Dokl. Akad. Nauk SSSR
Tom 223 (1975), No. 3

Soviet Math. Dokl.
Vol. 16 (1975), No. 4

A formula for the solution of a boundary value problem for the stationary equation of Brownian motion

Ju. P. Gor'kov*

The present note is devoted to the solution of the following boundary value problem: find a function $u(x, y)$, continuous and bounded in the region $E[x \leq 0, -\infty < y < \infty]$, satisfying for $x < 0$ the equation

$$Lu \equiv u_{yy} - yu_x = f(x, y) \tag{1}$$

and coinciding with the function $\phi(y)$ for $x = 0$, $y \leq 0$.

The equation (1) occurs in the study of Brownian motion, and it has been discussed in the papers [1, 2, 3] in connection with various mathematical problems. Let us also remark that the equation (1) belongs to the class of hypoelliptic second-order equations; boundary value problems for equations of this type have been studied in the papers [4, 5].

Suppose the following conditions are satisfied:

1. $\sup_{x,y} |f(x, y)| < \infty$, $(x, y) \in E$;
2. $f(x, y) \in C^{\alpha,\beta}(E)$, $1/3 < \alpha$, $0 < \beta$;
3. $\int_{-\infty}^{0} \int_{-\infty}^{\infty} |f(x, y)| \,(x^2 + y^2)\, dxdy < \infty$;
4. $\phi(y)$ is continuous and $|\phi(y)| < C_0 \left(1 + |y|^{\delta}\right)$, where $C_0 > 0$, $0 \leq \delta < 1/2$.

Let

$$\Phi(x, y, \xi, \eta) = \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{1}{t^2} \exp\left\{-\frac{(y-\eta)^2}{4t} - \frac{3}{t^3}\left(x - \xi + \frac{y+\eta}{2}t\right)^2\right\} \mathrm{d}t.$$

Then, because the function

$$z(x, y) = -\int_{-\infty}^{0} \int_{-\infty}^{\infty} f(\xi, \eta)\Phi(x, -y, \xi, -\eta) d\xi d\eta$$

satisfies equation (1) as well as condition 4) (for $x = 0$), it follows that the solution of the indicated problem has to be found only for $f \equiv 0$.

Let us proceed to the derivation of a formula for the solution of the problem. Suppose $u(x, y)$ satisfies (1) for $f(x, y) \equiv 0$ and the boundary condition

$$u(0, y) = \varphi(y), \quad y \leqslant 0. \tag{2}$$

---

* *AMS (MOS) subject classifications (1970).* Primary 35H05, 60J65; Secondary 35A22

Integrating both sides of the identity $\Phi(x,y,\xi,\eta)Lu=0$ over the region $x\le 0$, $-\infty<y<\infty$, we then get the equality

$$u(\xi,\eta)+\int_{-\infty}^{\infty} yu(0,y)\Phi(0,y,\xi,\eta)\,\mathrm{d}y=0 \tag{3}$$

Now let $\xi\to 0$ in this equality, denote $u(0,\eta)$ by $\mu(\eta)$, and use the relation

$$\Phi(0,y,0,\eta)=\frac{\sqrt{3}}{2\pi}\frac{1}{y^3+y\eta+\eta^2}$$

Then we get for $\mu(\eta)$ the equation

$$\mu(\eta)+\frac{\sqrt{3}}{2\pi}\int_0^{\infty}\frac{\tau}{\tau^2+\tau+1}\mu(\eta\tau)\,\mathrm{d}\tau=f_0(\eta),\quad f_0(\eta)=-\int_{-\infty}^{0} y\varphi(y)\Phi(0,y,0,\eta)\,\mathrm{d}y. \tag{4}$$

The solution of (4) can be found in explicit form. In fact, if one applies the Mellin transform to the left-hand right-hand sides of (4), then one obtains the new equation

$$\tilde{\mu}(s)+\frac{\sqrt{3}}{2\pi}\int_0^{\infty}\frac{\tau}{\tau^2+\tau+1}\left[\int_0^{\infty}\mu(\eta\tau)\eta^{s-1}\,\mathrm{d}\eta\right]\mathrm{d}\tau=\tilde{f}_0(s)$$
$$\left(\tilde{\mu}(s)=\int_0^{\infty}\mu(\eta)\cdot\eta^{s-1}\,\mathrm{d}\eta\right).$$

It follows that

$$\tilde{\mu}(s)+\frac{\sqrt{3}}{2\pi}\tilde{\mu}(s)\int_0^{\infty}\frac{\tau^{1-s}}{\tau^2+\tau+1}\,\mathrm{d}\tau=\tilde{f}_0(s).$$

Now let $s=1+i\gamma, -\infty<\gamma<\infty$. Then

$$\int_0^{\infty}\frac{\tau^{1-s}}{\tau^2+\tau+1}d\tau=\frac{2\pi}{\sqrt{3}}\frac{1}{1+2\cosh\frac{2}{3}\pi\gamma}$$

and

$$\tilde{\mu}(s)=\tilde{f}_0(s)-\frac{1}{4}\tilde{f}_0(s)\frac{1}{\cos^2\frac{1}{3}\pi(1-s)} \tag{5}$$

Applying the inverse Mellin transform to both sides of (5), one gets

$$\mu(\eta)=f_0(\eta)-\frac{1}{8\pi i}\int_{1-i\infty}^{1+i\infty}f_0(s)\frac{\eta^{-s}}{\cos^2\frac{1}{3}\pi(1-s)}\,\mathrm{d}s$$

If we now make use of the formula (13) of [7], §6.1, and take into account that

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{\eta^{-s}}{\cos^2\frac{1}{3}\pi s}\,\mathrm{d}s=\frac{3}{2\pi i}\eta^{\frac{3}{2}}\int_{\frac{5}{6}-i\infty}^{\frac{5}{6}+i\infty}\frac{\eta^{-3\gamma}}{\sin^2\pi\gamma}\,\mathrm{d}\gamma=\frac{9}{\pi^2}\eta^{\frac{3}{2}}\frac{\ln\eta}{\eta^3-1}$$

(see (20) in [7], §7.2), then we get

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}f_0(s)\frac{\eta^{-s}}{\cos^2\frac{1}{3}\pi(1-s)}\,\mathrm{d}s=\frac{9}{\pi^2}\int_0^{\infty}f_0(\eta\rho)\rho^{\frac{3}{2}}\frac{\ln\rho}{\rho^3-1}\,\mathrm{d}\rho$$

Hence

$$\mu(\eta)=f_0(\eta)-\frac{9}{4\pi^2}\int_0^\infty f_0(\eta\rho)\frac{\rho^{\frac{3}{2}}\ln\rho}{\rho^3-1}\,\mathrm{d}\rho \tag{6}$$

The expression (6) can be written in a simpler form if one makes use of the equality

$$\int_0^\infty \frac{1}{y^2+y\eta\rho+\eta^2\rho^2}\frac{\rho^{\frac{3}{2}}\ln\rho}{\rho^3-1}\,\mathrm{d}\rho=\frac{4\pi^2}{9}\frac{1}{y^2+y\eta+\eta^2}+\frac{4\pi^2}{3\sqrt{3}}\frac{|y|^{\frac{1}{2}}\eta^{\frac{1}{2}}}{y^3-\eta^3}$$

where $f_0(\eta)$ has been replaced by its explicit value (see above).

Finally we have

$$\mu(\eta)=\frac{3}{2\pi}\int_0^\infty\frac{\tau^{\frac{3}{2}}}{\tau^3+1}\varphi(-\tau\eta)\,\mathrm{d}\tau. \tag{7}$$

The relations (3) and (7) lead to the following formula for the solution of the problem (1), (2) (for $f(x,y)=0$):

$$u(x,y)=\int_{-\infty}^0\gamma\varphi(\gamma)G(x,y,\gamma)\,\mathrm{d}\gamma, \tag{8}$$

where

$$G(x,y,\gamma)=\frac{3}{2\pi}\int_0^\infty\frac{\tau^{\frac{3}{2}}}{\tau^3+1}\Phi(0,-\tau\gamma,x,y)\,\mathrm{d}\tau-\Phi(0,\gamma,x,y). \tag{9}$$

The function $u(x,y)$ given by (8) does indeed satisfy equation (1). This is an immediate consequence of the fact that $L\Phi(\xi,\eta,x,y)=0$ for $x\neq\xi$. In order to verify the formula (8) it is also necessary to check that $u(x,y)\to\phi\,(y_0)$ when $x\to-0, y\to y_0$ ($y_0\leq 0$). This can be done by use of the equality $G(0,y,\gamma)\equiv 0$ for $y<0$, $\gamma<0$.

Let us now write down the formula for the solution of the problem (1), (2):

$$\begin{aligned}u(x,y)=&\int_{-\infty}^0\gamma\varphi(\gamma)G(x,y,\gamma)\,\mathrm{d}\gamma\\&-\int_{-\infty}^0\int_{-\infty}^\infty f(\xi,\eta)\left[\Phi(x,y,\xi,\eta)-\int_{-\infty}^0\gamma\Phi(0,\gamma,\xi,\eta)G(x,y,\gamma)\,\mathrm{d}\gamma\right]\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{10}$$

If $f(x,y)$ is defined in the whole plane and if it satisfies the same conditions for $x\geq 0$ as it does for $x\leq 0$ (see 1)-3)), then the solution of the problem (1), (2) can be continued to the right half-plane across the semiaxis $y\geq 0$. The formula for the solution in the plane cut along the negative y-axis, including the origin, is written in similar fashion.

Now let $f(x,y)\equiv 0$ and $\phi(y)=|y|^\delta$, $0<\delta<\frac{1}{2}$. One can show that in this case the solution to the problem (1), (2) has the form

$$u(x,y)=\bar{\mu}(x,y)\left(|x|^{\frac{\delta}{3}}+y^\delta\right),$$

where $0<\mu_0\leq\bar{\mu}(x,y)\leq M_1$ ($x\leq 0$, $-\infty<y<\infty$). It follows easily from this fact that the problem (1), (2) has a unique solution in the class of functions growing not faster than $M\left(|x|^{\delta/3}+|y|^\delta\right)$.

Let us now consider the asymptotic behavior of the function $G(x,y,\gamma)$ as $x^2+y^2\to\infty$ for finite γ. First we remark that if $\phi(y)=c_1/y+c_2/y^2+O\left(1/y^3\right)$ as $y\to-\infty$, then

$$\mu(y)=-c_1/2y+c_2/y^2+O\left(1/y^{\frac{5}{2}}\right)\quad\text{as}\quad y\to+\infty$$

(see [7]). Further, if $\phi(y) = a_1 y + a_2 y^2 + a_3 y^3 + O\left(y^4\right)$ as $y \to -0$, then

$$\mu(y) = \frac{3}{2\pi} \int_{-\infty}^{0} \frac{\varphi(\zeta)}{|\zeta|^{\frac{3}{2}}} \, \mathrm{d}\zeta \cdot y^{\frac{1}{2}} + a_1 y - \frac{a_2}{2} y^2 + a_3 y^3 + O\left(y^{\frac{7}{2}}\right)$$

as $y \to +0$.

Taking these formula into account on easily deduces that $G(x, y, \gamma)$ has the representation

$$G(x, y, \gamma) = \frac{3}{2\pi} \int_{0}^{\infty} \frac{\Phi(0, \nu, \theta, 1) - \Phi(0, 0, \theta, 1)}{\nu^{\frac{3}{2}}} \, \mathrm{d}\nu + \frac{|\gamma|^{\frac{1}{2}}}{y^{\frac{5}{2}}} + O\left(\frac{1}{y^4}\right)$$

for $y > 0, |\theta| \le \theta_0$, $\theta = x/y^3$;

$$G(x, y, \gamma) = \frac{3}{2\pi} \int_{0}^{\infty} \frac{\Phi(1, \nu, 0, s) - \Phi(1, 0, 0, s)}{\nu^{\frac{3}{2}}} \, \mathrm{d}\nu \frac{|\gamma|^{\frac{1}{2}}}{|x|^{\frac{3}{2}}} + O\left(\frac{1}{|x|^{\frac{4}{3}}}\right)$$

for $|s| \le s_0$, $s = y/|x|^{\frac{1}{3}}$;

$$G(x, y, \gamma) = \frac{3}{2\pi} \int_{0}^{\infty} \frac{\Phi(0, \nu, r, -1) - \Phi(0, 0, r, -1)}{\nu^{\frac{3}{2}}} \, \mathrm{d}\nu \frac{|\gamma|^{\frac{1}{2}}}{|y|^{\frac{5}{2}}} + O\left(\frac{1}{y^4}\right)$$

for $y < 0$, $|r| \le r_0$, $r = x/|y|^3$; $\gamma < 0$, $|\gamma| \le \gamma_0$, $x \le 0$.

To conclude we will briefly consider the results from a probabilistic point of view, confining ourselves for simplicity to a homogeneous equation ($f(x, y) \equiv 0$). For a more natural probabilistic interpretation, it is convenient to rewrite the problem (1), (2) in the form

$$\begin{aligned} L_1 u \equiv \frac{1}{2} \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial x} = 0, \quad x < 0, \quad -\infty < y < \infty, \\ u\big|_{y \geqslant 0, x=0} = \varphi\left(-2^{\frac{1}{3}} y\right) \equiv \varphi_1(y). \end{aligned} \tag{1'}$$

The operator L_1 is the infinitesimal generator of a two-dimensional Markov process, which can be described by a system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}X(t) &= Y(t) \, \mathrm{d}t, \\ \mathrm{d}Y(t) &= \mathrm{d}\omega(t), \end{aligned} \tag{11}$$

where $\omega(t)$ is the standard Wiener process. The probabilistic representation of the solution of the problem ($1'$) is easily obtained from the general theory of Markov processes. More precisely, if we denote by $\mathrm{E}_{x,y}(\cdot)$ the mathematical expectation that corresponds to the solution of the system (11) with the initial condition $X(0) = x$, $Y(0) = y$, then (see [6])

$$u(x, y) = \mathbf{E}_{x,y} \phi_1(Y(\tau)) = \int_{0}^{\infty} p(x, y, \gamma) \varphi_1(\gamma) \, \mathrm{d}\gamma \tag{12}$$

where $p(x, y, \gamma)\Delta\gamma = P_{x,y}\{Y(\tau) \in [\gamma, \gamma + \Delta\gamma]\} + o(\Delta\gamma)$ as $\Delta\gamma \to 0$, and where τ is the moment of first passage of the process (11) on the half-line $y > 0$. It follows from the representation (8) and (12) that

$$p(x, y, \gamma) = -4^{\frac{1}{3}} \gamma G\left(x, -2^{\frac{1}{3}} y, -2^{\frac{1}{3}} \gamma\right).$$

I express my gratitude to A. M. Il'in for his interest in my work and his valuable remarks.

Institute of Mathematics and Mechanics
Ural Scientific Center
Academy of Sciences of the USSR

received 25/DEC/74

References

[1] A. Kolmogoroff. Zufällige Bewegungen. *Ann. Math. (2)*, 35:116–117, 1934.

[2] G. Uhlenbeck and L. Ornstein. On the theory of the brownian motion. *Physical review*, 36(5):823, 1930.

[3] S. Chandrasekhar. Stochastic problems in physics and astronomy. *Reviews of modern physics*, 15(1):1, 1943.

[4] O. A. Oleĭnik and E. V. Radkevich. Second order equations with nonnegative characteristic form. Itogi Nauki, Ser. Mat., Mat. Anal. 1969, 7-252., 1971.

[5] M. Derridj. Sur un problème aux limites non elliptique. *C. R. Acad. Sci., Paris, Sér. A*, 269:11–13, 1969.

[6] E. B. Dynkin. *Markov processes. Vols. I, II. Translated.*, volume 121/122. Springer, Cham, 1965.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. Tables of integral transforms. Vol. I. Bateman Manuscript Project. California Institute of Technology. New York: McGraw-Hill Book Co. XX, 391 p., 1954.