CALCULUS

INTEGRATION

Indefinite Integration

- Integration is the reverse process of differentiation.

$$x^2 \xrightarrow{D} 2x$$

$$x^2 \xleftarrow[I]{} 2x$$

$$\int f(x)\,dx = g(x) + C$$

1. $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c, \; n \neq -1$

2. $\int \dfrac{1}{x} \, dx = \log_e |x| + c$

3. $\int e^x \, dx = e^x + c$

4. $\int a^x \, dx = \dfrac{a^x}{\log_e a} + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

10 11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$

12. $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

13. $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + c$

Theorems of Integration

1. If $\int f(x)\,dx = F(x)$, then $\int f(ax \pm b)dx = \frac{1}{a} F(ax \pm b)$.

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sin(2x+3)\,dx = -\frac{1}{2}\cos(2x+3)$$

i. $\int \tan x \, dx = \ln(\sec x) + C$ **OR** $-\ln(\cos x) + C$

ii. $\int \cot x \, dx = \ln(\sin x) + C$

iii. $\int \sec x \, dx = \ln(\sec x + \tan x) + C$ **OR** $\ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

iv. $\int \operatorname{cosec} x \, dx = \ln(\operatorname{cosec} x - \cot x) + C$ **OR** $\ln\left|\tan\frac{x}{2}\right|$

Integration by Parts.

$$\int (uv)\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

For selecting u

I — Inverse

L — log.

A → Arithmatic

T → Trig.

E → Exp.

① $\int [f(x)]^n f'(x)\,dx$

$$= \frac{[f(x)]^{n+1}}{n+1} + C$$

② $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

DEFINITE INTEGRALS

$$\int_{x=a}^{x=b} f(x)\,dx = \Big[F(x)\Big]_a^b$$

$$= \Big[F(b) - F(a)\Big]$$

Properties of DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. If $a < c < b$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$6.\ \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & f(x) \text{ is even} \rightarrow f(-x) = f(x) \\ 0 & f(x) \text{ is odd} \rightarrow f(-x) = -f(x) \end{cases}$$

$$7.\ \int_{0}^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

$$8.\ \int_0^{2a} xf(x)dx = a\int_0^{2a} f(x)dx, \text{ if } f(2a-x) = f(x)$$

(1) $\int_0^{2\pi} \sin x \, dx = \left[-\cos x\right]_0^{2\pi}$

$f(x)$

$= \left[-\cos 2\pi + \cos 0\right]$

$= \left[-1 + 1\right] = 0$ ✓

$f(x) = \sin x$

$\boxed{2a = 2\pi}$

$f(2\pi - x) = \sin(2\pi - x) = -\sin x = -f(x)$ ✓

$\int_0^{2\pi} \sin x \, dx = 0$

Q. $\int_0^{\pi} x \sin x \, dx$ $= \pi$ $\equiv \int_0^{\pi} x f(x)\, dx$

$f(x) = \sin x$ $\quad 2a = \pi \Rightarrow a = \pi/2$

$f(2a-x) = f(\pi - x) = \sin(\pi - x)$

$= \sin x = f(x)$

$= a \int_0^{2a} f(x)\, dx$ $\quad = \frac{\pi}{2} \int_0^{\pi} \sin x \, dx$

$= \frac{\pi}{2} \left[-\cos x\right]_0^{\pi}$

$= \frac{\pi}{2}\left[-\cos \pi + \cos 0\right]$

$= \frac{\pi}{\cancel{2}}\left[1+1\right]$

$= \pi$

Q. $\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{1+\cos x}\,dx = 0$

$$\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{1+\cos x}\,dx$$

$$f(-x) = \frac{\sin 2(-x)}{1+\cos(-x)} = \frac{-\sin 2x}{1+\cos x} = -f(x)$$

$$\boxed{f(-x) = -f(x)} \rightarrow \underline{\underline{\text{odd}}}$$

Q. $\displaystyle\int_0^2 \left(\frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)} \right) dx$

Gate 2014 ME

Let $x-1 = t$

$dx = dt$

$x = 0$ $\quad$ $x = 2$

$t = -1$ $\quad$ $t = 1$

$$= \int_{-1}^{1} \frac{t^2 \sin t}{t^2 + \cos t}\, dt$$

$\to$ odd

$f(-t) = -f(t)$

$= \underline{\underline{0}}$

Q. $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\,dx = ?$ — (1)

Gate 2011

CE :- 2 marks

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-a+x}}\,dx = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} = I \quad -(2)$$

$$I + I = \int_0^a \frac{\cancel{\sqrt{x} + \sqrt{a-x}}}{\cancel{\sqrt{x} + \sqrt{a-x}}}\,dx$$

$$2I = \int_0^a dx \Rightarrow 2I = [x]_0^a \Rightarrow 2I = a$$

$$\Rightarrow \boxed{I = a/2}$$

Q. $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}\,dx = ?$

[GATE-2015, 2 MARKS]

$\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}\,dx$

Let $\frac{1}{x} = t$

$\Rightarrow -\frac{1}{x^2}dx = dt$

$x = \frac{1}{\pi}$ $\quad x = 2/\pi$

$t = \pi$ $\quad t = \pi/2$

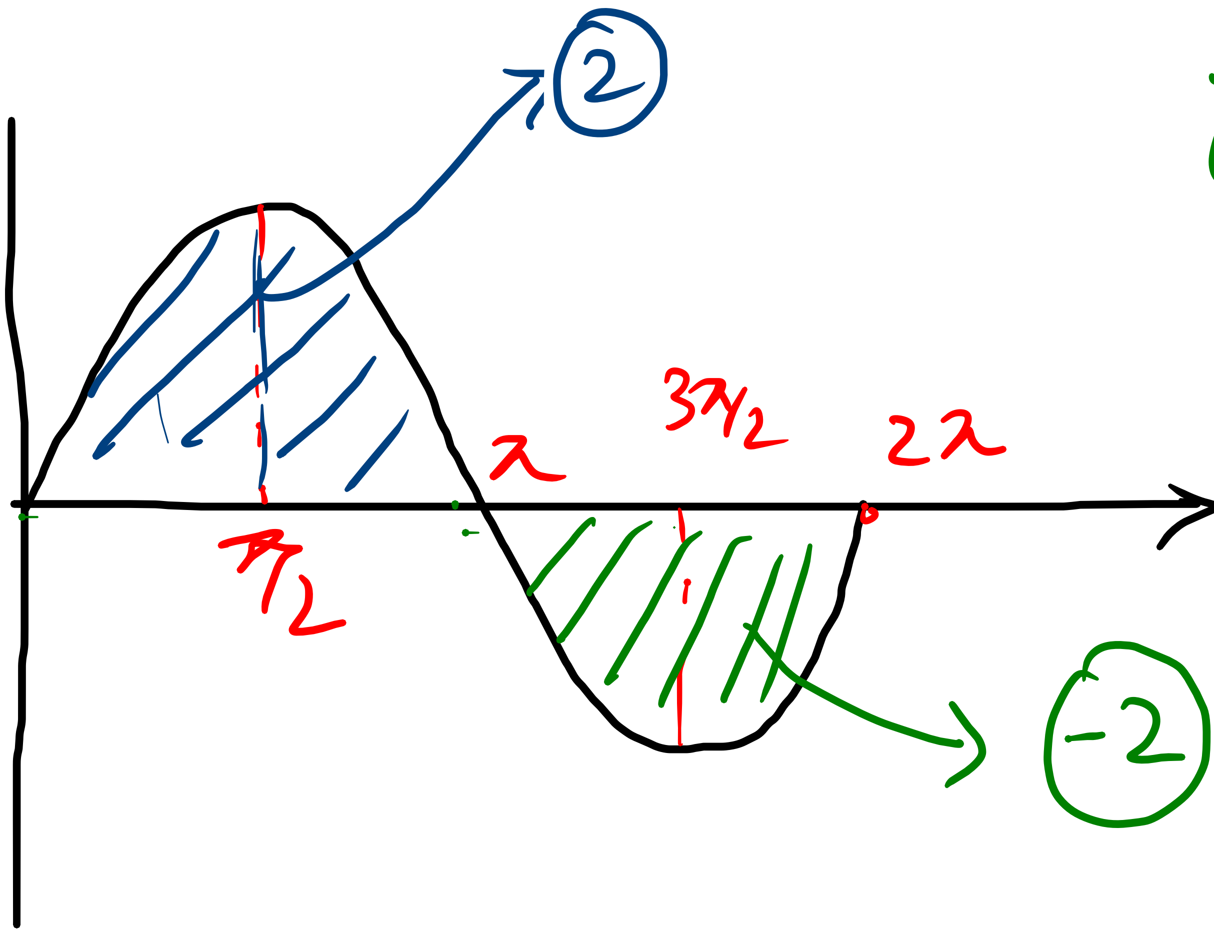
$\displaystyle\int_{\pi}^{\pi/2} -\cos t\,dt = \int_{\pi/2}^{\pi} \cos t\,dt$

$= [\sin t]_{\pi/2}^{\pi} = -1$

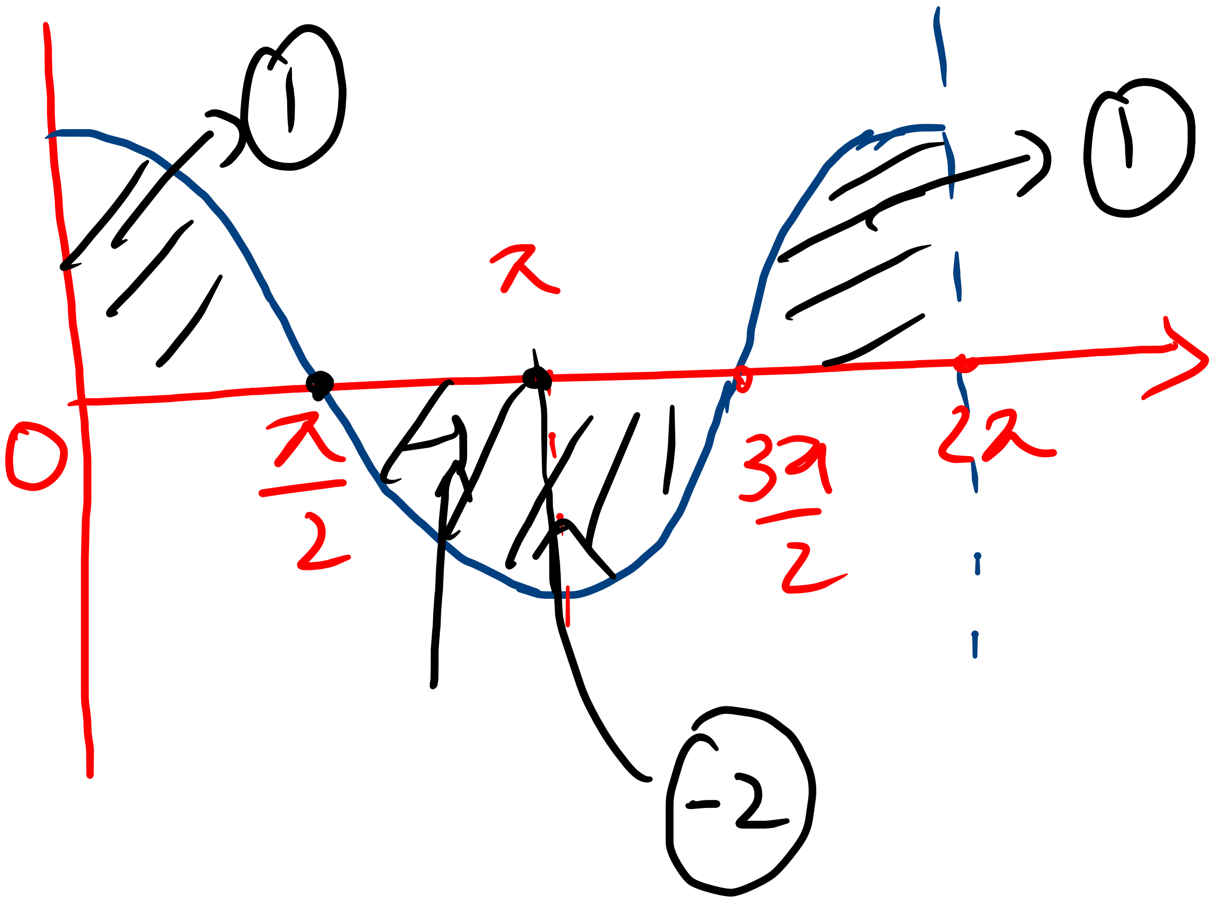
$$\int_0^{\pi} \sin x \, dx =$$

$$\int_0^{2\pi} \sin x \, dx = 0$$

$$\int_0^{3\pi/2} (\sin x) \, dx$$



$$\int_0^{\pi/2} \sin x \, dx = 1$$



$$\int_{\pi/2}^{3\pi/2} \cos x \, dx = -2$$

Q. $\int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx = ?$ = 1.2919 CE

[GATE-2017, 1 MARK]

Let $\sin^{-1} x = t \Rightarrow x = \sin t$

$\frac{1}{\sqrt{1-x^2}} dx = dt$

$x = 0$

$\Rightarrow \sin t = 0$

$\Rightarrow t = \sin^{-1} 0$

$= 0$

$x = 1$

$\Rightarrow \sin t = 1$

$\Rightarrow t = \pi/2$

$$\int_0^{\pi/2} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{\pi/2}$$

$$= \frac{\pi^3}{24} - 0 = \frac{\pi^3}{24}$$

Q. The value of the following definite integral is______.

(round off to three decimal places)

$$= \int_1^e (x \ln x)\, dx$$

$$\int_1^e (x \ln x)\, dx = \left(\ln x \int x\, dx - \int \left(\frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{2}\, dx \right) \right)_1^e$$

x: v, $\ln x$: u

$$= \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right)_1^e$$

$$= \left(\frac{e^2}{2}(1) - \frac{e^2}{4} + \frac{1}{4} \right) = \_\_\_\_$$

Q. $\int_{0}^{2\pi} |x \sin x|\, dx = k\pi,\ k = ?$ 4 CS

[GATE-2014, 1 MARK]

$$= \int_{0}^{2\pi} x|\sin x|\, dx = \int_{0}^{\pi} x \sin x + \int_{\pi}^{2\pi} -x \sin x\, dx$$

$$= \pi - (-3\pi)$$

$$= 4\pi = k\pi$$

$$\boxed{k = 4}$$

$$\int_{\pi}^{2\pi} x \sin x \, dx = \left[-x\cos x + \sin x\right]_{\pi}^{2\pi}$$

$u \quad v$

u	v
$+\ x$	$\sin x$
$-\ 1$	$-\cos x$
$+\ 0$	$-\sin x$

$$= \left[-2\pi(1) + 0 + \pi(-1) - 0\right]$$

$$= -3\pi$$

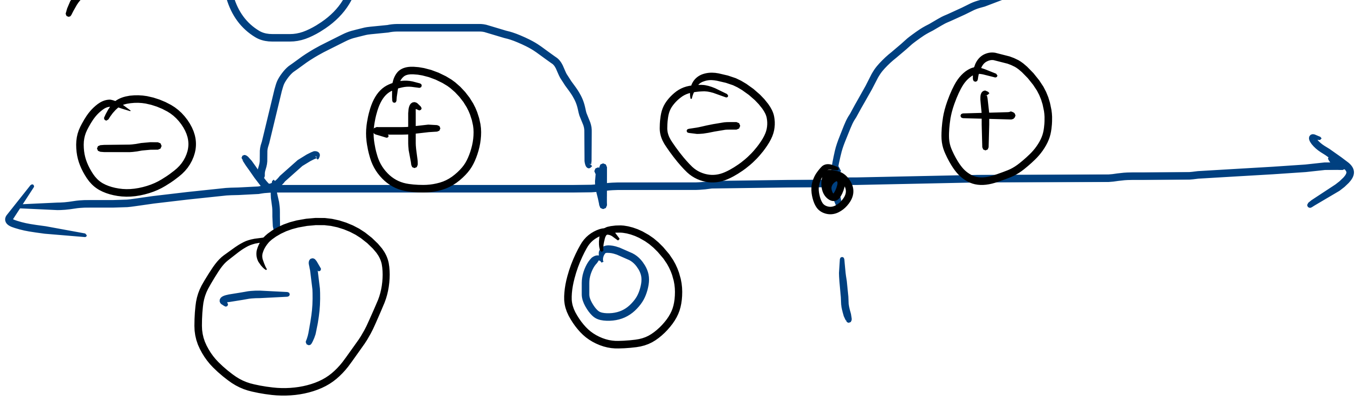
Q. $\int_{-1}^{2} |x^3 - x| \, dx = ?$

$$|x^3 - x| = (x^3 - x) \qquad x^3 - x \geq 0$$

$$= -(x^3 - x) \qquad x^3 - x < 0$$

$$x^3 - x \geq 0$$

$$\Rightarrow \quad x(x-1)(x+1) \geq 0$$



2

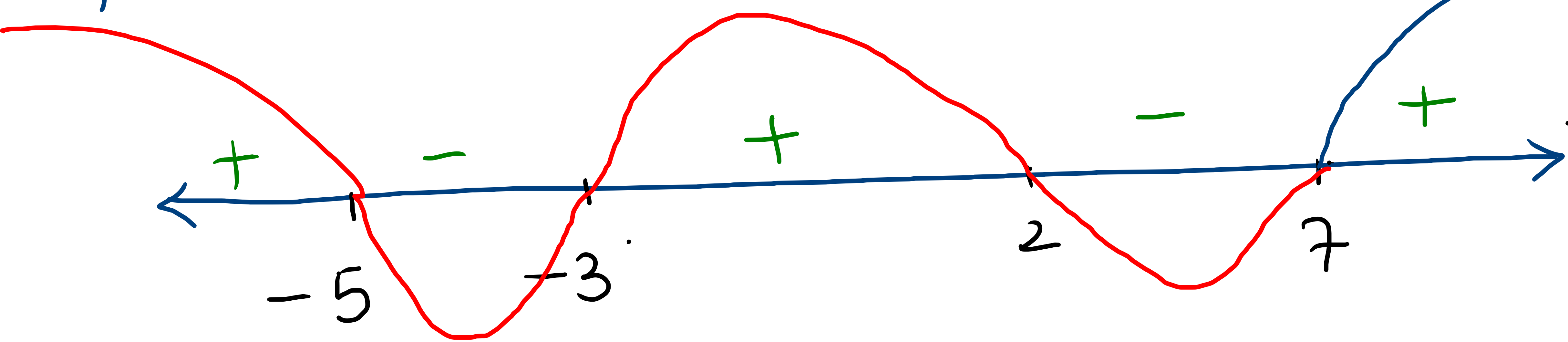
$$\int_{-1}^{2} (x^3 - x)\,dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

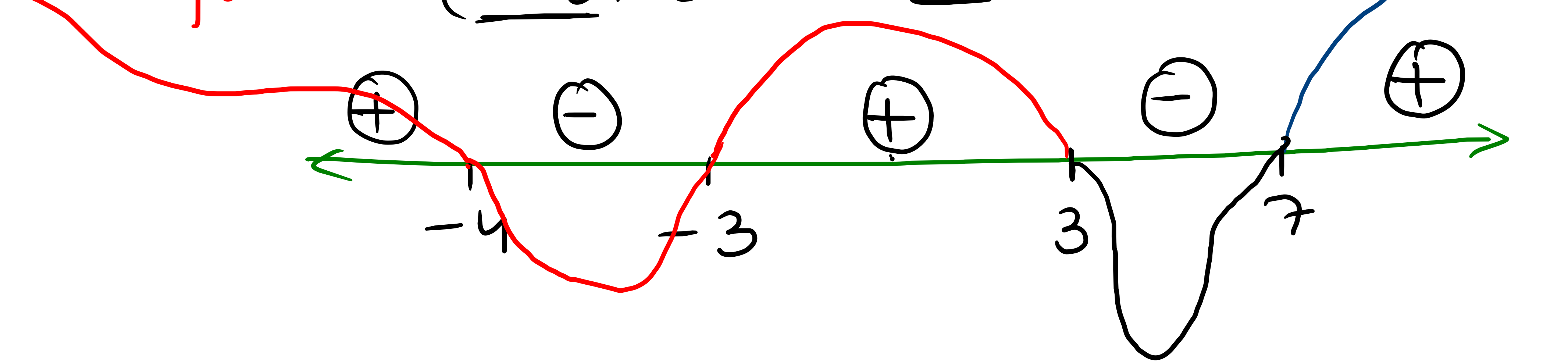
$$= \underline{\qquad}$$

Wavy Curve Method

$$f(x) = (x-2)(x+3)(x+5)(x-7) > 0$$



$f(x) \rightarrow (x-3)(x+4)(x-7)(x+3) \geq 0$



1 min

Q. Let f be a real-valued function of a real variable defined as $f(x) = x - [x]$, where [x] denotes [x] denotes the largest integer less than or equal to x. The value of $= \int_{0.25}^{1.25} f(x)\,dx$ is ______.

(up to 2 decimal places).

[GATE – 2018, 1 MARK]

EE

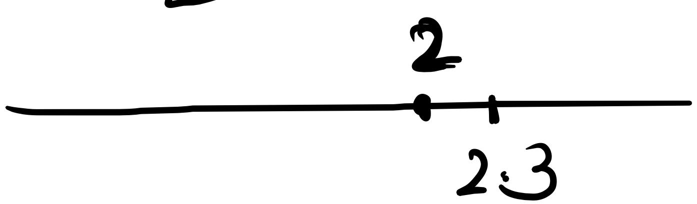
$$\int_{0.25}^{1.25} x - [x]\ dx$$

$$= \int_{0.25}^{1.25} x\,dx - \int_{0.25}^{1.25} [x]\,dx$$

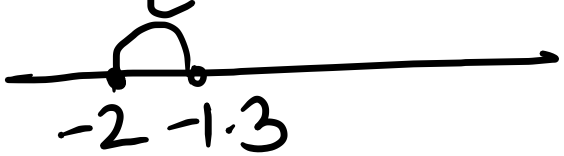
$$= \left[\frac{x^2}{2}\right]_{0.25}^{1.25} - (0.25)$$

$$= 0.5$$

$[2.3] = 2$



$[-1.3] = -2$



$$\int_{0.25}^{1.25}[x]\,dx = \int_{0.25}^{1}\cancelto{0}{[x]}\,dx + \int_{1}^{1.25}\cancelto{1}{[x]}\,dx$$

$$= 0 + \int_{1}^{1.25} 1\,dx = [x]_{1}^{1.25}$$

$$= 0.25$$

Double Factorial

$$n! = n(n-1)(n-2) \cdots\cdots 1$$

$$n!! = n(n-2)(n-4) \cdots\cdots 2 \text{ or } 1$$

$$8!! = 8 \times 6 \times 4 \times 2$$

$$7!! = 7 \times 5 \times 3 \times 1$$

Reduction Formulae

WALLI'S INTEGRALS

$$\int_0^{\pi/2} \sin^m x . \cos^n x \, dx = \frac{(m-1)!!\,(n-1)!!}{(m+n)!!} k \quad \text{where } k = \begin{cases} \pi/2, & \text{if m \& n are even} \\ 1, & \text{Otherwise} \end{cases}$$

$$W_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)!!}{(n)!!} k \quad \text{where } k = \begin{cases} \pi/2, & \text{if n is even} \\ 1, & \text{Otherwise} \end{cases}$$

$$\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/2} \cos^4 x \, dx$$

$$= \frac{(4-1)!!}{4!!} \times \pi/2$$

$$= \frac{3!!}{4!!} \times \pi/2 = \frac{3 \times 1}{4 \times 2} \times \pi/2$$

$$= \underline{\underline{\frac{3\pi}{16}}}$$

Q. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = ?$

$$\left(\frac{1-\cos 2x}{2}\right)$$

$$= 2\int_0^{\pi/2} \sin^2 x\,dx$$

$$= 2 \times \frac{(2-1)!!}{(2)!!} \times \frac{\pi}{2} = \frac{2 \times 1}{2} \times \frac{\pi}{2} = \frac{\pi}{2}$$

Q. $\int_0^{\pi} \sin^4 x \, dx = ?$

$2a = \pi$

$a = \pi/2$

$f(x) = \sin^4 x$

$f(\underline{2a-x}) = f(\pi - x) = \sin^4(\pi - x) = \sin^4 x = \underline{f(x)}$

$= 2\int_0^{\pi/2} \sin^4 x \, dx \quad = 2 \times \frac{3!!}{4!!} \times \pi/2$

$= \_\_\_$

Q. $\int_0^{\pi/2} \sin^4(x)\cos^7(x)\,dx = ?$

$$= \frac{3!!\ 6!!}{(11)!!} \times 1 =$$