On a Unified Theory of Boundary Value Problems for Elliptic-Parabolic Equations of Second Order

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The present lecture is concerned with the theory of boundary value problems for linear partial differential equations of second order. It consists of a proposal for a unified theory of boundary value problems for second order linear equations of elliptic-parabolic type, including the classical cases of totally elliptic equations, heat equation, and first order equations.

The theory does not claim to be complete at this time. Nevertheless it opens, in my opinion, many attractive fields for further research and, therefore, the results obtained up to the present time deserve to be expounded here.

The first results of this theory were obtained by the writer in 1956 [4]. A more complete treatment can be found in the mimeographed notes containing the lectures the writer delivered at the Istituto Italiano di Alta Matematica in Rome in 1957 [5].

In 1958, K. O. Friedrichs published a paper [7] on boundary value problems for linear differential equations independent of type. The Friedrichs theory covers equations that may be hyperbolic in one subregion of the domain where the problem is considered and elliptic in a different subregion. However, it does not seem possible to include in the class of equations considered by Friedrichs every elliptic-parabolic equation, since Friedrichs assumes a symmetry condition, that is not needed in the present treatment. A comparison between the Friedrichs theory and the present one would be – in the opinion of the writer – very interesting.

1 Hypotheses and statement of the general boundary value problem for second order linear elliptic-parabolic equations

Let us denote by $X^{(r)}$ the r-dimensional real Cartesian space. Let $x \equiv (x_1, \ldots, x_r)$ be a point in $X^{(r)}$ and B a connected open set in $X^{(r)}$. A function is said to belong to $C^m(B)$ (m nonnegative integer) if it is continuous in B together with its partial derivatives of order $h \leq m$. Let $a^{ij}(x)$ $(i, j = 1, \ldots, r)$, $b^i(x)$ $(i = 1, \ldots, r)$, $c(x)$ be real functions of $C^2(B)$, $C^1(B)$, $C^0(B)$ respectively.

We shall consider the following second order linear differential operator:

$$L(u) \equiv a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu \quad (1) \qquad \left(a^{ij} \equiv a^{ji}\right)$$

and suppose that for any $x \in B$ the quadratic form $a^{ij}(x)\lambda_i\lambda_j$ is positive definite or semi-definite, that is to say that L is a positive elliptic-parabolic operator in B. The case that perhaps for some points of B, $a^{ij}(x)\lambda_i\lambda_j \equiv 0$ is not excluded.

This means that in some subset of B – possibly coinciding with B – the differential operator might degenerate to a first order operator.

Let A be a bounded open set contained in B such that its boundary Σ coincides with the boundary of its closure $\bar{A}$.

In order to state the exact hypotheses to be assumed on Σ, we shall define first what we mean by a *regular* $(r-1)$-*cell* in $X^{(r)}$.

Let $x = x(t)$ be a function, defined in a closed domain D of the Cartesian $(r-1)$-space and with range in $X^{(r)}$, satisfying the following conditions: (a.) D is homeomorphic to an $(r-1)$-sphere, (b.) $x(t)$ maps D one-to-one onto a subet of $X^{(r)}$, (c.) the rank of the Jacobian matrix $\partial(x)/\partial(t)$ is $r-1$ at any point of D, (d.) the $(r-1)$-Lebesgue measure of the boundary of D is zero.

The range Σ_0 of $x(t)$ is an $(r-1)$-*regular cell of* $X^{(r)}$. The image of the boundary of D is called *the border* of Σ_0.

We shall suppose that Σ can be decomposed in a finite number of regular $(r-1)-$cells: $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_m$ such that any two of them have at most points of their borders in common.

In addition we shall suppose that the Green-Gauss identity, that transforms an r-integral over A into an $(r-1)$-integral over Σ, can be applied to $A \cup \Sigma$.

Let us denote by Σ_h' the subset of Σ_h $(h = 1, \ldots, m)$ such that at any point x of Σ_h' the tangent hyperplane to Σ_h is characteristic with respect to the operator L. This means that the direction cosines $n_1, \ldots, n_r$ of the normal to Σ_h at x satisfy the equation: $A^{ij}(x)n_i n_j = 0$. At any point x of Σ_h, not lying on the border of Σ_h, the function

$$b(x) = \left(b^i - a^{ij}_{x_j}\right) n_i,$$

is well defined, where $n_1, \ldots, n_r$ are the direction cosines of the inner normal to A at the point x of Σ_h.

For any fixed h we can extend the definition of $b(x)$ by continuity over all Σ_h (2). Let $\Sigma_h^{(1)}$ be the subset of Σ_h' defined by the condition $b(x) \geq 0$. Let us set:

$$\Sigma^{(1)} = \Sigma_1^{(1)} \cup \Sigma_2^{(1)} \cup \ldots \cup \Sigma_m^{(1)}$$
$$\Sigma^{(2)} = (\Sigma_1' \cup \Sigma_2' \cup \ldots \cup \Sigma_m') - \Sigma^{(1)}$$
$$\Sigma^{(3)} = \Sigma - \left(\Sigma^{(1)} \cup \Sigma^{(2)}\right).$$

The set Σ is decomposed in three disjoint subsets $\Sigma^{(1)}, \Sigma^{(2)}$, $\Sigma^{(3)}$. It must be observed that someone of these sets might be empty.

Let $\Sigma_D^{(3)}$ and $\Sigma_N^{(3)}$ be two disjoint sets such that

$$\Sigma^{(3)} = \Sigma_D^{(3)} \cup \Sigma_N^{(3)}.$$

The set $\Sigma_D^{(3)}$ or $\Sigma_N^{(3)}$ may be assumed to be empty. Of course both these sets are empty when $\Sigma^{(3)}$ is empty.

Let $f(x)$, $g(x)$, $h(x)$ be given functions defined respectively on A, on $\Sigma^{(2)} \cup \Sigma_D^{(3)}$ and on $\Sigma_N^{(3)}$.

The boundary value problem that we shall consider for a general elliptic-parabolic equation is the following:

$$\boxed{\begin{array}{ll} L(u) = f & \text{in } A, \\ u = g & \text{on } \Sigma^{(2)} \cup \Sigma_D^{(3)}, \\ A^{ij}(x) u_{x_i} n_j = h & \text{on } \Sigma_N^{(3)}. \end{array}} \tag{1.1}$$

2 Particular cases and examples

Let us suppose $a^{ij}(x)\lambda_i\lambda_j$ positive definite in any point of B, that is to say that L is a totally elliptic operator. In this particular case $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are both empty. If $\Sigma_N^{(3)}$ is assumed empty, the boundary value problem (1.1) is the classical Dirichlet problem. If $\Sigma_{\mathrm{D}}^{(3)}$ is empty, we have the Neumann problem for a general elliptic operator. In the case that $\Sigma_D^{(3)}$ and $\Sigma_N^{(3)}$ are not empty, problem (1.1) reduces to the mixed Dirichlet-Neumann boundary value problem for an elliptic operator. Let us now assume:

$$L(u) \equiv L_0(u) - u_{x_r}$$

where L_0 is an elliptic operator in the $r-1$ variables: $x_1, \ldots, x_{r-1}$. In this case L is the well known parabolic heat equation operator. $\Sigma^{(1)}$ is the part of Σ where the inner normal n is parallel, and opposite in direction to the x_r-axis (hatched line in Fig. 1), $\Sigma^{(2)}$ the part of Σ where n is parallel to the x_r-axis and oriented as this axis. $\Sigma^{(3)}$ is the part of the boundary where n and x_r form an angle different from 0 and π.

Let us consider the more general parabolic operator:

$$L(u) \equiv L_0(u) - b^r(x) u_{x_r}$$

where L_0 is defined as above (elliptic operator in the variables $x_1, \ldots, x_{r-1}$) and $b^r(x)$ is an arbitrary function, not necessarily of fixed sign, in $A \cup \Sigma$. We do not exclude that the coefficients of L_0 may depend also on the variable x_r.

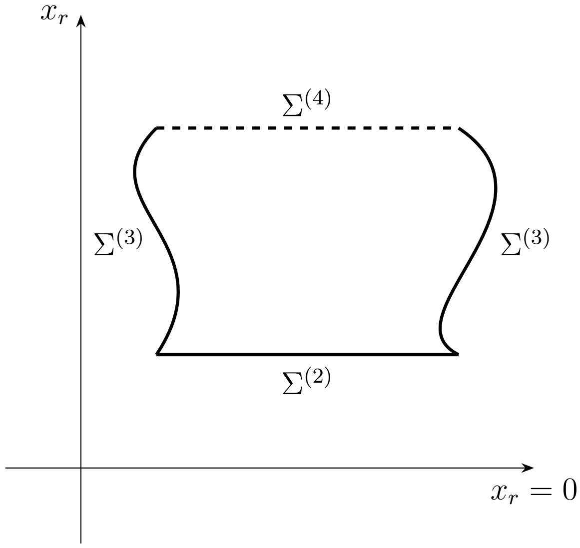


Fig. 1

Let W^+ (W^-) be the part (possibly empty) of Σ where the inner normal n is parallel and equiverse (opposite) to the x_r-axis. W_1^+ (W_1^-) be the subset of W^+ (W^-) where $b_r(x) \geq 0$ $(b_r(x) \leq 0)$ and W_2^+ (W_2^-) the remaining part. We have: $\Sigma^{(1)} \supset W_1^+ \cup W_1^-$; $\Sigma^{(2)} \supset W_2^+ \cup W_2^-$. It is easy to see how to distribute the points of Σ where the tangent hyperplane is not defined. As an example consider the parabolic equation

$$\frac{\partial^2 u}{\partial x_1^2} - \sin x_1 \frac{\partial u}{\partial x_2} = f\left(x_1, x_2\right),$$

in the domain indicated in Fig. 2.

The hatched line denotes the part $\Sigma^{(1)}$ of Σ where one must not prescribe any boundary condition.

As very simple and elementary examples consider the two first order equations:

$$u_{x_1} + c(x)u = f(x), \quad x_1 u_{x_1} + x_2 u_{x_2} + c(x)u = f(x)$$

In Fig. 3 and Fig. 4 the parts $\Sigma^{(1)}$ and $\Sigma^{(2)}$ of Σ are, respectively, shown ($\Sigma^{(1)} \equiv$ hatched line, $\Sigma^{(2)} \equiv$ continuous line).

As a more interesting example consider the first order equation,

$$b_1 u_{x_1} + b_2 u_{x_2} + cu = f,$$

in the interval $-a_1 \leq x_1 \leq a_1$, $-a_2 \leq x_2 \leq a_2$ $(a_1 > 0,\ a_2 > 0)$ and suppose that

$$b_1\left(-a_1, x_2\right) \geq 0, \quad b_1\left(a_1, x_2\right) \leq 0$$
$$b_2\left(x_1, -a_2\right) \geq 0, \quad b_2\left(x_1, a_2\right) \leq 0$$

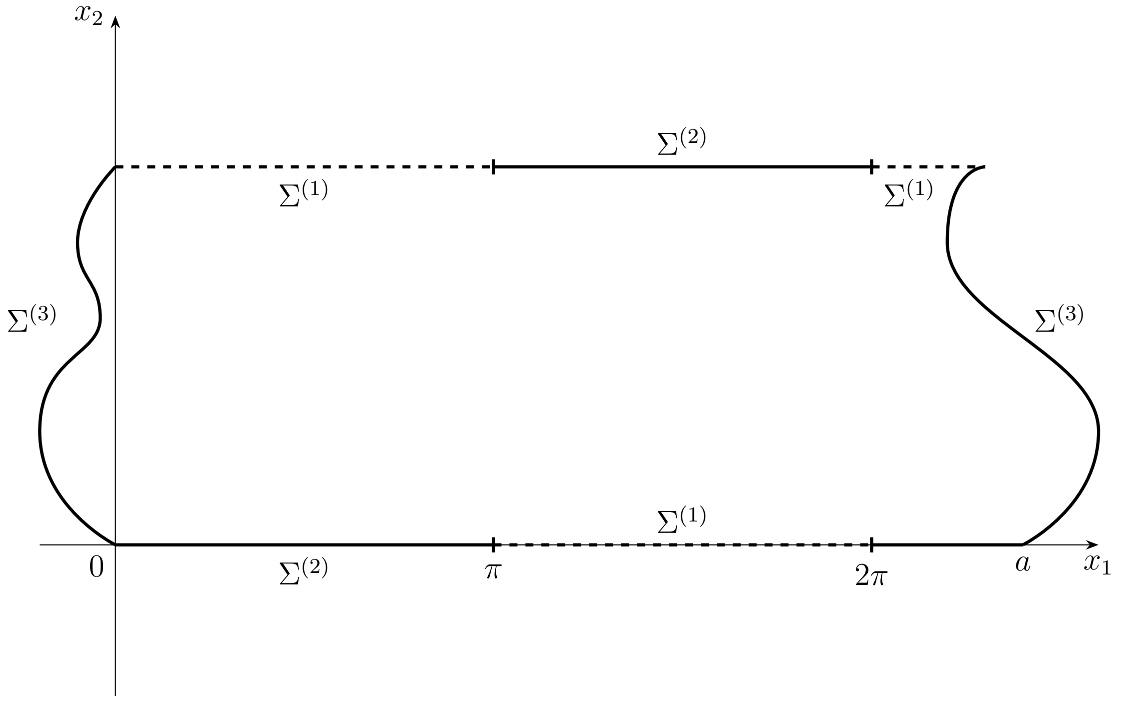


Fig. 2

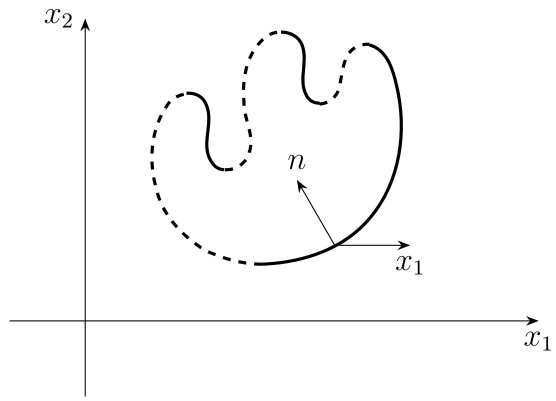


Fig. 3

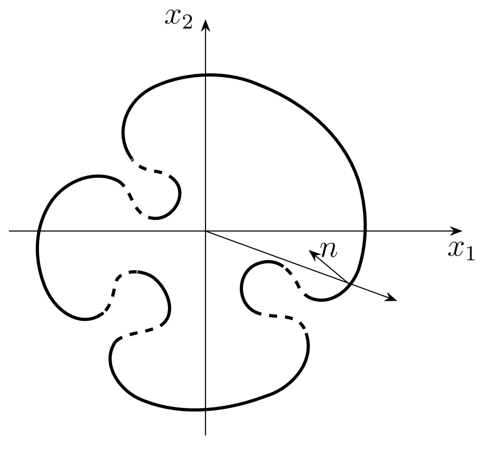


Fig. 4

Then $\Sigma^{(2)} \equiv \varnothing$ and $\Sigma \equiv \Sigma^{(1)}$. We have an example of problem without any boundary condition. This particular problem was considered by Picone in 1928 [14], and it was proved by him that in the case $c < 0$ a *regular* solution is determined by only satisfying the equation.

On the other hand if we consider the equation,

$$-b_1 u_{x_1} - b_2 u_{x_2} + cu = f$$

with b_1 and b_2 satisfying the above inequalities on Σ strictly, then $\Sigma \equiv \Sigma^{(2)}$ and $\Sigma^{(1)} \equiv \varnothing$. That is to say the value of u must be prescribed on the whole boundary.

Let us now consider some other examples concerning parabolic equations of second order. First we consider the equation,

$$\left(x_1^2 + x_2^2 - 1\right) u_{x_1 x_2} + x_2 \left(x_1^2 + 2x_2^2 - 2\right) u_{x_2} + cu = f.$$

Let A be the domain defined by the condition:

$$x_1^2 + x_2^2 > 1, \quad 0 < x_1 < 2, \quad -1 < x_2 < 1.$$

In this case $\Sigma^{(1)}$ is constituted by the two points $(0,1)$ and $(0,-1)$; $\Sigma^{(3)}$ is the segment $x_1 = 2$, $-1 < x_2 < 1$, $\Sigma^{(2)}$ the remaining part of the boundary. If we assume as A the part of the previous domain contained in the half plane $x_2 > 0$, then the part of the boundary lying on the x_1-axis is $\Sigma^{(1)}$ (Fig. 5).

For the equation:

$$x_2^2 u_{x_1 x_1} - 2x_1 x_2 u_{x_1 x_2} + x_1^2 u_{x_1 x_2} - 2x_1 u_{x_1} - 2x_2 u_{x_2} + cu = f$$

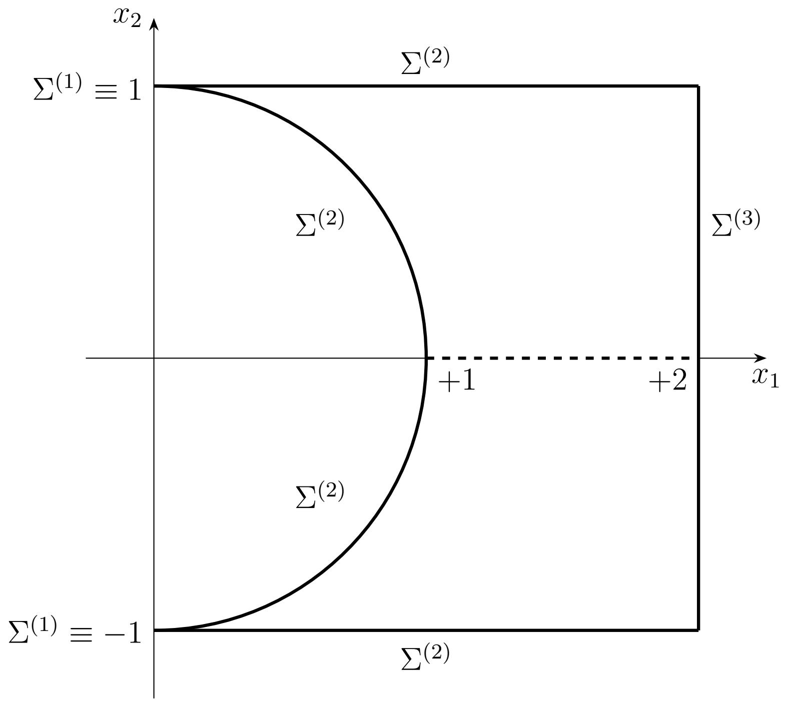
in a circular domain with center at the origin, the whole boundary is $\Sigma^{(1)}$. But in the star-shaped domain indicated in Fig. 6, the whole boundary is $\Sigma^{(3)}$. In the first case we have another example of a problem without boundary conditions and in the second case we can prescribe the value of u on the whole boundary.

As last example let us consider the mixed elliptic-parabolic equation:

$$k\left(x_2\right) u_{x_1 x_1} + u_{x_2 x_2} = f;$$

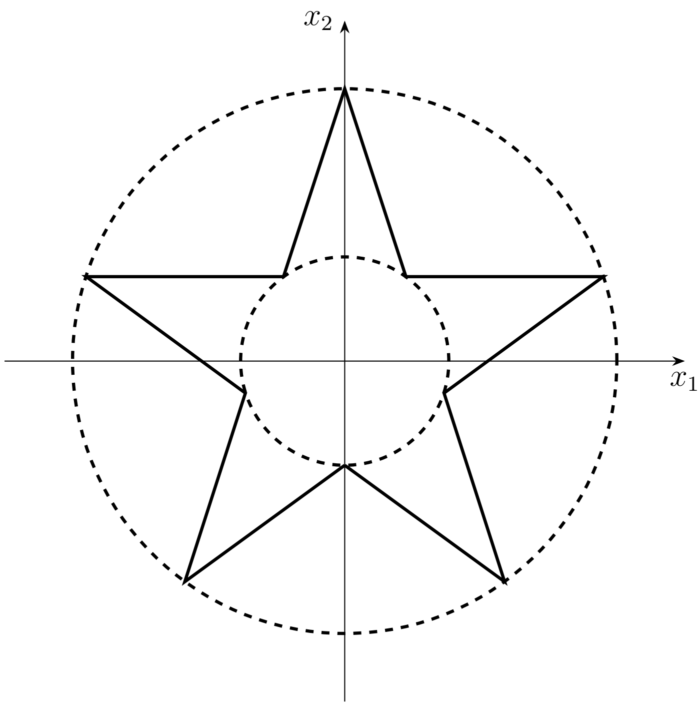
$k\left(x_2\right)$ is a function positive for $x_2 > 0$ and vanishing for $x_2 \leq 0$. Let A be the domain bounded by: (1.) a regular arc $\mathcal{C}_1$ joining the two points 0 and 1 of the x_1-axis and entirely lying in the halfplane $x_2 > 0$; (2.) an arc $\mathcal{C}_2$ (with $x_2 < 0$) joining the origin with the point $(1,-1)$ and with a tangent nowhere parallel to the x_2-axis; (3.) the segment $\mathcal{C}_3$ joining $(1,0)$ and $(1,-1)$. In this case we have $\Sigma^{(3)} = \mathcal{C}_1 \cup \mathcal{C}_2$, $\Sigma^{(1)} = \mathcal{C}_3$. We can indeed prescribe u on $\mathcal{C}_1$ and $\mathcal{C}_2$.

In the next sections it will be shown in what sense and under what further conditions there exist unique solutions of (1.1) (in particular, solutions of the example problems) which are continuously dependent on the data.



x_2
$\Sigma^{(2)}$
$\Sigma^{(1)} \equiv 1$
$\Sigma^{(2)}$
$\Sigma^{(3)}$
$+1$
$+2$
x_1
$\Sigma^{(2)}$
$\Sigma^{(1)} \equiv -1$
$\Sigma^{(2)}$

Fig. 5



x_2
x_1

Fig. 6

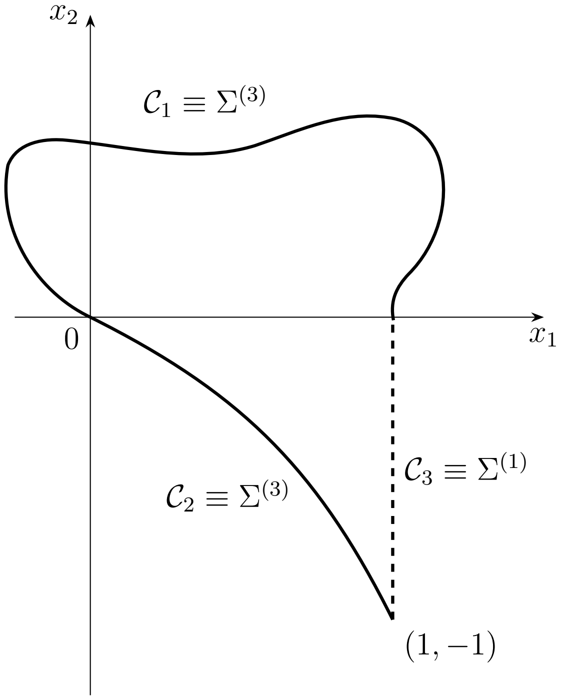


Fig. 7

3 Integral a priori estimates and maximum principles

Let us define as class C_L the family of all real functions u satisfying the following conditions:

a. u is continuous with its first derivatives in $A \cup \Sigma$, $[u \in C^1(A \cup \Sigma)]$,

b. u has continuous second derivatives in A, $[u \in C^2(A)]$,

c. the function $L(u)$ is bounded in A.

We shall denote by L^* the formal adjoint of the operator L:

$$\begin{aligned} L^*(u) &= (a^{ij})_{x_i x_j} - (b^i u)_{x_i} + cu \\ &\equiv a^{ij} u_{x_i x_j} + b^{*i} u_{x_i} + c^* u, \end{aligned}$$

where

$$b^{*i} = 2a^{ij}_{x_j} - b^i, \quad c^* = a^{ij}_{x_i x_j} - b^i_{x_i} + c.$$

I. *Let p be a real number such that $1 \le p < +\infty$ and let a function w exist belonging to $C^2(A \cup \Sigma)$ and satisfying the condition*

$$w \le 0, \quad L^*(w) + (p-1)cw > 0 \ \textit{in } A \cup \Sigma. \tag{3.1}$$

For any function u of C_L vanishing almost everywhere on $\Sigma^{(2)} \cup \Sigma^{(3)}$, the following inequality holds:

$$\boxed{\left(\int_A |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq p \frac{\max_{A\cup\Sigma} |w|}{\min_{A\cup\Sigma} [L^*(w) + (p-1)cw]} \left(\int_A |L(u)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.} \tag{3.2}$$

Proof. Let δ denote an arbitrary positive number. By using Green's integral theorem the following identity is shown:

$$\begin{aligned}
&\int_A \{(u^2+\delta)^{\frac{p}{2}} L^*(w) - w\big(p[(p-1)u^2+\delta](u^2+\delta)^{\frac{p}{2}-2} a^{hk} u_{x_h} u_{x_k} \\
&\quad + p(u^2+\delta)^{\frac{p}{2}-1} u L(u) + c(u^2+\delta)^{\frac{p}{2}-1}[(1-p)u^2+\delta]\big)\} \,\mathrm{d}x \\
&\quad = \int_{\Sigma^{(1)}\cup\Sigma^{(2)}} w(u^2+\delta)^{\frac{p}{2}} b \,\mathrm{d}\sigma + \int_{\Sigma^{(3)}} [p^2(u^2+\delta)^{\frac{p}{2}-1} w u a^{hk} u_{x_h} n_k \\
&\quad\quad - (u^2+\delta)^{\frac{p}{2}} (a^{hk} w_{x_h} n_k - bw)] \,\mathrm{d}\sigma.
\end{aligned} \tag{3.3}$$

Since a.e. on $\Sigma^{(1)}$, $b \geq 0$; and $a^{hk} u_{x_h} u_{x_k} \geq 0$, $w \leq 0$ in $A \cup \Sigma$, by the assumed hypotheses on u, it follows:

$$\begin{aligned}
&\int_A \left(u^2+\delta\right)^{\frac{p}{2}} \left\{L^*(w) - cw\left(u^2+\delta\right)^{-1}\left[(1-p)u^2+\delta\right]\right\} \mathrm{d}x \\
&\leq p \int_A \left(u^2+\delta\right)^{\frac{p}{2}-1} u w L(u) \,\mathrm{d}x + \delta^{\frac{p}{2}} \left[\int_{\Sigma^{(2)}} wb \,\mathrm{d}\sigma - \int_{\Sigma^{(3)}} \left(a^{hk} w_{x_h} n_k - bw\right) \mathrm{d}\sigma\right].
\end{aligned}$$

We observe that

$$\lim_{\delta\to 0} \left(u^2+\delta\right)^{\frac{p}{2}} \left\{L^*(w) - cw\frac{(1-p)u^2+\delta}{u^2+\delta}\right\} = \left(u^2\right)^{\frac{p}{2}} \left\{L^*(w) - (1-p)cw\right\},$$
$$\lim_{\delta\to 0} \left(u^2+\delta\right)^{\frac{p}{2}-1} u = \left(u^2\right)^{\frac{p}{2}-1} u,$$

uniformly with respect to x in $A \cup \Sigma$.

From (3.3) it follows

$$\int_A |u|^p \,\mathrm{d}x \leq p \frac{\max_{A\cup\Sigma} |w|}{\min_{A\cup\Sigma} [L^*(w) + (p-1)cw]} \int_A |u|^{p-1} |L(u)| \,\mathrm{d}x \tag{3.4}$$

This coincides with (3.2) for $p = 1$. For $p > 1$, (3.2) follows from (3.4) by applying the Schwarz-Hölder inequality to the integral on the right hand side. □

II. *If $c < 0$ in $A \cup \Sigma$, or $c^* < 0$ in $A \cup \Sigma$, p and w exist satisfying* (3.1).

Proof. In the first case ($c < 0$) we can assume $w \equiv -1$ and p large enough, that

$$\min_{A\cup\Sigma}\left[-a^{hk}_{x_h x_k} + b^h_{x_h} - pc\right] > 0 \quad \text{in } A\cup\Sigma.$$

In the second case ($c^* < 0$) we assume $w \equiv -1$ and p so small that

$$\min_{A\cup\Sigma}\left[(1-p)c - c^*\right] > 0$$

(in particular $p = 1$). $\square$

It is obvious that

III. *If $c < 0$ and $c^* < 0$ in $A \cup \Sigma$, then for any $p \geq 1$ the condition (3.1) is satisfied by assuming $w \equiv -1$.*

IV. *Let c be negative in $A \cup \Sigma$. Let w denote an arbitrary function of $C^2(A\cup\Sigma)$ negative in $A\cup\Sigma$ and p_0 a real number such that $L^*(w) + (p_0 - 1)\,cw \geq 0$ in $A\cup\Sigma$. For any p such that $p > p_0$, $p \geq 1$, and for any $u \in C_L$, vanishing a.e. on $\Sigma^{(2)} \cup \Sigma^{(3)}$, the following inequality holds:*

$$\boxed{\left(\int_A |u| cw \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{p-p_0}\left(\int_A \left|\frac{L(u)}{c}\right|^p cw\,\mathrm{d}x\right)^{\frac{1}{p}}.} \tag{3.5}$$

Proof. Since w satisfies for $p > p_0$ the conditions (3.1), from (3.3) for $p > p_0$, $p \geq 1$, it follows (for $\delta \to 0$) that

$$\int_A \left(u^2\right)^{\frac{p}{2}}\left[L^*(w) - (1-p)cw\right]\mathrm{d}x \leq p\int_A \left(u^2\right)^{\frac{p}{2}-1} uwL(u)\,\mathrm{d}x.$$

Since $L^*(w) - (1-p)cw \geq (p-p_0)\,w$, from the previous inequality it follows that

$$\int_A |u|^p cw\,\mathrm{d}x \leq \frac{p}{p-p_0}\int_A |u|^{p-1}|w||L(u)|\,\mathrm{d}x \tag{3.6}$$

which coincides with (3.5) for $p = 1$. In the case $p > 1$ we have

$$\begin{aligned}
&\int_A |u|^{p-1}|L(u)||w|\,\mathrm{d}x \\
&\leq \left(\int_A \left[|u|^{p-1}|c|^{\frac{p-1}{p}}\right]^{\frac{p}{p-1}}|w|\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_A \left[|L(u)||c|^{\frac{1-p}{p}}\right]^p|w|\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&= \left(\int_A |u|^p|c||w|\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_A \left|\frac{L(u)}{c}\right|^p|c||w|\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&= \left(\int_A |u|^p cw\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_A \left|\frac{L(u)}{c}\right|^p cw\,\mathrm{d}x\right)^{\frac{1}{p}}
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), the inequality (3.5) follows. $\square$

From (3.5) we deduce the following maximum principle:

V. *Let c be negative in $A \cup \Sigma$. For any $u \in C_L$ vanishing almost everywhere on $\Sigma^{(2)} \cup \Sigma^{(3)}$ the inequality holds (3):*

$$\boxed{\max_{A\cup\Sigma} |u| \leq \underset{A}{\mathrm{l.\,u.\,b.}} \left|\frac{L(u)}{c}\right|} \tag{3.8}$$

Proof. This follows from (3.5) for $p \to +\infty$. □

VI. *When hypotheses of theorems III or V are satisfied, a uniqueness theorem holds for the problem* (1.1) *with $\Sigma_N = \varnothing$ in the class C_L.*

It is evident that for particular classes of elliptic-operators L (as an instance first order operators) theorems I - VI remain valid with the same proof, assuming C_L to be defined in a more general way (not necessarily requiring the existence and the continuity of all the derivatives of u in A.)

We want now to consider some estimates in connection with problem (1.1) in the general case (i.e., Σ_N not empty).

VII. *Let p be a real number: $1 \leq p < +\infty$ and w a function of $C^2(A \cup \Sigma)$ such that:*

$$w \leq 0, \quad L^*(w) + (p-1)cw > 0 \quad on\ A \cup \Sigma \tag{3.1}$$

$$a^{hk} w_{x_h} n_k - bw \geq 0 \quad a.e.\ on\ \Sigma^{(3)} - \overline{\Sigma^{(3)}_D}. \quad (4)$$

For any $u \in C_L$ satisfying a.e. the conditions

$$u = 0\ on\ \Sigma^{(2)} \cup \overline{\Sigma^{(3)}_D}, \quad a^{hk} u_{x_h} n_k = 0\ on\ \Sigma^{(3)} - \overline{\Sigma^{(3)}_D}$$

the inequality (3.2) *holds.*

The proof is analogous to the proof of theorem I (5).

VIII. *Let c be negative in $A \cup \Sigma$ and a negative function w in $A \cup \Sigma$ exist such that $a^{hk} w_{x_h} n_k - bw \geq 0$ on $\Sigma^{(3)}_N$. Then $p \geq 1$ exists such that* (3.1) *is satisfied.*

The proof is obvious.

IX. *For the same hypotheses of the previous theorem, let p_0 be a real number such that $L^*(w) + (p_0 - 1)\, cw \geq 0$. For $p > p_0$, $p \geq 1$ and any u satisfying the conditions of theorem VII, the inequality* (3.5) *holds.*

This theorem is proved as theorem IV.

X. *For the same hypotheses of theorem VIII, the Inequality* (3.8) *holds for any u satisfying the conditions of theorem VII.*

From theorems VII and IX uniqueness theorems follow for the problem (1.1) in the class C_L.

XI. *For a fixed p $(1 \le p < +\infty)$ let a function $w \in C^2(A \cup \Sigma)$ exist satisfying the following conditions:*

$$w \le 0, \quad L^*(w) + (p-1)cw > 0 \quad \text{in } A \cup \Sigma,$$
$$w = 0 \text{ a.e. on } \overline{\Sigma_D^{(3)}}, \quad a^{hk} w_{x_h} n_k - bw \ge 0 \text{ a.e. on } \Sigma^{(3)} - \overline{\Sigma_D^{(3)}}.$$

For any $u \in C_L$, such that: $L(u) = 0$ in A, $a^{hk} u_{x_h} n_k = 0$ a.e. on $\overline{\Sigma^{(3)}} - \overline{\Sigma_D^{(3)}}$, the inequality holds

$$\boxed{\begin{aligned}\left(\int_A |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le &\left\{ \frac{\underset{\Sigma^{(2)}}{\mathrm{l.\,u.\,b.}} |bw|}{\min\limits_{A\cup\Sigma}[L^*(w) + (p-1)cw]} \right\}^{\frac{1}{p}} \left(\int_{\Sigma^{(2)}} |u|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \\ &+ \left\{ \frac{\underset{\overline{\Sigma_D^{(3)}}}{\mathrm{l.\,u.\,b.}} |a^{hk} w_{x_h} n_k|}{\min\limits_{A\cup\Sigma}[L^*(w) + (p-1)cw]} \right\}^{\frac{1}{p}} \left(\int_{\overline{\Sigma_D^{(3)}}} |u|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}}.\end{aligned}} \tag{3.9}$$

Proof. From (3.3) and the hypotheses for u and w, it follows that

$$\begin{aligned}&\int_A \left\{ \left(u^2+\delta\right)^{\frac{p}{2}} L^*(w) - cw\left(u^2+\delta\right)^{\frac{p}{2}-1}\left[(1-p)u^2+\delta\right]\right\} \mathrm{d}x \\ &\le \int_{\Sigma^{(2)}} bw\left(u^2+\delta\right)^{\frac{p}{2}} \mathrm{d}\sigma - \int_{\overline{\Sigma_D^{(3)}}} \left(u^2+\delta\right)^{\frac{p}{2}} a^{hk} w_{x_h} n_k \,\mathrm{d}\sigma.\end{aligned}$$

For $\delta \to 0$

$$\int_A \left(u^2\right)^{\frac{p}{2}} [L^*(w) + (p-1)cw] \,\mathrm{d}x \le \int_{\Sigma^{(2)}} w\left(u^2\right)^{\frac{p}{2}} b \,\mathrm{d}\sigma - \int_{\overline{\Sigma_D^{(3)}}} \left(u^2\right)^{\frac{p}{2}} a^{hk} w_{x_h} n_k \,\mathrm{d}\sigma;$$

(3.9) follows easily from this inequality. □

XII. *Let c be negative in $A \cup \Sigma$ and a function $w \in C^2(A \cup \Sigma)$ exist satisfying the conditions:*

$$w < 0 \text{ in } A \cup \Sigma, \quad a^{hk} w_{x_h} n_k - bw \ge 0 \text{ on } \Sigma^{(3)} - \overline{\Sigma_D^{(3)}}.$$

For any $u \in C_L$ satisfying the conditions of theorem XI the following maximum principle holds:

$$\boxed{\max_{A\cup\Sigma} |u| = \max_{\Sigma^{(2)} \cup \overline{\Sigma_D^{(3)}}} |u|.} \tag{3.10}$$

Proof. If p is an arbitrary even integer, we can use (3.3) with $\delta = 0$. Under the assumed hypotheses for w and u we then obtain

$$\begin{aligned}\min_{A\cup\Sigma}[L^*(w)+(p-1)cw]\int_A u^p\,\mathrm{d}x \le \operatorname*{l.\,u.\,b.}_{\Sigma^{(2)}}|bw|\int_{\Sigma^{(2)}} u^p\,\mathrm{d}\sigma \\ + p\max_{\overline{\Sigma^{(3)}_D}}\int_{\overline{\Sigma^{(3)}_D}}|u|^{p-1}|a^{hk}u_{x_h}n_k|\,\mathrm{d}\sigma + \operatorname*{l.\,u.\,b.}_{\overline{\Sigma^{(3)}_D}}|a^{hk}w_{x_h}n_k - bw|\int_{\overline{\Sigma^{(3)}_D}} u^p\,\mathrm{d}\sigma.\end{aligned} \tag{3.11}$$

Let us first suppose $u = 0$ a.e. on $\overline{\Sigma^{(3)}_D}$. Since two positive numbers P_1 and P_2 exist such that for p large enough, $P_1 \le \min_{A\cup\Sigma}[L^*(w)+(p-1)cw] \le P_2 p$, it follows that $\lim_{p\to\infty}\{\min_{A\cup\Sigma}[L^*(w)+(p-1)cw]\}^{\frac{1}{p}} = 1$. From (3.11) the inequality (3.10) follows easily.

Let us now consider the case that u does not vanish a.e. on $\overline{\Sigma^{(3)}_D}$. By considering the Schwarz-Hölder inequality,

$$\int_{\overline{\Sigma^{(3)}_D}}|u|^{p-1}|a^{hk}u_{x_h}n_k|\,\mathrm{d}\sigma \le \left(\int_{\overline{\Sigma^{(3)}_D}} u^p\,\mathrm{d}\sigma\right)^{\frac{p-1}{p}}\left(\int_{\overline{\Sigma^{(3)}_D}}|a^{hk}u_{x_h}n_k|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}},$$

from (3.11) we get

$$\min_{A\cup\Sigma}[L^*(w)+(p-1)cw]\int_A u^p\,\mathrm{d}\sigma \le K_p\int_{\Sigma^{(2)}\cup\overline{\Sigma^{(3)}_D}} u^p\,\mathrm{d}\sigma \tag{3.12}$$

where

$$\begin{aligned}K_p = \operatorname*{l.\,u.\,b.}_{\Sigma^{(2)}}|bw| + p\max_{\overline{\Sigma^{(3)}_D}}|w|\left(\int_{\overline{\Sigma^{(3)}_D}} u^p\,\mathrm{d}\sigma\right)^{-\frac{1}{p}}\left(\int_{\overline{\Sigma^{(3)}_D}}|a^{hk}u_{x_h}n_k|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}} \\ + \operatorname{l.\,u.\,b.}|a^{hk}w_{x_h}n_k - bw|.\end{aligned}$$

Let M be a positive number such that, for any even integer p,

$$\left(\int_{\overline{\Sigma^{(3)}_D}} u^p\,\mathrm{d}\sigma\right)^{-\frac{1}{p}}\left(\int_{\overline{\Sigma^{(3)}_D}}|a^{hk}u_{x_h}n_k|^p\right)^{\frac{1}{p}} \le M.$$

We get

$$\begin{aligned}K_p^{\frac{1}{p}} &\le \left[\operatorname*{l.\,u.\,b.}_{\Sigma^{(2)}}|bw| + p\max_{\overline{\Sigma^{(3)}_D}}|w|M + \operatorname*{l.\,u.\,b.}_{\overline{\Sigma^{(3)}_D}}|a^{hk}w_{x_h}n_k - bw|\right]^{\frac{1}{p}} \\ &\le p^{\frac{1}{p}}\left[\operatorname*{l.\,u.\,b.}_{\Sigma^{(2)}}|bw| + M\max_{\overline{\Sigma^{(3)}_D}}|w| + \operatorname*{l.\,u.\,b.}_{\overline{\Sigma^{(3)}_D}}|a^{hk}w_{x_h}n_k - bw|\right]^{\frac{1}{p}}\end{aligned}$$

and

$$\max_{p\to+\infty} \lim K_p^{\frac{1}{p}} \le 1.$$

Let p_0 be such that $\min_{A\cup\Sigma}[L^*(w) + (p-1)cw] \ge 0$ for $p > p_0$. For $p > p_0$, from (3.12) we obtain

$$\left(\min_{A\cup\Sigma}[L^*(w) + (p-1)cw]\right)^{\frac{1}{p}} \left(\int_A u^p \, d\sigma\right)^{\frac{1}{p}} \le K_p^{\frac{1}{p}} \left(\int_{\overline{\Sigma^{(2)} \cup \Sigma_D^{(3)}}} u^p \, d\sigma\right)^{\frac{1}{p}}$$

and for $p \to \infty$ we obtain (3.10). □

XIII. *Let c be negative in $A \cup \Sigma$. For any solution u of $L(u)$ of the class C_L not identically vanishing in A, the following maximum principle holds:*

$$\boxed{\max_A |u| < \max_{\overline{\Sigma^{(2)} \cup \Sigma^{(3)}}} |u|.}$$

Proof. Let us assume $\Sigma_D^{(3)} = \Sigma^{(3)}$. Then as a function w satisfying the hypotheses of the previous theorem we can select $w \equiv -1$. In this case the inequality (3.10) is the following:

$$\max_{A\cup\Sigma} |u| = \max_{\overline{\Sigma^{(2)} \cup \Sigma^{(3)}}} |u|.$$

Since $\max_{A\cup\Sigma} |u| = \max_{A\cup\Sigma} u$ or $\max_{A\cup\Sigma} |u| = -\min_{A\cup\Sigma} u$, it follows that u has in $A \cup \Sigma$ a positive maximum or a negative minimum.

From the condition $c < 0$, by a well known argument, it follows that such a positive maximum or negative minimum cannot be attained in a point of A (6). Therefore

$$|u(x)| < \max_{A\cup\Sigma} |u| \quad \text{for } x \in A.$$

It is evident that the same argument could be given to establish the maximum principle (as in theorem XIII) in the general case $\Sigma_N^{(3)} \neq \varnothing$. □

4 An abstract existence principle

In order to establish the necessary and sufficient condition for the existence of a weak solution of the problem (1.1), we shall make use of an abstract existence principle in Banach spaces.

Let $\mathscr{V}$ be an abstract manifold linear with respect to the real field and B_1 and B_2 real Banach spaces. Let M_i $(i = 1, 2)$ be a linear homomorphism of $\mathscr{V}$ into B_i; ϕ and ψ denote vectors of the adjoint spaces B_1^* and B_2^*, respectively. We consider for any $v \in \mathscr{V}$ the following functional equation:

$$\langle \phi, M_1(v) \rangle = \langle \psi, M_2(v) \rangle \,, \tag{4.1}$$

where ϕ is a given vector and ψ is the unknown. Let $\mathscr{V}_2$ be the kernel of the homomorphism M_2. The given vector ϕ must satisfy the following necessary conditions:

$$\langle \phi, M_1(v_2) \rangle = 0 \text{ for any } v_2 \in \mathscr{V}_2. \tag{4.2}$$

Let $M_1(\mathscr{V}_2)$ denote the image of $\mathscr{V}_2$ on B_1, for M_1 and $\overline{M_1(\mathscr{V}_2)}$ its closure. Let us consider the Banach factor-space: $Q = B_1/\overline{M_1(\mathscr{V}_2)}$. Let $\mathscr{M}_1$ be the homomorphism that maps $v \in \mathscr{V}$ in the equivalence class $[M_1(v)]$ of Q.

The following existence principle holds:

XIV. *A solution ψ of the functional equation* (4.1) *exists for any fixed ϕ satisfying* (4.2)*, when and only when a constant K exists such that for any $v \in \mathscr{V}$ the inequality holds*

$$\|\mathscr{M}_1(v)\|_Q \le K \|M_2(v)\|_{B_2}. \tag{4.3}$$

Let $\mathscr{A}$ be the (closed) subspace of B_2^*, consisting of the vectors ψ, solutions of the "homogeneous" problem,

$$\langle \psi, M_2(v) \rangle = 0, \text{ for any } v \in \mathscr{V}.$$

We denote by $\mathscr{F}$ the Banach factor-space: $\mathscr{F} = B_2^*/\mathscr{A}$. For any $\phi \in B_1^*$ satisfying the compatibility condition (4.2) (that is to say for any element of the adjoint space Q^*) an element Ψ of $\mathscr{F}$ is uniquely determined such that if ψ is any element in the equivalence class Ψ, then ψ is a solution of (4.1).

XV. *The element Ψ of $\mathscr{F}$ corresponding to $\phi \in B_1^*$ satisfying the compatibility conditions* (4.2) *satisfies the inequality*

$$\|\Psi\|_{\mathscr{F}} \le K \|\phi\|_{B_1^*}. \tag{4.4}$$

Inequality (4.4) is said to be the dual inequality of (4.3).

5 Existence of weak solutions for problem (1.1)

Let $u \in C_L$ and $v \in C_{L^*}$. Then the Green's identity holds

$$\int_A (vL(u) - uL^*(v))\,\mathrm{d}x = \int_{\Sigma^{(3)}} \left[ua^{hk}v_{x_h}n_k - va^{hk}u_{x_h}n_k\right]\mathrm{d}\sigma - \int_{\Sigma^{(1)}\cup\Sigma^{(2)}} uvb\,\mathrm{d}\sigma. \tag{5.1}$$

Let us suppose that u satisfies the boundary conditions

$$u = 0 \text{ a.e. on } \Sigma^{(2)} \cup \Sigma_D^{(3)}, \quad a^{hk}u_{x_h}n_k = 0 \text{ a.e. on } \Sigma_N^{(3)}. \tag{5.2}$$

From (5.1) it follows easily that

$$\int_A (uL^*(v) - vL(u))\,\mathrm{d}x = \int_{\Sigma^{(1)}} uvb\,\mathrm{d}\sigma + \int_{\overline{\Sigma_D^{(3)}}} va^{hk}u_{x_h}n_k\,\mathrm{d}\sigma - \int_{\Sigma^{(3)}\setminus\overline{\Sigma_D^{(3)}}} u(a^{hk}v_{x_h}n_k - bv)\,\mathrm{d}\sigma.$$

Let $\mathscr{V}$ be the linear manifold of the functions of C_{L^*} such that for any $u \in C_L$ satisfying (5.2)

$$\int_{\Sigma^{(1)}} uvb\,\mathrm{d}\sigma + \int_{\overline{\Sigma_D^{(3)}}} va^{hk}u_{x_h}n_k\,\mathrm{d}\sigma - \int_{\Sigma^{(3)}\setminus\overline{\Sigma_D^{(3)}}} u(a^{hk}v_{x_h}n_k - bv)\,\mathrm{d}\sigma = 0.$$

Let us set: $L(u) = f$. It follows that

$$\int_A vf\,\mathrm{d}x = \int_A uL^*(v)\,\mathrm{d}x \tag{5.3}$$

for any $v \in \mathscr{V}$.

We shall say that the problem (1.1) [with homogeneous boundary conditions $(g \equiv 0, h \equiv 0)$] admits an $\mathscr{L}^{(p)}$-weak solution if for every $f \in \mathscr{L}^{(p)}(A)$ $(p > 1)$, a function $u \in \mathscr{L}^{(p)}(A)$ exists satisfying (5.3) for any $v \in \mathscr{V}$.

From theorems XIV, XV and by the representation theorems of linear bounded functionals in the $\mathscr{L}^{(p)}$-spaces this general condition follows:

XVI. *An $\mathscr{L}^{(p)}$-weak solution of problem* (1.1) *exists for any $f \in \mathscr{L}^{(p)}(A)$ such that*

$$\int_A v_0 f\,\mathrm{d}x = 0 \quad (v_0 \in \mathscr{V}, L^*(v_0) = 0)$$

when and only when a constant K exists such that:

$$\underset{v_0\in\mathscr{V}_0}{\text{g.l.b.}} \left(\int_A |v+v_0|^{\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \leq K\left(\int_A |L^*(v)|^{\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \tag{5.4}$$

where $\mathscr{V}_0$ is the subset of $\mathscr{V}$ of all the functions which are solutions of the adjoint equation $L^*(v) = 0$.

When (5.4) *holds, any $\mathscr{L}^{(p)}$-weak solution satisfies the inequality,*

$$\underset{u_0\in\mathscr{U}_0}{\text{g.l.b.}} \left(\int_A |u+u_0|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leq K\left(\int_A |f|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5.5}$$

where $\mathscr{U}_0$ is the class of the $\mathscr{L}^{(p)}$-functions satisfying the conditions,

$$\int_A u_0 L^*(v)\,\mathrm{d}x = 0$$

for any $v \in \mathscr{V}$.

Inequality (5.5) shows that the solution u of problem (1.1) depends continuously – in the $\mathscr{L}^{(p)}$ norm – on the datum f, modulo the eigensolutions of the problem.

Therefore, when (5.4) has been proved, it seems reasonable from the last observation to consider problem (1.1) as a well posed boundary value problem.

The a priori estimates proved in Section 3 combined with theorem XV permit us to state that,

XVII. *Inequality* (5.4) *holds when the hypotheses of theorem VII are satisfied.*

The inequality (5.4) is deduced from the dual inequality of (3.2).

Other conditions for the validity of (5.4) can be obtained when it is possible to apply to the functions of $\mathscr{V}$ the results of Section 3 considered with respect to the operator L^*. For simplicity let us consider the particular case $\Sigma^{(3)} \equiv \Sigma_D^{(3)}$ and suppose that $\mathscr{V}$ coincides with the class of functions of C_{L^*} satisfying a.e. the boundary condition $v = 0$ on $\Sigma^{*(2)} \cup \Sigma^{(3)}$, where $\Sigma^{*(i)}$ $(i = 1, 2)$ is defined with respect to L^* as $\Sigma^{(i)}$ was defined with respect to L.

This hypothesis on $\mathscr{V}$ will be denoted as hypothesis a). We denote by $\Sigma_0^{(1)}$ the subset of $\Sigma^{(1)}$ where $b = 0$. The hypothesis a) is satisfied when the space spanned by the function η defined by

$$\eta = \begin{cases} u & \text{on } \Sigma^{(1)} \cup \Sigma_0^{(1)} \\ a^{hk} u_{x_h} n_k & \text{on } \Sigma^{(3)} \end{cases}$$

(for any $u \in C_L$ and satisfying (5.2)) is dense in $\mathscr{L}^{(1)}\left(\left[\Sigma^{(1)} - \Sigma_0^{(1)}\right] \cup \Sigma^{(3)}\right)$.

From theorem I we deduce that

XVIII. *If hypothesis a) is satisfied and a function w exists satisfying the conditions*

$$w \leq 0, \quad L(w) + \frac{1}{p-1} c^* w > 0 \quad \textit{in } A \cup \Sigma \quad (p > 1)$$

then for any $v \in \mathscr{V}$ the inequality holds

$$\left(\int_A |v|^{\frac{p}{p-1}} \, \mathrm{d}x\right)^{\frac{p-1}{p}} \leq K \left(\int_A |L^* w|^{\frac{p}{p-1}} \, \mathrm{d}x\right)^{\frac{p-1}{p}}$$

where

$$K = \frac{p}{p-1} \frac{\max\limits_{A \cup \Sigma} |u|}{\max\limits_{A \cup \Sigma} \left[L(w) + \frac{1}{p-1} c^* w\right]}.$$

In this case the $\mathscr{L}^{(p)}$-weak solution of problem (1.1) *exists for any fixed $f \in \mathscr{L}^{(p)}(A)$.*

Let us now suppose that a sequence of functions u_n exist, satisfying (5.2) and belonging to C_L, such that

$$\lim_{n\to\infty}\int_A |u-u_n|^p\,\mathrm{d}x = \lim_{n\to\infty}\int_A |L(u_n)-f|^p\,\mathrm{d}x = 0 \tag{5.6}$$

Then u is said to be (according to Friedrichs) an $\mathscr{L}^{(p)}$-strong solution of problem (1.1) (with homogeneous boundary condition). It is obvious that any $\mathscr{L}^{(p)}$-strong solution is a $\mathscr{L}^{(p)}$-weak solution.

We shall denote as problem (1.1*) the following one ($\Sigma^{(3)}=\Sigma^{(3)}_D$):

$$L^*(v)=f \text{ in } A, \quad v=0 \text{ on } \Sigma^{*(2)}\cup\Sigma^{*(3)}. \tag{1.1*}$$

XIX. *Let the hypothesis a) hold. If the $\mathscr{L}^{(p)}$-weak solution ($p>1$) solution of* (5.3) *is an $\mathscr{L}^{(p)}$-strong solution.*

For the existence and uniqueness of the $\mathscr{L}^{(p)}$-weak solution of (5.3) it follows that the inequality holds

$$\int_A |u|^p\,\mathrm{d}x \le K\int_A |f|^p\,\mathrm{d}x \tag{5.7}$$

Since only the zero function of $\mathscr{L}^{\left(\frac{p}{p-1}\right)}(A)$ is orthogonal to every $L(u)$ ($u\in C_L$ and satisfying (5.2)) a sequence u_n of such functions can be chosen in such a way that $L\left(u_n\right)$ converge to f; (5.6) then follows from (5.7).

6 The $\mathscr{H}$-weak solutions

We want now to consider a different approach to problem (1.1) that appears as a natural extension to our elliptic-parabolic problems of the well known method for elliptic equations founded on energy integral. We shall only consider, for the sake of simplicity, the case $\Sigma^{(3)}\equiv\Sigma^{(3)}_D$ with homogeneous boundary conditions

$$L(u)=f \text{ in } A, \quad u=0 \text{ on } \Sigma^{(2)}\cup\Sigma^{(3)}. \tag{1.1$_0$}$$

For a function $u\in C_L$ and a function $v\in C^1(A\cup\Sigma)$ the integral identity holds

$$\begin{aligned}\int_A vL(u)\,\mathrm{d}x = &-\int_A \left[a^{hk}v_{x_h}u_{x_k} + u\left(b^h - a^{hk}_{x_h}\right)v_{x_k} + \left(b^h_{x_h} - a^{hk}_{x_hx_k} - c\right)uv\right]\mathrm{d}x\\ &-\int_{\Sigma^{(3)}} va^{hk}u_{x_h}n_k\,\mathrm{d}\sigma - \int_{\Sigma^{(1)}} uvb\,\mathrm{d}\sigma.\end{aligned}$$

Let $\mathscr{W}$ be the class of functions belonging to $C^1(A \cup \Sigma)$ and vanishing on $\Sigma^{(3)}$ (when not empty). If $u \in C_L$ and vanishes a.e. on $\Sigma^{(2)} \cup \Sigma^{(3)}$, then for any $v \in \mathscr{W}$ the identity is satisfied

$$\int_A vL(u)\,\mathrm{d}x = -\int_A \left[a^{hk} v_{x_h} u_{x_k} + u\left(b^h - a^{hk}_{x_h}\right) v_{x_k} + \left(b^h_{x_h} - a^{hk}_{x_h x_k} - c\right) uv\right] \mathrm{d}x \\ - \int_{\Sigma^{(1)}} uvb\,\mathrm{d}\sigma.$$

Let us introduce a scalar product in $\mathscr{W}$ in the following way:

$$(u,v) = \int_A \left(a^{hk} u_{x_h} v_{x_k} + uv\right) \mathrm{d}x + \int_{\Sigma^{(1)} \cup \Sigma^{(2)}} uv|b|\,\mathrm{d}\sigma.$$

The space $\mathscr{H}$ is the Hilbert space obtained by functional completion from $\mathscr{W}$ with the introduced scalar product.

Let us consider for $u, v \in \mathscr{W}$ the bilinear form

$$B(u,v) = -\int_A \left[a^{hk} v_{x_h} u_{x_k} + u\left(b^h - a^{hk}_{x_k}\right) v_{x_h} + \left(b^h_{x_h} - a^{hk}_{x_h x_k} - c\right) uv\right] \mathrm{d}x \\ - \int_{\Sigma^{(1)}} uvb\,\mathrm{d}\sigma$$

It is easily seen that:

$$|B(u,v)| \le M \left(\int_A \left(|\operatorname{grad} v|^2 + v^2\right) \mathrm{d}x + \int_{\Sigma^{(1)}} |v|^2\,\mathrm{d}\sigma \right)^{\frac{1}{2}} \|u\|$$

where M is a constant depending on the coefficients of L. For any fixed $v \in \mathscr{W}$, $B(u,v)$ can be considered as a linear bounded functional of u defined on $\mathscr{H}$.

Given $f \in \mathscr{L}^{(2)}(A)$, we define as an $\mathscr{H}$-weak solution of the problem (1.1_0) a function u in $\mathscr{H}$ satisfying the equation

$$\int_A vf\,\mathrm{d}x = B(u,v),$$

for any $v \in \mathscr{W}$. Assuming hypothesis a) it is evident that any $\mathscr{H}$-weak solution is an $\mathscr{L}^{(2)}$-weak solution.

For the representation theorem of linear functionals in Hilbert space, For the representation theorem have for $u \in \mathscr{H}$, $v \in \mathscr{W}$:

$$B(u,v) = (u, T(v)).$$

$T(v)$ is a linear transformation defined in $\mathscr{W}$ and with range in $\mathscr{H}$.

For $u \in \mathscr{W}$, and $v \in \mathscr{W}$, we have

$$
B(u,v) = -\int_A \Big[a^{hk} v_{x_h} u_{x_k} + \frac{1}{2} u \left(b^h - a^{hk}_{x_k}\right) v_{x_h} - \frac{1}{2} v \left(b^h - a^{hk}_{x_k}\right) u_{x_h} + \left(\frac{1}{2} b^h_{x_h} - \frac{1}{2} a^{hk}_{x_h x_k} - c\right) uv\Big] \,\mathrm{d}x - \frac{1}{2}\int_{\Sigma^{(1)}} uvb \,\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma^{(2)}} uvb \,\mathrm{d}\sigma. \tag{6.1}
$$

Let us suppose that:

$$
\frac{1}{2} b^h_{x_h} - \frac{1}{2} a^{hk}_{x_h x_k} - c > m_0 > 0 \text{ in } A \cup \Sigma. \tag{6.2}
$$

This condition is satisfied if we assume c negative and $|c|$ large enough. If (6.2) is satisfied we easily get from (6.1) for $v \in \mathscr{W}$:

$$
|B(v,v)| = \int_A \left[a^{hk} v_{x_h} v_{k_k} + \left(\frac{1}{2} b^h_{x_h} - \frac{1}{2} a^{hk}_{x_h x_k} - c\right) v^2\right] \mathrm{d}x + \frac{1}{2}\int_{\Sigma^{(1)}} v^2 b \,\mathrm{d}\sigma - \frac{1}{2}\int_{\Sigma^{(2)}} v^2 b \,\mathrm{d}\sigma \geq \lambda_0 \|v\|^2 \qquad (\lambda_0 > 0).
$$

It follows that

$$
\begin{gathered}
\lambda_0 \|v\|^2 \leq |B(v,v)| \leq |(v, T(v))| \leq \|v\| \|T(v)\| \\
\left(\int_A v^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{1}{\lambda_0} \|T(v)\|.
\end{gathered} \tag{6.3}
$$

From theorem XIV we deduce:

XX. *If condition* (6.2) *is satisfied, for any* $f \in \mathscr{L}^{(2)}(A)$ *a* $\mathscr{H}$*-weak solution of problem* (1.1_0) *exists.*

The uniqueness of the $\mathscr{H}$-weak solution is an open question. It is connected with the continuity of the bilinear form $B(u,v)$ with respect to the pair (u,v). When this is the case then, since $B(u,v)$ can be extended by continuity in $\mathscr{H} \times \mathscr{H}$, from (6.3) uniqueness of the $\mathscr{H}$-weak solution follows easily.

$B(u,v)$ is continuous with respect to the pair (u,v) in the case that L be self-adjoint, i.e., $b^h \equiv a^{hk}_{x_k}$. This is easily seen from (6.1). In this case, if c is negative in $A \cup \Sigma$, a $\mathscr{H}$-weak solution exists for any given $f \in \mathscr{L}^{(2)}(A)$ and is unique.

Notes

1. The summation convention is assumed throughout this paper.
2. Extensions of $b(x)$ for different h's generally do not agree at the common points of the borders of two different Σ_h's.

3. C. Pucci (Rend. Acc. Nax. Lincel, 1957) proved (3.8), by a different method, under the hypothesis $u = 0$ on the whole of Σ. It must be observed that the set $\Sigma^{(2)} \cup \Sigma^{(3)}$, where u is required to vanish a.e. in order to establish (3.8), may be empty as some of the examples given in Section 2 prove.

4. $\overline{\Sigma_D^{(3)}}$ denotes the closure of $\Sigma_D^{(3)}$.

5. For details see [5].

6. Let x_0 be a point of minimum for u in A.

   Then $[L(w) - cu]_{x=x_0} = a^{hk}(x_0)\, u_{x_h x_k}(x_0)$. We have $\sum_{h,k}^{1,r} u_{x_h x_k}(x_0)\, \lambda_h \lambda_k \geq 0$ and $a^{hk}\lambda_h\lambda_k = \sum_{m=1}^{r} \left(g_m^h \lambda_h\right)^2$. We get $a^{hk} u_{x_h x_k} = \sum_{m=1}^{r} g_m^h g_m^k u_{x_h x_k} \geq 0$ for $x = x_0$. It follows from $c < 0$ that $u(x_0) \geq 0$. Analogously it is proved that $u(x_0) \leq 0$ if x_0 is a point of maximum for u in A.

7. For the proof of this theorem, see [3], [5].

8. See [3], [5].

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