

# 1 Inference rules

## 1.1 Formation of contexts, types, and their elements

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv B \text{ type}}{\Gamma \vdash A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv B \text{ type}}{\Gamma \vdash B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a:A}{\Gamma \vdash A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b:A}{\Gamma \vdash a:A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b:A}{\Gamma \vdash b:A}$$

## 1.2 Judgemental equality is an equivalence relation

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv B \text{ type}}{\Gamma \vdash B \equiv A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv B \text{ type} \quad \Gamma \vdash B \equiv C \text{ type}}{\Gamma \vdash A \equiv C \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A \quad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

### 1.3 Variable conversion

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x:A, \Delta \vdash B(x) \text{ type}}{\Gamma, x:A', \Delta \vdash B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x:A, \Delta \vdash \mathcal{J}}{\Gamma, x:A', \Delta \vdash \mathcal{J}}$$

Derived rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash a : A}{\Gamma \vdash a : A'}$$

### 1.4 Substitution

Metasyntactic notation (used in many places):

$$t[a/x] := t \text{ with all occurrences of } x \text{ replaced by } a$$

MLTT rule:

$$\frac{\Gamma \vdash a : A \quad \Gamma, x:A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]} S$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma, x:A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \equiv B[a'/x] \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma, x:A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv b[a'/x] : B[a/x]}$$

## 1.5 Weakening

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x:A, \Delta \vdash \mathcal{J}} W$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x:A \vdash B \text{ type}} W$$

## 1.6 Generic elements

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x:A \vdash x:A} \delta$$

## 1.7 Manipulating variables

Derived rule:

$$\frac{\Gamma, x:A, \Delta \vdash \mathcal{J}}{\Gamma, x':A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]} x'/x$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A, y:B, \Delta \vdash \mathcal{J}}{\Gamma, y:B, x:A, \Delta \vdash \mathcal{J}}$$

# 2 $\Pi$ types

## 2.1 Dependent functions

### 2.1.1 $\Pi$ formation

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \text{ type}} \Pi$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x:A \vdash B(x) \equiv B'(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \equiv \Pi_{(x:A')} B'(x) \text{ type}} \Pi\text{-eq}$$

### 2.1.2 $\Pi$ introduction

MLTT rule:

$$\frac{\Gamma, x:A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : \Pi_{(x:A)} B(x)} \lambda$$

MLTT rule:

$$\frac{\Gamma, x:A \vdash b(x) \equiv b'(x) : B(x)}{\Gamma \vdash \lambda x.b(x) \equiv \lambda x.b'(x) : \Pi_{(x:A)} B(x)} \lambda\text{-eq}$$

### 2.1.3 $\Pi$ elimination

MLTT rule:

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma, x:A \vdash f(x) : B(x)} \text{ev}$$

MLTT rule:

$$\frac{\Gamma \vdash f \equiv f' : \Pi_{(x:A)} B(x)}{\Gamma, x:A \vdash f(x) \equiv f'(x) : B(x)} \text{ev-eq}$$

### 2.1.4 $\Pi$ computation

MLTT rule:

$$\frac{\Gamma, x:A \vdash b(x) : B(x)}{\Gamma, x:A \vdash (\lambda y.b(y))(x) \equiv b(x) : B(x)} \beta$$

MLTT rule:

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma \vdash \lambda x.f(x) \equiv f : \Pi_{(x:A)} B(x)} \eta$$

## 2.2 Ordinary functions

### 2.2.1 Basic

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \text{ type}} \rightarrow$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \equiv \Pi_{(x:A)} B \text{ type}}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : A \rightarrow B} \lambda$$

Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow B}{\Gamma, x:A \vdash f(x) : B} \text{ev}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A \vdash b(x) : B}{\Gamma, x:A \vdash (\lambda y.b(y))(x) \equiv b(x) : B} \beta$$

Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow B}{\Gamma \vdash \lambda x.f(x) \equiv f : A \rightarrow B} \eta$$

### 2.2.2 Identity function

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{id}_A : A \rightarrow A}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \text{id}_A \equiv \lambda x.x : A \rightarrow A}$$

### 2.2.3 Composition

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma \vdash C \text{ type}}{\Gamma \vdash \text{comp} : (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma \vdash C \text{ type}}{\Gamma \vdash \text{comp} \equiv \lambda g.\lambda f.\lambda x.g(f(x)) : (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)}$$

Remark:

$$g \circ f \equiv \text{comp}(g, f) \equiv \lambda x.g(f(x))$$

Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash g : B \rightarrow C \quad \Gamma \vdash h : C \rightarrow D}{\Gamma \vdash (h \circ g) \circ f \equiv h \circ (g \circ f) : A \rightarrow D}$$

### 3 The natural numbers

#### 3.1 Basic

MLTT rule:

$$\frac{}{\vdash \mathbb{N} \text{ type}} \text{ N-form}$$

MLTT rule:

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}}$$

MLTT rule:

$$\frac{}{\vdash \text{succ}_{\mathbb{N}} : \mathbb{N}}$$

#### 3.2 Induction principle

Convention:

$$f(x, y) \quad := \quad f(x)(y) \equiv (f(x))(y)$$

Metasyntactic shortcut (only for this document):

$$\textcircled{\text{P}} \quad := \quad \Gamma, n : \mathbb{N} \vdash P(n) \text{ type}$$

MLTT rule:

$$\frac{\textcircled{\text{P}} \quad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \quad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \rightarrow P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_S) : \Pi_{(n:\mathbb{N})} P(n)} \text{ N-ind}$$

MLTT rule:

$$\frac{\textcircled{\text{P}} \quad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \quad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \rightarrow P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_S, 0_{\mathbb{N}}) \equiv p_0 : P(0_{\mathbb{N}})}$$

MLTT rule:

$$\frac{\textcircled{\text{P}} \quad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \quad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \rightarrow P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma, n : \mathbb{N} \vdash \text{ind}_{\mathbb{N}}(p_0, p_S, \text{succ}_{\mathbb{N}}(n)) \equiv p_S(n, \text{ind}_{\mathbb{N}}(p_0, p_S, n)) : P(\text{succ}_{\mathbb{N}}(n))}$$

Example (not in the textbook):

$$\frac{\textcircled{*} \quad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \quad \Gamma \vdash p_S : \Pi_{(n:\mathbb{N})} (P(n) \rightarrow P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_S, \text{succ}_{\mathbb{N}}(0_{\mathbb{N}})) \equiv p_S(0_{\mathbb{N}}, p_0) : P(\text{succ}_{\mathbb{N}}(0_{\mathbb{N}}))}$$

### 3.3 Addition

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) : \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) \equiv m : \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) \equiv \lambda n. \text{succ}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add}_{\mathbb{N}}(m) : \mathbb{N} \rightarrow \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \vdash \text{add}_{\mathbb{N}}(m) \equiv \text{ind}_{\mathbb{N}}(\text{add-zero}_{\mathbb{N}}(m), \text{add-succ}_{\mathbb{N}}(m)) : \mathbb{N} \rightarrow \mathbb{N}}$$

Remark:

$$m + 0_{\mathbb{N}} \equiv \text{add}_{\mathbb{N}}(m, 0_{\mathbb{N}}) \equiv m$$

Remark:

$$m + \text{succ}_{\mathbb{N}}(n) \equiv \text{add}_{\mathbb{N}}(m, \text{succ}_{\mathbb{N}}(n)) \equiv \text{succ}_{\mathbb{N}}(m + n)$$

## 4 Inductive types for...

### 4.1 Negation

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \equiv (A \rightarrow \emptyset) \text{ type}}$$

## 4.2 Coproducts

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A+B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \text{inl} : A \rightarrow A+B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \text{inr} : B \rightarrow A+B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A+B \vdash P(x) \text{ type}}{\Gamma \vdash \text{ind}_+ : (\Pi_{(x:A)} P(\text{inl}(x))) \rightarrow (\Pi_{(y:B)} P(\text{inr}(y))) \rightarrow (\Pi_{(z:A+B)} P(z))}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A+B \vdash P(x) \text{ type}}{\Gamma \vdash \text{ind}_+ : (\Pi_{(x:A)} P(\text{inl}(x))) \rightarrow (\Pi_{(y:B)} P(\text{inr}(y))) \rightarrow (\Pi_{(z:A+B)} P(z))}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A+B \vdash P(x) \text{ type}}{\Gamma \vdash \text{ind}_+ \equiv \lambda f. \lambda g. \lambda \text{inl}(x). f(x) : (\Pi_{(x:A)} P(\text{inl}(x))) \rightarrow (\Pi_{(y:B)} P(\text{inr}(y))) \rightarrow (\Pi_{(z:A+B)} P(z))}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A+B \vdash P(x) \text{ type}}{\Gamma \vdash \text{ind}_+ \equiv \lambda f. \lambda g. \lambda \text{inr}(y). g(y) : (\Pi_{(x:A)} P(\text{inl}(x))) \rightarrow (\Pi_{(y:B)} P(\text{inr}(y))) \rightarrow (\Pi_{(z:A+B)} P(z))}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma \vdash X \text{ type}}{\Gamma \vdash \text{ind}_+ : (A \rightarrow X) \rightarrow (B \rightarrow X) \rightarrow (A+B \rightarrow X)}$$



Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow A' \quad \Gamma \vdash g : B \rightarrow B'}{\Gamma \vdash f + g : A + B \rightarrow A' + B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow A' \quad \Gamma \vdash g : B \rightarrow B'}{\Gamma \vdash f + g \equiv \lambda \text{inl}(x). \text{inl}(f(x)) : A + B \rightarrow A' + B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \rightarrow A' \quad \Gamma \vdash g : B \rightarrow B'}{\Gamma \vdash f + g \equiv \lambda \text{inr}(y). \text{inr}(g(y)) : A + B \rightarrow A' + B'}$$

### 4.3 $\Sigma$ types

#### 4.3.1 Dependent pairs

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \Sigma_{(x:A)} B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_1 : (\Sigma_{(x:A)} B(x)) \rightarrow A}$$

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_1 \equiv \lambda(x, y). x : (\Sigma_{(x:A)} B(x)) \rightarrow A}$$

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_2 : \Pi_{(p:\Sigma_{(x:A)} B(x))} B(\text{pr}_1(p))}$$

MLTT rule:

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_2 \equiv \lambda(x, y). y : \Pi_{(p:\Sigma_{(x:A)} B(x))} B(\text{pr}_1(p))}$$

TODO many rules about dependent pairs are missing (hopefully, none of them is needed for the homework).

### 4.3.2 Ordinary pairs

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \equiv \Sigma_{(x:A)} B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A, y:B \vdash P(x, y) \text{ type}}{\Gamma \vdash \text{ind}_\times : (\Pi_{(x:A)} \Pi_{(y:B)} P(x, y)) \rightarrow (\Pi_{(z:A \times B)} P(z))}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type} \quad \Gamma, x:A, y:B \vdash P(x, y) \text{ type}}{\Gamma \vdash \text{ind}_\times \equiv \lambda g. \lambda(x, y). g(x, y) : (\Pi_{(x:A)} \Pi_{(y:B)} P(x, y)) \rightarrow (\Pi_{(z:A \times B)} P(z))}$$