1 Inference rules

1.1 Formation of contexts, types, and their elements

MLTT rule:

$$\frac{\Gamma,\, x:A \ \vdash\ B(x)\ \text{type}}{\Gamma \ \vdash\ A\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ A \equiv B\ \text{type}}{\Gamma \ \vdash\ A\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ A \equiv B\ \text{type}}{\Gamma \ \vdash\ B\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ a:A}{\Gamma \ \vdash\ A\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ a \equiv b:A}{\Gamma \ \vdash\ a:A}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ a \equiv b:A}{\Gamma \ \vdash\ b:A}$$

1.2 Judgemental equality is an equivalence relation

MLTT rule:

$$\frac{\Gamma \ \vdash\ A\ \text{type}}{\Gamma \ \vdash\ A \equiv A\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ A \equiv B\ \text{type}}{\Gamma \ \vdash\ B \equiv A\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash\ A \equiv B\ \text{type} \qquad \Gamma \ \vdash\ B \equiv C\ \text{type}}{\Gamma \ \vdash\ A \equiv C\ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

1.3 Variable conversion

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash B(x) \text{ type}}{\Gamma, x : A', \Delta \vdash B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : A', \Delta \vdash \mathcal{J}}$$

Derived rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'}$$

1.4 Substitution

Metasyntactic notation (used in many places):

$$t[a/x] := t \text{ with all occurences of } x \text{ replaced by } a$$

MLTT rule:

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\ S$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \equiv B[a'/x] \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv b[a'/x] : B[a/x]}$$

1.5 Weakening

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x\!:\!A, \Delta \vdash \mathcal{J}}\, W$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma, x\!:\!A \vdash B \text{ type}}\, W$$

1.6 Generic elements

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x\!:\!A \vdash x\!:\!A}\, \delta$$

1.7 Manipulating variables

Derived rule:

$$\frac{\Gamma, x\!:\!A, \Delta \vdash \mathcal{J}}{\Gamma, x'\!:\!A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}\, x'/x$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x\!:\!A, y\!:\!B, \Delta \vdash \mathcal{J}}{\Gamma, y\!:\!B, x\!:\!A, \Delta \vdash \mathcal{J}}$$

2 Π types

2.1 Dependent functions

2.1.1 Π formation

MLTT rule:

$$\frac{\Gamma, x\!:\!A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \text{ type}}\, \Pi$$

MLTT rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x\!:\!A \vdash B(x) \equiv B'(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \equiv \Pi_{(x:A')} B'(x) \text{ type}}\, \Pi\text{-eq}$$

2.1.2 Π introduction

MLTT rule:

$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x) : B(x)}{\Gamma\ \vdash\ \lambda x.b(x) : \Pi_{(x:A)}B(x)}\ \lambda$$

MLTT rule:

$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x) \equiv b'(x) : B(x)}{\Gamma\ \vdash\ \lambda x.b(x)\ \equiv\ \lambda x.b'(x) : \Pi_{(x:A)}B(x)}\ \lambda\text{-eq}$$

2.1.3 Π elimination

MLTT rule:

$$\frac{\Gamma\ \vdash\ f : \Pi_{(x:A)}B(x)}{\Gamma,\ x\!:\!A\ \vdash\ f(x) : B(x)}\ \text{ev}$$

MLTT rule:

$$\frac{\Gamma\ \vdash\ f \equiv f' : \Pi_{(x:A)}B(x)}{\Gamma,\ x\!:\!A\ \vdash\ f(x) \equiv f'(x) : B(x)}\ \text{ev-eq}$$

2.1.4 Π computation

MLTT rule:

$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x) : B(x)}{\Gamma,\ x\!:\!A\ \vdash\ (\lambda y.b(y))(x) \equiv b(x) : B(x)}\ \beta$$

MLTT rule:

$$\frac{\Gamma\ \vdash\ f : \Pi_{(x:A)}B(x)}{\Gamma\ \vdash\ \lambda x.f(x)\ \equiv\ f : \Pi_{(x:A)}B(x)}\ \eta$$

2.2 Ordinary functions

2.2.1 Basic

MLTT rule:

$$\frac{\Gamma\ \vdash\ A\ \text{type} \qquad \Gamma\ \vdash\ B\ \text{type}}{\Gamma\ \vdash\ A \to B\ \text{type}}\ \to$$

MLTT rule:

$$\frac{\Gamma\ \vdash\ A\ \text{type} \qquad \Gamma\ \vdash\ B\ \text{type}}{\Gamma\ \vdash\ A \to B \equiv \Pi_{(x:A)}B\ \text{type}}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x{:}A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : A \to B}\ \lambda$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x{:}A \vdash f(x) : B}\ \text{ev}$$

Derived rule:

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x{:}A \vdash b(x) : B}{\Gamma, x{:}A \vdash (\lambda y.b(y))(x) \equiv b(x) : B}\ \beta$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.f(x) \equiv f : A \to B}\ \eta$$

2.2.2 Identity function

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{id}_A : A \to A}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{id}_A \equiv \lambda x.x : A \to A}$$

2.2.3 Composition

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash C \text{ type}}{\Gamma \vdash \mathrm{comp} : (B \to C) \to (A \to B) \to (A \to C)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash C \text{ type}}{\Gamma \vdash \mathrm{comp} \equiv \lambda g.\lambda f.\lambda x.g(f(x)) : (B \to C) \to (A \to B) \to (A \to C)}$$

Convention:

$$g \circ f \quad := \quad \mathrm{comp}(g, f) \equiv \lambda x.g(f(x))$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to C \qquad \Gamma \vdash h : C \to D}{\Gamma \vdash (h \circ g) \circ f \equiv h \circ (g \circ f) : A \to D}$$

3 The natural numbers

3.1 Basic

MLTT rule:

$$\frac{}{\vdash \mathbb{N} \text{ type}} \ \mathbb{N}\text{-form}$$

MLTT rule:

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}}$$

MLTT rule:

$$\frac{}{\vdash \mathrm{succ}_{\mathbb{N}} : \mathbb{N}}$$

3.2 Induction principle

Convention:

$$f(x, y) \quad := \quad f(x)(y) \ \equiv \ (f(x))(y)$$

Metasyntactic shortcut (only for this document):

$$\text{Ⓟ} \quad := \quad \Gamma,\, n : \mathbb{N} \ \vdash \ P(n) \text{ type}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \ \vdash \ p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \ \vdash \ p_S : \Pi_{(n:\mathbb{N})}\, (P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma \ \vdash \ \mathrm{ind}_{\mathbb{N}}(p_0, p_S) : \Pi_{(n:\mathbb{N})} P(n)} \ \mathbb{N}\text{-ind}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \ \vdash \ p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \ \vdash \ p_S : \Pi_{(n:\mathbb{N})}\, (P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma \ \vdash \ \mathrm{ind}_{\mathbb{N}}(p_0, p_S, 0_{\mathbb{N}}) \equiv p_0 : P(0_{\mathbb{N}})}$$

MLTT rule:

$$\frac{\text{Ⓟ} \qquad \Gamma \ \vdash \ p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \ \vdash \ p_S : \Pi_{(n:\mathbb{N})}\, (P(n) \to P(\mathrm{succ}_{\mathbb{N}}(n)))}{\Gamma,\, n : \mathbb{N} \ \vdash \ \mathrm{ind}_{\mathbb{N}}(p_0, p_S, \mathrm{succ}_{\mathbb{N}}(n)) \equiv p_S(n, \mathrm{ind}_{\mathbb{N}}(p_0, p_S, n)) : P(\mathrm{succ}_{\mathbb{N}}(n))}$$

3.3 Addition

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) : \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-zero}_{\mathbb{N}}(m) \equiv m : \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add-succ}_{\mathbb{N}}(m) \equiv \lambda n.\, \text{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\frac{}{m : \mathbb{N} \vdash \text{add}_{\mathbb{N}}(m) \equiv \text{ind}_{\mathbb{N}}(\text{add-zero}_{\mathbb{N}}(m), \text{add-succ}_{\mathbb{N}}(m)) : \mathbb{N} \to \mathbb{N}}$$

Remark:

$$m + 0_{\mathbb{N}} \equiv \text{add}_{\mathbb{N}}(m, 0_{\mathbb{N}}) \equiv m$$

Remark:

$$m + \text{succ}_{\mathbb{N}}(n) \equiv \text{add}_{\mathbb{N}}(m, \text{succ}_{\mathbb{N}}(n)) \equiv \text{succ}_{\mathbb{N}}(m + n)$$

4 Inductive types for...

4.1 Negation

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \equiv (A \to \emptyset) \text{ type}}$$

4.2 Coproducts

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A{+}B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{inl} : A \to A{+}B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{inr} : B \to A{+}B}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A{+}B \vdash P(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{+} : \big(\Pi_{(x:A)} P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)} P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A{+}B \vdash P(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{+} : \big(\Pi_{(x:A)} P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)} P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A{+}B \vdash P(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{+} \equiv \lambda f.\lambda g.\lambda\, \mathrm{inl}(x).\, f(x) : \big(\Pi_{(x:A)} P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)} P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A{+}B \vdash P(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{+} \equiv \lambda f.\lambda g.\lambda\, \mathrm{inr}(y).\, g(y) : \big(\Pi_{(x:A)} P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)} P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)} P(z)\big)}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash X \text{ type}}{\Gamma \vdash \mathrm{ind}_{+} : (A \to X) \to (B \to Y) \to (A{+}B \to X)}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f + g : A + B \to A' + B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f + g \equiv \lambda \, \mathrm{inl}(x) \, . \, \mathrm{inl}(f(x)) : A + B \to A' + B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f + g \equiv \lambda \, \mathrm{inr}(y) \, . \, \mathrm{inr}(g(y)) : A + B \to A' + B'}$$

4.3 Σ types

4.3.1 Dependent pairs

MLTT rule:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \Sigma_{(x:A)} B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{pr}_1 : \big(\Sigma_{(x:A)} B(x)\big) \to A}$$

MLTT rule:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{pr}_1 \equiv \lambda(x, y) . x : \big(\Sigma_{(x:A)} B(x)\big) \to A}$$

MLTT rule:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{pr}_2 : \Pi_{\left(p \,:\, \Sigma_{(x:A)} B(x)\right)} B(\mathrm{pr}_1(p))}$$

MLTT rule:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{pr}_2 \equiv \lambda(x, y) . y : \Pi_{\left(p \,:\, \Sigma_{(x:A)} B(x)\right)} B(\mathrm{pr}_1(p))}$$

Disclaimer: Many rules about dependent pairs are missing (hopefully, none of them is needed for the homework.

4.3.2 Ordinary pairs

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \equiv \Sigma_{(x:A)} B \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A,\, y : B \vdash P(x,y) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\times} : \big(\Pi_{(x:A)} \Pi_{(y:B)} P(x,y)\big) \to \big(\Pi_{(z:A \times B)} P(z)\big)}$$

MLTT rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma,\, x : A,\, y : B \vdash P(x,y) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\times} \equiv \lambda g . \lambda (x,y) . g(x,y) : \big(\Pi_{(x:A)} \Pi_{(y:B)} P(x,y)\big) \to \big(\Pi_{(z:A \times B)} P(z)\big)}$$