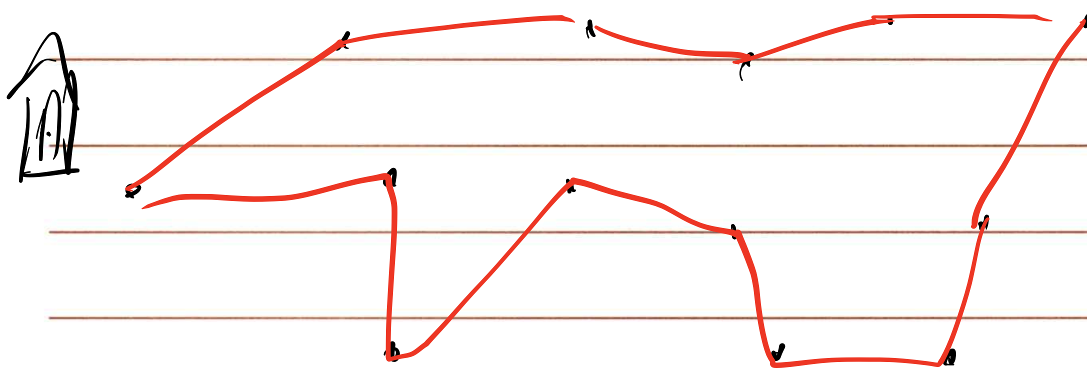
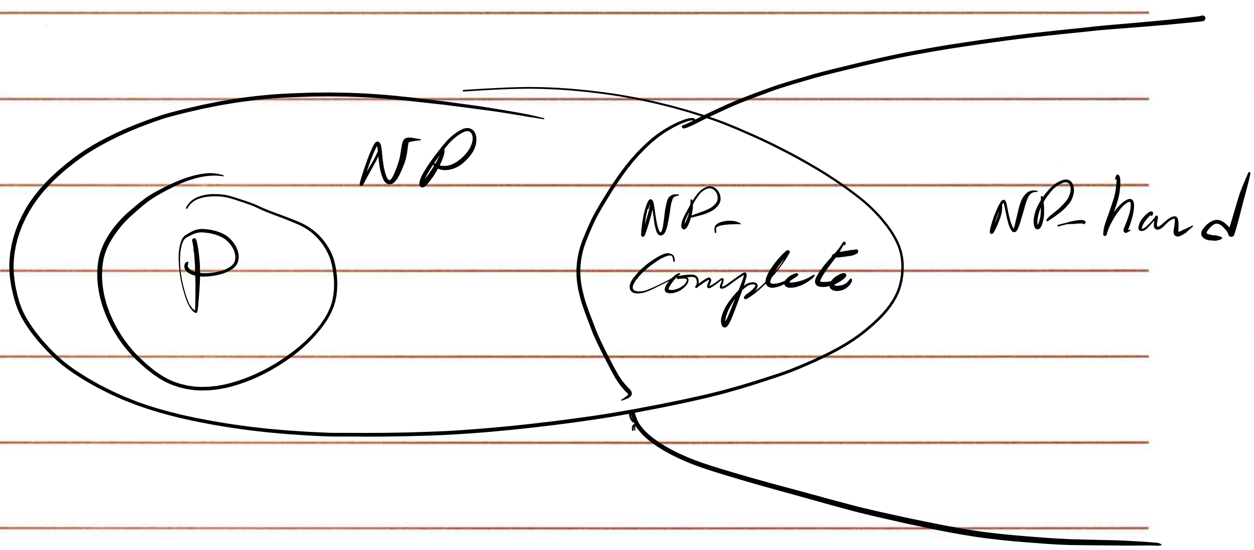
# Traveling Salesman Problem (TSP)
# &
# Hamiltonian Cycle





fully connected graph

## Problem Statement

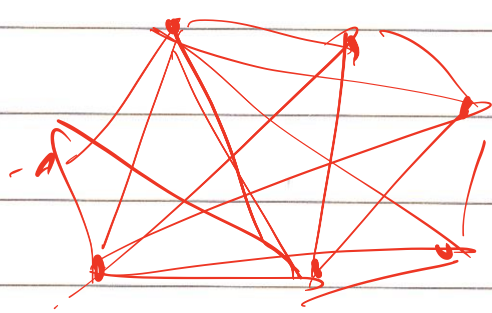
Given the set of distances, order $n$ cities in a tour $V_{i_1}, V_{i_2}, \ldots V_{i_n}$ with $i_1 = 1$, so it minimizes

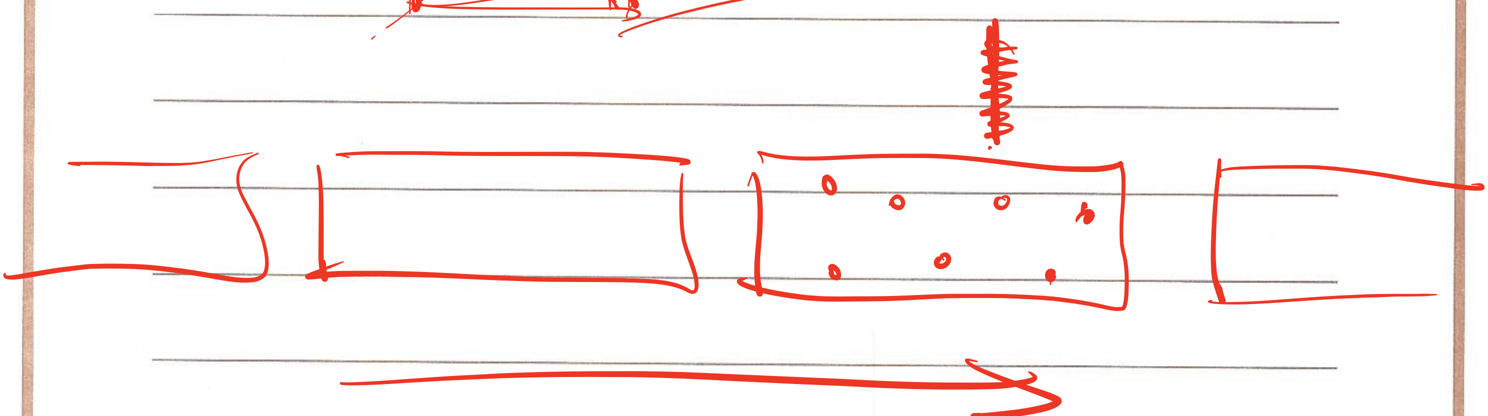
opt. versio

$$\sum d(V_{i_j}, V_{i_{j+1}}) + d(V_{i_n}, V_{i_1})$$

Vanilla

Choc-





## Decision version of TSP:

Given a set of distances on $n$ cities and a bound $D$, is there a tour of length/cost at most $D$?

**Def.** A cycle $C$ in $G$ is a Hamiltonian Cycle, if it visits each vertex exactly once.

**Problem Statement:**

Given an undirected graph $G$, is there a Hamiltonian cycle in $G$?

Show that the Hamiltonian Cycle Problem is NP-complete

1- Show that HC is in NP

a. polyn. length certificate

- ordered list of nodes on the HC.

b. polyn. time certifier

- Check that every node is in the list & no node is repeated
- Check that there is an edge between
  - every pair of adjacent nodes in the list
  - last node & first node.

2. Choose a problem that we already know is NP-Complete

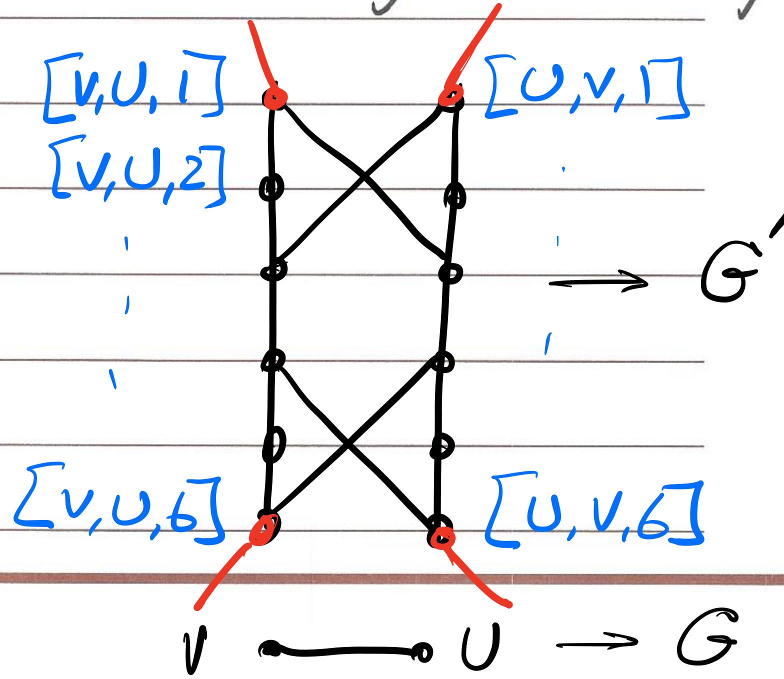
Choose vertex Cover

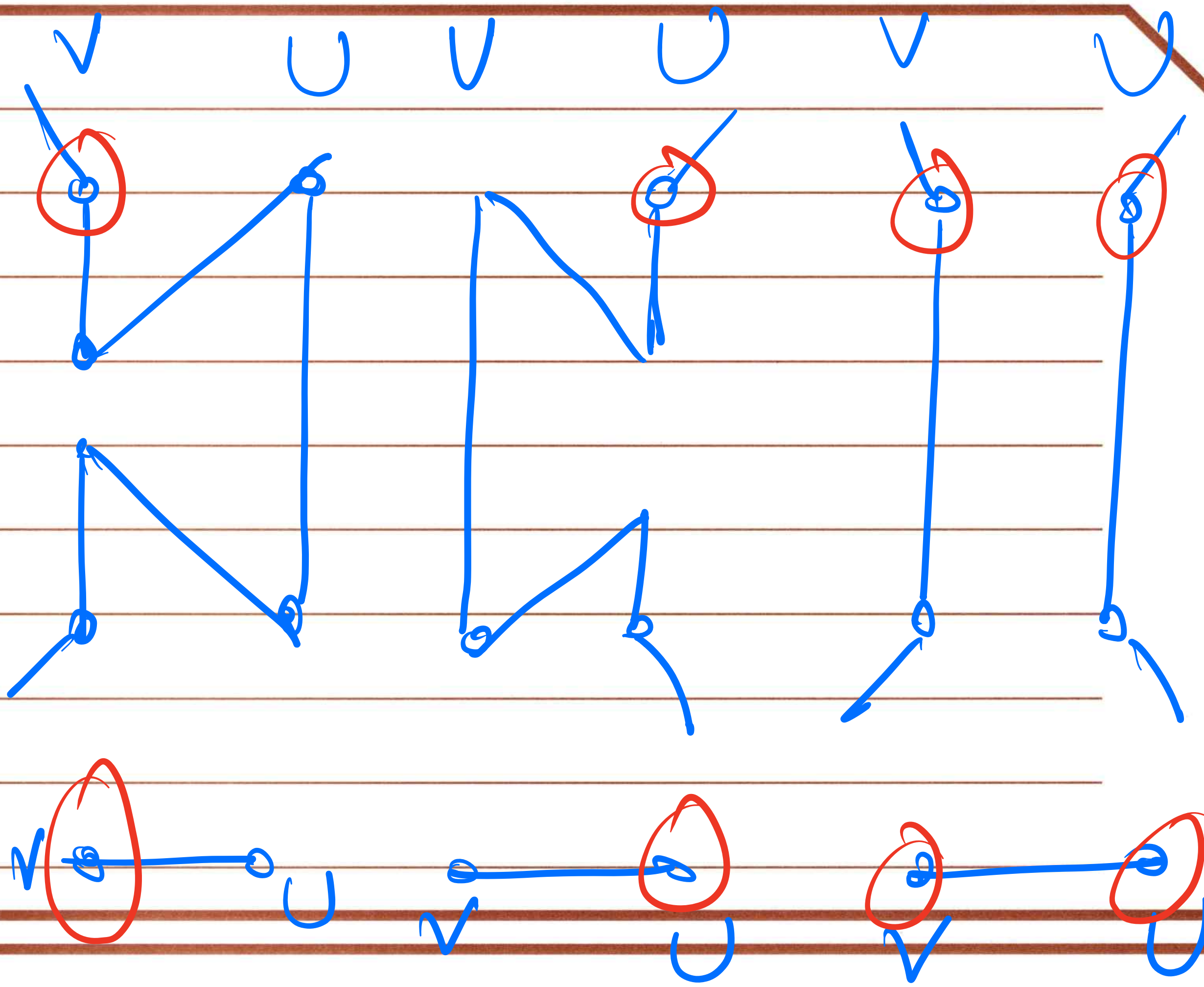
3. Show vertex Cover $\le_p$ HC

Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff $G$ has a vertex cover of size at most $\underline{k}$.

<u>Construction of $G'$</u>

For each edge $(V, U)$ in $G$, $G'$ will have one gadget $W_{VU}$ with following node labeling:



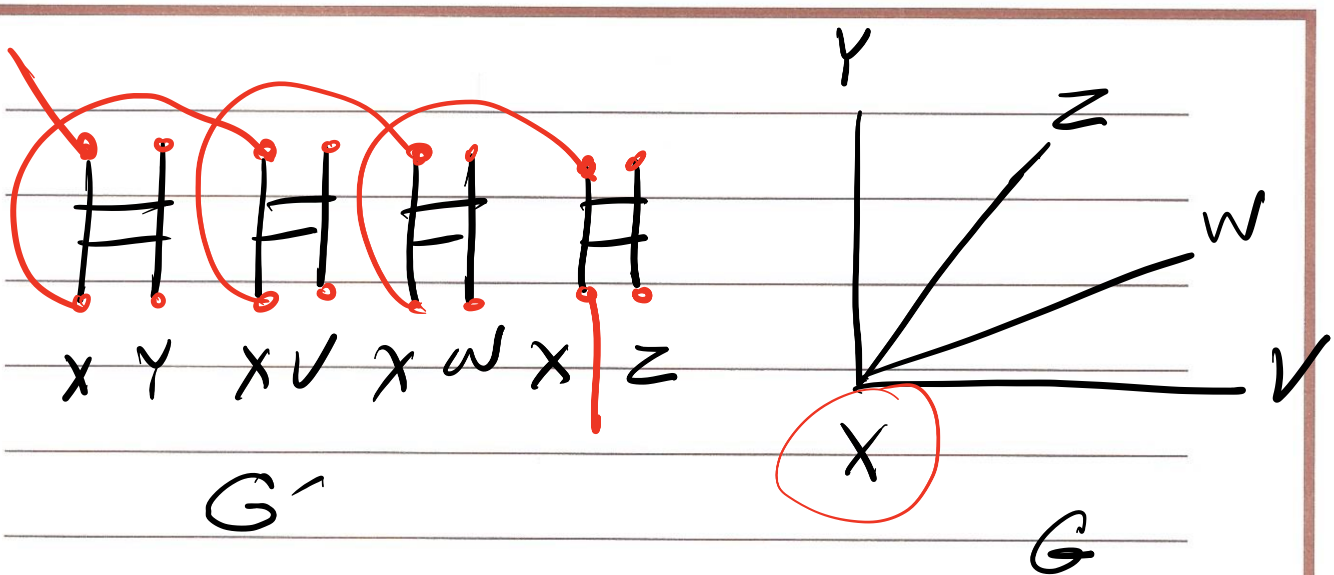


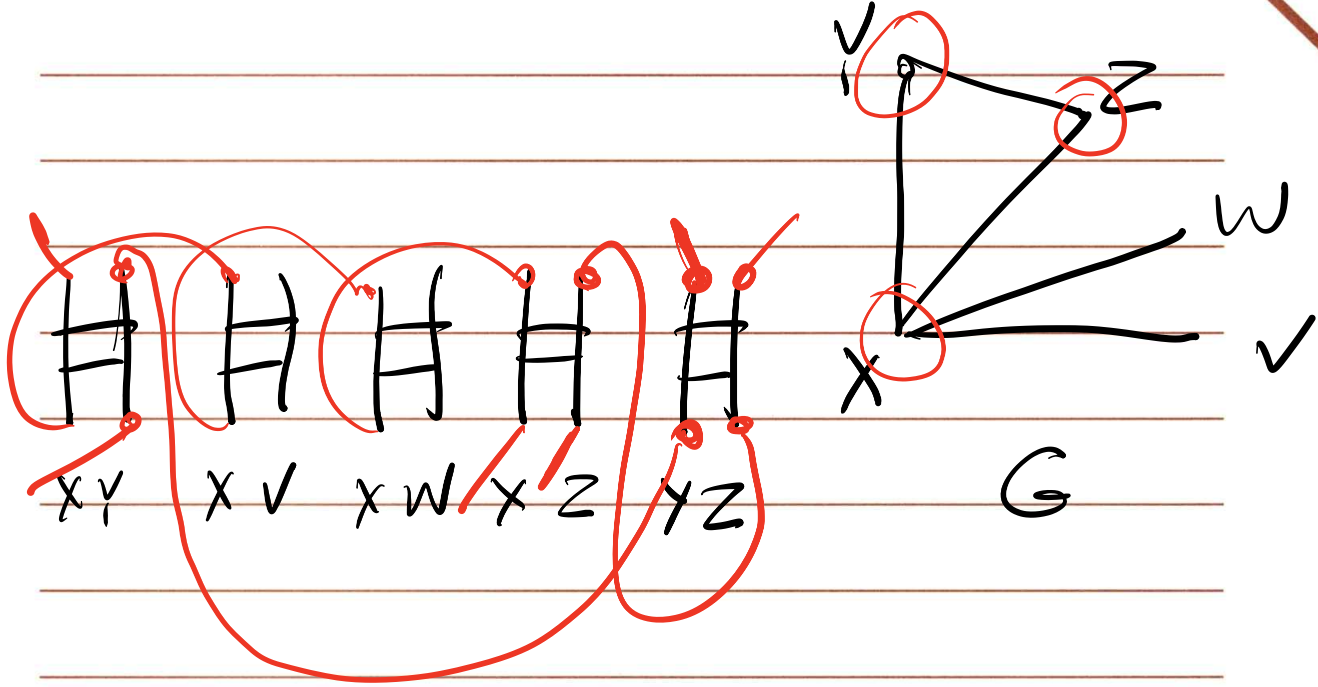
Other vertices in G′

- Selector vertices: There are k selector vertices in G′, $s_1, \dots, s_k$

Other edges in G′

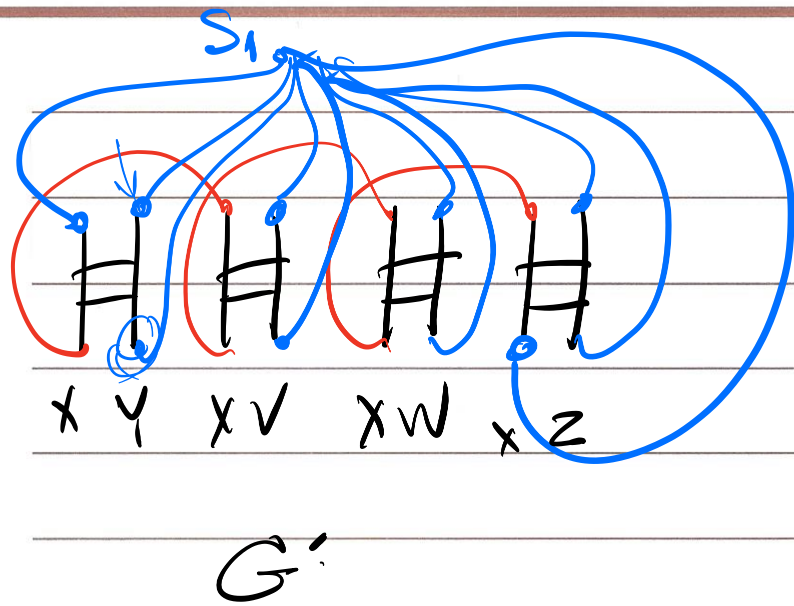
1. For each vertex $v \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on $v$ in G.

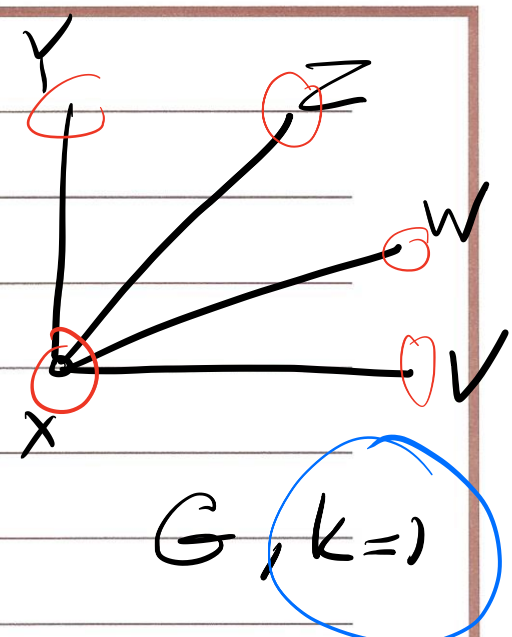


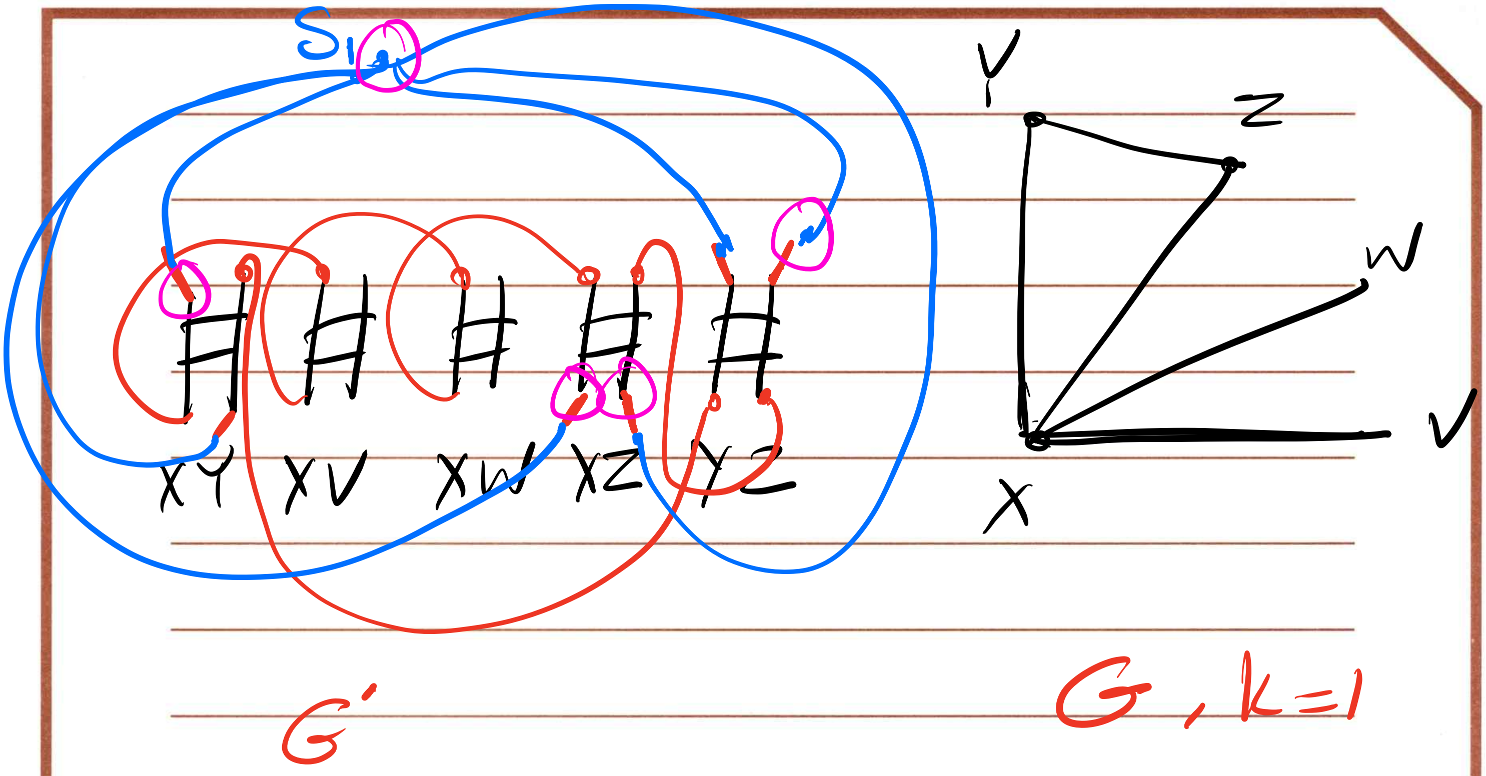


XY
XV
XW
XZ
YZ
Y
Z
W
V
X
G

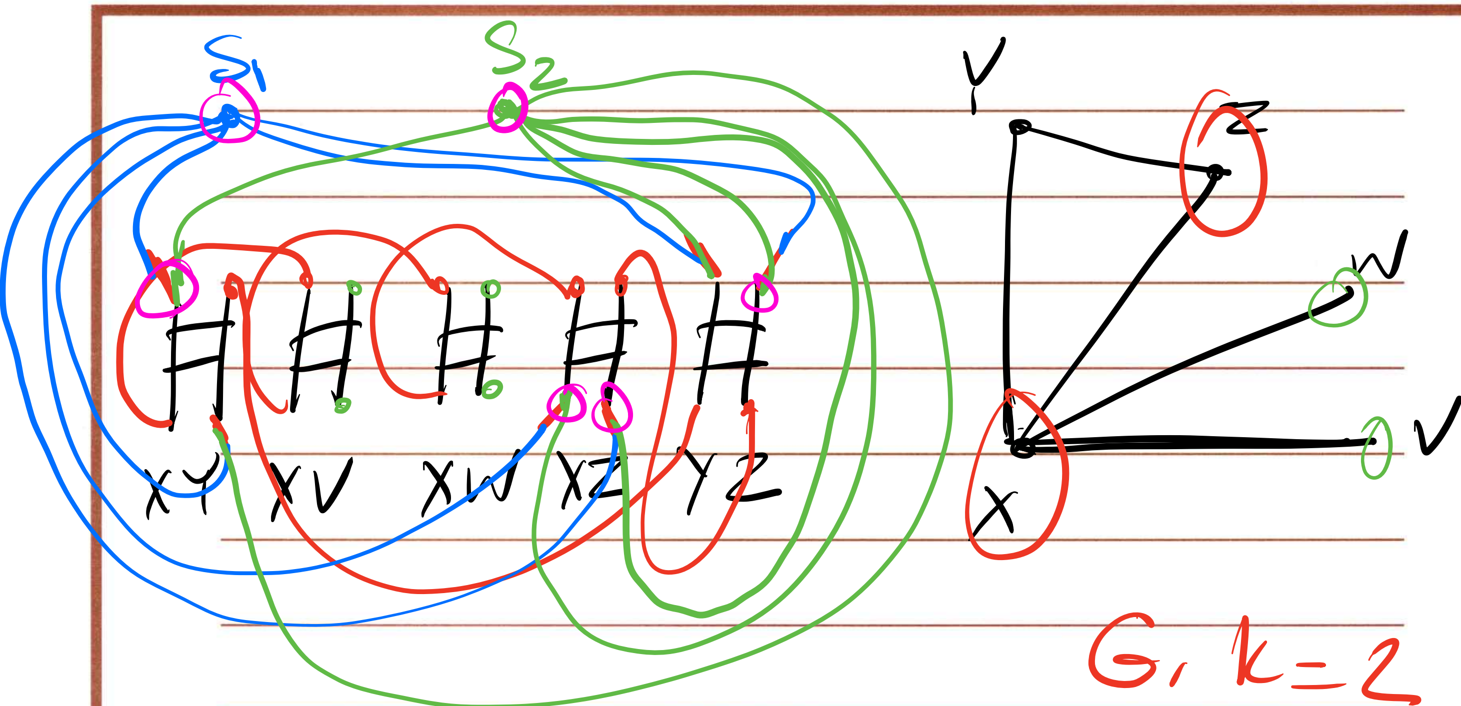
2- Final set of edges in G' join the first vertex [X, Y, 1] and last vertex [X, Y(deg(x)), 6] of each of these paths to each of the selector vertices.



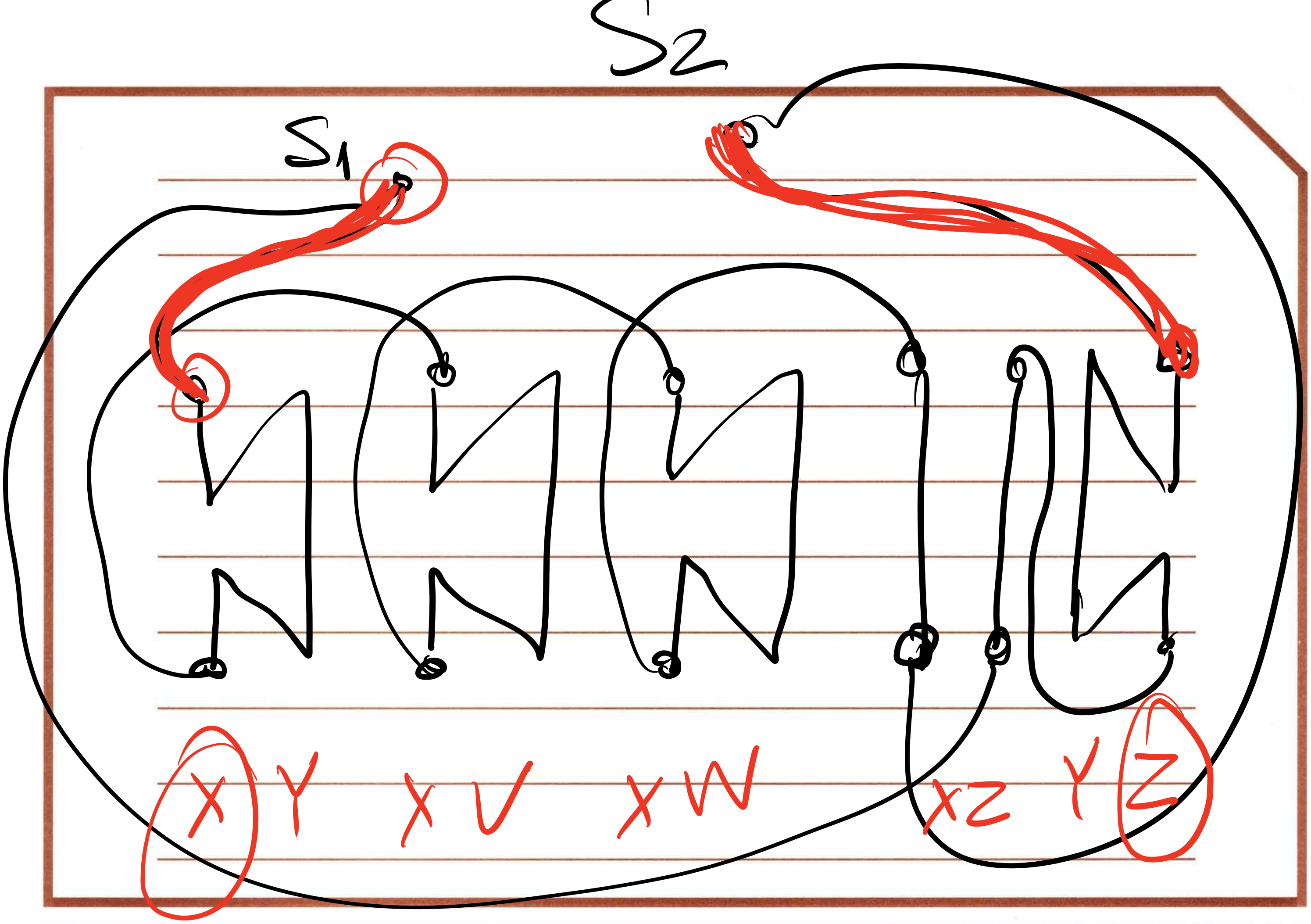





S1
Y
Z
W
V
X
XY
XV
XW
XZ
YZ
G'
G, k=1

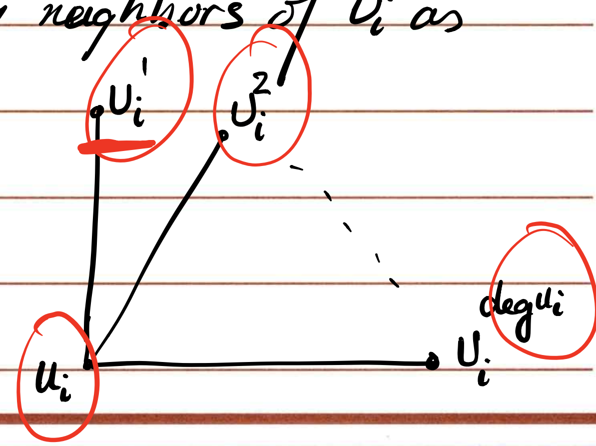
S1
S2
Y
Z
W
V
X
XY
XV
XW
XZ
YZ
G, k=2


S2
S1
X Y
X U
X W
X Z
Y Z

Proof: A) Suppose that $G = (V, E)$ has a vertex cover of size $k$. Let the vertex cover set be

$$S = \{u_1, u_2, \ldots, u_k\}$$

We will identify neighbors of $u_i$ as shown here:




Form a Ham. Cycle in $G'$ by following the nodes in $G$ in this order:

start at $s_1$ and go to

$$[u_1, u_1^1, 1] \; \ldots \; [u_1, u_1^1, 6]$$
$$[u_1, u_1^2, 1] \; \ldots \; [u_1, u_1^2, 6]$$
$$\vdots$$
$$[u_1, u_1^{\deg u_1}, 1] \; \ldots \; [u_1, u_1^{\deg u_1}, 6]$$

Then go to $S_2$ and follow the nodes

$[U_2, U_2^1, 1]$ . . . . . $[U_2, U_2^1, 6]$

$[U_2, U_2^2, 1]$ . . . . $[U_2, U_2^2, 6]$

$\vdots$

$[U_2, U_2^{\deg U_2}, 1]$ . . . $[U_2, U_2^{\deg U_2}, 6]$

Then go to $S_3$ . . . .

$\vdots$

$\vdots$

go to $S_k$

$[U_k, U_k^1, 1]$ . . . $[U_k, U_k^1, 6]$

$[U_k, U_k^2, 1]$ . . . $[U_k, U_k^2, 6]$

$\vdots$

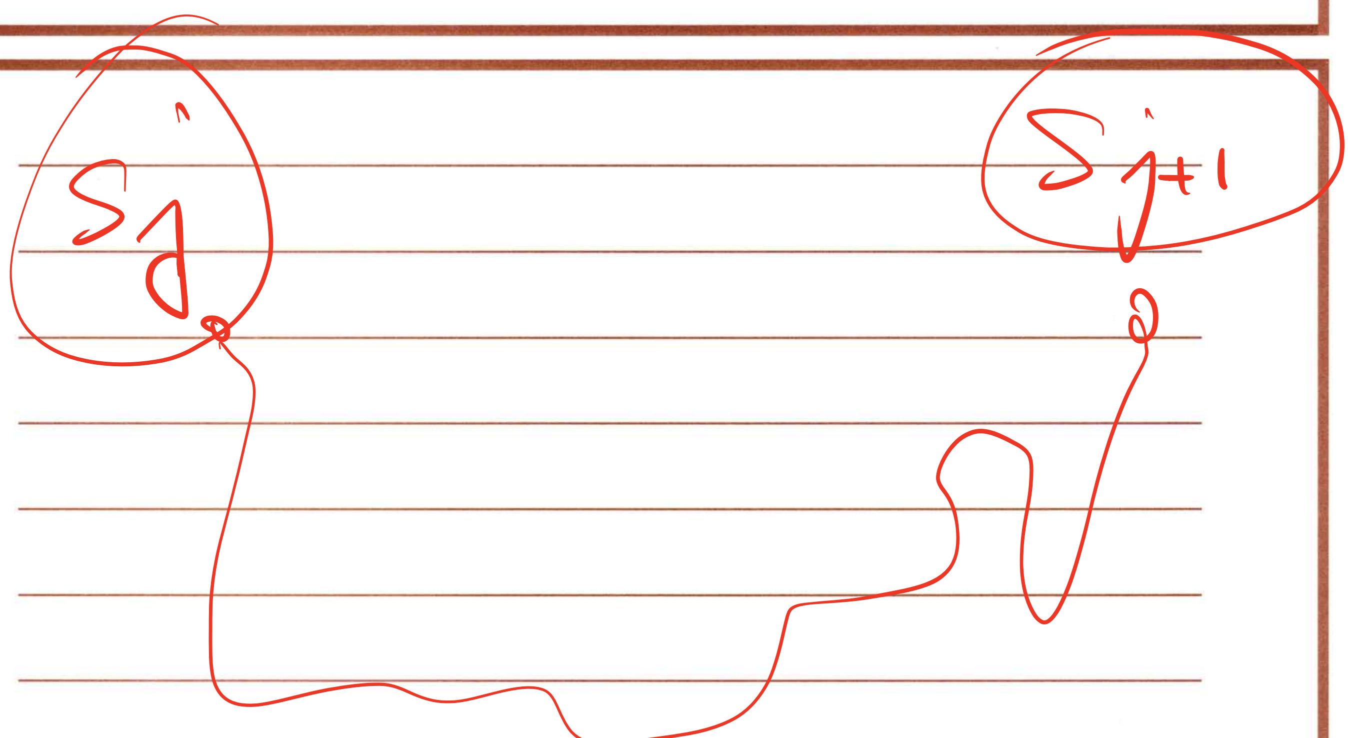
$[U_k, U_k^{\deg U_k}, 1]$ . . . $[U_k, U_k^{\deg U_k}, 6]$

Then return back to $S_1$.

B) Suppose $G'$ has a Hamiltonian cycle $C$, then the set

$$S = \{ u_j \in V : (s_j, [u_j, u_j', 1]) \in C \text{ for some } 1 \le j \le k \}$$

will be a vertex cover set in $G$.



We Prove that TSP is NP-Complete

1- Show that TSP $\in$ NP

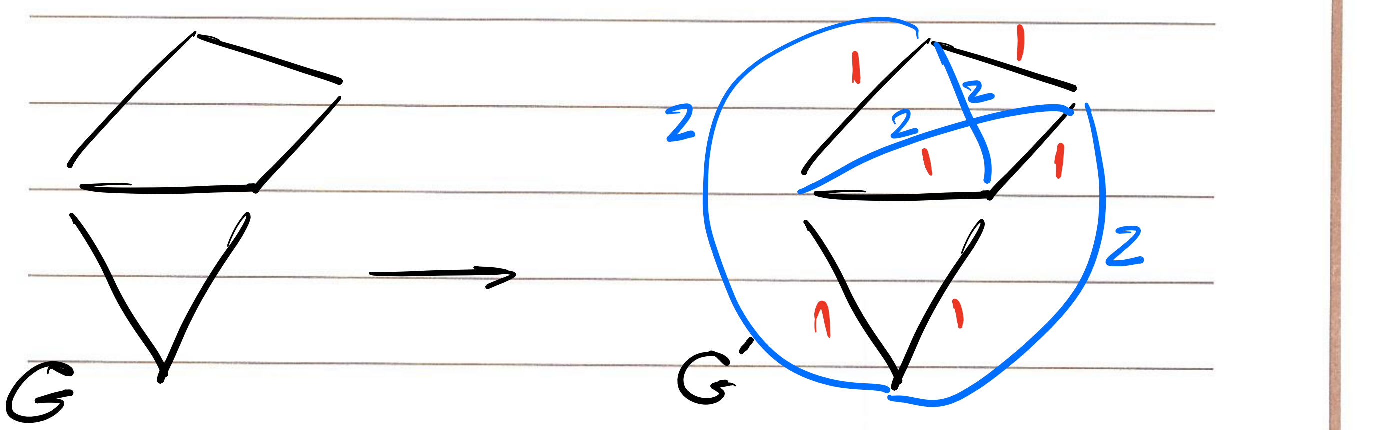
a. Certificate
a tour of cost at most $\underline{D}$.

b. certifier:

- all checks we did for HC
+ check that cost of tour $\leq D$

2- Choose an NP-Complete problem:
Hamiltonian Cycle.

3- Prove that Ham. Cycle $\leq_p$ TSP




$n$ = no. of nodes in $G$.

if there is a HC in G

→ we can have a tour of
Cost 1 in G'

if there is a tour of Cost 2
in G' → there is a HC
in G.

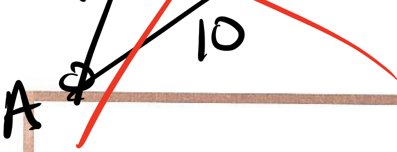
3SAT, indepSet, vertexCover,
set Cover, Ham. Cycle, TSP
0-1 knapsack, subsetsum

Given a set of $n$ items where item $i$ has weight $w_i$ and value $v_i$. is there a subset of these items with total weight $\leq W$ and total value $\geq V$

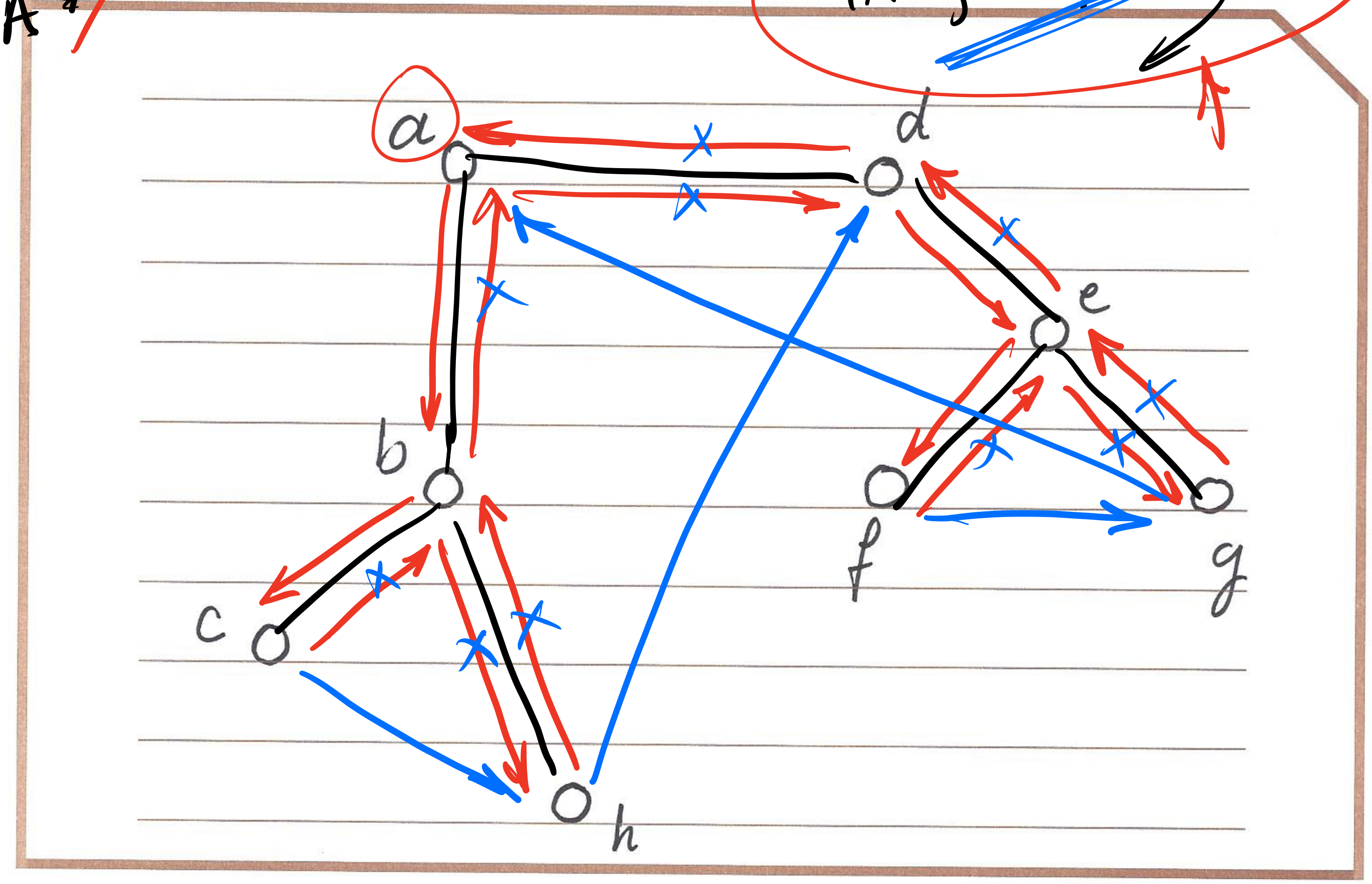
Decision version of 0-1 knapsack

B

ites




Triangle inequa




initial tour has cost = Cost of MST * 2

Cost of our approx. sol. $\leq$ 2 * Cost of MST

Cost of MST < Cost of opt TS tour

Cost of our approx. sol. < <u>2</u> * Cost of opt. tour

This is a 2-approximation alg.

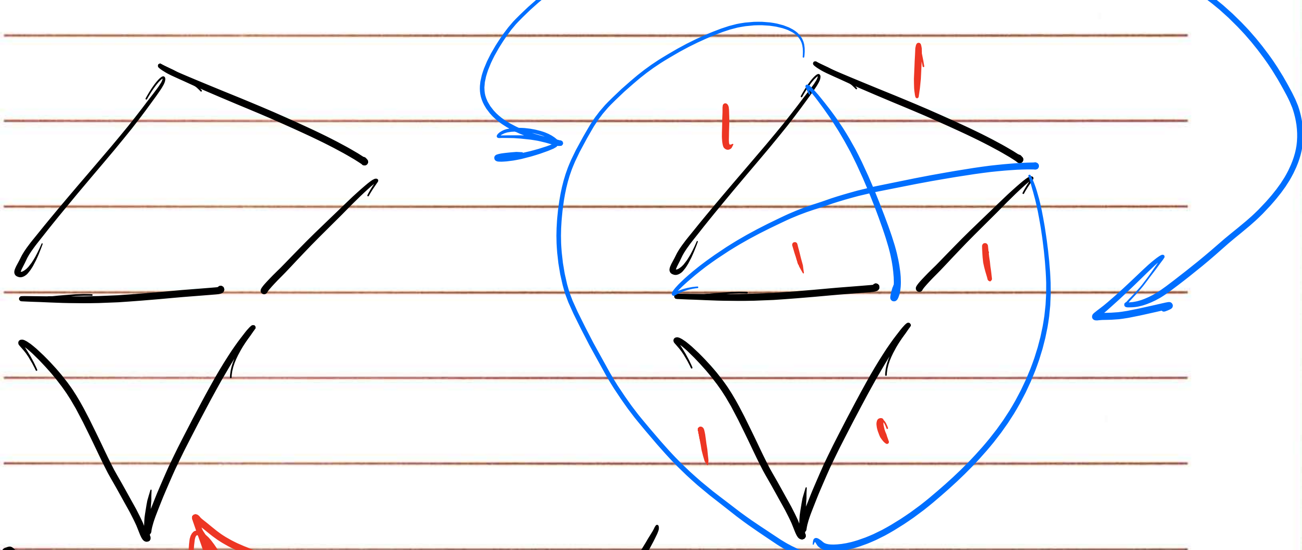
# General TSP

Theorem: if $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio $\rho$ for the general TSP

Plan: We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Given an instance of the HC problem on graph G, we will construct G' as follows:

- G' has the same set nodes as in G
- G' is a fully connected graph.
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$




G

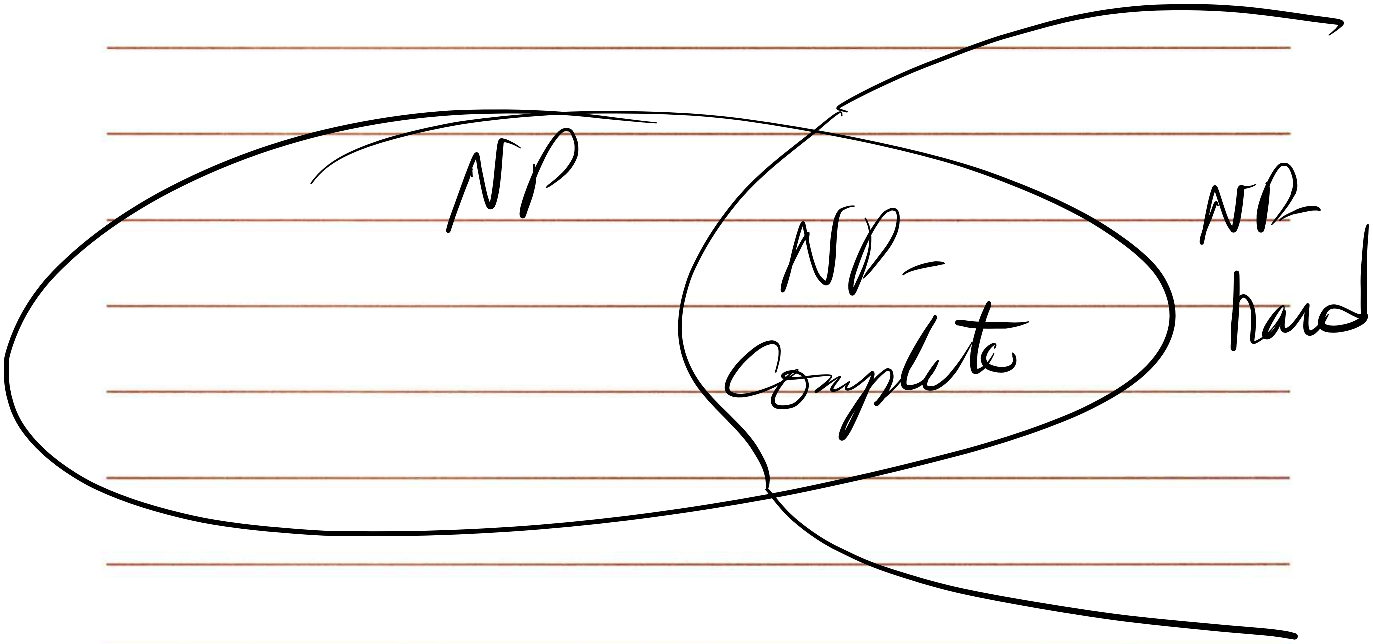
G'

- if G has a HC then G' will have a tour of cost $|V|$

if G' has a tour of cost $< f|V|$

$\longrightarrow$

G must have a HC.


NP
NP-
Complete
NP-
hard

# Discussion 11

**1.** In the *Min-Cost Fast Path* problem, we are given a directed graph *G=(V,E)* along with positive integer times $t_e$ and positive costs $c_e$ on each edge. The goal is to determine if there is a path *P* from *s* to *t* such that the total time on the path is at most *T* and the total cost is at most *C* (both *T* and *C* are parameters to the problem). Prove that this problem is **NP**-complete.

**2.** We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

**3.** Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

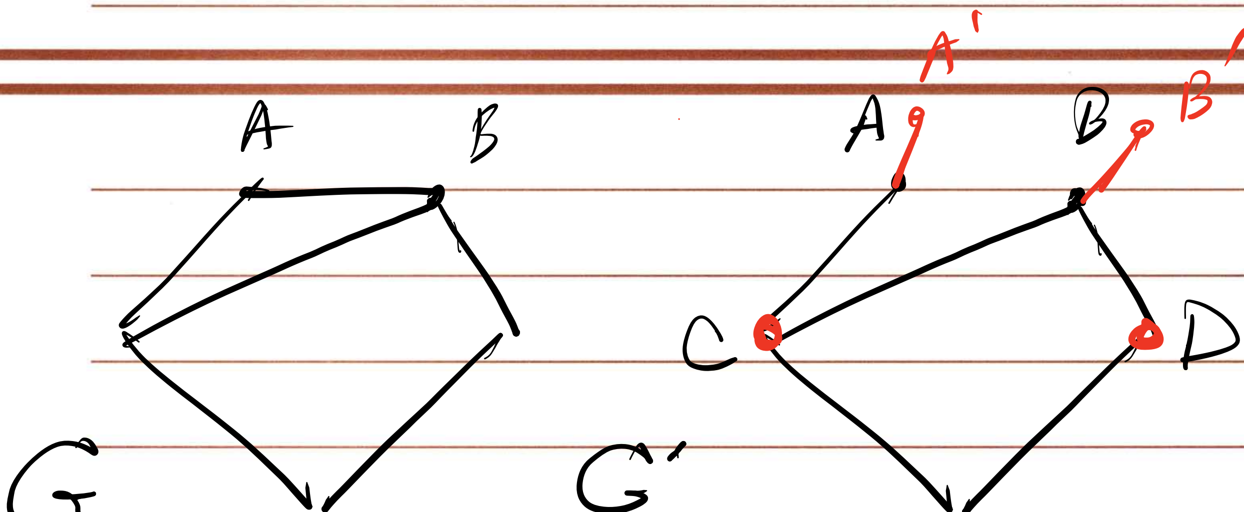
**2.** We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

no credit

1- Skip

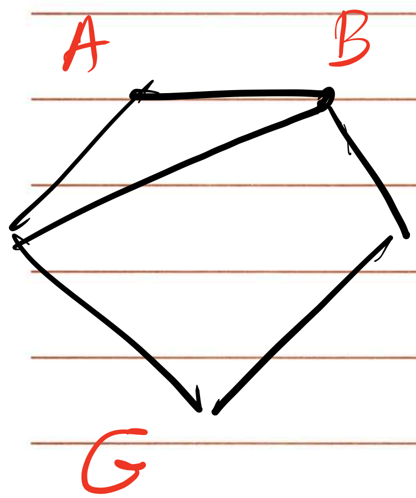
2- Choose HC.

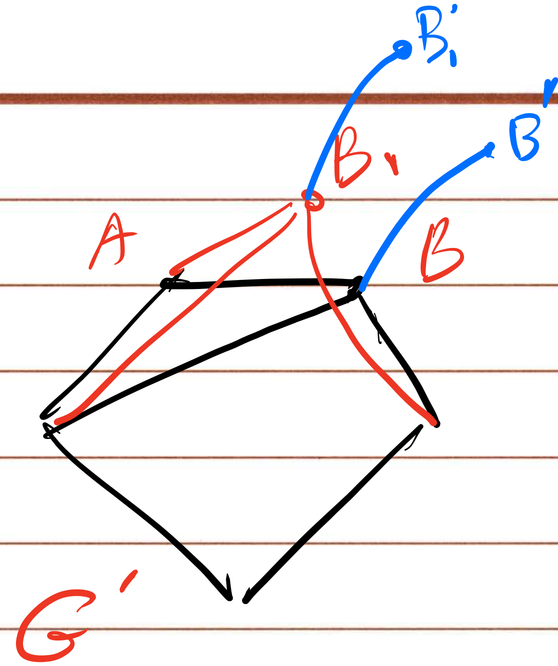
3- $HC \le_p HP$




repeat for every edge in G.

1




2

**1.** In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times $t_e$ and positive costs $c_e$ on each edge. The goal is to determine if there is a path $P$ from $s$ to $t$ such that the total time on the path is at most $T$ and the total cost is at most $C$ (both $T$ and $C$ are parameters to the problem). Prove that this problem is **NP**-complete.

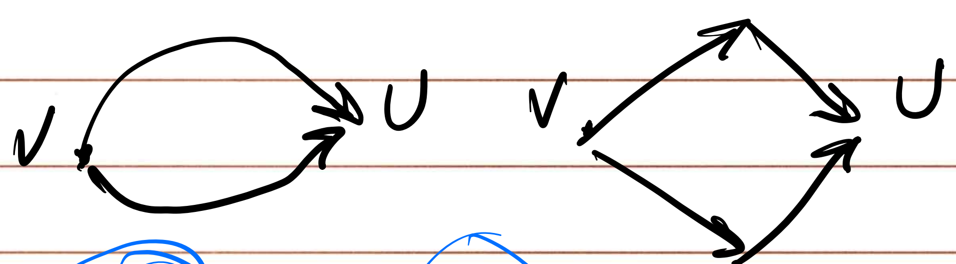
1- Skip

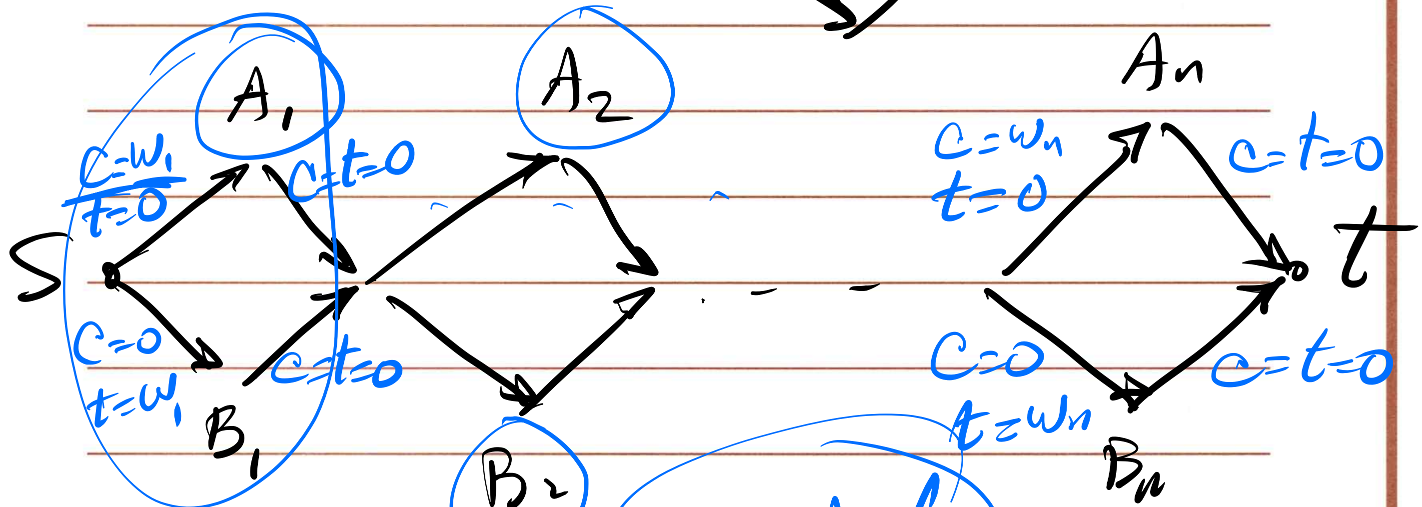
2- Choose SubsetSum

3- Subset sum $\leq_p$ MCFP

Given a set of items where item $i$ has weight $w_i$ if there a subset of them with total weight $\leq W$

& " " " $\geq M$




$C = W$

$T = \sum w_i - M$

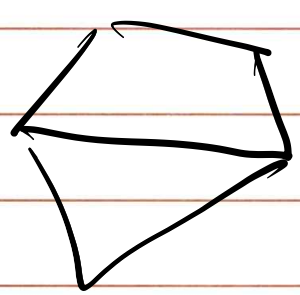
M

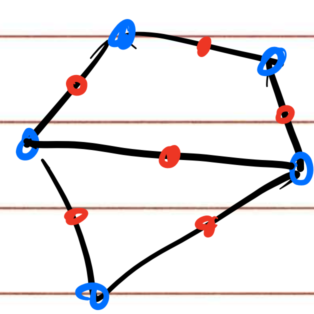
total weight of object

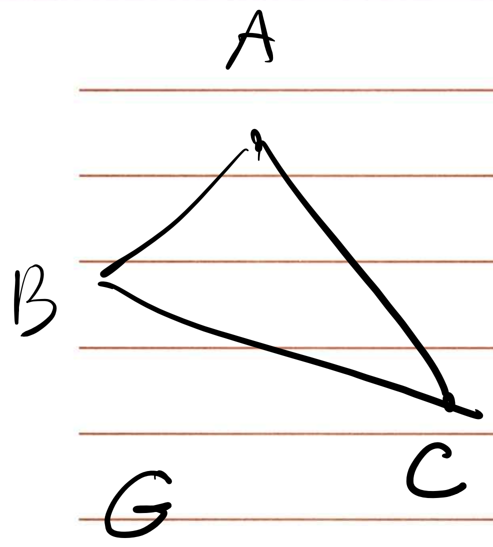
return to its starting point -- is also **NP**-complete.

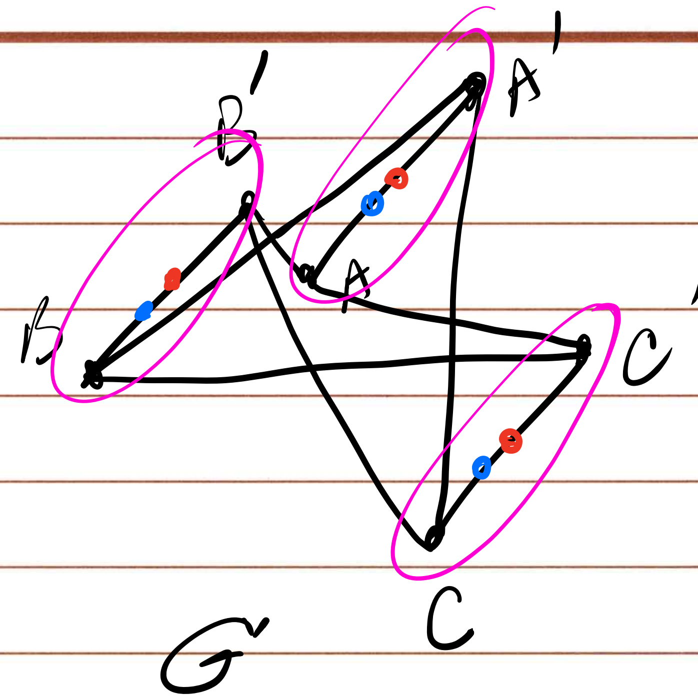
**3.** Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

1- Skip
2- choose HC.
3- HC $\leq_p$ HC in a Bipartite graph


G


G'



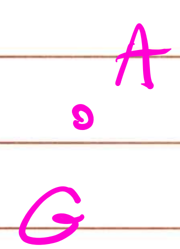

$A\,B\,C\,A$ $\qquad$ $A\,A'\,B\,B'\,C\,C'\,A$

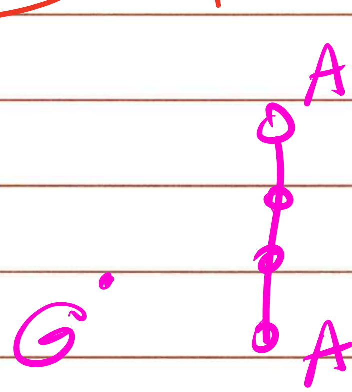
$B\,C\,A\,B$ $\qquad$ $B\,B'\,C\,C'\,A\,A'\,B$

?

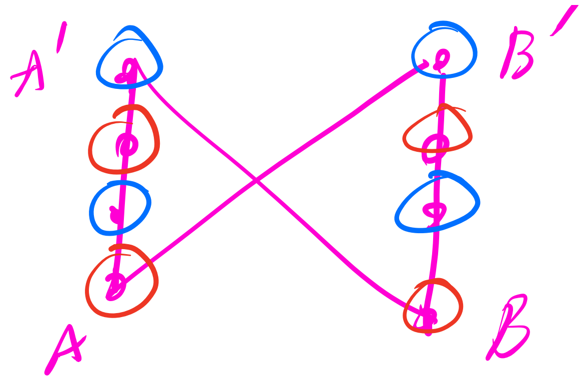
$B\,C'\,A\,B'\,C \cdots B$

node in G




A
B


A'
B'
A
B