MATH475 Weekly Work 6

Mihail Anghelici 260928404

October 19, 2020

Question 1

Let

$$\begin{aligned} \varphi(x,y) &:= \Phi(\tilde{x}-y) && (1)\\ &:= \Phi(y-\tilde{x}). && (2)\end{aligned}$$

We verify (1) using

$$\begin{cases} -\Delta_y \varphi(x,y) = 0 & \text{in } \mathbb{R}^3_+ := \{(y_1, y_2 \in \mathbb{R}, y_3 \in (0,\infty))\},\\ \varphi(x,\sigma) = \Phi(x-\sigma) = \frac{1}{4\pi|x-\sigma|} & \text{on } \partial\mathbb{R}^3_+ := \{(y_1,y_2,0) : y_1, y_2 \in \mathbb{R}\}.\end{cases}$$

We compute the gradient then the divergence and show it is equal to 0.

$$\begin{aligned}
\nabla\varphi(x,y) &= \nabla\left(\frac{1}{4\pi}\frac{1}{|(x_1,x_2,-x_3)-(y_1,y_2,y_3)|}\right)\\
&= \nabla\left(\frac{1}{4\pi}\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(-x_3-y_3)^2}}\right) \qquad (3)\\
&= \frac{1}{4\pi}\left(\frac{-(x_1-y_1)}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{3/2}}\right.\\
&\qquad ,\frac{-(x_2-y_2)}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{3/2}}\\
&\qquad \left.,\frac{(x_3+y_3)}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{3/2}}\right)\\
\nabla\cdot(\nabla\varphi(x,y)) &= \frac{1}{4\pi}\left(\frac{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)-3(x_1-x_2)^2}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{5/2}}\right.\\
&\qquad + \frac{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)-3(x_2-x_2)^2}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{5/2}}\\
&\qquad \left.+ \frac{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)-3(x_3+x_3)^2}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{5/2}}\right)
\end{aligned}$$

$$\Delta\varphi(x,y) = \frac{1}{4\pi}\left(\frac{3\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)-3\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{5/2}}\right)$$
$$= 0 \quad \checkmark$$

We perform the exact same computation for (2) which inevitably holds as well since the Laplacian operation is the same for a different sign inside the squared factors of the squared root in (3).

We now verify the boundary condition .Since by definition $G(x,y)=0$ on the boundary, then it follows that

$$G(x,y) = \Phi(x,y) - \Phi(\tilde{x}-y) = 0 \implies \Phi(x,y) \equiv \Phi(\tilde{x}-y),$$

hence on the boundary

$$\varphi(x,\sigma) = \Phi(\tilde{x}-\sigma) = \Phi(x-\sigma) = \frac{1}{4\pi|x-\sigma|},$$

which is precisely the the boundary condition defined for φ. Since $\Phi(\tilde{x}-y) = \Phi(y-\tilde{x})$, then it follows that the boundary condition is respected for the second component in the equality.

Question 2

By definition ,

$$u(x) = \int_{\Omega} f(y)G(x,y)\,\mathrm{d}y - \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}\,\mathrm{d}\sigma. \tag{4}$$

We compute the normal derivative of

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|\tilde{x}-y|}\right)$$

The reflection is with respect to the y_3 component and the normal vector is pointing inwards so it is negative, thence

$$-\left.\frac{\partial G}{\partial y_3}\right|_{y_3=0} = -\frac{1}{4\pi}\left(\frac{x_3-y_3}{|x-y|^3} - \frac{-x_3-y_3}{|\tilde{x}-y|^3}\right)$$
$$= -\frac{x_3}{2\pi|x-y|^3}.$$

We conclude, using (4) that the representation formula is

$$u(x) = \int_{\Omega} \frac{f(y)}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|\tilde{x}-y|}\right)\,\mathrm{d}y + \int_{\partial\Omega} g(y)\frac{x_3}{2\pi|x-y|^3}\,\mathrm{d}\sigma$$

Question 3

We show the claim through the following development

$$
\begin{aligned}
G(x,y) &= \Phi(x,y) - \Phi(\tilde{x} - y) \\
&= \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|\tilde{x}-y|}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}} - \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(-x_3-y_3)^2}}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(-1)^2(y_1-x_1)^2+(-1)^2(y_2-x_2)^2+(-1)^2(y_3-x_3)^2}}\right. \\
&\qquad \left. - \frac{1}{\sqrt{(-1^2)(y_1-x_1)^2+(-1)^2(y_2-x_2)^2+(-y_3-x_3)^2}}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{|y-x|} - \frac{1}{|\tilde{y}-x|}\right) \\
&= \Phi(y-x) - \Phi(\tilde{y}-x) \\
&= G(y,x) \quad \checkmark
\end{aligned}
$$