PHYS 350 Assignment 4

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Question 1

a)

We normalize to find N ,

$$\begin{pmatrix} -N^* i & 2N^* & 3N^* & -4N^* i \end{pmatrix} \begin{pmatrix} Ni \\ 2N \\ 3N \\ 4Ni \end{pmatrix} = |N|^2 + 4|N|^2 + 9|N|^2 + 16|N|^2 = 1 \implies N = \frac{1}{\sqrt{30}}.$$

b)

We know that

$$\hat{S}_x \xrightarrow{\text{Notes}} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

So then

$$\langle \varphi | \hat{S} | \varphi \rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} -i & 2 & 3 & -4i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} i \\ 2 \\ 3 \\ 4i \end{pmatrix} = \frac{4\hbar}{5}.$$

c)

Since

$$|\varphi\rangle = \frac{1}{\sqrt{30}} \left(i \left| \frac{3}{2}, \frac{3}{2} \right\rangle + 2 \left| \frac{3}{2}, \frac{1}{2} \right\rangle + 3 \left| \frac{3}{2}, -\frac{1}{2} \right\rangle + 4i \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \right),$$

then it follows that

$$|\langle z | \varphi \rangle|^2 = \left| \frac{2}{\sqrt{30}} \right|^2 = \frac{2}{15}.$$

Question 2

We know that

$$\hat{U}^{\dagger}(t)\hat{U}(t) = \mathrm{I} \implies \hat{U}^{\dagger}(\mathrm{d}t)\hat{U}(\mathrm{d}t) = \mathrm{I}.$$

Moreover,

$$\hat{U}^{\dagger}(\mathrm{d}t) = \left(\mathrm{I} - \frac{i}{\hbar}\hat{H}\mathrm{d}t\right)^{\dagger} = \mathrm{I} + \frac{i}{\hbar}\hat{H}^{\dagger}\mathrm{d}t = \hat{U}(-\mathrm{d}t).$$

Therefore,

$$\begin{aligned}\hat{U}^{\dagger}(dt)\hat{U}(\mathrm{d}t) &= \left(\mathrm{I} + \frac{i}{\hbar}\hat{H}^{\dagger}\mathrm{d}t\right)\left(I - \frac{i}{\hbar}\hat{H}\mathrm{d}t\right) \\ &= \mathrm{I} + \frac{i}{\hbar}(\hat{H}^{\dagger} - \hat{H})\mathrm{d}t + \underbrace{O(\mathrm{d}t)^2}_{\to 0} = 1 \\ \implies \frac{i}{\hbar}(\hat{H}^{\dagger} - \hat{H}) = 0 &\implies \hat{H}^{\dagger} = \hat{H},\end{aligned}$$

Hence the Hamiltonian is hermitian.

Question 3

We know that

$$\frac{\mathrm{d}\hat{U}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$$

Therefore, we have the Homogenous first order differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t) + \frac{i}{\hbar}\hat{H}\hat{U}(t) &= 0 \\ \implies \ln\left|\hat{U}(t)\right| &= -\int_0^t \frac{i}{\hbar}\hat{H}(t')\,\mathrm{d}t' \\ \implies \hat{U}(t) &= \exp\left\{-\int_0^t \frac{i}{\hbar}\hat{H}(t')\,\mathrm{d}t'\right\},\end{aligned}$$

which shows the requested claim.

Question 4

By definition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle\varphi(t)|[\hat{H},\hat{A}]|\varphi(t)\rangle + \cancel{\langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle} \quad 0 \text{ , since time independent}$$

The initial state is an energy eigenstate so

$$|\psi(0)\rangle = |E\rangle \implies |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|E\rangle = e^{-iEt}|E\rangle \implies \langle\psi(t)| = e^{iEt/\hbar}\langle E|$$
$$= \frac{i}{\hbar}e^{iEt/\hbar}\langle E|\,[\hat{H},\hat{A}]e^{-iEt/\hbar}|E\rangle$$

The exponential terms cancel one another, we're left off with

$$\begin{aligned}
&= \frac{i}{\hbar}\langle E|\hat{H},\hat{A}|E\rangle \\
&= \frac{i}{\hbar}\langle E|HA|E\rangle - \frac{i}{\hbar}\langle E|AH|E\rangle
\end{aligned}$$

By definition for the Hamiltoniana $H|E\rangle = E|E\rangle$ so then

$$= \frac{i}{\hbar}\langle E|EA|E\rangle - \frac{i}{\hbar}\langle E|AE|E\rangle$$

A is an energy eigenstate such that $AE = EA$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = 0,$$

so A does not change with time, the claim is proved.

Question 5

Let $|\varphi(0)\rangle = |+z\rangle$.We'll express that in the $|\pm x\rangle$ basis and apply the time operator,

$$\begin{aligned}
|\varphi(0)\rangle &= |+z\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle \\
|\varphi(t)\rangle &= \hat{U}(t)\,|\varphi(0)\rangle \\
&= e^{-i\hat{H}t/\hbar}\left(\frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle\right)
\end{aligned}$$

Since $\hat{U}(t)\,|+x\rangle = \exp\left\{\frac{-i}{\hbar}\hat{H}(t)\right\}|+x\rangle = \exp\left\{\frac{-i\omega_0 t}{2}\right\}|+x\rangle$, then it follows that

$$= \frac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|+x\rangle + \frac{1}{\sqrt{2}}e^{i\omega_0 t/2}|-x\rangle$$

We now transfer back to the $|\pm z\rangle$,

$$\begin{aligned}
&= \frac{1}{2}e^{-i\omega_0 t/2}\left(|+z\rangle + |-z\rangle\right) + \frac{1}{\sqrt{2}}e^{i\omega_0 t/2}\left(|+z\rangle - |-z\rangle\right) \\
&= \cos\frac{\omega_0 t}{2}|+z\rangle - i\sin\frac{\omega_0 t}{2}|-z\rangle
\end{aligned}$$

Then we find l_0 by setting $|\langle +z|\varphi(t)\rangle|^2 = 1/4$,

$$\frac{1}{4} = |\langle +z|\varphi(t)\rangle|^2 = \cos^2\frac{\omega_0 t}{2} \implies \frac{\pi}{3} = \frac{\omega_0 t}{2} \implies \frac{2\pi}{3} = \omega_0 t \xrightarrow{\times v_0} \frac{2\pi}{3} v_0 = \underbrace{\omega_0 l_0}_{\omega_0(t v_0)}$$

$$\therefore\ l_0 = \frac{2\pi}{3}\left(\frac{v_0}{\omega_0}\right).$$

Question 6

We'll work in the $|\pm z\rangle$ basis and then switch to $|\pm y\rangle$ near the end. Since $|\varphi(0)\rangle = |+z\rangle$ and since

$$|\psi(t)\rangle = a\left|\psi^+(t)\right\rangle + b\left|\psi^-(t)\right\rangle = ae^{-i\omega_{\text{eff}t}/2}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\omega t/2}\\ \sin\frac{\theta}{2}e^{i\omega t/2}\end{pmatrix} + be^{i\omega_{\text{eff}t}/2}\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\omega t/2}\\ -\cos\frac{\theta}{2}e^{i\omega t/2}\end{pmatrix},$$

then solving

$$|\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} + b\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix} \implies a = \cos\frac{\theta}{2} \quad \text{and} \quad b = \sin\frac{\theta}{2}.$$

So then we write

$$|\psi(t)\rangle = \cos\frac{\theta}{2}e^{-i\omega_{\text{eff}t}/2}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\omega t/2}\\ \sin\frac{\theta}{2}e^{it/2}\end{pmatrix} + \sin\frac{\theta}{2}e^{i\omega_{\text{eff}t}/2}\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\omega t/2}\\ -\cos\frac{\theta}{2}e^{it/2}\end{pmatrix}$$

We drop the $\exp\{\pm i\omega t/2\}$ since when we take the absolute value for the probability they vanish to 1.

$$\begin{aligned}
&= \left[\cos^2\frac{\theta}{2}e^{-i\omega_{\text{eff}}t/2} + \sin^2\frac{\theta}{2}e^{i\omega_{\text{eff}}t/2}\right]|+z\rangle \\
&\quad + \left[\frac{1}{2}\sin\theta e^{-i\omega_{\text{eff}}t/2} - \frac{1}{2}\sin\theta e^{i\omega_{\text{eff}}t/2}\right]|-z\rangle \\
&= \left[\sin^2\frac{\theta}{2}\left(\underbrace{e^{i\omega_{\text{eff}}t/2} - e^{-i\omega_{\text{eff}}t/2}}_{2i\sin(\omega_{\text{eff}}t)}\right)\right]|+z\rangle + \left[-\frac{1}{2}\sin\theta\left(\underbrace{e^{i\omega_{\text{eff}}t/2} - e^{-i\omega_{\text{eff}}t/2}}_{2i\sin(\omega_{\text{eff}}t)}\right)\right]|-z\rangle
\end{aligned}$$

We transform to the $|\pm y\rangle$ basis with $|+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$ and $|-z\rangle = \frac{-i}{\sqrt{2}}(|+y\rangle - |-y\rangle)$, which gives us

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega_{\text{eff}}t/2} + 2i\sin^2\frac{\theta}{2}\sin\left(\frac{\omega_{\text{eff}}t/2}{2}\right) - \sin\frac{\theta}{2}\sin\left(\frac{\omega_{\text{eff}}t/2}{2}\right)\right]|+y\rangle + (\dots)|-y\rangle$$

Finally we may compute the probability to find the particules in $S_y = \hbar/2$

$$|\langle +y|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left[e^{-i\omega_{\text{eff}}t/2} + \sin\left(\frac{\omega_{\text{eff}}t}{2}\right)\left(2i\sin^2\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\right]\right|^2$$

As a double check ,for $\theta \to 0$, $|\langle +y|\psi\rangle| \to \dfrac{1}{2}$ and for $\theta \to \frac{\pi}{2}$, $|\langle +y|\psi\rangle| \to \frac{1}{2}$ as it should.