Time Delay Estimation in Gravitationally Lensed Photon Stream Pairs

Michał Staniaszek
Supervisor: Peter Tiňo

For the degree of BSc Computer Science with Study Abroad

School of Computer Science
University of Birmingham

April 12, 2013

Abstract

In this report, we present a system for estimating the time delay Δ between pairs of photon streams from separate images of gravitationally lensed objects. Photon streams are simulated by generating arrival times of individual photons using non-homogeneous Poisson processes with a rate function $\lambda(t)$, which is randomly generated. We develop a linear estimator based on ordinary least squares regression, and a kernel density estimator, which we use to estimate the rate function of photon streams. Two time delay estimation methods are developed, using inter-function area or probability density functions to estimate the value of Δ based on estimates of the rate function. We compare the efficacy of the four possible combinations of estimation methods on sine functions and randomly generated functions, and show that there is no significant difference between the accuracy of the estimates they produce. However, the kernel density estimator combined with the inter-function area method appears to be the most consistent.

Keywords: Non-homogeneous Poisson process, gravitational lensing, machine learning, linear regression, kernel density estimation

Acknowledgements

I would like to thank Peter Tiňo for his clear and insightful supervision, without which this project would not have been possible. Thanks also to Jeremy Wyatt, Nick Hawes and Achim Jung for their inspirational teaching and enthusiastic support during my time at university, and for sparking my interest in artificial intelligence and mathematics.

Typeset in Linux Libertine using XƎLaTeX.

Contents

1 Introduction **1**

2 Background **2**
2.1 Gravitational Lensing . . . 2
2.2 Poisson Processes . . . 4
2.3 Regression . . . 5

3 Simulation of Photon Streams **6**
3.1 Function Generation . . . 6
3.2 Generating Streams from Functions . . . 7

4 Function Estimation **8**
4.1 Baseline Estimation . . . 8
4.1.1 Ordinary Least Squares . . . 8
4.1.2 Iterative Weighted Least Squares . . . 12
4.1.3 Piecewise Iterative Weighted Least Squares . . . 13
4.1.4 Baseline . . . 14
4.2 Kernel Density Estimation . . . 15

5 Time Delay Estimation **16**
5.1 Area Method . . . 17
5.2 Probability Density Function Method . . . 17

6 System **18**
6.1 Design . . . 19
6.2 Parameter Files . . . 21
6.2.1 System File . . . 21
6.2.2 Experiment File . . . 21
6.3 Libraries . . . 22
6.3.1 Parameter List . . . 22
6.3.2 Mathematics . . . 23
6.3.3 Input/Output Utilities . . . 23
6.3.4 General Utilities . . . 24
6.4 External Libraries and Tools . . . 24
6.5 Interface . . . 24
6.6 Function and Stream Generators . . . 25
6.7 Function Estimators . . . 26
6.8 Time Delay Estimators . . . 27
6.9 Experimenter . . . 28
6.10 Error Checking . . . 29

7 Development **29**
- 7.1 Development Process . . . 29
- 7.2 Development Methodologies . . . 30
- 7.3 Testing . . . 31
- 7.4 Version Control . . . 31
- 7.5 Project Management . . . 32

8 Experimentation **32**
- 8.1 Sine Functions . . . 34
  - 8.1.1 Preliminary Experiments . . . 34
  - 8.1.2 Refined Experiments . . . 35
- 8.2 Random Functions . . . 36
  - 8.2.1 Preliminary Experiments . . . 36
  - 8.2.2 Refined Experiments . . . 40

9 Conclusion **40**
- 9.1 Improvements and Future Work . . . 41
- 9.2 Individual Comments . . . 43

Appendix A Usage **46**
- A.1 Installation . . . 46
- A.2 General Usage . . . 46
  - A.2.1 Parameter files . . . 47
  - A.2.2 Generating Functions . . . 47
  - A.2.3 Estimating Functions and Time Delay . . . 48
  - A.2.4 Plotting Output . . . 48
- A.3 Running Experiments . . . 48
  - A.3.1 Creating Functions for Experimentation . . . 48
  - A.3.2 Generating Model Selection Data . . . 49
  - A.3.3 Experiment Parameter Setup . . . 49
  - A.3.4 Running Model Selection . . . 50
  - A.3.5 Time Delay Calculation . . . 50
  - A.3.6 Extracting Result Data . . . 51
  - A.3.7 Processing Result Data . . . 51
- A.4 CD Directory structure . . . 52

Appendix B Experimental Data **53**

1 Introduction

The continued advance of storage and sensing technology means that we are able to gather and store more data than ever. Gathering data is relatively simple in comparison to processing it, and with the volume of data available, choosing which data gets processed is becoming increasingly important. The Large Hadron Collider at CERN, for example, throws away tens of gigabytes of data each second, keeping only the most interesting data from the hundreds millions of collisions that occur [26].

The volume of data produced by modern telescopes, while not on the same scale as the LHC, is nonetheless overwhelming. Image sizes of one to two gigabytes are not uncommon, and deciding what data is actually relevant is not a trivial task [31]. Using intelligent filtering algorithms, it should be possible to flag up interesting-looking data for further study. While there are many areas in which such capabilities would be useful, we are interested in creating the base of a system for finding potential gravitationally lensed objects—stars or galaxies whose light is bent because of high mass stellar objects in its path, creating multiple images of the object. To find such objects, we need to study the rate of arrival of photons from light sources in the sky, and try to find two or more of these photon *streams* having the same characteristic function, which dictates how the arrival rate varies over time. Due to lensing effects, the streams from different images of the same object will have some time delay Δ between them.

Much work has been done in the astrophysics community to find more accurate estimates for the time delay of many lensed objects. An estimate of the possible range of time delays for the first observed lensed system, the quasar 0967+561A, was given by Dyer and Roeder in 1980, a year after its discovery, with a value of between 0.03 and 1.7 years [10]. Since then, there have been many estimates of the time delay using various methods such as χ^2 fitting and dispersion spectra, but the generally accepted value, determined by Kundić et al. in 1997 is 417 ± 3 days [20]. In this paper, a measurement of the global value of Hubble's constant is also made, indicating one useful application of the time delay. Increased accuracy of delay estimates increase the accuracy of the Hubble constant estimate. In addition, the time delay can be used to observe the distribution of dark matter in regions of space [30], and has also been proposed as a distance estimator [4]. For strongly lensed systems such as the above, time delays are generally large, and so daily flux measurements are sufficient to get a good idea of the characteristic function. In this project, we deal with time delays which are much shorter, on the order of minutes or hours, for which daily measurements do not provide enough data. For this reason, we consider the arrival times of individual photons.

This report details the development of a system to estimate the time delay between two photon streams, which implements two methods of function estimation, and two methods of time delay estimation. The first function estimator extends a method introduced by Massey et al. [23], performing piecewise continuous function estimates. The second function estimator is a simplified implementation of the kernel density estimator presented by Cuevas-Tello et al. [7]. We use two time delay estimation methods, one based on probability densities, and the other on the area between functions. We intend to perform a comparison of the efficacy of the various estimators implemented. One additional aim of the project is to create a system which might be useful in a larger system for automatically locating potential gravitationally lensed objects.

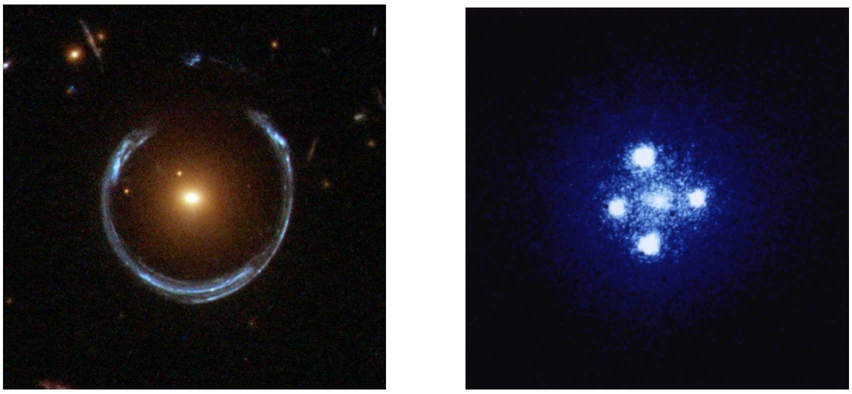
In section 2 we discuss the concepts underpinning the project in more detail. In section 3 we introduce our method of generating random functions from Gaussians, and photon streams using Poisson processes. Section 4 shows our approach to estimating the underlying function of a given stream of photons. Our methods of calculating the time delays between multiple photon streams are explained in section 5. Sections 6 and 7 give detailed information on the design and development of the system, including the software and project management aspects. Finally, in section 8 we present experimental data from both simulated and real data and discuss the relative effectiveness of our methods. We conclude in section 9 with a summary of the achievements of the project, suggestions for further work and potential improvements, and some individual comments.

2 Background

2.1 Gravitational Lensing

In an eight-year period starting in 1907 and ending in 1915 with the publication of a paper on the field equations of gravitation [11], Albert Einstein wrote many papers developing a new theory of gravitation, his general theory of relativity. This generalisation of special relativity and Newton's law of universal gravitation led to a revolution in physics, and remains one of the most important scientific advances to date. The theory describes how spacetime is affected by the presence of matter. The idea has many important consequences, but one is of particular relevance in the context of this report.

According to the theory, objects with mass, or *massive* objects, cause spacetime, a mathematical model which combines space and time into a single "object", to curve around them. A simple way to visualise this effect is to imagine dropping a ball onto a sheet of cloth which has been pulled taut. The ball will eventually come to a stop in the centre of the cloth, and cause it to sag. The sheet represents spacetime, and the ball represents anything from planets, to stars, or even entire galaxies. Depending on the weight of the ball, the shape of the cloth will be affected to different degrees—a ping pong ball will have hardly any effect at all, but if we drop a bowling ball onto the sheet, the effect will be very apparent. In a similar way, the amount that spacetime curves around an object depends on its mass. Objects with high mass curve the sheet a lot, and objects with low mass only a little. If a second ball, lighter than the first, is introduced to the system, what happens? With no initial velocity, it will roll in a straight line towards the first ball sitting at the centre of the sheet. This is one way of thinking about gravity and its relationship with spacetime—an object's gravitational attraction is a result of its mass curving spacetime, and the strength of the attraction is proportional to the mass. While objects with no mass, such as photons, cannot be affected by gravity directly, they *are* affected by the curvature of spacetime. This bending of light rays is known as *gravitational lensing*. In our example, lensing can be imagined as the change in the trajectory of a ball which is pushed at an angle towards a ball sitting in the centre of the cloth. The first person to study the effects of gravitational



(a) An Einstein ring   (b) Einstein's cross

Figure 1: Two examples of strong lensing effects. a) shows light from a distant blue galaxy being distorted by the central galaxy LRG 3-757 [8]. b) shows four images of a distant quasar being lensed by a foreground galaxy [16].

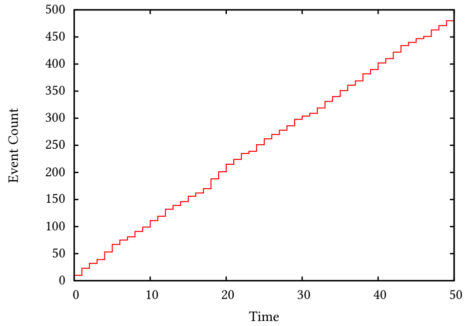
lensing was Orest Chvolson, publishing a short note to *Astronomische Nachrichten* in 1924 [6]. However, the concept was largely unknown until a short calculation by Einstein was published in *Science* in 1936 [12]. Interestingly, Chvolson's note appears directly above a note from Einstein[1], but there appears to be no evidence that Einstein had ever seen it [28]. The first gravitationally lensed object to be identified was the twin quasar SBS 0957+561, in 1979, and since then, over a hundred such objects have been discovered [33, 15]. The effect of gravitational lensing is, as the name suggests, similar to that of a lens, such as that of a camera. Unlike a camera lens, however, gravitational lenses do not have a focal point, but instead a focal line, resulting in images such as that shown in Figure 1a if the source (the object being lensed), the lensing object (the massive object around which the light is being bent) and the observer lie on a straight line. This effect is relatively rare, however, and in general rather than a ring, multiple images of the source can be observed. In these so called *strong* lensing effects, the distortion is very clearly visible. However, two other classes of lensing exist—*weak lensing* and *microlensing*. The effects of weak lensing cannot easily be observed visually, but statistical techniques can show the distortion produced. Microlensing works on even smaller scales than the other two classes, and can be used to detect planets and stars. It has also been proposed as a method to find objects such as black holes and brown dwarfs, which are otherwise difficult to detect [30].

2.2 Poisson Processes

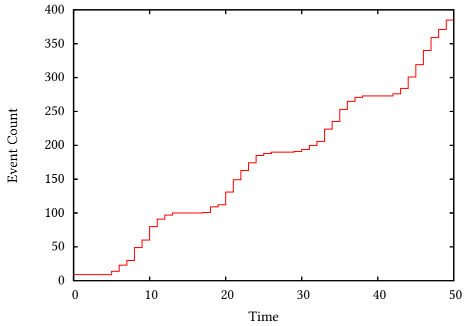
There are many situations in which one can benefit from a good model of the number of events of interest that occur in a given period. Poisson processes are *stochastic processes* that can be used to create such models. A stochastic process is a collection of random variables that represents the evolution of a system over time. Such processes do not evolve in a *deterministic* way. That is, the way they change as time passes is not predictable. Poisson processes are part of a family of stochastic processes which count the number of events and their time of occurrence in a given interval. They have been used to model the number of incoming requests to a server [2], as well as many other time-related systems. In their basic form, Poisson processes have the following important properties [29]:

1. $N(0) = 0$.
    - $N(t)$ represents the total number of events that occurred up until time t. Thus, if $N(0) = 0$, it follows that the process begins at $t = 0$.
2. The numbers of events occurring in disjoint time intervals are independent.
    - The *independent increment* assumption. This states that $N(t)$, the number of events that occur up to time t is *independent* of the number $N(t + s) - N(t)$, i.e. the number of events in the time interval between t and s. In other words, the number of events that occur in one interval does not have any effect on the number of events in any other time interval with which it does not overlap.
3. The probability distribution of the number of events that occur in a given interval is dependent only on the length of the interval.
    - The *stationary increment* assumption. The implication of this is that the probability distribution of $N(t + s) - N(t)$ is the same for all values of t. That is, the likelihood of n events occurring in the above time interval does not change, regardless of the value of t.
4. No counted occurrences are simultaneous.
    - For all events that occur in the duration of the process, no two events will occur at the same time.

Each Poisson processes is governed by a *rate parameter*, λ, which represents the number of events that occur in each time interval, on average. As we are counting events, it is clear that the rate parameter can never go below zero—there cannot be a negative number of events in a given time interval. There are two



(a) Homogeneous



(b) Non-homogeneous

Figure 2: Two examples of Poisson process paths. The homogeneous process has a rate parameter $\lambda = 10$, and the non-homogeneous rate parameter varies depending on the value of a sine function.

types of Poisson processes, *homogeneous* and *non-homogeneous*. In a homogeneous Poisson process (HPP), the rate parameter is constant for the entirety of the process. This means that in every interval, the same number of events are likely to occur. In contrast, a non-homogeneous Poisson process (NHPP) has a rate parameter which varies. This means that the rate at which events occur varies as the process evolves. As such, the value of λ becomes a function of time, written as $\lambda(t)$, called the *rate function*. As a result, the stationary increment assumption does not apply to NHPPs. Figure 2 shows some examples of how the Poisson process evolves over time.

2.3 Regression

Regression is a statistical technique used to fit lines or curves to data points in order to find some sort of relationship between them. The number of variables in the data is important. One of the variables is called a *dependent* variable. We want to find the relationship between this variable and the other variables, called *independent* variables. Consider the expression $y = f(x)$. If $f(x)$ is some function of the variable x, then we know that the value of y depends on the value of x, and so y is the dependent variable, and x is the independent variable. In linear regression, there can be multiple independent variables, but only a single dependent variable. In order to fit a line to data, a *predictor function* is used. This function predicts the value of the dependent variable, as we can never know its true value as we are only able to use a statistical sample of data from the random variable. Regression is used in fields ranging from epidemiology to economics. An example of its use is finding factors contributing to smoking initiation and cessation [22].

3 Simulation of Photon Streams

The first step in building the system was the development of a photon stream simulator. The ability to simulate photon streams means that the system can be tested on many different stream types, so that we are able to determine where its strengths and weaknesses lie. While many simulation tools are very complex, our system does not require simulation of the source objects or the movement of photons, as we are only interested in their arrival time. A source can be represented by a random variable which indicates its variability with time. Different types of sources will have different types of characteristic functions—the variation in a quasar will be very different to that of an individual star, for example. A NHPP is an ideal way to model this type of system. The function $\lambda(t)$ is the rate function of the process, and the times output by the process will represent the arrival times of the photons. $\lambda(t)$ provides a rate parameter at each time t for the duration of the simulation.

3.1 Function Generation

To generate random functions, we make use of Gaussians. The generation process involves four simple steps:

1. Pick some value Δt which represents the distance between the mean μ of successive Gaussians.

2. Define some value α, where the standard deviation σ of each Gaussian is determined by $\alpha \cdot \Delta t$.

3. For each Gaussian, choose some weight w_i, from a uniform distribution between -1 and 1, and scale it by some multiplier.

4. Using some step s, sum all the Gaussians at each point on the x-axis which we get from these s values.

The first step defines how spread out the Gaussians should be in the interval $[t_0, T]$ in which the function is to be generated. If the spread is large, then depending on the standard deviation of the Gaussians there will be many points in the interval where the value of the rate function is zero. On the other hand, with a low value of Δt, most points on the line should have some non-zero value.

The α parameter determines the standard deviation σ of all the Gaussians used to generate the function. The value of σ is the one that affects the final function the most. Low values will result in each Gaussian covering only a small interval, so if the Gaussians are sufficiently spread out, the variation in the function will be much larger than if higher values of σ are used.

With just the above two steps, the functions generated would be quite similar, because each Gaussian is assigned the same weight. With uniform Gaussians, there would be hills at each point where a Gaussian is centred, and very little to

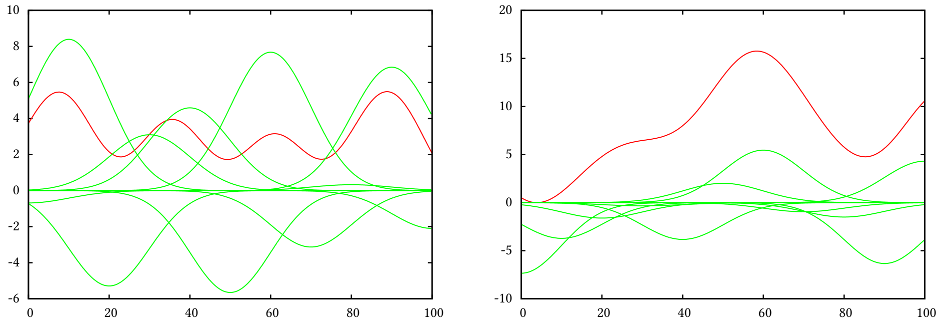


Figure 3: Two examples of functions randomly generated from Gaussians. The red function is constructed by summing the green gaussians. Generated with $\Delta t = 10$, $\alpha = 1$. The functions are shifted so that all points are ≥ 0.

speak of in between, and the height of the function would never exceed a certain value. To introduce more variation, a weight w_i sampled from the uniform distribution $U(-1, 1)$ is applied to each Gaussian. Uniform sampling simply means that any real number between -1 and 1 has an equal probability of being chosen. To further increase the variation in the functions that can be generated, a multiplier can be used, which scales the value of each weight.

The final step is to calculate the values which will make up the function. Starting at the beginning of the interval t_0, we sum the values of all the Gaussians at points along the line until the end of the interval, T, is reached. The points that are sampled are defined by $t_i = t_{i-1} + s$, where s is the sample resolution. The sum of all Gaussians at time t can be calculated by

$$f(t) = \sum_{g \in G} w_g \cdot e^{-(t-\mu_g)^2/2\sigma_g^2} \tag{1}$$

Where G is the set of Gaussians used to construct the function, and w_g, μ_g and σ_g are the weight, mean and standard deviation respectively of the Gaussian g. Figure 3 shows some examples of functions generated from Gaussians in this way. In addition to the random function generation, it may sometimes be useful to generate a function from a known expression, and the system includes this functionality as well, which is described in section 6.6.

3.2 Generating Streams from Functions

Once a function has been generated, we can use it as the rate function $\lambda(t)$ for a NHPP, which we can then use to generate event arrival times by building on the generation of event times from a HPP.

We use inverse transform sampling to generate event times from a HPP. This technique works by generating a uniform random number $U \sim U(0, 1)$, and finding the value on the x-axis at which the cumulative distribution function (CDF) of a probability distribution is equal to the value of U. To do so, it is necessary to invert the CDF. The waiting time to the next event in a Poisson process is an exponential function, which has CDF $1 - e^{\lambda x}$, which we can invert using the log function, giving us the waiting time t to the next event as [19]

$$t = -\frac{1}{\lambda} \log(U) \tag{2}$$

Using this calculation, it is possible to generate a realisation of a HPP for an interval of any finite length. This provides a base which can be extended to generate events from NHPPs.

To generate events from an NHPP with a rate function $\lambda(t)$, we use a technique called thinning, where we generate a large number of values, and then throw some of them away based on some criterion. In the case of the NHPP, we generate events from a HPP with a rate parameter λ, where $\lambda > \lambda(t)$ for $0 <= t <= T$. In other words, the homogeneous lambda value must be larger than the value of the rate function for the whole time the process runs. First, two random values are independently sampled from a uniform distribution, $U_1, U_2 \sim U(0,1)$. The first number, U_1, is used in (2) to find the next event time from the homogeneous process governed by λ. Using the time t generated from that, the value of $\lambda(t)$ is calculated. If $U_2 \leq \frac{\lambda(t)}{\lambda}$, then the event is kept. The closer $\lambda(t)$ is to λ, the more events will be kept, and so the number of events generated depends on the shape of the rate function.

4 Function Estimation

Once we have a photon stream, the next stage is to estimate its characteristic function. This section presents the two methods that we implemented to do this. The first method uses simple linear regression, and the second a kernel density estimator.

4.1 Baseline Estimation

In this section, we present the baseline method for function estimation. The core of the estimator is based on the iterative weighted least squares estimator derived by Massey et.al [23], and in the next two sections we attempt to explain it in simple terms. The subsequent sections detail our additions to the simple estimators in order to form the final baseline estimator.

4.1.1 Ordinary Least Squares

The ordinary least squares (OLS) estimator forms the core of the baseline estimator, estimating functions by minimising the sum of squared residuals. It is important to note the difference between errors and residuals. In statistical terms, an *error* is "The difference between the observed value of an index and its "true" value" [24], and a *residual* is "The difference between the observed value of a response variable and the value predicted by some model of interest" [13]. The true value of the function is unobservable—it is only possible to obtain a statistical sample. The residual, on the other hand, is the difference of the observation from some *estimate* of the function. This first estimator estimates a linear function of the form $y = ax + b$, or a straight line. While this is not directly useful for estimating characteristic functions, it was developed in order to gain a deeper understanding of the ideas behind regression, and we will extend it for our purposes in later sections.

In order to estimate the function, the stream of event times must first be converted into a form which is suitable for processing. To do this, we first pick a time interval $(0, T]$, and divide it into N sub-intervals, or *bins*. According to [23], the kth bin I_k is calculated by

$$I_k = \left(\frac{(k-1)T}{N}, \frac{kT}{N} \right], \ 1 \leq k \leq N \tag{3}$$

and the midpoint x_k of each bin is

$$x_k = \left(k - \frac{1}{2} \right) \frac{T}{N}, \ 1 \leq k \leq N \tag{4}$$

Due to the independent increments property of Poisson processes, splitting the interval leaves us with N bins, each of which is defined by an independent Poisson random variable [23] Y_k with mean

$$\lambda_k = \frac{T}{N}(a + bx_k) \tag{5}$$

where T/N is used to normalise the value of λ_k. The value of Y_k in our case is the number of photon arrival times for each bin. In order to perform regression on the data, we need a model, which is used to connect the random variables and the parameters, and describes how they are related. Our model becomes $Y = \alpha + \beta x + \epsilon$, or $Y_k = \alpha + \beta x_k + \epsilon_k$ [23]. The values α and β are the two regression parameters which we use to estimate the values of a and b in the characteristic function. We introduce a Poisson error ϵ which represents the error present in the data that we are trying to model. As mentioned before, this technique works by minimising the sum of squared residuals. The square of a residual can be computed by [18]

$$d_k^2 = (Y_k - [\alpha + \beta x_k])^2 \tag{6}$$

We use the regression parameters α and β rather than the unknowable values a and b, since we calculate residuals in relation to the estimate of the function. We introduce a weight w_k, initialised to 1, for each interval, which compensates for the Poisson error [23]. Introducing this weight into (6) and summing over all bins, we compute a weighted version of the residual sum of squares (RSS). We want to find the values of α and β for which the RSS is minimised.

$$\arg\min_{\alpha,\beta} \sum_{k=1}^{N} w_k (Y_k - [\alpha + \beta x_k])^2 \tag{7}$$

This is known as the *objective function.* Once the objective function is known, we can define estimators $\hat{\alpha}$ and $\hat{\beta}$, which we will use to estimate values of α and β to find the minimum [23].

$$\hat{\beta} = \frac{\sum_{k=1}^{N} w_k (x_k - \bar{x})(Y_k - \bar{Y})}{\sum_{k=1}^{N} w_k (x_k - \bar{x})^2} = \frac{\sum_{k=1}^{N} w_k (x_k - \bar{x}) Y_k}{\sum_{k=1}^{N} w_k (x_k - \bar{x})^2} \tag{8}$$

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x} \tag{9}$$

$$\text{where} \quad \bar{x} = \frac{1}{N}\sum_{k=1}^{N} w_k x_k \quad \text{and} \quad \bar{Y} = \frac{1}{N}\sum_{k=1}^{N} w_k Y_k \tag{10}$$

If we ignore the effect of adding the weight w_k for the time being, the calculation of $\hat{\beta}$ is equivalent to dividing the covariance of x with Y by the variance of x [18]. The covariance is a measure of the strength of the correlation between two or more random variables [34]. If high values of one variable occur when the other variable also has high values, then the covariance is positive. If high values of one variable occur when the other has low values, then it is negative. The variance, on the other hand, is a measure of the variation in values of a random variable. If all values are close to the mean, then the variance is small, and if there are large deviations from the mean value, then the variance is large. If the covariance is positive, then the values of Y increase as x increases. The variance of x depends only on the length of the interval—short intervals have low variance, and long intervals high variance. This is because the calculation of the variance is done by finding the distance to the midpoints of bins from the value of $\bar{x}$, which is the midpoint of the interval.

It is clear that the sign of $\hat{\beta}$ depends on whether the covariance is positive or negative, and this in turn defines the sign of the gradient. The steepness of the gradient is defined by the magnitude of the covariance. Since the value of the variance is constant, the larger the magnitude of the covariance, the steeper the gradient. Once we know the gradient of the line, the calculation of the intercept is simple, so long as we know the value of a point on the line. The point $(\bar{x}, \bar{Y})$ is one such point. We rearrange the equation $\bar{Y} = \hat{\alpha} + \hat{\beta}\bar{x}$ to solve for $\hat{\alpha}$. Notice that since the values of $\bar{x}$ and $\bar{Y}$ do not change, the line estimate pivots around the point defined by the mean values. The addition of the weights adds bias into the calculation, taking into consideration the variation of those bins which have a smaller estimated value of λ. The weight update calculation is discussed in the next section.

We normalise the values of $\hat{\alpha}$ and $\hat{\beta}$ by multiplying the resulting estimate by the number of bins over the interval length. The fewer bins used in the estimate, the larger the bin count will be for each bin, and consequently the larger the estimated values will be. To return the estimate to the correct scale, we set

$$\hat{a} = \frac{N}{T}\hat{\alpha} \quad \text{and} \quad \hat{b} = \frac{N}{T}\hat{\beta} \tag{11}$$

As we are dealing with a Poisson process with a rate function, it is natural to impose a constraint on the values of $\hat{a}$ and $\hat{b}$ which states that the rate function must be non-negative throughout the entire interval $[0, T]$, since it is not possible to have a negative rate [23].

$$\hat{a} \geq 0 \quad \text{and} \quad \hat{b} \geq -\hat{a}/T \tag{12}$$

There are two cases in which this constraint can be violated; when $a < 0$ or $b < -\hat{a}/T$ [23]. In the first case, we set

$$\begin{aligned} \hat{a} &= 0 \\ \hat{b} &= \frac{N}{T}\frac{\sum_{k=1}^{N} w_k x_k Y_k}{\sum_{k=1}^{N} w_k x_k^2} \end{aligned} \tag{13}$$

and in the second,

$$\begin{aligned} \hat{a} &= -\hat{b}T \\ \hat{b} &= -\frac{N}{T}\frac{\sum_{k=1}^{N} (T - x_k) Y_k}{\sum_{k=1}^{N} w_k (T - x_k)^2} \end{aligned} \tag{14}$$

Applying these adjustments in the cases in which the constraints are violated ensures that the final estimate is always within the required constraints. However, the adjustments introduce some non-linearity into the system [23]. With this set of equations, the fundamental structure of the OLS estimator is complete. In the next section, we discuss the addition of weight update rules and finding estimates of λ.

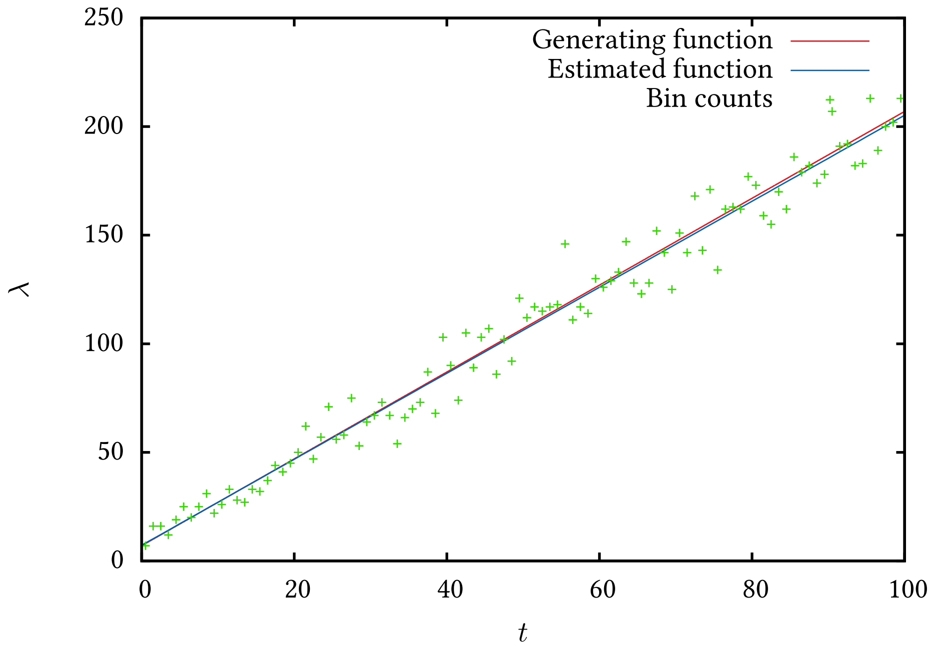


Figure 4: IWLS estimate of a photon stream with the characteristic function $\lambda(t) = a + bt$ where $a = 7$ and $b = 2$.

4.1.2 Iterative Weighted Least Squares

The iterative weighted least squares (IWLS) estimator builds upon the OLS estimator. As the name suggests, the extension introduces an iterative part. The OLS estimator performs a single estimate of the function, whereas IWLS estimator repeats the process multiple times, updating its estimates each time.

Perhaps the most important update to the estimator is the use of unequal weights, which change depending on the variances of the random variable which defines the bin which the weight is being applied to. A Poisson random variable has a variance that is equal to its mean—this means that a higher value of λ results in a larger variance. To compensate for this, we give higher weights to bins which have lower values of λ, as the variances will be lower. As shown in equation (5), the value of λ is easy to calculate, but the values of a and b must be known. In order to modify weights appropriately, we must be able to obtain estimates of λ, which can be done using [23]

$$\hat{\lambda}_k = \frac{T}{N}(\hat{a} + \hat{b}x_k) \tag{15}$$

The weights can then be updated by

$$\hat{w}_k = \frac{\dfrac{N}{\hat{\lambda}_k}}{\displaystyle\sum_{k-1}^{N} \left(\frac{1}{\hat{\lambda}_k}\right)} \tag{16}$$

Each iteration of the estimator updates these estimates of λ and the weight for each bin, and the process is stopped when the change in the estimates becomes negligible, which consistently happens in between two and five iterations [23]. Figure 4 shows an example of a function estimated using IWLS.

While this is an improvement on the OLS estimator, it is not sufficient for our purposes. The characteristic functions of stellar objects are not straight line functions, so we must extend this approach to give us some reasonable estimates of functions which are not straight lines.

4.1.3 Piecewise Iterative Weighted Least Squares

It is clear that the IWLS estimator alone is not sufficient to complete our task. In order to obtain a reasonable estimate of the characteristic function, we need to be able to estimate a function which is not a straight line. During the development process, we considered the possibility of approximating functions by multiple straight-line estimates, and this estimator is the result. This type of function is known as a piecewise linear function. Extending the approach presented in the previous two sections, we take the interval $[0, T]$, and split it into several sub-intervals. Then, the function underlying each of these sub-intervals is estimated using IWLS. We also add some minor extensions in an attempt to improve the quality of the estimates.

Sub-intervals are estimated starting from the first, and moving to the next once the process is complete. When the estimate is completed, a short interval after the sub-interval being estimated is checked to see how well the estimate for the previous sub-interval matches it. The extension interval is split into several bins. Using a probability density function (PDF), we evaluate the likelihood of obtaining the count Y_k for each bin given the estimate λ at that point. The PDF for a Poisson distribution is calculated by

$$P(Y_k = x) = \frac{\lambda^x e^{-\lambda}}{x!} \tag{17}$$

For each bin, $P(Y_k = x)$ must exceed some threshold. A lower threshold means that lines are less likely to be successfully extended. While this technique is an improvement on using straight lines to estimate functions which are curves, it is still not sufficient, as the resulting function estimate is piecewise disjoint—the estimate for each interval does not connect smoothly into the next, but jumps at the boundary between each sub-interval.

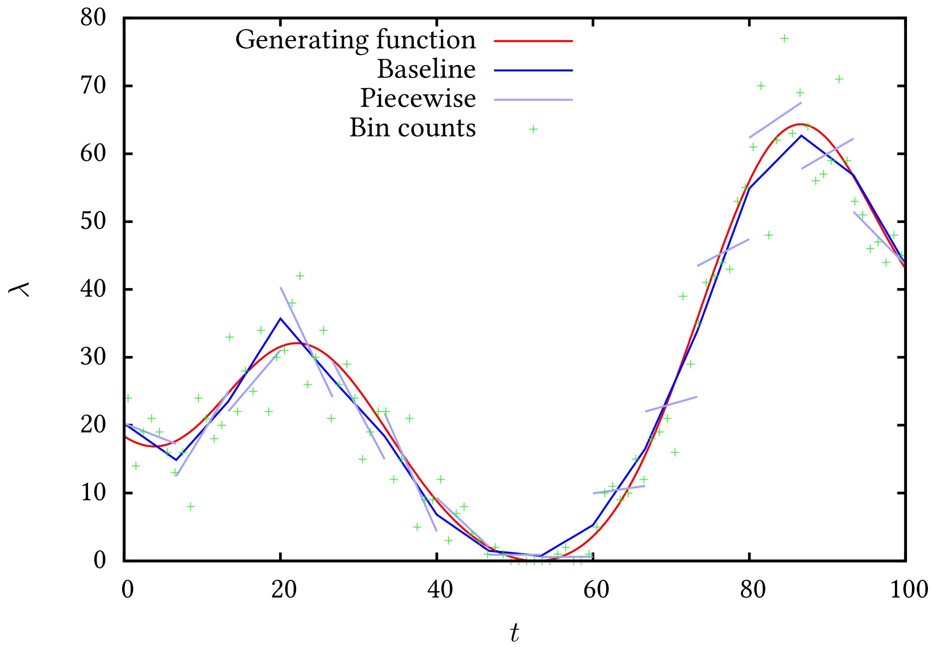


Figure 5: A comparison of the baseline and piecewise estimates on the same function. Note how the baseline estimate passes through the midpoint of the disjoint piecewise estimates at the breakpoints.

4.1.4 Baseline

As mentioned in the previous section, the piecewise IWLS estimator gives us a piecewise disjoint estimate of the function, but we would like one which is piecewise continuous. In order to do this, the end of each interval estimate must meet the start of the next. The estimate returned by the piecewise estimator has several breakpoints—points where the start of one sub-interval and the end of another meet. If there are L lines that make up the estimate, there will be $R = L - 1$ breakpoints. At each of these breakpoints r, we calculate the value of the previous and subsequent function estimates f, and find their midpoint m with

$$m_i = \frac{f_i(r_i) + f_{i+1}(r_i)}{2}, \quad 0 \leq i < R \tag{18}$$

The value of m is calculated for each breakpoint. Midpoints are not calculated at time 0 and time T. Instead, the function values at those points are used. Each sub-interval is now represented by a point p at the start and q at the end, each with an x and y coordinate. With these points, we can recalculate each sub-interval estimate f of the form $y = \hat{a} + \hat{b}x$ by replacing y with p_y and x with p_x, and recalculating the gradient $\hat{b}$ and intercept $\hat{a}$ with

$$\hat{b} = \frac{q_y - p_y}{q_x - p_x} \tag{19}$$

$$\hat{a} = p_y - \hat{b} \cdot p_x \tag{20}$$

In this way, each sub-interval estimate links points p and q, giving us a piecewise continuous function estimate, and this step completes the first function estimation method. Figure 5 shows an example of a piecewise and baseline estimate.

4.2 Kernel Density Estimation

The second function estimation method implemented was a kernel density estimator, which use *kernels* to estimate the probability density of a random variable. A kernel is simply a weighting function, which affects how much a given sample is considered when constructing the function estimate. Since the photon stream data is assumed to be generated by a source whose variability is defined by some random variable, the event times are a sample drawn from the PDF of that variable. We use a Gaussian kernel

$$K(t, \mu) = e^{-(t-\mu)^2/2\sigma^2} \tag{21}$$

to estimate the PDF, centring a kernel at each photon arrival time a by setting $\mu = a$. The width of the kernel depends on some fixed value σ. We perform a Gauss transform on the N kernels, finding the contribution of all the kernels at M points in time, from which we get an estimate $\hat{\lambda}(t)$ of the characteristic function.

$$\hat{\lambda}(t_i) = \sum_{j=1}^{N} K(t_i, \mu_j), \quad i = 1, \ldots, M \tag{22}$$

Using a larger M gives a higher resolution. Depending on the value of σ used, $\hat{\lambda}(t)$ will be some multiple of the actual function $\lambda(t)$. Thus, the final step is to normalise $\hat{\lambda}(t)$. We split the stream data into B bins with midpoints b and calculate the bin count x for each. We start with the normalisation constant η at a low value, and gradually increase it to some threshold, finding

$$\sum_{i=1}^{B} \log\left(\frac{\phi^x e^{-\phi}}{x!}\right), \quad \phi = \eta \cdot \hat{\lambda}(b_i) \tag{23}$$

for each value of η. The value of η which maximises this sum of log Poisson PDFs is used to normalise $\hat{\lambda}(t)$ in subsequent computations. Figure 6 shows an example of a kernel density estimate, and displays a weakness in the estimator. As one moves towards the start or end of the interval, fewer Gaussians make a noticeable contribution to the function calculation, resulting in a drop-off of the estimate.

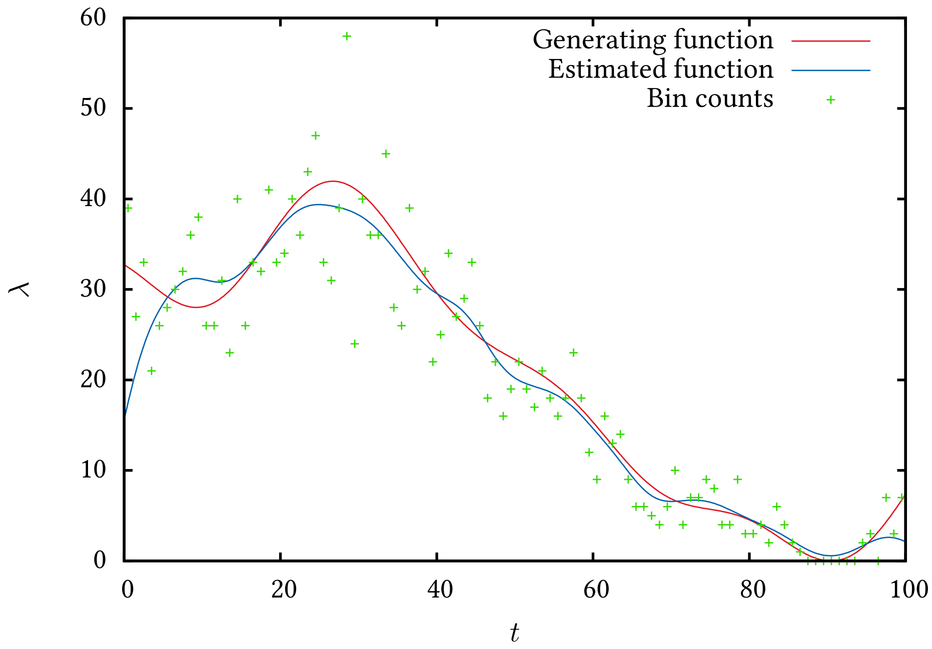


Figure 6: Estimate of a function using Gaussian kernels. The drop-off at the start and end of the interval is due to fewer Gaussians summed in those areas as no kernels are placed outside the interval.

5 Time Delay Estimation

Once we are able to estimate the characteristic function of photon streams, we can use these estimates to compute an estimate of the time delay between two streams. If the two streams come from the same source, then they should have the same characteristic function, but delayed by some value Δ. Our estimates of the characteristic function will differ for both streams due to the fact that the number of photon arrivals in each bin will be different for each stream, but each should look relatively similar. In this section we present two methods for estimating the time delay between a pair of streams based on their function estimates.

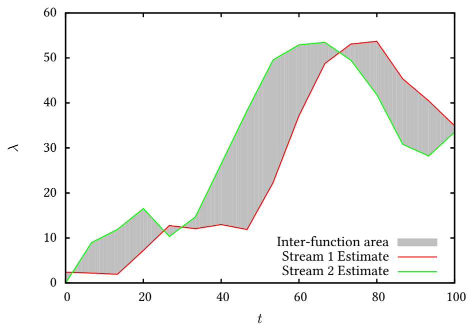
Both of the estimators work by starting Δ at $-\Delta_{\max}$, and increment it by some step until reach $+\Delta_{\max}$ is reached, using a metric to evaluate how good the estimate is with that value. It is important to note that the value of $\Delta_{\max}$ defines the interval in which the metric is computed. The need for calculation only in some specific interval should be clear—if one function is shifted by Δ, and both functions have the same time interval, then there will be an interval of length Δ at either end of the range in which only one of the function estimates has values. As such, the metric can only be computed in the overlapping area. Varying Δ changes the overlapping interval. Setting $\Delta = 0$ minimises the value, and $\Delta = \pm\Delta_{\max}$ maximises it. Performing calculations on different interval lengths would require the value of the metric for longer intervals to be scaled to that of the shortest. To make useful comparisons, we must perform calculations only on the interval in which the two functions overlap for all values of Δ. Imposing this constraint means that the value of Δ_{max} can never exceed the interval length T in which we are performing the estimate. We are left with the constraints $T_{\text{est}} = [t_0 + \Delta_{\text{max}}, T - \Delta_{\text{max}}]$, $\Delta_{\text{max}} < T$ on the interval and the maximum value of Δ.

5.1 Area Method

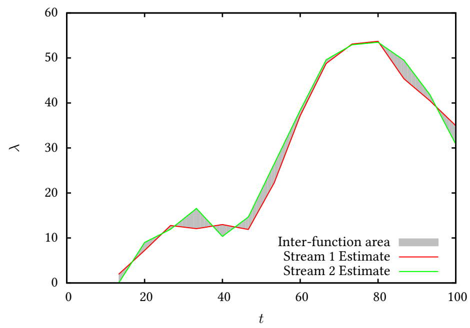
The first of the two methods uses a very simple metric to estimate the time delay. By taking the two function estimates, we can attempt to match up the two functions so that they "fit together" best. The goodness of fit can be determined by the area between the two functions $\hat{\lambda}_1$ and $\hat{\lambda}_2$, calculated by

$$\begin{aligned} d(\hat{\lambda}_1, \hat{\lambda}_2) &= \int (\hat{\lambda}_1(t) - \hat{\lambda}_2(t+\Delta))^2 \, dt \\ &\approx \frac{1}{N} \sum_{i=1}^{N} (\hat{\lambda}_1(t) - \hat{\lambda}_2(t+\Delta))^2 \end{aligned} \tag{24}$$

for each value of Δ. Our estimate of Δ is set to the value at which $d(\hat{\lambda}_1, \hat{\lambda}_2)$ is minimised. Rather than using an integral to get the exact area between the functions, we use a less computationally expensive discrete approximation. Figure 7 shows a visual representation of the logic behind the method.



(a) High area ($\Delta = 0$)



(b) Low area ($\Delta = 13.3$)

Figure 7: Comparison of area between functions for two different values of Δ. The first 13.3 time units must be ignored in (b). The value of Δ in (b) clearly results in a closer match and lower area between the two functions than the value in (a).

5.2 Probability Density Function Method

The second method of estimation is using probability density functions. As before, we guess a value of Δ between $-\Delta_{\text{max}}$ and $+\Delta_{\text{max}}$ and shift $\hat{\lambda}_2$ by that amount.

However, we know that there must be a single characteristic function, and we want to see how well our estimate of that matches the bin counts in each stream. We make an "average" function $\bar{\lambda}$ by combining the two function estimates we have, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (which is shifted by Δ).

$$\bar{\lambda}(t) = \frac{\hat{\lambda}_1(t) + \hat{\lambda}_2(t + \Delta)}{2} \tag{25}$$

The point on $\bar{\lambda}$ at time t is the midpoint between the values of the two estimates at that time. Once we have $\bar{\lambda}$, we can assign some score to the current estimate of the value of Δ.

$$\begin{aligned} \log P(S_A, S_B \mid \bar{\lambda}(t)) = \sum_{t=\Delta_{\max}}^{T-\Delta_{\max}} & \log P(S_A(t) \mid \bar{\lambda}(t)) \\ & + \log P(S_B(t+\Delta) \mid \bar{\lambda}(t)) \end{aligned} \tag{26}$$

Here, we calculate the probability that the function $\bar{\lambda}$ is the characteristic function of the two streams S_A and S_B. The streams are split into bins, and the log probability of the number of events in each bin given the value of λ calculated for that bin is computed and summed over all bins, as in Equation (23).

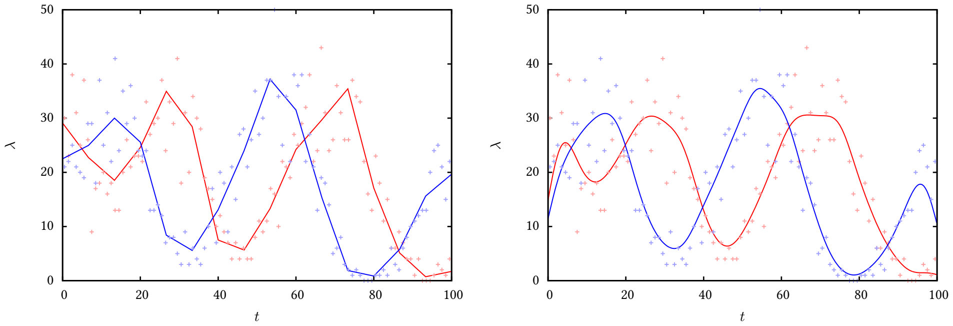
The calculation of λ is slightly more complicated than just taking its value at the midpoint of each bin. Since we are considering a number of events occurring in a given interval, we must consider the value of λ for the same interval. In order to do this, we use a discrete approximation of integrating $\lambda(t)$ over the interval.

$$\lambda_{a,b} = \int_a^b \lambda(t)\, dt \tag{27}$$

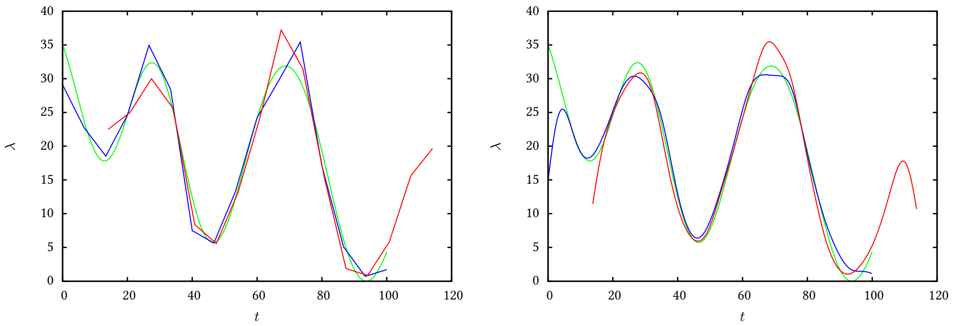
In the approximation t is incremented by some finite step for each successive value. The smaller the value of the step the more accurate the approximation of $\lambda_{a,b}$ becomes. As with the previous estimator, the estimate is made in two stages, first with a coarse pass over the values of delta to compute an initial estimate, and then a finer second pass around the first estimated value in order to refine the estimate. Figure 8 illustrates the whole estimation process.

6 System

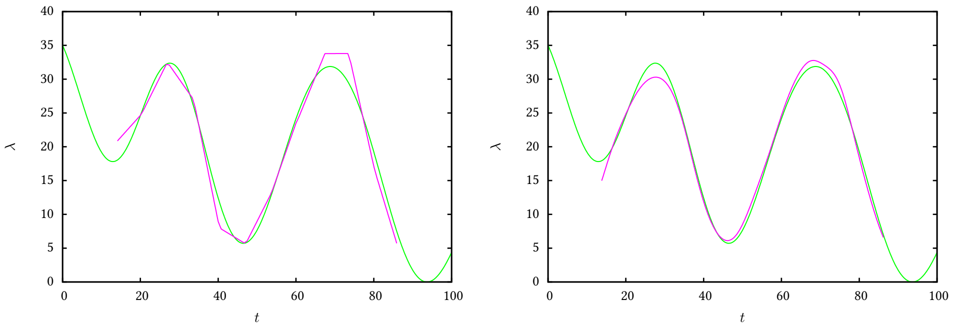
In this section we provide an overview of the system, and explain some of the details behind the implementation of the system. We also give some idea about the design decisions used in the implementation. Discussion of the programming methodologies and ideas used can be found in the 7 section. The system is very large (over 7000 lines of C code), and we therefore attempt to detail the key ideas behind each part of the implementation rather than an in depth discussion of the techniques. Each subsystem described in sections 3–5 also has its own section describing some of the important parts of its implementation.



(a) Estimate functions based on bin data



(b) Shift second (blue) function based on Δ estimate



(c) Combine functions to create final estimate

Figure 8: Illustration of the estimation process. Left column shows baseline method, right column Gaussian. The estimated value of Δ was 14.1 and 13.8 for baseline and Gaussian respectively, found using the PDF method. Actual value was 15. Points in (a) represent bin counts for the function of the same colour. The green line in (b) and (c) indicates the actual function, magenta is the final estimate.

6.1 Design

When designing the system, we made the decision to split the three main pieces of required functionality into two groups. The generation of streams and functions would make up one subsystem, and the function and time delay estimation would make up another. This is a logical way in which to divide the system, as they are linked only by the dependence of the estimators on data from the generators. It is not strictly necessary for the data to come from inside the system—as long as it has the structure required by the estimators it can be used. We use a single executable to launch both of the subsystems. Figure 9 gives an overview of the structure of the system.

As with any large program, there will inevitably be some code which has to be used in different places in the program. To make checking the correctness of the system and its modification easier, functions that are called more than once are put into libraries which are shared between all subsystems.

The input and output of the system is another important thing that must be considered. The system should be able to read data which follows some sort of structure. The structure should be simple, so that minimal effort is required to convert data into a form which the system can process. Input to the system is from simple text files, which are easy to construct, and easy to read in. Output from the system, both in terms of output to the interface, and also output files, also need to have some meaningful structure, and the results of calculations should be clear. Output files should not contain any unneeded information. The system can be made to output more detailed data, or not output anything at all, with the use of output flags. Output filenames are highly structured, which make reading in data much easier, and is of particular importance when doing experiments.

Once parameters are set, user interaction with the program is minimal. The output of the system is all numerical data. Textual output is simple to display, and there are many utility programs that can parse data files to draw graphs. As such, we decided not to use a command line interface over a graphical one. The development of a graphical interface is time consuming, and requires a lot of thought to be put into design. On the other hand, interaction with the command line is simply a question of reading text responses or parsing command line options. A graphical interface for the system would provide little benefit to the user in terms of additional information, as the system is a tool to use for data processing, not something that requires constant interaction with the user. Most scientists interact regularly with computers, and astronomers in particular regularly use data processing programs. As our intended user base is likely to have experience with command line interfaces, we feel that the lack of a graphical interface does not reflect negatively on the system.

In order to test the various methods developed, there has to be a way of running controlled experiments on the system. For this purpose, an experiment system which is a wrapper around the estimators forms the final subsystem. With it, multiple calls to the estimators can be made with different configuration parameters.

In addition to the core of the system, scripts are provided which can be used to plot the output data, and run more complex sets of experiments. Usage instructions can be found in Appendix A.3 and A.2.4.

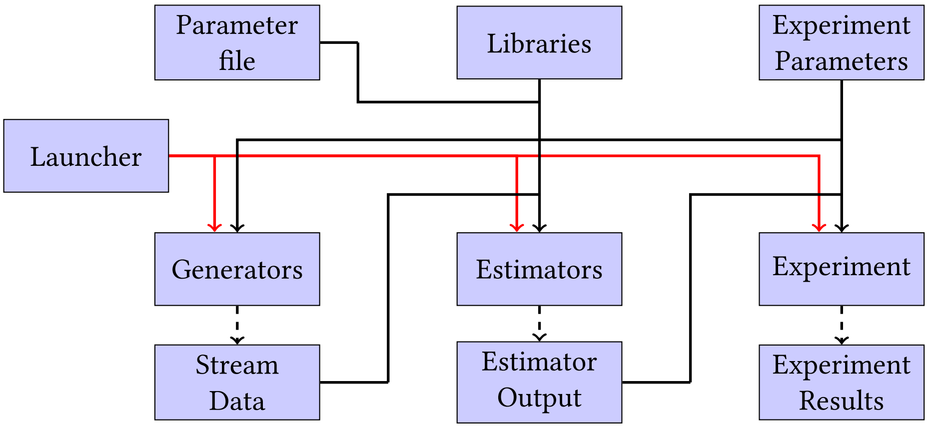


Figure 9: A simplified overview of the system structure. Solid lines indicate the dependencies of a given subsystem, and Dashed lines indicate output from a subsystem. The red lines indicate what the user can access through the launcher.

6.2 Parameter Files

The parameter files are used to configure the values of all parameters which affect the behaviour of the system. Separate files are used to configure the estimators and generators, and the experimenter. The files use a simple syntax. The # symbol defines a comment. A parameter is defined as an string of ASCII characters followed by a single space, followed by more ASCII characters. Each file is split into several sections, to aid the user in finding the parameters they are looking for. All parameters have comments describing their effect on the behaviour of the system, what values they can take, and other information relevant to the user. Functionality for generating parameter files with default settings are provided.

6.2.1 System File

This parameter file is the one which controls the behaviour of the estimators and generators, and is required for almost all operations. It facilitates the definition of output filenames, generation parameters for the stream generator, including the interval length, start time, and the expression used to generate the streams. The random function generator can be set up to change the multiplier applied to the Gaussians, change their resolution, and define how the standard deviation is set. The configuration of all the parameters used by estimators, both function and time delay, is also done here. The sections describing the implementation of parts of the system explain the exact parameters used and how they affect the behaviour.

6.2.2 Experiment File

A separate parameter file is used by the experimenter to prevent parameter duplication and allow greater flexibility with experiments. It contains parameters which affects the naming of output files, and allows the configuration of the intervals in which data is withheld in model selection. The most important parameters are those which define the names and parameters to test during the experiments.

6.3 Libraries

The system makes extensive use of custom libraries. Each library consists of a header file which contains the function prototypes and include information, along with a separate file for the functions, which are compiled by `libtool` into a convenience library. The advantage of using `libtool` over other ways of constructing libraries is that it can create both shared and static libraries. This means that if the library needs to be re-used elsewhere it is simple to take the shared object file created and compile the program including the library by passing the standard `-l[libname]` syntax to `gcc`. Due to some interdependencies between the lower level convenience libraries, they are merged into one main library, again functionality provided by `libtool`. The main purpose of the libraries is to provide a single place where oft-used functions can be defined once and used by all parts of the system.

6.3.1 Parameter List

The parameter list library defines a singly-linked list, used to store data parsed from the parameter files. These lists are required by many functions in the system to set their behaviour. The library provides functions for adding elements to the list and finding its length. A function for removal of elements is not provided, as there is no situation which should necessitate the removal of elements from the list. There is also functionality for checking whether a parameter with a given name exists, retrieving the value of a parameter, and setting the value of a parameter.

There are multiple retrieval functions, each of which retrieves values of different types. The parameter list is constructed in such a way that all values in it are stored as character arrays. This means that if a parameter value is required by some function, it must be converted into the type which that function requires. Since it is known inside the function which type is required, the relevant function can be called. Functions to read `double` and `int` types are provided, along with a function to retrieve the character array. In addition, some of the parameters in the files are comma-separated lists of integers or doubles, which must be parsed into arrays before they can be used. In order to reduce code duplication, the conversion of variables to the correct type is done inside the retrieval function.

Parameters are only parsed when they are required by a function. This reduces the complexity of the logic, as it is not necessary to deduce the type of the parameter—the function knows what type it requires. It also reduces the complexity of the data structure, as only character arrays need to be stored. In addition, some parameters are not required by some subsystems, so parsing every parameter in the file is unnecessary.

6.3.2 Mathematics

As the name implies, the mathematics library provides the mathematical functions required by the system which are not provided by the standard C library. Some of the library functions are based on functionality provided by the GNU Scientific Library [14], particularly those which calculate probability density functions or require random number generators. The most important part of the library is the functionality it provides for computations with Gaussians, in particular the discrete Gaussian transform. It also provides some basic functions, such as finding the minimum and maximum values in arrays, averaging, summing, adding to or multiplying arrays, and some implementations of statistical functions such as the root mean square error, standard deviation and the like.

The most challenging part of the implementation of the library was to get around the issues caused by double precision values. Functions which deal with calculations based on timings require a certain precision on the start and end times of intervals to work correctly. Due to the nature of their implementation, calculations with doubles often result in numbers which are only very close to the actual value. Particular problems were encountered when incrementing a value by a floating point number and comparing it to another. The floating point increments caused the value to be slightly (on the order of 1.0×10^{-20}) below the actual value, and this caused calculations to be incorrect and resulted in a cascade of erroneous calculations. To deal with this problem, functions for comparing doubles to a specific precision were implemented.

6.3.3 Input/Output Utilities

This library implements functionality for reading from and outputting to files, as well as for checking the state of files and directories on the system. It is also used to parse the parameter file into the system, and as such defines the syntax that the parameter file must follow. We were unable to find a library which provided similar functionality to the Java Properties class, which allows the structured reading and storage of parameters, and so implemented a simplified version in the form of the parameter files. This library also reads in event data files, which are needed as input to the estimators, and can retrieve either all events, or data in a specific interval.

As well as reading in data, the library also serves to output data from various data structures used within the system. This ranges from simple arrays to more complex data structures used to store representations of Gaussians or function estimates.

6.3.4 General Utilities

The final library is for functions which do not fit in anywhere else, such as memory allocation and freeing, printing structs, and error checking functions. There are also functions for generating default parameter files. This library makes the rest of the system much cleaner, as memory allocation and freeing for large structs can be done with a single function call.

6.4 External Libraries and Tools

The system uses a number of external libraries to augment the C standard libraries, and to reduce the need for us to write code which has already been written elsewhere. The GNU Scientific Library [14] provides the system with a larger variety of random number generators than the standard library provides, and also gives access to probability density function computations. The Check framework [5] is used to implement automated tests for the system, and is part of the GNU build system, which provides assistance for making source code packages portable to many Unix systems. Our system makes use of the `automake` and `libtool` frameworks to generate shared library files and makefiles, and directory structure follows that of the standard GNU package. The MuParser library [3] is used to parse expressions used to generate stream data. The Valgrind framework was used to debug memory errors [32].

6.5 Interface

Users interact with the system via a command line interface. Various flags passed to the executable change the behaviour of the system, but the majority of behaviour is controlled through the parameter file. The standard C libraries provide a useful function, `getopt`, specifically for the parsing of command line options. This function allows the parsing of short options, such as `-g`, or with the `getopt_long` function, longer options such as `--generate` can be parsed. Users familiar with *NIX systems will no doubt recognise such options, as they are used in almost every program which can be run from the command line. The parsing of options is done by the launcher, which is the only part of the system that the user interacts with directly. Each subsystem can be run by passing a specific option, and checks are made to ensure that only a single subsystem is being called. When an error occurs in the parsing of options, which can arise due to an option with a required parameter not having anything passed to it, or as a result of multiple subsystem calls, an error message is printed informing the user of the error.

As with many command line programs, instructions on what options are available, and some information on what they do can be displayed using the `-h` or `--help` options. The help information is also printed when there is some issue when parsing the parameters. To better facilitate the addition and removal of options, the value of each option is stored as a flag in a struct which is used to determine which subsystem to run. Instructions on how to use the system can be found in Appendix A.2.

6.6 Function and Stream Generators

The function and stream generation functions form the *generator* subsystem. The two different function generation methods use fundamentally different methods to generate functions. The random functions use Gaussians, which are represented in a struct containing the mean, standard deviation and weight of the Gaussian. We use another struct to store an array of Gaussians which represent the whole function. When one of these arrays is generated, its constituent Gaussians are output to a file as their mean, standard deviation and weight, so they can be used later if necessary. Once one of these sets of Gaussians is generated, it is passed to a function which implements the thinning procedure. The rate function $\lambda(t)$ is generated by summing the values of Gaussians in the set at time t using a Gauss transform. A two dimensional array is returned, containing the time of each event, and the value of $\lambda(t)$ at each time. Once the stream has been generated, depending on the requested output verbosity, the data is output to file in two columns. This process is repeated for the requested number of streams. Multiple different functions can be generated with one function call. Alternatively, a single function can be used to generate multiple independent stream pairs.

The generation of functions using expressions is done in a very similar way to the Gaussian generation, but since an expression is being used there is no need to store the representation of the function in a special way. Events are generated and thinned using a very similar function to the above, but use a `muparser` struct pointer which can be used to calculate values of the function it has parsed. This pointer is created in the setup function which reads data from the parameter file and parses the user-defined expression. If there is a syntax error in the expression, the program prints the location of the error using `muparser` functions and exits.

The generation in both cases is split into several stages. In the first stage, the parameters required by the function are read from the parameter list. If there are parameters that have been passed in as options to the command line, they take precedence. Once these parameters are checked, the top level function makes multiple calls to the second function, depending on how many functions are to be generated. The job of the second level function is to make calls to the function which actually performs stream generation, and output the resulting data to file.

This three-level structure is used throughout the system to separate the parameter retrieval and checking from the execution of the logic, and removes the need to re-parse the parameters for each call to the generator.

The configurable parameters for the generation functions include the value of Δ for each stream, the start time and length of the interval, the value of the homogeneous λ to use in the thinning procedure, and the expression to use to generate the function. In the case of the Gaussian generator, the distance between Gaussians, the sample resolution and the weight multiplier can be specified. In addition, the standard deviation can be set to be calculated using $\alpha \cdot \Delta t$, or simply taken from a specified value.

6.7 Function Estimators

Other than the libraries, the function estimators make up the largest portion of the system. As should be clear from what has been said above, the baseline estimator is built upon the IWLS estimator, and this is true in the code as well. The IWLS and OLS estimators form the base of the piecewise estimator, which is in turn used by the baseline estimator. The OLS estimator is implemented as a single iteration of the IWLS estimator; there is no separate code for OLS—calling the OLS function calls IWLS with a single iteration. The IWLS estimator first constructs arrays containing weights, bin counts and midpoints to be used in the estimation. At this stage, if there are no events in the interval that is being estimated, the estimator returns an empty estimate. The rest of the function is a large loop which performs the required weight and estimate updates, and outputs data when finished. The function returns a struct which contains the estimated values of a and b, and the start and end of the interval that was being estimated. With OLS, the number of sub-intervals can be configured. For IWLS, in addition to the number of sub-intervals, the number of iterations can be set.

The piecewise estimator uses a while loop to iterate through the given interval, which is split into sub-intervals by defining a maximum number of breakpoints. If the number of breakpoints is set to 4, then the maximum number of times the IWLS estimator will be called is 5—each breakpoint represents a point where the end of one interval meets the start of the next. During each iteration a function to extend the line estimated by IWLS is called. The process is hierarchical; if the initial extension fails, the function runs again, halving the interval length. If no extension is possible after a 5 iterations, then extension fails. If the extension is successful, then the next interval estimate starts directly after the end of the extended estimate rather than its expected start point. This process can lead to fewer sub-intervals than expected given the maximum number of breakpoints. Checking event data in the extension interval is necessary when extending the line. Rather than reading the event file each time, a function was written which can, given a set of event data, return an array containing events within a desired interval. The IWLS estimator returns an array of structs containing the estimate for each sub-interval.

The baseline estimator takes the struct from the piecewise estimate and modifies the estimates inside it to ensure that the function produced by combining them is piecewise continuous. Four functions perform the modifications—the first calculates a vector of breakpoints, the second computes function values at these breakpoints, the third computes the midpoints at the breakpoints, and the last adjusts the intercept and gradient of each sub-interval estimate. The baseline and piecewise estimators have the same configuration parameters. The iterations and sub-intervals for the IWLS estimator to use, the maximum extension length, the maximum number of breakpoints, and the threshold value for the probability density function can be specified.

The kernel density estimator is much simpler than the baseline estimator, using only two functions to perform all the operations required. The first stage is to generate an array of Gaussians using the event data—identical Gaussians centred at each event time, represented by their mean, standard deviation, and weight (set to 1). This array is then passed to a function which performs a Gauss transform on the array, by summing the Gaussians at points sampled at a given resolution. The function returns a two dimensional array containing the times of samples and the value of $\hat{\lambda}(t)$. A function which returns just the array of Gaussians is also used when all data on the Gaussians is required. The Gaussian estimator has only two parameters; the value of σ and the sampling resolution.

6.8 Time Delay Estimators

Both the area and PDF methods perform the same hierarchical estimate of the time delay. As always, the first stage of the process is to extract the required parameters. Once the initial estimate is received, the process is simply repeated with a slight change in the parameters to the function to make the second, finer pass over the data. Since both the methods may receive data from either of the two function estimation methods, they use a void pointer to receive the estimate data, and take a switch that is used to select the correct function to process the data. The estimate data is cast to the correct type before it is processed. Each of the functions returns a single double precision value of the estimate it makes.

To produce its estimate, the PDF estimator must combine the two function estimates into a single function. The different function estimates are stored in different data types, so a separate function is used for each type. The function can in theory combine any number of streams, but has only been tested to a maximum of 4. One of the parameters it takes is an array of time delays, which is used to shift the function in time before combination takes place.

The time delay estimation must somehow be combined with the function estimation. This is done by the `multi_estimate` function. Again, this is a two stage function, the first stage of which extracts the relevant parameters. Depending on the type of estimator, different parameters are retrieved. The function can do estimates of several functions with only a single call by using the standardised output filenames. The second stage of the function first estimates the characteristic function of each stream (tested up to 4 streams). If the kernel density method is being used, a normalisation constant is calculated. Finally, the time delay estimate is performed using the estimates and the normalisation constant (if required). Using the best scoring estimates between each stream, the functions for all streams are combined to make a single final estimate of the function, which is both output to file and returned to the caller.

The parameter file contains several parameters for configuring the time delay estimators. The estimation can be turned on or off, and the method can be chosen.

It is also possible to specify whether to use the hierarchical estimation method. A step for the first and second pass can be specified, as well as the range in which to check. The sample resolution must be specified for both the area and PDF estimators, and the PDF estimator also requires the number of bins into which it is to split the interval.

6.9 Experimenter

The purpose of the experimenter is to run the estimation subsystem multiple times, with different parameter settings. Its behaviour is modified by a separate parameter file. The code is designed in such a way that new experiments on different parameters can be added and removed with minimal effort on the part of the user.

A simple experiment can be set up by modifying just a few lines in a the parameter file. The experiment must be given a name, so that the system can reference it. Some parameters to experiment on must be set, and the type of estimator to use to estimate the function must also be specified. An additional parameter is used to specify whether an experiment with the given name should be run or not. To allow for greater flexibility, the parameter values to test can be defined as ranges. For example, entering `2,4,...,10` as the value for a parameter will result in values of 2, 4, 6, 8 and 10 being experimented on. There are two types of experiments that can be performed; the estimation of functions, or the estimation of the time delay. Function estimate experiments are used to determine optimal parameter settings for a given set of test data, using model selection. The experimenter can create copies of test data with events in certain intervals removed to use for this purpose.

With the modified data, the function estimators are run on the test set with different parameter combinations. Parameter settings are co-varied, which means that all possible combinations of parameters are tested. All possible values of parameters are stored in separate arrays for each parameter, and each has a pointer which indicates which value of the parameter should be used by the estimator. After each run of the estimator with a given set of parameters on all test data has been completed, the index of the last parameter is incremented by 1, and the process is repeated. Once the value of the index exceeds the length of the array, it is reset to 0, and the index on the second to last parameter is incremented by 1, and this process continues until all indices return to 0, similar to how a milometer works. After the experiment for a set of parameters is complete, the results of the estimates are analysed, and each is given a score based on a sum of log probabilities. The value of the function in each interval in which data was withheld is compared to the actual value from the original data. The closer the estimated and actual values are, the higher the score. Once all parameter combinations have been run, the best combination of parameters for each stream in the test data is written to file. Files are also produced in each sub-directory which give information about the parameters used for experiments in that directory.

Once the model selection is done, the optimum parameters can be extracted from the results and the time delay can be estimated. The time delay results are processed, with the estimate and error for each stream pair, and the mean, standard deviation and mean error of a set of stream pairs are output to a file. Functionality for running large numbers of experiments is provided by a number of shell scripts. Instructions on running experiments can be found in Appendix A.3.

6.10 Error Checking

Due to the large reliance on intervals in many parts of the system, a function for checking whether an interval is valid was implemented, along with a rigorous set of tests. Any function which works with intervals first checks their validity with this function. Error Other error checking mechanisms involve checking whether null pointers or other invalid values are received as parameters to a function. The system exits when something goes wrong, printing an error message indicating the function in which the error occurred. The parameter list parser and muparser functions also provide information about errors such as duplicate or missing parameters, or unparsable input.

7 Development

In this section, we discuss the programming methodologies and project management ideas used during the development of the project.

7.1 Development Process

The development process was made up of three key stages. First, before writing any code, the ideas behind the part of the system that was to be implemented were sketched out in a physical notebook. The details of this stage were specific to the needs of every bit of functionality, but generally consisted of the same decomposition of what was required. What parameters does it need? How does the input need to be processed? What should be output? For more complex parts of the system, we also planned out how it would connect to the main parts of the system. When more complex algorithms had to be implemented, we wrote a prototype on paper and tested it manually for a few simple cases to check its correctness.

Once we had a good idea of the structure of the code, we implemented a prototype which would have its own internal variables and would not actually return anything to the system, instead printing all its output to the terminal. The output was checked manually to verify its correctness. At this stage, automated tests were also written for many functions, particularly those which had an important role in mathematical calculations or error checking. By the end of this part of the process, we had a minimal working version of the function that we wanted to implement.

The final stage was to integrate the function or subsystem fully with the main system, abstracting out all the internal function variables to the parameter files, or taking them in as parameters to the function. More rigorous error checking was also implemented at this stage to ensure the correctness of parameters. Once integrated, tests were run again to confirm that no bugs had been introduced by the conversion.

7.2 Development Methodologies

We used a few principles of software development that we believed could guide us to create a better system. The Unix philosophy of operating system development has many ideas that can be used to develop much smaller systems. In *The Art of UNIX Programming*, Raymond abstracts some ideas behind the philosophy into a set of 17 short rules [27]. We found that a subset of these rules were applicable to our system:

Rule of Least Surprise In interface design, always do the least surprising thing.

Rule of Modularity Write simple parts connected by clean interfaces.

Rule of Optimising Prototype before polishing. Get it working before you optimise it.

Although the interface in our system involves minimal interaction, the rule of least surprise is still a good one to follow. When designing the behaviour of the launcher, we considered what the expected behaviour would be, and implemented the launcher in such a way to follow those expectations. One particular example is the presence of a help command which gives information about what the program does and what options it can parse. Entering `ls --help` on a Linux system gives an example of the contents of such a printout.

Our system is not so large as to have properly defined interfaces, but there is interaction between subsystems. During our implementation, we tried to follow the rule of modularity by making each part of the system as simple as possible. The functions which execute a particular task should be grouped together, and any functions which are not a direct part of that process should be grouped elsewhere. For example, the functions which call the estimators are very short, and are grouped together in one file. The estimators themselves are separate entities—they are not grouped together in one large file, but instead in their own dedicated files. Functions which are used by the baseline estimator, for example, are of no use to the iterative weighted least squares estimator, as their tasks are very different. Interactions between subsystems are made simpler by encapsulating data in structs.

As mentioned in the previous section, the rule of optimising was a key part of the development process. Moving from prototype to implementation to polishing means that time is not wasted optimising or trying to fix something that is fundamentally broken.

In *The Pragmatic Programmer*, Hunt and Thomas put forward their “DRY” (*D*on’t *R*epeat *Y*ourself) principle, which states that “Every piece of knowledge must have a single, unambiguous, authoritative representation in a system.” [17] We believe this to be the most important principle we have followed, as code duplication has many issues, mostly stemming from contradictions. The libraries are our attempt to ensure that there is one function for a single task, and the parameter files represent the single definition of control parameters in the system.

7.3 Testing

Any system requires testing to verify its correctness, and we have implemented a large number of tests for those functions which are central to the correct functioning of the system. Some functions, such as those which perform the estimation, it is not feasible to check, as the actual results that should be obtained for a normal input are not easily calculated without relying on the system itself. Those functions which perform mathematical computations and error checking are the ones which have undergone the most rigorous checks.

A total of 62 tests have been implemented, each of which contain multiple cases to check edge cases. Of these, 56 check library functions. Checks on functions in the mathematics library make up over half of those.

Tests are implemented using the Check framework [5], which is a unit testing framework designed for the C language. The main reason for its use is its integration into the GNU Autotools framework, which is used for automatic configuration and compilation of the code. The tests can be run by running `make check` from the top directory.

7.4 Version Control

The project was kept under version control using the `Git` and `SVN` revision control systems. All commits were made to the Git repository. The SVN repository was used as a backup, with tagged versions being committed for backup purposes.

A branching strategy was chosen, in an attempt to bring the project closer to one which might be performed in an industry environment. Several searches for a branching strategy led us to use one proposed by Driessen [9]. In this strategy, there are two main branches, `master` and `develop`. The state of `master` reflects the current version, and `develop` reflects the current state of development. There are two supporting branches, which deal with features, releases. For each new feature, or large change that was made to the system, we moved development to a new branch so as not to impact the main development branch. Branches were merged back to the main development branch when the feature was complete. When a large milestone in the project was completed, such as the completion of a subsystem, we branched into a separate branch for that release to make some modifications to information about the code, and then merged the release branch with `master` and `develop`.

Commits were made to the development branch when a small feature was completed, or some modifications were made. With this sort of regular commit activity, it would be easy to revert to a working version should a bug be found, and attempt to locate the root of the problem.

7.5 Project Management

As mentioned above, we kept detailed notes of algorithm prototypes and ideas about how to proceed with the implementation of the project in a physical notebook. This notebook also served the purpose of detailing mathematics and ideas that were relevant to the project, and how they might be used.

In addition to the notebook, we kept a change log of all the modifications made to the code in a text file which was updated with every commit to the repository. In this log we detailed which parts of the code were changed, what change was made, and if relevant, the reasoning behind the change. Not only the change log, but also each individual commit to the repository went into a reasonable amount of detail about the changes that were made. This log can be used to determine exactly when a specific change was made.

8 Experimentation

The experiments are done in two stages. First, the optimum parameter set for each function that is being experimented on is found using model selection. Model selection involves withholding some of the data from the estimator by removing the event data from intervals uniformly distributed across the interval being estimated. Each function is estimated, and the value of the function in the regions where data was removed is compared to the value that would be expected had all the data been present. The score is calculated using the Poisson PDF, as in Equation (23).

The Gaussian estimator was set to sample the kernels at a resolution of 0.3 time units, and the standard deviation of the kernels was varied. The baseline estimator was set to use 3 iterations of the IWLS estimator, and four other parameters were experimented on.

IWLS sub-intervals 2, 4, 6, 8, 10

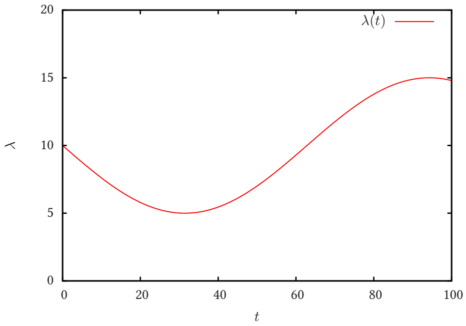
PDF threshold 0.01 to 0.15 with a step of 0.01

Maximum extension 5, 7, 9, 11, 13, 15, 17, 19, 20

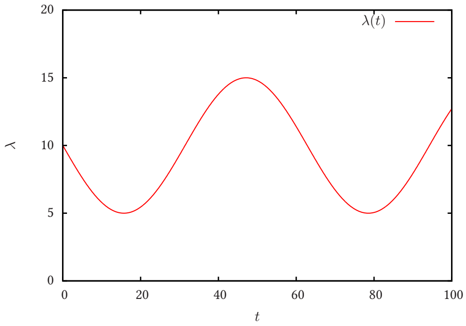
Maximum breakpoints As above

Gaussian standard deviation 0.5 to 20 with a step of 0.5

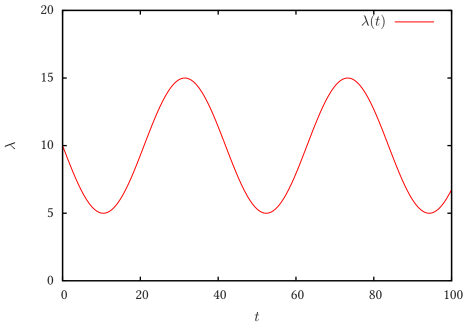
The parameters were co-varied, meaning that each value for one of the parameter settings was tested with all possible values of the other parameters, for a total of 6115 possible combinations.



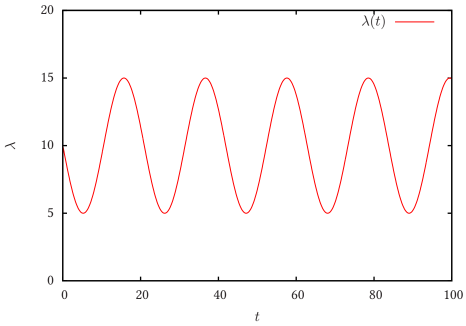
(a) $\alpha = 0.005$



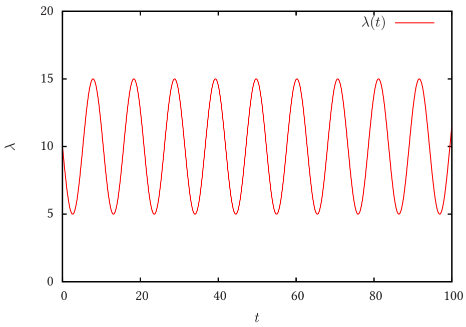
(b) $\alpha = 0.01$



(c) $\alpha = 0.015$



(d) $\alpha = 0.03$



(e) $\alpha = 0.06$

Figure 10: Functions used for preliminary experiments on sine functions, showing the different α values used. The generating function is $y = a - b\sin(\alpha t)$.

Once the optimum parameter set has been found, the time delay for the pair of streams is estimated, using all the data that is available. From this we receive estimates of the time delay on which it is possible to perform statistical analysis. The mean, standard deviation and error for each estimate on each function is calculated, and from this we can examine the effectiveness of the estimates. The aim of the experiments is to compare the effectiveness of the time delay estimation with four method combinations: Gaussian area, Gaussian pdf, baseline area and baseline pdf. Two statistical tests were done on the experimental results. A paired t-test was used to check whether one method was better than another. For the second test, we took the error values for the two methods combinations being compared, subtracted one from the other and performed a a one-sample t-test on the resulting set of values. The full set of statistical test results can be seen in Appendix B.

We assume that the distribution of the samples is Gaussian, but this may not be the case. However, full non-parametric testing is out of the scope of this project.

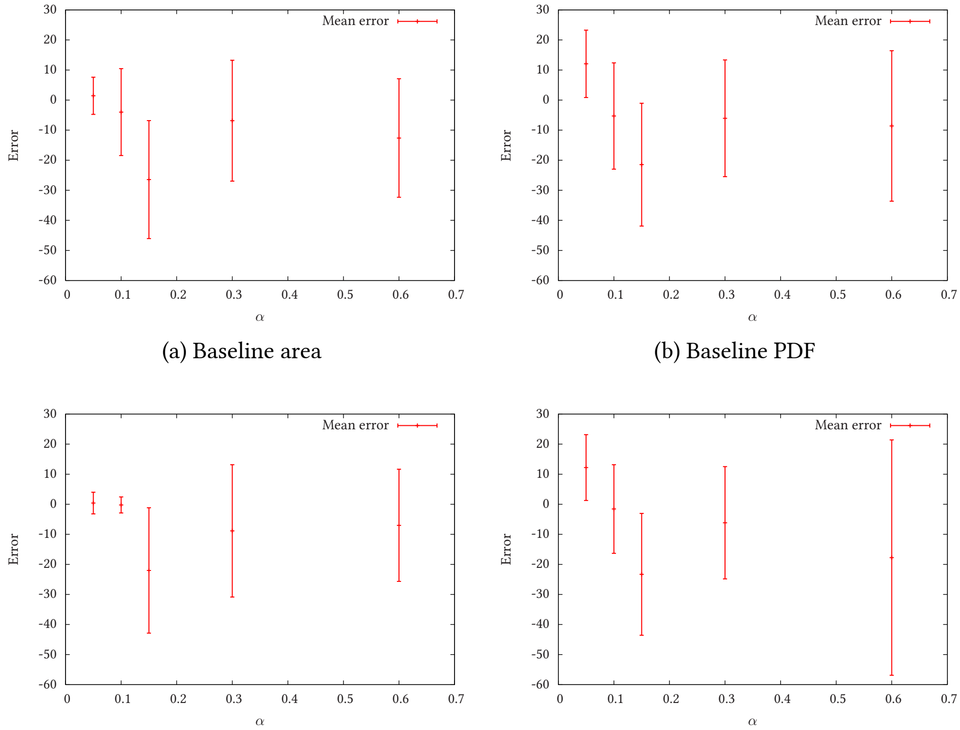
8.1 Sine Functions

The first experiment performed used stream data generated from functions of the form $y = a - b\sin(\alpha t)$. An increase in the value of α increases the oscillation frequency of the sine wave, and a decrease reduces it. The value of a indicates how much the wave is shifted along the y-axis, and b determines the amplitude of the wave. The values of a and b were set to 10 and 5 respectively.

8.1.1 Preliminary Experiments

In the first set of experiments, we investigate the performance of the estimators on five values of α: 0.05, 0.1, 0.15, 0.3 and 0.6. Figure 10 gives an indication of what the functions look like. For each value of α, 25 pairs of streams were independently generated, each with an interval of 100 time units and a time delay of 10 time steps between the two streams.

Figure 11 shows the error of the various estimator combination at each value of alpha. Performance appears optimal at low values of α, with large standard deviation with $\alpha > 0.1$. The area time delay estimator is significantly better than the PDF for both of the function estimators, with p-values of 0.00017 and 0.0000074 for the baseline and Gaussian method respectively at $\alpha = 0.05$. The difference between the two function estimation methods was not significant, with p-values in excess of 0.4 for comparisons between the baseline and Gaussian estimators for the same time delay estimators at $\alpha = 0.05$. Results from $\alpha > 0.005$ show no statistical significance in the difference between the various estimators, so we cannot say whether the area estimator is always better.



(a) Baseline area (b) Baseline PDF

(c) Gaussian area (d) Gaussian PDF

Figure 11: Mean error on the preliminary sine function experiments. Error bars show standard deviation of error. Performance appears to deteriorate when $\alpha > 0.1$.

8.1.2 Refined Experiments

Although the previous set of experiments provide some indication as to the performance of the estimators, we investigated their effectiveness on a smaller range of α values. In this set of experiments, we used the same parameters, but generated a new set of functions for values of α from 0.01 to 0.15, with a step of 0.01 between each successive set of stream pairs. 10 pairs of streams were generated for each α value. The time delay was set to 15 time units.

The result of this second set of experiments uncovered an interesting pattern in the performance of the estimators. Figure 12 shows the error for each combination of estimators for different values of α. It is clear to see from the graphs that there is a window of optimum performance where α is between 0.04 and 0.1. We believe this may be as a result of the shape of the functions that are being estimated. As with the previous set of experiments, the area estimator again outperforms the PDF estimator, which is visible in the graphs and Table 2. Within this window, the area method is significantly better than the PDF estimator in

	Gaussian	Baseline
Area	10.39 ± 3.60	11.43 ± 6.18
PDF	22.20 ± 10.94	22.06 ± 11.20

Table 1: Experimental results for $\alpha = 0.05$ for preliminary sine experiments. Actual time delay is 10. ($\mu \pm \sigma$)

	Gaussian	Baseline
Area	15.99 ± 3.10	15.95 ± 4.51
PDF	16.53 ± 11.80	15.72 ± 14.06

Table 2: Estimates for all method combinations at $\alpha = 0.07$ for refined sine experiments. Actual time delay is 15. ($\mu \pm \sigma$)

some cases, but this significance varies greatly with α, and we therefore cannot conclude that there is a definite increase in accuracy using the area method. As before, the Gaussian and baseline methods were not significantly different.

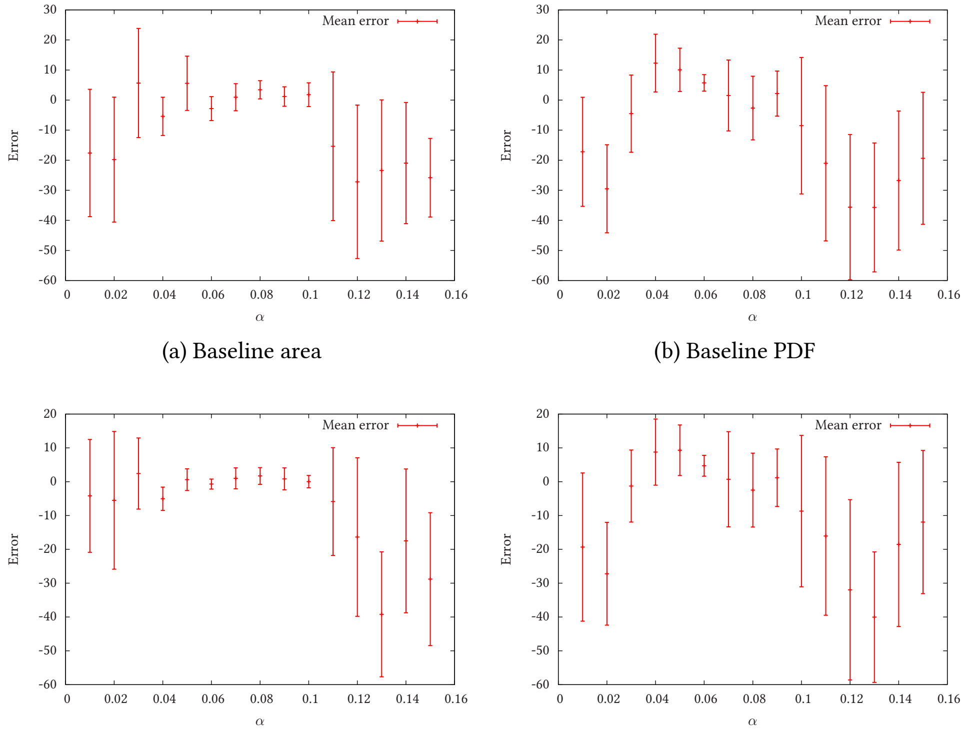
8.2 Random Functions

The experiments on sine functions did not yield any definitive result as to which methods were more effective, and so we also performed a series of experiments using random functions rather than sine curves. Evaluating the performance of the estimators on these functions is important, since functions from real lensed objects will be very unlikely to follow a sine curve, instead fluctuating somewhat randomly. This set of experiments should also allow us to investigate the window of optimum performance from the previous experiment. In order to test a variety of different functions, we varied the α parameter in the equation $\sigma = \alpha \cdot \Delta t$, where σ is the standard deviation of the Gaussians used to generate the random function. The weight of each Gaussian was set to 3, to produce a larger variation in the shape of functions. Figure 13 shows some randomly selected examples.

8.2.1 Preliminary Experiments

For the preliminary experiment, we chose to use five different values of α, 0.4, 0.8, 1, 2 and 3. While increasing the α parameter in the previous set of experiments would make the functions more difficult to estimate, in this case the opposite is true; larger values are easier to estimate, whereas smaller values are more difficult.

For the preliminary experiments we set the value of Δt to be 10, resulting in 11 Gaussians being spread uniformly across the 100 time unit interval. Given that α ranges from 0.4 to 3, the value of σ will be between 4 and 30 time units. Lower values of σ result in each Gaussian being spread over a smaller interval, which in turn means that when the Gaussians are summed to construct the function it will



(a) Baseline area

(b) Baseline PDF

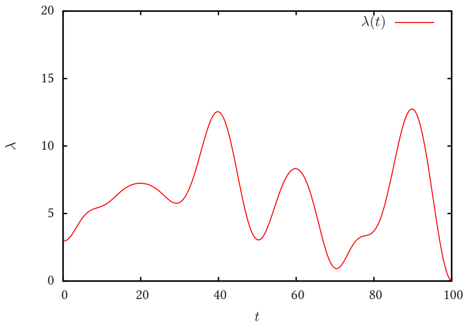
(c) Gaussian area

(d) Gaussian PDF

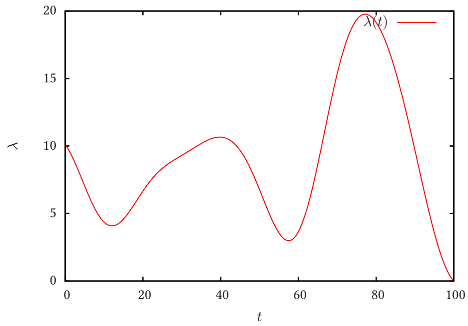
Figure 12: Error on the second set of sine function experiments. Error bars show standard deviation of error. Peak performance is in the window $0.04 \leq \alpha \leq 0.1$

have more variation than with large values. We generated 5 different functions for each value of α, and from each of these generated 5 pairs of photon streams. In this preliminary experiment, we wish to see whether there is a clear point of deterioration, and whether there is any evidence of a window in which the methods are most effective. The results from the experiment were very enlightening. Figure 14 shows the error of the estimators over all α values. The estimators performed much more consistently, that the window in the sine experiments was due to the shape of the functions being estimated. The methods that we use seem to be ineffective on functions which have a symmetrical shape, which sine functions are. The estimators appear to be much more stable, with the mean error deviating relatively little from zero, in comparison to the large variation in the sine function experiments.

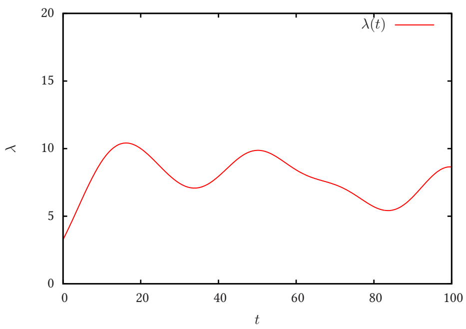
While the performance of the estimators in terms of the mean error was better, the difference between method combinations is still not significant. The large error at α is 1 is due to very large errors occurring in estimates of two functions in that data set. This indicates that while on average the estimators perform well, on functions with certain characteristics there are large differences in the perfor-



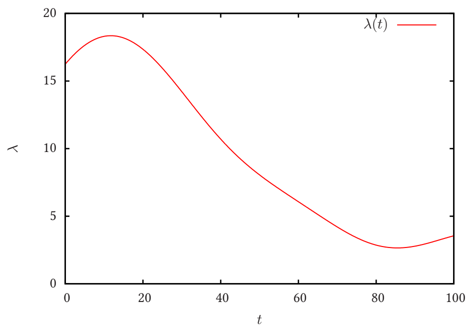
(a) $\alpha = 0.4$



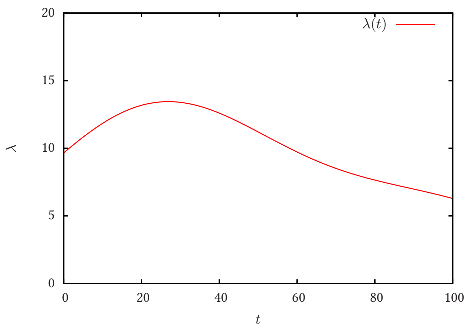
(b) $\alpha = 0.8$



(c) $\alpha = 1$



(d) $\alpha = 2$



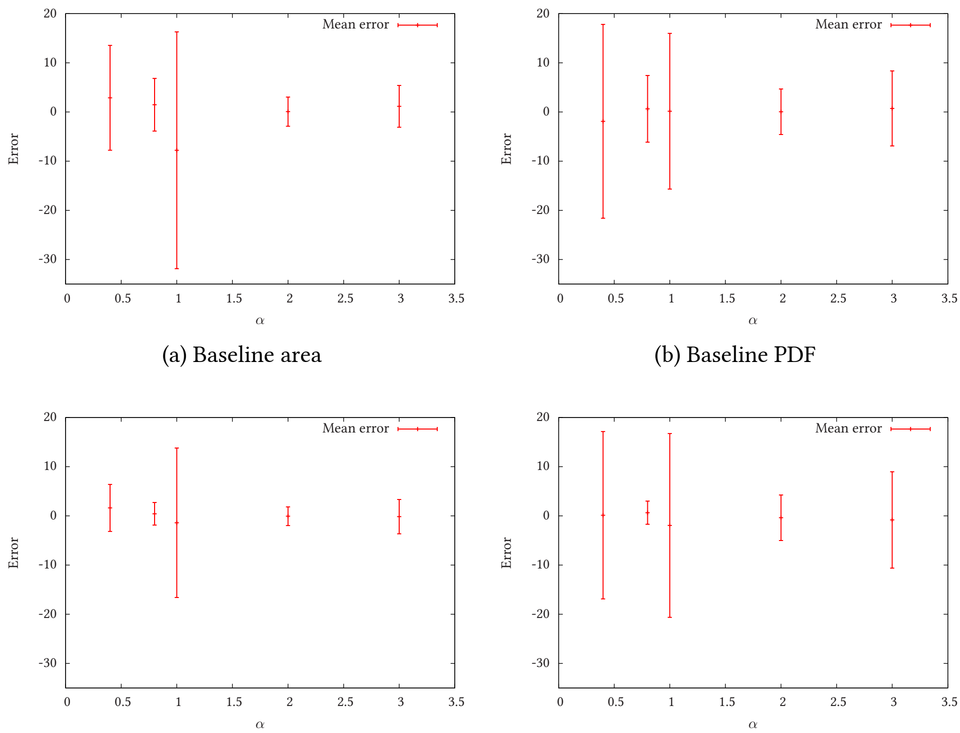
(e) $\alpha = 3$

Figure 13: Examples of random functions generated by different values of α. Oscillation of the functions decreases as α increases.

α	Baseline area	Baseline PDF	Gaussian area	Gaussian PDF
0.4	2.884 $\pm$ 6.7225	-1.904 $\pm$ 10.82	1.62 $\pm$ 1.9959	0.132 $\pm$ 11.392
0.8	1.472 $\pm$ 4.6097	0.64 $\pm$ 4.566	0.424 $\pm$ 1.5155	0.644 $\pm$ 0.61407
1.0	-7.784 $\pm$ 18.96	0.156 $\pm$ 8.9448	-1.4 $\pm$ 8.0519	-1.944 $\pm$ 9.1611
2.0	0.068 $\pm$ 0.64971	0.052 $\pm$ 3.0461	-0.068 $\pm$ 0.90184	-0.38 $\pm$ 4.2005
3.0	1.148 $\pm$ 0.99434	0.724 $\pm$ 5.9319	-0.156 $\pm$ 0.754	-0.824 $\pm$ 6.8956

Table 3: Error values from the first set of random experiments. Error standard deviation is higher when α is 0.4 and 1.0. The area method appears to have lower standard deviations than the PDF method. ($\mu_{\text{err}} \pm \sigma$)

mance. Both time delay estimation methods have worse performance when α is 0.4, but the estimate from the area method is clearly less affected. Table 3 gives a better idea of the differences in the errors.



(a) Baseline area

(b) Baseline PDF

(c) Gaussian area

(d) Gaussian PDF

Figure 14: Grand mean of error over 5 functions for each value of α for the preliminary random function experiments for each method combination.

8.2.2 Refined Experiments

In order to investigate the estimator performance further, we performed an additional experiment on a finer set of data, varying α from 0.1 to 1.5, with steps of 0.1. Going down to such a low value of α results in functions which have very large variations, with impulse-like peaks and troughs. The parameter ranges used were the same as in the previous experiment on random functions.

This experiment confirms our observation from the previous experiment that the Gaussian area method combination is the one which should be used to get the best estimates with the smallest errors, which is clear to see in Figure 15. Again, there was no pattern in the p-values that could be said to indicate that one method is significantly better than another, so we can not conclude with certainty that the Gaussian area method is indeed better than the others. However, this and previous experiments have shown that the size of the error from estimates with that combination is in the vast majority of cases smaller than that of other combinations. Errors appearing in combinations using the PDF method increase as values of α drop below 0.4, indicating that the method is more error-prone when functions have large variations. The PDF method in general has larger standard deviations on the error than the area method.

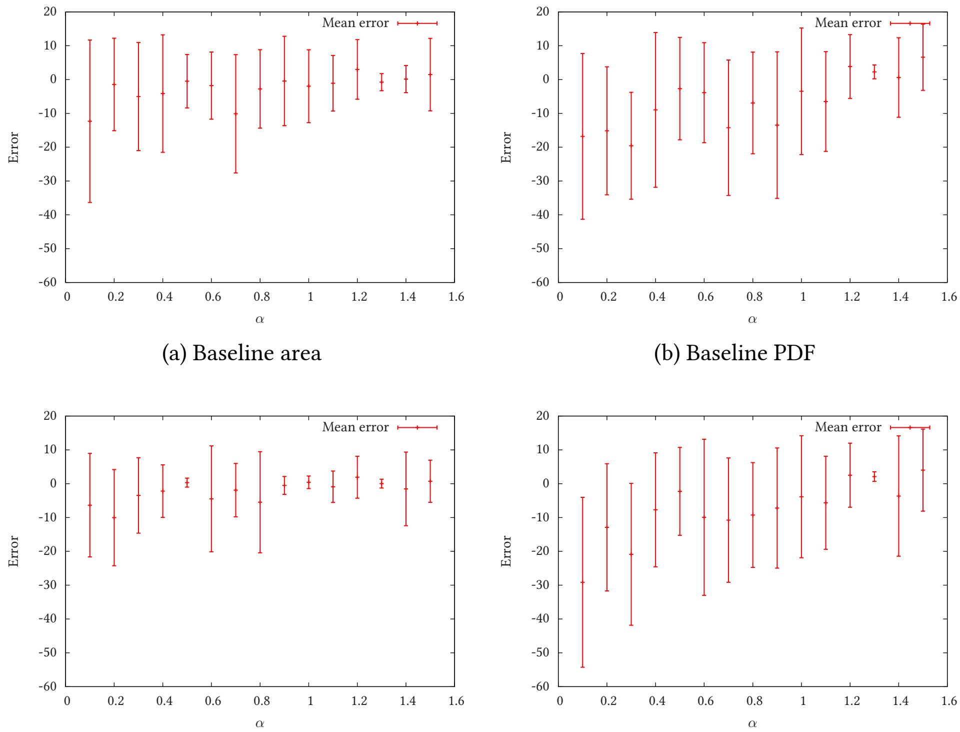
9 Conclusion

In this report, we have presented our system for estimating the time delay in gravitationally lensed photon stream pairs. We showed two methods for estimating the characteristic function of the stream; the baseline method, which is built upon the iterative weighted least squares method described by Massey et al. [23], and the Gaussian kernel density estimation method, a simplified method similar to the one presented by Cuevas-Tello et al. [7]. In addition, we presented two methods for time delay estimation, one using inter-function area, and another using probability density functions.

We performed two sets of experiments to determine how well the estimators performed on sine functions and randomly generated functions, and found that while the difference is not significant, the Gaussian kernel density function estimation method combined with the inter-function area time delay estimation method appears to perform best in most cases. Appendix B contains a selection of the experimental results. From the experiments, we also learnt that the estimators may have worse performance on functions which have some sort of symmetry or repeating pattern, like sine functions.

Having run the estimators with over 6000 different parameter combinations on multiple functions and with multiple stream pairs during experimentation, we are confident in the stability of the system. More formal checks of the correctness of the system are provided by unit tests for 62 functions which perform important tasks in the system.

We have created a system that can complete all the tasks that we set out



(a) Baseline area

(b) Baseline PDF

(c) Gaussian area

(d) Gaussian PDF

Figure 15: Grand mean of error over 5 functions for each value of α for the second set of random function experiments for each method combination.

to implement. It can generate functions from predefined expressions and randomised Gaussians, and use these functions to generate photon streams using non-homogeneous Poisson processes. Our two function estimation methods provide good estimates of the characteristic function of the photon streams, and using these estimates, our time delay estimation methods produce time delay estimates close to the actual value in the majority of cases. We believe that the system is a good starting point for a larger system for automatic detection of potential gravitationally lensed objects. In the next section we provide some ideas for improvements which may make it a better foundation.

9.1 Improvements and Future Work

The first improvement is in the simulation of photon streams. Currently, the λ parameter provided to the generator must be larger than the value of the function $\lambda(t)$, $0 \leq t \leq T$. This means that the maximum value of the function must be calculated before the program is run, or a value of λ must be chosen such that the function is unlikely to exceed it. In most cases this does not pose a real issue, and large values of λ can be chosen to no negative effect—the generation of data is still very fast. However, for ease of use of the system, implementing a method which does not require this parameter would be beneficial.Apart from the thinning method that we have used, there are many other methods of generating NHPPs [25, 21] which could be implemented to improve this.

There is also the potential for improvements to the baseline estimator. Currently, at each breakpoint only the midpoint is considered. A hierarchical search could improve the quality of estimates. Instead of considering only a single point, a search could be done along the line between the points to find the point at which the probability density function was maximised. If this was done for each breakpoint, then it should be possible to find a function which provides a better estimate than the current approach.

Using a fast Gauss transform function in the kernel density estimator could improve its running time. In addition, exploring ways to mitigate the drop-off at the start and end of the interval would give more accurate estimates at these points.

From our experiments, we discovered that the time delay estimators developed appear to perform worse on functions which have repeating patterns or are symmetrical in some way, such as the sine function. Adding confidence values to the time delay estimates would give more information to the user. Also, currently only the highest scoring value of Δ is reported. Reporting other peaks might provide information about periodicity.

Although we have performed several experiments, we were unable to obtain real photon stream data on which to test our estimators. To find out whether our system would be useful in real applications, it should be applied to some real world data.

As mentioned in the introduction, this system is intended to form a base for a system which can automatically identify potential gravitationally lensed objects. We believe that the current system provides a good foundation for such a system. However, given that its accuracy is limited, the ideal case is for this system to provide some sort of initial estimate, and then hand over to another system which is able to make more accurate estimates. We have identified three features that could be added as an extension to this system, or as separate systems:

1. Pull stream data from a database or some other form of storage

2. Compute likelihood of a pair of images coming from the same object based on estimates from our system

3. Keep track of which data has been processed and the confidence values of the estimates associated with that data

The combination of our system with a system or systems with these features would potentially create a system that could reasonably be applied to real-world problems.

9.2 Individual Comments

Although I have worked on several reasonably large projects during my time at university, this is the largest by far. Other projects of comparable size have been team projects, and as such I did not have to deal with the whole of the code base or management of the project. I believe that working alone on this project (other than weekly supervision meetings) has improved my abilities in many areas.

First and foremost, working on a project in a field which I have relatively little experience was quite a daunting task. Before starting I had some interest in machine learning, but my knowledge of problems and approaches to solving them was minimal. Developing the function estimators was particularly challenging, with literature on the subject being quite heavy on mathematics with which I was unfamiliar. I had to study the papers on which the function estimators are based for some time before I felt confident that I understood the important points. I have come to understand the techniques much better than I did initially, but there is still much to learn. Statistical testing was another challenging part of the project, requiring me to understand how various statistical techniques work, and which approaches are valid for what sort of data. Processing and analysing the results of the experiments was also new to me, but was a good learning experience which will be useful for any scientific projects I may encounter in the future.

In addition to being in an unfamiliar field with new mathematical concepts, I also chose to write the project in C, a language which I had studied for only a short time before starting the project. Attempting to implement a complex system in a language which one is new to is difficult, and it took a few months before I was able to add new features and modify old code with the confidence and speed with which I can do so in other languages. C has a rather small set of standard libraries, and so I had to implement many features that are commonly available in the standard libraries of other languages. For more complex functionality, in order to save time I had to find libraries to use, and work out how to use a system with relatively sparse documentation and information available. I think that forcing myself into an uncomfortable situation in terms of unknown environments has paid off, as I am now confident in my C skills.

During the course of the project, I had to make several decisions about the structure of the code, and make sweeping changes to the code base. One example is the point at which I made the switch from the use of pointer arrays to store estimate data to using structs. This required the modification of some of the fundamentals of the system and required great care to implement without introducing bugs. While in team environments it is possible to discuss structural changes and how to go about implementing a new feature, I had to rely on my own judgement to do both, which required a lot of time considering the benefits of one approach over another.

I have learnt a lot from working on the project, and I hope to make good use of not only the technical knowledge, but also the experience of working on a large and challenging project in the future.

References

[1] SAO/NASA Astrophysics Data System (ADS). *Über eine mögliche Form fiktiver Doppelsterne.* URL: `http://articles.adsabs.harvard.edu/full/1924AN....221..329C`.

[2] Martin F Arlitt and Carey L Williamson. "Internet web servers: Workload characterization and performance implications". In: *IEEE/ACM Transactions on Networking (ToN)* 5.5 (1997), pp. 631–645.

[3] Ingo Berg. *MuParser - a fast math parser library.* URL: `http://muparser.beltoforion.de/`.

[4] V Bozza and L Mancini. "Time delay in black hole gravitational lensing as a distance estimator". In: *General Relativity and Gravitation* 36.2 (2004), pp. 435–450.

[5] *Check: A unit testing framework for C.* URL: `http://sourceforge.net/projects/check/`.

[6] Orest Chwolson. "Über eine mögliche Form fiktiver Doppelsterne". In: *Astronomische Nachrichten* 221 (June 1924), p. 329.

[7] Juan C. Cuevas-Tello, Peter Tiňo, and Somak Raychaudhury. "How accurate are the time delay estimates in gravitational lensing?" In: *Astronomy and Astrophysics* 454.3 (2006), pp. 695–706.

[8] NASA Astronomy Picture of the Day. *A Horseshoe Einstein Ring from Hubble.* URL: `http://apod.nasa.gov/apod/ap111221.html`.

[9] Vincent Driessen. *A successful Git branching model.* URL: `http://nvie.com/posts/a-successful-git-branching-model/`.

[10] CC Dyer and RC Roeder. "A range of time delays for the double quasar 0957+ 561 A, B". In: *The Astrophysical Journal* 241 (1980), pp. L133–L136.

[11] Albert Einstein. "Die Feldgleichungen der Gravitation". In: *Preußische Akademie der Wissenschaften, Sitzungsberichte.* 1915.

[12] Albert Einstein. "Lens-Like Action of a Star by the Deviation of Light in the Gravitational Field". In: *Science* 84 (Dec. 1936), pp. 506–507.

[13] B.S. Everitt and A. Skrondal. *The Cambridge Dictionary of Statistics.* Cambridge University Press, 2010.

[14] Free Software Foundation. *GSL - GNU Scientific Library.* URL: `http://www.gnu.org/software/gsl/`.

[15] CFA-Arizona Space Telescope Lens Survey of Gravitational Lenses. *CASTLeS Survey.* URL: `http://www.cfa.harvard.edu/castles/`.

[16] HubbleSite. *First ESA Faint Object Camera Science Images the Gravitational Lens G2237 + 0305.* URL: `http://hubblesite.org/newscenter/archive/releases/1990/20/image/a/`.

[17] A. Hunt and D. Thomas. *The Pragmatic Programmer: From Journeyman to Master*. Pearson Education, 1999.

[18] J.F. Kenney and E.S. Keeping. *Mathematics of Statistics, Pt. 1, 3rd ed.* Van Nostrand, 1962. Chap. 15, pp. 252–285.

[19] Donald E. Knuth. *The Art Of Computer Programming, Volume 2: Seminumerical Algorithms, 3/E.* 1998. Chap. 3.4.1.

[20] Tomislav Kundić et al. "A robust determination of the time delay in 0957+ 561A, B and a measurement of the global value of Hubble's constant". In: *The Astrophysical Journal* 482.1 (1997), p. 75.

[21] P.A.W. Lewis and G.S. Shedler. "Simulation of nonhomogeneous Poisson processes with log linear rate function". In: *Biometrika* 63.3 (1976), pp. 501–505.

[22] A Jeanne M van Loon et al. "Determinants of smoking status: cross-sectional data on smoking initiation and cessation". In: *The European Journal of Public Health* 15.3 (2005), pp. 256–261.

[23] William A Massey, Geraldine A Parker, and Ward Whitt. "Estimating the parameters of a nonhomogeneous Poisson process with linear rate". In: *Telecommunication Systems* 5.2 (1996), pp. 361–388.

[24] *OECD Glossary of Statistical Terms*. OECD Publishing, 2008.

[25] Raghu Pasupathy. "Generating Nonhomogenous Poisson Processes". In: *Wiley Encyclopedia of Operations Research and Management Science* (2011).

[26] Worldwide LHC Computing Grid Project. *What is WLCG: Data Processing*. 2011. URL: `http://lcg-archive.web.cern.ch/lcg-archive/public/data-processing.htm`.

[27] E.S. Raymond. *The Art of UNIX Programming*. Pearson Education, 2003.

[28] Jürgen Renn and Tilman A Sauer. *Eclipses of the stars*. Max-Planck-Inst. für Wissenschaftsgeschichte, 2000.

[29] Sheldon M Ross. *Simulation*. Academic Press, 1997.

[30] Peter Schneider, Christopher Kochanek, and Joachim Wambsganss. *Gravitational Lensing: Strong, Weak and Micro: Saas-Fee Advanced Course 33*. Vol. 33. Springer, 2006.

[31] Jean-Luc Starck and Fionn Murtagh. *Handbook of Astronomical Data Analysis*. 2002.

[32] *Valgrind*. URL: `http://valgrind.org`.

[33] D Walsh, RF Carswell, and RJ Weymann. "0957+ 561 A, B- Twin quasistellar objects or gravitational lens?" In: *Nature* 279.5712 (1979), pp. 381–384.

[34] Eric W. Weisstein. *Covariance*. URL: `http://mathworld.wolfram.com/Covariance.html`.

Appendices

Appendix A   Usage

A.1   Installation

This installation guide is intended for users of Linux distributions, particularly those which are Ubuntu based. The program has been tested on Linux Mint 13 and 14, but should work on most Linux distributions. First, download the latest version of the program from https://github.com/heuristicus/final-year-project/tags and extract it with your favourite program. Alternatively, clone the current version of the repository with

```
git clone https://github.com/heuristicus/final-year-project.git
```

Before the program can be configured, we must install some libraries without which the program will not run. Download the latest muParser package from http://sourceforge.net/projects/muparser/files/latest/download (must be $\geq$ v2.2.3). Then, run the following commands

```
unzip muparser_v_[your_version]
cd muparser_v_[your_version]
./configure --prefix=/usr
make && make install // may require sudo
```

This will install muParser so that the header files it uses can be found in `/usr/include`. Your system must have the `g++` package installed for the `configure` command to complete, and you may also require the `autoconf` package. We must also install the GNU Scientific Library and the Check test framework. All the required packages can be installed with

```
apt-get install libgsl0-dev check g++ autoconf
```

Once this is done run `./configure` followed by `make` in the program directory.

A.2   General Usage

The executable for the program can be found in the `src` directory, and is named `deltastream`. It can be run from the top level directory with

```
src/deltastream [OPTIONS]
```

To find out what options are available, call the executable with the `-h` or `--help` options. We will detail some of the options below. All parameters which govern the behaviour of the system are defined in the parameter files, which have information about what the effect of each is.

A.2.1 Parameter files

Some parameter files are provided with the program, but if for some reason they are deleted, then additional ones can be created using

```
deltastream -d paramfile.txt // default
deltastream -d paramfile.txt -x a // experiment
```

A.2.2 Generating Functions

The `-g` switch is used to run all generation functions. Generating a random function can be done in one of two ways. Using

```
deltastream -g params.txt -r -c 1
```

We can generate a file containing a Gaussian representation of a random function which we can use to generate streams. Changing the number passed to the `-c` switch changes the number of functions generated. To generate streams from the functions, we use

```
deltastream -g params.txt -f rand -n 2 -i random_function_0.dat
```

This takes the data in the file `random_function_0.dat`, generated in the previous step, and generates two streams. Modifying the number passed to the `-n` option will generate different numbers of streams. Another way to generate random functions is with

```
deltastream -g params.txt -f rand -c 3 -n 2
```

The `-c` switch defines how many functions should be generated. After the functions are generated, two streams are generated from each. If you wish to generate multiple different pairs of streams from the same function, use

```
deltastream -g params.txt -f rand -c 3 -n 2 -u
```

The first function generated will be copied into multiple files, and streams will be generated from those copied files. The `-t` switch can be used to specify more or less verbose output. For example, passing a value of 3 will output bin counts for the streams, and a file containing the sum of Gaussians which make up the random function.

The generation of streams from expressions is rather simpler. The following two commands are equivalent.

```
deltastream -g params.txt -n 2
deltastream -g params.txt -f mup -n 2
```

The generator defaults to generating streams from the expression defined in the parameter file. Multiple pairs can be generated using the `-c` switch.

A.2.3 Estimating Functions and Time Delay

Estimates of functions are done using the `-e` switch. The most important parameters are defined in the parameter file. Once streams have been generated, we can estimate them using the baseline estimator

```
deltastream -e params.txt -a base -n 2
```

If the streams were generated from a random function, the `-r` switch must be added to indicate this fact. Again, if there are multiple functions to estimate at once, use the `-c` switch to specify the number. The `-a` option has 5 possible arguments (`ols`, `iwls`, `pc`, `base` and `gauss`), each of which use a different estimator to produce an estimate. Passing a value larger than 1 to the `-n` option will result in an estimate of the time delay. To estimate only the function, simply omit the switch.

A.2.4 Plotting Output

The `scripts/plot.sh` script can be used to plot various data which is output from the system. Calling it with the `-h` option will output information about what plots can be made. The script generates a `.tex` file using `gnuplot`, which it then processes into a `.pdf` and displays using `evince`. After doing a function estimate with the baseline estimator, the generating function can be plotted along with the bin data and estimate using

```
scripts/plot.sh -f output random_function_0_sum.dat est_out.dat
 random_function_0_output_stream_0_bins.dat
```

A.3 Running Experiments

A.3.1 Creating Functions for Experimentation

Using the `genfunc_rand.sh` script found in the `scripts` directory, random functions can be generated, conforming to certain parameters. In this file, we specify the directory to which to output by modifying the `OUTPUT_DIR` parameter. The `LAUNCHER_LOC` parameter specifies the location of the `deltastream` executable used to run the program. The `PARAM_FILE` parameter defines the location of the parameter file to use to generate the functions.

Once these have been set, we specify the values to use to generate the function. The values in the the AVALS parameter define what values of α will be used to generate the functions. The DIVISOR parameter specifies what to divide the values in AVALS by when modifying the α parameter in the parameter file. This can be set to 1 to just use the values inside the array. The values in the AVALS array are also used to create directories, so the divisor is also used to prevent creation of directories such as alpha_0.3. The NFUNCS parameter defines how many different functions to generate. NPAIRS defines the number of pairs of streams that will be generated from each function. Streams generated will be copies of the function. For example, when NPAIRS is set to 5, a function $f(a)$ is generated, along with two streams. Then, four more streams are generated from the same function $f(a)$. This allows for multiple trials on similar data. The FPREF and APREF define the text that is prepended to the directories. Setting FPREF to function_ and APREF to alpha_ will put each set of functions in a directory structure like alpha_1/function_1.

A.3.2 Generating Model Selection Data

Next, we use the stutter_batch.sh script to generate streams with data removed in certain intervals to use for model selection. Here, we set the INDIR parameter to the directory which we set as the output directory in the previous script, and make sure to set the AVALS, NFUNCS and NPAIRS parameters to the same values. We must also define the EXP_PFILE parameter, which tells the script where to look for the experimental parameters. In this file, we must set up which data should be removed. Modifying values in the setup section of the experiment parameter file will allow the choosing of various intervals. To generate a default experiment parameter file, use deltastream -d [filename] -x a. Once this is set up, we run the script, and it generates a new set of files in the same location as the original data which has data in some intervals removed, with names something like random_function_0_output_stream_0_stuttered.dat.

A.3.3 Experiment Parameter Setup

Now, we set up the experiments that we wish to perform on the data. In the experiment parameter file, there are various options which control how the experiments are run. The most important is the experiment_names parameter, which defines the names of the experiments that you wish to run. Once the names are set, we must define four parameters that are used to run the experiment.

```
experiment_names exp_1,exp_2 // Name the experiments
// These parameters will be varied during the experiments
exp_1_params base_max_breakpoints,base_max_extension
exp_2_params gauss_est_stdev
test_exp_1 yes // We want to experiment on this
test_exp_2 no // This will not be experimented on
// Set the estimator to use for the experiment
```

```
exp_1_estimator base
ext_2_estimator gauss
// Estimate the function or the time delay
exp_1_type function
exp_2_type delay

// Set the parameter values for experiments
base_max_extension 3,6,...,11
base_max_breakpoints 4,5,...,10
gauss_est_stdev 1,2,3,4

// This is important! Set the time delay between streams
// Used later to analyse the results
timedelta 0,15
```

When setting the parameter values, `...` can be used to specify a range. In the example, the `base_max_extension` parameters would be 3, 6, 9 and 11. The `timedelta` parameter is important as well—it provides the program with the actual value of the time delay between streams, which is used to determine the score of certain parameter settings. Information about the parameters used to generate streams can be found in the output directories in the `gen_params.txt` file.

A.3.4 Running Model Selection

Once the parameters are set up, we run model selection on the generated streams using the `runexp_batch.sh` script. Here we again set the various parameters needed, and specify a new output directory into which the experiment data is output. Depending on the number of experiments being run, the data can take up a lot of space (on the order of gigabytes), so choose a disk with plenty of free space. It is also a good idea to run a small subset of the experiments before running them all, just to make sure that you are outputting to the correct directory—**data in the output directories from previous experiments is overwritten**. Once you are sure that everything is good to go, run the script. Time taken depends on the number of parameter combinations and number of functions you are running the experiments on. A reasonably large set of data (approximately 151,000 experiments) took approximately two hours on an Intel i5 processor.

A.3.5 Time Delay Calculation

Once the experiments have completed, we use the best parameter settings from the model selection stage to run time delay estimators on the data again, this time with all data available to the estimators. First, we use the `get_goodness.sh` script to extract the experiment numbers of the highest scoring parameter settings. Inside the `runtd_exp.sh` file, we modify the relevant parameters, setting the parameter files to read from, the directory from which to read the parameter data—the directory set as the output directory for the model selection, the location in which the files output from the `get_goodness.sh` script, and the place where we wish to put the files produced by this stage of the process. When the script is run, it performs a time delay estimation on the streams with the best parameters for each function and α value. Inside each directory, a file `results.txt` is produced, which contains the some data about the performance of the estimators with that combination of methods on the given α value for that function. In the next step, we extract this data into a more usable form.

A.3.6 Extracting Result Data

In the `extract_results.sh` script, we set up the parameters so that `INDIR` is set to read from the top level of the time delay results directory, and `OUTDIR` is set to the location to which we wish to output the aggregated results. There are three different flags that can be set to produce data in different forms for processing. The `TT` flag makes the script output error data in a form in which it can be processed by other scripts to run t-tests. The `DV` flag outputs data which can be used to calculate the mean value of the time delay estimate across all functions. Usually, the means are calculated on a per-function basis, but setting this flag outputs data in a form which groups data from all the functions for one value of α into one set which can then be easily processed as a single set of estimates. The `EV` flag does a similar thing to the `DV` flag, but for error data. The error values are grouped by α value, and the resulting files can be used to find the aggregate error for each value of α for a specific method combination. Running the `extract_results.sh` script will output the data. Next, we will explain how to process the resulting files.

A.3.7 Processing Result Data

Inside the results directory, the top level contains files which detail the mean estimate, standard deviation and mean error for each function for each value of α. The `results` directory contains directories with files which are used to produce different data. The `data` directory contains copies of all results files, with the filenames showing what experiment the file was taken from.

T-tests To create data for t-tests, we use the files in the `alpha_errors` directory. With this data we will be able to compare the errors of one combination of method to another. The `ttest_columnate_agg.sh` and `ttest_columnate_individual.sh` scripts are used to process the data further into files readable by the `ttest.m` script. The first script groups data so that when the t-tests are run, results from all functions for one value of α for one method are compared to the same set of functions for the same value of α, but with a different method combination. The second script processes data so that results for individual functions are compared, rather than an aggregate set of data. T-test data will be output to a directory `ttest` in the directory specified in the script. In each file, there will be columns of data used for the t-test, as well as some information about where the data was taken from.

Using the `ttest.m` script, we can run t-tests on the data. The script was written using GNU Octave, but should also be compatible with Matlab. The `read_start_x`, `read_start_y`, `read_end_x` and `read_end_y` must be modified to match the data before the script is run. These values specify the range used by the `dlmread` command to parse in data from the files. In the case of 4 columns with 25 lines each, the values are set to

```
read_start_x=0
read_start_y=0
read_end_x=24
read_end_y=3
```

When run, the script produces a set of t-tests from the data. The `paired_tests` matrix contains the results of two-tailed paired t-tests on the data, and the `single_sample` matrix contains the results of single sample t-tests on the error values calculated by subtracting one set of data from the other. The comparisons array indicates which columns were compared to produce each column of the matrix. In general, 1 refers to the baseline area method, 2 to the baseline PDF method, 3 to the Gaussian area method, and 4 to the Gaussian PDF method.

Mean and Standard Deviation of Estimates Using the `multifunc_mean.sh` script, the mean and standard deviation of estimates from different combinations of methods can be generated. Setting the `INDIR` variable to point to the `results/estimates` directory will perform the computations using a short Octave script, and output the results to a file, which will additionally contain tables for use in Emacs' `org-mode`. Tables 4 and 7 are examples of these tables converted into LaTeXusing the export functionality built into `org-mode`.

Error of Estimates Being able to display the error of combinations of methods, such as the graphs in Figures 11 and 12 is also useful, and data to do this can be produced by the `multifunc_errmean.sh` script. The script will produce files for each combination of methods, which can then be plotted with a program such as gnuplot. One way to plot the data using gnuplot is

```
plot "baseline_area_err.txt" using 1:2:3 with errorbars
```

A.4 CD Directory structure

The `code` directory contains the scripts, tests, source code and documents for the project. The `experiments` directory contains experimental data and results. Data is contained within zip files in the subdirectories. Files with the `time_delay` suffix contain the results of the time delay estimation after model selection was performed. The `streams` suffix contains the functions that were estimated, and the `results` suffix contains some processed results.

Appendix B Experimental Data

In this appendix we present the full set of experimental data. BA=baseline area, GA=Gaussian area, BP=baseline PDF, GP=Gaussian PDF. BA/GA indicates a test comparing the Gaussian area method against the baseline area method.

α	Baseline area	Gaussian area	Baseline PDF	Gaussian PDF
0.05	11.432 $\pm$ 6.18	10.388 $\pm$ 3.60	22.064 $\pm$ 11.20	22.20 $\pm$ 10.94
0.10	6.008 $\pm$ 14.46	9.76 $\pm$ 2.67	4.712 $\pm$ 17.68	8.42 $\pm$ 14.73
0.15	-16.44 $\pm$ 19.62	-12.024 $\pm$ 20.85	-11.472 $\pm$ 20.41	-13.308 $\pm$ 20.26
0.30	3.152 $\pm$ 20.09	1.14 $\pm$ 22.01	3.94 $\pm$ 19.40	3.84 $\pm$ 18.67
0.60	-2.62 $\pm$ 19.71	3.00 $\pm$ 18.65	1.404 $\pm$ 25.02	-7.744 $\pm$ 39.14

Table 4: Table of mean estimate and standard deviation for combinations of methods on the first set of sine experiments ($\mu \pm \sigma$, $n = 25$). The actual time delay is 10.

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.05	0.47815	0.00017369	0.00011531	1.2894×10^{-5}	7.3895×10^{-6}	0.96623
0.10	0.21721	0.78218	0.56963	0.17294	0.66295	0.43371
0.15	0.45352	0.39424	0.58888	0.92654	0.82960	0.75580
0.30	0.74226	0.89066	0.90272	0.64223	0.64879	0.98556
0.60	0.31540	0.53891	0.56946	0.80325	0.23073	0.33954

Table 5: Table of paired t-test p-values for preliminary sine experiments

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.05	0.40134	3.4334×10^{-5}	2.7901×10^{-5}	3.7237×10^{-6}	4.5120×10^{-6}	0.87340
0.10	0.19016	0.77691	0.58516	0.16046	0.65594	0.35306
0.15	0.44337	0.39762	0.54968	0.92600	0.84756	0.72240
0.30	0.77304	0.90010	0.89556	0.60185	0.54686	0.98399
0.60	0.25471	0.49996	0.56251	0.75755	0.22830	0.27032

Table 6: Table of p-values for preliminary sine experiments for one sample t-test performed on error values

α	Baseline area	Gaussian area	Baseline PDF	Gaussian PDF
0.01	-2.60 $\pm$ 21.17	10.81 $\pm$ 16.69	-2.21 $\pm$ 18.15	-4.32 $\pm$ 21.91
0.02	-4.80 $\pm$ 20.77	9.49 $\pm$ 20.36	-14.51 $\pm$ 14.65	-12.22 $\pm$ 15.20
0.03	20.67 $\pm$ 18.17	17.41 $\pm$ 10.52	10.49 $\pm$ 12.84	13.72 $\pm$ 10.65
0.04	9.58 $\pm$ 6.36	9.97 $\pm$ 3.44	27.3 $\pm$ 9.61	23.75 $\pm$ 9.77
0.05	20.58 $\pm$ 9.03	15.61 $\pm$ 3.21	25.06 $\pm$ 7.20	24.30 $\pm$ 7.48
0.06	12.17 $\pm$ 3.97	14.30 $\pm$ 1.48	20.72 $\pm$ 2.74	19.71 $\pm$ 3.09
0.07	15.95 $\pm$ 4.51	15.99 $\pm$ 3.10	16.53 $\pm$ 11.80	15.72 $\pm$ 14.06
0.08	18.42 $\pm$ 3.03	16.70 $\pm$ 2.46	12.35 $\pm$ 10.60	12.52 $\pm$ 10.93
0.09	16.19 $\pm$ 3.24	15.83 $\pm$ 3.25	17.16 $\pm$ 7.50	16.17 $\pm$ 8.51
0.10	16.79 $\pm$ 3.95	15.01 $\pm$ 1.82	6.48 $\pm$ 22.69	6.31 $\pm$ 22.38
0.11	-0.36 $\pm$ 24.73	9.13 $\pm$ 15.92	-6.03 $\pm$ 25.82	-1.06 $\pm$ 23.42
0.12	-12.19 $\pm$ 25.52	-1.36 $\pm$ 23.43	-20.62 $\pm$ 24.16	-16.97 $\pm$ 26.66
0.13	-8.42 $\pm$ 23.48	-24.21 $\pm$ 18.47	-20.71 $\pm$ 21.45	-25.04 $\pm$ 19.28
0.14	-5.96 $\pm$ 20.16	-2.49 $\pm$ 21.27	-11.75 $\pm$ 23.12	-3.53 $\pm$ 24.27
0.15	-10.83 $\pm$ 13.07	-13.80 $\pm$ 19.64	-4.36 $\pm$ 21.96	3.07 $\pm$ 21.17

Table 7: Table of mean estimate and standard deviation for combinations of methods on the second set of sine experiments ($\mu \pm \sigma$, $n = 10$). The actual time delay is 15.

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.01	0.15295	0.96699	0.86740	0.13054	0.11670	0.82642
0.02	0.157811	0.266797	0.398533	0.010178	0.019558	0.748621
0.03	0.64686	0.18670	0.33527	0.22703	0.46908	0.56859
0.04	0.87326	0.0002154	0.0018482	7.5319×10^{-5}	0.00085570	0.44706
0.05	0.1371851	0.2597158	0.3539836	0.0020559	0.0049413	0.8286646
0.06	0.14919	4.719×10^{-5}	0.00027971	7.8065×10^{-6}	0.00016456	0.47247
0.07	0.98273	0.89192	0.96325	0.89579	0.95576	0.89615
0.08	0.20268	0.11576	0.13599	0.24583	0.27772	0.97364
0.09	0.81655	0.72573	0.99482	0.63121	0.91209	0.79640
0.10	0.23567	0.19600	0.18339	0.27572	0.26019	0.98741
0.11	0.34587	0.63994	0.95152	0.15109	0.29466	0.67393
0.12	0.36082	0.48103	0.70215	0.10321	0.20355	0.76434
0.13	0.13020	0.26141	0.11813	0.71497	0.92673	0.65779
0.14	0.72652	0.57817	0.81986	0.38812	0.92404	0.47133
0.15	0.710077	0.457267	0.110994	0.349136	0.096720	0.474311

Table 8: Table of paired t-test p-values for second set of sine experiments.

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.01	0.238363	0.963181	0.834164	0.109943	0.083662	0.718245
0.02	0.012781	0.337071	0.318196	0.020011	0.011247	0.693379
0.03	0.70689	0.28411	0.42725	0.17888	0.27976	0.46476
0.04	0.85129	5.6159×10^{-5}	0.0016052	7.1618×10^{-5}	0.00051425	0.11740
0.05	0.13016	0.29164	0.38843	0.00076639	0.0017663	0.20065
0.06	0.12141	0.00027185	0.00054176	3.4371×10^{-5}	0.00034019	0.058177
0.07	0.96101	0.89812	0.96606	0.89112	0.95491	0.65016
0.08	0.093847	0.128048	0.147634	0.223662	0.252733	0.634744
0.09	0.71169	0.73272	0.99485	0.66951	0.91808	0.36155
0.10	0.28488	0.22685	0.21387	0.29118	0.27635	0.66831
0.11	0.23385	0.66956	0.95495	0.10105	0.18555	0.61964
0.12	0.42867	0.60176	0.75116	0.11260	0.20142	0.67466
0.13	0.16087	0.33354	0.14625	0.48521	0.91090	0.37568
0.14	0.74527	0.64935	0.84987	0.33823	0.91812	0.15872
0.15	0.74025	0.52892	0.17196	0.40520	0.18736	0.39848

Table 9: Table of p-values for second set of sine experiments for one-sample t-test performed on error values

α	Baseline area	Baseline PDF	Gaussian area	Gaussian PDF
0.4	17.884 ± 10.465	13.096 ± 19.351	16.62 ± 4.6861	15.132 ± 16.708
0.8	16.472 ± 5.2632	15.64 ± 6.6625	15.424 ± 2.259	15.644 ± 2.3131
1.0	7.216 ± 23.638	15.156 ± 15.542	13.6 ± 14.92	13.056 ± 18.349
2.0	15.068 ± 2.911	15.052 ± 4.5451	14.932 ± 1.8674	14.62 ± 4.5364
3.0	16.148 ± 4.246	15.724 ± 7.6279	14.844 ± 3.4838	14.176 ± 9.7994

Table 10: Results for preliminary random function experiments. Values shown are calculated by aggregating estimate data from 5 functions with 5 estimates for each α value. The actual time delay is 15. ($\mu \pm \sigma$, $n = 25$)

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.4	0.29071	0.59093	0.4965	0.38927	0.69758	0.67568
0.8	0.63269	0.37356	0.48299	0.88083	0.99779	0.73982
1	0.17468	0.26783	0.34286	0.7245	0.67007	0.91057
2	0.98845	0.84775	0.68509	0.90508	0.74266	0.75625
3	0.80916	0.24102	0.3605	0.60221	0.53605	0.74949

Table 11: Paired t-test results for preliminary random function experiments, calculated by aggregating results for each function for each method and comparing. $n = 25$

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.4	0.28246	0.6203	0.43163	0.39593	0.68339	0.69719
0.8	0.524	0.23804	0.44777	0.86289	0.99737	0.7035
1	0.15115	0.12471	0.24086	0.69366	0.48648	0.85042
2	0.988	0.84899	0.67334	0.89557	0.31721	0.71896
3	0.78673	0.13475	0.3649	0.5442	0.1848	0.70261

Table 12: One sample t-test results for preliminary random function experiments, calculated by aggregating results for each function for each method and computing the error difference on estimates. $n = 25$

α	Baseline area	Baseline PDF	Gaussian area	Gaussian PDF
0.1	2.668 $\pm$ 24.011	-1.808 $\pm$ 24.508	8.656 $\pm$ 15.301	-14.152 $\pm$ 25.117
0.2	13.54 $\pm$ 13.682	-0.16 $\pm$ 18.927	4.964 $\pm$ 14.226	2.1 $\pm$ 18.803
0.3	9.976 $\pm$ 15.983	-4.576 $\pm$ 15.793	11.528 $\pm$ 11.16	-5.872 $\pm$ 20.965
0.4	10.848 $\pm$ 17.376	6.032 $\pm$ 22.883	12.808 $\pm$ 7.7668	7.284 $\pm$ 16.856
0.5	14.508 $\pm$ 7.9035	12.312 $\pm$ 15.138	15.316 $\pm$ 1.3203	12.732 $\pm$ 12.987
0.6	13.212 $\pm$ 9.9238	11.116 $\pm$ 14.794	10.524 $\pm$ 15.65	5.072 $\pm$ 23.078
0.7	4.88 $\pm$ 17.486	0.74 $\pm$ 20.025	13.096 $\pm$ 7.8901	4.236 $\pm$ 18.376
0.8	12.224 $\pm$ 11.602	8.076 $\pm$ 15.033	9.523 $\pm$ 14.948	5.748 $\pm$ 15.487
0.9	14.568 $\pm$ 13.218	1.524 $\pm$ 21.66	14.488 $\pm$ 2.6585	7.8 $\pm$ 17.765
1.0	13.024 $\pm$ 10.781	11.544 $\pm$ 18.706	15.42 $\pm$ 1.8552	11.148 $\pm$ 18.037
1.1	13.9 $\pm$ 8.1991	8.496 $\pm$ 14.743	14.1 $\pm$ 4.6522	9.348 $\pm$ 13.755
1.2	17.988 $\pm$ 8.8131	18.852 $\pm$ 9.433	16.912 $\pm$ 6.1955	17.512 $\pm$ 9.4673
1.3	14.228 $\pm$ 2.5246	17.264 $\pm$ 2.0459	15.028 $\pm$ 1.3014	17.096 $\pm$ 1.4202
1.4	15.128 $\pm$ 4.0093	15.588 $\pm$ 11.763	13.46 $\pm$ 10.885	11.348 $\pm$ 17.795
1.5	16.46 $\pm$ 10.726	21.592 $\pm$ 9.7962	15.724 $\pm$ 6.2363	19.004 $\pm$ 12.125

Table 13: Results for second set of random function experiments. Values shown are calculated by aggregating estimate data from 5 functions with 5 estimates for each α value. The actual time delay is 15. ($\mu \pm \sigma$, $n = 25$)

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.1	0.51733	0.29827	0.019342	0.076423	0.084991	0.00032037
0.2	0.0051315	0.034784	0.017553	0.28463	0.67378	0.54648
0.3	0.0021849	0.69233	0.0042038	0.00012929	0.80605	0.00062073
0.4	0.40614	0.60899	0.46525	0.16734	0.8266	0.14323
0.5	0.52329	0.61644	0.56189	0.32788	0.91659	0.32726
0.6	0.55908	0.47181	0.11175	0.89125	0.27578	0.33316
0.7	0.44001	0.037343	0.89952	0.0060815	0.52319	0.031529
0.8	0.28021	0.47903	0.10078	0.73421	0.59217	0.38478
0.9	0.013321	0.97645	0.13302	0.0046355	0.26822	0.068792
1.0	0.73328	0.27892	0.65733	0.30772	0.93958	0.24459
1.1	0.11578	0.91596	0.16168	0.076168	0.83356	0.10831
1.2	0.73935	0.61978	0.85478	0.39434	0.61843	0.79203
1.3	2.4441×10^{-5}	0.16549	9.5411×10^{-6}	2.9926×10^{-5}	0.73738	2.2806×10^{-6}
1.4	0.85395	0.47565	0.30533	0.50993	0.32528	0.61501
1.5	0.083675	0.76805	0.43587	0.014871	0.41057	0.23495

Table 14: Paired t-test results for second set of random function experiments, calculated by aggregating results for each function for each method and comparing. $n = 25$

α	BA/GA	BA/BP	BA/GP	GA/BP	GA/GP	BP/GP
0.1	0.53491	0.29591	0.023309	0.13562	0.015965	0.0024401
0.2	0.0091442	0.018017	0.0072404	0.22424	0.55365	0.5514
0.3	0.0033213	0.53154	0.0015812	0.00045591	0.80415	0.0012471
0.4	0.44784	0.61693	0.50809	0.16338	0.72314	0.15106
0.5	0.53795	0.5997	0.57453	0.34868	0.87381	0.33991
0.6	0.57406	0.47553	0.12949	0.89505	0.1207	0.15509
0.7	0.37818	0.020646	0.87337	0.0073325	0.11969	0.026935
0.8	0.18908	0.38286	0.056598	0.59594	0.23452	0.23195
0.9	0.004523	0.97677	0.04792	0.0070827	0.12395	0.075853
1.0	0.7347	0.25313	0.52376	0.30931	0.92957	0.22829
1.1	0.098855	0.89295	0.12298	0.081173	0.77935	0.10247
1.2	0.66062	0.52511	0.79827	0.31073	0.23649	0.75251
1.3	6.6396×10^{-7}	0.044196	4.2502×10^{-8}	1.978×10^{-8}	0.48723	6.8121×10^{-10}
1.4	0.8409	0.54461	0.28351	0.42718	0.14969	0.48467
1.5	0.098748	0.63584	0.33856	0.0080649	0.3598	0.21506

Table 15: Single sample t-test results for second set of random function experiments, calculated by aggregating results for each function for each method and computing the error difference on estimates. $n = 25$