# Probability & Statistics in one go

Four fair coins are tossed simultaneously. The probability that at least one head and one tail turn up is :

(A) 1/16

(B) 1/8

(C) 7/8

(D) 15/16

HHHH, TTTT

$$1 - \frac{2}{16} = \frac{14}{16} = \frac{7}{8}$$

**Answer: (C)**

The probability that a number selected at random between 100 and 999 (both inclusive) will not contain the digit 7 is:

A. $\frac{16}{25}$ B. $\left(\frac{9}{10}\right)^3$ C. $\frac{27}{75}$ D. $\frac{18}{25}$

$\underline{\ }\ \underline{\ }\ \underline{\ }$

$\downarrow\ \downarrow\ \downarrow$

2 $\quad 8 \quad 9 \quad 9$

$$\frac{\cancel{8}^{2} \times 9 \times \cancel{9}}{\cancel{9} \times \cancel{10}_{5} \times \cancel{10}_{5}}$$

$$= \frac{18}{25}$$

A bag contains $10$ blue marbles, $20$ green marbles and $30$ red marbles. A marble is drawn from the bag, its colour recorded and it is put back in the bag. This process is repeated $3$ times. The probability that no two of the marbles drawn have the same colour is

A. $\left(\frac{1}{36}\right)$

B. $\left(\frac{1}{6}\right)$

C. $\left(\frac{1}{4}\right)$

D. $\left(\frac{1}{3}\right)$

BGR $\times$ 3!

$$\frac{1\cancel{0}}{6\cancel{0}} \times \frac{\cancel{20}}{\cancel{60}_{\cancel{3}}} \times \frac{\cancel{30}}{\cancel{60}_{\cancel{2}}} \times \cancel{6}\cancel{3}$$

**Example:** 500 students are taking one or more courses out of chemistry, physics and mathematics. Registration records indicate course enrolment as follows: chemistry (329), physics (186), mathematics (295), chemistry and physics (83), chemistry and mathematics (217), and physics and mathematics (63), How many students are taking all 3 subjects?

$|C \cup P \cup M| = 500$

$|C| = 329$, $|P| = 186$, $|M| = 295$

$|C \cap P| = 83$, $|C \cap M| = 217$, $|P \cap M| = 63$

a. 37
b. 43
c. 47
d. 53

$|C \cap P \cap M| = ?$

Answer (d)

$|C \cup P \cup M| = |C| + |P| + |M| - |C \cap P| - |P \cap M| - |C \cap M| + |C \cap P \cap M|$

$500 = 329 + 186 + 295 - 83 - 217 - 63 + |C \cap P \cap M|$

$|C \cap P \cap M| = 53$

# Different Types of Events

- Mutually Exclusive Events
- Exhaustive Events
- Equally Likely Events
- Independent Events
- Dependent Events

(1) A & B are m e

$$A \cap B = \phi$$

(2) $A \cup B = S$

(3) $P(A) = P(B) \rightarrow$ A & B are equally likely

(4) A & B are ind events

not ind ← { A → getting an odd no ; B → getting an even no } ??

$$P(A \cap B) = P(A)\, P(B)$$

$$\boxed{\frac{1}{36} = \frac{6}{36} \times \frac{6}{36}}$$

Roll two dice simultaneously

ind events but not mutually exc ← {
A getting 2 on first dice $= \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$
B getting 3 on 2nd dice $= \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$
}

GATE DA 2024

Three fair coins are tossed independently. T is the event that two or more tosses result in heads. S is the event that two or more tosses result in tails. What is the probability of the event $T \cap S$ ?

(a) 0

(b) 0.5

(c) 0.25

(d) 1

$T = \{(TT$

$T \cap S = \phi$

**Answer: (A)**

# Conditional Probability

$\{HH, HT, TH, TT\}$

$\to \{HT, TH, TT\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)\,P(B)$$

Independence

$P(A|B)$ $P(A)$

P and Q are considering to apply for job. The probability that p applies for job is 1/4. The probability that P applies for job given that Q applies for the job 1/2 and The probability that Q applies for job given that P applies for the job 1/3.The probability that P does not apply for job given that Q does not apply for the job .

**(A)** 4/5
**(B)** 5/6
**(C)** 7/8
**(D)** 11/12

**Answer: (A)**

$P(P) = \frac{1}{4}$

$P(P|Q) = 1/2 \Rightarrow \frac{P(P\cap Q)}{P(Q)} = 1/2$

$P(Q|P) = 1/3$

$\Rightarrow \frac{P(Q\cap P)}{P(P)} = 1/3$

$P(Q\cap P) = 1/12$

$\frac{1/12}{P(Q)} = 1/2$

$P(Q) = \frac{1}{6}$

$P(P'/Q') = \frac{P(P'\cap Q')}{P(Q')} = \frac{1 - P(P\cup Q)}{1 - \frac{1}{6}}$

$= \frac{1 - [P(P) + P(Q) - P(P\cap Q)]}{5/6}$

$= \frac{1 - [\frac{1}{4} + \frac{1}{6} - \frac{1}{12}]}{5/6}$

$= \frac{\frac{12 - 3 - 2 + 1}{\cancel{12}\ 2}}{5/6} = \frac{4}{5}$

# Law of total probability & Bayes's formula

$$A_i \cap A_j = \phi$$

$$\cup A_i = S$$


$$P(B) = P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2) + \cdots + P(B|A_n)\,P(A_n)$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

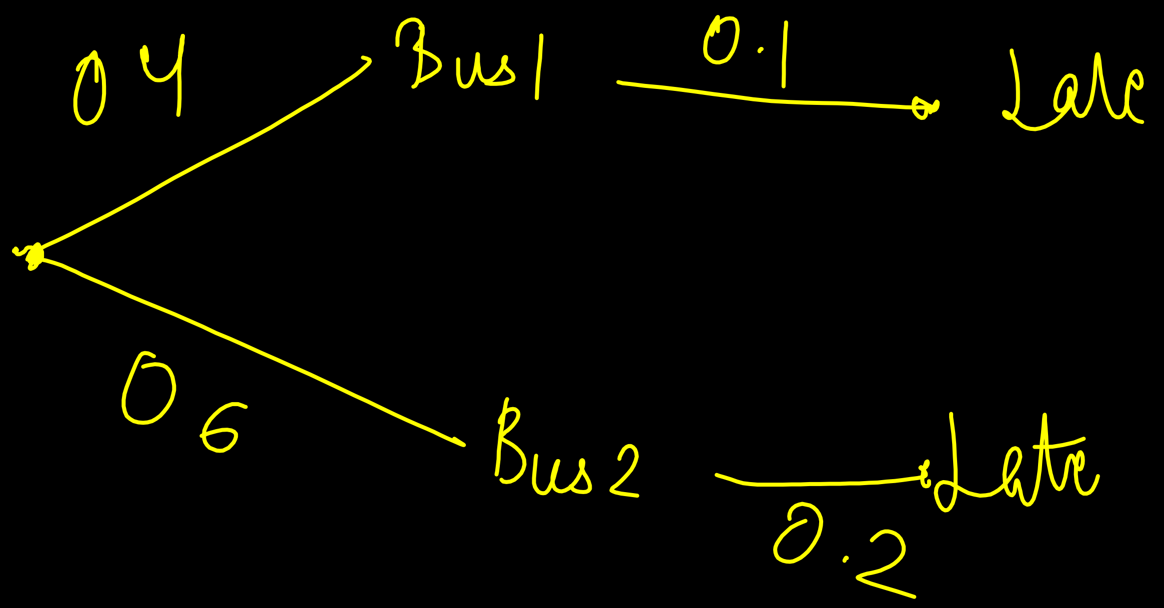
Suppose that if I take bus 1, then I am late with probability 0.1. If I take bus 2, I am late with probability 0.2. The probability that I take bus 1 is 0.4, and the probability that I take bus 2 is 0.6. What is the probability that I am late?

Answer: 0.~~14~~

$$P(\text{Late}) = P(\text{Late}|\text{Bus 1})\, P(\text{Bus 1}) + P(\text{Late}|\text{Bus 2})\, P(\text{Bus 2})$$

$$= 0.1 \times 0.4 + 0.2 \times 0.6$$

$$= 0.16$$




$$= 0.4 \times 0.1 + 0.6 \times 0.2$$

GATE DA Sample paper 2024

A class contains 60% students who are incapable of changing their opinions about anything, and 40% of students are changing their minds at random, with probability 0.3, between subsequent votes on the same issue. Then, the probability of a student randomly chosen voted twice in the same way is ________.

$P(\text{Student voted twice in a same way})$

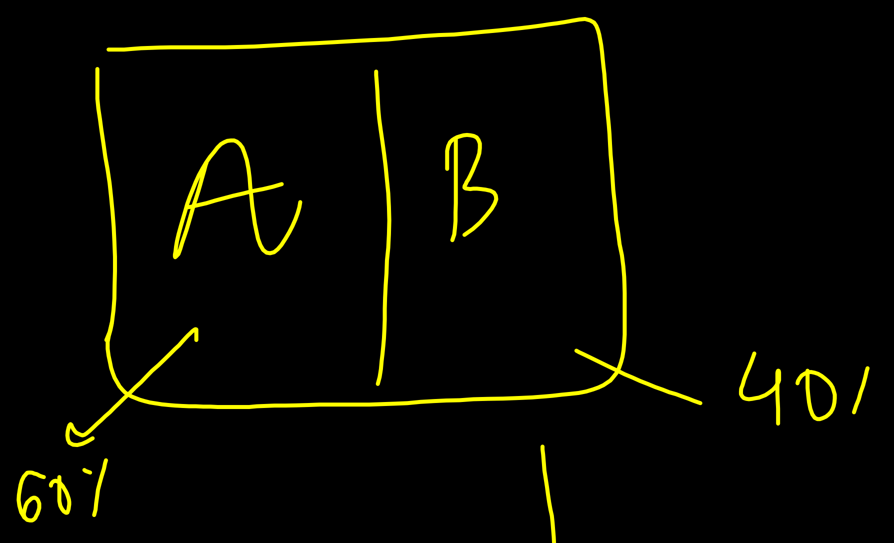
$= P(\text{Not change})$

$= P(A)\, P(\text{Not change} \mid A) + P(B) \cdot P(\text{not change} \mid B)$

$= 0.6 \times 1 + 0.4 \times 0.7$

$= 0.6 + 0.28$

$= 0.88$




$P(\text{change} \mid B) = 0.3$

HW

GATE DA Sample paper 2024

Let {O1, O2, O3, O4} represent the outcome of a random experiment, with P({O1})=P({O2})=P({O3})=P({O4}). Consider the following events: P={O1,O2}, Q={O2,O3}, R={O3,O4},S={O1,O2,O3}. Then, which of the following statements is true?

(A) P and Q are independent
(B) P and Q are not independent
(C) R and S are independent
(D) Q and S are not independent

HW

Consider two events $T$ and $S$. Let $\bar{T}$ denote the complement of the event $T$. The probability associated with different events are given as follows:

$$P(\bar{T}) = 0.6, \qquad P(S|T) = 0.3, \qquad P(S|\bar{T}) = 0.6$$

Then, $P(T|S)$ is ______ (rounded off to **two** decimal places).

# Random Variable

$X : S \to \mathbb{R}$

$S = \{HH, HT, TH, TT\}$

$X$ : no of heads

$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$

$X = 0, 1, 2$

$P(X=0) = \frac{1}{4}$

$P(X=1) = \frac{2}{4}$

$P(X=2) = \frac{1}{4}$

$$P(X=x) = \begin{cases} 1/4 & x = 0, 2 \\ 1/2 & x = 1 \end{cases}$$

$$\underbrace{\begin{array}{l} (i)\ 0 \le p_i \le 1 \\ (ii)\ \sum p_i = 1 \end{array}}_{\text{Discrete}}$$

$f(x) \to$ prob dist

$\downarrow$

Continuous RV

$(i)\ 0 < f < 1$

$(ii)\ \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$P(a < x < b) = \int_a^b f(x)\,dx$$

# Discrete Random Variable:

# Continuous Random Variable:

**Example:- Compute the value of P (1 < X < 2).**

Such that

$$f(x) = \begin{cases} kx^3, & 0 \le x \le 3 \\ 0, & otherwise \end{cases}$$

Where, f(x) is a density function

find cdf

$f(x) = \begin{cases} \frac{4}{81}x^3, & 0 < x < 3 \\ 0, & otherwise \end{cases}$

cdf

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{0}^{3} kx^3\,dx = 1$

$\boxed{k = \frac{4}{81}}$

$F(x) = \begin{cases} 0 & x < 0 \\ x^4/81 & 0 < x < 3 \\ 1 & x > 3 \end{cases}$

$x < 0$

$F(x) = P(X \le x) = \int 0\,dx$

$= 0$

$0 < x < 3$

$F(x) = P(X \le x) = \int_0^x \frac{4}{81}x^3\,dx$

$= \frac{x^4}{81}$

$x > 3$

$F(x) = P(X \le x)$

$= \int_{-\infty}^{0} 0 + \int_0^3 \frac{4}{81}x^3 + \int_3^x 0$

$= 1$

$P(1 < X < 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{4}{81}x^3\,dx = \frac{4}{81}\left[\frac{x^4}{4}\right]_1^2 = \frac{15}{81}$

$P(3 < X < 5) = \int_3^5 f(x)\,dx = 0$

## Cumulative Density function

$F(x) = P(X \leq x)$

Roll a dice

$P(X=1) = 1/6$

$P(X=2) = 1/6$

$P(X=3) = 1/6$

$P(X=4)$

$F(3) = P(X \leq 3)$

$= P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

Consider the density function

$$h(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find cdf.

# Expectation

$$E(x) = \sum x f(x)$$

$$E(x) = \int x f(x) \, dx$$

$$Var = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum x^2 f(x)$$

$$E(x^2) = \int x^2 f(x)$$

A fair die with faces {1,2,3,4,5,6} is thrown repeatedly till ′3′ is observed for the first time. Let X denote the number of times the dice is thrown. The expected value of X is ________.

**(GATE ECE 2015 Set 3)**

1. 5
2. 6
3. 10
4. 15

Answer (b)

## Properties of Expectation

i. Let g and h be functions, and let a and b be constants. For any random variable X (discrete or continuous),

$E\{ag(X) + bh(X)\} = aE\{g(X)\} + bE\{h(X)\}$.

In particular, $E(aX + b) = aE(X) + b$.

ii. Let X and Y be ANY random variables (discrete, continuous, independent, or non-independent). Then $E(X + Y) = E(X) + E(Y)$.

iii. Let X and Y be independent random variables, and g, h be functions.

Then $E(XY) = E(X)E(Y)$

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

# Variance

***Example:*** Let $X$ be a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x^{-2} & \text{for } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Solve

**Properties of Variance**

(i) Let g be a function, and let a and b be constants.

For any random variable X (discrete or continuous),

$\mathrm{Var}\{ ag(X) + b \} = a^2 \mathrm{Var}\{ g(X) \}$.

In particular, $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$.

(ii) Let X and Y be independent random variables.

Then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

(iii) If X and Y are NOT independent,

then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{cov}(X, Y)$. cov??

GATE DA Sample paper 2024

X is a uniformly distributed random variable from 0 to 1

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The variance of X is

(A) 1/2

(B) 1/3

(C) 1/4

(D) 1/12

# Joint Probability Distributions

$\{1, 2, 3, 4, 5, 6\}$

$X, Y$

Example 1: Roll two dice. Let X be the value on the first die and let Y be the value on the second die

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

In example 1, describe the event B = ‘Y − X ≥ 2’ and find its probability.

X

Joint pdf

Y

| $X\backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

$f(2,3)$
$= P(X=2, Y=3)$
$= \frac{1}{36}$

$Y=3, X=1$
$Y=4, X=1$
$X=2$

$Y=5, X=1$
$X=2$
$X=3$

$Y=6$ $X=1$
$X=2$
$X=3$
$X=4$

2, 3
3, 2
1, 4
4, 1

$P(Y-X \geq 2) = \frac{10}{36} = \frac{5}{18}$

$P(X+Y=5) = \frac{4}{36}$

## Joint Probability Mass Function (Joint PMF)

The function f(x, y) is a joint probability function, or probability mass function of the discrete random variables X and Y if

i). $f(x, y) \geq 0$ for all $(x, y)$,

ii). $\sum_x \sum_y f(x, y) = 1$,

iii). $P(X = x, Y = y) = f(x, y)$.

Let X be a coin flip, Y be a dice. Find the joint PMF.

| | X/Y | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | H | 1/12 | 1/12 | 1/12 | 1/12 | 1/12 | 1/12 |
| 0 | T | 1/12 | 1/12 | 1/12 | 1/12 | 1/12 | 1/12 |

$A = (X + Y = 3)$

$B = (\min(X, Y) = 1)$

$P(A) = P(X=1, Y=2) + P(X=0, Y=3)$

$= \frac{1}{6}$

$P(B) = P(X=1, Y=1) + P(X=1, Y=2) + \ldots P(X=1, Y=6)$

$= 6 \times \frac{1}{12} = \frac{1}{2}$

In the previous example, Let's take head as 1 and tail as 0, define A = {X + Y = 3} and B = {min(X, Y ) = 1}. Find P[A] and P[B].

| | Y = 1 | Y = 2 | Y = 3 | Y = 4 | Y = 5 | Y = 6 |
|---|---|---|---|---|---|---|
| $X = 0$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |
| $X = 1$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |

**Marginal Probability Mass Function (Marginal PMF)**

The marginal distributions of X alone and Y along are, respectively

$$g(x) = \sum_{y} f(x,y) \quad \text{and} \quad h(y) = \sum_{x} f(x,y).$$

$$f_X(x) = \sum_{y} f(x,y) \qquad f_Y(y) = \sum_{x} f(x,y)$$

Example 2: Data supplied by a company in Duluth, Minnesota, resulted in the contingency table displayed as below for number of bedroom and number of bathrooms for 50 homes currently for sale. Suppose that one of these 50 homes is selected at random. Let X and Y denote the number of bedrooms and the number of bathrooms, respectively, of the home obtained.

| | | $x$ | | |
|---|---|---|---|---|
| | | 2 | 3 | 4 |
| $y$ | 2 | 3 | 14 | 2 |
| | 3 | 0 | 12 | 11 |
| | 4 | 0 | 2 | 5 |
| | 5 | 0 | 0 | 1 |

(i) Obtain the joint PMF of X and Y.

(ii) Find the probability that the home obtained has the same number of bedrooms and bathrooms, i.e., P(X = Y). $\rightarrow \frac{3}{50} + \frac{12}{50} + \frac{5}{50} = \frac{20}{50} = \frac{2}{5}$

(iii) Find the marginal distribution of X alone.

(iv) Find the marginal distribution of Y alone.

| | | x | | |
|---|---|---|---|---|
| | | 2 | 3 | 4 |
| y | 2 | 3 | 14 | 2 |
| | 3 | 0 | 12 | 11 |
| | 4 | 0 | 2 | 5 |
| | 5 | 0 | 0 | 1 |

| | | x | | | $f_Y(y)$ |
|---|---|---|---|---|---|
| | | 2 | 3 | 4 | |
| y | 2 | 3/50 | 14/50 | 2/50 | 19/50 |
| | 3 | 0 | 12/50 | 11/50 | 23/50 |
| | 4 | 0 | 2/50 | 5/50 | 7/50 |
| | 5 | 0 | 0 | 1/50 | 1/50 |
| | | 3/50 | 28/50 | 19/50 | |

$$\rightarrow f_X(x) = \begin{cases} 3/50 & x=2 \\ 28/50 & x=3 \\ 19/50 & x=4 \end{cases}$$

$$f(x|y=2) = \frac{f(x, y=2)}{f_Y(y=2)} = \frac{f(x,2)}{19/50}$$

| $f(x\|y=2)$ | 2 | 3 | 4 |
|---|---|---|---|
| | 3/19 | 14/19 | 2/19 |

# Conditional Probability Mass Function (Conditional PMF)

$$f(x/y) = \frac{f(x,y)}{f_Y(y)}$$

$$f(y/x) = \frac{f(x,y)}{f_Y(y)}$$

Example 2: Data supplied by a company in Duluth, Minnesota, resulted in the contingency table displayed as below for number of bedroom and number of bathrooms for 50 homes currently for sale. Suppose that one of these 50 homes is selected at random. Let X and Y denote the number of bedrooms and the number of bathrooms, respectively, of the home obtained.

| | | $x$ | | |
|---|---|---|---|---|
| | | 2 | 3 | 4 |
| $y$ | 2 | 3 | 14 | 2 |
| | 3 | 0 | 12 | 11 |
| | 4 | 0 | 2 | 5 |
| | 5 | 0 | 0 | 1 |

EXAMPLE . Refer to Example 2.

(a) Find the distribution of X|Y = 2?

(b) Use the result to determine f(3|2) = P(X = 3|Y = 2). $= \frac{14}{19}$

| | | x | | |
|---|---|---|---|---|
| | | 2 | 3 | 4 |
| | 2 | 3 | 14 | 2 |
| | 3 | 0 | 12 | 11 |
| y | 4 | 0 | 2 | 5 |
| | 5 | 0 | 0 | 1 |

## Independence

Jointly-distributed random variables X and Y are independent if their joint pmf is the product of the marginal pmf's:

$$\boxed{f(x, y) = f_X(x) f_Y(y)}$$

$f(x)$

$f_X(x)$

$f_{X,Y}(x, y)$

$\downarrow$

$f(x, y)$

## Joint Probability Density Function (Joint PDF)

The function $f(x,y)$ is a *joint probability density function* of the continuous random variables $X$ and $Y$ if

i). $f(x,y) \geq 0$ for all $(x,y)$,

ii). $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 1$,

iii). $\mathrm{P}[(X,Y) \in A] = \int\int_{A} f(x,y)\ dx\ dy$, for any region $A$ in the $xy$ plane.

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_{c}^{d}\int_{a}^{b} f(x,y)\, dx\, dy$$

Suppose X and Y both take values in [0,1] with density f(x, y) = 4xy.
Show f(x, y) is a valid joint pdf,
visualize the event A = 'X < 0.5 and Y > 0.5' and find its probability.

$$f(x, y) = 4xy \qquad 0 \le x, y \le 1$$

$$P(X < 0.5, Y > 0.5)$$

$$= \int_{0.5}^{1} \int_{0}^{0.5} 4xy \, dx \, dy$$

Determine $k$ so that

$$f(x,y) = \begin{cases} k(2-x)(1-y) & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a joint probability density function.

Solve:

Suppose that the random variables X and Y have a joint density function given by

$$f(x,y)=\begin{cases} c\,(2x+y) & 2<x<6,\ 0<y<5\\ 0 & \text{otherwise}\end{cases}$$

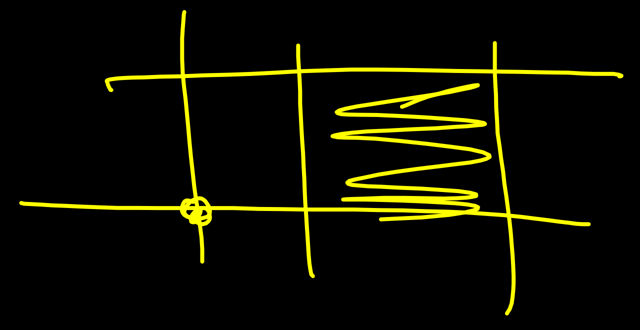
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^5 c(2x+y)\,dy$$

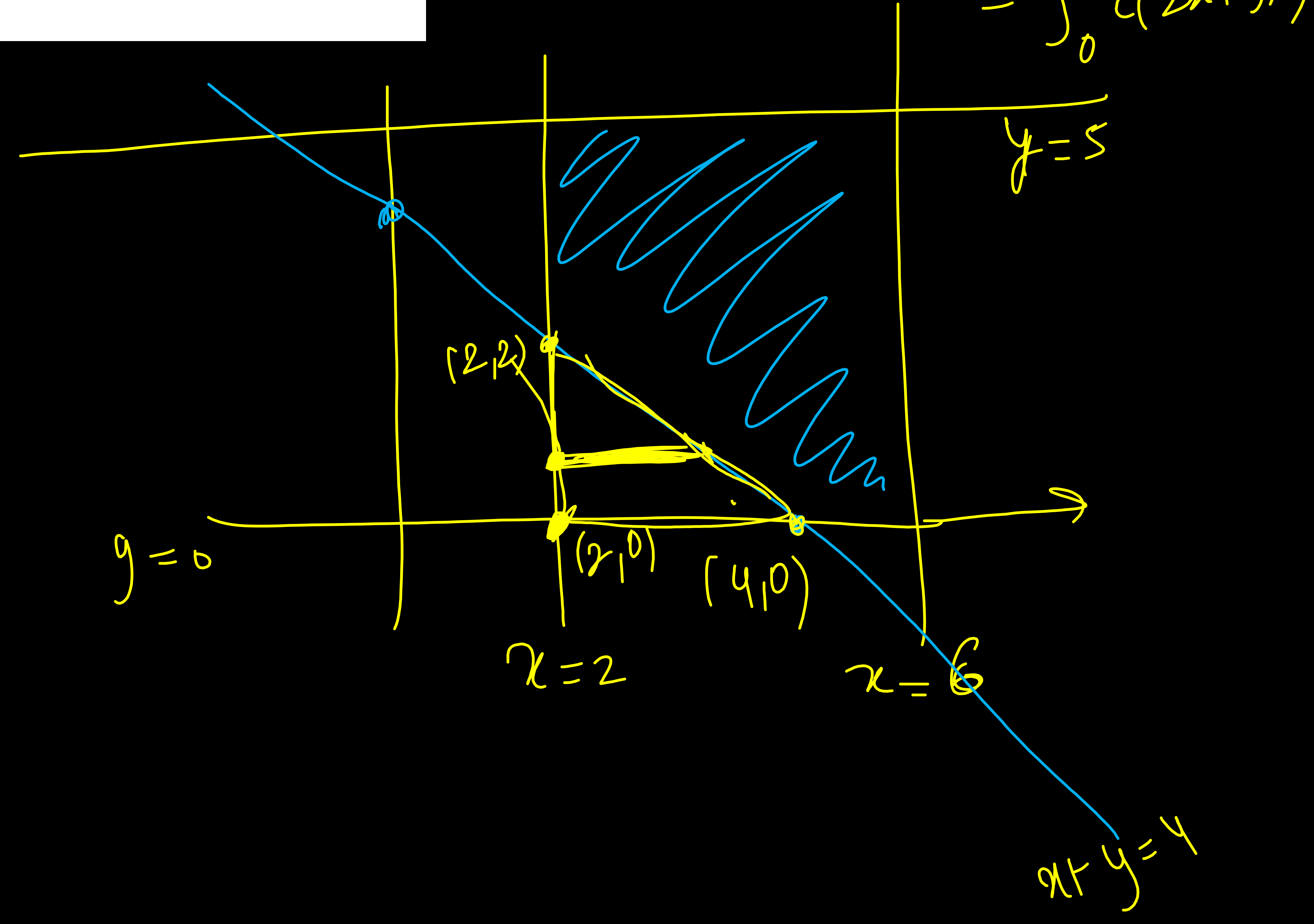
Find

(a) the constant c,

(b) P(X + Y > 4)

$$\boxed{\begin{matrix} 2 < x < 4 \\ 0 < y < 2 \end{matrix}}$$






$$P(X+Y>4)$$

$$= 1 - P(X+Y<4)$$

$$= 1 - \int_0^2 \int_2^{4-y} f(x,y)\,dx\,dy$$

## Marginal Probability Density Function (Marginal PDF)

The *marginal* distributions of $X$ alone and $Y$ along are, respectively,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

and

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Consider the joint density function of the random variables X and Y:

$$f(x, y) = \begin{cases} 1/y & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < x < y < 1$

a. Find marginal pdfs
b. Find P(X>1/2)

$f_X(x) = \int_x^1 \frac{1}{y}\, dy = [\ln y]_x^1 = -\ln x$

$0 < x < 1$

$P(X > 1/2)$

$P(a < X < b)$

$P(X > 1/2) = \int_{1/2}^1 f_X(x)\, dx$

$= \int_{1/2}^1 -\ln x\, dx$

$f_Y(y) = \int_0^y \frac{1}{y}\, dx$

## Conditional Probability Density Function (Conditional PDF)

Let $X$ and $Y$ be two continuous random variables. The conditional distribution of $Y|X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \quad \text{provided } g(x) > 0.$$

Similarly, the conditional distribution of $X|Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \quad \text{provided } h(y) > 0.$$

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

HW

Consider the joint probability density function of the random variables X and Y:

$$f(x,y) = \begin{cases} \frac{3x-y}{9}, & \text{if } 1 < x < 3,\ 1 < y < 2 \\ 0, & \text{elsewhere} \end{cases}.$$

(a) Verify that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

(b) Find the marginal distribution of $X$ alone.

(c) Find $\mathrm{P}(1 < X < 2)$.

(d) Find the conditional distribution of $Y|X = x$.

(e) Find $\mathrm{P}(0.5 < Y < 1|X = 2)$.

# Expectation and Variance in Joint Random Variable

$$E(g(x,y)) = \iint g(x,y)\, f(x,y)\, dx\, dy$$

$$E(g(x)) = \iint g(x)\, f(x,y)\, dx\, dy$$

$f_X(x)$

$$E(g(x)) = \int g(x) f_X(x)\, dx$$

$$E(h(y)) = \iint h(y)\, f(x,y)\, dx\, dy$$

$f_Y(y)$

$$E(h(y)) = \int h(y) f_Y(y)\, dy$$

$$V = E(X)^2 - (E(X))^2$$

**Example:** $X$ and $Y$ have joint density

$$f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z = X + Y$. Find the mean and variance of $Z$.

$$E(Z) = E(X+Y)$$

$$= \iint (x+y)\, f(x,y)\, dx\,dy$$

$$= \int_0^1\int_0^1 (x+y)^2\, dx\,dy$$

$$E(Z^2) = E((X+Y)^2)$$

$$= \iint (x+y)^2 f(x,y)\, dx\,dy$$

$$= \int_0^1\int_0^1 (x+y)^3\, dx\,dy$$

$$V(Z) = E(Z^2) - (E(Z))^2$$

## Covariance

$$\longrightarrow \text{cov}(X, Y) = E(XY) - E(X)E(Y) \Rightarrow$$

not corr

$\text{cov}(X, Y) = 0$

$$\underbrace{E(XY) = E(X)E(Y)}$$

$\Downarrow$

$X$ & $Y$ are indep

$$\text{corr} = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}}$$

$$-1 \le \text{corr} \le 1$$

The covariance of X and Y for the given joint probability distribution is:

| X/Y | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 3/24 | 8/24 | 1/24 |
| 1 | 2/24 | 4/24 | 2/24 |
| 2 | 1/24 | 1/24 | 2/24 |

$X$ and $Y$ are discrete random variables with joint PDF.

$$f(x,y) = \begin{cases} \frac{2x+y}{60}, & 0 < x < 3,\ 0 < y < 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance $\operatorname{cov}(x,y)$.

**(Properties of Covariance)** *Let* $X, Y, Z$ *be random variables, and let c be a constant. Then:*

1. *Covariance-Variance Relationship:* $\text{Var}[X] = \text{Cov}[X, X]$
2. *Pulling Out Constants:*

$$\text{Cov}[cX, Y] = c \cdot \text{Cov}[X, Y]$$

$$\text{Cov}[X, cY] = c \cdot \text{Cov}[X, Y]$$

3. *Distributive Property:*

$$\text{Cov}[X + Y, Z] = \text{Cov}[X, Z] + \text{Cov}[Y, Z]$$

$$\text{Cov}[X, Y + Z] = \text{Cov}[X, Y] + \text{Cov}[X, Z]$$

4. *Symmetry:* $\text{Cov}[X, Y] = \text{Cov}[Y, X]$
5. *Constants cannot covary:* $\text{Cov}[X, c] = 0$.

If Var(X + 2Y ) = 40 and Var(X − 2Y ) = 20, what is Cov(X, Y )?

$$\text{cov}(X+2Y, X+2Y) = \text{var}(X+2Y)$$

$$\text{cov}(X,X) + \text{cov}(X,2Y) + \text{cov}(2Y,X) + \text{cov}(2Y,2Y) = 40$$

$$\text{Var}(X) + 2\,\text{cov}(X,Y) + 2\,\text{cov}(Y,X) + 4\,\text{cov}(Y,Y) = 40$$

$$\text{var}X + 4\,\text{cov}(X,Y) + 4\,\text{var}(Y) = 40 \quad \text{—} \; (1)$$

$$\text{Cov}(X-2Y, X-2Y) = \text{var}(X-2Y)$$

$$\text{Cov}(X,X) + \text{cov}(X,-2Y) + \text{cov}(-2Y,X) + \text{cov}(-2Y,-2Y) = 20$$

$$\text{Var}(X) - 4\,\text{cov}(X,Y) + 4\,\text{var}(Y) = 20 \quad \text{—} \; (2)$$

$$(1) - (2)$$

$$8\,\text{cov}(X,Y) = 20$$

$$\text{Cov}(X,Y) = 2.5$$

If $\sigma_x = \sigma_y$ and x, y are related by u = x + y; v = x - y, what is the cov(u,v)?

$$\mathrm{cov}(x+y,\ x-y)$$

$$= \mathrm{cov}(x,x) + \mathrm{cov}(x,-y) + \mathrm{cov}(y,x) + \mathrm{cov}(y,-y)$$

$$= \cancel{\sigma_x} - \cancel{\mathrm{cov}(x,y)} + \cancel{\mathrm{cov}(x,y)} - \cancel{\sigma_y}$$

$$= 0$$

# Correlation

What is the correlation between $x$ and $a-x$?

## Conditional Expectation and Variance

$$\mu_{X \mid Y=y} = \mathbb{E}(X \mid Y = y) = \sum_x x f_{X \mid Y}(x \mid y).$$

$$E(X^2/Y) = \int x^2 f(x/y)\, dx$$

$$E(X/Y) = \int x\, f(x/y)\, dx$$

$$E(Y/x) = \int y\, f(y/x)\, dy$$

$$Var(X/Y) = E(X^2/Y)^2 - (E(X/Y))^2$$

Let $X$ and $Y$ be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional expectation of Y given X?

Let $X$ and $Y$ be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional variance of Y given X?

Consider a joint probability density function of two random variables $X$ and $Y$

$$f_{X,Y}(x,y) = \begin{cases} 2xy, & 0 < x < 2, \quad 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

Then, $E[Y \mid X = 1.5]$ is ________

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2xy}{x^3} = \frac{2y}{x^2}$$

$$f_X(x) = \int_0^x 2xy\,dy = x^3$$

$$E(y|X=1.5) = \int_0^{1.5} y\, f(y|x=1.5)\,dy$$

$$f(y|X=1.5) = \frac{2y}{(1.5)^2}$$

$$= \int_0^{1.5} y \times \frac{2y}{(1.5)^2}\,dy = 1$$

## GATE DA 2024

$X = 1 \longrightarrow 1/4$

$X = 0 \longrightarrow 3/4$ $\Big\} \longrightarrow \frac{1}{4}$

Two fair coins are tossed independently. X is a random variable that takes a value of 1 if both tosses are heads and 0 otherwise. Y is a random variable that takes a value of 1 if at least one of the tosses is heads and 0 otherwise.

The value of the covariance of X and Y is _______ (rounded off to three decimal places).

$\sum\sum xy\,f(x,y)$

$X = 1 \Rightarrow HH$

$X = 0 \Rightarrow HT, TH, TT$

$Y = 1$ HH, HT, TH

$Y = 0$ TT

Answer: 0.0625

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 1/4 | 2/4 |
| 1 | 0 | 1/4 |

$0 \times 0 \times \frac{1}{4} + 0 \times 1 \times \frac{2}{4} + 1 \times 0 \times 0 + 1 \times 1 \times \frac{1}{4} = \frac{1}{4}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{1}{4} - \frac{1}{4} \times \frac{3}{4} = \frac{1}{16} = 0.0625$

| | |
|---|---|
| | Consider the following joint distribution of random variables X and Y: $$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 \le x \le 2, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$ Which one or more of the following statements is/are correct? |
| (A) | X and Y are mutually uncorrelated. |
| (B) | X and Y are mutually independent. |
| (C) | The mean of X is 1. |
| (D) | The mean of Y is 0.5 |

$E(XY) = E(X)E(Y)$

$f(x,y) = f_X(x) f_Y(y)$

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# Probability Distributions

# Thank you