[C2-001] 기초수학

Lecture 04: Matrix Inverse

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Topics

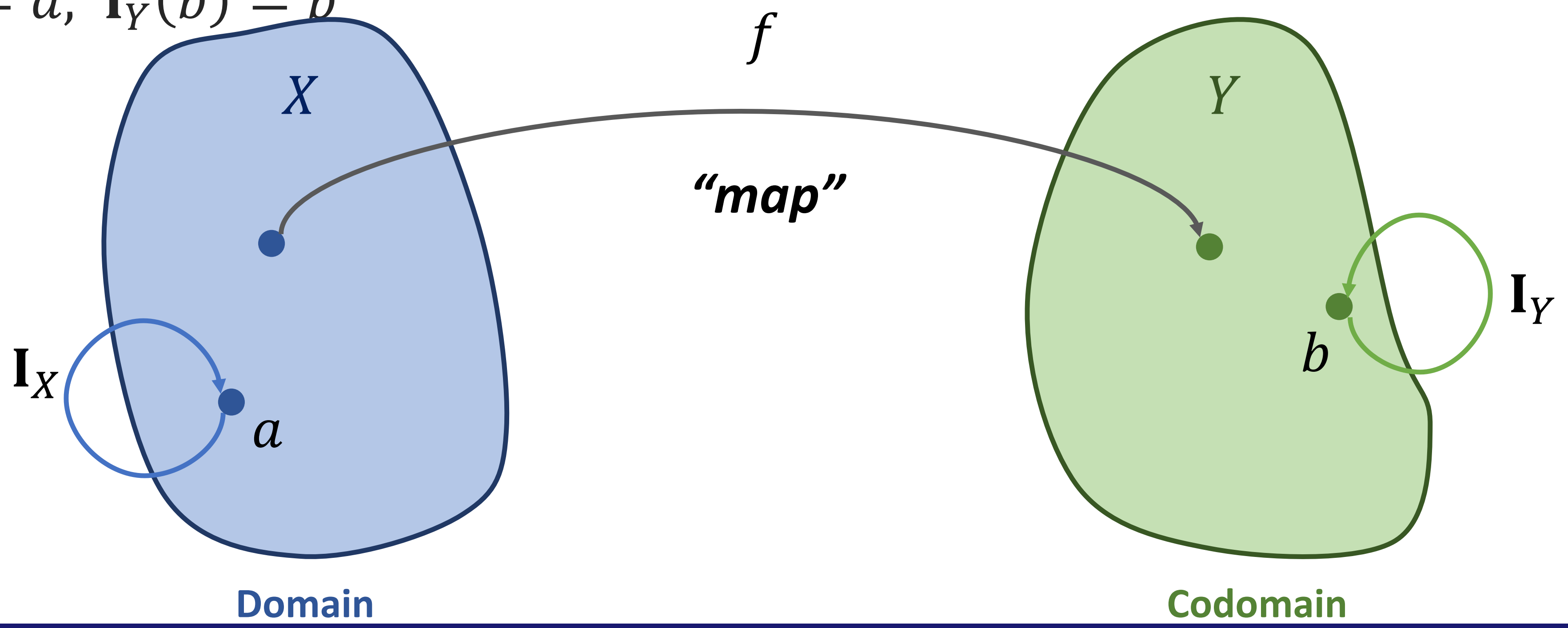
- Matrix Inverse

Introduction: **The Inverse of a Function**

- **Identity Function**

— An identity function (identity relation; map; identity transformation) is a function that always returns the same value that was used as its argument

- $\mathbf{I}_X: X \to Y,\ a \in X,\ b \in Y,$
- $\mathbf{I}_X(a) = a,\ \mathbf{I}_Y(b) = b$



Introduction: **The Inverse of a Function**

- **Inverse of a Function**

– An inverse function is a function that reverse another function.

– If the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x.
  - $f(x) = y \iff g(y) = f^{-1}(y) = x$

– $f: X \to Y$ is invertible $\Leftrightarrow$ there exists a function, $f^{-1}: Y \to X$ such that $f^{-1} \circ f = \mathbf{I}_X$ and $f \circ f^{-1} = \mathbf{I}_Y$

Introduction: **The Inverse of a Function**

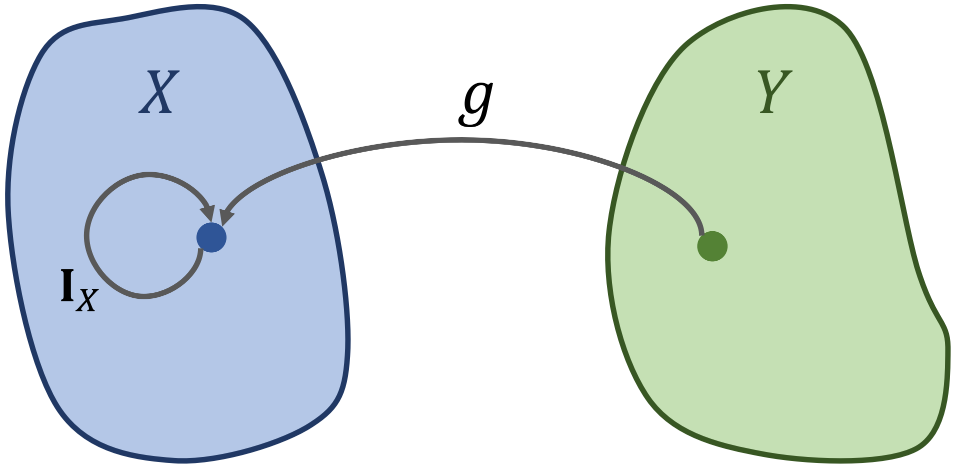
- **Inverse of a Function**

— **Q)** f is invertible, then is f^{-1} unique?

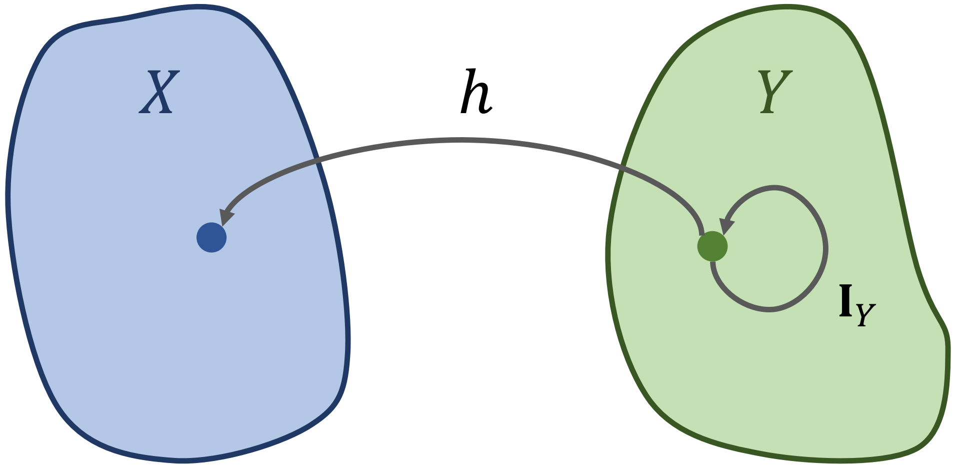
— **A)** $g: Y \to X,\ \ g \circ f = \mathbf{I}_X$ and $f \circ g = \mathbf{I}_Y$

$h: Y \to X,\ \ h \circ f = \mathbf{I}_X$ and $f \circ h = \mathbf{I}_Y$

$\longrightarrow\ \ g = \mathbf{I}_X \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ \mathbf{I}_Y = h$



$\mathbf{I}_X \circ g$

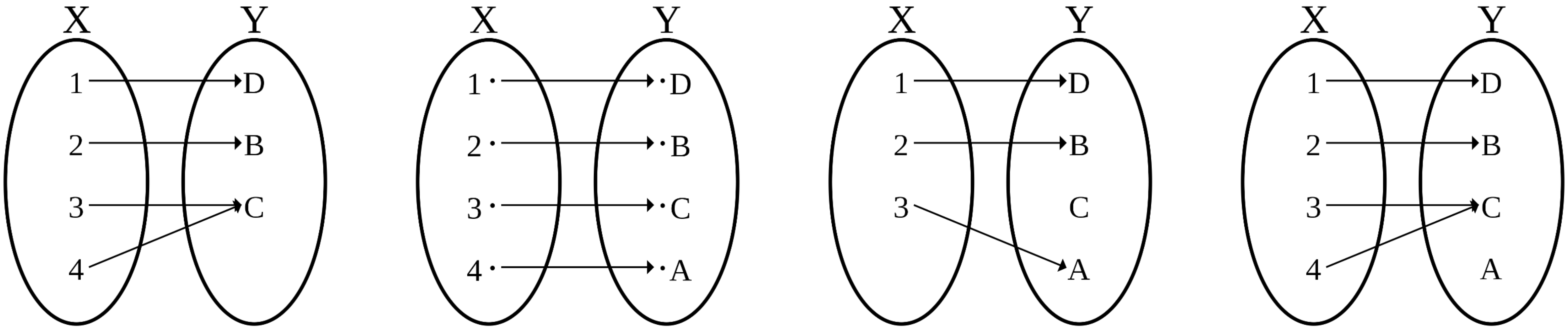


$h \circ \mathbf{I}_Y$

Surjective Function **(Onto)**

— A surjective function (onto function) is a function f that maps an element x to every element y.

— If $f: X \rightarrow Y$, then f is said to be *surjective* if $\forall y$ (every y) $\in Y$,
$\exists$ *at least one* $x \in X$ (there exists at least one x) such that $f(x) = y$

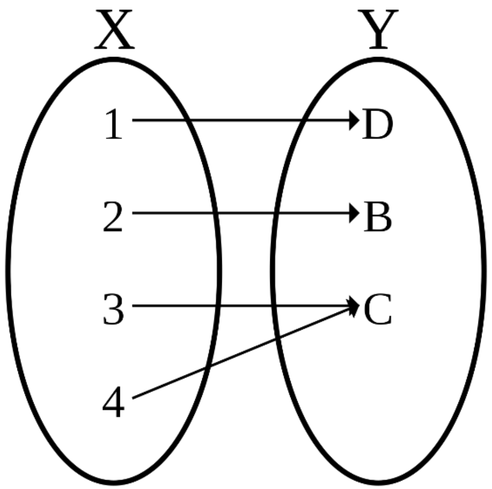


(a) **Surjective** (b) **Surjective** (c) **Non-surjective** (d) **Non-surjective**

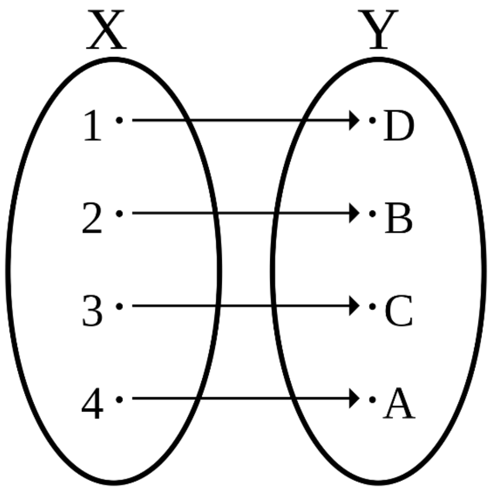
Injective Function **(One-to-one)**

— A injective function (one-to-one function) is a function f that maps an distinct element x to distinct element y.

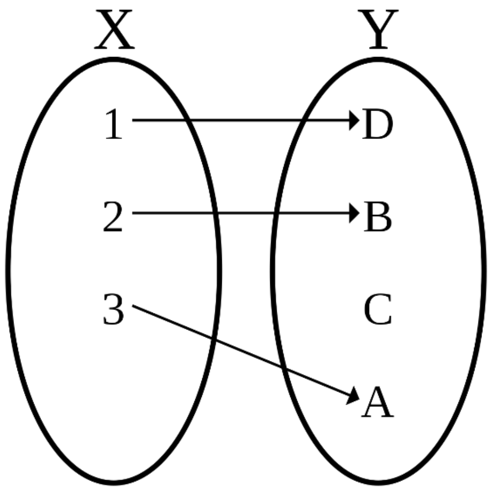
— If $f: X \rightarrow Y$, then f is said to be *injective* if $\forall y$ (every y) $\in Y$,
$\exists$ *at most one* $x \in X$ such that $f(x) = y$ (i.e., $a \neq b \Leftrightarrow f(a) \neq f(b)$)



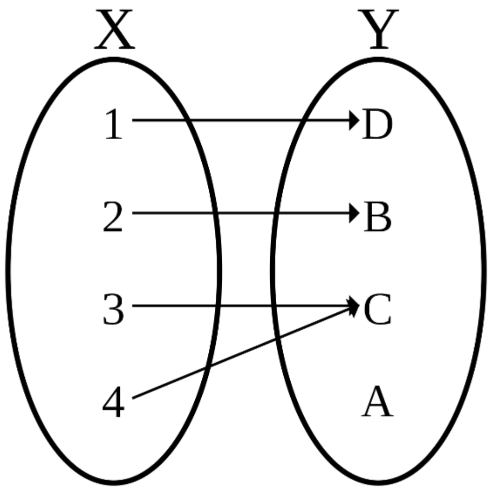
(a)
Non-injective



(b)
Injective



(c)
Injective

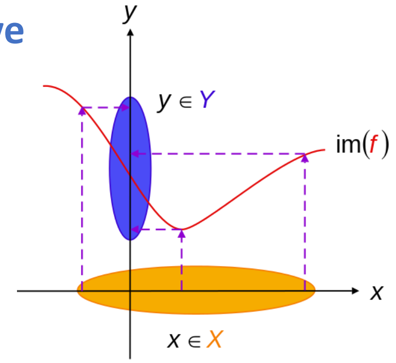


(d)
Non-injective

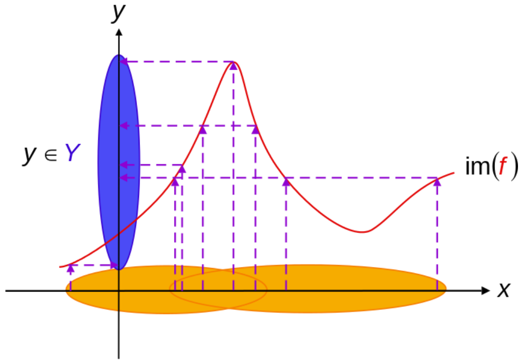
Surjective & Injective Functions

- Surjective Function

Surjective



$f : X \to Y$
$y = f(x)$

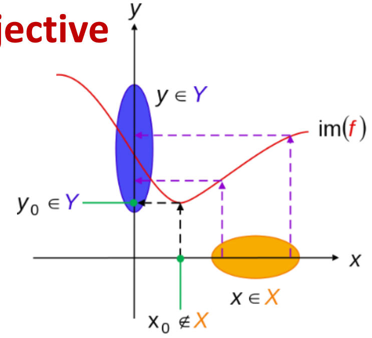


$x \in X_1$ $\quad$ $x \in X_2$

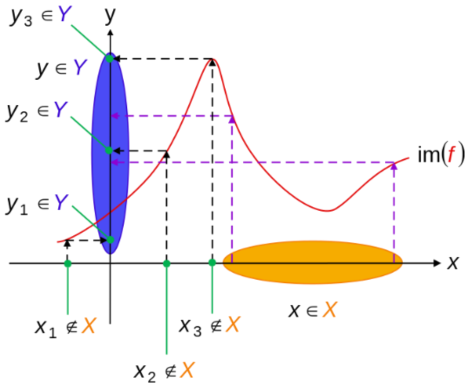
$f : X_1 \to Y_1$ $\quad$ $f : X_2 \to Y_2$

$y = f(x)$

Non-surjective



$f : X \to Y$
$y = f(x)$



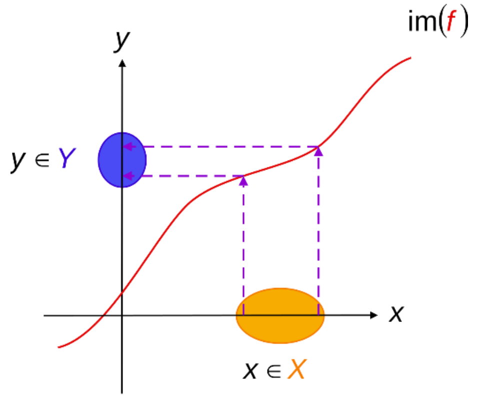
$x \in X_1$ $\quad$ $x \in X_2$

$f : X_1 \to Y_1$ $\quad$ $f : X_2 \to Y_2$

$y = f(x)$

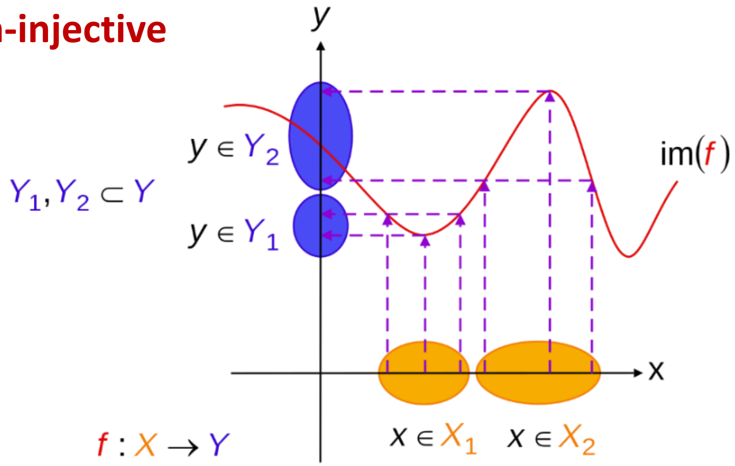
- Injective Function

Injective



$f : X \to Y$
$y = f(x)$

Non-injective



$f : X \to Y$
$y = f(x)$

$x \in X_1$ $\quad$ $x \in X_2$

$X_1, X_2 \subset X$

https://en.wikipedia.org/wiki/Surjective_function
https://en.wikipedia.org/wiki/Injective_function

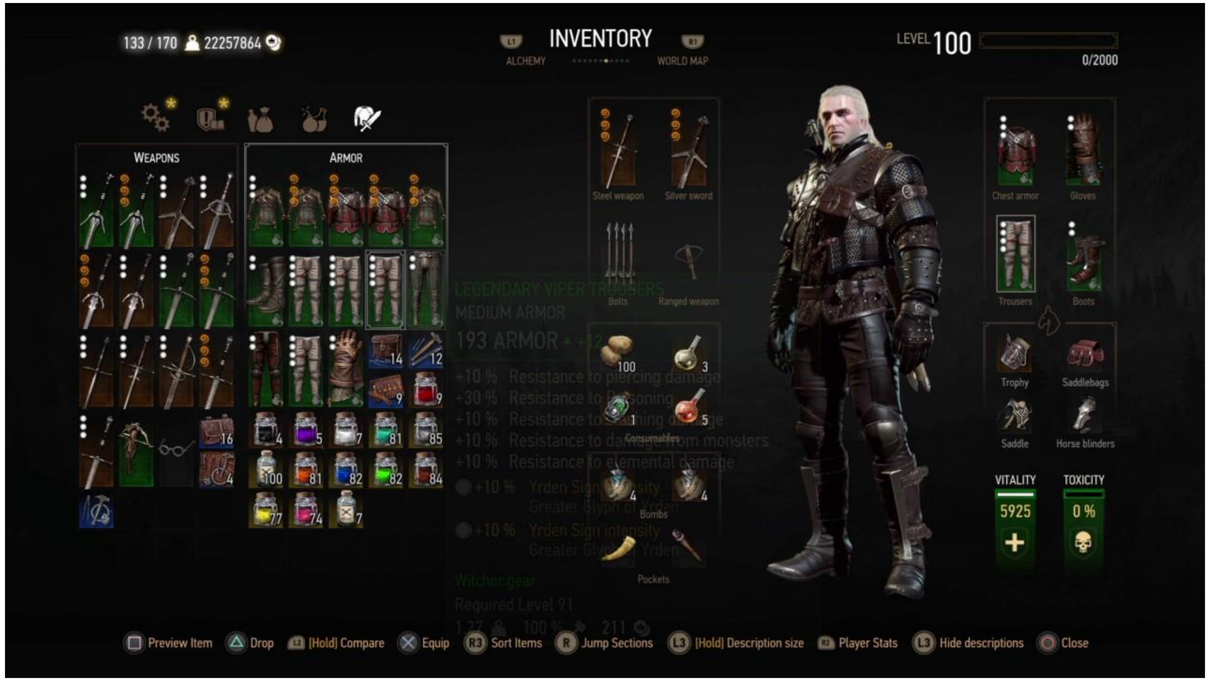
Examples of Surjective & Injective Functions

- Example of Surjective Function

3D modeling

Projection
onto 2D monitor

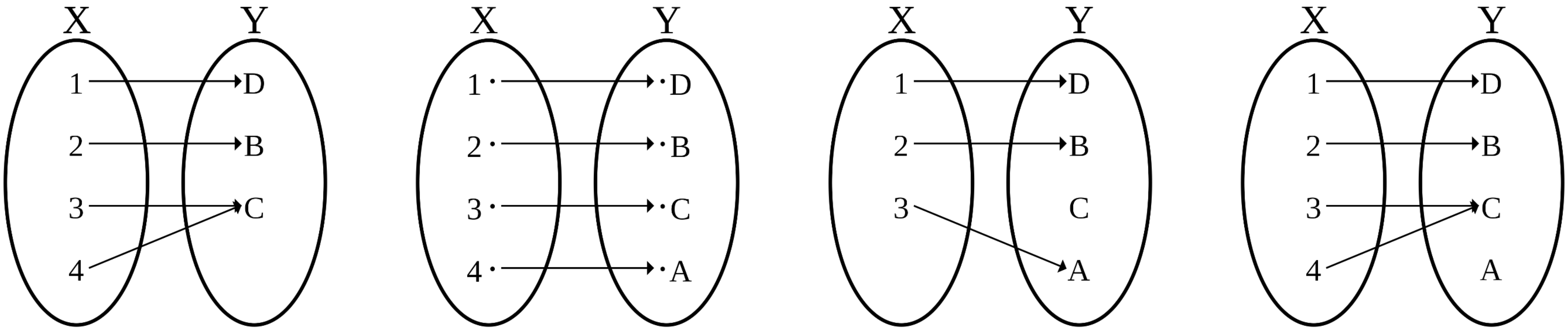
- Example of Injective Function



A character – Each item

Relating Invertibility to being Onto and One-to-one

- f is invertible $\Longleftrightarrow$ for every $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$
  - For every $y \in Y$: Surjective (Onto)
  - A unique $x \in X$: Injective (One-to-one)



(a) **Non-invertible**

(b) **Invertible**

(c) **Non-invertible**

(d) **Non-invertible**

Invertibility of Linear Transformation

- Matrix Condition for Onto Transformation

– $\mathcal{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \longrightarrow Span(\mathbf{v}_1, \cdots, \mathbf{v}_n) = C(\mathbf{A}) = \mathbb{R}^m$

– $\mathcal{T}$ is onto $\Leftrightarrow C(\mathbf{A}) = \mathbb{R}^m \longrightarrow rref(\mathbf{A})$ has a pivot entry in every row

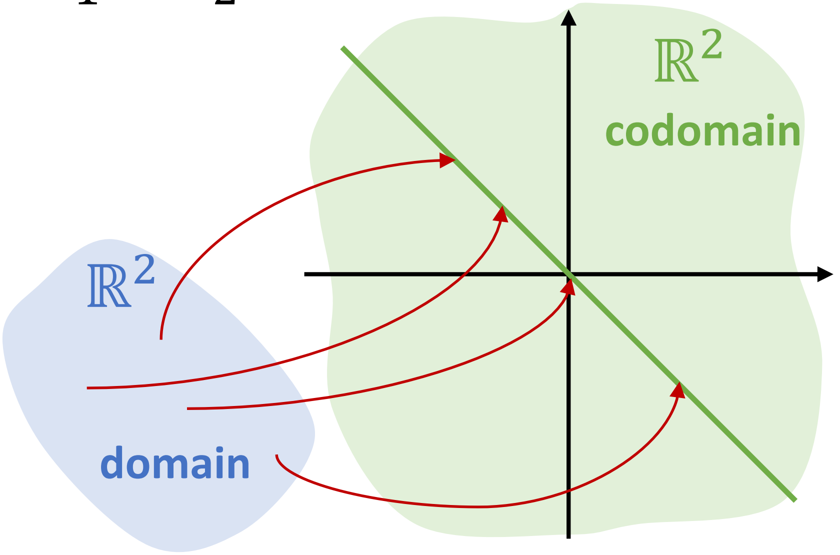
1) m pivot entries
2) $rank(\mathbf{A}) = m$ $\left(rank(\mathbf{A}) = dim\big(C(\mathbf{A})\big) = \#\text{ of basis for } C(\mathbf{A})\right)$

- $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^3, \mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$
- $rank(\mathcal{T}) = 2 \neq 3(\because \mathbb{R}^3) \longrightarrow \mathcal{T}$ is not onto. $\longrightarrow \mathcal{T}$ is not invertible.

Invertibility of Linear Transformation

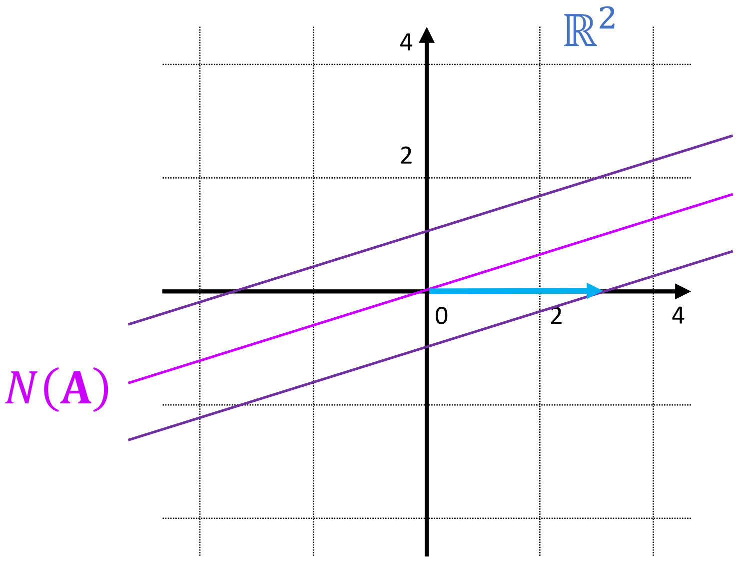
- Exploring the Particular Solution Set of Ax=b
  - $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\mathbf{x} \longrightarrow \mathbf{Ax} = \mathbf{b}$

    $\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ -1 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right]$: **Non-surjective**
  - $\mathcal{T}$ is not surjective because $\mathrm{Im}(\mathcal{T})$ is not the entire codomain.
  - Only members $\mathbf{b} \in \mathbb{R}^2$ that have solutions are the ones $b_1 + b_2 = 0$.
  - Constraint: $x_1 - 3x_2 = b_1 \longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



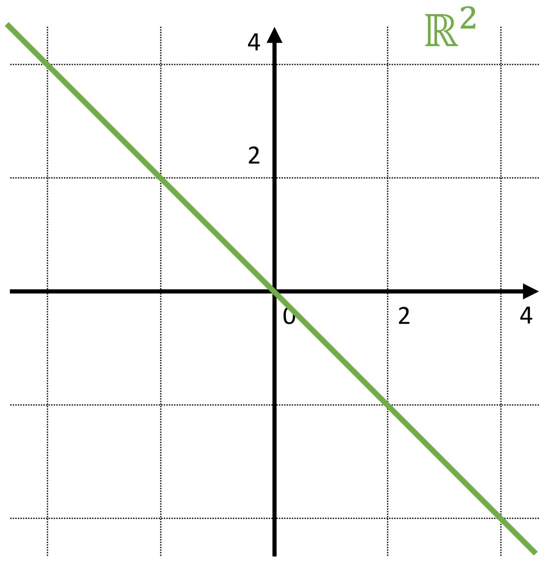
Invertibility of Linear Transformation

- Exploring the Particular Solution Set of $\mathbf{Ax} = \mathbf{b}$
  - For a particular $\mathbf{b}$ that has a solution $\mathbf{Ax} = \mathbf{b}$, the solution set is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
  - Assuming that $\mathbf{Ax} = \mathbf{b}$ has a solution: the solution set $\{\mathbf{x}_p + \boldsymbol{n} \mid \boldsymbol{n} = N(\mathbf{A})\}$.
  - One-to-one $\longrightarrow$ At most one solution: $N(\mathbf{A})$ has to just have the zero vector, $\mathbf{0}$



Domain

$\mathcal{T}$
$: \mathbb{R}^2 \to \mathbb{R}^2$
$\mathbf{x} \to \mathbf{Ax} = \mathbf{b}$



Codomain

Invertibility of Linear Transformation

- Matrix Condition for One-to-one Transformation
  - Any solution to the inhomogeneous system, $\mathbf{Ax} = \mathbf{b}$, will take the form, $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$

  - $\mathbf{Ax} = \mathbf{0} \longrightarrow [\mathbf{A} \quad | \quad \mathbf{0}] \rightarrow [rref(\mathbf{A}) \quad | \quad \mathbf{0}]$

    $\longrightarrow \mathbf{x}_h = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = N(\mathbf{A}) = Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$
  - $\mathbf{Ax} = \mathbf{b} \longrightarrow [\mathbf{A} \quad | \quad \mathbf{b}] \rightarrow [rref(\mathbf{A}) \quad | \quad \mathbf{b}']$

    $\longrightarrow \mathbf{x} = \mathbf{b}' + \mathbf{0} = \mathbf{b}' + x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{x}_p + \mathbf{x}_h$

  - $\mathcal{T}: X \rightarrow Y$, if $\mathcal{T}$ is one-to-one transformation, $\forall \mathbf{b} \in Y$, $\exists$ *at most one solution* to $\mathbf{Ax} = \mathbf{b}$

    $\longrightarrow \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ can be only one solution $\longrightarrow \mathbf{x}_h = N(\mathbf{A}) = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \{\mathbf{0}\}$

    $\longrightarrow x_1 = x_2 = \cdots = x_n = 0$: *Linearly independent*

    $\longrightarrow \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$: Basis for $C(\mathbf{A}) \longrightarrow \therefore dim\big(C(\mathbf{A})\big) = rank(\mathbf{A}) = n$

Invertibility of Linear Transformation

- Invertibility of Linear Transformation

— Onto (Surjective) $\Leftarrow rank(\mathbf{A}) = m$

— One-to-one (Injective) $\Leftarrow rank(\mathbf{A}) = n$

— Onto & One-to-one (invertible) $\Rightarrow rank(\mathbf{A}) = m = n \Rightarrow n \times n$ square mtx

  - $\mathbf{A}_{n\times n} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]_{n\times n} \longrightarrow rref(\mathbf{A}) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n} = \mathbf{I}_n$

  - $\therefore \mathcal{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ \mathcal{T}(\mathbf{x}) = \mathbf{A}\mathbf{x},\ \mathcal{T}$ is only invertible if $rref(\mathbf{A}) = \mathbf{I}_n$

Finding **Inverse**

- Example of Finding Inverse Matrix, $\mathbf{A}^{-1}$
  - $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{\mathcal{T}_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{\mathcal{T}_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\mathcal{T}_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
    - $\mathcal{T}_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_3 - x_1 \end{bmatrix}, \mathcal{T}_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_3 - 2x_2 \end{bmatrix}, \mathcal{T}_3\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 - 2x_3 \\ x_3 \end{bmatrix}$
    - $\mathcal{T}_1(\mathbf{x}) = \mathbf{S}_1\mathbf{x}, \quad \mathcal{T}_2(\mathbf{x}) = \mathbf{S}_2\mathbf{x}, \quad \mathcal{T}_3(\mathbf{x}) = \mathbf{S}_3\mathbf{x}$
    - $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad \mathbf{S}_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$
  - $[\mathbf{A} \mid \mathbf{I}] \to [\mathbf{S}_1\mathbf{A} \mid \mathbf{S}_1\mathbf{I}] \to [\mathbf{S}_2\mathbf{S}_1\mathbf{A} \mid \mathbf{S}_2\mathbf{S}_1\mathbf{I}]$

    $\to [\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1\mathbf{A} \mid \mathbf{S}_3\mathbf{S}_2\mathbf{S}_1\mathbf{I}] = [\mathbf{I} \mid \mathbf{A}^{-1}]$

Finding **Inverse**

- Example of Finding Inverse Matrix, $\mathbf{A}^{-1}$
  - $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \longrightarrow \mathbf{A}^{-1} = ?$
  - $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \rightarrow$

    $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & -1 \\ 0 & 1 & 0 & 7 & 5 & 2 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right] = [rref(\mathbf{A}) \quad | \quad \mathbf{A}^{-1}]$

Determinant

- 2 x 2 Determinant
  - $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
    - $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\mathcal{T}_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right] \xrightarrow{\mathcal{T}_2} \left[\begin{array}{cc|cc} (ad-bc)a & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]$

      $\xrightarrow{\mathcal{T}_3} \left[\begin{array}{cc|cc} 1 & 0 & \frac{ad}{(ad-bc)a} & \frac{-ab}{(ad-bc)a} \\ 0 & 1 & \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{array}\right] \longrightarrow \therefore \mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Definition of Determinant
  - $ad - bc \neq 0 \Leftrightarrow \mathbf{A}$ is invertible
  - $det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
    - $\therefore \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Determinant

- 3 x 3 Determinant
  - $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow det(\mathbf{A}) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- Example of Finding 3 x 3 Determinant
  - $\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix} \longrightarrow det(\mathbf{B}) = 1 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$

    $= (-1 - 0) - 2(2 - 12) + 4(0 + 4) = 35$: Invertible

Determinant

- *n* x *n* Determinant
  - $\mathbf{A}_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\widetilde{\mathbf{A}}_{ij} = (n-1)\times(n-1)$ matrix
  - $det(\mathbf{A}) = a_{11}\left|\widetilde{\mathbf{A}}_{11}\right| - a_{12}\left|\widetilde{\mathbf{A}}_{12}\right| + a_{13}\left|\widetilde{\mathbf{A}}_{13}\right| - \cdots \pm a_{1n}\left|\widetilde{\mathbf{A}}_{1n}\right|$

    $= a_{11}det\left(\widetilde{\mathbf{A}}_{11}\right) - a_{12}det\left(\widetilde{\mathbf{A}}_{12}\right) + a_{13}det\left(\widetilde{\mathbf{A}}_{13}\right) - \cdots \pm a_{1n}det\left(\widetilde{\mathbf{A}}_{1n}\right)$
- Example
  - $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix} = 1\begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix}$
  - $= 1\cdot\left(-2\cdot(-9)\right) - 2\cdot\left(-2\cdot(-6)\right) + 3\cdot\left(1\cdot(-9)\right) - 4\cdot\left(1\cdot(-6) + 2\cdot(-2)\right) = 7$

Determinant

- Determinant along Other Rows/Columns
  - $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix} = -2 \begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix}$
  - $= -2 \cdot (2 \cdot 2) + 3 \cdot \big(-2 \cdot (-4) + 3 \cdot (-1)\big) = -8 + 3 \cdot (8 - 3) = 7$

- Sign of Determinant
  - $sign(i,j) = (-1)^{(i+j)},\ (e.g.,)\ 4 \times 4\text{:} \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Determinant

- Rule of Sarrus for Determinant
  - $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

    $= a(ei - fh) - b(di - fg) + c(dh - eg)$

    $= aei + bfg + cdh - afh - bdi - ceg$

  - $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \longrightarrow \left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - afh - bdi - ceg$

More Determinant

- Determinant When Row Multiplied by Scalar
  - $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = det(\mathbf{A}),\ \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k(ad - bc) = k \cdot det(\mathbf{A}) = det(\mathbf{A}')$
  - $k\mathbf{A} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow det(k\mathbf{A}) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2(ad - bc) = k^2 det(\mathbf{A})$
  - $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix} = det(\mathbf{B})$
  - $\begin{vmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{vmatrix} = -kd\begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke\begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf\begin{vmatrix} a & b \\ g & h \end{vmatrix} = kdet(\mathbf{B}) = det(\mathbf{B}')$
  - $\therefore det(\mathbf{A}') = kdet(\mathbf{A}),\ det(k\mathbf{A}) = k^n det(\mathbf{A})$

More Determinant

- Determinant When Row is Added
  - $\mathbf{X} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ x_1 + y_1 & x_2 + y_2 & \cdots & x_n + y_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$
  - $det(\mathbf{X}) = \sum_{j=1}^{n} (-1)^{(i+j)} x_j \left|\widetilde{\mathbf{A}}_{ij}\right|$
  - $det(\mathbf{Y}) = \sum_{j=1}^{n} (-1)^{(i+j)} y_j \left|\widetilde{\mathbf{A}}_{ij}\right|$
  - $det(\mathbf{Z}) = \sum_{j=1}^{n} (-1)^{(i+j)} \left(x_j + y_j\right) \left|\widetilde{\mathbf{A}}_{ij}\right| = det(\mathbf{X}) + det(\mathbf{Y})$

More Determinant

- Example of Determinant When Row is Added
  - $\mathbf{X} = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} a & b \\ x_1 + y_1 & x_2 + y_2 \end{bmatrix}$
  - $det(\mathbf{X}) = ax_2 - bx_1$
  - $det(\mathbf{Y}) = ay_2 - by_1$
  - $det(\mathbf{Z}) = a(x_2 + y_2) - b(x_1 + y_1) = ax_2 - bx_1 + ay_2 - by_1 = det(\mathbf{X}) + det(\mathbf{Y})$

More Determinant

- Determinant with Duplicate Row
  - $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \longrightarrow \mathbf{S}_{ij} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \longrightarrow det(\mathbf{S}_{ij}) = -det(\mathbf{A})$
  - If $\mathbf{r}_i = \mathbf{r}_j$, $\mathbf{S}_{ij} = \mathbf{A} \longrightarrow det(\mathbf{S}_{ij}) = det(\mathbf{A}) = -det(\mathbf{A}) = 0$ **: Not invertible**

  - Matrix $\mathbf{A}$ is invertible $\Leftrightarrow rref(\mathbf{A}) = \mathbf{I}_n$
  - Duplicate row $\longrightarrow$ Never get $rref(\mathbf{A}) = \mathbf{I}_n \longrightarrow$ Not invertible $\longrightarrow det(\mathbf{A}) = 0$

More Determinant

- Determinant After Row Operations
  - $\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j - c\mathbf{r}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \longrightarrow det(\mathbf{B}) = \begin{vmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{vmatrix} + \begin{vmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ -c\mathbf{r}_i \\ \vdots \\ \mathbf{r}_n \end{vmatrix} = \begin{vmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{vmatrix} - c \begin{vmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_n \end{vmatrix} = det(\mathbf{A})$

More Determinant

- Upper/Lower Triangular Determinant
  - $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \longrightarrow det(\mathbf{A}) = ad$
  - $\mathbf{B} = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \longrightarrow det(\mathbf{B}) = a\begin{vmatrix} e & f \\ 0 & i \end{vmatrix} - b\begin{vmatrix} 0 & f \\ 0 & i \end{vmatrix} + c\begin{vmatrix} 0 & e \\ 0 & 0 \end{vmatrix} = aei$
  - $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \longrightarrow det(\mathbf{A}) a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$

    $= a_{11}a_{22}\cdots a_{nn}$

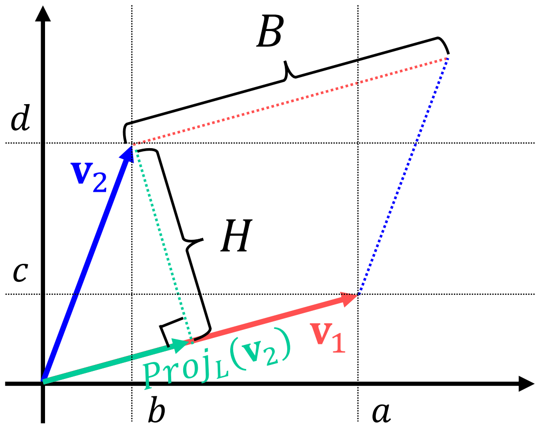
More Determinant

- Simpler 4 x 4 Determinant
  - Our goal is to make a given matrix the upper triangular matrix form.
  - $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix}$
  - $det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{vmatrix} \to -\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 6 & -4 & 4 \end{vmatrix}$

$$\to -\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -6 & 4 \end{vmatrix} \to -\begin{vmatrix} \mathbf{1} & 2 & 2 & 1 \\ 0 & \mathbf{3} & 1 & 0 \\ 0 & 0 & \mathbf{2} & 1 \\ 0 & 0 & 0 & \mathbf{7} \end{vmatrix} = -42$$

More Determinant: **Area of a Parallelogram**

- Determinant and Area of a Parallelogram
  - $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2]$, $\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$



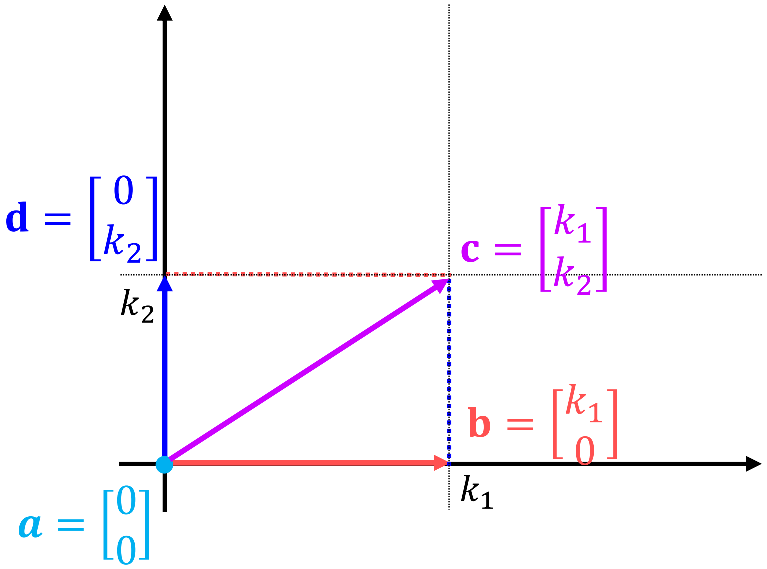
  - Area of a parallelogram
    - $B = \|\mathbf{v}_1\| \longrightarrow B^2 = \|\mathbf{v}_1\|^2 = \mathbf{v}_1 \cdot \mathbf{v}_1$
    - $H^2 = \|\mathbf{v}_2\|^2 - \|Proj_L(\mathbf{v}_2)\|^2 = \mathbf{v}_2 \cdot \mathbf{v}_2 - \left\|\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\right\|^2 = \mathbf{v}_2 \cdot \mathbf{v}_2 - \left(\frac{(\mathbf{v}_2 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_1)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{v}_1)}\mathbf{v}_1 \cdot \mathbf{v}_1\right)$
    - $S = BH \longrightarrow S^2 = B^2H^2 = (\mathbf{v}_1 \cdot \mathbf{v}_1)\left(\mathbf{v}_2 \cdot \mathbf{v}_2 - \left(\frac{(\mathbf{v}_2 \cdot \mathbf{v}_1)^2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\right) = (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1)^2$

      $= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 = a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - (a^2b^2 + 2abcd + c^2d^2)$

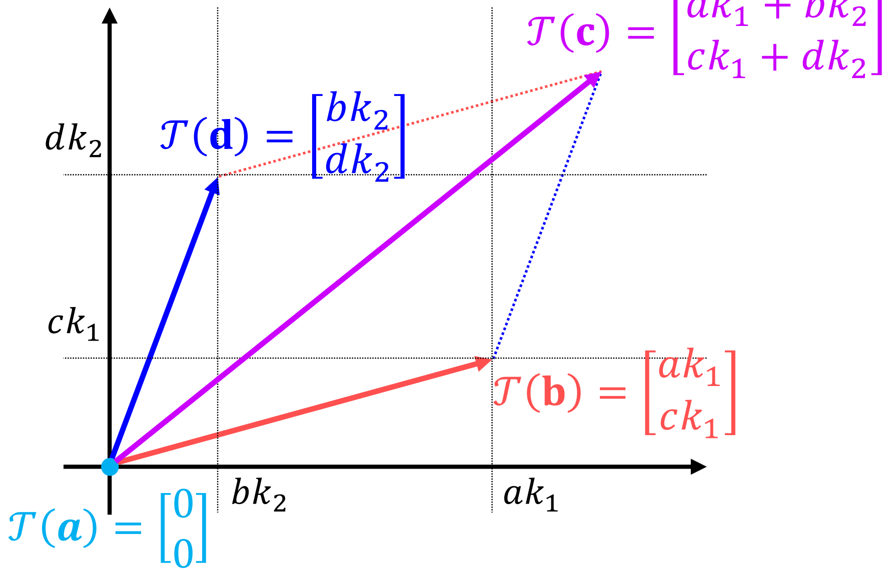
      $= a^2d^2 - 2abcd + c^2d^2 = (ad - bc)^2 = \left(det(\mathbf{A})\right)^2$
    - $\therefore S = |det(\mathbf{A})|$

More Determinant: **Scaling**

- Determinant as Scaling Factor
  - $\mathcal{T}:\ \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x} = \mathbf{A}\mathbf{x}$
  - $\mathbf{R} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \xrightarrow{\mathcal{T}} \mathcal{T}(\mathbf{R}) = \begin{bmatrix} ak_1 & bk_2 \\ ck_1 & dk_2 \end{bmatrix} = \mathbf{P}$
  - $det(\mathbf{P}) = |k_1k_2ad - k_1k_2bc| = |k_1k_2(ad-bc)| = |k_1k_2 \cdot det(\mathbf{A})|$



$\mathcal{T}$
$: \mathbb{R}^2 \to \mathbb{R}^2$



Next Lecture

- Affine Transformation