NOTES ON SIMULATING MAXWELL'S EQUATIONS

JOHN SHUMWAY

1. Introduction

Maxwell's equations are: Gauss's law,

$$\oiint \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}, \tag{1}$$

Gauss's law for magnetism,

$$\oiint \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0, \tag{2}$$

Maxwell–Faraday equation (Faraday's law of induction),

$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_S(\mathbf{B})}{\partial t}. \tag{3}$$

and Ampère's circuital law (with Maxwell's correction),

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_S(\mathbf{E})}{\partial t}. \tag{4}$$

The first two laws relate fields to the sources (charges), and once they are satisfied, the dynamics causes them to always be satisfied. The last two laws can be rewritten to look more like equations of motion, Faraday's law of induction tells us how magnetic fields change,

$$\frac{\partial}{\partial t} \Phi_S(\mathbf{B}) = -\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{5}$$

and Ampère's circuital law with Maxwell's correction tells us how electric fields change,

$$\frac{\partial \Phi_S(\mathbf{E})}{\partial t} = \frac{1}{\mu_0 \varepsilon_0} \oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} - \frac{I_S}{\varepsilon_0}. \tag{6}$$

2. Discretizing the fields and sources

We construct a three dimensional grid of cubes. The comments of the electric fields, (E_x, E_y, and E_z), live on the edges of the cubes. Charge densities (ρ) live on the vertices of the cubes, and current densities (J_x, J_y, and J_z) live on the edges of the cubes.

We construct a second three-dimensional grid of cubes, this one shifted so that the cubes of the second grid are centered on the vertices of the first grid. The components of the magnetic field (B_x, B_y, and B_z) live on the edges of the second grid.

Denote the edge length of the grids by a, so that the length of a cube edge is a, the area of a cube face is a^2, and the volume of a cube is a^2.

3. Geometric interpretation of fluxes , curls, and currents

Now the integrals in Maxwell's equations take on simple geometric and algebraic meanings.

The electric flux through a surface, $\Phi_S(\mathbf{E})$ is simply the electric field component E_i on the first grid that pierces a face S on the second grid, times the area of the face a^2. For example a face on the +x side of a cube has flux,

$$\Phi_S(\mathbf{E}) = a^2 E_x, \tag{7}$$

where E_x can be read off an edge of the first grid.

The magnetic flux is defined in a similar manner,

$$\Phi_S(\mathbf{B}) = a^2 B_x. \tag{8}$$

The electric curl around a face on the first grid is just the sum of the electric fields circulating around the face times the length of an edge, a,

$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = a(E_y + E_z - E_y - E_z) \tag{9}$$

The magnetic curl is defined in a similar manner,

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = a(B_y + B_z - B_y - B_z) \tag{10}$$

Finally, the current I_S through a face S on the second grid is just the current density J_i on the edge that pierces that face times the area of the face, a^2,

$$I_S = a^2 J_x. \tag{11}$$

4. Integrating Maxwell's equations forward in time

To integrate Maxwell's equations, choose a small time step, $\Delta t < a/c$, where c is the speed of light. First Faraday's law of induction to advance the magnetic fields,

$$B_x^{(\mathrm{new})} = B_x^{(\mathrm{old})} - \frac{\Delta t}{a}(E_y + E_z - E_y - E_z) \tag{12}$$

Then calculate the current densities using $\mathbf{J} = \sigma \mathbf{E}$ anywhere there is conducting materials. Finally update the electric fields using Ampère's law with Maxwell's correction.

$$E_x^{(\mathrm{new})} = E_x^{(\mathrm{old})} + \frac{c^2 \Delta t}{a}(B_y + B_z - B_y - B_z) - \frac{\Delta t}{\varepsilon_0} J_x. \tag{13}$$

Here we have used $\epsilon_0 \mu_0 = 1/c^2$ to simplify the expression a bit.