CSCI 561
Foundation for Artificial Intelligence

11-12: The First-Order Logic

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Review: Logic and Reasoning

- Knowledge Base (KB): contains a set of <u>sentences</u> expressed using a **knowledge representation language**
  - TELL: operator to add a sentence to the KB
  - ASK: to query the KB
- Logics are KRLs where conclusions can be drawn
  - Syntax
  - Semantics
- Entailment: KB $\models \alpha$ iff α is true in all worlds where KB is true
- Inference: KB $\vdash_i \alpha$ = sentence α can be derived from KB using procedure i
  - Sound: whenever KB $\vdash_i \alpha$, then KB $\models \alpha$ is true
  - Complete: whenever KB $\models \alpha$, then KB $\vdash_i \alpha$

Last Time: Syntax of propositional logic

Propositional logic is the simplest logic-

The proposition symbols P_1, P_2 etc are sentences

If S is a sentence, $\neg S$ is a sentence

If S_1 and S_2 is a sentence, $S_1 \wedge S_2$ is a sentence

If S_1 and S_2 is a sentence, $S_1 \vee S_2$ is a sentence

If S_1 and S_2 is a sentence, $S_1 \Rightarrow S_2$ is a sentence

If S_1 and S_2 is a sentence, $S_1 \Leftrightarrow S_2$ is a sentence

Last Time: Semantics of Propositional logic

Each model specifies true/false for each proposition symbol

E.g.	A	B	C
	$True$	$True$	$False$

Rules for evaluating truth with respect to a model m:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true <u>and</u>	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true <u>or</u>	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false <u>or</u>	S_2	is true
i.e.,	is false iff	S_1	is true <u>and</u>	S_2	is false
$S_1 \Leftrightarrow S_2$	is true iff	$S_1 \Rightarrow S_2$	is true <u>and</u>	$S_2 \Rightarrow S_1$	is true

Last Time: Inference rules for propositional logic

◇ **Modus Ponens** or **Implication-Elimination**: (From an implication and the premise of the implication, you can infer the conclusion.)

$$\frac{\alpha \Rightarrow \beta, \qquad \alpha}{\beta}$$

◇ **And-Elimination**: (From a conjunction, you can infer any of the conjuncts.)

$$\frac{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}{\alpha_i}$$

◇ **And-Introduction**: (From a list of sentences, you can infer their conjunction.)

$$\frac{\alpha_1, \ \alpha_2, \quad \ldots, \quad \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}$$

◇ **Or-Introduction**: (From a sentence, you can infer its disjunction with anything else at all.)

$$\frac{\alpha_i}{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n}$$

◇ **Double-Negation Elimination**: (From a doubly negated sentence, you can infer a positive sentence.)

$$\frac{\neg\neg\alpha}{\alpha}$$

◇ **Unit Resolution**: (From a disjunction, if one of the disjuncts is false, then you can infer the other one is true.)

$$\frac{\alpha \vee \beta, \qquad \neg\beta}{\alpha}$$

◇ **Resolution**: (This is the most difficult. Because β cannot be both true and false, one of the other disjuncts must be true in one of the premises. Or equivalently, implication is transitive.)

$$\frac{\alpha \vee \beta, \qquad \neg\beta \vee \gamma}{\alpha \vee \gamma} \quad \text{or equivalently} \quad \frac{\neg\alpha \Rightarrow \beta, \qquad \beta \Rightarrow \gamma}{\neg\alpha \Rightarrow \gamma}$$

This time

- **First-order logic**
  - Syntax
  - Semantics
  - Wumpus world example

- **Ontology** (ont = ‘to be’; logica = ‘word’): kinds of things one can talk about in the language

Why first-order logic?

- We saw that propositional logic is limited because it only makes the ontological commitment that the world consists of **facts**.

- Difficult to represent even simple worlds like the Wumpus world:

  e.g.,

  "don't go forward if the Wumpus is in front of you" takes 64 rules

First-order logic (FOL)

- Ontological commitments:
  - **Objects**: wheel, door, body, engine, seat, car, passenger, driver
  - **Relations**: Inside(car, passenger), Beside(driver, passenger)
  - **Functions**: ColorOf(car)
  - **Properties**: Color(car), IsOpen(door), IsOn(engine)

- Differences between Functions and Relations
  - Relations return True/False
  - Functions return an object

Semantics

There is a correspondence between

- functions, which return values (objects)
- predicates, which are true or false

Function: fatherOf(Mary) = Bill

Predicate: fatherOf(Mary, Bill)   [true or false]

Examples:

- “One plus two equals three”
  Objects:
  Relations:
  Properties:
  Functions:

- “Squares neighboring the Wumpus are smelly”
  Objects:
  Relations:
  Properties:
  Functions:

Examples:

- “One plus two equals three”

Objects:	one, two, three, one plus two
Relations:	equals
Properties:	--
Functions:	plus (“one plus two” is the name of the object obtained by applying function plus to one and two; three is another name for this object)

- “Squares neighboring the Wumpus are smelly”

Objects:	Wumpus, square
Relations:	neighboring
Properties:	smelly
Functions:	--

FOL: Syntax of basic elements

- **Constant symbols:** 1, 5, A, B, USC, JPL, Alex, Manos, …
- **Predicate symbols:** >, Friend, Student, Colleague, …
- **Function symbols:** +, sqrt, SchoolOf, TeacherOf, ClassOf, …
- **Variables:** *x, y, z, next, first, last, …*
- **Connectives:** $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- **Quantifiers:** $\forall$, $\exists$
- **Equality:** =

FOL: Atomic sentences

AtomicSentence → Predicate(Term, …) | Term = Term

Term → Function(Term, …) | Constant | Variable

- Examples:
  - SchoolOf(Manos)
  - Colleague(TeacherOf(Alex), TeacherOf(Manos))
  - >((+ x y), x)

FOL: Complex sentences

Sentence → AtomicSentence
| Sentence Connective Sentence
| Quantifier Variable, … Sentence
| ¬ Sentence
| (Sentence)

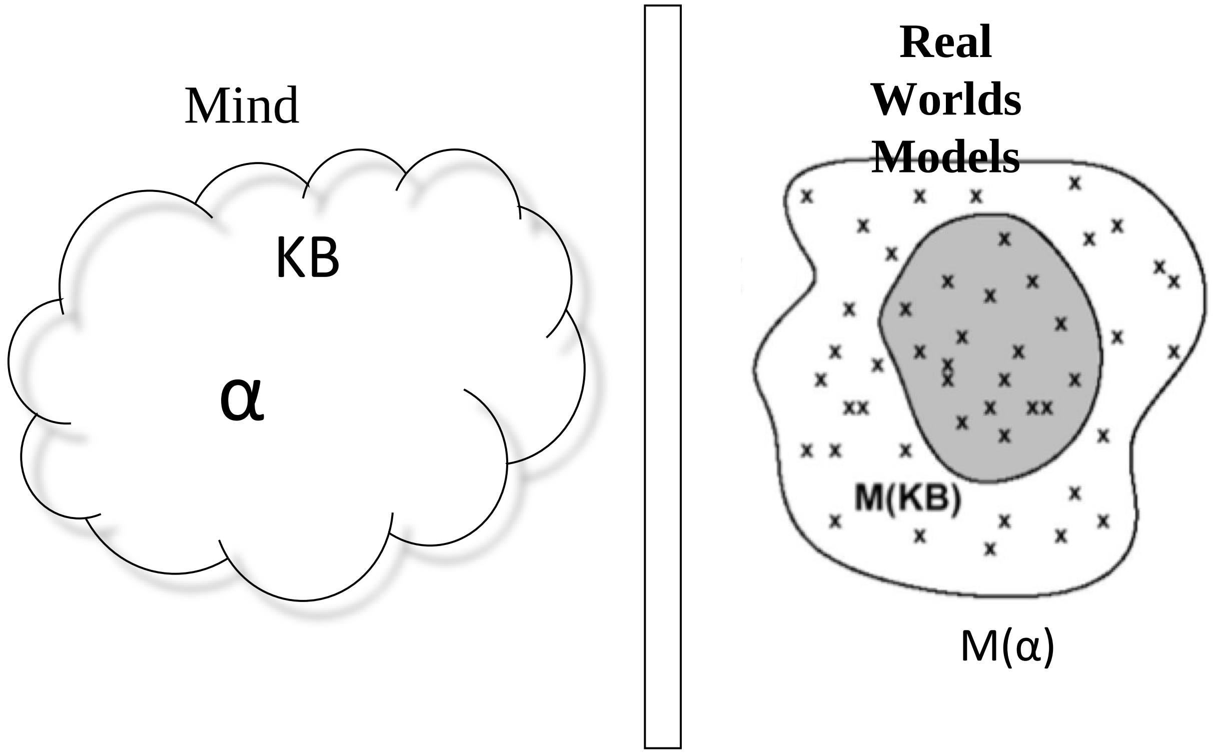
- Examples:
  - S1 ∧ S2, S1 ∨ S2, (S1 ∧ S2) ∨ S3, S1 ⇒ S2, S1⇔ S3
  - Colleague(Paolo, Maja) ⇒ Colleague(Maja, Paolo)
    Student(Alex, Paolo) ⇒ Teacher(Paolo, Alex)

Semantics of atomic sentences

- Sentences in FOL are interpreted with respect to a model
- Model contains objects and relations among them
- Terms: refer to objects (e.g., Door, Alex, StudentOf(Paolo))
  - Constant symbols: refer to objects
  - Predicate symbols: refer to relations
  - Function symbols: refer to functions

- An atomic sentence *predicate*($term_1, \ldots, term_n$) is **true** iff the relation referred to by *predicate* holds between the objects referred to by $term_1, \ldots, term_n$

Logic (mind) and Models (real worlds)



"The semantic wall"

Logic Representation of the Real World

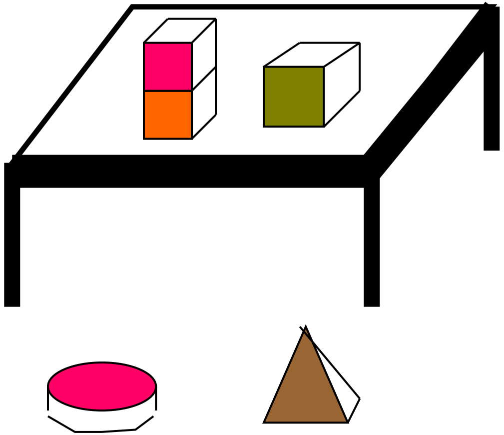
Symbol Systems

Objects:
+BLOCKA+
+BLOCKB+
+BLOCKC+
……

Relations:
ON (+BLOCKA+ +BLOCKB+)
IS_RED (+BLOCKA+)
……

"Knowledge Base"
"Robot Brain"

The Real World



"The semantic wall"

Example model

- Objects: John, James, Marry, Alex, Dan, Joe, Anne, Rich
- Relation: sets of tuples of objects
  {<John, James>, <Marry, Alex>, <Marry, James>, ...}
  {<Dan, Joe>, <Anne, Marry>, <Marry, Joe>, ...}
- E.g.:
  Parent relation -- {<John, James>, <Marry, Alex>, <Marry, James>}

then the predicate Parent(John, James) is true
the predicate Parent(John, Marry) is false

Mind

Quantifiers

- Expressing sentences about **collections** of objects without enumeration (naming individuals)

- E.g., All Trojans are clever

  Someone in the class is sleeping

- Universal quantification (for all): ∀

- Existential quantification (there exists): ∃

Universal quantification (for all): $\forall$

$\forall$ *<variables> <sentence>*

- *"Everyone in the* CS561 *class is smart"*:
  $\forall x \;\; In(x, CS561) \Rightarrow Smart(x)$

- **$\forall$ P corresponds to the conjunction of instantiations of P**
  (In(Manos, CS561) $\Rightarrow$ Smart(Manos)) $\wedge$
  (In(Dan, CS561) $\Rightarrow$ Smart(Dan)) $\wedge$
  …
  (In(Bush, CS561) $\Rightarrow$ Smart(Bush))

Universal quantification (for all): ∀

- ⇒ is a natural connective to use with ∀

- Common mistake: to use ∧ in conjunction with ∀
  e.g.: $\forall x \ \ \text{In}(x, \text{CS561}) \land \text{Smart}(x)$
  means *"everyone is in* CS56*1 and everyone is smart"*

Existential quantification (there exists): ∃

∃ *<variables> <sentence>*

- *"Someone in the CS561 class is smart"*:
  ∃ x In(x, CS561) ∧ Smart(x)

- **∃ P corresponds to the disjunction of instantiations of P**
  (In(Manos, CS561) ∧ Smart(Manos)) ∨
  (In(Dan, CS561) ∧ Smart(Dan)) ∨
  …
  (In(Bush, CS561) ∧ Smart(Bush))

Existential quantification (there exists): ∃

- $\wedge$ is a natural connective to use with ∃

- Common mistake: to use $\Rightarrow$ in conjunction with ∃
  e.g: $\exists\, x \;\; In(x, CS561) \Rightarrow Smart(x)$
  is true if there is anyone that is not in CS561! // $\sim In(x,CS561) \vee Smart(x)$

  remember:

  false $\Rightarrow$ true is valid

  P=>Q is the same as $\sim P \vee Q$

Properties of quantifiers

$\forall x\ \forall y$ is the same as $\forall y\ \forall x$ (why??)

$\exists x\ \exists y$ is the same as $\exists y\ \exists x$ (why??)

$\exists x\ \forall y$ is not the same as $\forall y\ \exists x$

$\exists x\ \forall y\ \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y\ \exists x\ \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Not "all loved by one person" but "each is loved by at least one"

Quantifier duality: each can be expressed using the other

$\forall x\ \ Likes(x, IceCream) \qquad \neg\exists x\ \ \neg Likes(x, IceCream)$

$\exists x\ \ Likes(x, Broccoli) \qquad \neg\forall x\ \ \neg Likes(x, Broccoli)$

Proof?

Proof

Variable: x
Constants: $X_1, X_2, \ldots X_n$

In general, we want to prove:

$\forall x \; P(x) \Leftrightarrow \neg\exists x \; \neg P(x)$

- $\forall x \; P(x) = \neg(\neg(\forall x \; P(x))) = \neg(\neg(P(X_1) \wedge P(X_2) \wedge \ldots \wedge P(X_n)))$
  $= \neg(\neg P(X_1) \vee \neg P(X_2) \vee \ldots \vee \neg P(X_n)))$
- $\exists x \; \neg P(x) = \neg P(X_1) \vee \neg P(X_2) \vee \ldots \vee \neg P(X_n)$
- $\neg\exists x \; \neg P(x) = \neg(\neg P(X_1) \vee \neg P(X_2) \vee \ldots \vee \neg P(X_n))$

Example sentences

- Brothers are siblings

  ▪

- Sibling is transitive

  ▪

- One’s mother is one’s sibling’s mother

  ▪

- A first cousin is a child of a parent’s sibling

  ▪

Example sentences

- Brothers are siblings

  $\forall\, x, y \quad Brother(x, y) \Rightarrow Sibling(x, y)$

- Sibling is transitive

  $\forall\, x, y, z \quad Sibling(x, y) \land Sibling(y, z) \Rightarrow Sibling(x, z)$

- One’s mother is one’s sibling’s mother

  $\forall\, m, c, d \quad Mother(m, c) \land Sibling(c, d) \Rightarrow Mother(m, d)$

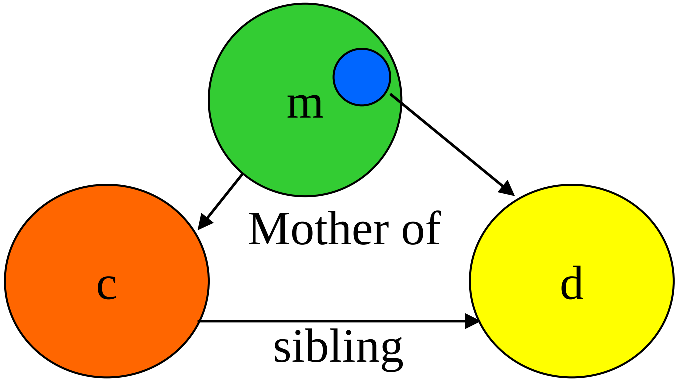
- A first cousin is a child of a parent’s sibling

  $\forall\, c, d \quad FirstCousin(c, d) \Leftrightarrow \exists\, p, ps \;\; Parent(p, d) \land Sibling(p, ps) \land Child(c, ps)$

Example sentences

- One’s mother is one’s sibling’s mother
  $\forall$ m, c, d   Mother(m, c) $\wedge$ Sibling(c, d) $\Rightarrow$ Mother(m, d)

- $\forall$ c, d **$\exists$m** Mother(m, c) $\wedge$ Sibling(c, d) $\Rightarrow$ Mother(m, d)



Translating English to FOL

- Every gardener likes the sun.
  $\forall$ x gardener(x) => likes(x,Sun)

- You can fool some of the people all of the time.
  $\exists$ x $\forall$ t (person(x) ^ time(t)) => can-fool(x,t)

Translating English to FOL

- You can fool all of the people some of the time.

  `∀x (person(x) => ∃t (time(t) ^ can-fool(x,t)))`

- All purple mushrooms are poisonous.

  `∀x (mushroom(x) ^ purple(x) => poisonous(x))`

Caution with nested quantifiers

- $\forall x\ \exists y\ P(x,y)$ is the same as $\forall x\ (\exists y\ P(x,y))$ which means
  "for every x, it is true that there exists y such that P(x,y)"

- $\exists y\ \forall x\ P(x,y)$ is the same as $\exists y\ (\forall x\ P(x,y))$ which means
  "there exists a y, such that it is true that for every x P(x,y)"

Translating English to FOL…

- No purple mushroom is poisonous.

```
¬ ∃x ( purple(x) ^ mushroom(x) ^ poisonous(x))
```

or, equivalently,

```
∀x ((mushroom(x) ^ purple(x)) => ¬poisonous(x))
```

Translating English to FOL…

- There are exactly two purple mushrooms.

$\exists x\ \exists y\ mushroom(x) \wedge purple(x) \wedge mushroom(y) \wedge purple(y) \wedge \neg(x=y) \wedge \forall z((mushroom(z) \wedge purple(z)) \Rightarrow ((x=z) \vee (y=z)))$

- Deb is not tall.

$\neg tall(Deb)$

Translating English to FOL…

- X is above Y iff X is directly on top of Y or else there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.

```
(∀ x)(∀ y) above(x,y) <=> (on(x,y) v (∃ z)
  (on(x,z) ^ above(z,y)))
```

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge$$
$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Higher-Order Logic?

- First-order logic allows us to quantify over objects (= the first-order entities that exist in the world)

- Higher-order logic also allows quantification over relations and functions.
  e.g., "two objects are equal iff all properties applied to them are equivalent":

  $$\forall\, x,y \quad (x=y) \Leftrightarrow (\forall\, p,\ p(x) \Leftrightarrow p(y))$$

- Higher-order logics are more expressive than first-order; however, so far we have little understanding on how to effectively reason with sentences in higher-order logic

Logical Agents for the Wumpus World

Remember: generic knowledge-based agent:

```
function KB-AGENT(percept) returns an action
    static: KB, a knowledge base
            t, a counter, initially 0, indicating time

    TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
    action ← ASK(KB, MAKE-ACTION-QUERY(t))
    TELL(KB, MAKE-ACTION-SENTENCE(action, t))
    t ← t + 1
    return action
```

1. TELL KB what was perceived
   Uses a KRL to insert new sentences, representations of facts, into KB

2. ASK KB what to do
   Uses logical reasoning to examine actions and select the best

Using the FOL Knowledge Base

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$\text{TELL}(KB, Percept([Smell, Breeze, None], 5))$
$\text{ASK}(KB, \exists a \;\; Action(a, 5))$

I.e., does the KB entail any particular actions at $t = 5$?

Answer: $Yes, \;\; \{a/Shoot\}$ $\leftarrow$ substitution (binding list) // set of solutions

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$\text{ASK}(\text{KB}, \text{S})$ returns some/all σ such that $KB \models S\sigma$

Wumpus World, FOL Knowledge Base

"Perception"

$\forall b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$

$\forall s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

Deducing Hidden Properties

Properties of locations:

$\forall l, t \;\; At(Agent, l, t) \land Smelt(t) \;\Rightarrow\; Smelly(l)$

$\forall l, t \;\; At(Agent, l, t) \land Breeze(t) \;\Rightarrow\; Breezy(l)$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall y \;\; Breezy(y) \;\Rightarrow\; \exists x \;\; Pit(x) \land Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y \;\; Pit(x) \land Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:

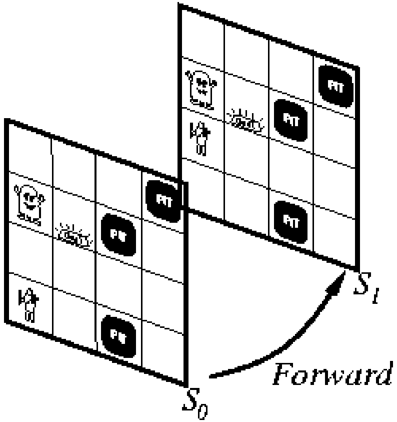
$$\forall y \;\; Breezy(y) \;\Leftrightarrow\; [\exists x \;\; Pit(x) \land Adjacent(x, y)]$$

Situation Calculus

Facts hold in situations, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:
Adds a situation argument to each non-eternal predicate
E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s



Describing Actions (in Situation Calculus)

"Effect" axiom—describe changes due to action

$\forall s \;\; AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe non-changes due to action

$\forall s \;\; HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$

Frame problem: find an elegant way to handle non-change

(a) representation—avoid frame axioms

(b) inference—avoid repeated "copy-overs" to keep track of state

// May result in too many frame axioms

Challenges

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

Describing Actions (cont'd)

Successor-state axioms solve the representational frame problem

Each axiom is "about" a predicate (not an action per se):

P true afterwards $\Leftrightarrow$ [an action made P true

$\lor$ P true already and no action made P false]

For holding the gold:

$$\forall a, s \;\; Holding(Gold, Result(a, s)) \;\Leftrightarrow$$
$$[(a = Grab \land AtGold(s))$$
$$\lor\ (Holding(Gold, s) \land a \neq Release)]$$

Planning (by Situation Calculus)

Initial condition in KB:

$At(Agent, [1,1], S_0)$
$At(Gold, [1,2], S_0)$

Query: $\text{ASK}(KB, \exists s \;\; Holding(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Generating Action Sequences

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $\text{Ask}(KB, \exists p \;\; Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$\forall s \;\; PlanResult([], s) = s$ [ ] = empty plan

$\forall a, p, s \;\; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Recursively continue until it gets to empty plan [ ]

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB