PHYS356 Assignment 8

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Question 1

a)

$$\langle u\rangle = \frac{\int ue^{-u/kT}\sin\theta \mathrm{d}\theta\mathrm{d}\varphi}{e^{-u/kT}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi} = \frac{2\pi}{2\pi}\frac{\int ue^{-u/kT}\sin\theta\mathrm{d}\theta}{e^{-u/kT}\sin\theta\mathrm{d}\theta}$$

We want to solve these integrals with respect to u which is given by the problem , so let

$$\theta = u = -pE\theta \implies \mathrm{d}u = pE\sin\theta\mathrm{d}\theta,$$

The integration bounds then change accordingly with $-pE \rightarrow pE$.

$$\begin{aligned}
&= \frac{\int_{-pE}^{pE} ue^{-u/kT}\mathrm{d}u}{\int_{pE}^{pE} e^{-u/kT}\mathrm{d}u} \\
&= \frac{\left[ukT\left(-e^{-u/kT}\right)\Big|_{-pE}^{pE} - \left[(kT)^2e^{-u/kT}\right|_{-pE}^{pE}}{\left[-kTe^{-u/kT}\right|_{-pE}^{pE}} \\
&= \frac{-(pE)(kT)e^{-pE/kT} + (pE)(kT)e^{pE/kT} - (kT)^2\left(e^{-pE/kT} - e^{pE/kT}\right)}{-kT\left(e^{-pE/kT} - e^{pE/kT}\right)} \\
&= \frac{pEe^{-pE/kT} + pEe^{pE/kT} + (kT)\left(e^{-pE/kt} - e^{pe/kT}\right)}{e^{-pE/kt} - e^{pe/kT}} \\
&= \frac{pE\left(e^{-pE/kt} + e^{pe/kT}\right)}{e^{-pE/kt} - e^{pe/kT}} + kT = -pE\frac{\left(e^{-pE/kt} + e^{pe/kT}\right)}{\left(e^{-pE/kt} - e^{pe/kT}\right)} + kT
\end{aligned}$$

$$\therefore \langle u \rangle = kT - pE \coth\left(\frac{pE}{kT}\right).$$

Using the information from the question we express this with the polarizability.

$$\langle u \rangle = -pE\left(\coth\frac{pE}{kT} - \frac{kT}{pE}\right) = -pE\langle \cos\theta \rangle \implies \langle \cos\theta \rangle = \left(\coth\frac{pE}{kT} - \frac{kT}{pE}\right)$$
$$\therefore P = Np\langle \cos\theta \rangle = Np\left(\coth\frac{pE}{kT} - \frac{kT}{pE}\right).$$

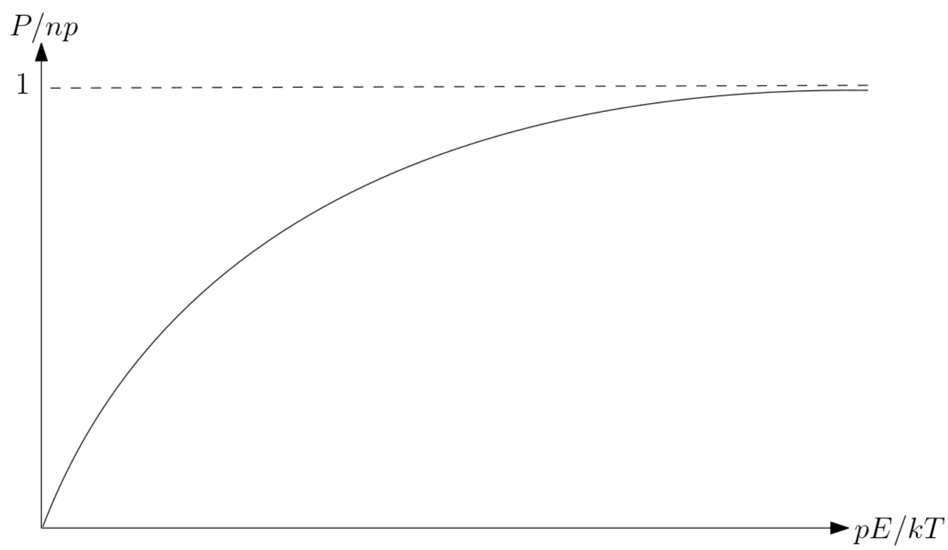

Figure 1: Given the expression found, we note that $\coth(\infty) \to 1$, hence the horizontal asymptote at $P/np = 1$.

b)

We consider the $kT >> pE$ case, so let us Taylor expand

$$\coth\left(\frac{pE}{kT}\right) - \frac{kT}{pE} = \frac{pE}{3kT} - \left(\frac{pE}{kT}\right)^3 \frac{1}{45} + \ldots,$$

so then

$$\frac{P}{Np} \approx \frac{pE}{3kT} \implies P \approx \frac{Np^2E}{3kT}.$$

By definition , $P = \epsilon_0 \chi_e E$ and so we can solve for χ_e;

$$\chi_e = \frac{Np^2}{3kT\epsilon_0}.$$

We know that water has a permanent dipole moment of 1.85 Debie , which is equivalent to 6.1×10^{-30} C m. Then for 18 g of water (1 mol),

$$\frac{1}{m_p} = \frac{N}{N_A} \implies N = \frac{1}{18\text{ g}}(6 \times 10^{26}\text{ kg}) = 3.3 \times 10^{28}.$$

Thus,

$$\chi_e = \frac{(3.3 \times 10^{28})(6.1 \times 10^{-30})^2}{3(8.85 \times 10^{-12})(1.38 \times 10^{-23})(293.15)} \approx 11.43,$$

which is much lower than $\chi_e = 79$ from the table.

For the steam, we use $PV = NkT$,

$$PV = NkT \implies N = \frac{PV}{kT} = \frac{(10^5)}{(1.38 \times 10^{-23})(373.15)} = 1.94 \times 10^{25}$$
$$\implies \chi_e = \frac{(1.94 \times 10^{25})(6.1 \times 10^{-30})^2}{3(8.85 \times 10^{-12})(1.48 \times 10^{-23})(373.15)} \approx 0.00528,$$

this result is in good agreemint with the tabulated value of 0.00589.

Question 2

a)

By definition $\vec{F} = q(\vec{E} + (\vec{v} \times \vec{B}))$. Because there is no deflection there is essentialy no net force on the particle such that

$$\vec{F} = q(\vec{E} + (\vec{v} \times \vec{B})) = 0 \implies \vec{E} = -\vec{v} \times \vec{B}.$$

Let the electric field be in the $\hat{x}$ direction, the magnetic field in the $\hat{y}$ direction and the speed in $-\hat{z}$ direction, all of which are perpendicular to one another. Then,

$$\vec{E}(\hat{x}) = -vB(-\hat{z} \times \hat{y}) = -vB(-\hat{x}) \implies E = vB \implies v = \frac{E}{B}.$$

b)

From lecture notes, $v = rBq/m$ so then

$$\frac{q}{m} = \frac{v}{rB} \overset{v=E/b}{\implies} \frac{q}{m} = \frac{E}{rB^2}.$$

Question 3

a)

We first note that the vertical and horizontal components do not contribute to a magnetic field at the point P. We find the magnetic field of the $r = a$ segment contribution. In cylindrical coordinates,

$$d\vec{l} = a\mathrm{d}\varphi\hat{\varphi}; \quad \hat{r} = -\hat{s}; \quad r^2 = a^2.$$

$$\begin{aligned} B(\vec{r})_{r=a} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2} &= \frac{\mu_0 I}{4\pi}\int_0^{\pi/2} \frac{a\mathrm{d}\varphi(\hat{\varphi}\times -\hat{s})}{a^2} \\ &= \frac{\mu_0 I}{4\pi}\int_0^{\pi/2} \frac{\mathrm{d}\varphi}{a^2}\hat{z} = \frac{\mu_0 I}{8a}\hat{z}. \end{aligned}$$

By symmetry, it is obvious that for the $r = b$ segment

$$B(\vec{r})_{r=b} = \frac{\mu_0 I}{8b}(-\hat{z}).$$

Therefore we conclude

$$B_p = \frac{mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{z},$$

where $\hat{z}$ points out of the page.

b)

We use once again cylindrical coordinates,

$$d\vec{l} = R\mathrm{d}\varphi\hat{\varphi}; \quad \hat{r} = -\hat{s}; \quad r^2 = R^2.$$

So we have the contribution from the semi-circle at point P given by

$$B(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{\pi} = \frac{R\mathrm{d}\varphi(\varphi \hat{\times} -\hat{s})}{R^2} = \frac{\mu_0 I}{4R}(-\hat{z}).$$

For the bottom infinite line contribution we use same procedure outlined in the lecture notes ;

$$d\vec{l}\times\hat{r} = dl\cos\theta; \quad d\vec{l} = \frac{s}{\cos\theta}\mathrm{d}\theta \implies \frac{1}{r^2} = \frac{\cos^2\theta}{r}.$$

So then the magnetic field at point P .

$$B(\vec{R}) = \frac{\mu_0 I}{4\pi}\int_0^{\pi/2} \frac{\cos\theta}{R}\mathrm{d}\theta$$

Where the integration bounds were found by extending the line to infinity from which we find that $\theta \in (0, \pi/2)$ (semi-infinite line).

$$\therefore B(\vec{r}) = \frac{\mu_0 I}{4\pi R}(-\hat{z}).$$

Again, by symmetry, the contribution from the top line should be the exact same quantity in the same direction , and so we multiply the bottom line contribution by a factor of 2. The net magnetic field at point P is then

$$B(\vec{r})_P = \frac{\mu_0 I}{4R}(-\hat{z}) + \frac{mu_0 I}{2\pi R}(-\hat{z}) = \frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)(-\hat{z}),$$

with the direction pointing inside the page.

Question 4

We first note that $\theta_2 \leq \theta \leq \theta_1$. So we will integrate along the solenoid with respect to θ. We consider an infinitesimal horizontal portion of the solenoid of thickness dx. Then,

$$n' \equiv n\mathrm{d}x \implies I' \equiv n'I = nI\mathrm{d}x.$$

Then, using the result of Example 5.6 ,

$$B(x) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \rightarrow \mathrm{d}\vec{B} = \frac{\mu_0 nI\mathrm{d}x}{2}\frac{a^2}{(x^2+a^2)^{3/2}}$$
$$\implies B = \int \mathrm{d}B = \frac{\mu_0 nIa^2}{2}\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}}$$

Since we want to integrate over θ, we perform a change of variables.

$$\tan\theta = a/x \implies x = a\cot\theta \implies \mathrm{d}x = -a\csc^2\theta\mathrm{d}\theta.$$
$$= \frac{\mu_0 nIa^2}{2}\int_{\theta_1}^{\theta_2}\frac{-a\csc^2\theta\mathrm{d}\theta}{\left(a^2\left(\cot^2\theta+1\right)\right)^{3/2}}$$
$$= -\frac{\mu_0 nIa^2}{2a^2}\int_{\theta_1}^{\theta_2}\frac{\csc^2\theta}{\left(\cot^2\theta+1\right)}\mathrm{d}\theta$$
$$= -\frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2}\frac{1}{\csc\theta}\mathrm{d}\theta$$
$$\therefore B(\vec{r}) = \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1).$$

We note that if the cylinder were to stretch infinitely, the θ bounds would become $\theta_1 = \pi$ and $\theta_2 = 0$ which would give us

$$B(\vec{r}) = \mu_0 nI.$$

Question 5

From previous assignments, we know the electric field from an infinite charged line, and the magnetic field from a charged line from Equation 5.38:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}; \qquad \vec{B} = \frac{\mu_0 I}{2\pi r}.$$

Therefore, given $\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] = 0$ (since the forces balance out) we get

$$qR = qvB \implies \frac{\lambda}{2\pi\epsilon_0} = v\frac{\mu_0 I}{2\pi r} \implies v = \frac{\lambda}{\epsilon_0\mu_0 I}.$$

Now since by definition of a current, $I = \lambda v$ we conclude

$$v^2 = \frac{1}{\epsilon_0\mu_0} \implies v = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$