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A problem of random accelerations

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The partial differential equation $yu_x + u_{yy} = 0$ is solved in a vertical strip. The method is first to solve the equation in the upper and lower halves of the strip. By matching u and u_y across the x-axis a singular integral equation for $u_x(x, 0)$ is obtained. This is converted into an equation with Cauchy kernel whose solutions are explicitly known.

1

In this report we consider the linear partial differential equation

$$yu_x + u_{yy} = 0 \tag{1.1}$$

in a vertical strip $0 < x < 1$, $-\infty < y < \infty$, together with the boundary data

$$\begin{aligned} u(0, y) &= U_0(y) \quad \text{if } y < 0, \\ u(1, y) &= U_1(y) \quad \text{if } y > 0. \end{aligned} \tag{1.2}$$

This problem arises in the study of a randomly accelerated particle moving in the interval $(0, 1)$. Let $\xi(t)$ be the position of the particle at time t, and t_1 the first time when $\xi(t) = 0$ or 1. The velocity $v(t)$ is given by a Brownian motion process,

$$E\{\Delta v\} = 0, \quad E\{(\Delta v)^2\} = \sigma \Delta t,$$

where $E\{\ \}$ denotes expected value and we normalize by taking $\sigma^2 = 2$. Equation (1.1) is the steady state form of the backward equation for the vector Markov process $(\xi(t), v(t))$. The probabilistic interpretation of $u(x, y)$ is as a conditional expected value:

$$u(x, y) = E\{U_2[v(t_1)] | \xi(0) = x, v(0) = y\}, \tag{1.3}$$

where $U_2(y) = U_0(y)$ if $y < 0$, $U_2(y) = U_1(y)$ if $y > 0$.

Our object is to give a fairly explicit formula for $u(x, y)$. The method is first to solve (1.1) separately in the upper and lower halves of the strip. This is done in §'s 2 and 3 using Fourier transforms and constructing Green's functions. By matching u and u_y across the x-axis a singular integral equation for $u_x(x, 0)$ is obtained in §4. This is converted into an integral equation of the second kind with Cauchy kernel whose solutions are known explicitly. In §5 a uniqueness theorem is proved. By probabilistic methods it can be shown that the solution u of (1.1)-(1.2) which we get actually is given by (1.3). However, this is not done here.

This report was initiated by a conversation at the Mathematics Research Center between S. Agmon and the author. Agmon later made several more helpful suggestions, and in particular pointed out that the integral equation (4.3) can be reduced to an equation of the second kind with Cauchy kernel.

2

In this section, we shall consider the equation

$$-yv_x + v_{yy} = 0 \tag{2.1}$$

in the upper half-plane $y > 0$, with the boundary data

$$v(x, 0) = \psi(x), \quad -\infty < x < \infty. \tag{2.2}$$

It is assumed that ψ is continuous with compact support, that $\psi(x) = 0$ for $x \leq 0$, and that ψ' is continuous except at 0 and 1. Let $\hat{v}$ be the Fourier transform in the variable x:

$$\hat{v}(\tau, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x, y) e^{i\tau x} \, \mathrm{d}x, \quad y > 0.$$

This changes (2.1) into the ordinary differential equation

$$\hat{v}_{yy} + i\tau y \hat{v} = 0 \tag{2.3}$$

whose solutions are given in terms of Bessel functions of orders $\pm 1/3$ as follows. Let

$$z = \frac{2}{3} \left(i\tau y^3 \right)^{\frac{1}{2}}.$$

Occasionally we must consider not merely real τ but complex $\tau = \tau_1 + i\tau_2$ in the upper half-plane. Let us agree that

$$\frac{\pi}{2} \leq \arg i\tau \leq \frac{3\pi}{2}, \quad \arg(i\tau a)^{\lambda} = \lambda \arg i\tau a$$

for any $a > 0$ and λ. In particular, z belongs to the quadrant $Q : \pi/4 \leq \arg z \leq 3\pi/4$. The general solution of (2.3) is

$$\hat{v} = (i\tau y)^{\frac{1}{2}} \left[A(\tau) J_{-\frac{1}{3}}(z) + B(\tau) J_{\frac{1}{3}}(z) \right]$$

We want a solution of (2.1) which is 0 for $x \leq 0$ and tends rapidly to 0 as $x \to \infty$. Therefore, we choose the coefficients $A(\tau)$ and $B(\tau)$ so that

$$\hat{v}(\tau, y) = \hat{\psi}(\tau) H(z), \tag{2.4}$$

where the function H is defined as follows. For $z \in Q$

$$J_n(z) = \frac{1}{(2\pi z)^{\frac{1}{2}}} \left\{ e^{c\pi i} W_{0,n}(2iz) + e^{-c\pi i} W_{0,n}(-2iz) \right\},$$

where $c = \frac{1}{2}\left(n + \frac{1}{2}\right)$ and $W_{0,n} = W_{0,-n}$ is Whittaker's funciton, which has the asymptotic expansion as $z \to \infty$ in Q

$$W_{0,n}(2iz) = e^{-iz} \left\{ 1 + \sum_{j=1}^{m} a_j z^{-j} + O(z^{-m}) \right\}, \tag{2.5}$$

$$j!(8i)^j a_j = \prod_{l=1}^{j} \left\{ 4n^2 - (2l-1)^2 \right\}.$$

See [WW, p. 362]. Writing $W = W_{0,\frac{1}{3}}$ for short,

$$J_{-\frac{1}{3}}(z) + e^{\frac{2\pi i}{3}} J_{\frac{1}{3}}(z) = \frac{e^{\frac{i\pi}{4}} + e^{-\frac{i\pi}{12}}}{(2\pi z)^{\frac{1}{2}}} W(-2iz).$$

Let

$$H(z) = C_1 z^{-\frac{1}{6}} W(-2iz), \tag{2.6}$$

where the constant C_1 is chosen so that $H(0) = 1$. Then $H(z)$ tends exponentially to 0 as $z \to \infty$ in Q, and since H is analytic the same is true of each derivative $H^{(k)}(z)$. The function H has the convergent expansion about 0

$$H(z) = 1 + \text{power series in } z^2 + C_2 z^{\frac{2}{3}} (\text{power series in } z^2).$$

Since $\hat{v}$ is a product in (2.4), v is the convolution (in x) of ψ and the function G whose transform $\hat{G} = H$. By the inversion formula,

$$G(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(z) e^{-ix\tau} \, d\tau, \quad y > 0. \tag{2.7}$$

The function G is of class C^∞ and satisfies (2.1).

Since $|H(z)| \leq C_3 e^{-|z|}$ for every $z \in Q$, $\int_{-\infty}^{\infty} |H|^2 \, d\tau_1$ is bounded independent of $\tau_2 \geq 0$ (the bound depends on y). Therefore [PW, p. 8] $G(x, y) = 0$ for $x < 0$. Using the formula $(2\pi)^{-\frac{1}{2}} \widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2$,

$$v(x, y) = \frac{1}{\sqrt{2\pi}} \int_0^x \psi(\xi) G(x - \xi, y) \, d\xi, \quad y > 0. \tag{2.8}$$

The integral is from 0 to x since both ψ and G vanish for $x < 0$. The function v is of class C^∞, satisfies (2.1), and $v(x, y) = 0$ for $x \le 0$. It remains to verify (2.2) and find a formula for the normal derivative $v_y(x, 0^+)$. Let

$$g(x) = \frac{1}{\sqrt{2\pi}} G(x, 1). \tag{2.9}$$

The substitution $\sigma = \tau y^3$ in (2.7) shows that

$$y^{-3} g(y^{-3} x) = \frac{1}{\sqrt{2\pi}} G(x, y).$$

If we set $g_a(x) = a^{-1} g(a^{-1} x)$, then

$$v(x, y) = \psi * g_a, \quad a = y^3. \tag{2.8'}$$

The function g is bounded. Setting $y = 1$ in (2.7) and integrating by parts,

$$x g(x) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_\tau e^{-i\tau x} \,\mathrm{d}\tau.$$

[Unless otherwise indicated τ is real.] Since $H_\tau = O(\tau^{-\frac{2}{3}})$ as $\tau \to 0$ and H_τ tends rapidly to 0 as $|\tau| \to \infty$, H_τ belongs to $L^p(-\infty, \infty)$ for $1 \le p < 3/2$. Therefore $xg(x)$ is bounded and belongs to $L^q(-\infty, \infty)$ for $q > 3$ [Z, 12.41]. By Hölder's inequality, $g \in L^1(-\infty, \infty)$, and from the inversion formula

$$1 = H(0) = \int_{-\infty}^{\infty} g(x) \,\mathrm{d}x.$$

Thus the functions g_a form an approximate identity as $a \to 0^+$.

Lemma. *Let f be an integrable function with compact support and A an open interval on which f is continuous. Then*

$$f(x) = \lim_{a \to 0^+} (f * g_a)(x)$$

for every $x \in A$. The convergence is uniform on closed subintervals of A and at ∞.

This result is well known. Applying the lemma with $f = \psi$, $v(x, y) \to \psi(x)$ uniformly as $y \to 0^+$. (Actually, $g \ge 0$. Since v is 0 at infinity the maximum principle for parabolic equations shows that $v \ge 0$ for any $\psi \ge 0$, and this implies $g \ge 0$.)

Applying this lemma with $f = \psi'$, $v_x(x, y) \to \psi'(x)$ as $y \to 0^+$, uniformly on any closed interval not containing 0 or 1. In view of (2.1), $v_{yy} \to 0$ uniformly on such closed intervals, and hence $v_y(x, y)$ also tends uniformly on such intervals to a limit $v_y(x, 0^+)$.

Let us now show that

$$v_y(x, 0^+) = C_4 \int_0^x \psi'(\xi)(x - \xi)^{-\frac{1}{3}} \,\mathrm{d}\xi, \quad 0 < x < 1. \tag{2.10}$$

Now $\hat{v}_y = \hat{\psi}\hat{H}_y$ from (2.4), and hence

$$\hat{v}_y(\tau, y) = \left(\frac{2}{3}\right)^{-\frac{1}{3}} \hat{\psi}(\tau)(i\tau)^{\frac{1}{3}} z^{\frac{1}{3}} H'(z), \quad y > 0.$$

From the expansion of $H(z)$ about 0,

$$\lim_{y \to 0^+} \hat{v}_y(\tau, y) = \left(\frac{2}{3}\right)^{\frac{2}{3}} C_2 \hat{\psi}(\tau)(i\tau)^{\frac{1}{3}} \tag{2.11}$$

for every τ. But $z^{\frac{1}{3}} H'(z)$ is bounded; and from the assumptions about ψ, $\hat{\psi}$ is bounded and $\hat{\psi}(\tau) = O(\tau^{-1})$ as $|\tau| \to \infty$. Hence $\int_{-\infty}^{\infty} |\hat{v}_y|^2 \, d\tau$ is bounded and the convergence in (2.11) is also in L^2-norm. By Parseval's formula, $v_y(x, y)$ tends to $v_y(x, 0^+)$ in L^2-norm as $y \to 0^+$.

Let $x_+^a = x^a$ if $x > 0$, and $x_+^a = 0$ if $x < 0$. For $a < -1$, x_+^a is to be interpreted as a generalized function or Schwartz distribution. Its Fourier transform is

$$\widehat{x_+^a} = \Gamma(a)(-i\tau)^{-a-1}.$$

In our case $a = 1/3$. Using a theorem about Fourier transforms of distributions of class $\mathcal{D}'_{Lp}$ [S, p. 57, 126], we obtain from (2.11)

$$v_y(x, 0^+) = C_5 \psi * x_+^{-\frac{4}{3}}$$

But $x_+^{-\frac{4}{3}}$ is the derivative of $-3x_+^{-\frac{1}{3}}$, and since the derivative can be applied to either factor in a convolution,

$$\psi * x_+^{-\frac{4}{3}} = -3\psi' * x_+^{-\frac{1}{3}}.$$

If $0 < x < 1$ the right side is given by the ordinary convolution integral and we have the desired formula (2.10).

3

Let us next find a solution V of (2.1) in the quarter-plane $x > 0$, $y > 0$ with the boundary data

$$\begin{aligned} V(0, y) &= U(y), \quad y > 0, \\ V(x, 0) &= 0, \quad x > 0. \end{aligned} \tag{3.1}$$

The method is to construct a Green's function $F(x - \xi, y, \eta)$ for the operator $L(V) = -yV_x + V_{yy}$ and the upper half-plane. It must satisfy the equation $L(F) = \delta(x - \xi, y - \eta)$ in the variables (x, y) and the adjoint equation $M(F) = \delta(x-\xi, y-\eta)$ in (ξ, η) where δ is Dirac's delta function. Moreover, $F(x, y, \eta) = 0$ for $x \le 0$ and all $y, \eta > 0$ and for $y = 0, \infty$ and all $x, \eta > 0$. From the formula

$$VM(F) - FL(V) = \eta(VF)_\xi + VF_{\eta\eta} - FV_{\eta\eta}$$

and integration by parts, one gets formally

$$V(x,y)=\int_0^\infty \eta F(x,y,\eta)U(\eta)\,\mathrm{d}\eta, \quad y\geq 0. \tag{3.2}$$

Let us proceed to find F and conditions on U such that (3.2) in fact gives a solution to (3.1). Since the method is well known certain details will be merely indicated. The Fourier transform $\hat{F}(\tau,y,\eta)$ must satisfy the transformed equation

$$\hat{F}_{yy}+i\tau y\hat{F}=\delta(y-\eta).$$

$\hat{F}$ must be continuous at $y=\eta$ and

$$\hat{F}_y(\tau,\eta^+,\eta)-\hat{F}_y(\tau,\eta^-,\eta)=1.$$

For each τ the function $H(z)$ is a solution of (2.3) vanishing at $y=\infty$, and

$$K(z)=z^{\frac{1}{3}}J_{\frac{1}{3}}(z)$$

is a solution vanishing at $y=0$. Let us take

$$\begin{aligned}\hat{F}&=a(\tau)K(z)H(\zeta), \quad \text{if } 0\leq y<\eta,\\ \hat{F}&=a(\tau)H(z)K(\zeta), \quad \text{if } 0\leq \eta<y,\end{aligned}$$

where $\zeta=\frac{2}{3}(i\tau\eta^3)^{\frac{1}{2}}$. By a short calculation, using the identity $z(J_{\frac{1}{3}}J'_{-\frac{1}{3}}-J_{-\frac{1}{3}}J'_{\frac{1}{3}})=-\sqrt{3}/\pi$, the two conditions at $y=\eta$ give for a suitable constant c_1

$$\hat{F}=c_1(i\tau)^{-\frac{1}{3}}K(z)H(\zeta), \quad \text{if } 0\leq y<\eta, \tag{3.2'}$$

and $\hat{F}(\tau,\eta,y)=\hat{F}(\tau,y,\eta)$. Equation (3.2') can be rewritten

$$\hat{F}=\left(\frac{2}{3}\right)^{\frac{2}{3}}c_1yz^{-\frac{1}{3}}J_{\frac{1}{3}}(z)K(\zeta), \quad 0\leq y<\eta. \tag{3.2''}$$

From (3.2'') and the asymptotic expansion (2.5),

$$|\hat{F}|\leq c_2y\exp(|z|-|\zeta|), \quad 0\leq y<\eta.$$

The inverse transform F is of class C^∞ in (x,y,η) so as long as $y\neq\eta$ (also for $y=\eta$, $x>0$ from (3.3) below), and F satisfies (2.1) in (x,y). It vanishes for $x\leq 0$, and $F(x,\eta,y)=F(x,y,\eta)$. Moreover, $(xF)_x=i\tau\hat{F}_\tau$ from which together with (3.2'') and the formulas

$$i\tau\frac{\partial z}{\partial\tau}=\frac{i}{2}z, \quad i\tau\frac{\partial\zeta}{\partial\tau}=\frac{i}{2}\zeta$$

we get the estimate

$$\widehat{(xF)}_x\leq c_3y\exp(|z|-|\zeta|), \quad 0\leq y<\eta.$$

If $|y-\eta|$ is bounded away from 0, then F and $(xF)_x$ are bounded. Hence so is $xF_x = (xF)_x - F$.

Let Ω be the fundamental solution of the heat equation:

$$\begin{aligned} \Omega(x,y) &= \frac{1}{\sqrt{4\pi x}} e^{-\frac{y^2}{4x}}, && x > 0 \\ &= 0, && x \le 0, \end{aligned}$$

whose Fourier transform (in x) is

$$\hat{\Omega}(\tau, y) = \frac{1}{2i(i\tau)^{\frac{1}{2}}} e^{i(i\tau)^{\frac{1}{2}}|y|}.$$

Let

$$Y = \frac{2}{3}\left(\eta^{\frac{3}{2}} - y^{\frac{3}{2}}\right), \quad (i\tau)^{\frac{1}{2}} Y = \zeta - z.$$

A short calculation taking account of (2.5) and (2.6) shows that

$$\hat{F}(\tau, y, \eta) = y^{-\frac{1}{4}} \eta^{-\frac{1}{4}} [\hat{\Omega}(\tau, Y) + \hat{\Psi}(\tau, y, \eta)]$$

where $\hat{\Psi}$ and $\widehat{(x\Psi)}_x = i\tau\hat{\Psi}_\tau$ are integrable in τ, uniformly with respect to (y,η) so long as y and η are bounded away from 0. Then

$$F(x,y,\eta) = y^{-\frac{1}{4}} \eta^{-\frac{1}{4}} [\Omega(x, Y) + \Psi(x, y, \eta)], \tag{3.3}$$

where Ψ and $x\Psi_x$ are bounded so long as y and η are bounded away from 0. Moreover given y, Ψ and its partial derivatives are $O[\exp(-\eta^3)]$ as $\eta \to \infty$, uniformly with respect to x.

Now let U be a continuous function such that, for some $\epsilon > 0$, $U(\eta) = O[\exp(\eta^{3-\epsilon})]$ as $\eta \to \infty$. In $\eta U(\eta)$ is integrable on any finite interval $[0,a]$, then (3.2) defines for $x > 0$, $y > 0$ a solution of (2.1). Using (3.3) standard reasoning shows that if $\eta_0 > 0$

$$U(\eta_0) = \lim_{(x,y)\to(0,\eta_0)} V(x,y).$$

It remains to examine the behavior of V and its derivatives as $y \to 0^+$. If $U(\eta) = 0$ in some neighborhood of 0, then V is of class C^∞ across the x-axis and $V(x,0) = 0$. Therefore it suffices to consider the case when $U(x) = 0$ for all x outside some finite interval $[0,a]$.

Let us assume that $U(\eta) = O(\eta^2)$ as $\eta \to 0^+$. The substitution $\sigma = \tau\eta^3$ shows that

$$\begin{aligned} F(x,y,\eta) &= \eta^{-2} F(\eta^{-3}x, \eta^{-1}y, 1), \\ V(x,y) &= \int_0^a F(\eta^{-3}x, \eta^{-1}y, 1)\eta^{-1} U(\eta)\, \mathrm{d}\eta. \end{aligned}$$

Outside the interval $\frac{1}{2} \le \eta^{-1}y \le 2$ the integrand tends uniformly to 0 as $y \to 0^+$. From (3.3)

$$|V(x,y)| \le c_4 \int_{\frac{y}{2}}^{2y} \Omega(\eta^{-3}x, \eta^{-\frac{3}{2}}Y)\eta^{\frac{1}{2}}\, \mathrm{d}\eta + O(1),$$

and since $\Omega(\eta^{-3}x, \eta^{-\frac{3}{2}}Y) = \eta^{\frac{3}{2}}\Omega(x, Y)$, $\mathrm{d}Y = \eta^{\frac{1}{2}}\,\mathrm{d}\eta$,

$$|V(x,y)| \leq c_4(2y)^{\frac{3}{2}} \int_{-\infty}^{\infty} \Omega(x,Y)\,\mathrm{d}Y + O(1).$$

Since the last integral is 1, $V(x,y)$ tends uniformly to 0 as $y \to 0^+$.

Since $xF_x(x,y,1)$ is bounded on any interval $x \geq \delta > 0$,

$$V_x = \int_0^a F_x(\eta^{-3}x, \eta^{-1}y, 1)\eta^{-4}U(\eta)\,\mathrm{d}\eta$$

is continuous for $x > 0$, $y \geq 0$. Since V satisfies (2.1), V_y and V_{yy} are continuous on the x-axis for $x > 0$, and

$$V_y(x, 0^+) = \int_0^a F_y(\eta^{-3}x, 0, 1)\eta^{-2}U(\eta)\,\mathrm{d}\eta. \tag{3.4}$$

From (3.2") $\hat{F}_y(\tau, 0, \zeta)$ is a constant times $H(\zeta)$. Therefore

$$V_y(x, 0^+) = c_5 \int_0^a g(\eta^{-3}x)\eta^{-2}U(\eta)\,\mathrm{d}\eta \tag{3.5}$$

where g was defined in §2. Since the integrand is bounded and $g(0) = 0$, by Lebesgue's convergence theorem $V_y(x, 0^+)$ tends to 0 as $x \to 0^+$. For $x > 0$ we have

$$V_{yx}(x, 0^+) = c_5 \int_0^a g'(\eta^{-3}x)\eta^{-5}\,\mathrm{d}\eta.$$

Making the substitution $r = \eta^{-3}x$ and using the estimate $U(\eta) = O(\eta^2)$

$$|V_{yx}(x, 0^+)| \leq c_6 x^{-\frac{2}{3}} \int_0^{\infty} |g'(r)| r^{-\frac{1}{3}}\,\mathrm{d}r.$$

Since

$$\widehat{xg'} = i\tau\widehat{xg} = \tau H_\tau$$

from estimates in §2 xg' is integrable; and consequently, $g'x^{-\frac{1}{3}}$ is integrable. Therefore $x^{\frac{2}{3}}V_{yx}(x, 0^+)$ is bounded.

4

Let us return to the problem in §1. We seek a solution u of (1.1) in the open strip which is continuous in the closed strip and satisfies the boundary data (1.2). Let us assume that U_0, U_1 are continuous, and that the second derivatives $U_0''(0)$, $U_1''(0)$ exist. Moreover, for some $\epsilon > 0$

$$\begin{aligned} U_0(y) &= O[\exp|y|^{3-\epsilon}] && \text{as } y \to -\infty, \\ U_1(y) &= O[\exp y^{3-\epsilon}] && \text{as } y \to +\infty. \end{aligned}$$

Let $\phi(x)$ be continuous on $[0,1]$ with $\phi(0) = U_0(0)$, $\phi(1) = U_1(0)$, and ϕ' continuous on $(0,1)$. Using §'s 2 and 3 let us find solutions u^+, u^- of (1.1) in the upper and lower halves of the strip, such that

$$\begin{gathered} u^+(1,y) = U_1(y), \quad u^-(0,y) = U_0(y), \\ u^+(x,0) = u^-(x,0) = \phi(x). \end{gathered}$$

The linear function $w^+(y) = U_1(0) + U_1'(0)y$ satisfies (2.1). By §3 there is a solution of (2.1) $V^+(x,y)$ with

$$\begin{aligned} &V^+(0,y) = U_1(y) - w^+(y), && y \geq 0 \\ &V^+(x,0) = 0, && x \geq 0. \end{aligned}$$

Let $\psi^+(x) = \phi(1-x) - U_1(0)$ for $0 \leq x \leq 1$, and define ψ^+ arbitrarily outside $[0,1]$ subject to the conditions in §2. Let $v^+(x,y)$ be the solution of (2.1) constructed there with $\psi = \psi^+$, and

$$\begin{gathered} u^+(x,y) = v^+(1-x,y) + V^+(1-x,y) + w^+(y), \\ 0 \leq x \leq 1, y \geq 0. \end{gathered}$$

In the same way we find v^-, V^-, w^-, solutions of (2.1) in the quadrant $x > 0$, $y > 0$, and set

$$\begin{gathered} u^-(x,y) = v^-(x,-y) + V^-(x,-y) + w^-(-y), \\ 0 \leq x \leq 1, y \leq 0. \end{gathered}$$

Let $u = u^+$ for $y \geq 0$ and $u = u^-$ for $y \leq 0$. The function u is continuous in the closed strip and satisfies (1.1) except at $(0,0)$, $(1,0)$. Note that for $y = 0$ each term in (1.1) is 0.

It remains to choose ϕ so that $u_y^+(x,0^+) = u_y^-(x,0^+)$. Therefore we must have

$$v_y^+(1-x,0^+) = -v_y^-(x,0^+) + \mu(x),$$

where

$$\mu(x) = -[V_y^+(1-x,0^+) + V_y^-(x,0^+) + U_1'(0) - U_0'(0)].$$

According to (2.10)

$$v_y^+(1-x,0^+) = C_4 \int_0^{1-x} {\psi^+}'(1-x-\xi)^{-\frac{1}{3}}\,\mathrm{d}\xi,$$

and since ${\psi^+}'(\xi) = -\phi'(1-\xi)$,

$$v_y^+(1-x,0^+) = -C_4 \int_x^1 \phi'(\xi)(\xi - x)^{-\frac{1}{3}}\,\mathrm{d}\xi.$$

Similarly

$$v_y^-(x,0^+) = C_4 \int_0^x \phi'(\xi)(x-\xi)^{-\frac{1}{3}}\,\mathrm{d}\xi.$$

Therefore, ϕ must satisfy the integral equation

$$\int_0^x \phi'(\xi)(x-\xi)^{-\frac{1}{3}}\,\mathrm{d}\xi = \int_x^1 \phi'(\xi)(\xi-x)^{-\frac{1}{3}}\,\mathrm{d}\xi + C_4^{-1}\mu(x), \qquad 0<x<1. \tag{4.2}$$

Let us treat (4.2) as a particular case of the equation

$$\int_0^x f(\xi)(x-\xi)^{\alpha-1}\,\mathrm{d}\xi = \int_x^1 f(\xi)(\xi-x)^{\alpha-1}\,\mathrm{d}\xi + \int_0^x p(\xi)(\xi-x)^{\alpha-1}\,\mathrm{d}\xi, \qquad 0<x<1, \tag{4.3}$$

where $\alpha \in (0,1)$. Let us assume that $x^m(1-x)^m p(x)$ is Hölder continuous on $[0,1]$, where

$$m = \frac{\alpha+1}{2}.$$

Notice that except for a factor $\Gamma(\alpha)^{-1}$ the first and last integrals are Riemann-Liouville fractional integrals of order α. Consider the equation with Cauchy kernel

$$F(x) - \frac{k}{\pi}\int_0^1 \frac{F(\xi)\,\mathrm{d}\xi}{\xi - x} = cP(x), \tag{4.4}$$

where $P(x) = x^\alpha p(x)$ and the constants k, c will be chosen later. Let F be a solution of (4.4) such that $x^{\gamma-\alpha}(1-x)^\gamma F(x)$ is Hölder continuous on $[0,1]$ for some $\gamma < 1$. Let us show that $f(x) = x^{-\alpha}F(x)$ solves (4.3).

Multiplying by $\xi^{-\alpha}(x-\xi)^{\alpha-1}$ and integrating,

$$\int_0^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi - \frac{k}{\pi}\int_0^x (x-\xi)^{\alpha-1}\xi^{-\alpha}\int_0^1 \frac{F(\eta)\,\mathrm{d}\eta}{\eta-\xi} = c\int_0^x (x-\xi)^{\alpha-1}p(\xi)\,\mathrm{d}\xi.$$

Near 0 the inner integral defines a function of the form $a\xi^{\alpha-\gamma} + b\xi^{\alpha-\lambda}\chi(\xi)$, where $\lambda > \Gamma$ and χ is Hölder continuous [Mu, p. 75]. Let us write the inner integral on the left as the sum of integrals from 0 to x and from x to 1, and interchange the order of integration. The second of these iterated integrals is absolutely convergent. The interchange of order $\int_0^x \ldots\,\mathrm{d}\xi \int_0^x \ldots\,\mathrm{d}\eta = \int_0^x \ldots\,\mathrm{d}\eta \int_0^x \ldots\,\mathrm{d}\xi$ is easily justified if F is Hölder continuous on $[0,x]$, and then by a passage to the limit argument for F satisfying the present assumptions.

Making the substitutions

$$x = \frac{1}{1+t}, \quad \xi = \frac{1}{1+s}, \quad \eta = \frac{1}{1+r},$$

$$\int_0^x \frac{\xi^{-\alpha}(x-\xi)^{\alpha-1}}{\eta-\xi}\,\mathrm{d}\xi = \frac{r+1}{(t+1)^{\alpha-1}}\int_t^\infty \frac{(s-t)^{\alpha-1}}{s-r}\,\mathrm{d}s.$$

If $x < \eta < 1$ (i.e. $r < t$), the substitution $s - t = (t-r)q$ shows that

$$\int_t^\infty \frac{(s-t)^{\alpha-1}}{s-r}\,\mathrm{d}s = (t-r)^{\alpha-1}\int_0^\infty \frac{q^{\alpha-1}\,\mathrm{d}q}{q+1} = (t-r)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.$$

If $0 < \eta < x$, the substitution $s - t = (r - t)q$ shows that

$$\int_t^\infty \frac{(s-t)^{\alpha-1}}{s-r}\,\mathrm{d}s = (r-t)^{\alpha-1}\int_0^\infty \frac{q^{\alpha-1}\,\mathrm{d}q}{q-1} = -(r-t)^{\alpha-1}\pi\cot\pi\alpha.$$

The value $-\pi\cot\pi\alpha$ for the last integral may be found by contour integration. Bu

$$\frac{(r+1)(r-t)^{\alpha-1}}{(t+1)^{\alpha-1}} = (x-\eta)^{\alpha-1}\eta^{-\alpha},$$

and hence

$$\int_0^x \phi(\eta)\,\mathrm{d}\eta \int_0^x \frac{(x-\xi)^{\alpha-1}\xi^{-\alpha}}{\eta-\xi}\,\mathrm{d}\xi = -\pi\cot\pi\alpha\int_0^x (x-\eta)^{\alpha-1} + f(\eta)\,\mathrm{d}\eta,$$

$$\int_x^1 \phi(\eta)\,\mathrm{d}\eta \int_0^x \frac{(x-\xi)^{\alpha-1}\xi^{-\alpha}}{\eta-\xi}\,\mathrm{d}\xi = \frac{\pi}{\sin\pi\alpha}\int_x^1 (\eta-x)^{\alpha-1} f(\eta)\,\mathrm{d}\eta,$$

$$(1+k\cot\pi\alpha)\int_0^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi = \frac{k}{\sin\pi\alpha}\int_0^1 (\xi-x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi + c\int_0^x (x-\xi)^{\alpha-1} p(\xi)\,\mathrm{d}\xi.$$

If we take

$$k = \frac{\sin\pi\alpha}{1-\cos\pi\alpha} = -\tan\pi m,$$

and $\mathrm{c}^{-1} = 1 - \cos\pi\alpha$, then f is a solution of (4.3). It is known [M, p. 130] that

$$F(x) = \frac{c}{1+k^2}P(x) + \frac{kc}{(1+k^2)\,\pi}x^{m-1}(1-x)^{-m}\int_0^1 \frac{\xi^{1-m}(1-\xi)^m P(\xi)\,\mathrm{d}\xi}{\xi - x} + bx^{m-1}(1-x)^{-m}$$

is a solution of (4.4), where b is an arbitrary constant. Since $\alpha - m = m - 1$,

$$f(x) = \frac{c}{1+k^2}p(x) + \frac{kc}{(1+k^2)\pi}x^{-m}(1-x)^{-m}\int_0^1 \frac{\xi^m(1-\xi)^m p(\xi)\,\mathrm{d}\xi}{\xi - x} + bx^{-m}(1-x)^{-m} \tag{4.5}$$

is a solution of (4.3). The assumption made about F above is satisfied for any $\gamma > m$.

P. M. Anselone pointed out that the solution $f(x) = bx^{-m}(1-x)^{-m}$ of the homogeneous form of (4.3) can be verified directly. Let $A(x)$ denote the left side of (4.3). When $f(x) = f(1-x)$ and $p(x) = 0$, (4.3) becomes $A(x) = A(1-x)$. Let us suppose that f has this special form. By writing $A(x)$ as an integral in s on $(0, \infty)$ as before and making the substitution $\sigma = (s-t)/t$,

$$A(x) = bB(m-\alpha, \alpha)x^{m-1}(1-x)^{m-1},$$

which is symmetric in x and $1 - x$.

In the present problem $\alpha = 2/3$, $m = 5/6$, $f(x) = \phi'(x)$ and $p(x)$ is $[C_4\Gamma(\alpha)]^{-1}$ times the fractional derivative of order $2/3$ of $\mu(x)$. Then

$$p(x) = [C_4\Gamma(\alpha)]^{-1} I_{\frac{1}{3}}\mu'(x),$$

where $I_{1-\alpha}$ is the integral of order $1-\alpha$. From the estimate for $V_{yx}(x, 0^+)$ at the end of §3,

$$|\mu'(x)| \le C x^{-\frac{2}{3}}(1-x)^{-\frac{2}{3}},$$

and from this it is not difficult to show that $x^m(1-x)^m p(x)$ is Hölder continuous. The constant b in (4.5) is determined from the condition

$$U_1(0) - U_0(0) = \int_0^1 \phi'(x)\,\mathrm{d}x.$$

In particular let $U_1(y) = 1$, $U_0(0) = 0$. Then $u(x,y)$ represents the probability of reaching 1 before 0 starting at x with velocity y. In this case $\mu(x) = 0$ and

$$\phi'(x) = b x^{-\frac{5}{6}}(1-x)^{-\frac{5}{6}}, \quad b^{-1} = B\left(\frac{5}{6}, \frac{5}{6}\right).$$

5

Let us now prove a uniqueness theorem. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of (1.1) with the same boundary data (1.2), and let $w = u_1 - u_2$. Let us assume that w is bounded and continuous in the closed strip and that w_y is square integrable over the strip. Then for any rectangle $R: 0 \le x \le 1$, $|y| \le a$,

$$0 = \iint_R w(yw_x + w_{yy})\,\mathrm{d}x\,\mathrm{d}y = -\iint_R w_y^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial R}\left(\frac{1}{2}yw^2\,\mathrm{d}y - ww_y\,\mathrm{d}x\right),$$

and since w_y^2 is integrable there exist a_n, $n = 1, 2, \ldots$ tending to ∞ such that $\int_{\partial R_n} ww_y\,\mathrm{d}x$ tends to 0. Moreover $\int_{\partial R_n} \frac{1}{2}yw^2\,\mathrm{d}y \le 0$ since $w(1,y) = 0$ for $y > 0$ and $w(0,y) = 0$ for $y < 0$. From this

$$\iint_R w_y^2\,\mathrm{d}x\,\mathrm{d}y = 0,$$

and then $w_y \equiv 0$. Then $-yw_x = w_{yy} = 0$, and since every horizontal line has a point where $w = 0$, $w \equiv 0$.

6

Let us mention a more general problem in which the random process ξ satisfies (formally) the linear constant-coefficient stochastic differential equation

$$\ddot{\xi} + a\dot{\xi} + b\xi = \dot{\rho}, \quad a \ge 0, \quad b \ge 0, \tag{6.1}$$

where $\rho(t)$ is a Brownian motion process normalized in the same way as §1. In engineering language $\dot{\rho}(t)$ is a "white noise". The steady state form of the backward equation for the vector Markov process $(\xi(t), \dot{\xi}(t))$ is now

$$yu_x + u_{yy} - (ay + bx)u_y = 0. \tag{6.2}$$

We have considered the case $a = b = 0$. When $a > 0$, $b = 0$, (6.1) describes the Ornstein-Uhlenbeck process, which is a more refined model for Brownian motion. When $a = 0$, $b > 0$, (6.1) describes a randomly accelerated harmonic oscillator. See the articles in [W] by Chandrasekhar, Uhlenbeck-Ornstein and Wang-Uhlenbeck.

The equation $-yv_x + v_{yy} = f(x, y)$ has under suitable assumptions on f, the particular solution in the quarter-plane $x > 0$, $y > 0$

$$v_p(x, y) = \int_0^x \int_0^\infty F(x - \xi, y, \eta) f(\xi, \eta)\, \mathrm{d}\eta\, \mathrm{d}\xi$$

which is 0 when $y = 0$ or $x = 0$. Suppose that u satisfies (6.2) in the strip and take $f(x, y) = [ay + b(1 - x)]u_y(1 - x, y)$. Then if $\phi(x) = u(x, 0)$,

$$u(1 - x, y) = \phi(1 - x) * g_a + V^+(x, y) + w^+(y) + v_p(x, y).$$

Integrating by parts (formally) we obtain for $0 < x < 1$, $y \geq 0$,

$$\begin{aligned} u(1 - x, y) = \phi(1 - x) * g_a + V^+(x, y) + w^+(y) \\ - \int_0^x \int_0^\infty [a\eta + b(1 - \xi)F]_\eta u(1 - \xi, \eta)\, \mathrm{d}\eta\, \mathrm{d}\xi \end{aligned}$$

with a similar expression for $u(x, -y)$. From these two integral equations for u, together with the equation obtained by matching $u_y(x, 0^+)$ and $u_y(x, 0^-)$ one should be able to get some information about solutions of (6.2) at least for small values of a and b.

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