The local realization of generating series of finite Lie rank*

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1 Introduction

Realization of nonlinear systems by state-space is a classical problem in control theory. This problem has been completely solved by Kalman [10] in the case of linear systems. Similarly, it was solved for bilinear systems (see Brockett [2], d'Alessandro, Isidori, Ruberti [1], Fliess [3], Sussmann [12]). In the general case, let us mention the work of Sussmann [13], Hermann, Krener [7] and Jakubczyk [9]: they assume that the solutions are regular at any time and for any inputs. This restriction lead Fliess to study local realization of nonlinear systems [5].

The present paper deals with the latter subject; there are no new results here, but we give elementary proofs of Fliess' results [5]. He shows that to these systems correspond generating series (which are noncommutative formal power series) which have the property that their Lie rank is finite[1]. Then he shows existence and unicity of locally reduced (or minimal) realizations and shows that minimality is equivalent to the two following properties: weak local controllability and weak local observability.

We want to give here a simple proof of these results, especially the proof of unicity, for which Fliess uses in [5] sophisticated results on Lie groups and algebras. The present work extends the "syntactic" approach of series with finite Lie rank, as studied in [5], and of nonlinear systems. Hopefully, this will lead to find a realization, which is minimal (in some sense) and which is given directly by the generating series; this would be analogue to the bilinear case, where the minimal state-space is directly given by the Hankel matrix. Our main tool here is the theorem of Poincaré-Birkhoff-Witt. We treat only the "analytic" case of [5]. The "formal" case is simpler and is obtained by omitting in the sequel everything dealing with convergence or majoration of coefficients.

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[1]Bilinear systems correspond to generating series whose rank is finite.

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2 Generating series of finite Lie rank

We consider a system of the following form

$$\begin{cases} \dot{q}(t) = A_0(q) + \sum_{i=1}^{n} u_i(t) A_i(q) \\ y(t) = h(q) \end{cases} \tag{1}$$

where the state q belongs to a connected analytic $\mathbb{R}$-variety Q, where the A_i's are analytic vector fields and h a real analytic function defined in a neighborhood of the given initial state $q(0)$. The inputs $u_1, \ldots, u_n$ are real and piecewise continuous.

By Fliess [4] th.III.2, the output y of system (1) is given by the following formula (for small enough time and inputs):

$$y(t) = h|_{q(0)} + \sum_{\nu \geq 0} \sum_{j_0,\ldots,j_\nu \geq 0}^{n} \left(A_{j_0} \ldots A_{j_\nu} h|_{q(0)} \right) \int_0^t \mathrm{d}\xi_{j_\nu} \ldots \mathrm{d}\xi_{j_0} \tag{2}$$

where $|_q(0)$ means evaluation in $q(0)$ and where the iterated integrals $\int \mathrm{d}\xi_{j_\nu} \ldots \mathrm{d}\xi_0$ are defined by the formulas (u_0 is the constant function equal to 1):

$$\int_0^t \mathrm{d}\xi_j = \int_0^t u_j(\tau)\mathrm{d}\tau, \text{ and if } \nu \geq 1$$
$$\int_0^t \mathrm{d}\xi_{j_\nu} \ldots \mathrm{d}\xi_{j_0} = \int_0^t \left[u_{j_\nu}(\tau)\mathrm{d}\tau \int_0^\tau \mathrm{d}\xi_{j_{\nu-1}} \ldots \mathrm{d}\xi_{j_0} \right]$$

Actually, the input-output behaviour of system (1) is completely defined by its **generating series** ([4] p.12), which is a noncommutative formal power series in the variables $x_0, \ldots, x_n$:

$$g = h|_{q(0)} + \sum_{\nu \geq 0} \sum_{j_0,\ldots,j_\nu \geq 0}^{n} \left(A_{j_0} \ldots A_{j_\nu} h|_{q(0)} \right) x_{j_\nu} \ldots x_{j_0} \tag{3}$$

Recall that a formal power series in the noncommutative variables $x_0, \ldots, x_n$ is a mapping g from the free monoid X^* generated by $X = \{x_0, \ldots x_n\}$ into $\mathbb{R}$, denoted by $w \to (g, w)$, for any **word** (= element of X^*) w, including the empty word, denoted by 1. The formal series g will also be denoted by

$$g = \sum_{w \in X^*} (g, w) w$$

The set of formal series is denoted by $\mathbb{R}\langle\langle X \rangle\rangle$. Following [5], we give a definition, inspired by (3). First, let us say that a (commutative) formal series in

$\mathbb{R}[[q]] = \mathbb{R}[[q_1, \dots, q_d]]$ (where the q_i's are commutative variables) is **convergent** if it converges in a neighborhood of 0; similarly, we say that a **formal vector field**, that is, an operator of $\mathbb{R}[[q]]$ of the form[2]

$$A = \sum_{1 \le k \le d} \theta_k \left(q_1, \dots, q_d\right) \frac{\partial}{\partial q_k}$$

where the θ_k are in $\mathbb{R}[[q]]$, is **convergent**, if the θ_k's are convergent formal series. Because of the reverse order in (3) of the A's and the x's, we let the formal vector fields operate at the right of the formal series.

Definition 1. *A formal series $g \in \mathbb{R}\langle\langle X\rangle\rangle$ is **produced differentially** if there exists an integer d, an homomorphism μ from the free monoid into the multiplicative monoid of the endomorphisms of $\mathbb{R}[[q]]$ such that μx is a formal vector field for any x in X, a convergent formal series h in $\mathbb{R}[[q]]$ such that*

$$\forall w \in X^*, \quad (g, w) = h(\mu w)\big|_0$$

(that is, (g, w) is the constant term of the series which is the image of h under the operator μw).

*We call the couple (μ, h) a **differential representation** of g, of **dimension** d.*

It is now obvious that if g is the generating series of system (1), then it is produced differentially (take local coordinates around $q(0)$ in (3)). Conversely, if a series g is produced differentially, then one may associate to it a system of type (1), whose generating series is g. Thus, **the study of differential representations of series is equivalent to the study of local realizations of systems like** (1).

Before stating the main result, we need some notations. Denote by $\mathbb{R}\langle X\rangle$ the set of noncommutative **polynomials**, that is, formal series having only a finite number of nonzero coefficients. Then $\mathbb{R}\langle\langle X\rangle\rangle$ is isomorphic to the dual of $\mathbb{R}\langle X\rangle$ (because $\mathbb{R}\langle X\rangle \simeq \mathbb{R}^{(X^*)}$ and $\mathbb{R}\langle\langle X\rangle\rangle \simeq \mathbb{R}^{X^*}$, as vector spaces), with duality

$$\begin{aligned} &\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle X\rangle \to \mathbb{R} \\ &(S, P) \mapsto (S, P) = \sum_{w \in X^*} (S, w)(P, w) \end{aligned}$$

The set $\mathbb{R}\langle X\rangle$ possesses an associative product, which extends linearly the concatenation of words in X^*. Thus, the algebra $\mathbb{R}\langle X\rangle$ acts naturally at the left and at the right of $\mathbb{R}\langle\langle X\rangle\rangle$, in the following way

$$\begin{aligned} S \circ P &= \sum_{w \in X^*} (S, Pw) w \\ P \circ S &= \sum_{w \in X^*} (S, wP) w \end{aligned} \tag{4}$$

[2] A formal vector field may also be defined as a derivation of the algebra $\mathbb{R}[[q]]$, which is continuous for the usual topology of $\mathbb{R}[[q]]$.

These two actions are associative (that is, $S \circ PQ = (S \circ P) \circ Q$ and $PQ \circ S = P \circ (Q \circ S)$) and commute each to another (that is $P \circ (S \circ Q) = (P \circ S) \circ Q$).

We denote by $L\langle X \rangle$ the Lie algebra generated by the elements of X: an element of $L\langle X \rangle$ is called a **Lie polynomial**.

Definition 2. *The **Lie rank** of a formal series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is the dimension of the vector space*

$$\{P \circ g \mid P \in L\langle X \rangle\}$$

*We say that a formal series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ satisfies to the **convergence hypothesis*** (C) *if: for any Lie polynomials $P_1, \ldots, P_q$, there exist constants α and C such that*

$$\forall i_1, \ldots, i_q \in \mathbb{N}, \quad \left| \left(g, P_1^{i_1} \ldots P_q^{i_q} \right) \right| \leq \alpha C^{i_1 + \ldots + i_q} i_1! \ldots i_q! \tag{C}$$

Note that one may replace $i_1! \ldots i_q!$ by $(i_1 + \ldots + i_q)!$, because these numbers satisfy to

$$i_1! \ldots i_q! \leq (i_1 + \ldots + i_q)! \leq q^{i_1 + \ldots + i_q} i_1! \ldots i_q!$$

Now, we can state the following fundamental result.

Theorem (Fliess [5])**.** *A series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it satisfies to the convergence hypothesis. In this case, its Lie rank d is equal to the smallest dimension of all its differential representations. If (μ, h) and (μ', h') are two differential representations of dimension d of g (with the same $\mathbb{R}[[q]]$), then there exists a continuous and convergent*[3] *automorphism of $\mathbb{R}[[q]]$ such that $h' = \phi(h)$ and $\phi(k\mu w) = \phi(k)\mu' w$ for any word w and any series k in $\mathbb{R}[[q]]$.*

Note that the fact that ϕ is bijection is equivalent to the well-known Jacobian condition

$$\left| \frac{\partial \phi\,(q_j)}{\partial q_i}(0) \right| \neq 0$$

The third part of this paper is devoted to the proof of the theorem. We need before some definition and results.

On $\mathbb{R}\langle\langle X \rangle\rangle$ is defined another product, the **shuffle**, which is associative and commutative and which will be important in the sequel; this is not a surprise, as the shuffle intervenes already for systems of the form (1): indeed, by Fliess ([4] prop. II.4), if two systems admit the generating series g_1 and g_2, then the system whose output is the product of the outputs of these two systems admits as generating series the shuffle of g_1 and g_2.

If w is a word of length $|w| = p$ ($w = y_1 \ldots y_p$, $y_i \in X$) and $I \subset \{1, \ldots, p\}$, denote by $w|I$ the word $y_{i_1} \ldots y_{i_r}$, with $I = \{i_1 < \ldots < i_r\}$. Then the shuffle $u \star v$ of two words u and v is defined as

$$u \star v = \sum w(I, J)$$

[3] that is, for any q_i, $\phi\,(q_i)$ is convergent and without constant term.

where the sum is extended to all partitions (I, J) of $(1, \ldots, |u| + |v|)$ with $\mathrm{Card}(I) = |u|$, $\mathrm{Card}(J) = |v|$ and where the word $w(I, J)$, of length $|u| + |v|$, is defined by

$$w(I, J)|I = u, \quad w(I, J)|J = v$$

Note that there are $\binom{|u|+|v|}{|u|}$ such words.

Example. $y_1y_2 \star y_2y_3 = 2y_1y_2^2y_3 + y_1y_2y_3y_2 + y_2y_1y_2y_3 + y_2y_1y_3y_2 + y_2y_3y_1y_2$.

The shuffle $S \star T$ of two series in $\mathbb{R}\langle\langle X\rangle\rangle$ is then defined by

$$S \star T = \sum_{u,v \in X^*} (S, u)(T, v) u \star v$$

(extension by linearity and continuity). **From now on, we denote $S \star T$ simply by** ST: there will be no ambiguity because we consider here only the shuffle structure of $\mathbb{R}\langle\langle X\rangle\rangle$[4]. However, $\mathbb{R}\langle X\rangle$ will be considered with its concatenation structure.

Lemma 1. *Let $c_p : \mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle^{\otimes p}$ be the concatenation homomorphism defined by*

$$c_p(x) = x \otimes 1 \cdots \otimes 1 + 1 \otimes x \cdots \otimes 1 + 1 \otimes 1 \cdots \otimes x$$

For any series $S_1, \ldots, S_p$ in $\mathbb{R}\langle\langle X\rangle\rangle$ and any polynomial P one has

$$(S_1 \cdots S_P, P) = (S_1 \otimes \cdots \otimes S_p, c_p(P))$$

where the duality between $\mathbb{R}\langle\langle X\rangle\rangle$ and $\mathbb{R}\langle X\rangle$ is extended to $\mathbb{R}\langle\langle X\rangle\rangle^{\otimes p}$ and $\mathbb{R}\langle X\rangle^{\otimes p}$, that is

$$(S_1 \otimes \cdots \otimes S_p, P_1 \otimes \cdots \otimes P_p) = (S_1, P_1) \ldots (S_p, P_p)$$

Proof. It is enough to prove the lemma when the S's and the P's are words; in this case, it is a simple consequence of the definition of the shuffle. □

Lemma 2. *If P is a Lie polynomial, then*

$$c_p(P) = P \otimes 1 \cdots \otimes 1 + 1 \otimes P \cdots \otimes 1 + 1 \otimes 1 \cdots \otimes P$$

Proof. This is true if $P = x \in X$. Moreover, if it is true for P and Q, then also for $[P, Q] = PQ - QP$, as is easily verified. So, the lemma follows. □

We shall use the following classical result.

Theorem (Poincaré-Birkhoff-Witt)**.** *Let $P_1, \ldots, P_n, \ldots$ be a basis of $L\langle X\rangle$. Then the polynomials*

$$P_{i_1}^{j_1} \ldots P_{i_r}^{j_r}, \quad r \geq 0, \quad i_1, \ldots, i_r, j_1, \ldots, j_r \geq 1, \quad i_1 < \ldots < i_r$$

form a basis of $\mathbb{R}\langle X\rangle$.

For a proof, see [8], corollary C p. 92.

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[4] $\mathbb{R}\langle\langle X\rangle\rangle$ possesses also the noncommutative product which extends the concatenation of words.

3 Proof of theorem

a) Let g be a series having the differential representation (μ, g) of dimension d. We show that the Lie rank of g is $\leq d$ and that g satisfies (C). For the first assertion, we do as in [5] II.a. Let T_i $(1 \leq i \leq d)$ be the series

$$T_i = \sum_w \left. \frac{\partial(h\mu w)}{\partial q_i} \right|_0 w$$

Let P be a Lie element. We extend $\mu : X^* \to \mathrm{End}(\mathbb{R}[[q]])$ to an algebra homomorphism from $\mathbb{R}\langle X \rangle$ into $\mathrm{End}(\mathbb{R}[[q]])$. Then $\mu(P)$ is a continuous derivation (because derivations form a Lie algebra) of $\mathbb{R}[[q]]$, hence

$$\begin{aligned}(P \circ S, w) = (S, wP) &= h\mu(wP)|_0 = (h\mu w)\mu P|_0 \\ &= \left. \left[\sum_{1 \leq i \leq d} \frac{\partial(h\mu w)}{\partial q_i} (q_i \mu P) \right] \right|_0 = \left. \sum_i (T_i, w)\,(q_i \mu P) \right|_0\end{aligned}$$

Thus $P \circ S = \sum_i (q_i \mu P)|_0 \, T_i$ is a linear combination of $T_1, \ldots, T_d$.

We show now that g satisfies to (C)[5]. By assumption, the series h and $q_i \mu x_j$ $(1 \leq i \leq d,\ 0 \leq j \leq n)$ are convergent in a neighborhood of 0. We thus may find constants α and C such these series are all bounded by the same series

$$f = \alpha \sum_r C^r q^r = \alpha(1 - Cq)^{-1}$$

in the sense that the coefficient of $q_1^{i_1} \ldots q_d^{i_d}$ is bounded by $\alpha C^{i_1 + \cdots + i_d}$. It is then easily shown that $h\mu w$ (w of length p) is bounded by the series $f\Delta^p$, where Δ is the differential operator

$$\Delta = d \frac{\alpha}{1 - Cq} \frac{\partial}{\partial q}$$

A simple computation shows that

$$f\Delta^p = \frac{\alpha(d\alpha C)^p 1 \cdot 3 \cdots (2p-1)}{(1 - Cq)^{2p+1}}$$

Hence, we obtain

$$|(g, w)| = |h\mu w|_0| \leq f\Delta^p|_0 = \frac{\alpha(d\alpha C)^p (2p)!}{2^p p!}$$

As $\binom{2p}{p}$ is bounded by 2^p, we obtain

$$|(g, w)| \leq \alpha(2d\alpha C)^p p!$$

[5] We follow Gröbner [6], chap. 1.

Now let $P_1, \dots, P_q$ be Lie polynomials, $y_1, \dots, y_q$ new letters and $g' \in \mathbb{R}\langle\langle y_1, \dots, y_q \rangle\rangle$ the series having the differential representation (μ', h) with $\mu' y_i = \mu P_i$. The previous paragraph implies that if w is a word of length p in the y's, then one has an inequality of the form

$$|(g', w)| \leq \beta D^p p!$$

which proves (C), in view of the remark following the definition of (C).

b) We come now to the converse, which is the essential part of the theorem. Let g be a series of Lie rank d and which satisfies to the convergence hypothesis.

Let $L_0 = \{P \in L\langle X \rangle \mid P \circ g = 0\}$. By assumption, L_0 is of finite codimension d in $L\langle X \rangle$. Moreover, it is a sub-Lie-algebra of $L\langle X \rangle$. Let $P_1, \dots, P_n, \dots$ be a basis of $L\langle X \rangle$ such that $P_{d+1}, \dots, P_n, \dots$ is a basis of L_0. Let $S_1, \dots, S_d$ the series defined by $(S_i, P_j) = \delta_{i,j}$ and $\left(S_i, P_{i_1}^{j_1} \dots P_{i_r}^{j_r}\right) = 0$ if $r = 0$ or if $j_1 + \dots + j_r \geq 2$ (use the P-B-W theorem). Then

$$g = \sum_{i_1, \dots, i_d \geq 0} \frac{\left(g, P_1^{i_1} \dots P_d^{i_d}\right)}{i_1! \dots i_d!} S_1^{i_1} \dots S_d^{i_d} \tag{1}$$

Note that the S_i's have zero constant term, which ensures that the sum is well-defined. In fact, we shall prove a more general result.

Proposition 1. *Let L_0 be a sub-Lie-algebra of $L\langle X \rangle$ of codimension d. Let $P_1, \dots, P_d$ be a basis of $L\langle X \rangle$ modulo L_0 and $S_1, \dots, S_d$ series without constant term such that $(S_i, P_j) = \delta_{i,j}$ and which vanish on the left ideal $J = \mathbb{R}\langle X \rangle L_0$. Then*

$$J^{\perp} = \{S \mid (S, P) = 0, \forall P \in J\} = \mathbb{R}\,[[S_1, \dots, S_d]]$$

Moreover, any $S \in \mathbb{R}\,[[S_1, \dots, S_d]]$ has a unique expression as series in the S_i's.

Proof. **1)** We show that $J^{\perp}$ contains $\mathbb{R}[[S_1, \dots, S_d]]$. As $J^{\perp}$ is closed for the usual topology of $\mathbb{R}\langle\langle X \rangle\rangle$ and closed for the operation $T \to T \circ P$ ($P \in \mathbb{R}\langle X \rangle$), because J is a left ideal, it is enough to show that it is also closed for the shuffle. Let S, T in $J^{\perp}$: it suffices to show that $(ST) \circ w$ vanishes on L_0 for any word w (it will imply that ST vanishes on $X^* L_0$, hence on J).

Lemma 3. *$T \to T \circ x$ is a derivation for the shuffle.*

By the lemma (which is well-known), $(ST) \circ w$ is a linear combination of series of the form $(S \circ u)(T \circ v)$. As S, T vanish on J, we obtain that $S \circ u$, $T \circ v$ vanish on L_0.

Lemma 4. *Let $i > j$, $T_1, \dots, T_i$ series without constant term and $Q_1, \dots, Q_j$ be Lie polynomials. Then*

$$(T_1 \dots T_i, Q_1 \dots Q_j) = 0$$

Proof. The lemma follows from Lemmas 1 and 2: write that $(T_1 \ldots T_i, Q_1 \ldots Q_j) = (T_1 \otimes \ldots \otimes T_i, c_i\,(Q_1 \ldots Q_j))$ and note that $c_i\,(Q_1 \ldots Q_j)$, which is the product from 1 to j of

$$Q_\ell \otimes 1 \ldots \otimes 1 + 1 \otimes Q_\ell \ldots \otimes 1 + 1 \otimes 1 \ldots \otimes Q_\ell$$

is a sum of tensors each of which has a 1 as factor; as $(T_k, 1) = 0$, we obtain Lemma 4. □

By this lemma, the shuffle of two series which vanish on L_0 is still vanishing on L_0. Hence each $(S \circ u)(T \circ v)$ vanishes on L_0, and so does also $(ST) \circ w$.

2) We prove that $J^\perp \subset \mathbb{R}\,[[S_1, \ldots, S_d]]$. Let $S \in J^\perp$. We have to find coefficients $a_{i_1,\ldots,i_d}$ such that

$$S = \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} S_1^{i_1} \ldots S_d^{i_d} \tag{2}$$

For $I = (i_1, \ldots, i_d)$, let $a(I) = a_{i_1,\ldots,i_d}$, $S(I) = S_1^{i_1} \ldots S_d^{i_d}$, $P(I) = P_1^{i_1} \ldots P_d^{i_d}$, $|I| = i_1 + \ldots + i_d$ and $I! = i_1! \ldots i_d!$. By the P-B-W theorem, we have to show that both sides of (2) have the same value on any polynomial of the form

$$P_1^{i_1} \ldots P_r^{i_r}, \quad r \geq 0, \quad 1 \leq i_1 < \ldots < i_r, \quad 1 \leq j_1, \ldots j_r$$

But, if $r > d$, then this polynomial is in J, hence both sides map it to zero. Hence, we have to find coefficients $a(I)$ such that

$$\forall J, \quad (S, P(J)) = \sum_I a(I)(S(I), P(J)) \tag{3}$$

Lemma 5. *(i) If $|I| > |J|$ or if $|I| = |J|$ and $I \neq J$, then $(S(I), P(J)) = 0$*
(ii) $(S(I), P(I)) = |I|!$

Proof. If $|I| > |J|$, use Lemma 4. Otherwise, use Lemmas 1 and 2 to prove that

$$(T_1 \ldots T_n, Q_1 \cdots Q_n) = \sum_{\sigma \in \mathfrak{S}_n} \left(T_1, Q_{\sigma(1)}\right) \ldots \left(T_n, Q_{\sigma(n)}\right)$$

which is true for any series T_i without constant term and any Lie polynomials Q_j. □

Lemma 5 shows that (3) is a triangular system of linear equations in the a's, with $I!$ on the diagonal. Hence, it admits one and only one solution, which proves the proposition. □

(1) is proved by using the fact that in this case one has: $|I| < |J|$ implies $(S(I), P(J)) = 0$ (use Lemmas 1 and 2).

(1) gives almost the differential representation of g. Indeed, g is given in (1) as a commutative series in $S_1, \ldots, S_d$, and by Proposition 1, $\mathbb{R}\,[[S_1, \ldots, S_d]]$ is isomorphic to an algebra of commutative formal power series in d variables. We have to define μ and h. We let $h = g$ and define μw $(w \in X^*)$ as $T \to T \circ w$.

By Lemma 3, μx is a continuous derivation, which maps $\mathbb{R}[[S_1, \ldots, S_d]]$ into itself (by the proposition). Moreover

$$(g, w) = (g \circ w, 1) = (h\mu w, 1)$$

and the constant term of $h\mu w$ is also the constant term of $h\mu w$ when expressed as a series in the S_i's (because the latter are without constant term). We still have to show that the operators μx are convergent. The series $P_1 \circ g, \ldots, P_d \circ g$ being linearly independent, we may find polynomials $Q_1, \ldots, Q_d$ such that

$$(P_i \circ g, Q_j) = \delta_{i,j}$$

Let $T_1, \ldots, T_d$ be defined by

$$T_i = g \circ Q_i - (g, Q_i) \tag{4}$$

The T_i's are without constant term, vanish on J and we have

$$(T_i, P_j) = (g \circ Q_i, P_j) = (g, Q_i P_j) = (P_j \circ g, Q_i) = \delta_{j,i}$$

Hence by the proposition

$$\mathbb{R}\,[[T_1, \ldots, T_d]] = J^{\perp} = \mathbb{R}\,[[S_1, \ldots, S_d]]$$

As for g, we have relations of the form

$$T_j = \sum_I \frac{(T_j, P(I))}{I!} S(I) \tag{5}$$

Moreover by (4), the T_j's satisfy to the convergence hypothesis. Thus, by (5), the T_j's may be written as convergent series in the S_i's. We use now the following classical result.

Theorem (of implicit functions)**.** *Let $t_1, \ldots, t_d$ convergent series in $\mathbb{R}\,[[s_1, \ldots, s_d]]$ without constant term and such that $\mathbb{R}[[s]] = \mathbb{R}[[t]]$. Then each s_i may be written as a convergent series in $t_1, \ldots, t_d$.*

By this theorem, each S_i is a convergent series in $T_1, \ldots, T_d$. As previously, the series

$$T_j \mu x = T_j \circ x$$

satisfy to (C) and are thus convergent series in the S_i's; hence, they are also convergent series when expressed as series in the T_i's. All this shows that (μ, h) is a differential representation of g.

c) Now, let g be a series of Lie rank d and (μ, h) be a differential representation of dimension d of g. We use the notations of paragraph b).

Lemma 6. *The mapping* $\eta : \mathbb{R}[[q]] \to \mathbb{R}\langle\langle X\rangle\rangle$ *which maps* k *onto* $\sum_w \left(k\mu w|_0\right) w$ *is a continuous homomorphism (for the shuffle), such that for any word* w *one has* $\eta(k\mu w) = \eta(k) \circ w$.

Proof. This lemma is a simple consequence of [4] prop. III. 1. □

Lemma 7. *The mapping* $\theta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle$ *which maps* S *onto* $\sum_I \frac{(S,P(I))}{I!} S(I)$ *is a continuous shuffle homomorphism.*

Proof. In order to prove this lemma, note that if $S_1, \ldots, S_n, \ldots$ are the series which are the elements of the dual basis of the P-B-W basis constructed on $P_1, \ldots, P_n, \ldots$, which correspond to $P_1, \ldots, P_n, \ldots$ then $\mathbb{R}\langle\langle X\rangle\rangle = \mathbb{R}[[S_1, \ldots, S_n, \ldots]]$ and the mapping of Lemma 7 is just a projection: $S_1 \to S_1, \ldots, S_d \to S_d, S_n \to 0$ if $n > d$. □

By Lemma 6, $\eta(\mathbb{R}[[q]])$ contains $g = \eta(h)$ and is closed for the operations $T \to T \circ w$. Hence, it contains the T_i's defined by (4), hence also $\mathbb{R}[[T_i]] = \mathbb{R}[[S_i]]$. As the restriction of θ to $R\left[[S_i]\right]$ is the identity, the mapping $\phi = \theta \circ \eta : \mathbb{R}[[q]] \to \mathbb{R}[[S_i]]$ is surjective. As it is a continuous homomorphism from an algebra of formal power series in d commutative variables into another, ϕ is also injective. We deduce that η is also a bijection $\mathbb{R}[[q]] \to \mathbb{R}[[S_i]]$: first, η is injective (otherwise $\phi = \theta \circ \eta$ is not); moreover we may find series $k_1, \ldots, k_d$ in $\mathbb{R}[[q]]$ such that $q_i \to k_i$ is a continuous automorphism of $\mathbb{R}[[q]]$ and such that $\phi\left(k_i\right) = S_i$. As $\eta(R[[q]])$ contains $\mathbb{R}[[S_i]]$ and $\mathbb{R}[[q]] = \mathbb{R}[[k_i]]$, we have that S_i is a series in the $\eta\left(k_i\right)$'s: $S_i = s(\eta(k_1), \ldots, \eta(k_d))$.

Apply θ : then $S_i = \theta\left(S_i\right) = s\left(S_1, \ldots, S_d\right)$, which shows that s has only the term S_i, hence $S_i = \eta\left(k_i\right)$. This shows that $\eta(\mathbb{R}[[q]]) = \eta(\mathbb{R}[[k_i]]) = \mathbb{R}[[\eta(k_i)]] = \mathbb{R}[[S_i]]$ and η is a bijection as claimed.

We still have to prove the assertions about convergence (we have already seen that any differential representation of dimension d of g is isomorphic to the one defined by $S_1, \ldots, S_d$). By assumption, the series g and the operators $T \to T \circ x$ of $\mathbb{R}[[S_i]]$ are convergent when expressed as series in the $\eta\left(q_i\right)$'s (as h and μx are convergent, when expressed in the q_i's). Hence the series T_i of (4) are convergent in the $\eta(q_i)$'s. Hence η is a convergent isomorphism from $\mathbb{R}[[q]]$ onto $\mathbb{R}[[T_i]]$. This ends the proof of the theorem.

References

[1] P. d'Alessandro, A. Isidori, A. Ruberti: Realization and structure theory of bilinear systems, Siam J. Control 12 (1974) 517-535.

[2] R.W. Brockett: On the algebraic structure of bilinear systems, In: Theory and Application of Variable Structure Systems (Mohler, Ruberti, ed.), Acad. Press (1972) 153-168.

[3] M. Fliess: Sur la réalisation des systèmes dynamiques bilinéaires, C.R. Acad. Sci. Paris A 277 (1973) 923-926.

[4] M. Fliess: Fonctionnelles causales non linéaires et indéterminées non commutatives, Bull. Soc. Math. France 109 (1981) 3-40.

[5] M. Fliess: Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives, Invent. Math. 71 (1983) 521-537.

[6] W. Gröbner: Die Lie Reihen und ihre Anwendungen, Berlin, VEB Deutscher Verlag der Wissenschaften (1967).

[7] R. Hermann, A.J. Krener: Nonlinear controllability and observability, IEEE Trans. Automat. Control 22 (1977) 728-740.

[8] J.E. Humphreys: Introduction to Lie algebras and representation theory, Springer Verlag (1980).

[9] B. Jakubczyk: Existence and uniqueness of realizations of nonlinear systems, SIAM J. Control Optimiz 18 (1980) 455-471.

[10] R.E. Kalman: Mathematical description of linear dynamical systems, SIAM J. Control 1 (1963) 152-162.

[11] M. Lothaire, Combinatorics on words, Addison Wesley (1983).

[12] H.J. Sussmann: Minimal realizations and canonical forms for bilinear systems, J. Franklin Inst. 301 (1976) 593-604.

[13] H.J. Sussmann: Existence and uniqueness of minimal realizations of nonlinear systems, Math. Systems Theory 10 (1977) 263-284.