Identities in the Lie Algebra of Vector Fields on the Real Line[*]

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Abstract

In the present study, it is proved that many identities that are satisfied in the Lie algebra of vector fields on the real line are a consequence of just one of these identities: $T_4 \equiv 0$. The article discusses the hypothesis that the algebra $\mathfrak{h}_k$ general by k vector fields on the real line is isomorphic to a universal T_4-algebra with k generators $\mathfrak{G}_k$.

Introduction

The concept of the functional dimension of a representation is well known in the theory of finite-dimensional representations of Lie groups (cf. [1, 2]). This concept is less self-evident for the case in which the Lie group is itself infinite-dimensional, and there are a number of different approaches to a definition (cf. [3]). One approach is associated with the algebraic structure of Lie algebras of vector fields on finite-dimensional manifolds. It was discovered comparatively recently that nontrivial identities are satisfied in these Lie algebras (cf. [4] and the literature cited there). That is, suppose that

$$T_k = T_k(x_1, x_2, \ldots, x_k; y) = \mathrm{Alt}(\mathrm{ad}\, x_1 \cdots \mathrm{ad}\, x_k)y \tag{1}$$

where Alt denotes the alternating sum over all permutations of the variables.

Simple calculations (cf., for example, [4]) show that the identity $T_k \equiv 0$ is satisfied in the algebra $\mathrm{Vect}(M^n)$ of vector fields on an n-dimensional manifold M^n when $k \geq (n+1)^2$.

Correspondingly, the identity $T_4 \equiv 0$ is satisfied in the Lie algebra $\mathrm{Vect}(\mathbb{R}^1)$ of vector fields on the real line. Note that if M is an arbitrary manifold (possibly infinite-dimensional) and if P is a one-dimensional bundle on M, all the identities of the Lie algebra $\mathrm{Vect}(\mathbb{R}^1)$, in particular, the identity $T_4 \equiv 0$, are satisfied in the Lie algebra $\mathrm{Vect}(M, P)$ of all vector fields on M that are tangent to the fibers of P.

Let us consider the category A_k of all Lie algebras with k generators that may be realized as subalgebras in $\mathrm{Vect}(M, P)$, where P is a one-dimensional bundle. (Morphisms in this category are homomorphisms of the Lie algebra that carries generators into generators.) Let us prove that there exists a universal

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object $\mathfrak{h}_k$ in A_k. We introduce a countable number of independent variables φ_i^j, $1 \le j \le k-1$, $i = 0, 1, 2, \ldots$, and define the operator $\mathscr{D}$ in $\mathbb{C}[\varphi_i^j]$ by the formula

$$\mathscr{D} = \sum_{j=1}^{k-1} \sum_{i=1}^{\infty} \varphi_i^j \frac{\partial}{\partial \varphi_{i-1}^j}. \tag{2}$$

The operator $\mathscr{D}$ may be interpreted as a linear vector field on an infinite-dimensional space M with coordinates φ_i^j. Suppose that P is a one-dimensional bundle generated by the field $\mathscr{D}$. We denote by $\mathfrak{h}_k$ the Lie algebra generated by the k vector fields

$$\mathscr{D}, \varphi_0^1 \mathscr{D}, \ldots, \varphi_0^{k-1} \mathscr{D}.$$

By construction, $\mathfrak{h}_k$ is an object of the category A_k. If $\mathfrak{h}$ is another object of A_k generated by the fields

$$\xi_0, \xi_1, \ldots, \xi_{k-1} \in \operatorname{Vect}(\tilde{M}, \tilde{P}),$$

then there will exist a homomorphism $\tau : \mathfrak{h}_k \to \mathfrak{h}$ that carries $\mathscr{D}$ into ξ_0 and $\varphi_0^i \mathscr{D}$ into ξ_i, $1 \le i \le k - 1$. In fact, we need only verify that for any Lie polynomial $L(x_1, \ldots, x_k)$, it follows from the equality $L(\mathscr{D}, \varphi_0^1 \mathscr{D}, \ldots, \varphi_0^{k-1} \mathscr{D}) = 0$ that $L(\xi_0, \ldots, \xi_{k-1}) = 0$. It may be assumed that, in appropriate coordinates $(x_0, x_1, \ldots)$ on $\tilde{M}$, the bundle $\tilde{P}$ is locally rectified and is generated by the field $\xi_0 = \partial/\partial x_0$. (We will not go into the details of the theory of vector fields on infinite-dimensional manifolds, since this is not essential to the purely algebraic discussions that follow.) The fields ξ_i assume the form $\xi_i = y_i \partial/\partial x_0$, where y_i are certain smooth functions on $\tilde{M}$. The field $L(\xi_0, \ldots, \xi_{k-1}) = 0$ may be written in the form

$$P\left(y_1, \ldots, y_{k-1}; \frac{\partial y_1}{\partial x_0}, \ldots, \frac{\partial y_{k-1}}{\partial x_0}, \ldots, \frac{\partial^m y_{k-1}}{\partial x_0^m}\right) \frac{\partial}{\partial x_0},$$

where P is a polynomial. At the same time, $L(\mathscr{D}, \varphi_0^1 \mathscr{D}, \ldots, \varphi_0^{k-1} \mathscr{D})$ may be written in the form

$$P(\varphi_0^1, \ldots, \varphi_0^{k-1}, \varphi_1^1, \ldots, \varphi_1^{k-1}, \ldots, \varphi_m^{k-1}) \mathscr{D}.$$

By definition, this expression must equal 0, whence $P \equiv 0$, and the assertion is proved. Thereby we have also proved the universality of $\mathfrak{h}_k$.

Now suppose that $\mathscr{B}_k$ is the category of Lie algebras with k generators in which the identity $T_4 \equiv 0$ is satisfied. We will call the objects of $\mathscr{B}_k$ Lie T_4-algebras. There is also a universal object in this category, which we denote by $\mathfrak{G}_k$. It is the quotient algebra of a free algebra with k generators in each ideal generated by elements of the form $T_4(x_1, x_2, x_3, x_4; y)$.

The algebras $\mathfrak{h}_k$ and $\mathfrak{G}_k$ share a number of features (cf. Theorems 1 to 4 below), which lead us to the hypothesis that they might be isomorphic. More precisely, we suppose that the canonical epimorphism $\pi : \mathfrak{G}_k \to \mathfrak{h}_k$, which exists by virtue of the definition of $\mathfrak{G}_k$, is in fact an isomorphism.

The algebras of vector fields on manifolds are differentiation algebras of the commutative associative algebras of functions on these manifolds. We begin our study of $\mathfrak{G}_k$ by constructing for it a commutative associative algebra $A(\mathfrak{G}_k) \subset \operatorname{End}(\mathfrak{G}_k)$ (by analogy with the algebras of vector fields) such that $\mathfrak{G}_k$ acts on

the algebra $A(\mathfrak{G}_k)$ by means of differentiations, and moreover, the mapping $\mathfrak{G}_k \to \operatorname{Der} A(\mathfrak{G}_k)$ specified by this action is a morphism of $A(\mathfrak{G}_k)$-modules.

In Section 2 we present a "free algebra theorem" for the Lie algebras $\mathfrak{h}_k$, which may be considered an analog of the well-known Shirshov theorem [6] on the subalgebras of a free Lie algebra.

Proofs of these results are given below in Section 3.

1 Statement of fundamental results

The following fact will guide our discussion in the construction of the algebra $A(\mathfrak{G}_k)$.

If $x_i = x_i(t)\mathrm{d}/\mathrm{d}t$, $y = y(t)\mathrm{d}/\mathrm{d}t$ are elements of $\operatorname{Vect}(\mathbb{R}^1)$, then it may be verified directly that

$$T_3(x_1, x_2, x_3; y) = -2 \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} \cdot y. \tag{3}$$

This means that the operator

$$a(x_1, x_2, x_3) = \operatorname{Alt}(\operatorname{ad} x_1 \operatorname{ad} x_2 \operatorname{ad} x_3)$$

is an operator for multiplication by a function in $\operatorname{Vect}(\mathbb{R}^1)$.

Theorem 1. *For any Lie T_4-algebra $\mathfrak{G}$, the operators $a(x_1, x_2, x_3)$, $x_i \in \mathfrak{G}$, generate a commutative subalgebra $A(\mathfrak{G})$ in $\operatorname{End}(\mathfrak{G})$.*

The following lemma reveals to some extent the structure of the subalgebra $A(\mathfrak{G})$ in the case $\mathfrak{G} = \mathfrak{h}_k$.

Lemma 1. *The quotient field $B(\mathfrak{h}_k)$ of the algebra $A(\mathfrak{h}_k)$ consists of all rational functions of φ_i^j, $1 \leq j \leq k-1$, $i = 0, 1, \ldots$.*

Proof. Suppose that $a \in A(\mathfrak{h}_k)$ is such that $\mathscr{D}(a) \neq 0$ (for example, $a = a(\mathscr{D}, \varphi_0^1 \mathscr{D}, \varphi_1^1 \mathscr{D})$). Then

$$(\varphi_i^j \mathscr{D})(a) = \varphi_i^j(\mathscr{D}(a)) \in A(\mathfrak{h}_k).$$

Hence

$$\varphi_i^j = (\varphi_i^j \mathscr{D}(a))(\mathscr{D}(a))^{-1} \in B(\mathfrak{h}_k). \qquad \square$$

Theorem 2. *Suppose that $\mathfrak{G}$ is a Lie T_4-algebra and that $\xi \in \mathfrak{G}$ and $a, a_1, a_2 \in A(\mathfrak{G})$. Then:*

1. *$[\operatorname{ad}\xi, a] \in A(\mathfrak{G})$,*

2. *$[\operatorname{ad}\xi, a_1 a_2] = [\operatorname{ad}\xi, a_1]a_2 + a_1[\operatorname{ad}\xi, a_2]$.*

Thus, the action of $\mathfrak{G}$ on $A(\mathfrak{G})$ may be determined by the formula

$$\xi(a) = [\operatorname{ad}\xi, a] \tag{4}$$

and the action operators will constitute differentiations of the algebra $A(\mathfrak{G})$. In the case $\mathfrak{G} = \operatorname{Vect}(\mathbb{R}^1)$, this action coincides with the ordinary action of vector fields on functions.

Suppose that A is an arbitrary commutative algebra, and that $\operatorname{Der} A$ is the Lie differentiation algebra of A. Clearly, $\operatorname{Der} A$ is an A-module relative to the operation

$$(a\xi)(b) = a \cdot \xi(b). \tag{5}$$

We suppose that $\mathfrak{p} \subset \operatorname{Der} A$ is simultaneously a Lie subalgebra and an A-submodule. We will call $\mathfrak{p}$ an A-subalgebra. Then for $\xi, \eta \in \mathfrak{p}$, $a, b \in A$, we have the following equalities:

$$\xi(a)\eta = [\operatorname{ad}\xi, a]\eta, \tag{6}$$
$$[a\xi, b\eta] = a\xi(b)\eta - b\eta(a)\xi + ab[\xi, \eta]. \tag{7}$$

Indeed,

$$[\operatorname{ad}\xi, a]\eta = \operatorname{ad}\xi(a\eta) - a\operatorname{ad}\xi(\eta) = [\xi, a\eta] - a[\xi, \eta] = \xi a\eta - a\eta\xi - a[\xi, \eta]$$
$$= \xi(a)\eta + a\xi\eta - a\eta\xi - a[\xi, \eta] = \xi(a)\eta,$$

and (7) follows from (6):

$$[\xi, b\eta] = \operatorname{ad}\xi(b\eta) = [\operatorname{ad}\xi, b]\eta + b\operatorname{ad}\xi(\eta) = \xi(b)\eta + b[\xi, \eta].$$

Theorem 3. *For any Lie T_4-algebra $\mathfrak{G}$, the mapping of $\mathfrak{G}$ into $\operatorname{Der} A(\mathfrak{G})$ specified by (4) is a morphism of $A(\mathfrak{G})$-modules, so that its image $\mathfrak{p}$ is an $A(\mathfrak{G})$-subalgebra in $\operatorname{Der} A(\mathfrak{G})$.*

As a module over an algebra of functions, the Lie algebra $\operatorname{Vect}(M, P)$ possesses yet another property. That is, any two elements of $\operatorname{Vect}(M, P)$ are linearly dependent over an algebra of functions.

Theorem 4. *Suppose that $\mathfrak{G}$ is a Lie T_4-algebra. For any $\xi, \eta \in \mathfrak{G}$ and any $a \in A(\mathfrak{G})$,*

$$\xi(a)\eta = \eta(a)\xi. \tag{8}$$

Note that (8) is self-evident for $\mathfrak{h}_k$:

$$P(\varphi)\mathscr{D}(a)Q(\varphi)\mathscr{D} = Q(\varphi)\mathscr{D}(a)P(\varphi)\mathscr{D}.$$

2 Free algebra theorem

The fundamental result of the present section is the following assertion.

Theorem 5. *Any subalgebra in $\mathfrak{h}_k$ generated by two linearly independent elements is isomorphic to $\mathfrak{h}_2$.*

Proof. Suppose that the subalgebra $\mathfrak{h} \subset \mathfrak{h}_k$ is generated by the elements $P(\varphi)\mathscr{D}$ and $Q(\varphi)\mathscr{D}$, where

$$\varphi = (\varphi_0^1, \ldots, \varphi_0^{k-1}, \ldots; \ldots \varphi_N^{k-1}),$$

and that the polynomials P and Q are linearly independent. Our goal is to prove that the mapping $\tau : \mathfrak{h}_2 \to \mathfrak{h}$, which exists by virtue of the universality of $\mathfrak{h}_2$, possesses the property that $\operatorname{Ker}\tau = \{0\}$. Let us represent τ in the form of a composition of two mappings. For this purpose, we consider the Lie algebra $\mathfrak{G}$

of expressions of the form $R(\varphi)\mathscr{D}$, where R is a rational function with a natural law of commutativity:

$$[R_1(\varphi)\mathscr{D}, R_2(\varphi)\mathscr{D}] = (R_1\mathscr{D}R_2 - R_2\mathscr{D}R_1)\mathscr{D}.$$

Suppose that τ_1 is a homomorphism of $\mathfrak{h}_2$ into $\mathfrak{G}$ that carries $\mathscr{D}$ into $\mathscr{D}$ and $\varphi_0^1\mathscr{D}$ into $S(\varphi)\mathscr{D}$, where $S(\varphi)$ is a not constant function.

Lemma 2. $\operatorname{Ker}\tau_1 = \{0\}$.

Proof. Suppose that $L(x_1, x_2)$ is a Lie polynomial that does not vanish under the substitution $x_1 = \mathscr{D}$, $x_2 = \varphi_0^1\mathscr{D}$. Then

$$L(\mathscr{D}, \varphi_0^1\mathscr{D}) = P_L(\varphi_0^1, \varphi_1^1, \ldots, \varphi_N^1)\mathscr{D},$$

where P_L is some polynomial. Therefore,

$$L(\mathscr{D}, S(\varphi)\mathscr{D}) = P_L(\psi_0, \ldots, \psi_N)\mathscr{D},$$

where $\psi_k = \mathscr{D}^k S$.

It remains for us to check the algebraic independence of the variables $\psi_0, \ldots, \psi_N$. For this purpose, we arrange the variables φ_i^j so that the effect of $\mathscr{D}$ is to increase the ordinal number of the variable. (Lexicographic ordering, for example, would be appropriate.) Suppose that $n(S)$ is the greatest ordinal number such that $\partial S/\partial \varphi_n \neq 0$. Then

$$n(\psi_0) = n(S) < n(\psi_1) < \cdots < n(\psi_N). \qquad \square$$

Now let us consider the homomorphism τ_2 of the algebra $\tau_1(\mathfrak{h}_2)$ into $\mathfrak{G}$ that carries elements of $\mathscr{D}$ and $S\mathscr{D}$ into $P\mathscr{D}$ and $PS\mathscr{D}$, respectively. The existence of τ_2 is guaranteed by Lemma 2 and the universality of $\mathfrak{h}_2$.

Lemma 3. *If $P \neq 0$, then* $\operatorname{Ker}\tau_2 = \{0\}$.

Proof. The proof of the lemma is similar to the preceding proof. If

$$L(\mathscr{D}, S\mathscr{D}) = P_L(\psi_0, \ldots, \psi_N)\mathscr{D},$$

then

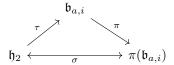$$L(P\mathscr{D}, PS\mathscr{D}) = P_L(\theta_0, \ldots, \theta_N)P\mathscr{D},$$

where $\theta_k = (P(\varphi)\mathscr{D})^k S$. The algebraic independence of the variables $\theta_0, \ldots, \theta_N$ may be verified as above. The lemma is proved. $\qquad \square$

The assertion of the theorem follows from Lemmas 2 and 3 if we set $S = Q/P$. $\qquad \square$

By means of Theorems 1 to 5, we may find numerous subalgebras in $\mathfrak{G}_k$ that are isomorphic to $\mathfrak{h}_2$.

Lemma 4. *If $a \in A(\mathfrak{G}_k)$ so that $\pi(a) \neq 0$, then the subalgebra $\mathfrak{b}_{a,i}$ in $\mathfrak{G}_k$ generated by ξ_i and a_{ξ_i} is isomorphic to the Lie algebra $\mathfrak{h}_2$.*

Proof. Under the canonical homomorphism $\pi : \mathfrak{G}_k \to \mathfrak{h}_k$,

$$
\begin{array}{ccc}
 & \mathfrak{b}_{a,i} & \\
\tau \nearrow & & \searrow \pi \\
\mathfrak{h}_2 & \xleftarrow{\hspace{1.5em}} \xrightarrow[\sigma]{\hspace{1.5em}} & \pi(\mathfrak{b}_{a,i})
\end{array}
$$

the algebra $\mathfrak{b}_{a,i}$ is mapped onto the algebra $\pi(\mathfrak{b}_{a,i})$ that is generated by the linearly independent elements $\varphi_0^{i-1}\mathscr{D}$, $\pi(a)\varphi_0^{i-1}\mathscr{D}$, and consequently, by Theorem 5, the Lie algebra $\pi(\mathfrak{b}_{a,i})$ is isomorphic to the Lie algebra $\mathfrak{h}_2$. Since ξ_i is a differentiation of the algebra $A(\mathfrak{G}_k)$, and since the commutative algebra

$$B(\mathfrak{h}_2) = \mathbb{C}[\varphi_0, \varphi_1, \ldots, \varphi_N, \ldots]$$

is mapped homomorphically into $A(\mathfrak{G}_k)$,

$$\varphi_0 \to a, \ldots, \varphi_n \to \xi_i^n(a),$$

so that $A(\mathfrak{h}_2) \overset{\text{onto}}{\to} A(\mathfrak{b}_{a,i})$, then the mapping of generators $\tau : \mathscr{D} \to \xi_i$, $\varphi_0 \mathscr{D} \to a\xi_i$ is continued to a homomorphism of the Lie algebra $\mathfrak{h}_2$ onto $\mathfrak{b}_{a,i}$.

All three mappings τ, π, and σ carry generators into generators, and consequently $\sigma \circ \pi \circ \tau = \text{id}$, and also τ is an isomorphism of the Lie algebras $\mathfrak{h}_2$ and $\mathfrak{b}_{a,i}$. The lemma is proved. $\qquad\square$

From this lemma we obtain the following assertion.

Corollary. *To prove the isomorphism between $\mathfrak{G}_2$ and $\mathfrak{h}_2$, it is necessary and sufficient to prove that $\mathfrak{G}_2$ is canonically isomorphic to two of its own subalgebras, namely those that are generated by ξ_1, $[\xi_1, \xi_2]$ and $\xi_1, [[\xi_1, \xi_2], \xi_2]$, respectively.*

The necessity follows from Theorem 5; and sufficiency, from Lemma 4, applied to

$$a = a(\xi_1, \xi_2, [\xi_1, \xi_2]).$$

3 Proofs of Theorems 1 to 4

To avoid cumbersome notation, we will denote alternation by the index of the sum of products of operators of $\operatorname{ad}\xi$, omitting the symbols Alt and ad, i.e., the notation $\xi_1\xi_2\eta\xi_3$ will denote the operator $\operatorname{Alt}_\xi(\operatorname{ad}\xi_1 \operatorname{ad}\xi_2 \operatorname{ad}\eta \operatorname{ad}\xi_3)$ and similarly

$$\xi_1\xi_2[\eta\xi_3](\zeta) := \operatorname{Alt}_\xi(\operatorname{ad}\xi_1 \operatorname{ad}\xi_2 \operatorname{ad}[\eta, \xi_3])(\zeta) = \operatorname{Alt}_\xi[\xi_1[\xi_2[[\eta\xi_3]\zeta]]].$$

To prove Theorem 1, we will require an equivalent form of the identity $T_4 \equiv 0$ which is of independent interest.

Proposition 1. *The identity $T_4 \equiv 0$ is equivalent to*

$$\eta\xi_1\xi_2\xi_3 - 2\xi_1\xi_2\eta\xi_3 \equiv 0. \tag{9}$$

Proof. Writing $T_4(\eta, \xi_1, \xi_2, \xi_3; \zeta) = (\eta\xi_1\xi_2\xi_3 - \xi_1\eta\xi_2\xi_3 + \xi_1\xi_2\eta\xi_3 - \xi_1\xi_2\xi_3\eta)(\zeta)$, using Jacobi's identity and anticommutativity, we insert η at the rightmost position, replacing ζ:

$$-\xi_1\xi_2\xi_3\eta(\zeta) = \xi_1\xi_2\xi_3\zeta(\eta),$$
$$\xi_1\xi_2\eta\xi_3(\zeta) = \xi_1\xi_2[\eta,\xi_3](\zeta) + \xi_1\xi_2\xi_3\eta(\zeta)$$
$$= (\xi_1\xi_2\zeta\xi_3 - \xi_1\xi_2\xi_3\zeta)(\eta)$$
$$-\xi_1\eta\xi_2\xi_3(\zeta) = -\xi_1[[\eta,\xi_2],\xi_3](\zeta) - \xi_1\xi_2\xi_3\eta(\zeta)$$
$$= (-\xi_1\zeta\xi_2\xi_3 + \xi_1\xi_2\xi_3\zeta)(\eta).$$

In this last case and in what follows, we express η at once in terms of two variables, since ξ_1 and ξ_2 are alternating, while the other two terms are reduced:

$$\eta\xi_1\xi_2\xi_3(\zeta) = [[\eta,\xi_1],\xi_2]\xi_3(\zeta) + \xi_1\xi_2\eta\xi_3(\zeta)$$
$$= [[[\eta,\xi_1],\xi_2],\xi_3](\zeta) + \xi_3[[\eta,\xi_1],\xi_2](\zeta) + \xi_1\xi_2\eta\xi_3(\zeta)$$
$$= (-\zeta\xi_1\xi_2\xi_3 + \xi_1\zeta\xi_2\xi_3 + \xi_1\xi_2\zeta\xi_3 - \xi_1\xi_2\xi_3\zeta)(\eta).$$

Adding together these equalities, term by term, yields the desired formula. $\square$

Corollary. *In a Lie T_4-algebra, $\xi_1\xi_2\eta\xi_3\xi_4 = 0$, since*

$$0 = \eta\xi_1\xi_2\xi_3\xi_4 = 2\xi_1\xi_2\eta\xi_3\xi_4. \tag{10}$$

Now let us prove Theorem 1. In accordance with our stipulations about notation, the commutator of the two operators $a(\xi_1, \xi_2, \xi_3)$ and $a(\eta_1, \eta_2, \eta_3)$ has the form

$$\xi_1\xi_2\xi_3\eta_1\eta_2\eta_3 - \eta_1\eta_2\eta_3\xi_1\xi_2\xi_3.$$

Let us consider "randomizing" permutations of this expression, that is, operators that may be constructed by alternating the two triples of elements

$$\mathrm{ad}\,\xi_1, \mathrm{ad}\,\xi_2, \mathrm{ad}\,\xi_3 \text{ and } \mathrm{ad}\,\eta_1, \mathrm{ad}\,\eta_2, \mathrm{ad}\,\eta_3$$

within each of which an alternation occurs. It is easier to prove that all these operators are equal to zero simultaneously than to prove that just the first one is equal to zero.

There are precisely 10 pairs:

   I. $\xi_1\xi_2\xi_3\eta_1\eta_2\eta_3 - \eta_1\eta_2\eta_3\xi_1\xi_2\xi_3$

  II. $\xi_1\xi_2\eta_1\xi_3\eta_2\eta_3 - \eta_1\eta_2\xi_1\eta_3\xi_2\xi_3$

 III. $\xi_1\xi_2\eta_1\eta_2\xi_3\eta_3 - \eta_1\eta_2\xi_1\xi_2\eta_3\xi_3$

 IV. $\xi_1\xi_2\eta_1\eta_2\eta_3\xi_3 - \eta_1\eta_2\xi_1\xi_2\xi_3\eta_3$

  V. $\xi_1\eta_1\xi_2\xi_3\eta_2\eta_3 - \eta_1\xi_1\eta_2\eta_3\xi_2\xi_3$

 VI. $\xi_1\eta_1\xi_2\eta_2\xi_3\eta_3 - \eta_1\xi_1\eta_2\xi_2\eta_3\xi_3$

VII. $\xi_1\eta_1\xi_2\eta_2\eta_3\xi_3 - \eta_1\xi_1\eta_2\xi_2\xi_3\eta_3$

VIII. $\xi_1\eta_1\eta_2\xi_2\xi_3\eta_3 - \eta_1\xi_1\xi_2\eta_2\eta_3\xi_3$

 IX. $\xi_1\eta_1\eta_2\xi_2\eta_3\xi_3 - \eta_1\xi_1\xi_3\eta_2\xi_3\eta_3$

  X. $\xi_1\eta_1\eta_2\eta_3\xi_2\xi_3 - \eta_1\xi_1\xi_2\xi_3\eta_2\eta_3$

Like identities (9) and (10), the identity $T_4 \equiv 0$ yields a linear dependence between them:

$\mathrm{I} + \mathrm{IV} = 0$ since $\xi_1\xi_2\xi_3\eta_1\eta_2\eta_3 = 2\xi_1\xi_2\eta_1\eta_2\xi_3\eta_3$ (identity (9), written for $\eta_1, \eta_2, \eta_3, \xi_3$) $= 2 \circlearrowleft_\eta \xi_1\xi_2[\eta_1, \eta_2]\xi_3\eta_3 = \eta_1\eta_2\xi_1\xi_2\xi_3\eta_3$ (identity (9) for $\xi_1, \xi_2, \xi_3, [\eta_1, \eta_2]$) and analogously $\eta_1\eta_2\eta_3\xi_1\xi_2\xi_3 = \xi_1\xi_2\eta_1\eta_2\eta_3\xi_3$

$\mathrm{I} - 2 \cdot \mathrm{III} = 0$ since $\xi_1\xi_2\xi_3\eta_1\eta_2\eta_3 = 2\xi_1\xi_2\eta_1\eta_2\xi_3\eta_3$

$\mathrm{I} - 2 \cdot \mathrm{VIII} = 0$ since $\xi_1\xi_2\xi_3\eta_1\eta_2\eta_3 = \circlearrowleft_\xi \xi_1[\xi_2, \xi_3]\eta_1\eta_2\eta_3 = 2 \circlearrowleft_\xi \xi_1\eta_1\eta_2[\xi_2, \xi_3]\eta_3 = 2\xi_1\eta_1\eta_2\xi_2\xi_3\eta_3$ (identity (9) for $\eta_1, \eta_2, \eta_3, [\xi_2, \xi_3]$)

$\frac{5}{2} \cdot \mathrm{I} - \mathrm{II} = 0$ since $\mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} = 0$ ($T_4(\xi_3, \eta_1, \eta_2, \eta_3; \dots) = 0$)

$2 \cdot \mathrm{II} + \mathrm{X} = 0$ since $2\xi_1\xi_2\eta_1\xi_3\eta_2\eta_3 = \eta_1\xi_1\xi_2\xi_3\eta_2\eta_3$ (identity (9) for $\xi_1, \xi_2, \xi_3, \eta_1$)

$\frac{13}{2} \cdot \mathrm{I} - \mathrm{V} = 0$ since $\mathrm{I} - \mathrm{II} + \mathrm{V} + \mathrm{X} = 0$ ($T_4(\xi_1, \xi_2, \xi_3, \eta_1; \dots) = 0$)

$\mathrm{IV} - 2 \cdot \mathrm{IX} = 0$ since $\xi_1\xi_2\eta_1\eta_2\eta_3\xi_3 = 2\xi_1\eta_1\eta_2\xi_2\eta_3\xi_3$ (identity (9) for $\eta_1, \eta_2, \eta_3, \xi_2$)

$6 \cdot \mathrm{I} - \mathrm{VI} = 0$ since $\mathrm{I} - \mathrm{II} + \mathrm{III} + \mathrm{V} - \mathrm{VI} + \mathrm{VIII} = 0$ ($T_4(\xi_2, \xi_3, \eta_1, \eta_2; \dots) = 0$)

$\frac{7}{2} \cdot \mathrm{I} - \mathrm{VII} = 0$ since $\mathrm{IV} - \mathrm{VII} + \mathrm{IX} - \mathrm{X} = 0$ ($T_4(\eta_1, \xi_2, \eta_2, \eta_3; \dots) = 0$)

Identity (10) yields one more linear equation independent of the preceding equations:

$$0 = \mathrm{II} - \mathrm{III} - \mathrm{VIII} - \mathrm{VII} = \left(\frac{5}{2} - \frac{1}{2} - \frac{1}{2} - \frac{7}{2}\right)\mathrm{I} = -2\mathrm{I}.$$

Hence the assertion of Theorem 1 follows.

Proof of Theorem 2. Let us prove that

$$[\mathrm{ad}\,\xi, a(\eta_1, \eta_2, \eta_3)] = a([\xi, \eta_1], \eta_2, \eta_3) + a(\eta_1, [\xi, \eta_2], \eta_3) + a(\eta_1, \eta_2, [\xi, \eta_3]). \quad (11)$$

In fact,

$$\begin{aligned}
\xi\eta_1\eta_2\eta_3 - \eta_1\eta_2\eta_3\xi &= \xi\eta_1\eta_2\eta_3 - \eta_1\xi\eta_2\eta_3 + \eta_1\xi\eta_2\eta_3 - \eta_1\eta_2\xi\eta_3 \\
&\quad + \eta_1\eta_2\xi\eta_3 - \eta_1\eta_2\eta_3\xi \\
&= [\xi, \eta_1]\eta_2\eta_3 + \eta_1[\xi, \eta_2]\eta_3 + \eta_1\eta_2[\xi, \eta_3].
\end{aligned}$$

Here alternation occurs with respect to η_1, η_2, η_3. But if we group the terms containing

$$[\xi, \eta_1], \quad [\xi, \eta_2], \quad [\xi, \eta_3]$$

we obtain (11), and consequently, Assertion 1 of Theorem 2 for the generators of the algebra $A(\mathfrak{G})$. Assertion 2 is a corollary of the fact that the commutator is a differentiation. In our case,

$$\begin{aligned}
[\mathrm{ad}\,\xi, a_1 a_2] &= \mathrm{ad}\,\xi a_1 a_2 - a_1 a_2\,\mathrm{ad}\,\xi \\
&= \mathrm{ad}\,\xi a_1 a_2 - a_1\,\mathrm{ad}\,\xi a_2 + a_1\,\mathrm{ad}\,\xi a_2 - a_1 a_2\,\mathrm{ad}\,\xi \\
&= [\mathrm{ad}\,\xi, a_1] a_2 + a_1 [\mathrm{ad}\,\xi, a_2].
\end{aligned}$$

Hence Assertion 1 follows for an arbitrary $a \in A(\mathfrak{G})$. $\square$

Proof of Theorem 3. It is necessary and sufficient to prove that Equation (5) is satisfied for the Lie T_4-algebra $\mathfrak{G}$ and both the algebras $A(\mathfrak{G}) \subset \operatorname{End} \mathfrak{G}$ defined in Section 1 and the action of $\mathfrak{G}$ on $A(\mathfrak{G})$; that is, to prove that

$$a(\xi)(b) = a\xi(b)$$

for $\xi \in \mathfrak{G}, a, b \in A(\mathfrak{G})$ under the condition that

$$\xi(a) := [\operatorname{ad} \xi, a].$$

It is sufficient to verify this fact for $a = a\,(\zeta_1, \zeta_2, \zeta_3)$:

$$
\begin{aligned}
[\operatorname{ad} a(\xi), b] &= [\operatorname{ad}[\zeta_1[\zeta_2[\zeta_3, \xi]]], b] \quad (\text{since } \operatorname{ad}[\xi, \eta] = [\operatorname{ad} \xi, \operatorname{ad} \eta]) \\
&= [[\operatorname{ad} \zeta_1[\operatorname{ad} \zeta_2[\operatorname{ad} \zeta_3, \operatorname{ad} \xi]]], b] \\
&= [\zeta_1\zeta_2\zeta_3\xi, b] - [\zeta_1\zeta_2\xi\zeta_3, b] - [\zeta_1\xi\zeta_2\zeta_3, b] + [\xi\zeta_1\zeta_2\zeta_3, b] \\
&= a[\operatorname{ad} \xi, b] + [\zeta_1[\zeta_2\xi]\zeta_3, b].
\end{aligned}
$$

But

$$
\begin{aligned}
\xi(a) &= [\operatorname{ad} \xi, a] = \xi\zeta_1\zeta_2\zeta_3 - \zeta_1\zeta_2\zeta_3\xi - T_4\,(\xi, \zeta_1, \zeta_2, \zeta_3) \\
&= \zeta_1\xi\zeta_2\zeta_3 - \zeta_1\zeta_2\xi\zeta_3 = \zeta_1[\xi, \zeta_2]\zeta_3 \in A(\mathfrak{G}),
\end{aligned}
$$

which consequently commutes with $b \in A(\mathfrak{G}) \Rightarrow [\operatorname{ad} a(\xi), b] = a[\operatorname{ad} \xi, b]$. Theorem 3 is proved. $\qquad\square$

Proof of Theorem 4. For this proof, we will use only the equality

$$\xi(a) = \zeta_1[\xi\zeta_2]\zeta_3$$

for $\xi \in \mathfrak{G}, a = a(\zeta_1, \zeta_2, \zeta_3) \in A(\mathfrak{G})$. Then

$$\xi(a)\eta = \zeta_1[\xi\zeta_2]\zeta_3(\eta)$$

by Jacobi's identity, and also

$$
\begin{aligned}
&= \zeta_1[[\xi\zeta_2]\zeta_3](\eta) + \zeta_1\zeta_3[\xi\zeta_2](\eta) \\
&= -\zeta_1\eta\zeta_3\zeta_2(\xi) + \zeta_1\zeta_3\eta\zeta_2(\xi) \\
&= (\zeta_1\eta\zeta_2\zeta_3 - \zeta_1\zeta_2\eta\zeta_3)(\xi) = \zeta_1[\eta\zeta_2]\zeta_3(\xi) \\
&= \eta(a)\xi. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\square
\end{aligned}
$$

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