Calculus GATE DA

Functions

Limit

Continuity

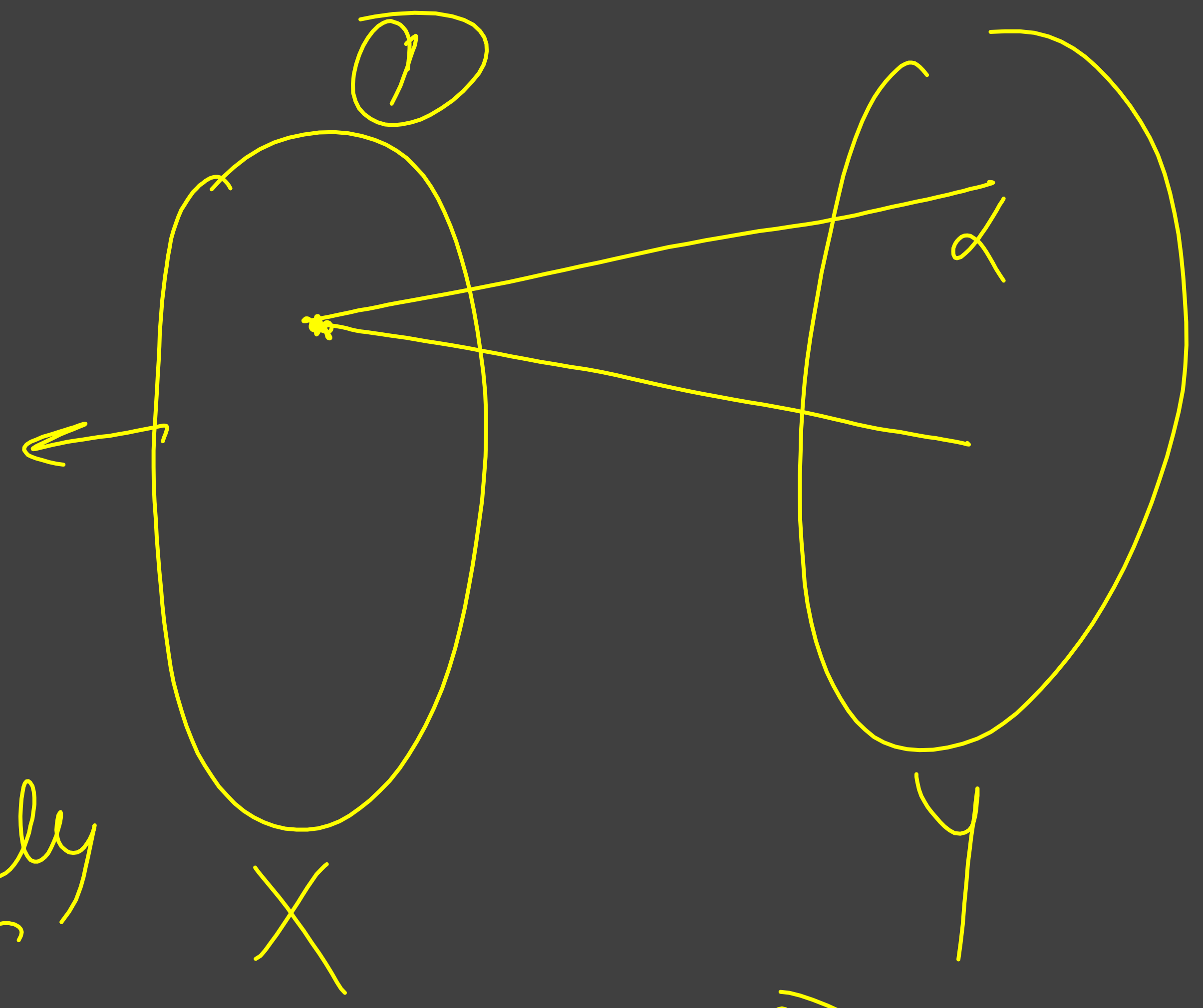
Differentiability

Taylor's thm

Maxima & Minima

$f: X \longrightarrow Y$

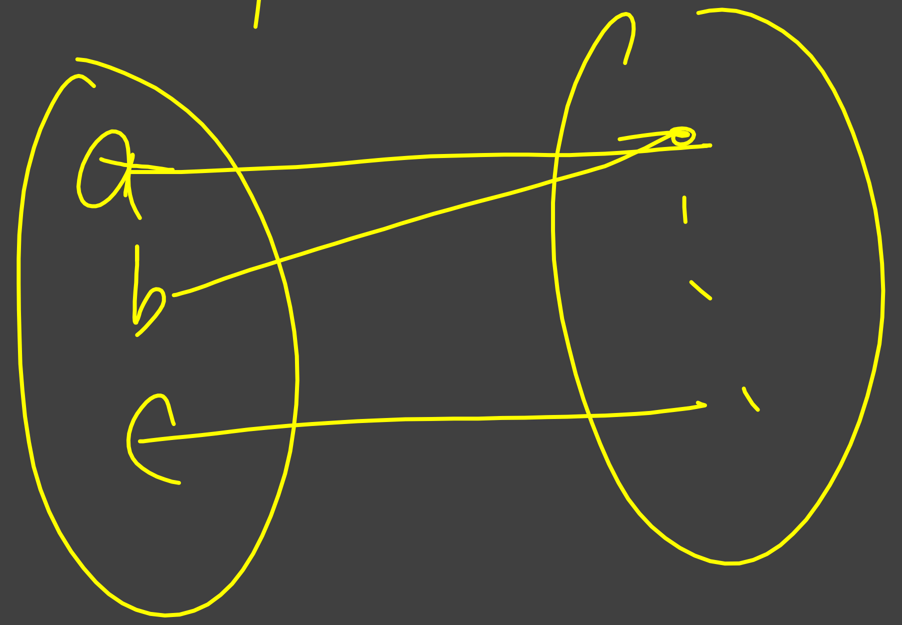
Function

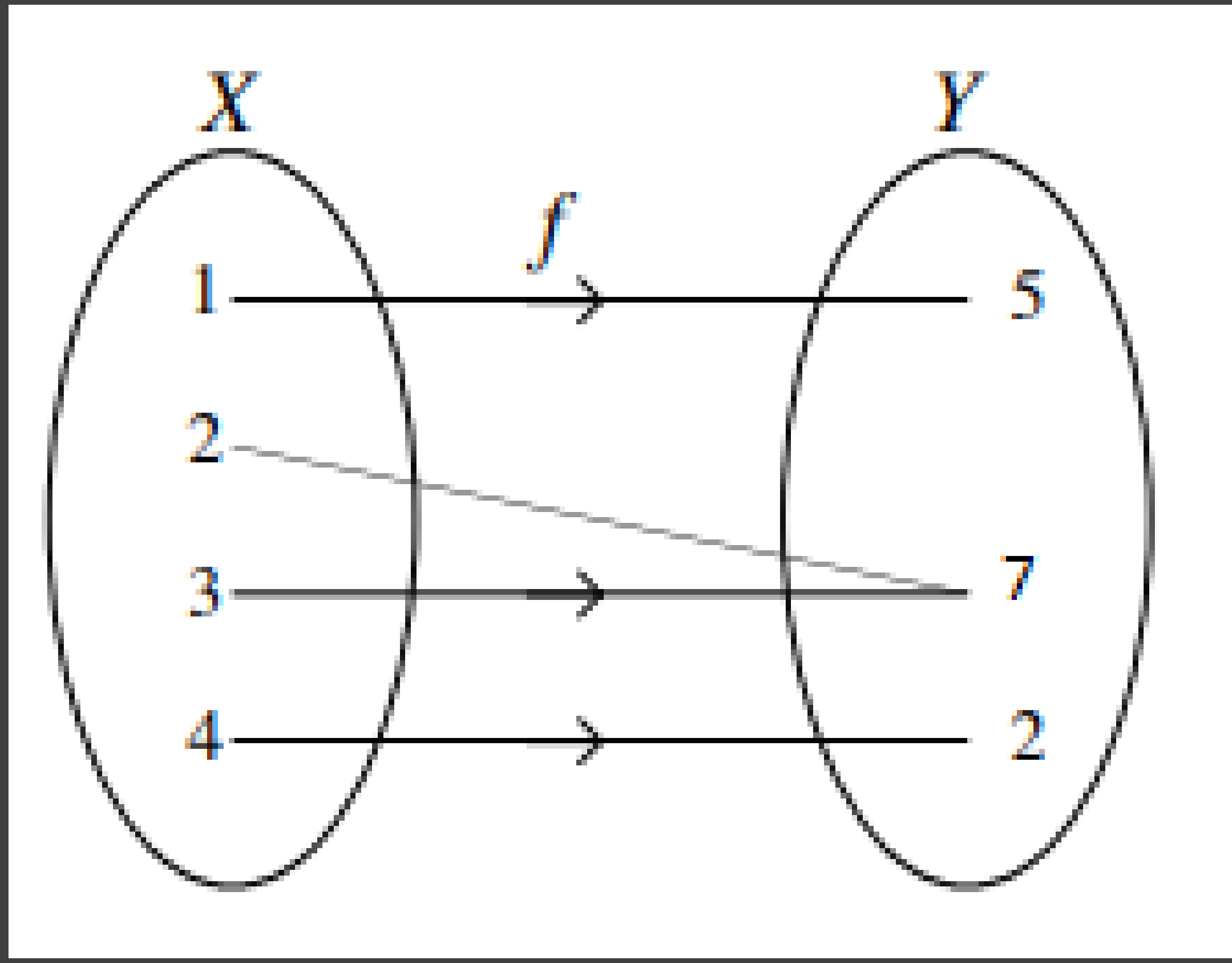


2 left hand side should completely used up

$f(a) = 4$

$f(a) = 5$



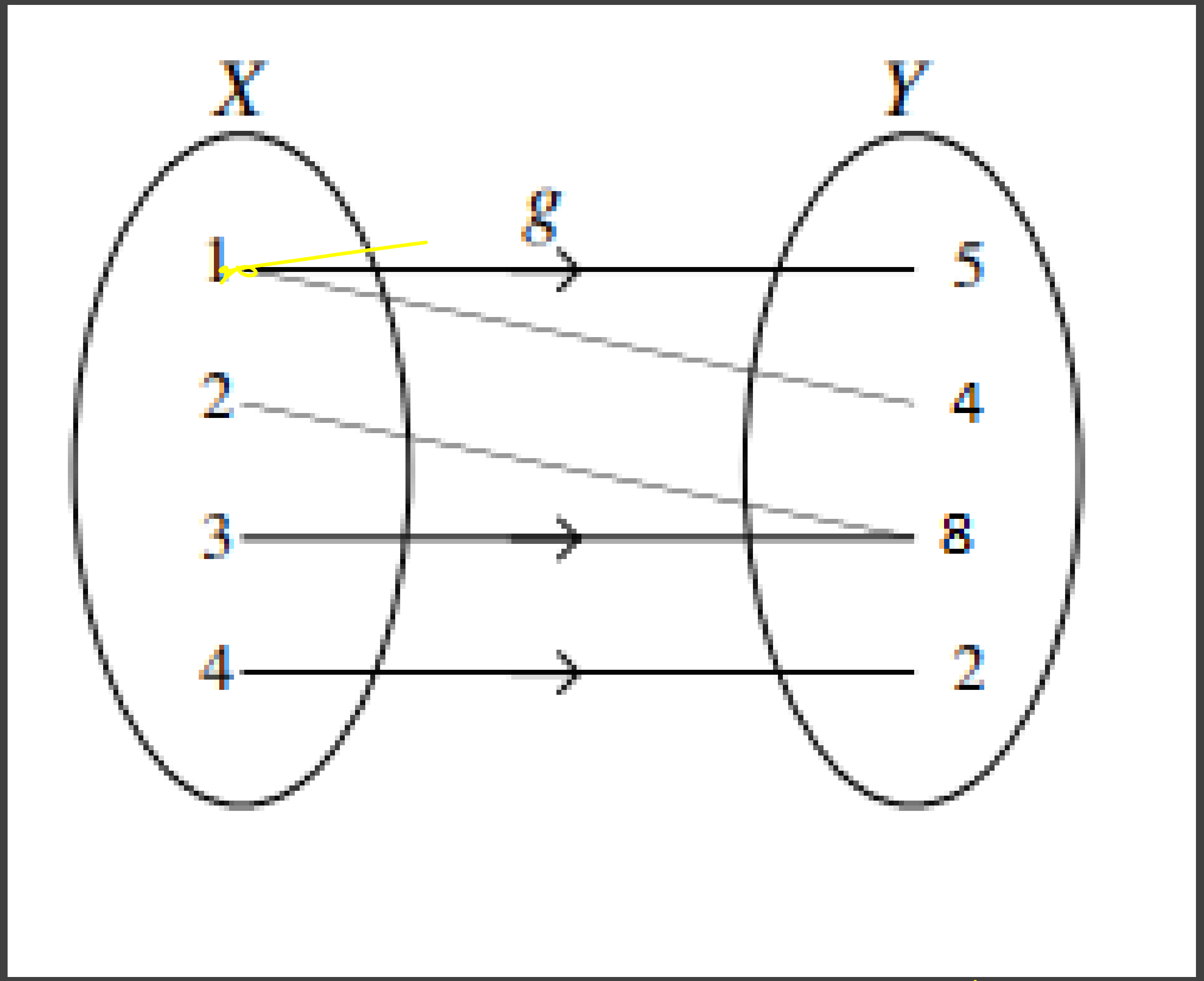


$f \quad X \to Y$

2 ✓

1 ✓

f is function



Not g(1) ≠

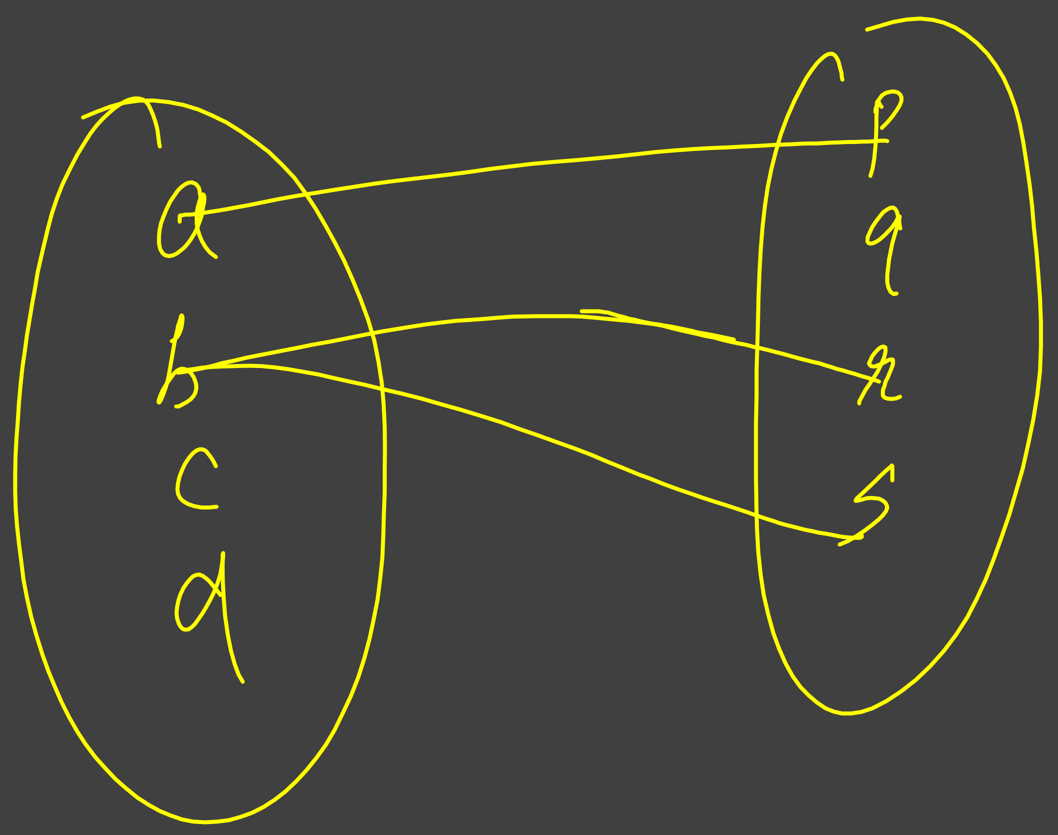
Let A = {a, b, c, d }, B = { p, q, r, s } denote sets.

R : A –> B, R is a function from A to B. Then which of the following relations are not functions ?

(i) { (a, p) (b, q) (c, r) }

(ii) { (a, p) (b, q) (c, s) (d, r) }

(iii) { (a, p) (b, s) (b, r) (c, q) }



(A) (i) and (ii) only

(B) (ii) and (iii) only

(C) (i) and (iii) only

(D) None of these

$f: X \to Y$

Terms related to functions:

- Domain ($\to X$) and co-domain ($\to Y$)
- Range $= \{ f(x) \mid x \in A,\ f: A \to B \}$
- Image and Pre-Image

$f(a) = b$

* b is an image a
* a is preimage of b

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Range =

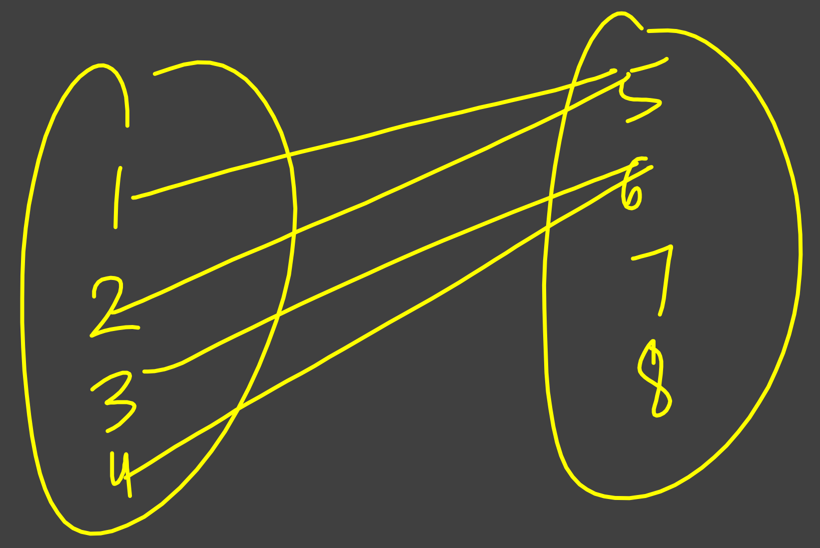
(i) $\mathbb{R}^+$

(ii) $\mathbb{R}$

~~(iii)~~ $[0, \infty)$

(iv) $(0, \infty)$

$f: A \to B$



Domain $= \{1, 2, 3, 4\}$

Codomain $= \{5, 6, 7, 8\}$

Range $= \{5, 6\}$

$f(25) = f(30) = f(15) = f(20)$
$= f(10) = f(5)$

A function $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$, defined on the set of positive integers $\mathbb{N}^+$, satisfies the following properties:

$$f(n) = f(n/2) \qquad \textbf{if } n \text{ is even}$$
$$f(n) = f(n+5) \qquad \textbf{if } n \text{ is odd}$$

$\{a$

Let $R = \{i \mid \exists j : \underline{f(j) = i}\}$ be the set of distinct values that f takes. The maximum possible size of R is ____2____ .

$R = \{a, b\}$

Range

$f(1) = f(6) = f(3) = f(8) = f(4) = f(2) = f(1) = a$

$f(2) = f(1) = a$

$f(3) = f(1) = a$

$f(4) = f(1) = a$

$f(5) = f(10) = f(5) = b$

$f(6) = a$

$f(7) = f(12) = f(6) = a$

$f(8) = a$

$f(9) = f(14) = f(7) = a$

$f(10) = b$

Function Arithmetic:

$f(x) = \sin x$

$g(x) = \cos x$

- The sum of f and g, denoted f + g, is the function defined by the formula
  $$(f + g)(x) = f(x) + g(x)$$
  $(f+g)(x) = \sin x + \cos x$

- The difference of f and g, denoted f − g, is the function defined by the formula
  $$(f - g)(x) = f(x) - g(x)$$

- The product of f and g, denoted fg, is the function defined by the formula
  $$(fg)(x) = f(x)g(x)$$

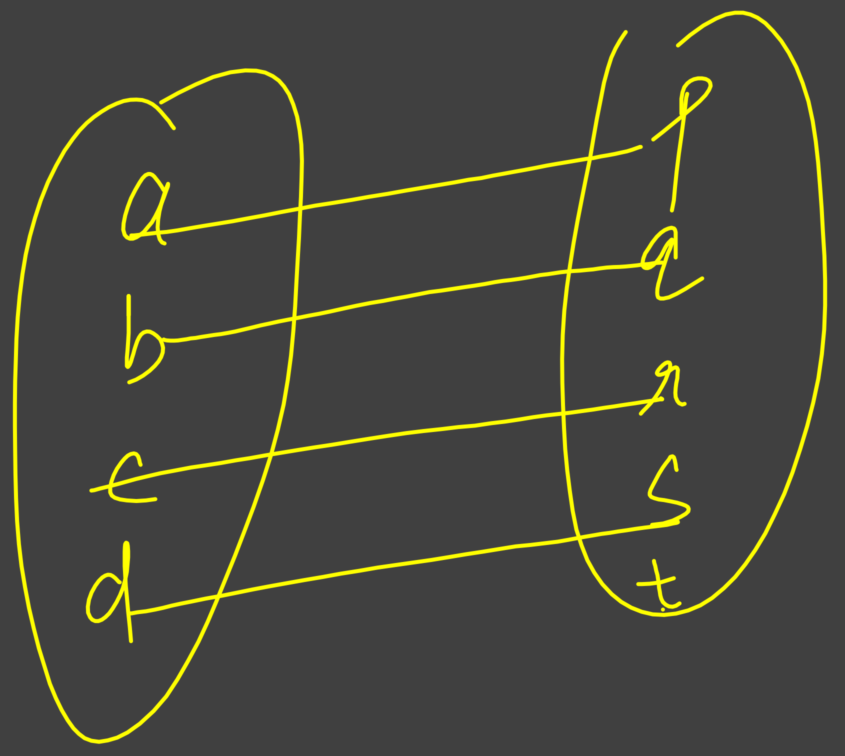
- The quotient of f and g, denoted $\frac{f}{g}$, is the function defined by the formula
  $$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$
  , provided $g(x) \neq 0$.

Types of functions

① Injective functions (one – one)

each element has unique image

one – one

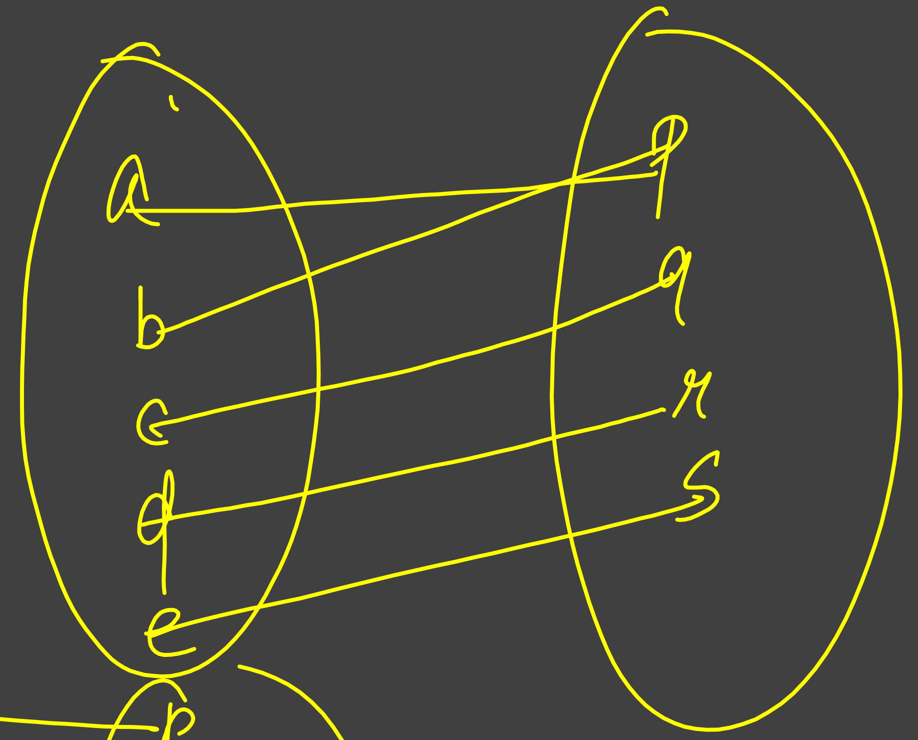


② Surjective (onto)

Codomain = Range

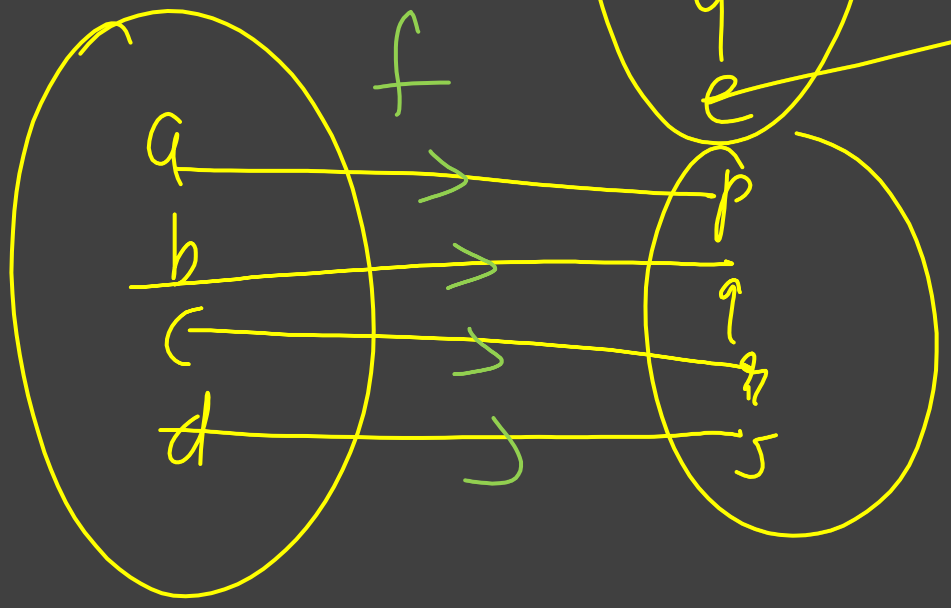
$\rightarrow \boxed{\begin{array}{l} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = x^2 \end{array}} \rightarrow$ not onto

$\left\{ \begin{array}{l} f : \mathbb{R} \rightarrow [0, \infty) \text{ onto} \\ f(x) = x^2 \end{array} \right.$



③ Bijective function

(One-one & onto both)



$$2 \times \frac{1}{2} = 1$$

$$f^{-1}(f(a)) = f^{-1}(p) = a$$

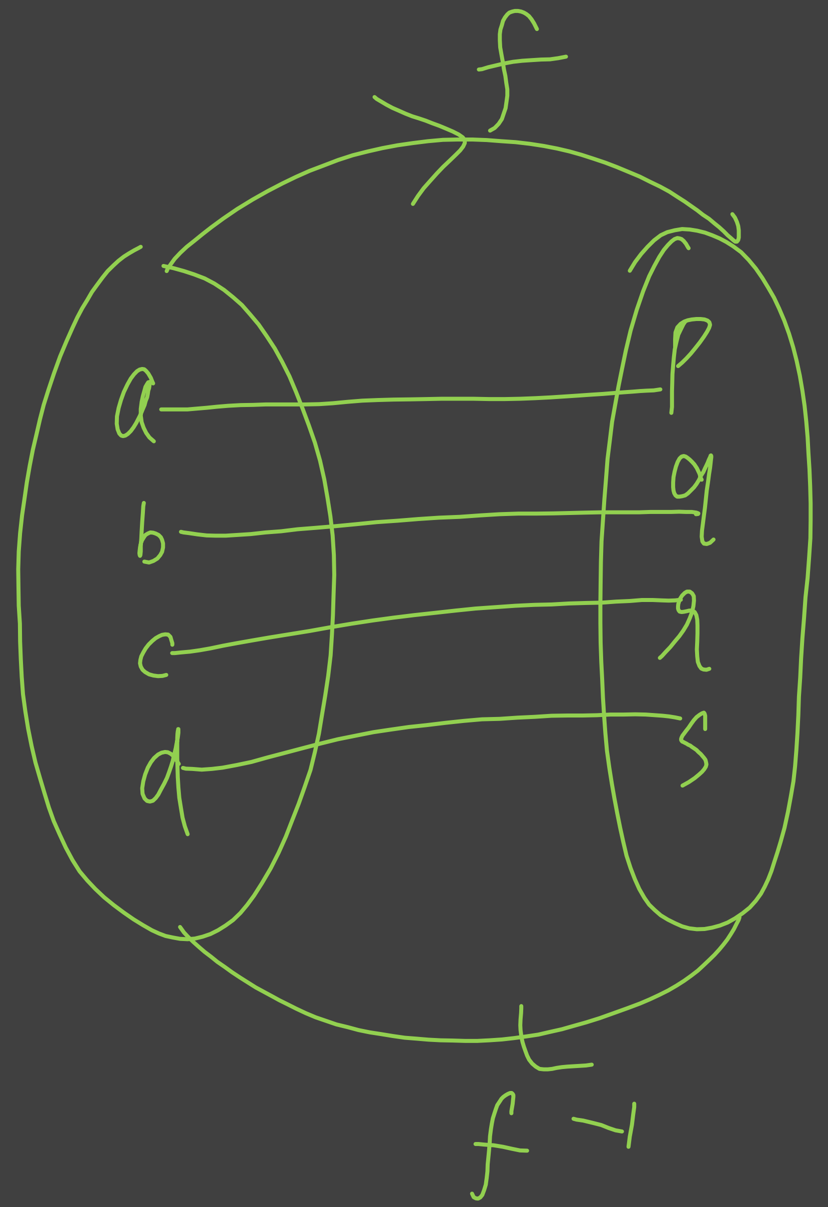
$$f(x)$$

$$f^{-1}(x)$$

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$

$$f^{-1}(p) = a$$



Inverse Functions

$$f(x) = 2x+5$$

$$f^{-1}(x) = y$$

$$x = f(y)$$

$$x = 2y+5$$

$$\frac{x-5}{2} = y$$

$$f^{-1}(x) = \frac{x-5}{2}$$

$$f^{-1}(f(4)) = f^{-1}(13)$$

$$= 4$$

$$f(x) = x^4$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x^4$$

$$= 1^4$$

$$= 1$$

$$f(x) = x^2$$

$$\lim_{x \to 2} f(x) = 2^2$$

$$= 4$$

Let R denote the set of real numbers. Let $f:R\times R\rightarrow R\times R$ be a bijective function defined by $f(x,y)=(x+y,x-y)$. The inverse function of f is given by

a) $f^{-1}(x,y)=\left(\frac{1}{x+y},\frac{1}{x-y}\right)$

b) $f^{-1}(x,y)=(x-y,x+y)$

c) $f^{-1}(x,y)=\left(\frac{x+y}{2},\frac{x-y}{2}\right)$

d) $f^{-1}(x,y)=\left(2\left(x-y\right),2\left(x+y\right)\right)$

$$f^{-1}(x,y)=\left(\frac{x+y}{2},\frac{x-y}{2}\right)$$

$$f(x,y)=(x+y,x-y)$$

$$f^{-1}(x,y)=(a,b)$$

$$(x,y)=f(a,b)$$

$$(x,y)=(a+b,a-b)$$

$$x=a+b$$

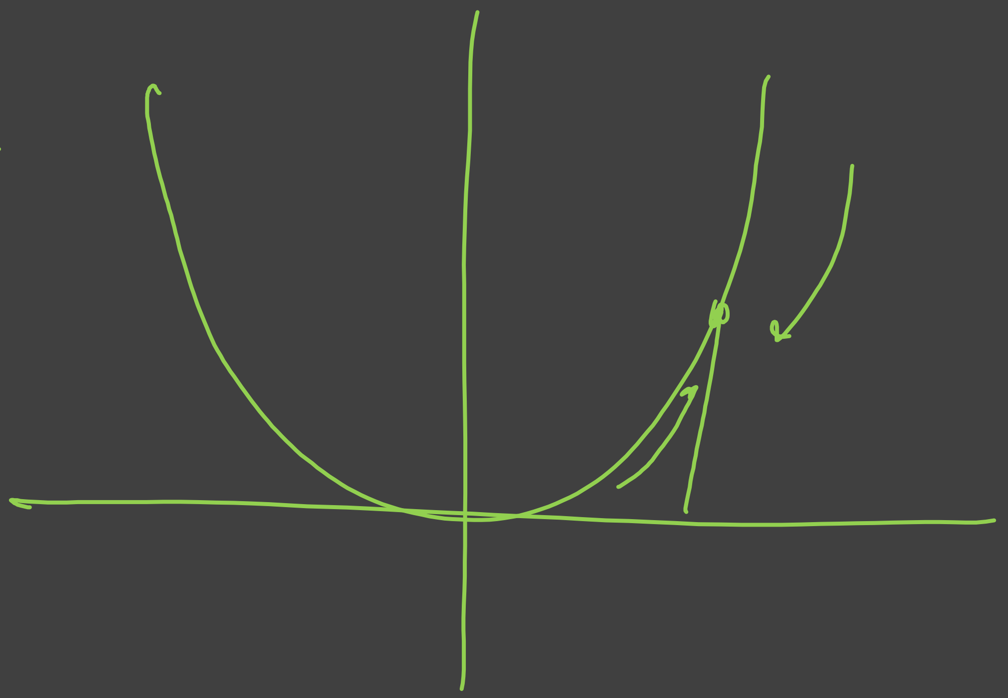
$$y=a-b$$

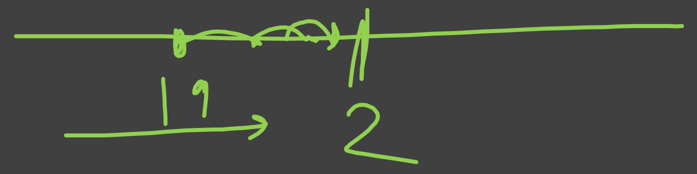
$$\frac{x+y}{2}=a$$

$$b=\frac{x-y}{2}$$

Limit

$f(x) = x^2$

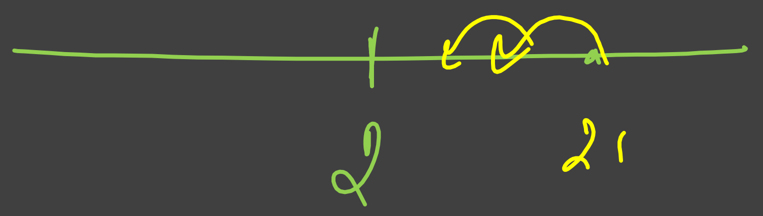




$$f(1.9) = 3.61$$
$$f(1.99) = 3.9601$$
$$f(1.999) = 3.996001$$
$$f(1.9999) = 3.99960001$$

$\downarrow$ 2 $\qquad \downarrow$ 4

11'35



$$f(2.1) = 4.41$$
$$f(2.01) = 4.0401$$
$$f(2.001) = 4.004001$$

$\downarrow$ 2 $\qquad \downarrow$ 4

$$LHL = \lim_{x \to 2^-} f(x) = 4$$

$$RHL = \lim_{x \to 2^+} f(x) = 4$$

$$LHL = RHL$$

$\downarrow$

limit exist

$\& \lim_{x \to 2} f(x) = 4$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x \to 0} f(x)$$

$$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = -0 = 0$$

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$$

$$LHL = RHL$$

limit exists

$$g(x) = \begin{cases} x^2+1, & x \geq 1 \\ 2x+1, & x < 1 \end{cases}$$

$$\lim_{x \to 1} g(x)$$

$LHL = 3$

$RHL = 2$

not exist

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x^2+1 = 2^2+1 = 5$$

$f(x) = x/|x|$ $= \begin{cases} \frac{x}{x} , & x > 0 \\ \frac{x}{-x} , & x < 0 \end{cases}$

$\lim_{x \to 0} f(x)$ $= \begin{cases} 1 , & x > 0 \\ -1 , & x < 0 \end{cases}$

$LHL = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0} -1$

$= -1$

$RHL = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0} 1$

$= 1$

Q $g(x) = \frac{1}{x}$

$\lim_{x \to 0} g(x) = \infty$

limit doesn't exist

$|x| = \begin{cases} -x , & x < 0 \\ x , & x \geq 0 \end{cases}$

$f(x) = 1/x$

L'Hospital Rule –

0^∞ , 1^∞ , $\boxed{\frac{0}{0}, \frac{\infty}{\infty}}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \text{ — } \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{\cos x}{1}$$

$$= 1$$

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \cos x = 1$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$
- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to \infty} (1 + \frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim_{x \to 0} \frac{(a^x - 1)}{x} = \ln a$
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to 0} \frac{\ln (1 + x)}{x} = 1$
- $\lim_{x \to \infty} x^{\frac{1}{x}} = 1$

$$\lim_{x\to 0} \frac{x\cos(x)-\sin(x)}{x^2\sin(x)} \quad \text{——} \quad \frac{0}{0}$$

$$\lim_{x\to 0} \frac{-x\sin x + \cancel{\cos x} - \cancel{\cos x}}{2x\sin x + x^2\cos x}$$

$$\lim_{x\to 0} \frac{-\cancel{x}\sin x}{\cancel{x}(2\sin x + x\cos x)} \quad \text{——} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{-\cos x}{2\cos x + \cos x - x\sin x}$$

$$= \frac{-1}{2+1-0}$$

$$= -1/3$$

$\lim_{x\to 0} \frac{a^{mx}-b^{nx}}{\sin kx}$ — $\left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{ma^{mx}\log a - nb^{nx}\log b}{k\cos kx}$$

$$= \frac{m\log a - n\log b}{k}$$

$$= \frac{\log a^m - \log b^n}{k}$$

$$= \frac{1}{k}\log\left(\frac{a^m}{b^n}\right)$$

$$= \log\left(\frac{a^m}{b^n}\right)^{1/k} = \log\left(\frac{a^{m/k}}{b^{n/k}}\right)$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}-x}{\left(\sqrt{x^2+1}+x\right)}\times\left(\sqrt{x^2+1}+x\right)$$

$$\lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x}$$

$$\frac{1}{\infty+\infty}$$

$$=\frac{1}{\infty}=0$$

$R-\infty$

$$\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \frac{2 \sin x \cos x}{1}$$

$$= 0$$

(a) 0
(b) inf
(c) 1
(d) -1

Answer (a)

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

(a) 1
(b) Limit does not exist
(c) 53/12
(d) 108/7

$$\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$$

$$\lim_{x \to 1} \frac{7x^6 - 10x^4}{3x^2 - 6x}$$

(a) 0

(b) -1

(c) 1

(d) Does not exists

$$= \frac{-3}{-3} = 1$$

Answer(c)

$$\lim_{x \to 0} \frac{a^{mx} - b^{mx}}{\sin(kx)}$$

(A) $\frac{1}{k} \ln \frac{a^m}{b^m}$

(B) $\frac{1}{k} \ln \frac{b^m}{a^m}$

(C) $\frac{2}{k} \ln \frac{a^m}{b^m}$

(D) $\frac{2}{k} \ln \frac{b^m}{a^m}$

Evaluate

$$\lim_{y \to 7} \frac{y^2 - 4y - 21}{3y^2 - 17y - 28}$$

(A) 2/7
(B) 3/5
(C) 3/7
(D) 2/5

Answer: (D)

$$\lim_{x\to 0}\frac{\ln((x^2+1)\cos x)}{x^2} = \_\_\_\_\_\_.$$

$$\lim_{x\to 0}\frac{\ln(x^2+1)+\ln\cos x}{x^2}$$

$$\lim_{x\to 0}\frac{\frac{2x}{x^2+1}+\frac{1}{\cos x}(-\sin x)}{2x}$$

$$\lim_{x\to 0}\frac{\frac{\cancel{2x}}{(x^2+1)\cancel{2x}}-\frac{\sin x}{\cos x\,(2x)}}{1}$$

$$= \frac{1}{1}-\frac{1}{2} = 1/2$$

$\infty \quad 0$

$e^{\infty} = \infty$

$e^{-\infty} = 0$

The value of $\lim_{x \to \infty} (1 + x^2) e^{-x}$ is

(A) 0
(B) 1/2
(C) 1
(D) ∞

$\lim_{x \to \infty} \frac{1 + x^2}{e^x}$

$\lim_{x \to \infty} \frac{2x}{e^x}$

$\lim_{x \to \infty} \frac{2}{e^x}$

$= 0$

Answer: (A)

$\lim_{x\to\infty} x^{1/x}$ *is*

(A) ∞

(B) 0

(C) 1

(D) Not Defined

Answer: (C)

$y = x^{1/x}$

$\log y = \frac{1}{x}\log x$

$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\log x}{x}$

$= \lim_{x\to\infty} \frac{1/x}{1}$

$\lim_{x\to\infty} \log y = 0$

$\lim_{x\to\infty} y = e^0$

$= 1$

$e^{\lim_{x\to 0} \cancel{x} \times \frac{2}{\cancel{x}}} = e^2$

$\lim_{x\to 0} (1+x)^{2/x} = e^2$

1^∞

$\lim_{x\to 0} (1+f(x))^{g(x)}$

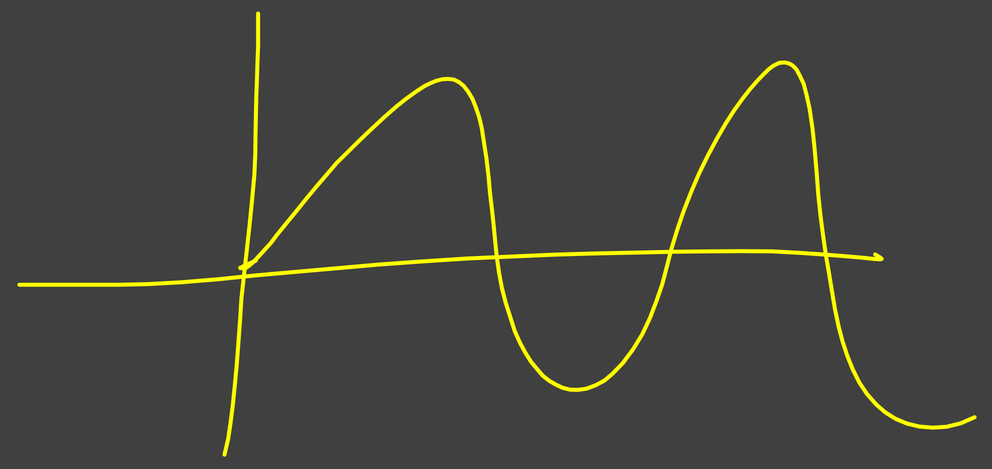
$= e^{\lim_{x\to 0} f(x)\, g(x)}$

$$\lim_{x\to\infty} \frac{x-\sin x}{x+\cos x} \text{ equals}$$

(A) 1
(B) -1
(C) INF
(D) -INF

$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$



$$\lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = \frac{1}{1} = 1$$

Answer: (A)

$$\lim_{x\to\infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2} = \_\_\_\_\_\_.$$

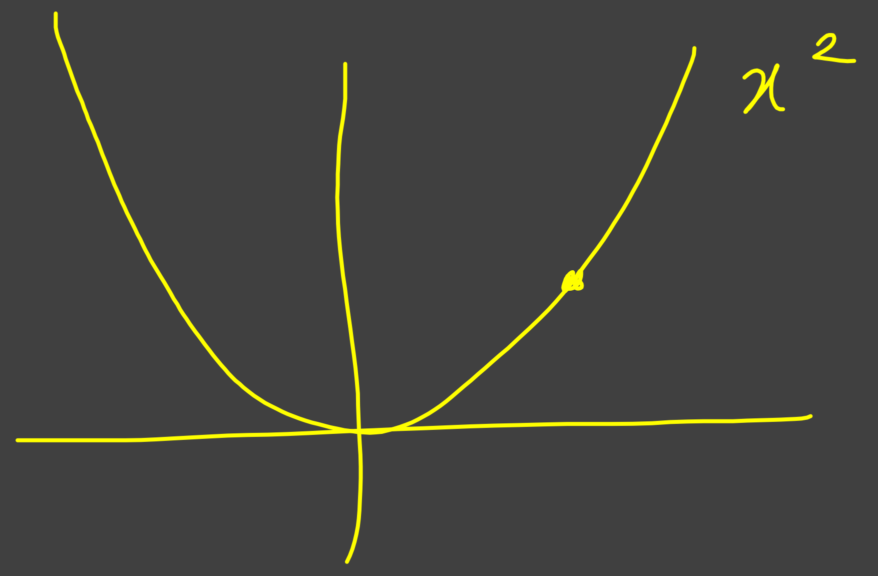
$$\lim_{x\to\infty} \frac{3x^2 + \sin x}{2x + 2\sin x \cos x}$$

$$\lim_{x\to\infty} \frac{3x^2 + \sin x}{2x + \sin 2x}$$

$$\lim_{x\to\infty} \frac{6x + \cos x}{2 + \cos 2x}$$

$$= \infty$$

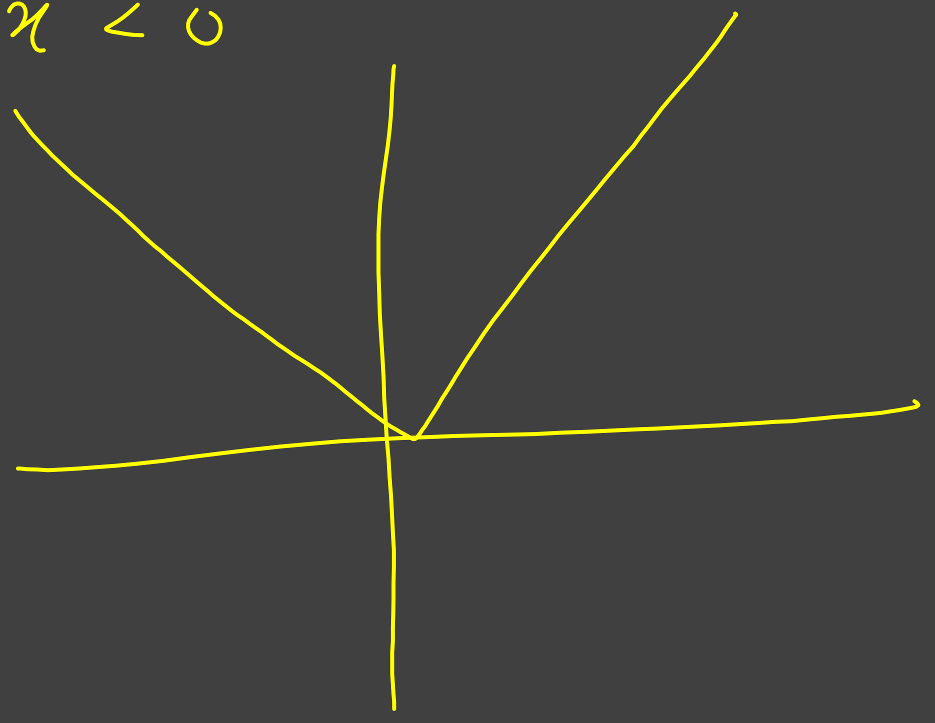
Continuity



$$LHL = RHL = f(a)$$

Eg $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\begin{cases} LHL = 0 \\ RHL = 0 \\ f(0) = 0 \end{cases}$



Q $\frac{|x|}{x}$ at $x = 0$??

No

$LHL = -1$

$RHL = 1$

$f(x) = \begin{cases} 3x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$

$LHL = RHL = 1$

$f(0) = 2$

Example –

$$f(x) = \begin{cases} 0; & x = 0 \\ \frac{1}{2} - x; & 0 < x < \frac{1}{2} \\ \frac{1}{2}; & x = \frac{1}{2} \\ \frac{3}{2} - x; & \frac{1}{2} < x < 1 \\ 1; & x \geq 1 \end{cases}$$

not cont at $0, \frac{1}{2}, 1$

f is $x = 4$??

cont

Let x be a real number.

The **floor function** of x, denoted by $\lfloor x \rfloor$ or floor(x), is defined to be the greatest integer that is less than or equal to x.

The **ceiling function** of x, denoted by $\lceil x \rceil$ or ceil(x), is defined to be the least integer that is greater than or equal to x.

$\lfloor 5.1 \rfloor = 5$

$\lfloor 4.23 \rfloor = 4$

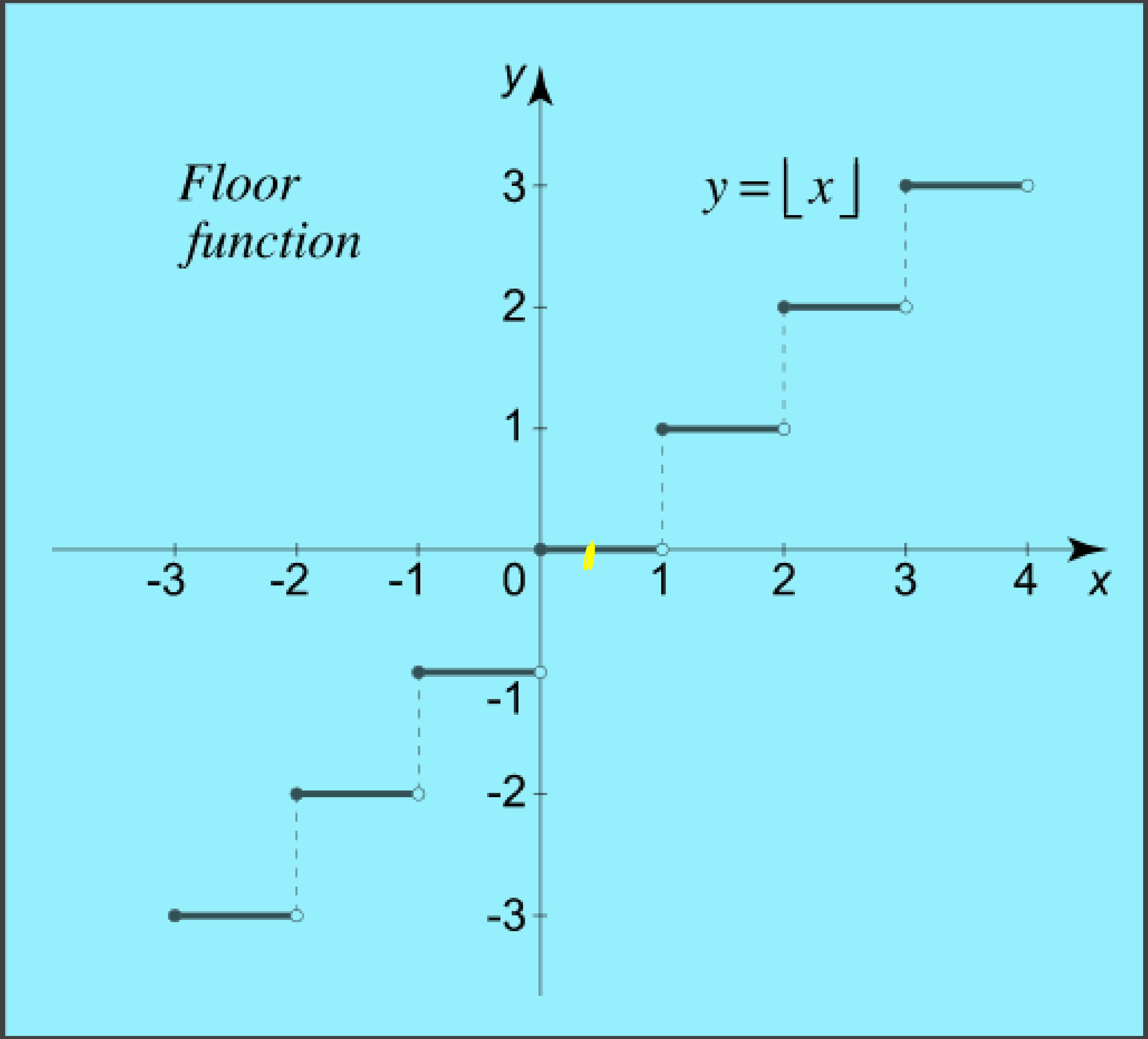
$\lfloor -2.1 \rfloor = -3$

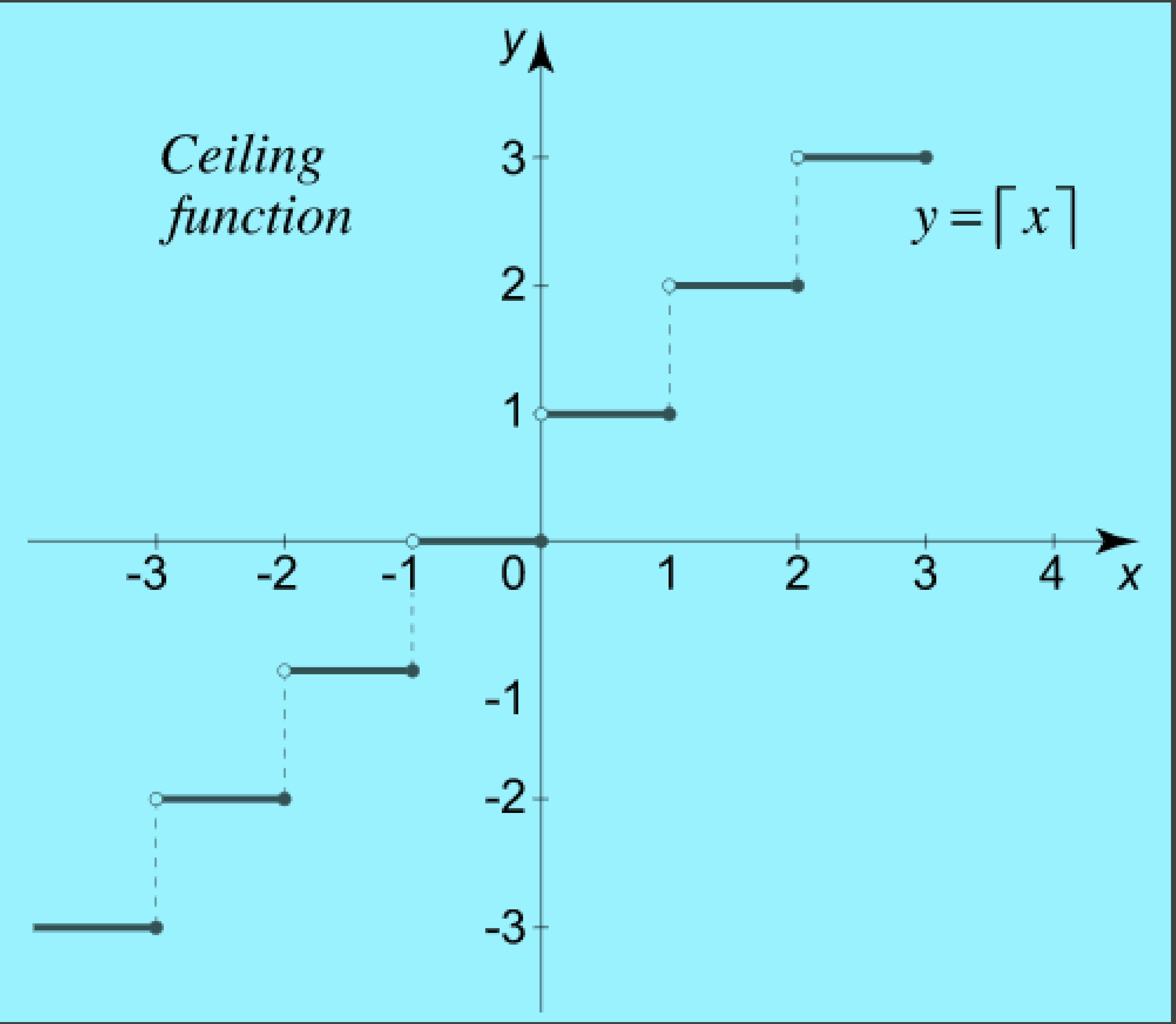
$\lceil 5.1 \rceil = 6$

$\lceil 4.23 \rceil = 5$

$\lceil -2.1 \rceil = -2$

$-3 \quad -2 \quad -1$


Floor function
$y = \lfloor x \rfloor$
y
x
3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3
4



Ceiling function
$y = \lceil x \rceil$
y
x
3
2
1
0
-1
-2
-3
1
2
3
4
-1
-2
-3

Example – For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2), & \text{if } x \leq 0 \\ 4x+1, & \text{otherwise} \end{cases}$$

continuous at x = 0 ?

LHL = RHL = f(0)

$$\lim_{x \to 0} \lambda(x^2 - 2) = \lim_{x \to 0} 4x + 1$$

$$\lambda(-2) = 1$$

$$\boxed{\lambda = -\frac{1}{2}}$$

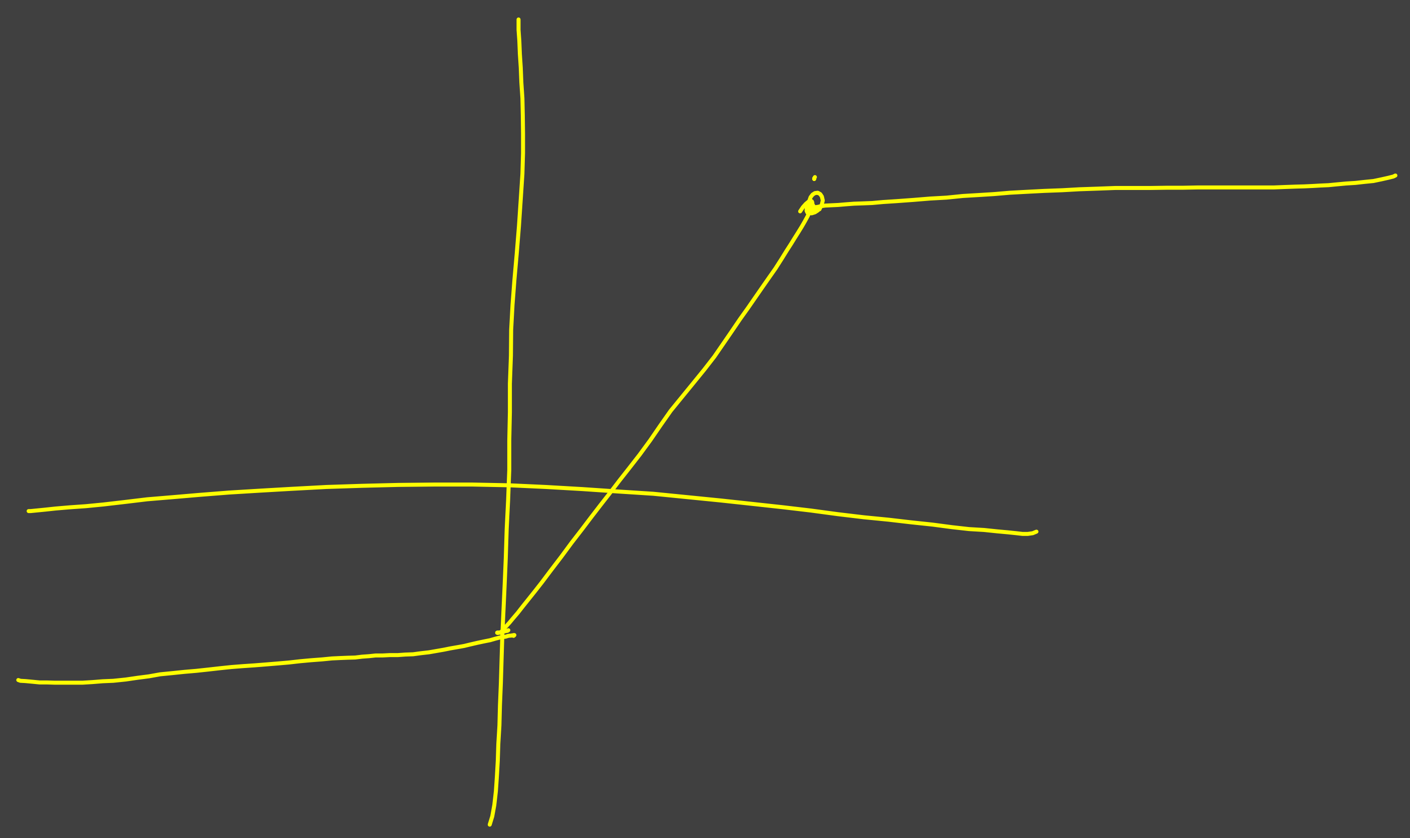
Example – Find all points of discontinuity of the function f(x) defined by –

$$f(x) = |x| - |x - 1|$$

$$f(x) = \begin{cases} -x-(-(x-1)), & x \le 0 \\ x-(-(x-1)), & 0 < x < 1 \\ x-(x-1), & x > 1 \end{cases}$$

$$= \begin{cases} -1, & x \le 0 \\ 2x-1, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$

Cont



Example – $f(x) = \frac{|x|}{x}$

LHL = −1

RHL = 1

Example – $f(x) = \frac{x^2-3x-4}{x^2+3x-4}$ is not continuous at x = ?

$$= \frac{x^2-4x+x-4}{x^2+4x-x-4}$$

$$= \frac{(x-4)(x+1)}{\boxed{(x+4)(x-1)}}$$

$$\Downarrow$$

$$(x+4)(x-1) = 0$$

$$x = -4, 1$$

Example – $f(x) = \frac{x^2-9}{x-3}$ is not continuous at x = ?

$= x+3$ ✓

GATE CS 2013

Which one of the following functions is continuous at x = 3?

(A) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x - 1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(B) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8 - x & \text{if } x \neq 3 \end{cases}$

(C) $f(x) = \begin{cases} x + 3, & \text{if } x \leq 3 \\ x - 4 & \text{if } x > 3 \end{cases}$

(D) $f(x) = \frac{1}{x^3 - 27}, \quad \text{if } x \neq 3$

Example: What should be the value of λ such that the function defined below is continuous at $x = \frac{\pi}{2}$?

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\frac{\pi}{2} - x}, & if \;\; x \neq \dfrac{\pi}{2} \\ 1, & if \;\; x = \dfrac{\pi}{2} \end{cases}$$

(A) 0

(B) $\frac{2}{\pi}$

(C) 1

(D) $\frac{\pi}{2}$

Answer : (C)

$$\lim_{x \to \pi/2} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = 1$$

$$\lim_{x \to \pi/2} \frac{\lambda(-\sin x)}{-1} = 1$$

$$\lambda = 1$$

Differentiability

$$LHD = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

$$RHD = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

$$LHD = RHD$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$LHD = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = -1$$

$$RHD = \lim_{x \to 0^+} \frac{x - 0}{x - 1} = 1$$

$$LHD \neq RHD$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$= \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Example : g(x) = $x^{1/3}$

$x = 0$??

$g'(x) = \frac{1}{3} x^{-2/3}$

$= \infty$

$\Downarrow$

not diff

$f(x) = \begin{cases} 3x^2+1, & x \geq 1 \\ 6x-2, & x < 1 \end{cases}$

$f'(x) = \begin{cases} 6x, & x \geq 1 \\ 6, & x < 1 \end{cases}$

diff

Example: $g(x) = \begin{cases} x+1, & x \leq 1 \\ 3x-1, & x > 1 \end{cases}$

$g'(x) = \begin{cases} 1, & x \leq 1 \\ 3, & x > 1 \end{cases}$

not diff at $x = 1$

$g(x)$ $x = 5$??

What Is the Difference Between Differentiable and Continuous Function?

Diff $\Rightarrow$ cont

cont $\not\Rightarrow$ diff

$|x|$

Let $f:\mathbb{R} \to \mathbb{R}$ be the function $f(x) = \dfrac{1}{1+e^{-x}}$.

The value of the derivative of f at x where $f(x) = 0.4$ is ______ (rounded off to **two** decimal places).

$$f'(x) = \frac{-e^{-x}}{(1+e^{-x})^2}$$

Q.37	Let $f:\mathbb{R}\to\mathbb{R}$ be a function. **Note:** $\mathbb{R}$ denotes the set of real numbers. $$f(x)=\begin{cases}-x, & \text{if } x<-2\\ ax^2+bx+c, & \text{if } x\in[-2,2]\\ x, & \text{if } x>2\end{cases}$$ Which **ONE** of the following choices gives the values of a,b,c that make the function f continuous and differentiable?
(A)	$a=\frac{1}{4}, b=0, c=1$
(B)	$a=\frac{1}{2}, b=0, c=0$
(C)	$a=0, b=0, c=0$
(D)	$a=1, b=1, c=-4$

$+2 = 4a-2b+c$

$2 = 4a+2b+c$

$$f'(x)=\begin{cases}-1, & x<-2\\ 2ax+b, & -2\le x\le 2\\ 1, & x>2\end{cases}$$

$x=-2$

$-1 = -4a+b$

$x=2$

$1 = 4a+b$

$$\boxed{\begin{aligned} b&=0\\ a&=1/4\\ c&=1\end{aligned}}$$

Let f be a function defined by

$$f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ ax^2 + bx + c & \text{for } 1 < x \le 2 \\ x + d & \text{for } x > 2 \end{cases}$$

$1 = a + b + c$

$2 + d = 4a + 2b + c$

Find the values for the constants a, b, c and d so that f is continuous and differentiable everywhere on the real line.

$$f'(x) = \begin{cases} 2x & , x \le 1 \\ 2ax + b & , 1 < x \le 2 \\ 1 & , x > 2 \end{cases}$$

$a = -0.5, b = 3, c = -1.5, d = 0.5$

$2 = 2a + b$

$1 = 4a + b$

$a = -1/2 \quad b = 3$

$c = -3/2$

$1 - 3 + \frac{1}{2}$

$\frac{3}{2} - 3$

$-3/$

$f(x)$ $x=a$

Taylor Series Theorem

$\boxed{1.50}$

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots$$

$$f(x) = e^x \quad , \; x = 0$$

$$f(0) = e^0 = 1$$

Maclaurin Series

$$f'(0) = 1 \quad , \; f''(0) = 1$$

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Find the Maclaurin series for f(x) = sin x.

$f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - -$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - - - -$$

Write the Taylor series for f(x) = 1/x centered at a=1

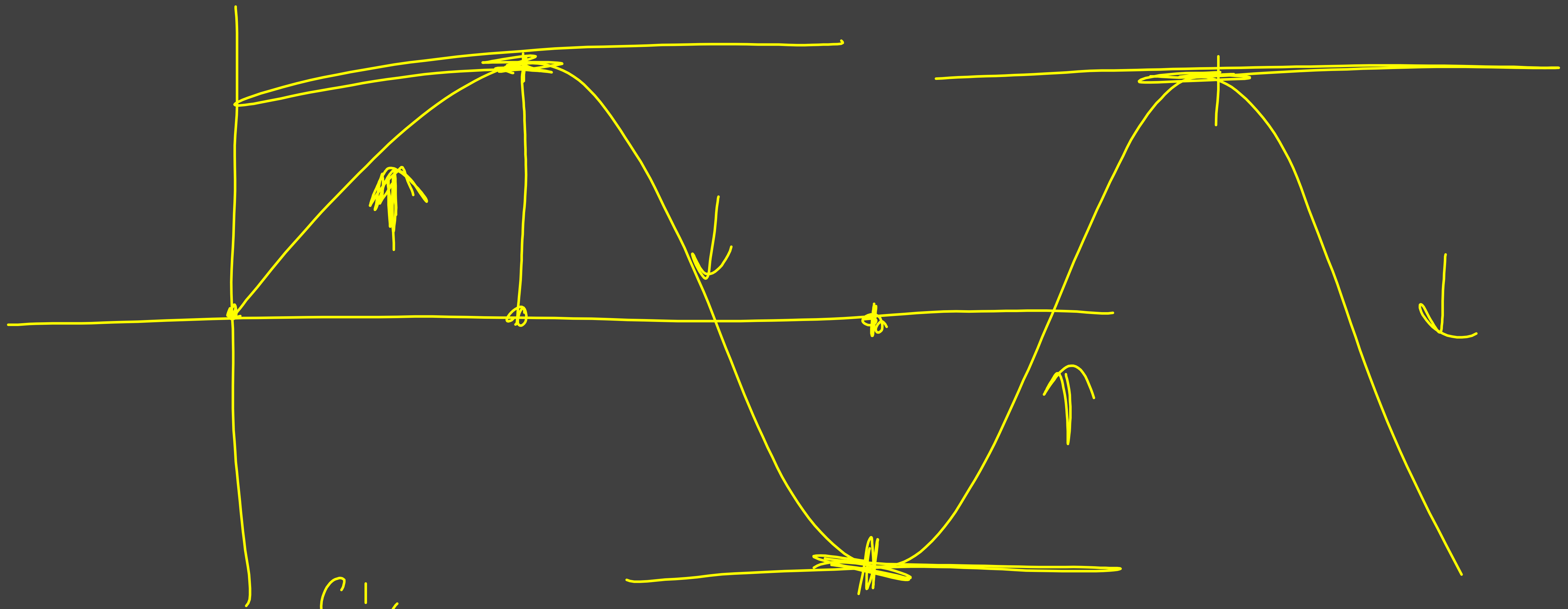
$$f'(x) = -\frac{1}{x^2} \qquad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \qquad f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4} \qquad f'''(1) = -6$$

$$f(x) = 1 - 1(x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots$$

Increasing and Decreasing function



$f'(x) = 0$
$\hookrightarrow$ critical points

$f'(x) > 0 \rightarrow \uparrow$

$f'(x) < 0 \rightarrow \downarrow$

$$f(x) = 2x^3 + 3x^2 - 36x$$

$$f'(x) = 6x^2 + 6x - 36 = 0$$

$$f'(x) = 6(x^2 + x - 6)$$

$$(x^2 + x - 6) = 0$$

$$(x+3)(x-2) = 0$$

$$x = -3, 2$$

$(-\infty, -3)$ increasing

$(-3, 2)$ decreasing

$(2, \infty)$ Increasing

$$f(x) = x^3 + x^2 - x + 2$$

$$f'(x) = 3x^2 + 2x - 1 = 0$$

$$3x^2 + 3x - x - 1 = 0$$

$$(3x-1)(x+1) = 0$$

$$x = -1, 1/3$$

$(-\infty, -1)$ increasing

$(-1, 1/3)$ decreasing

$(1/3, \infty)$ increasing

$f'(x) > 0 \uparrow$

$f'(x) < 0 \downarrow$

Consider the functions

I. e^{-x}

II. $x^2 - \sin x$

III. $\sqrt{x^3 + 1}$

Which of the above functions is/are increasing everywhere in [0, 1] ?

(A) III only

(B) II only

(C) II and III only

(D) I and III only

II $f'(x) = 2x - \cos x$

$\downarrow$ $[0, 1]$

$f'(0) = 0 - 1 < 0$

$f'(1) = 2 - \cos 1 > 0$

I $f'(x) = -e^{-x} < 0$

III $f'(x) = \dfrac{3x^2}{2\sqrt{x^3+1}} > 0$

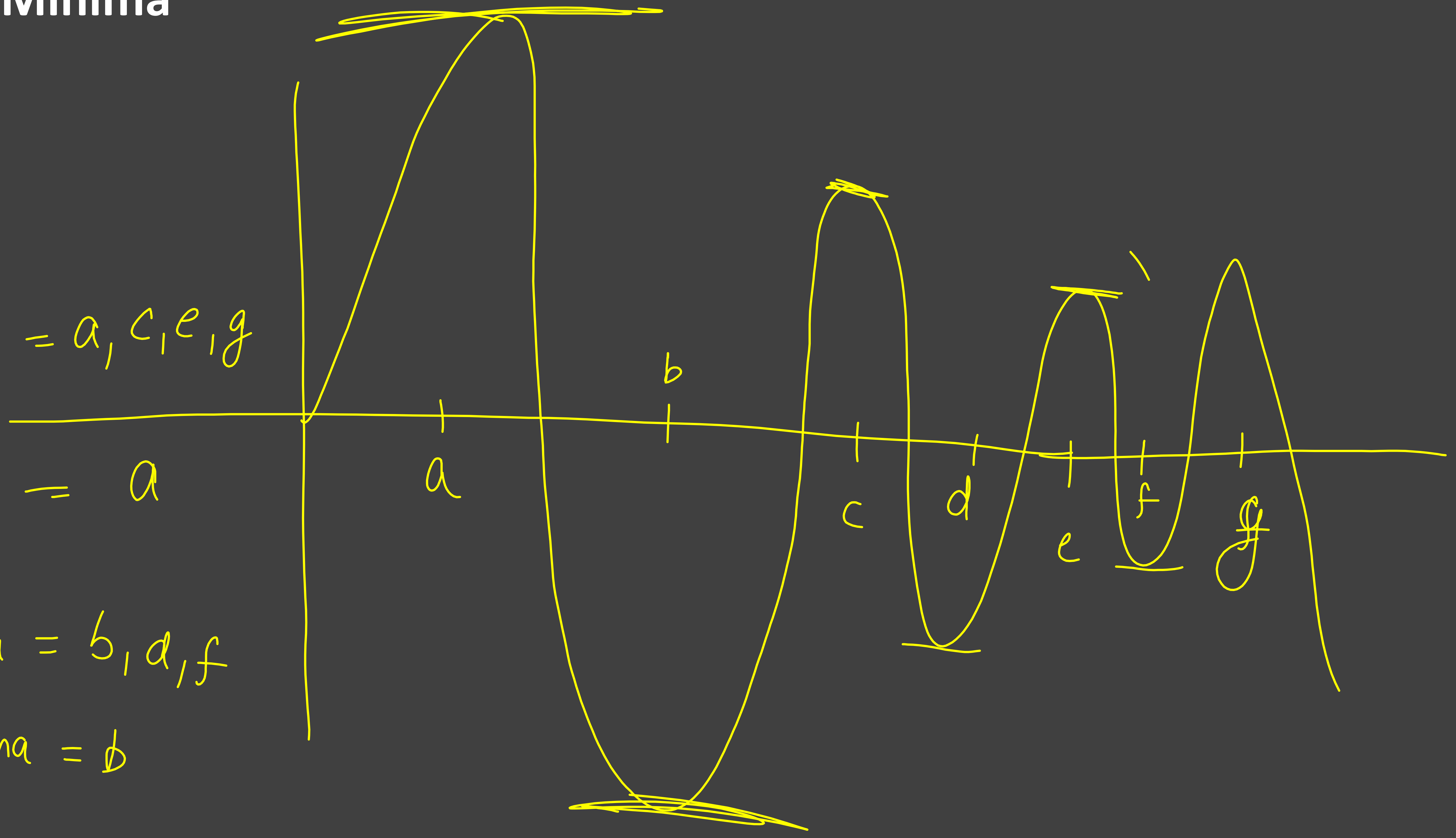
Maxima and Minima

Local Maxima = a, c, e, g

Global Maxima = a

Local Minima = b, d, f

Global Minima = b



Example : Find the stationary point of the function y = $x^2 - 2x - 3$ and hence determine the nature of this point.

$f'(x) = 2x - 2 = 0$

$x = 1$

2nd derivative test

$f''(x) < 0 \longrightarrow$ maxima

$f''(x) > 0 \longrightarrow$ minima

$f''(x) = 0 \longrightarrow$ saddle point

$f''(x) = 2$

$f''(1) = 2 > 0$

minima

Which of the following points is a local maximum of the function y = $2x^3 - 15x^2 - 36x + 6$?
(a) (−3, 15), (b) (−6, 25), (c) (−2, 25), (d) (−1, 25).

(x, y)

$f''(x) = 6x^2 - 30x - 36 = 0$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6, -1$

$x = -1$

$y = 2(-1)^3 - 15(-1)^2 - 36(-1) + 6$

$= 25$

$f''(x) = 6(2x-5)$

$f''(-1) = -42 < 0 \rightarrow$ maxima

$f''(\ ) = 42 > 0 \longrightarrow$ minima

Which of the following statements is true for the function $y = x + 1 + \frac{1}{x}$?

(a) (1, 3) is a local maximum
(b) (1, −3) is a local minimum
(c) (−1, 1) is a local minimum
(d) (−1, −1) is a local maximum

$f'(x) = 1 + 0 - \frac{1}{x^2} = 0$

$x = \pm 1$

$f''(x) = \frac{+2}{x^3}$

$f''(1) = 2 > 0 \rightarrow$ minima

$f''(-1) = -2 < 0 \longrightarrow$ maxima

(1, 3) is local minima
(−1, −1) is local max

Which of the following statements is true for the function y = $\frac{x^2}{2} - \cos x$?

(a) $\left(\frac{\pi}{2}, \frac{\pi^2}{8}\right)$ is a local minimum

(b) $\left(\frac{\pi}{2}, \frac{\pi^2}{8}\right)$ is a local maximum

(c) (0, −1) is a local minimum

(d) (0, −1) is a local maximum.

Consider the function f(x) = sin(x) in the interval $[\frac{\pi}{4}, \frac{7\pi}{4}]$. The number and location(s) of the local minima of this function are

(A) One, at $\frac{\pi}{2}$

(B) One, at $\frac{3\pi}{2}$

(C) Two, at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

(D) Two, at $\frac{\pi}{4}$ and $\frac{3\pi}{2}$

$f'(x) = \cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\frac{\pi}{4}, \frac{7\pi}{4}$

$f''(x) = -\sin x$

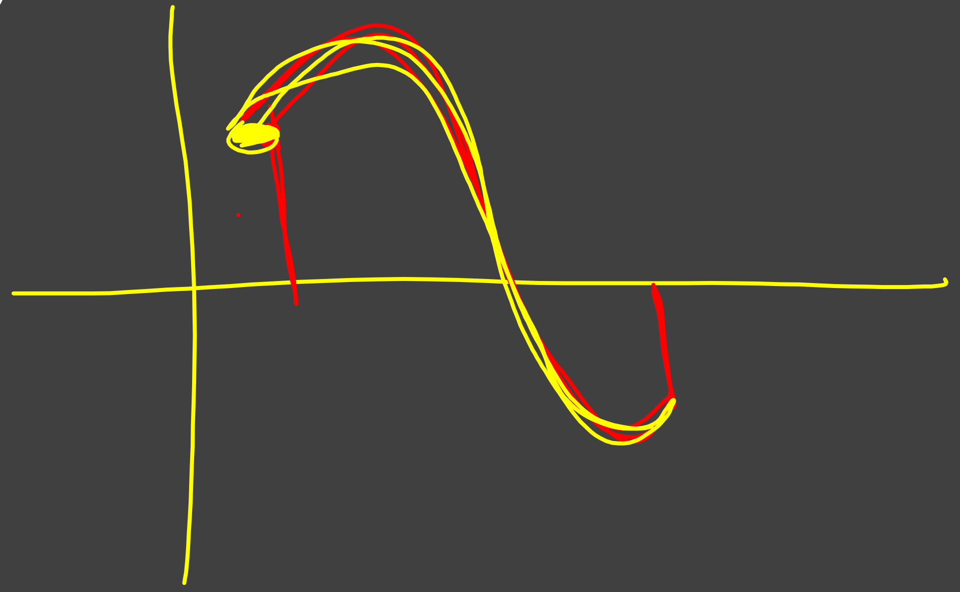
$f''(\pi/2) = -\sin \pi/2 = -1 < 0$

maxima

$f''(3\pi/2) = 1 > 0$

minima

Answer: (D)



A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct extrema for the curve $3x^4 – 16x^3 + 24x^2 + 37$

(A) 0

(B) 1

(C) 2

(D) 3

Answer: (B)

If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then

(A) $\alpha = -6$, $\beta = -1/2$

(B) $\alpha = 2$, $\beta = -1/2$

(C) $\alpha = 2$, $\beta = 1/2$

(D) $\alpha = -6$, $\beta = 1/2$

What is the maximum value of the function $f(x) = 2x^2-2x+6$ in the interval [0,2] ?

(A) 6

(B) 10

(C) 12

(D) 5.5

Answer: (B)

Consider the function $f:\mathbb{R}\to\mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers. $$f(x)=\frac{x^4}{4}-\frac{2x^3}{3}-\frac{3x^2}{2}+1$$ Which of the following statements is/are **TRUE**?
$x = 0$ is a local maximum of f
$x = 3$ is a local minimum of f
$x = -1$ is a local maximum of f
$x = 0$ is a local minimum of f

Q.15	For any twice differentiable function $f : \mathbb{R} \to \mathbb{R}$, if at some $x^* \in \mathbb{R}$, $f'(x^*) = 0$ and $f''(x^*) > 0$, then the function f necessarily has a ______ at $x = x^*$. **Note**: $\mathbb{R}$ denotes the set of real numbers.
(A)	local minimum
(B)	global minimum
(C)	local maximum
(D)	global maximum

Thank you