Theory and Applications of Nonlinear Control Systems
C.I. Byrnes and A. Lindquist (editors)

On the local controllability of a scalar-input control system

Gianna Stefani

Dipartimento di Sistemi e Informatica
Università di Firenze
Firenze, Italia

A necessary condition of local controllability of a scalar-input control system is proven. The proof is based on a general result also proven in this work on the construction of a "suitable chart".

The necessary condition of local controllability is extended to a necessary condition of local controllability along a reference trajectory.

1 Introduction

Let us consider the following control system

$$\begin{cases} \dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t)) \\ x(0) = x_0 \end{cases} \tag{1}$$

where the state $x(t)$ belongs to a C^ω manifold M, f_0 and f_1 are C^ω vector fields such that $f_0(x_0) = 0$ and $f_1(x_0) \neq 0$. The control map $t \to u(t)$ belongs to the collection U of the integrable maps defined on $[0, 1]$ with values in $[-1, 1]$.

The system (1) is said to be *locally controllable* if, for each $t > 0$, x_0 belongs to $\operatorname{int} R(x_0, t)$ (= the interior of the reachable set from x_0 at time t). To the author's knowledge the necessary conditions of local controllability are the following ones (see [2, 3])

$$L(x_0) = T_{x_0}M \tag{2}$$

(L is the Lie algebra generated by f_0 and f_1)

$$\operatorname{ad}^2_{f_1} f_0(x_0) \in S^1(x_0) \tag{3}$$

$\operatorname{ad}^k_{f_1} f_0$ is defined by $\operatorname{ad}_{f_1} f_0 = [f_1, f_0]$, $\operatorname{ad}^k_{f_1} f_0 = [f_1, \operatorname{ad}^{k-1}_{f_1} f_0]$, and S^k is the subspace of L consisting of all the brackets containing f_1 at most k times.

In this paper the following generalization of (3) will be proven.

Theorem 1. *A necessary condition for the local controllability of the system* (1) *is*

$$\forall r \text{ even}, \quad \mathrm{ad}_{f_1}^r f_0(x_0) \in S^{r-1}(x_0). \tag{4}$$

The proof of the theorem is based on a general result also proven in the paper on the construction of a "suitable chart".

To be more precise, let $F = \{F_i\}_{i \geq 0}$ be an increasing filtration at x_0 of L, that is a sequence of linear subspaces of L such that

i) $F_i \subseteq F_{i+1}$

ii) $F_\infty \equiv \bigcup_{i \geq 0} F_i = L$

iii) $[F_i, F_j] \subseteq F_{i+j}$

iv) $\forall f \in F_0, \quad f(x_0) = 0.$

For each $f \in L$ the weight (with respect to F) is defined by $w(f) = \min(i : f \in F_i)$. Obviously $w([f, g]) \leq w(f) + w(g)$, $\forall f, g \in L$.

Identifying a vector field f with the derivation L_f, the associative, non commutative algebra of differential operators A, generated on $\mathbb{R}$ by L, acting on the C^ω functions defined on M, can be considered. The filtration F induces a filtration $A = \{A_i\}_{i \geq 0}$ of the algebra A by $A_i = \mathrm{span}\{D = L_{f_1} \circ \ldots \circ L_{f_j} : \sum_{k=1}^{j} w(f_k) \leq i\}$. The definition of weight can be extended to each $D \in A$ in the obvious way $w(D) = \min\{i : D \in A_i\}$.

Theorem 2. *Let $\varphi : M \to \mathbb{R}$ be a C^ω function such that $L_f \varphi(x_0) = 0$ for each $f \in F_r$. There is a C^ω function $\tilde{\varphi}$ such that $j_1 \tilde{\varphi}(x_0) = j_1 \varphi(x_0)$ (i.e. φ and $\tilde{\varphi}$ coincide up to the first order) and $D\tilde{\varphi}(x_0) = 0$ for each $D \in A_r$.*

Remark 1.1. *If $r = +\infty$ so that $L(x_0) \neq T_{x_0} M$, the construction of $\tilde{\varphi}$ provides an iterative method to construct the* local *integral manifold of L through x_0 (see Remark 2.1).*

Finally if $f_0(x_0) \neq 0$, (4) becomes a necessary condition of local controllability along a reference trajectory in the following sense

Theorem 3. *Let $(t, x_0) \to \exp t f_0(x_0)$ be the local flow of f_0. If* (4) *doesn't hold, then there is $T > 0$ such that $\forall t \leq T$, $\exp t f_0(x_0) \notin \mathrm{int}\, R(x_0, t)$.*

2 Proof of Theorem 2

Let $x = \{x_1, \ldots, x_n\}$ be a chart at x_0 adapted to F in the following sense

i) $x(x_0) = 0 \in \mathbb{R}^n$

ii) $\mathrm{span}\left\{\frac{\partial}{\partial x_1}(x_0), \ldots, \frac{\partial}{\partial x_{m_i}}(x_0)\right\} = F_i(x_0)$

m_i being the dimension of $F_i(x_0)$.

Starting from any chart at x_0, an adapted chart can be obtained by a linear change of coordinates.

The monomial $x_1^{\nu_1} x_2^{\nu_2} \dots x_n^{\nu_n}$, $\nu_i \in \mathbb{N}$, will be denoted by x^ν. For each $r = 1, \dots, n$, the weight w_r is defined as $w_r = \min\left\{i : \frac{\partial}{\partial x_r}(x_0) \in F_i(x_0)\right\}$ ($w_r = +\infty$ if $\frac{\partial}{\partial x_r}(x_0) \notin L(x_0)$) and the monomial x^ν is said to have degree $|\nu| = \sum_{i=1}^n \nu_i$ and weight $\|\nu\| = \sum_{i=1}^n \nu_i w_i$.

Let $m = \dim L(x_0)$ ($m = n$ if L has full rank at x_0). Let $g_1, \dots, g_m \in L$ be such that $g_i(x_0) = \frac{\partial}{\partial x^i}(x_0)$, $i = 1, \dots, m$. The differential operator $L_{g_m}^{\nu_m} \circ \dots \circ L_{g_1}^{\nu_1} \in A$ is denoted by D^ν and $|\nu|$ ($\|\nu\|$) is the degree (the weight) of D^ν.

Moreover the set of all multiindex of "degree s and weight r" is denoted by $\chi(s,r)$ i.e.

$$\chi(s,r) = \{(\nu_1, \dots, \nu_m) : \nu_i \geq 0, |\nu| = s, \|\nu\| \leq r\}.$$

Notice that $\chi(s,r) = \emptyset$ if $s > r$.

Let φ be as in the Theorem 2, that is $L_f \varphi(x_0) = 0, \forall f \in F_r$. Without loss of generality we can suppose $\varphi(x_0) = 0$.

Let us define recursively for $s \leq r$:

$$\varphi_1 = \varphi, \quad \varphi_s = \varphi_{s-1} - \sum_{\nu \in \chi(s,r)} \frac{1}{\nu!} D^\nu \varphi_{s-1}(x_0)\, x^\nu,$$

where $\nu! = \nu_1! \dots \nu_n!$. It is clear that φ and φ_s coincide up to the first order.

a) <u>$r < +\infty$</u>

Let us prove by induction on $s \leq r$ that

$$L_{f_1} \circ \dots \circ L_{f_s} \varphi_s(x_0) = 0 \quad \text{if} \quad \sum_{j=1}^s w(f_j) \leq r \tag{5}$$

For $s = 1$, (5) follows by the hypothesis.

Let $f_1, \dots, f_{s+1}$ be such that $\sum_{j=1}^{s+1} w(f_j) \leq r$.

$$L_{f_1} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0) = L_{f_2} \circ L_{f_1} \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0) + L_{[f_1,f_2]} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0).$$

Being $\varphi_{s+1} = \varphi_s +$ (polynomial of order $s+1$), we get $L_{[f_1,f_2]} \cdot \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0) = L_{[f_1,f_2]} \circ \dots \circ L_{f_{s+1}} \varphi_s(x_0) = 0$ by the inductive hypothesis, so that $L_{f_1} \circ L_{f_2} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0) = L_{f_2} \circ L_{f_1} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0)$. In an analogous way it is possible to show that $L_{f_1} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0) = L_{f_{\sigma(1)}} \circ \dots \circ L_{f_{\sigma(s+1)}} \varphi_{s+1}(x_0)$ for each permutation σ of $[1, \dots, s+1]$. This means that $L_{f_1} \circ \dots \circ L_{f_{s+1}} \varphi_{s+1}(x_0)$ is a linear combination of terms of type $D^\nu \varphi_{s+1}(x_0)$ with $|\nu| = s+1$ and $\|\nu\| \leq r$.

If $|\nu'| = s+1$, $D^\nu x^{\nu'}(x_0) = \begin{cases} 0 & \text{if } \nu' \neq \nu \\ \nu! & \text{if } \nu' = \nu \end{cases}$.

Hence $D^\nu \varphi_{s+1}(x_0) = D^\nu \varphi_s(x_0) - \sum_{\nu' \in \chi(s+1,r)} \frac{D^{\nu'} \varphi_s(x_0)}{\nu'!} D^\nu x^{\nu'} = 0$.

Let us define $\tilde{\varphi} = \varphi_r$.

The previous arguments show that $D\tilde{\varphi}(x_0) = 0$ if D is a differential operator of degree and weight at most r. To get the proof it is sufficient to prove by induction on i that $L_{f_1} \circ \dots \circ L_{f_{r+i}} \tilde{\varphi}(x_0) = 0$ if $\sum_{j=1}^{r+i} w(f_j) \leq r$. The last condition implies that there is $k \in \{1, \dots, r+i\}$ such that $f_k \in F_0$. We get $L_{f_1} \circ \dots \circ L_{f_{r+i}} \tilde{\varphi}(x_0) = L_{f_k} \circ D_1 \tilde{\varphi}(x_0) + D_2 \tilde{\varphi}(x_0)$, where D_1 and D_2 are differential operator of degree $r+i-1$ and weight at most r. Hence

$L_{f_k} \circ D_1\tilde{\varphi}(x_0) = 0$ by $f_k(x_0) = 0$ and $D_2\tilde{\varphi}(x_0) = 0$ by the induction hypothesis and the theorem is proven.

b) $\underline{r = +\infty}$

Let N be the *local* integral manifold of L through x_0. By a Nagano's theorem N exists and its dimension is m.

Let $\{x_1, \ldots, x_n\}$ be adapted to F and such that $x_{m+1} = \ldots = x_n = 0$ are the equations of N.

Let the function ψ be defined by $\psi(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_m, 0 \ldots 0)$. As $\psi_{|N} = \varphi_{|N}$ and each $f \in L$ is tangent to N it is sufficient to prove that the sequence $\{\varphi_s\}_{s\geq 1}$ converges to $\tilde{\varphi} = \varphi - \psi$, in fact $L_f\varphi_{|N} = L_f\psi_{|N}, \forall f \in L$. Let Ψ_s be the Taylor expansion of ψ up to order s. By $L_f\varphi(x_0) = 0 \forall f \in L$, it follows that $\psi_1 = 0$ and hence $\varphi_1 = \varphi - \psi_1$. By the inductive hypothesis, let $\varphi_s = \varphi - \psi_s$. We get $\varphi_{s+1} = \varphi - \psi_s - \sum_{|\nu|=s+1} D^\nu(\varphi - \psi_s)(x_0)/\nu! x^\nu$. But $D^\nu\varphi(x_0) = D^\nu\psi(x_0)$, so that $D^\nu(\varphi - \psi_s)(x_0) = D^\nu(\psi - \psi_s)(x_0) = \frac{\partial^{s+1}}{\partial x^\nu}\psi(x_0)$ and $\psi_s + \sum_{|\nu|=s+1} D^\nu(\varphi - \psi_s)(x_0)/\nu! x^\nu = \psi_{s+1}$.

Remark 2.1. *The previous result gives an iterative method to construct the equations of N in the following sense.*

Starting from any chart at x_0 it is possible, by a linear change of coordinates, to get a chart $\{x_1, \ldots, x_n\}$ adapted to F and such that $L_f x_i(x_0) = 0$, $i = m+1, \ldots, n$. Applying the theorem and denoting $x_i - \tilde{x}_i(x_1, \ldots, x_m)$ by ψ_i, we get that $x_i = \psi_i(x_1, \ldots, x_m), i = m+1, \ldots, n$, are parametric equations of N, where the parameters are "the coordinates of the tangent space $T_{x_0}N$".

Example 2.1. *Let $M = \mathbb{R}^2$, $f_0(x, y) = (x, 2y)$, $f_1(x, y) = (1, x + y)$. Let the filtration F be defined by: $F_0 = \operatorname{span}\{f_0\}$, $F_i = F_0 + S^i$. We get $[f_1, f_0](x, y) = (1, x)$, $\operatorname{ad}^k_{f_0}(f_1)(x, y) = (-1)^k(1, x)$ and $[f_1, [f_0, f_1]](x, y) = (0, x)$, so that $F_1(0, 0) = F_2(0, 0) = \operatorname{span}\{(1, 0)\}$. Applying the Theorem 2 for $r = 2$, we get: $\tilde{y} = y - \frac{1}{2}f_1^2 y(0, 0)x^2 = y - \frac{1}{2}x^2$ and $L_{f_1} \circ L^h_{f_0} \circ L_{f_1} \circ L^k_{f_0}\tilde{y}(0, 0) = 0 \quad \forall h, k \geq 0$.*

Example 2.2. *Let $M = \{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| < 1\}$, $f_0(x, y, z) = (0, x, x^2/(1 + xy))$, $f_1(x, y, z) = (1, y, (y + xy)/(1 + xy))$.*

Let the filtration F be defined by $F_0 = \{0\}$, $F_1 = \operatorname{span}\{f_0, f_1\}$, $F_2 = F_1 + \operatorname{span}\{[f_0, f_1]\}$, $F_3 = L$. We get $L(0) = F_2(0) = \{(\lambda, \mu, 0) : (\lambda, \mu) \in \mathbb{R}^2\}$. It is easy to see that $L_{f_0}(z - \log(1 + xy)) = L_{f_1}(z - \log(1 + xy)) = 0$. In this case $\tilde{z} = z - \log(1 + xy)$ and $z - z_s = \sum_{i=1}^{s}(-1)^{i-1}\frac{(xy)^i}{i}$ is the Taylor expansion of $\log(1 + xy)$ up to order s.

3 Proof of Theorem 1

Let $r > 0$ be even and let $\operatorname{ad}^r_{f_1} f_0(x_0) \notin S^{r-1}(x_0)$. There is a C^ω function φ such that $\varphi(x_0) = 0$, $L_{\operatorname{ad}^r_{f_1} f_0}\varphi(x_0) = 1$ and $L_f\varphi(x_0) = 0, \forall f \in S^{r-1}$. By the Theorem 2 applied to the filtration $F_0 = \operatorname{span}(f_0)$, $F_i = F_0 + S^i$, there is $\tilde{\varphi}$ with the same properties and $D\tilde{\varphi}(x_0) = 0$ for each $D \in A_{r-1}$. In what follows we set $\varphi = \tilde{\varphi}$. Moreover let ψ be a C^ω function such that $L_{f_1}\psi(x_0) = 1$.

We shall prove that there is $T > 0$ with the following property:

$$\text{If } x \in R(x_0, t),\ t \leq T, \text{ is such that } \psi(x) = 0, \text{ then } \varphi(x) \geq 0. \tag{P}$$

This shows that $x_0 \notin \operatorname{int} R(x_0, t)$. To get the proof we need some technical results on integral inequalities.

Let the integral operator $I : U \to U$ be defined by $Iu(t) = \int_0^t u(s)\mathrm{d}s$, $\forall u \in U$. Moreover for each $u \in U$ we define $I_u : U \to U$ by $I_u w = I(uw)$, $\forall w \in U$. Let $v = Iu$. It is not difficult to see that

$$I^{k+1}w(t) = \int_0^t \frac{(t-s)^k}{k!} w(s)\mathrm{d}s \quad \forall w \in U \tag{6}$$

$$I_u^k I w(t) = \int_0^t \frac{(v(t)-v(s))^k}{k!} w(s)\mathrm{d}s \quad \forall w \in U \tag{7}$$

Let $h = (h_1, \ldots, h_j)$ be a multiindex with $h_1, \ldots, h_j \geq 0$, we denote $|h| = h_1 + \ldots + h_j$. Moreover, in what follows we denote by $\mathfrak{i} : [0,1] \to [0,1]$ the identity map $\mathfrak{i}(t) \equiv 1$ and for sake of simplicity we define

$$\left(I_u^{k_j} \circ I^{h_j} \circ \cdots \circ I_u^{k_1} \circ I^{h_1}\right) \mathfrak{i}(t) = p(k,h,u,t) \tag{8}$$

It is not difficult to see that $\forall u \in U$,

$$|p(k,h,u,t)| \leq \frac{2^\ell}{\ell!} \ell^2 t^{\ell-2} \int_0^t |v(s)|\mathrm{d}s \tag{9}$$

if $|k| \geq 1$, $|k| + |h| = \ell \geq h_1 + 2$, $v = Iu$.

Lemma 3.1. *Let $j \geq 1$; $h_1, k_j \geq 0$; $k_1, \ldots, k_{j-1}, h_2, \ldots, h_j \geq 1$; $|k| = r \geq 1$; $|k| + |h| = \ell \geq r+1$; $v = Iu$. Then*

$$|p(k,h,u,t)| \leq \frac{r^r (2r)!(r+1)^{\ell+1}}{\ell!} \sum_{i=0}^r |v(t)|^{r-i} t^{\ell-r-i/r} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}. \tag{10}$$

Proof. Let $h_1 \geq 1$. Then

$$\begin{aligned}
|p(k,h,u,t)| &= |(I_u^{k_j} \circ I^{h_j} \ldots\ldots I_u^{k_1} \circ I^{h_1})\mathfrak{i}(t)| \\
&\leq \int_0^t \frac{|v(t)-v(s)|^{k_j}}{k_j!} |(I^{h_j-1} \circ \ldots \circ I_u^{k_1} \circ I^{h_1})\mathfrak{i}(s)|\mathrm{d}s \\
&\leq \left(\int_0^t \frac{(|v(t)|+|v(s)|)^{k_j}}{k_j!}\mathrm{d}s\right) \frac{t^{h_j-1}}{(h_j-1)!} \left(\int_0^t \frac{(|v(t)|+|v(s)|)^{k_{j-1}}}{k_{j-1}!}\mathrm{d}s\right) \cdots \\
&\quad \cdots \frac{t^{h_2-1}}{(h_2-1)!} \left(\int_0^t \frac{(|v(t)|+|v(s)|)^{k_1}}{k_1!}\mathrm{d}s\right) \frac{t^{h_1-1}}{(h_1-1)!} \\
&\leq \frac{t^{\ell-j-r}}{(h_1-1)!\cdots(h_j-1)!} \sum_{i_1=0}^{k_1} \cdots \sum_{i_j=0}^{k_j} \binom{k_1}{i_1} \cdots \binom{k_j}{i_j} \frac{|v(t)|^{r-i_1-\cdots-i_j}}{k_1!\cdots k_j!} \left(\int_0^t |v(s)|^{i_1}\mathrm{d}s\right) \cdots \\
&\quad \cdots \left(\int_0^t |v(s)|^{i_j}\mathrm{d}s\right).
\end{aligned}$$

Using the Hölder inequality

$$\int_0^t |v(s)|^i \mathrm{d}s \leq t^{1-i/r} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}, \quad i \leq r \tag{11}$$

we get

$$|p(k,h,u,t)| \leq \frac{t^{\ell-j-r}}{(h_1-1)!\cdots(h_j-1)!}\sum_{i_1=0}^{k_1}\cdots$$

$$\cdots\sum_{i_j=0}^{k_j}\frac{|v(t)|^{r-i_1-\cdots-i_j}}{i_1!\cdots i_j!}t^{1-i_1/r}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i_1/r}\cdots t^{1-i_j/r}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i_j/r}$$

Using the equality

$$\sum_{\substack{i_1,\cdots,i_j\geq 0\\ i_1+\cdots+i_j=i}}\frac{i!}{i_1!\cdots i_j!} = j^i \quad \forall j,i\geq 1 \tag{12}$$

we get

$$\begin{aligned}|p(k,h,u,t)| &\leq \sum_{i=0}^{r}\frac{j^i}{i!}|v(t)|^{r-i}\frac{t^{\ell-r-i/r}}{(h_1-1)!\cdots(h_j-1)!}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}\\ &\leq \frac{r^r(2r)!(r+1)^{\ell+1}}{\ell!}\sum_{i=0}^{r}|v(t)|^{r-i}t^{\ell-r-i/r}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}.\end{aligned}$$

If $h_1=0$, in an analogous way we obtain

$$\begin{aligned}|p(k,h,u,t)| &\leq \sum_{i=0}^{r-k_1}\frac{(j-1)^i}{i!}\frac{|v(t)|^{r-k_1-i}}{k_1!}\frac{t^{\ell-r-(k_1+i)/r}}{(h_2-1)!\cdots(h_j-1)!}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{(i+k_1)/r}\\ &= \sum_{i=k_1}^{r}\frac{(j-1)^{i-k_1}}{(i-k_1)!}\frac{|v(t)|^{r-i}}{k_1!}\frac{t^{\ell-r-i/r}}{(h_2-1)!\cdots(h_j-1)!}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}\\ &\leq (r-1)^{r-k_1}(2r-k_1-1)!\frac{(r+1)^{\ell+1}}{\ell!}\sum_{i=k_1}^{r}|v(t)|^{r-i}t^{\ell-r-i/r}\left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r}\end{aligned}$$

Hence (10) is true also in this case. □

Corollary 3.1. *Let $j\geq 1$; $h_1,k_j\geq 0$; $k_1,\ldots,k_{j-1},h_2,\ldots,h_j\geq 1$; $|h|+|k|=\ell$. If $|k|>r\geq 1$ or $|k|=r\geq 1$ and $k_j=0$, then*

$$|p(k,h,u,t)| \leq (2r)!r^{r+1}\frac{(2r+2)^\ell}{(\ell-1)!}t^{\ell-r-1}\int_0^t |v(s)|^r \mathrm{d}s \tag{13}$$

Proof. There is $j'\geq 1$, $k'_{j'}\leq k_{j'}$ and $k'=\left(k_1,\ldots,k_{j'-1},k'_{j'}\right)$ such that $|k'|=r$. Hence setting $h' = (h_1, \dots, h_{j'})$ we get $\ell - r - |h'| \geq 1$ and

$$
\begin{aligned}
|p(k,h,u,t)| &\leq \frac{t^{\ell-r-|h'|-1}}{(l-r-|h'|-1)!} \int_0^t |p(k',h',u,s)|\,\mathrm{d}s \\
&\leq \frac{t^{\ell-r-|h'|-1}}{(\ell-r-|h'|-1)!} \frac{r^r(2r)!(r+1)^{r+|h'|+1}}{(r+|h'|)!} \sum_{i=0}^{r} t^{|h'|-i/r} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r} \int_0^t |v(s)|^{r-i}\mathrm{d}s \\
&\leq (\text{by (11) and (12)}) \; \frac{2^{\ell-1}}{(l-1)!} r^r(2r)!(r+1)^{r+|h'|+1} t^{\ell-r-1} \sum_{i=0}^{r} t^{\frac{i}{r}-\frac{i}{r}} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{\frac{i}{r}+\frac{r-i}{r}} \\
&\leq (2r)! r^{r+1} \frac{(2r+2)^l}{(l-1)!} t^{\ell-r-1} \int_0^t |v(s)|^r \mathrm{d}s.
\end{aligned}
$$

□

Corollary 3.2. *Let u be such that $|v(t)| \leq A \int_0^t |v(s)|\mathrm{d}s$ for some $A \geq 1$ and let $j \geq 1$; $h_1 \geq 0$; $k_1, \dots, k_j, h_2, \dots, h_j \geq 1$; $|h| + |k| = \ell \geq r+1$; $|k| = r$. Then*

$$
|p(k,h,u,t)| \leq (2r)! r^{r+1} A^r \frac{(r+1)^{\ell+1} t^{\ell-r-1}}{\ell!} \int_0^t |v(s)|^r \mathrm{d}s \tag{14}
$$

Proof.

$$
\begin{aligned}
|p(k,h,u,t)| &\leq \frac{r^r(2r)!(r+1)^{\ell+1}}{\ell!} \sum_{i=0}^{r} A^{r-i} \left(\int_0^t |v(s)|\mathrm{d}s\right)^{r-i} t^{\ell-r-i/r} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{i/r} \\
&\leq (\text{by (11)}) \frac{r^r(2r)!(r+1)^{\ell+1}}{\ell!} A^r \sum_{i=0}^{r} t^{(1-\frac{1}{r})(r-i)} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{1-\frac{i}{r}} t^{\ell-r-\frac{i}{r}} \left(\int_0^t |v(s)|^r \mathrm{d}s\right)^{\frac{i}{r}} \\
&\leq (2r)! r^{r+1} A^r \frac{(r+1)^{\ell+1} t^{\ell-r-1}}{\ell!} \int_0^t |v(s)|^r \mathrm{d}s,
\end{aligned}
$$

using $t^{r-i} \leq 1$ since $t \leq 1$. □

Now let us prove the property (P). Let $x(u,t)$ be the solution at time t relative to the control map u. It is known [1, 3] that there is $T^* > 0$ such that $\forall t \leq T^*$

$$
\varphi(x(u,t)) = \sum_{|h|+|k| \geq 0} \left(D(k,h)\varphi\right)(x_0)\, p(k,h,u,t) \tag{15}
$$

where the differential operator $D(k,h)$ is defined by

$$
D(k,h) = L_{f_0}^{h_1} \circ L_{f_1}^{k_1} \cdots \circ L_{f_0}^{h_j} \circ L_{f_1}^{k_j} \tag{16}
$$

and $p(k,h,u,t)$ is defined by (8).

By the properties of φ, taking into account that $f_0(x_0) = 0$ and

$$
\left(I_u^{r-i} \circ I \circ I_u^i\right) \mathrm{i}(t) = \sum_{j=0}^{r-i} \binom{r-i}{j} v^j(t) \int_0^t (-1)^{r-i-j} \frac{v^{r-j}}{(r-i)!i!}(\tau)\mathrm{d}\tau \tag{17}
$$

and

$$\mathrm{ad}_{f_1}^r f_0 = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} L_{f_1}^i \circ L_{f_0} \circ L_{f_1}^{r-i} \tag{18}$$

we get

$$\begin{aligned}
|\varphi(x(u,t))| = \Big| & L_{f_1}^r \varphi(x_0)\, v^r(t) + \sum_{i=0}^{r} (-1)^{r-i} L_{f_1}^i \circ L_{f_0} \circ L_{f_1}^{r-i} \varphi(x_0) \int_0^t \frac{v^r(\tau)\mathrm{d}\tau}{i!(r-i)!} \\
& + \sum_{i=0}^{r} L_{f_1}^i \circ L_{f_0} \circ L_{f_1}^{r-i} \varphi(x_0) \sum_{j=1}^{r-i} (-1)^{r-i-j} \binom{r-i}{j} \frac{v^j(t)}{(r-i)!i!} \int_0^t v^{r-j}(\tau)\mathrm{d}\tau \\
& + L_{f_1}^{r+1} \varphi(x_0)\, v^{r+1}(t) + \sum_{\substack{|k|\geq r \\ |k|+|h|\geq r+2}} D(k,h)\varphi(x_0)\, p(k,h,u,t) \Big| \\
& \geq \text{(being } r \text{ even)} \quad \frac{1}{r!} \int_0^t v^r(\tau)\mathrm{d}\tau - |\text{all other terms}|
\end{aligned}$$

Being φ, f_0, f_1 analytic there is $K_\varphi \geq 1$ such that

$$|D(k,h)\varphi(x_0)| \leq K_\varphi^{|k|+|h|}(|k|+|h|)! \tag{19}$$

(see [1, 3]). Hence if t and u are such that

$$|v(t)| \leq A \int_0^t |v(\tau)|\mathrm{d}\tau \text{ for some } A \geq 1 \tag{20}$$

we get by (13) and (14)

$$|\varphi(x(u,t))| \geq \int_0^t v^r(\tau)\mathrm{d}\tau \left(\frac{1}{r!} - \eta(t) \right)$$

where

$$\begin{aligned}
\eta(t) \leq & (AK_\varphi)^r\, r! t^{r-1} + (AK_\varphi)^{r+1} (r+1)! t^r + \sum_{i=0}^{r} \sum_{j=1}^{r-i} \frac{t^j}{j!i!} (AK_\varphi)^{r+1} (r+1)! \\
& + \sum_{\substack{|k|\geq r \\ \ell=|k|+|h|\geq r+2}} (2r)! r^{r+1} A^r \frac{(2r+2)^{\ell+1}}{(\ell-1)!} t^{\ell-r-1} K_\varphi^\ell \ell! \\
& \leq (2r)! (AK_\varphi r)^{r+1}\, t \left(t^{r-2} + t^{r-1} + \sum_{i=0}^{r} \sum_{j=1}^{r-i} \frac{t^{j-1}}{j!i!} + \sum_{\ell \geq r+2} 2^\ell (2r+2)^{\ell+1} K_\varphi^\ell \ell t^{\ell-r-2} \right).
\end{aligned}$$

The series $\sum_{\ell\geq r+2} (2(2r+2)K_\varphi)^{\ell+1} \ell t^{\ell-r-2}$ converges to a C^ω function for $|t| < \frac{1}{2(2r+2)K_\varphi}$, so that there is $T > 0$ such that for each $t < T$ and each u satisfying (20), $\eta(t) < \frac{1}{r!}$. Hence we get the proof of the property (P) by the following

Lemma 3.2. *Let ψ be an analytic function such that $L_{f_1}\psi(x_0) = 1$. There is $T^\star > 0$ and $A \geq 1$ such that if $t < T^*$ and u satisfies $\psi(x(u,t)) = 0$, then $|v(t)| \leq A \int_0^t |v(\tau)| \mathrm{d}\tau$.*

Proof. By (15) and the fact that $f_0(x_0) = 0$ we get

$$\begin{aligned}\psi(x(u,t)) &= L_{f_1}\psi(x_0)\, v(t) + \sum_{i\geq 1} L_{f_1} \circ L^i_{f_0}\psi(x_0)\,(I^i \circ I_u)\mathfrak{i}(t) \\ &+ \sum_{|k|\geq 2} D(k,h)\psi(x_0)\, p(k,h,u,t) = v(t) + \rho(t).\end{aligned}$$

Hence if $\psi(x(u,t)) = 0$, by (13) and (19) we get

$$\begin{aligned}|v(t)| = |\rho(t)| &\leq \sum_{\ell=|k|+|h|\geq 2} 2^\ell 2 \frac{4^\ell}{(\ell-1)!} K^\ell_\psi \ell! t^{\ell-2} \int_0^t |v(\tau)| \mathrm{d}\tau \\ &\leq \int_0^t |v(\tau)| \mathrm{d}\tau \cdot \sum_{\ell\geq 2} 2(8K_\psi)^\ell \ell t^{\ell-2}.\end{aligned}$$

The series $\sum_{\ell\geq 2} (8K_\psi)^\ell \ell t^{\ell-2}$ converges to an increasing C^ω function for $0 \leq t < \frac{1}{8K_\psi}$ and the lemma is proven. ☐

4 Proof of Theorem 3

The proof is a slight modification of the one of the Theorem 1. First of all we need a C^ω map $\tilde{\varphi}$ such that

i) $L_{\mathrm{ad}^r_{f_1} f_0}\tilde{\varphi}(x_0) = 1$

ii) $L^{h_1}_{f_0} \circ L^{k_1}_{f_1} \circ \cdots \circ L^{h_j}_{f_0} \circ L^{k_1}_{f_1}\tilde{\varphi}(x_0) = 0$ if $|k| \leq r-1$.

To get such a $\tilde{\varphi}$ let us start by a φ such that

$$L_{\mathrm{ad}^r_{f_1} f_0}\varphi(x_0) = 1, \quad L_f\varphi(x_0) = 0 \quad \forall f \in S^{r-1}$$

and $L_{f_0}\varphi \equiv c$ (constant in a neighbourhood of x_0).

Let us apply Theorem 2 to the filtration F given by $F_0 = \{0\}$, $F_i = S^i$, starting from an adapted chart with the following properties:

a) If $f_0(x_0) \notin S^{r-1}(x_0)$ we choose $x_1, \ldots, x_n$ such that $L_{f_0}x_i \equiv 0 \quad i = 1, \ldots, m_{r-1}$

b) If $f_0(x_0) \in S^{r-1}(x_0)$, let $k = \max\{j : f_0(x_0) \notin S^j(x_0)\}$; we choose $x_1, \ldots, x_n$ such that $L_{f_0}x_{m_k+1} \equiv 1$ and $L_{f_0}x_j \equiv 0 \quad j \neq m_k + 1$.

In both cases a) and b) we get

$$L_{f_0}\tilde{\varphi} = L_{f_0}\varphi = c \tag{21}$$

In the case a) $\tilde{\varphi} = \varphi +$ (a function depending only on $x_1, \ldots, x_{m_r}$) and (21) can be easily proven.

Concerning the case b) we shall prove $L_{f_0}\varphi_s = L_{f_0}\varphi$ by induction on s.

$$L_{f_0}\varphi_1 = L_{f_0}\varphi = c. \quad L_{f_0}\varphi_{s+1} = \text{(see Section 2) } L_{f_0}\varphi_s - \sum_{\nu\in\chi(s+1,r-1)} \frac{D^\nu \varphi_s(x_0)}{\nu!} L_{f_0} x^\nu.$$

If $\nu_k = 0$, then $L_{f_0}x^\nu = 0$. If $\nu_k \neq 0$, then

$$\begin{aligned} D^\nu \varphi_s(x_0) &= L_{g_k} \circ L_{g_{m_r}}^{\nu_{m_r}} \circ \cdots \circ L_{g_k}^{\nu_k - 1} \circ \cdots \circ L_{g_1}^{\nu_1}\varphi_s(x_0) \\ &= \text{(by the choice of the chart) } L_{f_0} \circ D^{\nu'}\varphi_s(x_0) \\ &= D^{\nu'} \circ f_0\varphi_s(x_0) + D^{\nu''}\varphi_s(x_0) \end{aligned}$$

where $D^{\nu'}, D^{\nu''} \in A^{r-1}$. Hence $D^{\nu'} \circ f_0\varphi_s(x_0) = D^{\nu'}c = 0$, $D^{\nu''}\varphi_s(x_0) = 0$ and (21) is proven.

By (21) the proof of ii) follows easily.

In what follows we set $\varphi = \tilde{\varphi}$.

Let ψ be a C^ω function such that $L_{f_1}\psi(x_0) = 1$. We shall prove that there is $T > 0$ with the following property:

$$\begin{aligned} &\text{If } x \in R(x_0, t), t \le T, \text{ is such that } \psi(x) = \psi(\exp t f_0(x_0)), \\ &\quad \text{then } \varphi(x) \ge \varphi(\exp t f_0(x_0)). \end{aligned} \tag{P'}$$

In fact $\varphi(x(u,t)) - \varphi(\exp t f_0(x_0)) = \sum_{|k|\ge r} D(k,h)\varphi(x_0)p(k,h,u,t)$ and we get the property (P') as the property (P) in Section 3 by the following

Lemma 4.1. *Let ψ be an analytic function such that $L_{f_1}\psi(x_0) = 1$. There is $T^* > 0$ and $A \ge 1$ such that if $t < T^*$ and u satisfies $\psi(x(u,t)) = \psi(\exp t f_0(x_0))$, then $|v(t)| \le A\int_0^t |v(\tau)| \mathrm{d}\tau$.*

Proof. By (15) we get

$$\begin{aligned} &\psi(x(u,t)) - \psi(\exp t f_0(x_0)) = \sum_{i\ge 0} L_{f_0}^i \circ L_{f_1}\psi(x_0) \int_0^t \frac{s^i}{i!} u(s)\mathrm{d}s \\ &+ \sum_{i\ge 1} L_{f_1} \circ L_{f_0}^i \psi(x_0) \left(I^i \circ I_u\right) \mathrm{i}(t) + \sum_{|k|\ge 2} D(k,h)\psi(x_0)\, p(k,h,u,t) \\ &= v(t) \sum_{i\ge 0} L_{f_0}^i \circ L_{f_1}\psi(x_0) \frac{t^i}{i!} - \sum_{i\ge 1} L_{f_0}^i \circ L_{f_1}\psi(x_0) \int_0^t \frac{s^{i-1}}{(i-1)!} v(s)\mathrm{d}s + \text{(the other terms)}. \end{aligned}$$

Hence by (13) and (19) we get

$$\left| v(t) \left(1 + \sum_{i\ge 1} L_{f_0}^i \circ L_{f_1}\psi(x_0) \frac{t^i}{i!}\right)\right| \le \sum_{\substack{|k|\ge 1 \\ \ell = |h| + |k| \ge 2}} K_\psi^\ell \ell! 2 \frac{4^\ell}{(\ell-1)!} t^{\ell-2} \int_0^t |v(\tau)|\mathrm{d}\tau.$$

The series $\sum_{i\ge 1} L_{f_0}^i \circ L_{f_1}\psi(x_0) \frac{t^i}{i!}$ converges to a C^ω function $\rho(t)$ for $0 \le t < \frac{1}{K_\psi}$. $\rho(0) = 0$, so there is T_1 such that $\rho(t) > -\frac{1}{2}$ for each $t \le T_1$.

We get for each $t \leq T_1$

$$|v(t)| \leq 2 \sum_{\ell \geq 2} 2^\ell 2 K_\psi^\ell 4^\ell \ell t^{\ell-2} \int_0^t |v(\tau)| \mathrm{d}\tau = 4 \sum_{\ell \geq 2} (8 K_\psi)^\ell \ell t^{\ell-2} \int_0^t |v(\tau)| \mathrm{d}\tau.$$

The proof follows as in Lemma 3.2. □

References

[1] Fliess, M. – "Fonctionnelles causales non linéaires et indéterminées non commutatives", Bull. Soc. Math. France, 109 (1981), 3-40.

[2] Sussmann, H.J. and Jurdjevic, V. – "Controllability of non linear systems", J. Diff. Eq., 12 (1972), 95-116.

[3] Sussmann, H.J. – "Lie brackets and local controllability: a sufficient condition for scalar-input systems", SIAM J. Control Optim, 21 (1983), 686-713.