1 Inference rules

1.1 Formation of contexts, types, and their elements

MLTT rule:
$$\frac{\Gamma,\, x\!:\!A \;\vdash\; B(x) \text{ type}}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; B \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; a\!:\!A}{\Gamma \;\vdash\; A \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; a \equiv b\!:\!A}{\Gamma \;\vdash\; a\!:\!A}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; a \equiv b\!:\!A}{\Gamma \;\vdash\; b\!:\!A}$$

1.2 Judgemental equality is an equivalence relation

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; A \equiv A \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type}}{\Gamma \;\vdash\; B \equiv A \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \equiv B \text{ type} \qquad \Gamma \;\vdash\; B \equiv C \text{ type}}{\Gamma \;\vdash\; A \equiv C \text{ type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a : A}{\Gamma \ \vdash \ a \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a \equiv b : A}{\Gamma \ \vdash \ b \equiv a : A}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a \equiv b : A \qquad \Gamma \ \vdash \ b \equiv c : A}{\Gamma \ \vdash \ a \equiv c : A}$$

1.3 Variable conversion

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \equiv A' \ \text{type} \qquad \Gamma,\ x : A,\ \Delta \ \vdash \ B(x) \ \text{type}}{\Gamma,\ x : A',\ \Delta \ \vdash \ B(x) \ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \equiv A' \ \text{type} \qquad \Gamma,\ x : A,\ \Delta \vdash \mathcal{J}}{\Gamma,\ x : A',\ \Delta \vdash \mathcal{J}}$$

Derived rule:

$$\frac{\Gamma \ \vdash \ A \equiv A' \ \text{type} \qquad \Gamma \ \vdash \ a : A}{\Gamma \ \vdash \ a : A'}$$

1.4 Substitution

Metasyntactic notation (used in many places):

$$t[a/x] := t \text{ with all occurences of } x \text{ replaced by } a$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a : A \qquad \Gamma,\ x : A,\ \Delta \vdash \mathcal{J}}{\Gamma,\ \Delta[a/x] \ \vdash \ \mathcal{J}[a/x]} \ S$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a \equiv a' : A \qquad \Gamma,\ x : A,\ \Delta \ \vdash \ B \ \text{type}}{\Gamma,\ \Delta[a/x] \ \vdash \ B[a/x] \equiv B[a'/x] \ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ a \equiv a' : A \qquad \Gamma,\ x : A,\ \Delta \ \vdash \ b : B}{\Gamma,\ \Delta[a/x] \ \vdash \ b[a/x] \equiv b[a'/x] : B[a/x]}$$

1.5   Weakening

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \ \text{type} \qquad \Gamma, \Delta \ \vdash \ \mathcal{J}}{\Gamma, \ x{:}A, \ \Delta \ \vdash \ \mathcal{J}} \ W$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \ \text{type} \qquad \Gamma \ \vdash \ B \ \text{type}}{\Gamma, \ x{:}A \ \vdash \ B \ \text{type}} \ W$$

1.6   Generic elements

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \ \text{type}}{\Gamma, \ x{:}A \ \vdash \ x{:}A} \ \delta$$

1.7   Manipulating variables

Derived rule:

$$\frac{\Gamma, \ x{:}A, \ \Delta \ \vdash \ \mathcal{J}}{\Gamma, \ x'{:}A, \ \Delta[x'/x] \ \vdash \ \mathcal{J}[x'/x]} \ x'/x$$

Derived rule:

$$\frac{\Gamma \ \vdash \ B \ \text{type} \qquad \Gamma, \ x{:}A, \ y{:}B, \ \Delta \ \vdash \ \mathcal{J}}{\Gamma, \ y{:}B, \ x{:}A, \ \Delta \ \vdash \ \mathcal{J}}$$

2 Π types

2.1   Dependent functions

2.1.1 Π formation

MLTT rule:

$$\frac{\Gamma, \ x{:}A \ \vdash \ B(x) \ \text{type}}{\Gamma \ \vdash \ \Pi_{(x:A)}B(x) \ \text{type}} \ \Pi$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \equiv A' \ \text{type} \qquad \Gamma, \ x{:}A \ \vdash \ B(x) \equiv B'(x) \ \text{type}}{\Gamma \ \vdash \ \Pi_{(x:A)}B(x) \equiv \Pi_{(x:A')}B'(x) \ \text{type}} \ \Pi\text{-eq}$$

2.1.2 Π introduction

MLTT rule:
$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x):B(x)}{\Gamma\ \vdash\ \lambda x.b(x):\Pi_{(x:A)}B(x)}\ \lambda$$

MLTT rule:
$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x)\equiv b'(x):B(x)}{\Gamma\ \vdash\ \lambda x.b(x)\ \equiv\ \lambda x.b'(x):\Pi_{(x:A)}B(x)}\ \lambda\text{-eq}$$

2.1.3 Π elimination

MLTT rule:
$$\frac{\Gamma\ \vdash\ f:\Pi_{(x:A)}B(x)}{\Gamma,\ x\!:\!A\ \vdash\ f(x):B(x)}\ \text{ev}$$

MLTT rule:
$$\frac{\Gamma\ \vdash\ f\equiv f':\Pi_{(x:A)}B(x)}{\Gamma,\ x\!:\!A\ \vdash\ f(x)\equiv f'(x):B(x)}\ \text{ev-eq}$$

2.1.4 Π computation

MLTT rule:
$$\frac{\Gamma,\ x\!:\!A\ \vdash\ b(x):B(x)}{\Gamma,\ x\!:\!A\ \vdash\ (\lambda y.b(y))(x)\equiv b(x):B(x)}\ \beta$$

MLTT rule:
$$\frac{\Gamma\ \vdash\ f:\Pi_{(x:A)}B(x)}{\Gamma\ \vdash\ \lambda x.f(x)\ \equiv\ f:\Pi_{(x:A)}B(x)}\ \eta$$

2.2 Ordinary functions

2.2.1 Basic

MLTT rule:
$$\frac{\Gamma\ \vdash\ A\ \text{type}\qquad \Gamma\ \vdash\ B\ \text{type}}{\Gamma\ \vdash\ A\!\to\!B\ \text{type}}\ \to$$

MLTT rule:
$$\frac{\Gamma\ \vdash\ A\ \text{type}\qquad \Gamma\ \vdash\ B\ \text{type}}{\Gamma\ \vdash\ A\!\to\!B\equiv\Pi_{(x:A)}B\ \text{type}}$$

Derived rule:

$$\frac{\Gamma \;\vdash\; B \text{ type} \qquad \Gamma,\, x\!:\!A \;\vdash\; b(x) : B(x)}{\Gamma \;\vdash\; \lambda x.b(x) : A \to B} \; \lambda$$

Derived rule:

$$\frac{\Gamma \;\vdash\; f : A \to B}{\Gamma,\, x\!:\!A \;\vdash\; f(x) : B} \; \text{ev}$$

Derived rule:

$$\frac{\Gamma \;\vdash\; B \text{ type} \qquad \Gamma,\, x\!:\!A \;\vdash\; b(x) : B}{\Gamma,\, x\!:\!A \;\vdash\; (\lambda y.b(y))(x) \equiv b(x) : B} \; \beta$$

Derived rule:

$$\frac{\Gamma \;\vdash\; f : A \to B}{\Gamma \;\vdash\; \lambda x.f(x) \equiv f : A \to B} \; \eta$$

2.2.2 Identity function

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type}}{\Gamma \;\vdash\; \text{id}_A : A \to A}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; \text{id}_A \equiv \lambda x.x : A \to A}$$

2.2.3 Function composition

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma \;\vdash\; C \text{ type}}{\Gamma \;\vdash\; \text{comp} : (B \to C) \to (A \to B) \to (A \to C)}$$

MLTT rule:

$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma \;\vdash\; C \text{ type}}{\Gamma \;\vdash\; \text{comp} \equiv \lambda g.\lambda f.\lambda x.g(f(x)) : (B \to C) \to (A \to B) \to (A \to C)}$$

Convention:

$$g \circ f \quad := \quad \text{comp}(g, f) \equiv \lambda x.g(f(x))$$

Derived rule:

$$\frac{\Gamma \;\vdash\; f : A \to B \qquad \Gamma \;\vdash\; g : B \to C \qquad \Gamma \;\vdash\; h : C \to D}{\Gamma \;\vdash\; (h \circ g) \circ f \equiv h \circ (g \circ f) : A \to D}$$

3 The natural numbers

3.1 Basic

MLTT rule:

$$\frac{}{\vdash \mathbb{N} \text{ type}} \; \mathbb{N}\text{-form}$$

MLTT rule:

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}}$$

MLTT rule:

$$\frac{}{\vdash \text{succ}_{\mathbb{N}} : \mathbb{N}}$$

3.2 Induction principle

Convention (used in many places):

$$f(x, y) \quad := \quad f(x)(y) \; \equiv \; (f(x))(y)$$

MLTT rule:

$$\frac{\Gamma, \, n : \mathbb{N} \; \vdash \; P(n) \text{ type} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n : \mathbb{N})} \, (P(n) \to P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_S) : \Pi_{(n : \mathbb{N})} P(n)} \; \mathbb{N}\text{-ind}$$

MLTT rule:

$$\frac{\Gamma, \, n : \mathbb{N} \; \vdash \; P(n) \text{ type} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n : \mathbb{N})} \, (P(n) \to P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_S, 0_{\mathbb{N}}) \equiv p_0 : P(0_{\mathbb{N}})}$$

MLTT rule:

$$\frac{\Gamma, \, n : \mathbb{N} \; \vdash \; P(n) \text{ type} \qquad \Gamma \vdash p_0 : P(0_{\mathbb{N}}) \qquad \Gamma \vdash p_S : \Pi_{(n : \mathbb{N})} \, (P(n) \to P(\text{succ}_{\mathbb{N}}(n)))}{\Gamma, \, n : \mathbb{N} \; \vdash \; \text{ind}_{\mathbb{N}}(p_0, p_S, \text{succ}_{\mathbb{N}}(n)) \equiv p_S(n, \text{ind}_{\mathbb{N}}(p_0, p_S, n)) : P(\text{succ}_{\mathbb{N}}(n))}$$

3.3 Addition

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add-zero}_{\mathbb{N}}(m) : \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add-zero}_{\mathbb{N}}(m) \equiv m : \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add-succ}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add-succ}_{\mathbb{N}}(m) \equiv \lambda n.\, \text{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add}_{\mathbb{N}}(m) : \mathbb{N} \to \mathbb{N}}$$

MLTT rule:

$$\overline{m : \mathbb{N} \ \vdash \ \text{add}_{\mathbb{N}}(m) \equiv \text{ind}_{\mathbb{N}}(\text{add-zero}_{\mathbb{N}}(m), \text{add-succ}_{\mathbb{N}}(m)) : \mathbb{N} \to \mathbb{N}}$$

Remark:

$$m + 0_{\mathbb{N}} \equiv \text{add}_{\mathbb{N}}(m, 0_{\mathbb{N}}) \equiv m$$

Remark:

$$m + \text{succ}_{\mathbb{N}}(n) \equiv \text{add}_{\mathbb{N}}(m, \text{succ}_{\mathbb{N}}(n)) \equiv \text{succ}_{\mathbb{N}}(m + n)$$

4 Inductive types for. . .

4.1 Negation

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \ \text{type}}{\Gamma \ \vdash \ \neg A \ \text{type}}$$

MLTT rule:

$$\frac{\Gamma \ \vdash \ A \ \text{type}}{\Gamma \ \vdash \ \neg A \equiv (A \to \emptyset) \ \text{type}}$$

4.2 Coproducts

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; A{+}B \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; \mathrm{inl} : A \to A{+}B}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; \mathrm{inr} : B \to A{+}B}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma, x : A{+}B \vdash P(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_+ : \big(\Pi_{(x:A)}P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)}P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)}P(z)\big)}$$

MLTT rule:
$$\frac{\begin{array}{c}\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma, z : A{+}B \;\vdash\; P(z) \text{ type} \\ \Gamma \;\vdash\; f : \Pi_{(x:A)}P(\mathrm{inl}(x)) \qquad \Gamma \;\vdash\; g : \Pi_{(y:B)}P(\mathrm{inr}(y)) \qquad \Gamma \;\vdash\; a : A\end{array}}{\Gamma \;\vdash\; \mathrm{ind}_+(f,g,\mathrm{inl}(a)) \equiv f(a) : P(\mathrm{inl}(a))}$$

MLTT rule?
$$\frac{\begin{array}{c}\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma, z : A{+}B \;\vdash\; P(z) \text{ type} \\ \Gamma \;\vdash\; f : \Pi_{(x:A)}P(\mathrm{inl}(x)) \qquad \Gamma \;\vdash\; g : \Pi_{(y:B)}P(\mathrm{inr}(y))\end{array}}{\Gamma, a : A \;\vdash\; \mathrm{ind}_+(f,g,\mathrm{inl}(a)) \equiv f(a) : P(\mathrm{inl}(a))}$$

MLTT rule!
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma, z : A{+}B \;\vdash\; P(z) \text{ type}}{\Gamma, f : \Pi_{(x:A)}P(\mathrm{inl}(x)),\, g : \Pi_{(y:B)}P(\mathrm{inr}(y)),\, a : A \;\vdash\; \mathrm{ind}_+(f,g,\mathrm{inl}(a)) \equiv f(a) : P(\mathrm{inl}(a))}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma, x : A{+}B \;\vdash\; P(x) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_+ \equiv \lambda f.\lambda g.\lambda\,\mathrm{inr}(y).g(y) : \big(\Pi_{(x:A)}P(\mathrm{inl}(x))\big) \to \big(\Pi_{(y:B)}P(\mathrm{inr}(y))\big) \to \big(\Pi_{(z:A+B)}P(z)\big)}$$

Derived rule:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash X \text{ type}}{\Gamma \vdash \text{ind}_+ : (A \to X) \to (B \to Y) \to (A+B \to X)}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f+g : A+B \to A'+B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f+g \equiv \lambda \,\text{inl}(x)\,.\,\text{inl}(f(x)) : A+B \to A'+B'}$$

Derived rule:

$$\frac{\Gamma \vdash f : A \to A' \qquad \Gamma \vdash g : B \to B'}{\Gamma \vdash f+g \equiv \lambda \,\text{inr}(y)\,.\,\text{inr}(g(y)) : A+B \to A'+B'}$$

4.3 Σ types

4.3.1   Dependent pairs

MLTT rule:

$$\frac{\Gamma,\, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \Sigma_{(x:A)}B(x) \text{ type}}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_1 : \big(\Sigma_{(x:A)}B(x)\big) \to A}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_1 \equiv \lambda(x,y).x : \big(\Sigma_{(x:A)}B(x)\big) \to A}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_2 : \Pi_{\big(p\,:\,\Sigma_{(x:A)}B(x)\big)}B(\text{pr}_1(p))}$$

MLTT rule:

$$\frac{\Gamma,\, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \text{pr}_2 \equiv \lambda(x,y).y : \Pi_{\big(p\,:\,\Sigma_{(x:A)}B(x)\big)}B(\text{pr}_1(p))}$$

Disclaimer: Many rules about dependent pairs are missing (hopefully, none of them is needed for the homework.

4.3.2 Ordinary pairs

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; A \times B \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type}}{\Gamma \;\vdash\; A \times B \equiv \Sigma_{(x:A)} B \text{ type}}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma,\, x\!:\!A,\, y\!:\!B \;\vdash\; P(x,y) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_\times : \big(\Pi_{(x:A)} \Pi_{(y:B)} P(x,y)\big) \to \big(\Pi_{(z:A \times B)} P(z)\big)}$$

MLTT rule:
$$\frac{\Gamma \;\vdash\; A \text{ type} \qquad \Gamma \;\vdash\; B \text{ type} \qquad \Gamma,\, x\!:\!A,\, y\!:\!B \;\vdash\; P(x,y) \text{ type}}{\Gamma \;\vdash\; \mathrm{ind}_\times \equiv \lambda g.\lambda(x,y).g(x,y) : \big(\Pi_{(x:A)} \Pi_{(y:B)} P(x,y)\big) \to \big(\Pi_{(z:A \times B)} P(z)\big)}$$