[C2-001] 기초수학

Lecture 03: Linear Transformation II

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Recap: Linear System

- Solving Linear Systems using Reduced Row Echelon Form (RREF)
  - $$\begin{matrix} x_1 + 2x_2 + x_3 + x_4 = 7 \\ x_1 + 2x_2 + 2x_3 - x_4 = 12 \\ 2x_1 + 4x_2 + \quad +6x_4 = 4 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & |\,7 \\ 1 & 2 & 2 & -1 & |12 \\ 2 & 4 & 0 & 6 & |\,4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |\;7 \\ 0 & 0 & -1 & 2 & |-5 \\ 0 & 0 & -2 & 4 & |-10 \end{bmatrix} \rightarrow$$
  - $$\begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |\;7 \\ 0 & 0 & \mathbf{1} & -2 & |\;5 \\ 0 & 0 & -2 & 4 & |-10 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 1 & 1 & |7 \\ 0 & 0 & \mathbf{1} & -2 & |5 \\ 0 & 0 & 0 & 0 & |0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 & |2 \\ 0 & 0 & \mathbf{1} & -2 & |5 \\ 0 & 0 & 0 & 0 & |0 \end{bmatrix} : rref(\mathbf{A})$$
  - $$\begin{matrix} x_1 = 2 - 2x_2 - 3x_4 \\ x_3 = 5 + 2x_4 \end{matrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Recap: Linear Transformation I: **Null Space**

- Null Space

— The null space is the set all vectors in V that map to **0**

– $N = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$: Null space of $\mathbf{A}$, $N(\mathbf{A})$

- $\mathbf{Ax} = \mathbf{0}$: Homogeneous eq., $N = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$ is the subspace of **A**?
  - $\mathbf{A0} = \mathbf{0}$
  - $\mathbf{v}_1,\ \mathbf{v}_2 \in N,\ \mathbf{Av}_1 = \mathbf{0},\ \mathbf{Av}_2 = \mathbf{0}\ \rightarrow \mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{Av}_1 + \mathbf{Av}_2 = \mathbf{0}$
  - $\mathbf{v}_1 \in N,\ c \in \mathbb{R},\ \mathbf{A}(c\mathbf{v}_1) = c\mathbf{Av}_1 = \mathbf{0}$

Recap: Linear Transformation I: **Null Space**

- Example of Null Space
  - $\mathbf{Ax} = \mathbf{0} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} : N(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^4 \mid \mathbf{Ax} = \mathbf{0}\}$
  - $\begin{matrix} x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 & = 0 \\ 4x_1 + 3x_2 + 2x_3 + x_4 & = 0 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 0 & -1 & -2 & 0 \\ 0 & \mathbf{1} & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] : rref(\mathbf{A})$$

Recap: Linear Transformation I: **Null Space**

- Example of Null Space
  - $\begin{bmatrix} \color{red}{\mathbf{1}} & 0 & -1 & -2 & | & 0 \\ 0 & \color{red}{\mathbf{1}} & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} : rref(\mathbf{A})$
  - $\begin{matrix} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{matrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$
  - $N(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) = N\big(rref(\mathbf{A})\big)$

Recap: Linear Transformation I: Null Space

- Relation between Null Space and Column Vector of Matrix $\mathbf{A}$
  - $\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$
  - $\mathbf{Ax} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$
  - $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent

    $\Leftrightarrow x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ for x_i, only solution is $x_i = 0\ for\ 1 \le i \le n$

    $\Leftrightarrow N(\mathbf{A}) = \{\mathbf{0}\} \quad \leftarrow x_1, x_2, \cdots, x_n = 0$

Recap: Linear Transformation I: **Column Space**

- The column space is the vector space spanned by the matrix's column vectors
  - $\mathbf{A} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n],\ \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^m \ \rightarrow C(\mathbf{A}) = Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$
  - $\{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \mid x_1, x_2, \cdots, x_n \in \mathbb{R}\} = Span(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) = C(\mathbf{A})$
  - $\mathbf{Ax} = \mathbf{b}_1,\ if\ \mathbf{b}_1 \notin C(\mathbf{A}) \ \Longrightarrow \mathbf{Ax} = \mathbf{b}_1$ has no solution
  - $\mathbf{Ax} = \mathbf{b}_2,\ if\ \mathbf{b}_2 \in C(\mathbf{A}) \ \Longrightarrow \mathbf{Ax} = \mathbf{b}_2$ has at least one solution

Linear Transformation I: **Column Space**

- Basis for Column Space & Null Space
  - $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \rightarrow C(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$

- These vectors are basis of $C(\mathbf{A})$?

— If these vectors are *linearly independent*, they would be the basis of $C(\mathbf{A})$

— Linearly independent $\Leftrightarrow N(\mathbf{A}) = \{\mathbf{0}\} \Leftrightarrow N\big(rref(\mathbf{A})\big) = \{\mathbf{0}\}$

Linear Transformation I: **Column Space**

- Basis for Column Space & Null Space

  - $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 4 & 3 & 0 \\ 3 & 4 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \rightarrow$

  $$\left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 3 & 2 & 0 \\ 0 & \mathbf{1} & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] : rref(\mathbf{A})$$

  - $\begin{matrix} x_1 = -3x_3 - 2x_4 \\ x_2 = +2x_3 + x_4 \end{matrix} \rightarrow N(\mathbf{A}) = N\big(rref(\mathbf{A})\big) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Linear Transformation I: **Column Space**

- Find the Basis for Column Space
  - $\mathbf{A}\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \mathbf{0}$
  - $N(\mathbf{A}) = \begin{cases} x_1 = -3x_3 - 2x_4 \\ x_2 = +2x_3 + x_4 \end{cases}$
  - If $x_3 = 0,\ x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = -x_4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \rightarrow x_4 = -1: \begin{cases} x_1 = -3 \cdot 0 - 2 \cdot (-1) = 2 \\ x_2 = +2 \cdot 0 + 1 \cdot (-1) = -1 \end{cases}$
  - $2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$: Linearly dependent

Linear Transformation I: **Column Space**

- Find the Basis for Column Space

  - If $x_4 = 0,\ x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = -x_3 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \rightarrow x_3 = -1: \begin{cases} x_1 = -3 \cdot (-1) - 2 \cdot 0 = 3 \\ x_2 = +2 \cdot (-1) + 1 \cdot 0 = -2 \end{cases}$

  - $3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$: Linearly dependent

  - $C(\mathbf{A}) = Span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right)$: Linearly independent $\Longrightarrow \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right\}$: a basis for $C(\mathbf{A})$

Dimension of **Null Space**

- Nullity

— The dimension of null space (= The number of free variables in $rref(\mathbf{A})$)

- $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix}$, $N(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^5 \mid \mathbf{Ax} = \mathbf{0}\} = N(rref(\mathbf{A}))$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 & 2 & | & 0 \\ 1 & 1 & 3 & 1 & 4 & | & 0 \end{bmatrix}$$

Dimension of **Null Space**

- Nullity
  — The dimension of null space (= The number of free variables in $rref(\mathbf{A})$)
  — Dimension of a subspace: The number of vectors in a basis for the subspace.
  – $dim(\mathbf{A}) = nullity(\mathbf{A}) = 3$

- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad N(\mathbf{A}) = N(rref(\mathbf{A})) = Span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$

Dimension of **Column Space**

- Rank

— The dimension of the vector space generated (or spanned) by its columns

- $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix} \rightarrow C(\mathbf{A}) = Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$: Basis for $C(\mathbf{A})$?

Topics

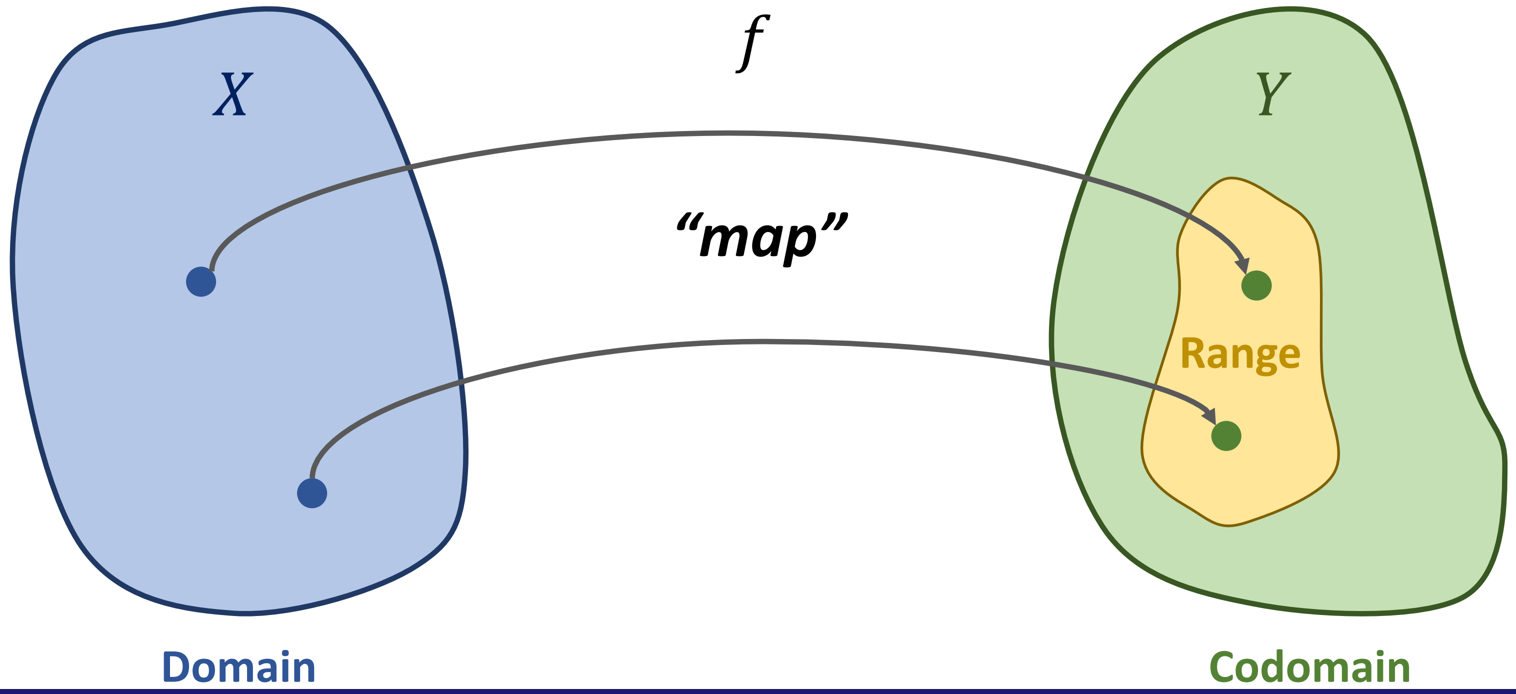
- Linear Transformation II

Function

- A function f is a relation where every value in the first set X (domain) maps to one and only one value in the second set Y (Codomain)

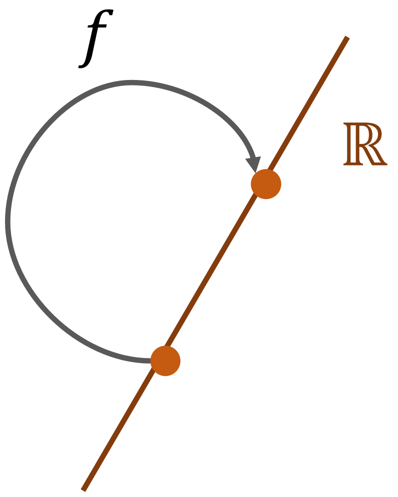
– Range: A subset of codomain that the function actually maps to

– $f: X \rightarrow Y$

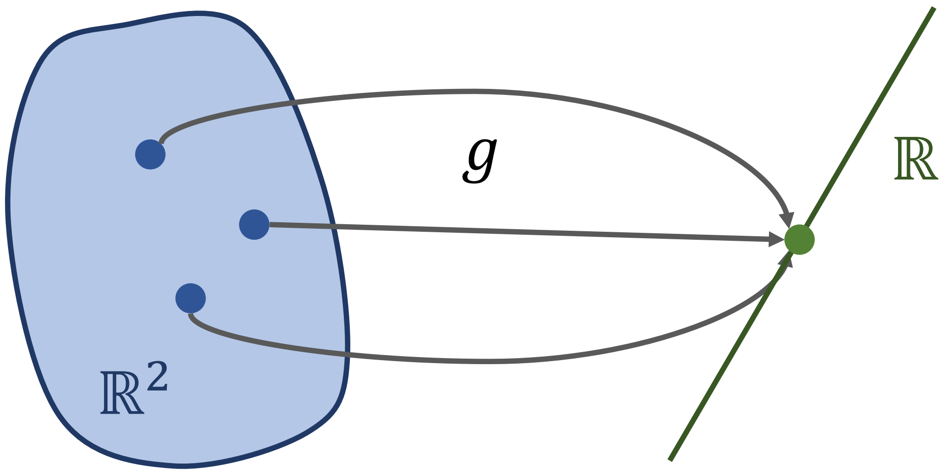


Function

- Example of Function
  - $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \qquad \Leftrightarrow f: x \longmapsto x^2$
  - $g: \mathbb{R}^2 \to \mathbb{R}, \; g(x_1, x_2) = 2 \Leftrightarrow g: x_1, x_2 \longmapsto 2$



$f(x) = x^2$

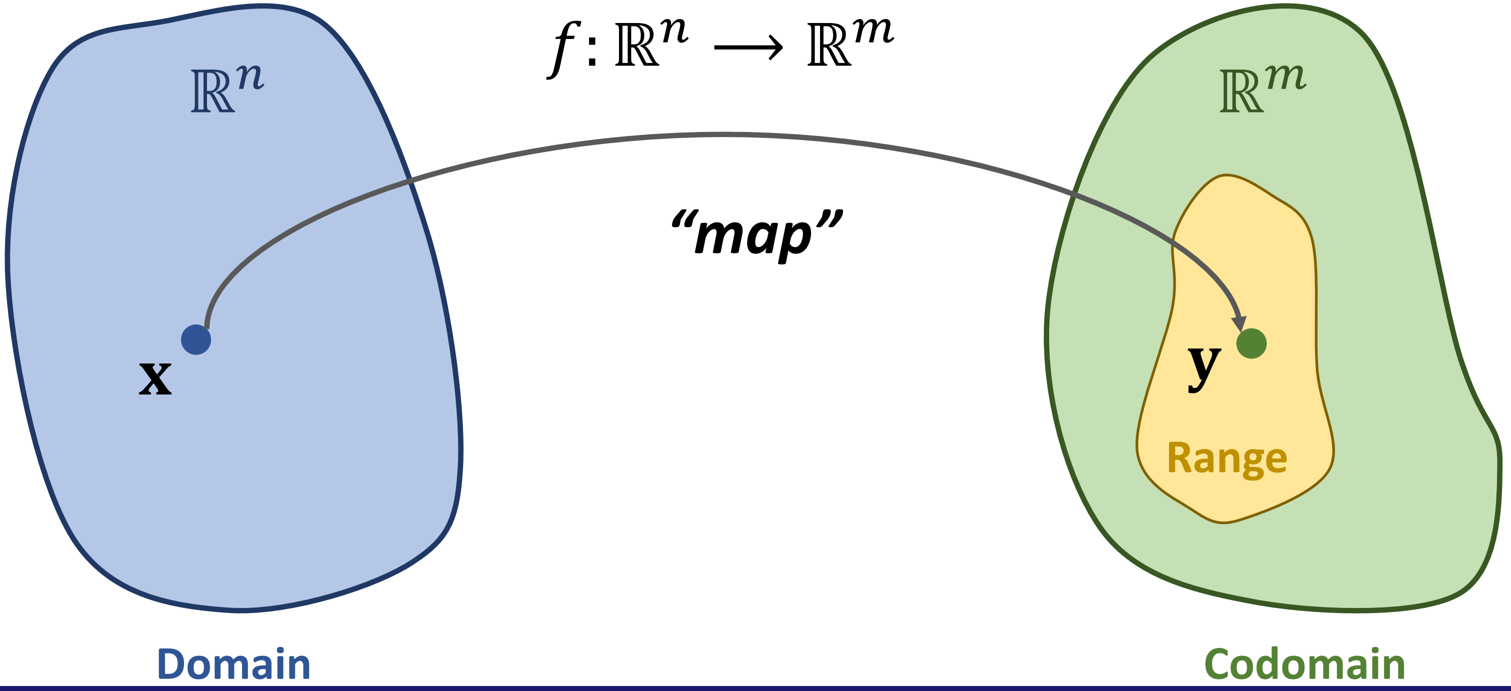


$g(x_1, x_2) = 2$

Transformation

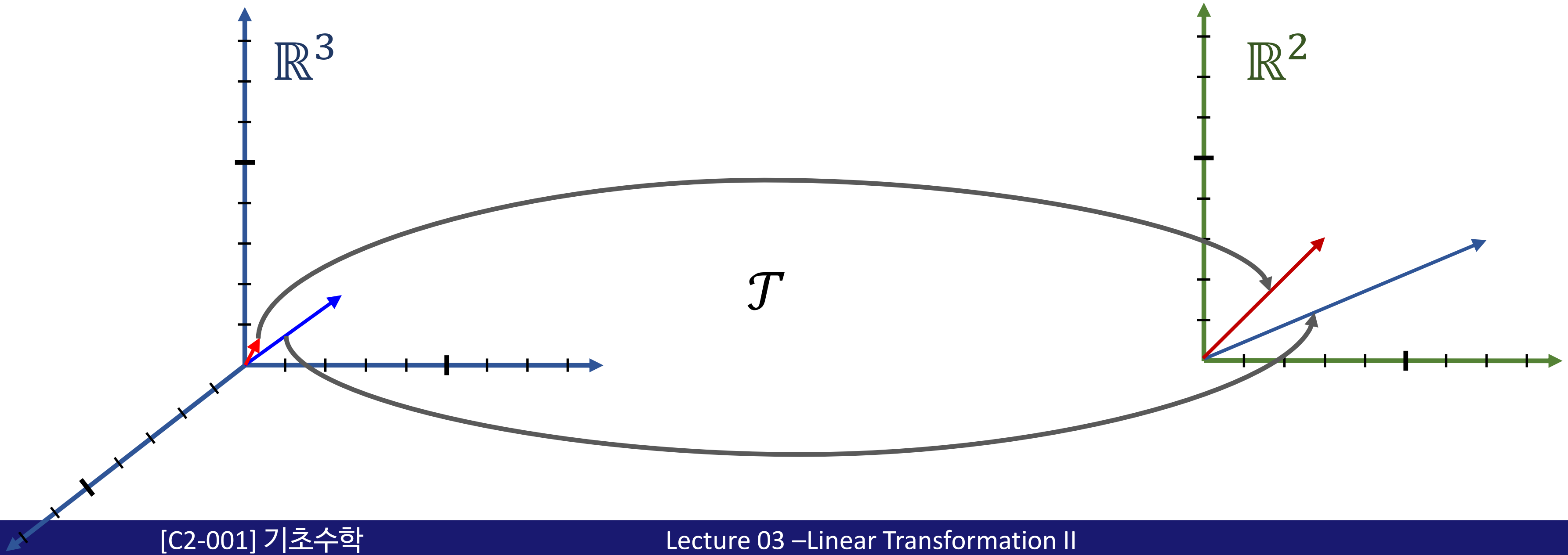
- A transformation is known as a function whose domain is an n-dimensional space ($\mathbb{R}^n$) and whose range is an m -dimensional space ($\mathbb{R}^m$) (i.e., *function operation of vectors*)

– $\mathbb{R}^n = \{n - tuple,\ \mathbf{x} = (x_1, x_2, \cdots, x_n) \mid x_1, x_2, \cdots, x_n \in \mathbb{R}\},\ \mathbf{x} \in \mathbb{R}^n$



Transformation

- Example of Transformation
  - $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2,\ f(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3) \longrightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 3x_3 \end{bmatrix}$
  - $f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix},\ f\left(\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$



Linear Transformation

- A linear transformation $\mathcal{T}$ is mapping between two vector spaces $X \subseteq \mathbb{R}^n$and $Y \subseteq \mathbb{R}^m$, where for all vectors in $\mathbb{R}^n$ and for all scalars c:

① $\mathcal{T}(\mathbf{a}+\mathbf{b}) = \mathcal{T}(\mathbf{a}) + \mathcal{T}(\mathbf{b})$

② $\mathcal{T}(\mathrm{c}\mathbf{a}) = \mathrm{c}\mathcal{T}(\mathbf{a})$

- Linear Transformation OR Not?
  - $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2,\ f(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3) \longrightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 3x_3 \end{bmatrix}$

Linear Transformation

- Example of Linear Transformation

  - $\mathcal{T}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \mathcal{T}(x_1, x_2) = (x_1 + x_2, 2x_1) \longrightarrow \mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$① \mathcal{T}(\mathbf{a} + \mathbf{b}) = \mathcal{T}\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 2a_1 + 2b_1 \end{bmatrix}$$

$$\mathcal{T}(\mathbf{a}) = \mathcal{T}\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ 2a_1 \end{bmatrix},\ \mathcal{T}(\mathbf{b}) = \mathcal{T}\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 \\ 2b_1 \end{bmatrix}$$

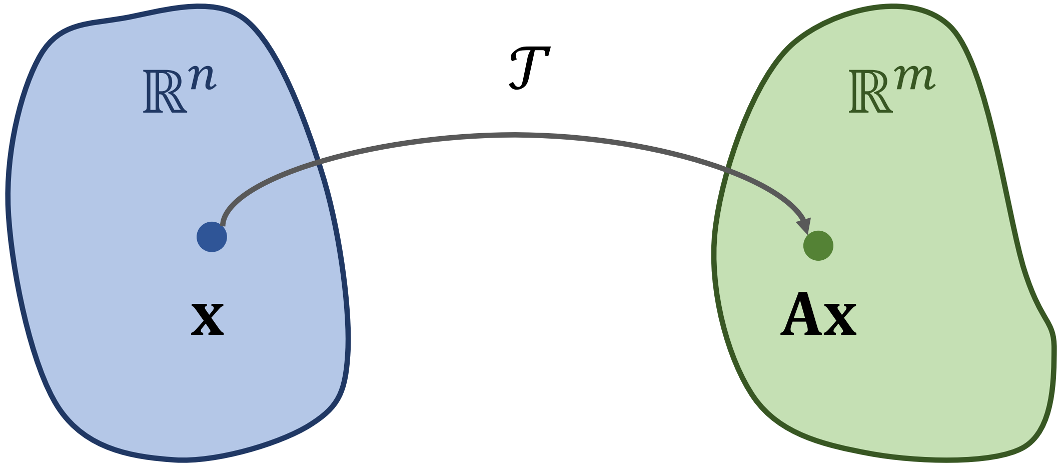
$$\longrightarrow \mathcal{T}(\mathbf{a}) + \mathcal{T}(\mathbf{b}) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 2a_1 + 2b_1 \end{bmatrix}$$

$$② c\mathbf{a} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix} \longrightarrow \mathcal{T}(c\mathbf{a}) = \mathcal{T}\left(\begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}\right) = \begin{bmatrix} ca_1 + ca_2 \\ 2ca_1 \end{bmatrix} = c\begin{bmatrix} a_1 + a_2 \\ 2a_1 \end{bmatrix} = c\mathcal{T}(\mathbf{a})$$

Linear Transformation and Basis Vectors

- Matrix-Vector Products As Linear Transformation
  – $\mathbf{A}_{m\times n} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n],\ \mathcal{T}: \mathbb{R}^n \to \mathbb{R}^m,\ \mathcal{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$
  – $\mathbf{A}\mathbf{x} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \in \mathbb{R}^m$

- Example of Matrix-Vector Products As Linear Transformation
  - $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix},\ \mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \mathbf{B}\mathbf{x}$
  - $\mathcal{T}(\mathbf{x}) = \mathbf{B}\mathbf{x} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$
  - $\mathcal{T}(x_1, x_2) = (2x_1 - x_2, 3x_1 + 4x_2,)$



Linear Transformation and Basis Vectors

- Linear Transformation As Matrix-Vector Products
- Standard basis for $\mathbb{R}^n$: ① $Span(\cdot) = \mathbb{R}^n$, ② Linearly independent

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n] \longrightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

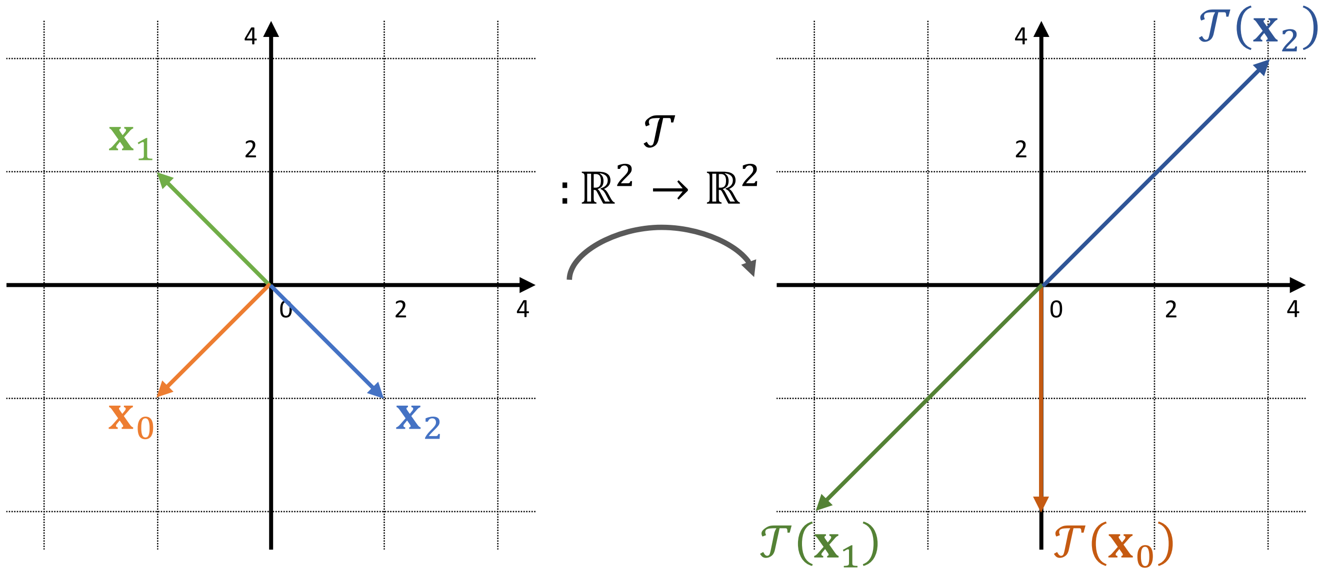
- Representation of Linear Transformation with standard basis
  - $\mathcal{T}(\mathbf{x}) = \mathcal{T}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = \mathcal{T}(x_1\mathbf{e}_1) + \mathcal{T}(x_2\mathbf{e}_2) + \cdots + \mathcal{T}(x_n\mathbf{e}_n)$
    $= x_1\mathcal{T}(\mathbf{e}_1) + x_2\mathcal{T}(x_2\mathbf{e}_2) + \cdots + x_n\mathcal{T}(\mathbf{e}_n)$
  - $\mathcal{T}(\mathbf{x}) = [\mathcal{T}(\mathbf{e}_1)\ \mathcal{T}(\mathbf{e}_2)\ \cdots\ \mathcal{T}(\mathbf{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Linear Transformation and Basis Vectors

- Example of Linear Transformation As Matrix-Vector Products
  - $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^3,\ \mathcal{T}(x_1, x_2) = (x_1 + 3x_2, 5x_2 - 4x_1, 4x_1 + x_2) \longrightarrow \mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 5x_2 - 4x_1 \\ 4x_1 + x_2 \end{bmatrix}$
  - $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{cases} \mathcal{T}(\mathbf{e}_1) = \mathcal{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\ \mathcal{T}(\mathbf{e}_2) = \mathcal{T}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \end{cases} \longrightarrow \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 1 \end{bmatrix}$
  - $\therefore \mathcal{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Image of Transformation

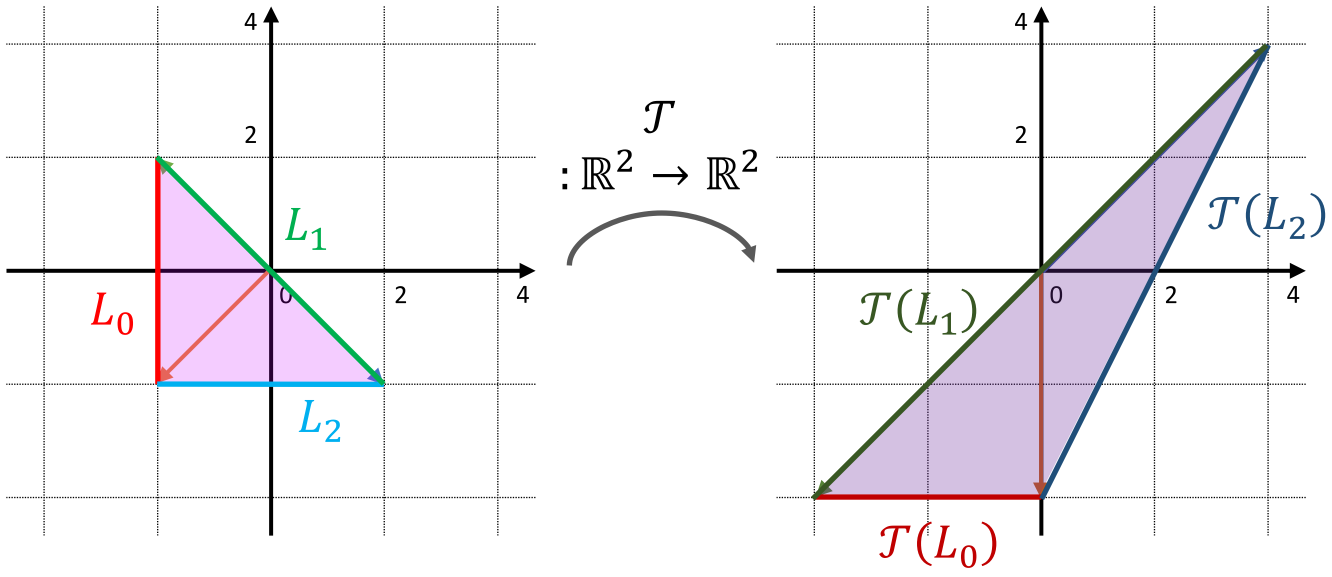
- Image of a Subset Under Transformation
  - $S = \{L_0, L_1, L_2\}$
    - $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
    - $L_0 = \{\mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \mid 0 \le t \le 1\}$
    - $L_1 = \{\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1) \mid 0 \le t \le 1\}$
    - $L_2 = \{\mathbf{x}_2 + t(\mathbf{x}_0 - \mathbf{x}_2) \mid 0 \le t \le 1\}$



  - $\mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
    - $\mathcal{T}(\mathbf{x}_0) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, $\mathcal{T}(\mathbf{x}_1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$, $\mathcal{T}(\mathbf{x}_2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$
    - $\mathcal{T}(L_0) = \{\mathcal{T}(\mathbf{x}_0) + t(\mathcal{T}(\mathbf{x}_1) - \mathcal{T}(\mathbf{x}_0)) \mid 0 \le t \le 1\}$
    - $\mathcal{T}(L_1) = \{\mathcal{T}(\mathbf{x}_1) + t(\mathcal{T}(\mathbf{x}_2) - \mathcal{T}(\mathbf{x}_1)) \mid 0 \le t \le 1\}$
    - $\mathcal{T}(L_2) = \{\mathcal{T}(\mathbf{x}_2) + t(\mathcal{T}(\mathbf{x}_0) - \mathcal{T}(\mathbf{x}_2)) \mid 0 \le t \le 1\}$

Image of Transformation

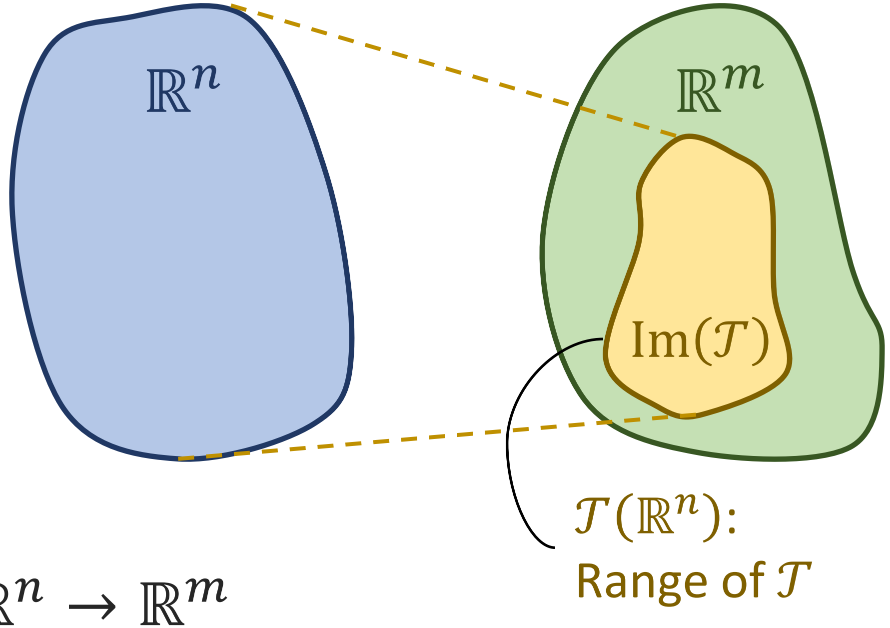
- Image of a Subset Under Transformation
  - $S = \{L_0, L_1, L_2\}$
    - $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
    - $L_0 = \{\mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \mid 0 \le t \le 1\}$
    - $L_1 = \{\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1) \mid 0 \le t \le 1\}$
    - $L_2 = \{\mathbf{x}_2 + t(\mathbf{x}_0 - \mathbf{x}_2) \mid 0 \le t \le 1\}$



  - $\mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
    - $\mathcal{T}(\mathbf{x}_0) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, $\mathcal{T}(\mathbf{x}_1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$, $\mathcal{T}(\mathbf{x}_2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$
    - $\mathcal{T}(L_0) = \{\mathcal{T}(\mathbf{x}_0) + t(\mathcal{T}(\mathbf{x}_1) - \mathcal{T}(\mathbf{x}_0)) \mid 0 \le t \le 1\}$
    - $\mathcal{T}(L_1) = \{\mathcal{T}(\mathbf{x}_1) + t(\mathcal{T}(\mathbf{x}_2) - \mathcal{T}(\mathbf{x}_1)) \mid 0 \le t \le 1\}$
    - $\mathcal{T}(L_2) = \{\mathcal{T}(\mathbf{x}_2) + t(\mathcal{T}(\mathbf{x}_0) - \mathcal{T}(\mathbf{x}_2)) \mid 0 \le t \le 1\}$

Image of Transformation

- Image of a Subset Under Transformation
  - V: Subspace in : $\mathbb{R}^n$
  - $\mathcal{T}: \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{T}(V)$: Image of V under $\mathcal{T}$ & Subspace
    - $\mathcal{T}(\mathbf{a}), \mathcal{T}(\mathbf{b}) \in \mathcal{T}(V)$, $\begin{cases} \mathcal{T}(\mathbf{a}) + \mathcal{T}(\mathbf{b}) = \mathcal{T}(\mathbf{a}+\mathbf{b}) \in \mathcal{T}(V) \\ c\mathcal{T}(\mathbf{a}) \in \mathcal{T}(V) \\ \mathbf{0} \in \mathcal{T}(V) \end{cases}$



  - $\mathcal{T}(\mathbb{R}^n)$: Image of $\mathbb{R}^n$ under $\mathcal{T}$ $= \{\mathcal{T}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\}$, $\mathcal{T}: \mathbb{R}^n \to \mathbb{R}^m$
    - Range of $\mathcal{T}$ $=$ Image of $\mathbb{R}^n$ under $\mathcal{T}$

  - $\mathcal{T}(\mathbf{x}) = \mathbf{Ax}$: Image of $\mathbb{R}^n$ under $\mathcal{T}$: $\mathcal{T}(\mathbb{R}^n)$

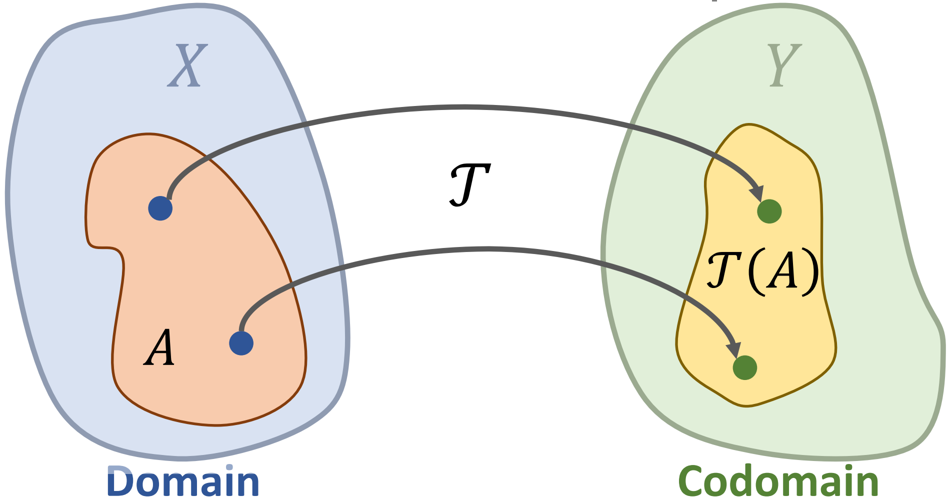
    $=$ Image of $\mathcal{T}$ $= \mathrm{Im}(\mathcal{T}) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\} = C(\mathbf{A}) = Span(\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n)$

Preimage and Kernel

- Preimage of a Set Under Transformation
  - Image of subset A of domain under $\mathcal{T}$ is the set of all output values
  - Preimage (inverse image) of subset B of codomain under $\mathcal{T}$ is the set of all elements of the domain that map to the members of B

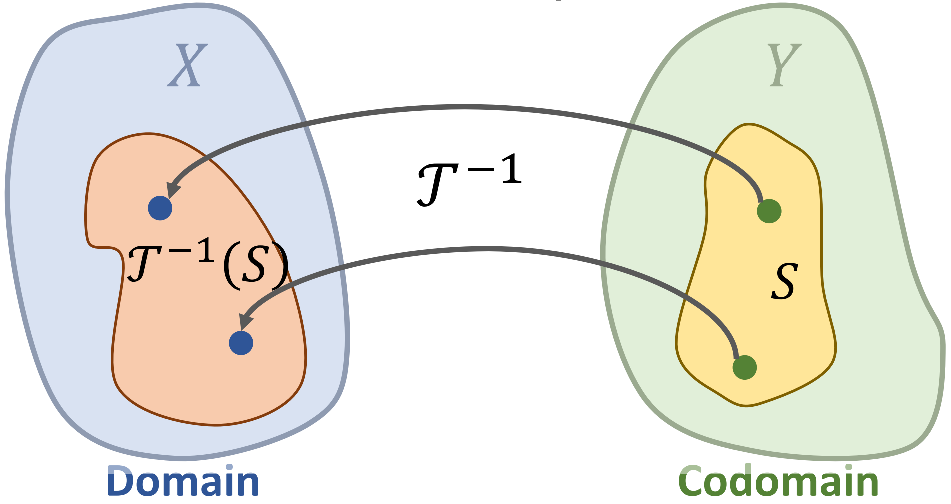
- $A \subseteq X$: Subset of X
- $\mathcal{T}(A) \subseteq Y$: Image of A under $\mathcal{T}$

$$= \{\mathcal{T}(\mathbf{x}) \in Y \mid \mathbf{x} \in A\}$$



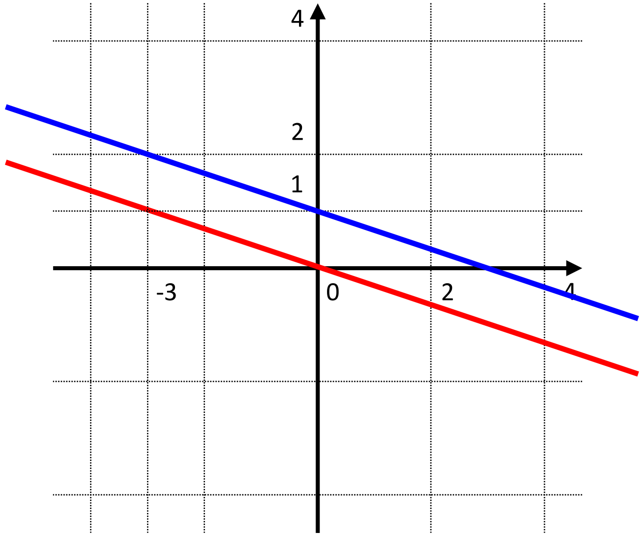
- $S \subseteq Y$: Subset of Y
- $\mathcal{T}^{-1}(S)$: Preimage of S under $\mathcal{T}$

$$= \{\mathbf{x} \in X \mid \mathcal{T}(\mathbf{x}) \in S\}$$



Preimage and Kernel

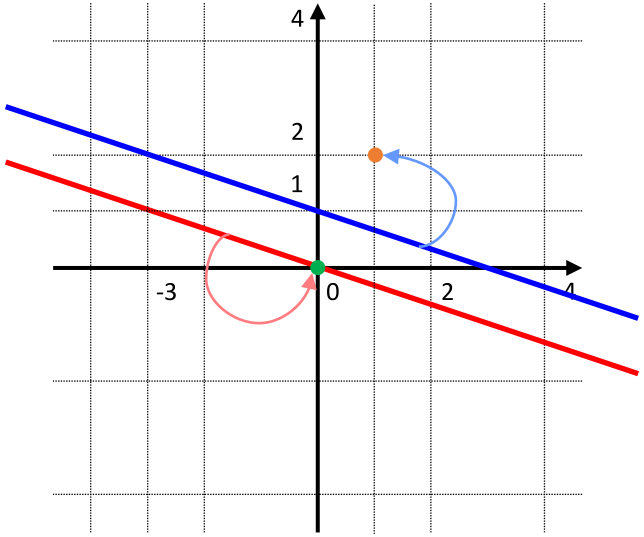
- Kernel of Transformation
  - Kernel is the preimage (inverse image) of 0
  - Kernel of a matrix is the null space of the matrix



- $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Ax}$
- $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \to \mathcal{T}^{-1}(S) = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{Ax} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ OR } \mathbf{Ax} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + 3x_2 = 0 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, t \in \mathbb{R} \cdots C$
- $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + 3x_2 = 1 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, t \in \mathbb{R} \cdots D$
- $\mathcal{T}(D) = \{\mathbf{0}\}$: Kernel of $\mathcal{T}$, $Ker(\mathcal{T}) = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathcal{T}(\mathbf{x}) = \{\mathbf{0}\} \right\} = N(\mathbf{A})$

Preimage and Kernel

- Kernel of Transformation
  - Kernel is the preimage (inverse image) of 0
  - Kernel of a matrix is the null space of the matrix

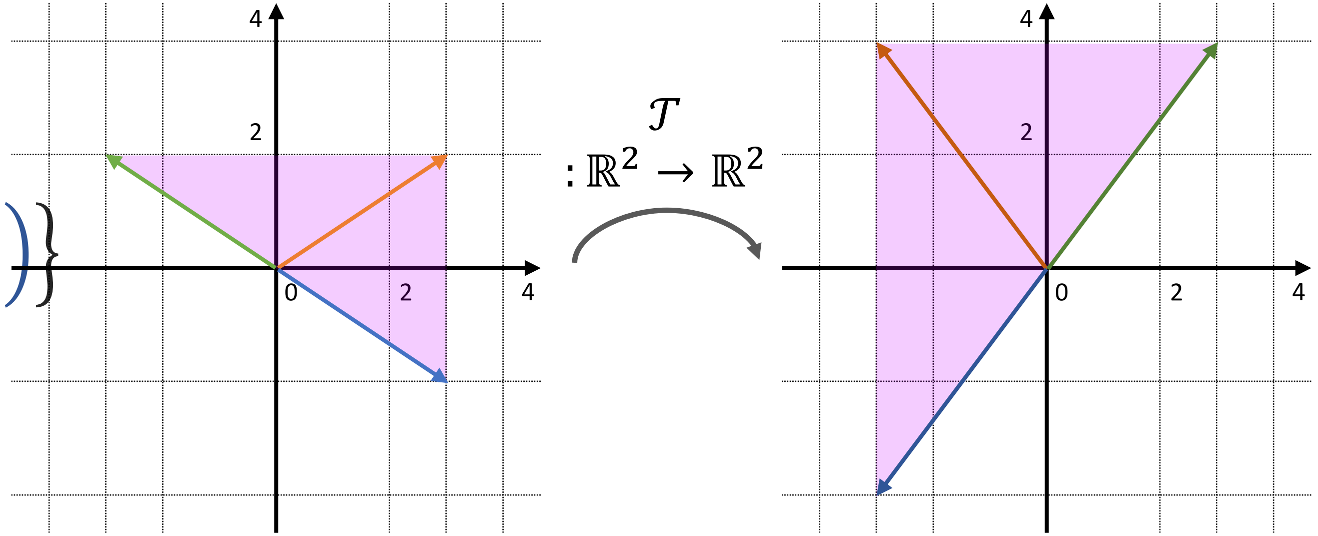


- $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{T}(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Ax}$
- $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \to \mathcal{T}^{-1}(S) = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{Ax} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ OR } \mathbf{Ax} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + 3x_2 = 0 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, t \in \mathbb{R} \cdots C$
- $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + 3x_2 = 1 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, t \in \mathbb{R} \cdots D$
- $\mathcal{T}(D) = \{\mathbf{0}\}$: Kernel of $\mathcal{T}$, $Ker(\mathcal{T}) = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathcal{T}(\mathbf{x}) = \{\mathbf{0}\} \right\} = N(\mathbf{A})$

Linear Transformation: **Scaling and Reflection**

- Scaling & Reflection
  - $\mathcal{T}: \mathbb{R}^n \rightarrow \mathbb{R}^m,\ \mathcal{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} = [\mathcal{T}(\mathbf{e}_1)\, \mathcal{T}(\mathbf{e}_2)\ \cdots\ \mathcal{T}(\mathbf{e}_n)]$

- Reflect around y-axis & Stretch $\times$ 2 in y direction
  - $\mathcal{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ 2y \end{bmatrix} \rightarrow \mathbf{A} = \left[\mathcal{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\ \mathcal{T}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \mathcal{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
  - $S = \left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}$
  - $\mathcal{T}(S) = \left\{\mathcal{T}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right), \mathcal{T}\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right), \mathcal{T}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)\right\}$
    $= \left\{\begin{bmatrix} -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \end{bmatrix}\right\}$

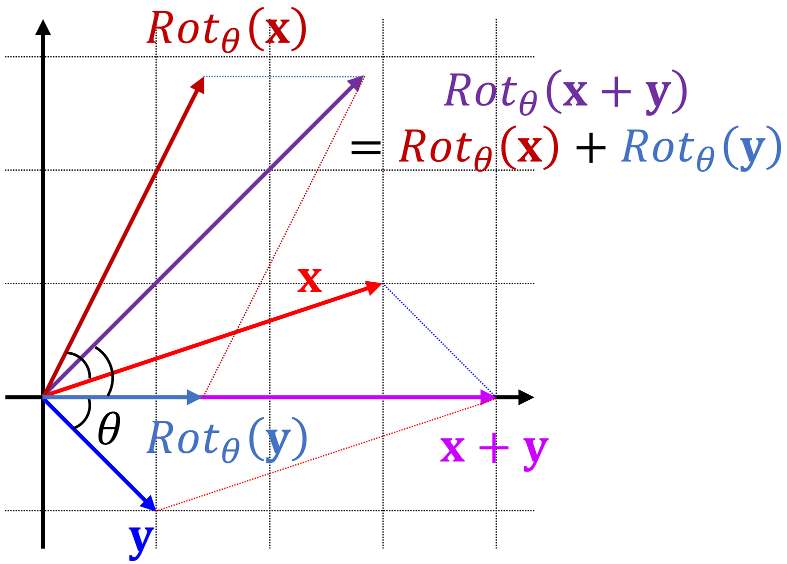


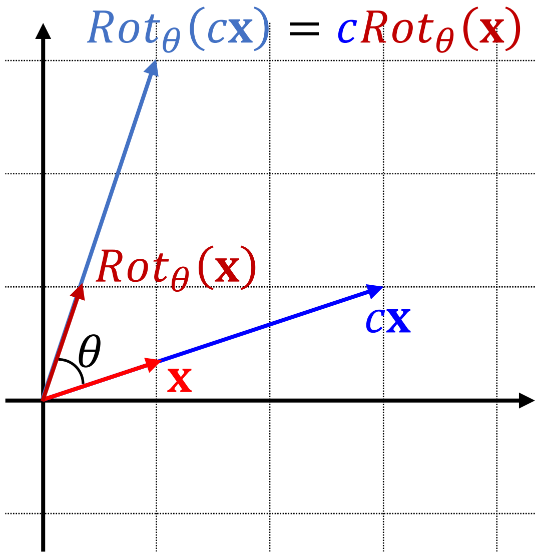
Linear Transformation: **Rotation in** $\mathbb{R}^2$

- $Rot_\theta(\mathbf{x})$: Counter clockwise θ degree rotation of $\mathbf{x}$

① $Rot_\theta(\mathbf{x}+\mathbf{y}) = Rot_\theta(\mathbf{x}) + Rot_\theta(\mathbf{y})$

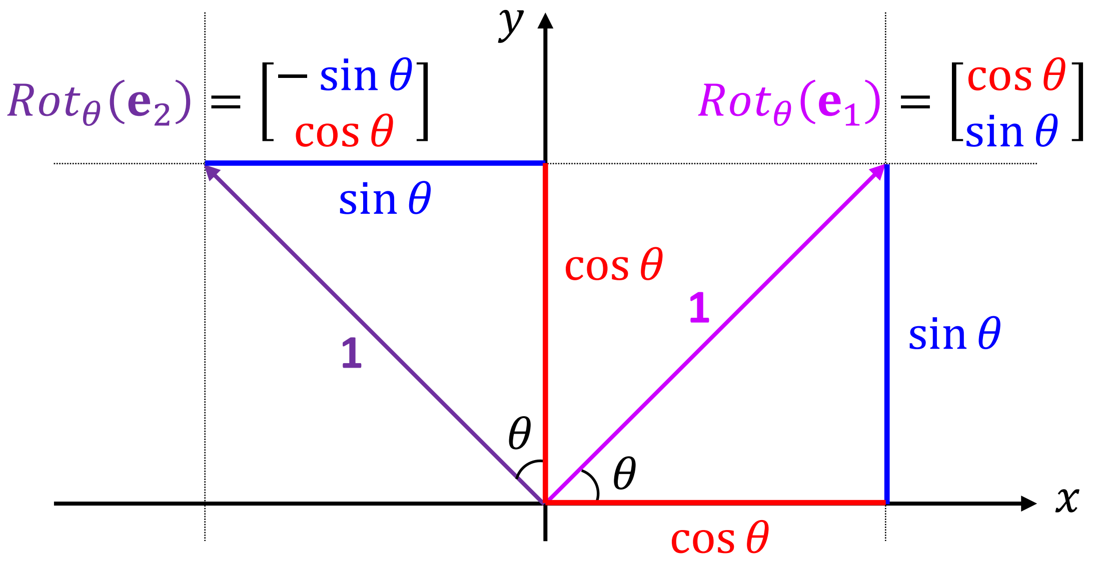
② $Rot_\theta(c\mathbf{x}) = cRot_\theta(\mathbf{x})$





Linear Transformation: **Rotation in $\mathbb{R}^2$**

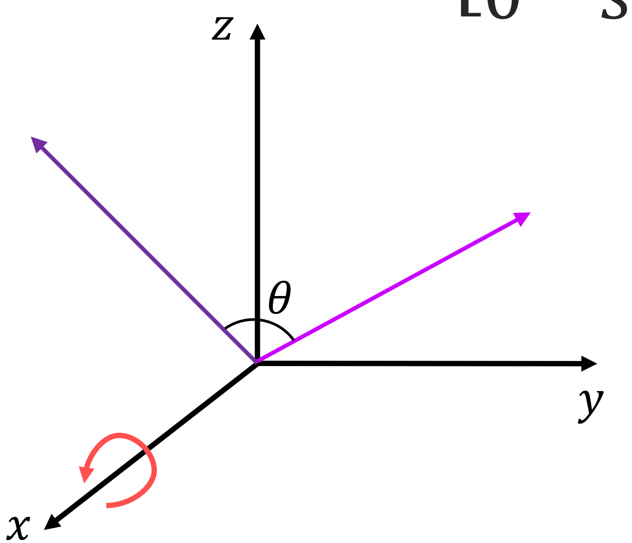
- $Rot_\theta: \mathbb{R}^2 \to \mathbb{R}^2,\ Rot_\theta(\mathbf{x}) = \mathbf{Ax},\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2]$
- $Rot_\theta(\mathbf{x}) = \mathbf{Ax} = [Rot_\theta(\mathbf{e}_1) \quad Rot_\theta(\mathbf{e}_2)]\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\mathbf{x}$

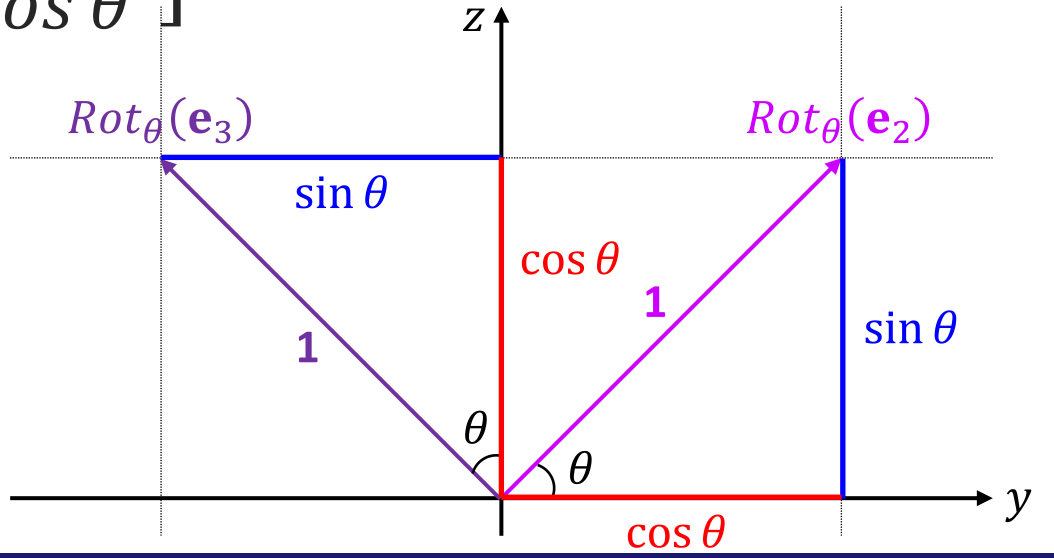


Linear Transformation: **Rotation in $\mathbb{R}^3$ (x-axis)**

- $Rot_\theta: \mathbb{R}^3 \to \mathbb{R}^3,\ Rot_\theta(\mathbf{x}) = \mathbf{A}\mathbf{x},\ \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$
- $Rot_\theta(\mathbf{x}) = \mathbf{A}\mathbf{x} = [Rot_\theta(\mathbf{e}_1) \quad Rot_\theta(\mathbf{e}_2) \quad Rot_\theta(\mathbf{e}_3)]\mathbf{x}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\mathbf{x}$$





Remind: **Unit Vector**

- Unit Vector (Normalized Vector)

— Unit vector, $\hat{\mathbf{u}}$, is the vector has length of "1"

$$- \ \mathbf{u} \in \mathbb{R}^n \longrightarrow \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \ \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = 1$$

$$- \ \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \longrightarrow \mathbf{u}: \begin{cases} ① \text{ Same Direction} \\ ② \ \|\mathbf{u}\| = 1 \end{cases} \longrightarrow \mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \hat{\mathbf{u}}$$

Linear Transformation: **Projection**

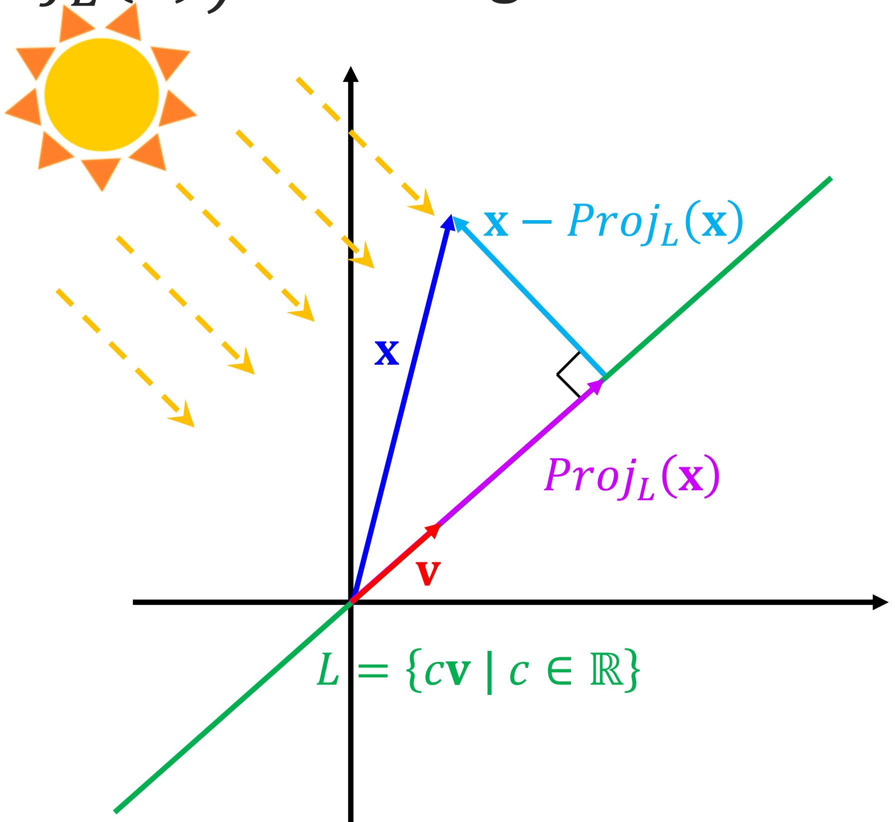
- Introduction To Projection

① $\mathrm{Proj}_{\mathrm{L}}(\mathbf{x})$: Shadow of $\mathbf{x}$ on L

② $Proj_L(\mathbf{x})$: Same vector in L where $\left(\mathbf{x} - Proj_L(\mathbf{x})\right)$ is orthogonal to $L = c\mathbf{v}$

- $(\mathbf{x} - c\mathbf{v}) \cdot \mathbf{v} = 0 \longrightarrow \mathbf{x} \cdot \mathbf{v} - c\mathbf{v} \cdot \mathbf{v} = 0$
  $\longrightarrow \ \mathbf{x} \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} \longrightarrow \ c = \frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$

- $Proj_L(\mathbf{x}) = c\mathbf{v} = \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$



Linear Transformation: **Projection**

- Projection As Matrix-Vector Product
  - $Proj_L: \mathbb{R}^n \to \mathbb{R}^n,\ Proj_L(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{x}\cdot\mathbf{v})\mathbf{v}$ $(\mathbf{v}\text{: unit vector}, \hat{\mathbf{u}})$
  - $Proj_L(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{x}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$

- Linear Transform of Projection

① $Proj_L(\mathbf{a}+\mathbf{b}) = \big((\mathbf{a}+\mathbf{b})\cdot\hat{\mathbf{u}}\big)\hat{\mathbf{u}} = (\mathbf{a}\cdot\hat{\mathbf{u}} + \mathbf{b}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = (\mathbf{a}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{b}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ $= Proj_L(\mathbf{a}) + Proj_L(\mathbf{b})$

② $Proj_L(c\mathbf{a}) = \big((c\mathbf{a})\cdot\hat{\mathbf{u}}\big)\hat{\mathbf{u}} = c(\mathbf{a}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = cProj_L(\mathbf{a})$

  - $Proj_L(\mathbf{x}) = (\mathbf{x}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{A}\mathbf{x} = \left[\left(\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix} \quad \left(\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}\right]\mathbf{x}$

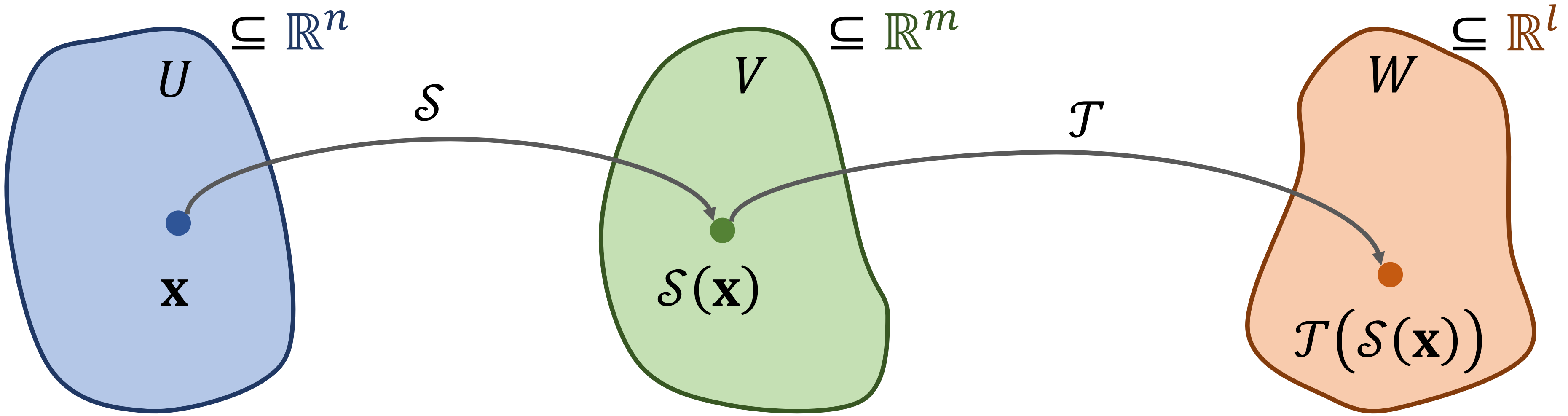
$$= \left[u_1\begin{bmatrix}u_1\\u_2\end{bmatrix} \quad u_2\begin{bmatrix}u_1\\u_2\end{bmatrix}\right]\mathbf{x} = \begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}\mathbf{x}$$

Linear Transformation: **Composition**

- Composition of Linear Transformation
  - $\mathcal{S}: U \longrightarrow V,\ \mathcal{S}(\mathbf{x}) = \mathbf{Ax}$ and $\mathcal{T}: V \longrightarrow W,\ \mathcal{T}(\mathbf{x}) = \mathbf{Bx}$
  - $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m,\ W \subseteq \mathbb{R}^l$
  - $\mathcal{T} \circ \mathcal{S}: U \longrightarrow W$: The composition of $\mathcal{T}$ with $\mathcal{S}$
  - $\mathcal{T} \circ \mathcal{S}(\mathbf{x}) = \mathcal{T}\big(\mathcal{S}(\mathbf{x})\big)$

    ① $\mathcal{T} \circ \mathcal{S}(\mathbf{x}+\mathbf{y}) = \mathcal{T}\big(\mathcal{S}(\mathbf{x}+\mathbf{y})\big) = \mathcal{T}\big(\mathcal{S}(\mathbf{x}) + \mathcal{S}(\mathbf{y})\big) = \mathcal{T}\big(\mathcal{S}(\mathbf{x})\big) + \mathcal{T}\big(\mathcal{S}(\mathbf{y})\big) = \mathcal{T} \circ \mathcal{S}(\mathbf{x}) + \mathcal{T} \circ \mathcal{S}(\mathbf{y})$

    ② $\mathcal{T} \circ \mathcal{S}(c\mathbf{x}) = \mathcal{T}\big(\mathcal{S}(c\mathbf{x})\big) = \mathcal{T}\big(c\mathcal{S}(\mathbf{x})\big) = c\mathcal{T}\big(\mathcal{S}(\mathbf{x})\big) = c\mathcal{T} \circ \mathcal{S}(\mathbf{x})$



Linear Transformation: **Composition**

- Composition of Linear Transformation
  - $\mathcal{T} \circ \mathcal{S}(\mathbf{x}) = \mathcal{T}\big(\mathcal{S}(\mathbf{x})\big) = \mathcal{T}(\mathbf{Ax}) = \mathbf{B}(\mathbf{Ax}) = \mathbf{Cx}$
  - $\mathbf{C} = [\mathbf{B}(\mathbf{Ae}_1)\ \mathbf{B}(\mathbf{Ae}_2)\ \cdots\ \mathbf{B}(\mathbf{Ae}_n)] = [\mathbf{Ba}_1\ \mathbf{Ba}_2 \cdots\ \mathbf{Ba}_n]$
  - $\mathcal{T} \circ \mathcal{S}(\mathbf{x}) = \mathcal{T}(\mathbf{Ax}) = \mathbf{B}(\mathbf{Ax}) = [\mathbf{Ba}_1\ \mathbf{Ba}_2 \cdots\ \mathbf{Ba}_n]\mathbf{x} = \mathbf{BAx}$

Next Lecture

- Matrix Inversion