1 Inference rules

1.1 Formation of contexts, types, and their elements

MLTT rule:

$$\frac{\Gamma,\, x : A \;\vdash\; B(x) \text{ type}}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; a : A}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; a \equiv b : A}{\Gamma \;\vdash\; a : A}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; a \equiv b : A}{\Gamma \;\vdash\; b : A}$$

1.2 Judgemental equality is an equivalence relation

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; A \equiv A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; B \equiv A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type} \qquad \Gamma \;\vdash\; B \equiv C \text{ type}}{\Gamma \;\vdash\; A \equiv C \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

1.3 Variable conversion

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash B(x) \text{ type}}{\Gamma, x : A', \Delta \vdash B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : A', \Delta \vdash \mathcal{J}}$$

Derived rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'}$$

1.4 Substitution

Metasyntactic notation (used in many places):

$$t[a/x] := t \text{ with all occurences of } x \text{ replaced by } a$$

MLTT rule:

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\ S$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \equiv B[a'/x] \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv b[a'/x] : B[a/x]}$$

1.5 Weakening

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x\!:\!A, \Delta \vdash \mathcal{J}}\, W$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma, x\!:\!A \vdash B \text{ type}}\, W$$

1.6 Generic elements

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x\!:\!A \vdash x\!:\!A}\, \delta$$

1.7 Manipulating variables

Derived rule:

$$\frac{\Gamma, x\!:\!A, \Delta \vdash \mathcal{J}}{\Gamma, x'\!:\!A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}\, x'/x$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x\!:\!A, y\!:\!B, \Delta \vdash \mathcal{J}}{\Gamma, y\!:\!B, x\!:\!A, \Delta \vdash \mathcal{J}}$$

2 Π types

2.1 Dependent functions

2.1.1 Π formation

MLTT rule:

$$\frac{\Gamma, x\!:\!A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \text{ type}}\, \Pi$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x\!:\!A \vdash B(x) \equiv B'(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \equiv \Pi_{(x:A')} B'(x) \text{ type}}\, \Pi\text{-eq}$$

2.1.2 Π introduction

MLTT rule:

$$\frac{\Gamma,\, x\!:\!A \;\vdash\; b(x) : B(x)}{\Gamma \;\vdash\; \lambda x.b(x) : \Pi_{(x:A)}B(x)}\;\lambda$$

MLTT rule:

$$\frac{\Gamma,\, x\!:\!A \;\vdash\; b(x) \equiv b'(x) : B(x)}{\Gamma \;\vdash\; \lambda x.b(x) \;\equiv\; \lambda x.b'(x) : \Pi_{(x:A)}B(x)}\;\lambda\text{-eq}$$

2.1.3 Π elimination

MLTT rule:

$$\frac{\Gamma \;\vdash\; f : \Pi_{(x:A)}B(x)}{\Gamma,\, x\!:\!A \;\vdash\; f(x) : B(x)}\;\text{ev}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; f \equiv f' : \Pi_{(x:A)}B(x)}{\Gamma,\, x\!:\!A \;\vdash\; f(x) \equiv f'(x) : B(x)}\;\text{ev-eq}$$

2.1.4 Π computation

MLTT rule:

$$\frac{\Gamma,\, x\!:\!A \;\vdash\; b(x) : B(x)}{\Gamma,\, x\!:\!A \;\vdash\; (\lambda y.b(y))(x) \equiv b(x) : B(x)}\;\beta$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; f : \Pi_{(x:A)}B(x)}{\Gamma \;\vdash\; \lambda x.f(x) \;\equiv\; f : \Pi_{(x:A)}B(x)}\;\eta$$

2.2 Ordinary functions

2.2.1 Basic

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; A \to B \text{ type}}\;\to$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; A \to B \equiv \Pi_{(x:A)}B \text{ type}}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma,\, x\!:\!A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : A \to B}\ \lambda$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma,\, x\!:\!A \vdash f(x) : B}\ \mathrm{ev}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma,\, x\!:\!A \vdash b(x) : B}{\Gamma,\, x\!:\!A \vdash (\lambda y.b(y))(x) \equiv b(x) : B}\ \beta$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.f(x) \equiv f : A \to B}\ \eta$$

The symbol $\to$ is right-associative:

$$X \to Y \to Z \quad := \quad X \to (Y \to Z)$$

2.2.2 Identity function

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{id}_A : A \to A}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{id}_A \equiv \lambda x.x : A \to A}$$

2.2.3 Function composition

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash C \text{ type}}{\Gamma \vdash \mathrm{comp} : (B \to C) \to (A \to B) \to (A \to C)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash C \text{ type}}{\Gamma \vdash \mathrm{comp} \equiv \lambda g.\lambda f.\lambda x.g(f(x)) : (B \to C) \to (A \to B) \to (A \to C)}$$

Convention:

$$g \circ f \quad := \quad \mathrm{comp}(g, f) \equiv \lambda x. g(f(x))$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to C \qquad \Gamma \vdash h : C \to D}{\Gamma \vdash (h \circ g) \circ f \equiv h \circ (g \circ f) : A \to D}$$

3 The natural numbers

3.1 Basic

MLTT rule:

$$\frac{}{\vdash \mathbb{N} \text{ type}} \ \mathbb{N}\text{-form}$$

MLTT rule:

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}}$$

MLTT rule:

$$\frac{}{\vdash \mathrm{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}}$$

3.2 Induction principle

Convention (used in many places):

$$f(x, y) \quad := \quad f(x)(y) \equiv (f(x))(y)$$

Metasyntactic shortcut (only for this document):

$$\text{Ⓟ} \quad := \quad \Gamma, n : \mathbb{N} \vdash P(n) \text{ type}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(p_0, p_S) : \Pi_{(n:\mathbb{N})} P(n)} \ \mathbb{N}\text{-ind}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(p_0, p_S, 0_{\mathbb{N}}) \equiv p_0 : P(0_{\mathbb{N}})}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})}\,(P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma,\, n:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(p_0, p_S, \mathrm{succ}_{\mathbb{N}}(n)) \equiv p_S(n, \mathrm{ind}_{\mathbb{N}}(p_0, p_S, n)) : P(\mathrm{succ}_{\mathbb{N}}(n))}$$

3.3 Addition

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) : \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) \equiv m : \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) \equiv \lambda n.\, \mathrm{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \mathrm{add}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \mathrm{add}_{\mathbb{N}}(m) \equiv \mathrm{ind}_{\mathbb{N}}(\text{add-zero}_{\mathbb{N}}(m), \text{add-succ}_{\mathbb{N}}(m)) : \mathbb{N} \to \mathbb{N}}$$

Remark:

$$m + 0_{\mathbb{N}} \equiv \mathrm{add}_{\mathbb{N}}(m, 0_{\mathbb{N}}) \equiv m$$

Remark:

$$m + \mathrm{succ}_{\mathbb{N}}(n) \equiv \mathrm{add}_{\mathbb{N}}(m, \mathrm{succ}_{\mathbb{N}}(n)) \equiv \mathrm{succ}_{\mathbb{N}}(m + n)$$

4 More inductive types

4.1 The unit type

MLTT rule:

$$\frac{}{\vdash \mathbf{1} \text{ type}}$$

MLTT rule:

$$\frac{}{\vdash \bigstar : \mathbf{1}}$$

MLTT rule:

$$\frac{\Gamma,\, x\!:\!\mathbf{1} \;\vdash\; P(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_{\mathbf{1}} : P(\bigstar) \to \Pi_{(x:\mathbf{1})} P(x)}$$

MLTT rule:

$$\frac{\Gamma,\, x\!:\!\mathbf{1} \;\vdash\; P(x) \text{ type} \qquad \Gamma \;\vdash\; p : P(\bigstar)}{\Gamma \;\vdash\; \mathrm{ind}_{\mathbf{1}}(p, \bigstar) \equiv p : P(\bigstar)}$$

Derived rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_{\mathbf{1}} : A \to (\mathbf{1} \to A)}$$

Derived rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_{\mathbf{1}} \;\equiv\; \lambda a.\lambda x.a : A \to (\mathbf{1} \to A)}$$

4.2 The empty type

MLTT rule:

$$\frac{}{\vdash \emptyset \text{ type}}$$

MLTT rule:

$$\frac{\Gamma,\, x\!:\!\emptyset \;\vdash\; P(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_{\emptyset} : \Pi_{(x:\emptyset)} P(x)}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; \text{ex-falso} : \emptyset \to A}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; \text{ex-falso} \equiv \mathrm{ind}_{\emptyset} : \emptyset \to A}$$

4.3 Negation

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \equiv (A \to \emptyset) \text{ type}}$$

4.4 Coproducts

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A+B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{inl} : A \to A+B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{inr} : B \to A+B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, z : A+B \vdash P(z) \text{ type}}{\Gamma \vdash \mathrm{ind}_+ : \big(\Pi_{(x:A)} P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)} P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, z : A+B \vdash P(z) \text{ type}}{\Gamma,\, f : \Pi_{(x:A)} P(\mathrm{inl}(x)),\, g : \Pi_{(y:B)} P(\mathrm{inr}(y)),\, a : A \vdash \mathrm{ind}_+(f, g, \mathrm{inl}(a)) \equiv f(a) : P(\mathrm{inl}(a))}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, z : A+B \vdash P(z) \text{ type}}{\Gamma,\, f : \Pi_{(x:A)} P(\mathrm{inl}(x)),\, g : \Pi_{(y:B)} P(\mathrm{inr}(y)),\, b : B \vdash \mathrm{ind}_+(f, g, \mathrm{inr}(b)) \equiv g(b) : P(\mathrm{inr}(b))}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash X \text{ type}}{\Gamma \vdash \mathrm{ind}_+ : (A \to X) \to (B \to Y) \to (A+B \to X)}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A' \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash B' \text{ type}}{\Gamma,\, f : A \to A',\, g : B \to B' \;\vdash\; f{+}g \;:\; A{+}B \to A'{+}B'}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A' \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash B' \text{ type}}{\Gamma,\, f : A \to A',\, g : B \to B',\, x{:}A \;\vdash\; (f{+}g)(\mathrm{inl}(x)) \;\equiv\; \mathrm{inl}(f(x)) : A'{+}B'}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A' \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash B' \text{ type}}{\Gamma,\, f : A \to A',\, g : B \to B',\, y{:}B \;\vdash\; (f{+}g)(\mathrm{inr}(y)) \;\equiv\; \mathrm{inr}(g(y)) : A'{+}B'}$$

4.5 Σ types

4.5.1 Dependent pairs

MLTT rule:

$$\frac{\Gamma,\, x{:}A \;\vdash\; B(x) \text{ type}}{\Gamma \;\vdash\; \Sigma_{(x:A)} B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \;\vdash\; B(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{pr}_1 : \big(\Sigma_{(x:A)} B(x)\big) \to A}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \;\vdash\; B(x) \text{ type}}{\Gamma,\, (x,y){:}\,\Sigma_{(x:A)} B(x) \;\vdash\; \mathrm{pr}_1((x,y)) \;\equiv\; x : A}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \;\vdash\; B(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{pr}_2 : \Pi_{\left(p\,:\,\Sigma_{(x:A)} B(x)\right)} B(\mathrm{pr}_1(p))}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \;\vdash\; B(x) \text{ type}}{\Gamma,\, (x,y){:}\,\Sigma_{(x:A)} B(x) \;\vdash\; \mathrm{pr}_2((x,y)) \;\equiv\; y : B(x)}$$

Disclaimer: Many rules about dependent pairs are missing (hopefully, none of them is needed for the homework.

4.5.2 Ordinary pairs

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \equiv \Sigma_{(x:A)} B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A,\, y : B \vdash P(x,y) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\times} : \big(\Pi_{(x:A)} \Pi_{(y:B)} P(x,y)\big) \to \big(\Pi_{(z:A\times B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A,\, y : B \vdash P(x,y) \text{ type}}{\Gamma,\, g : \Pi_{(x:A)} \Pi_{(y:B)} P(x,y),\, (x,y) : A \times B \vdash \mathrm{ind}_{\times}(g, (x,y)) \equiv g(x,y) : P(x,y)}$$