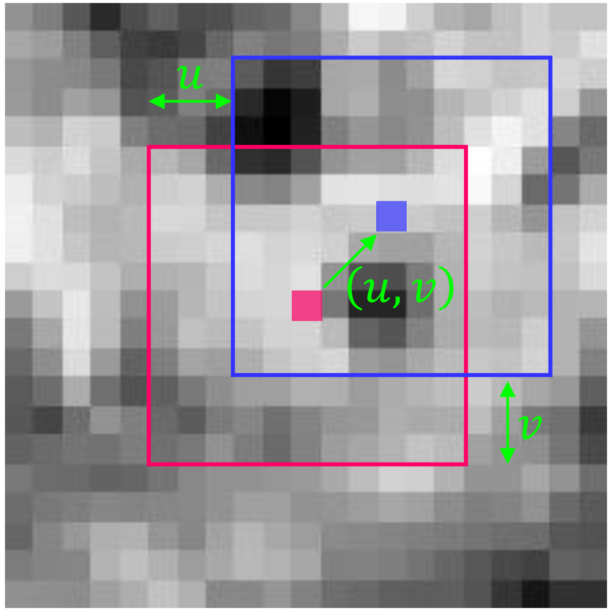


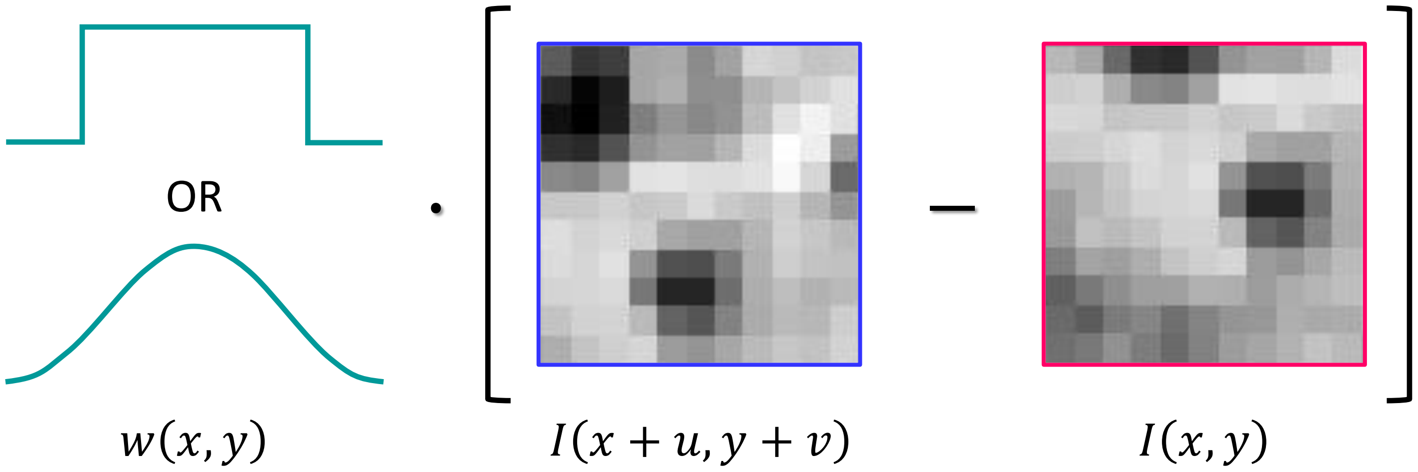
$I(x+u,y+v)$ $\qquad$ $I(x,y)$

Harris Corner Detection: ② Covariance Matrix

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} \underbrace{\boxed{w(x,y)}}_{\text{Window function}} [\underbrace{\boxed{I(x+u,y+v)}}_{\text{Neighboring pixels}} - \underbrace{\boxed{I(x,y)}}_{\text{Center pixel}}]^2$$





Harris Corner Detection: ② **Covariance Matrix**

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} w(x,y)[I(x+u,y+v) - I(x,y)]^2$$

$$\approx \sum_{(x,y)\in P} w(x,y)\left[\left(\frac{\partial I}{\partial x}u\right)^2 + \left(\frac{\partial I}{\partial y}v\right)^2 + 2\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}uv\right]$$

Harris Corner Detection: ② **Covariance Matrix**

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} w(x,y)[I(x+u, y+v) - I(x,y)]^2$$

$$\approx \sum_{(x,y)\in P} w(x,y)\left[\left(\frac{\partial I}{\partial x}u\right)^2 + \left(\frac{\partial I}{\partial y}v\right)^2 + 2\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}uv\right]$$

$$= [u \quad v]\begin{bmatrix} \sum_{p\in P}\left(\frac{\partial I}{\partial x}\right)^2 & \sum_{p\in P}\frac{\partial I}{\partial x}\frac{\partial I}{\partial y} \\ \sum_{p\in P}\frac{\partial I}{\partial x}\frac{\partial I}{\partial y} & \sum_{p\in P}\left(\frac{\partial I}{\partial y}\right)^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

where $w(x,y) = 1$

Harris Corner Detection: ② **Covariance Matrix**

- Change of intensity for the shift value $[u, v]$, error function:

$$E(u,v) = \sum_{(x,y)\in P} w(x,y)[I(x+u,y+v) - I(x,y)]^2$$

$$\approx \sum_{(x,y)\in P} w(x,y)\left[\left(\frac{\partial I}{\partial x}u\right)^2 + \left(\frac{\partial I}{\partial y}v\right)^2 + 2\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}uv\right]$$

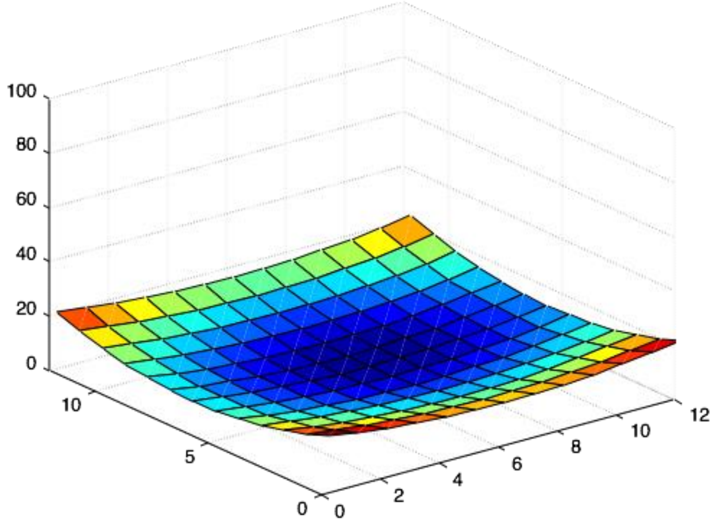
$$= [u \quad v]\begin{bmatrix} \sum_{p\in P}\left(\frac{\partial I}{\partial x}\right)^2 & \sum_{p\in P}\frac{\partial I}{\partial x}\frac{\partial I}{\partial y} \\ \sum_{p\in P}\frac{\partial I}{\partial x}\frac{\partial I}{\partial y} & \sum_{p\in P}\left(\frac{\partial I}{\partial y}\right)^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Covariance matrix

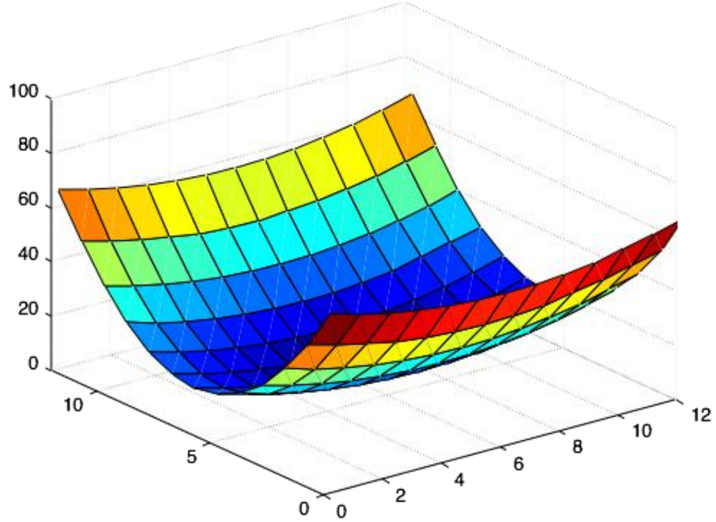
where $w(x,y) = 1$

Harris Corner Detection: ② Covariance Matrix

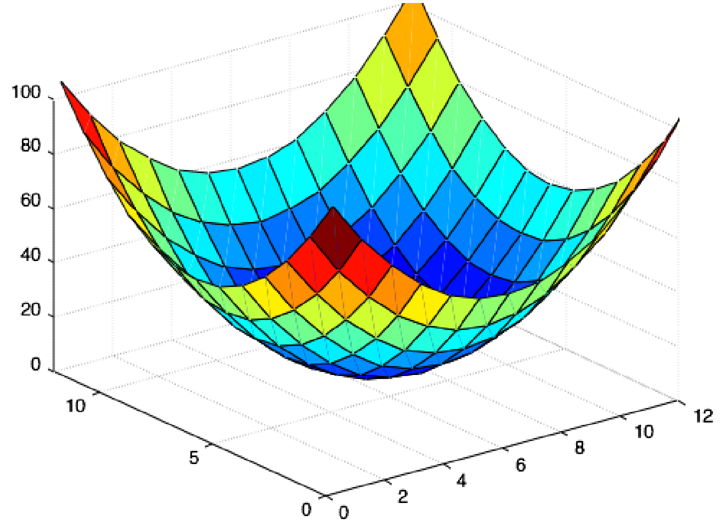
- Visualization of a quadratic error surface, $E(u, v)$



Flat



Edge



Corner

Harris Corner Detection: ③ **Eigenvalues & Eigenvectors**

- Compute **eigenvalues** and **eigenvectors**

- Given a square matrix $\mathbf{A}$, a scalar λ is called an eigenvalue of $\mathbf{A}$ if there exists a non-zero vector $\mathbf{v}$ that satisfies:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

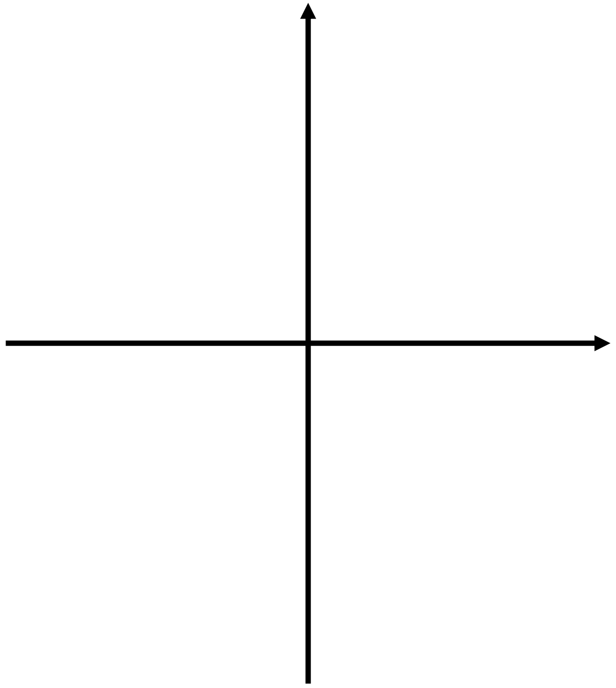
- The vector $\mathbf{v}$ is called an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue λ
- The eigenvalues of $\mathbf{A}$ can be obtained by solving

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \longrightarrow \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Harris Corner Detection: ③ **Eigenvalues & Eigenvectors**

- **Visualization** as an **Ellipse**

$$\mathbf{M} = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^{-1}$$

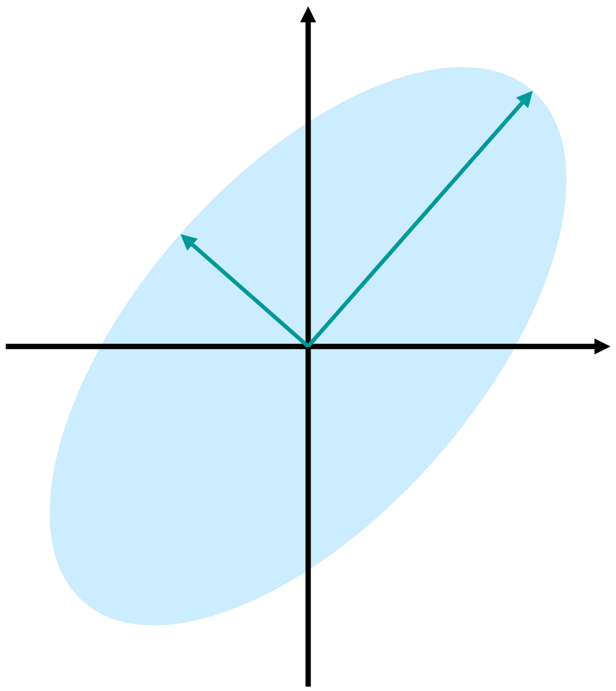


Harris Corner Detection: ③ Eigenvalues & Eigenvectors

- **Visualization** as an **Ellipse**

— **Eigenvectors** determines the **orientation of the ellipse**

$$\mathbf{M} = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} = \underbrace{[\mathbf{v}_1 \quad \mathbf{v}_2]}_{\textbf{Eigenvectors}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^{-1}$$



Harris Corner Detection: ③ Eigenvalues & Eigenvectors

- **Visualization** as an **Ellipse**

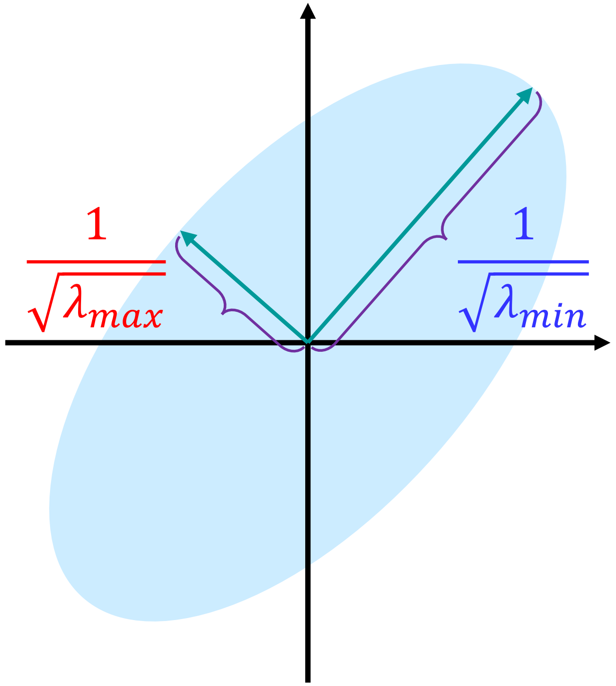
— **Eigenvectors** determines the **orientation of the ellipse**

— **Eigenvalues** determines the axis **lengths of the ellipse**

  - $\frac{1}{\sqrt{\lambda_{max}}}$: The direction of the fastest change
  - $\frac{1}{\sqrt{\lambda_{min}}}$: The direction of the slowest change

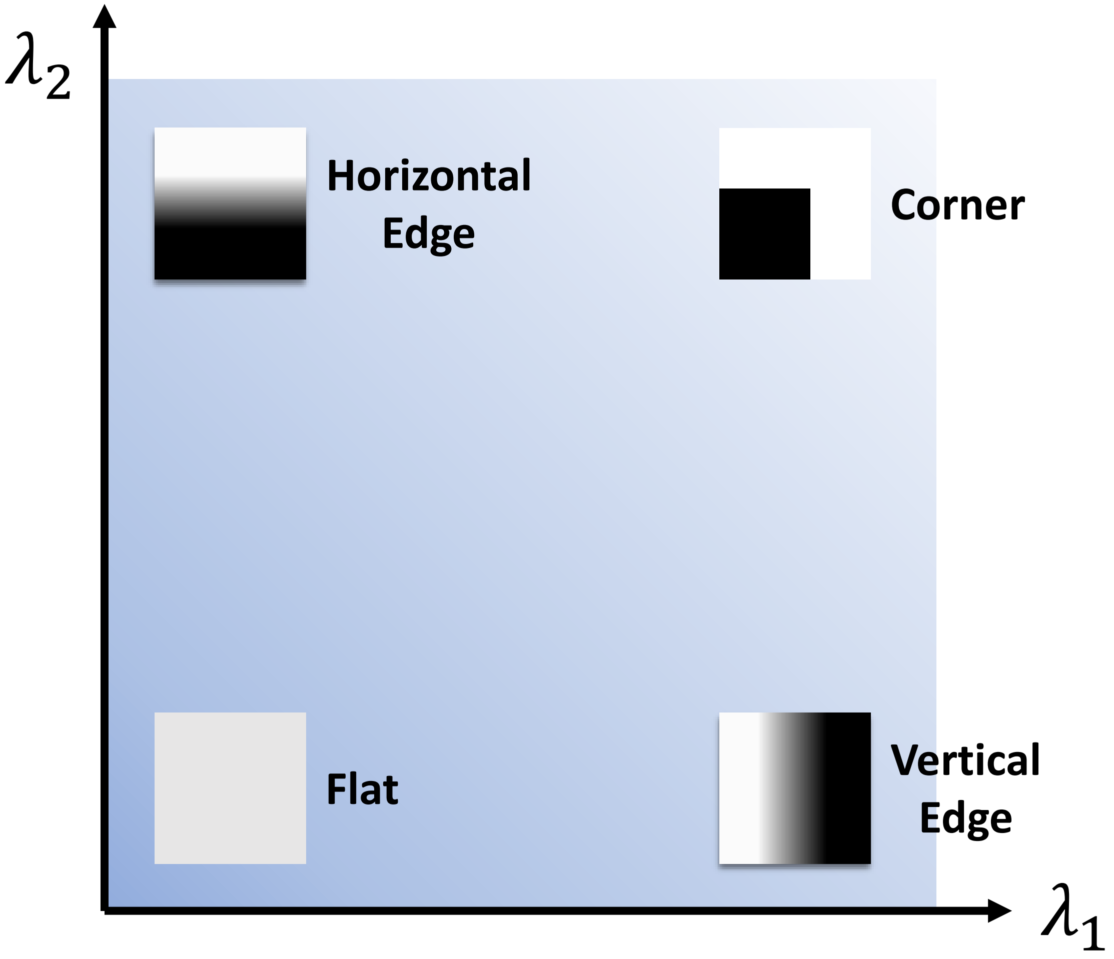
$$\mathbf{M} = \begin{bmatrix} \sum_{p\in P} I_x I_x & \sum_{p\in P} I_x I_y \\ \sum_{p\in P} I_y I_x & \sum_{p\in P} I_y I_y \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^{-1}$$

Eigenvectors **Eigenvalues**



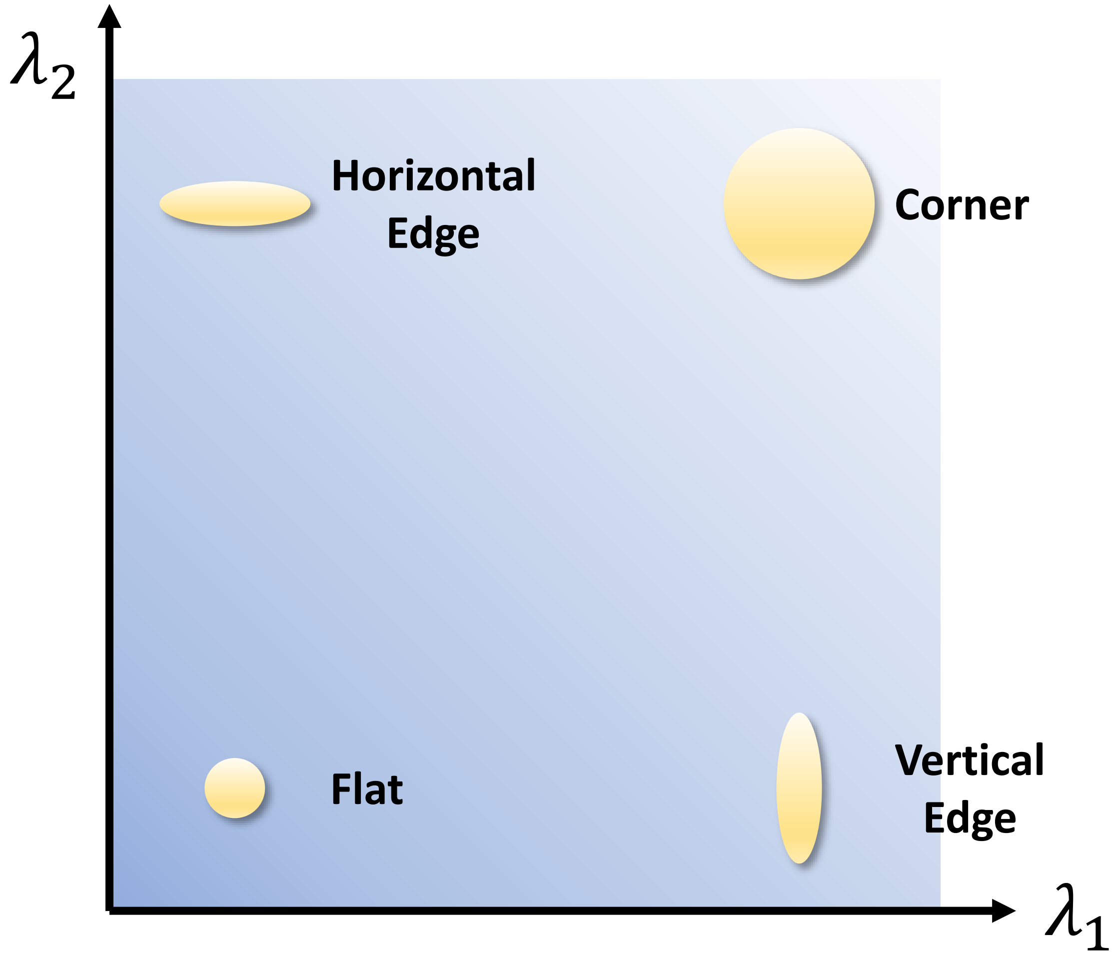
Harris Corner Detection: ③ Eigenvalues & Eigenvectors

- **Classification of Image Points** using Eigenvalues of **C**



Harris Corner Detection: ③ **Eigenvalues & Eigenvectors**

- **Classification of Image Points** using Eigenvalues of **C**



Harris Corner Detection: ④ Eigenvalue Thresholding

- Use **threshold on eigenvalues** to detect corners

- **Cornerness:**

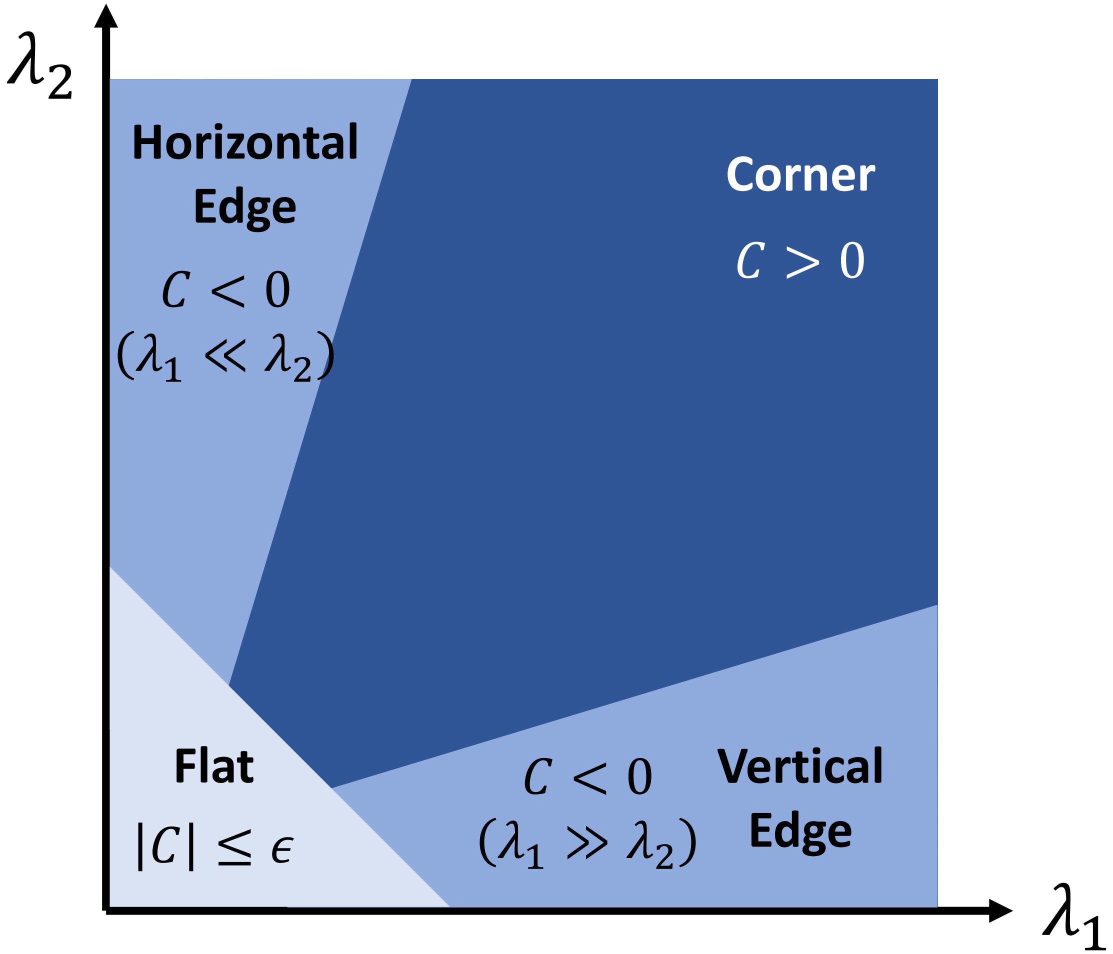
$$C = \lambda_1\lambda_2 - \alpha(\lambda_1 + \lambda_2)^2$$
$$= \det(\mathbf{M}) - \alpha \cdot \mathrm{tr}(\mathbf{M})^2$$

$0.04 \le \alpha \le 0.06$

— Flat region: $|C| \le \epsilon$

— Edge: $C < 0$

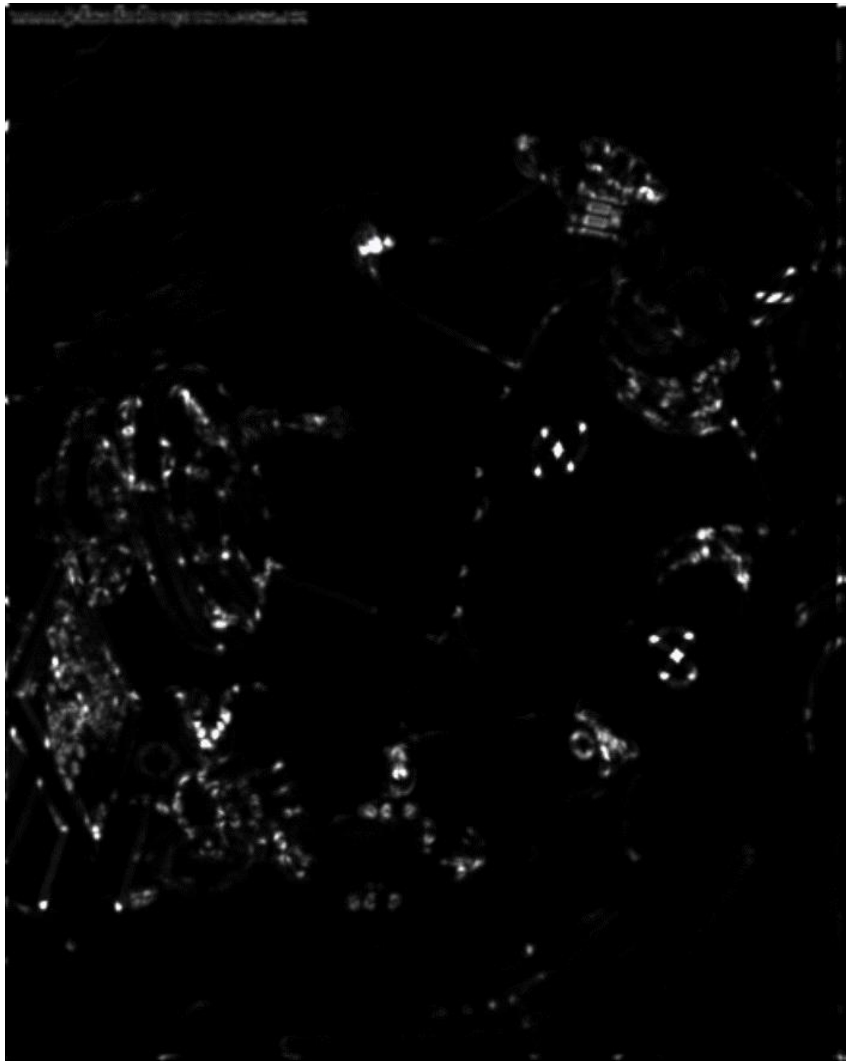
— Corner: $C > 0$



Example: Harris Corner Detection

Example: Harris Corner Detection

Example: Harris Corner Detection



Corner response map

Corner detection result ($\sigma = 1$)

Wish You All The Best!

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