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**Algorithm 1: Measurement off policy (MOP) SMC:**


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1 Initialize filter particles: simulate  $X_{0,j}^{F,\theta} \sim f_{X_0}(\cdot; \theta)$  for  $j$  in  $1:J$ 
2 Initialize filter weights:  $w_{0,j}^{F,\theta} = 1/J$  for  $j$  in  $1:J$ 
3 for  $n$  in  $1:N$  do
4   Simulate for prediction:  $X_{n,j}^{P,\theta} \sim f_{X_n|X_{n-1}}(\cdot | X_{n-1,j}^F; \theta)$  for  $j$  in  $1:J$ 
5   Evaluate measurement density:  $g_{n,j}^\theta = f_{Y_n|X_n}(g_{n,j}^* | X_{n,j}^{P,\theta}; \theta)$  for  $j$  in  $1:J$ 
6   Update weights to compensate for resampling:  $w_{n,j}^{P,\theta} = \frac{w_{n-1,j}^{F,\theta} g_{n,j}^\theta}{g_{n,j}^\phi}$  for  $j$  in  $1:J$ 
7   Conditional likelihood under  $\phi$ :  $L_n^\phi = \frac{1}{J} \sum_{m=1}^J g_{n,m}^\phi$ 
8   Normalize weights:  $\tilde{g}_{n,j}^\phi = \frac{g_{n,j}^\phi}{J L_n^\phi}$  for  $j$  in  $1:J$ 
9   Apply systematic resampling to select indices  $k_{1:J}$  with  $\text{Prob}(k_j = m) = \tilde{g}_{n,m}^\phi$ 
10  Resample: set  $X_{n,j}^{F,\theta} = X_{n,k_j}^{P,\theta}$  and  $w_{n,j}^{F,\theta} = w_{n,k_j}^{P,\theta}$  for  $j$  in  $1:J$ 
11 end

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- MOP-SMC requires that the algorithm is first run at  $\phi$ , for which it is a vanilla particle filter.  $g_{n,j}^\phi$  and  $\tilde{g}_{n,j}^\phi$  are computed at this first pass. Then it is run at  $\theta$ , with the seed kept fixed.
- Here, the resampling rule for particle  $j$  depends on  $j$  (and therefore  $\phi$  and  $X_{n,j}^{P,\phi}$ ) but not on  $\theta$  or  $X_{n,j}^{P,\theta}$ . This does not have the standard form for a weighted particle filter, for which we can have general reweighting rules but they are usually a function of the location of the particle and the model parameter,  $\theta$ . That may be why it is harder to see that this is a properly weighted filter. Nevertheless, inspection of the algorithm shows that each particle is properly reweighted to account for its resampling probability, in step 6, so it should be properly weighted.
- The final estimate of the likelihood is either

$$L(\theta) = \left( \frac{1}{J} \sum_{j=1}^J w_{N,j}^{F,\theta} \right) \prod_{n=1}^N L_n^\phi \quad (1)$$

or

$$L'(\theta) = \left( \frac{1}{J} \sum_{j=1}^J w_{N,j}^{P,\theta} g_{N,j}^\theta \right) \prod_{n=1}^{N-1} L_n^\phi \quad (2)$$

with  $L'(\theta)$  presumably having slightly lower variance.

- Weighted samples representing the filter distribution,  $f_{X_n|Y_{1:n}}(x_n | y_{1:n}^*; \theta)$  are either  $\{(X_{n,j}^{F,\theta}, w_{n,j}^{F,\theta}), j \text{ in } 1:J\}$  or  $\{(X_{n,j}^{P,\theta}, g_{n,j}^\theta w_{n,j}^{P,\theta}), j \text{ in } 1:J\}$ , meaning that an expectation over  $f_{X_n|Y_{1:n}}(x_n | y_{1:n}^*; \theta)$  is consistently estimated by a corresponding weighted average of the filter or prediction particles.
- As long as **rprocess** is a continuously differentiable function of  $\theta$  for fixed seed, and **dmeasure** is a continuously differentiable function of  $\theta$ , and  $g_{n,j}^\phi \neq 0$ , we see that MOP is a continuously differentiable function of  $\theta$  for fixed seed. Since it also provided an unbiased estimate of the likelihood, this justifies exchanging the order of differentiation and integration to ensure that its derivative is an unbiased estimate of the derivative of the likelihood.
- Taking the derivative with respect to  $\theta$  at  $\theta = \phi$ , step 6 looks very much like the stop gradient approach.