MATH475 Assignment 2

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1 Question 1

Let us multiply $-\Delta u$ by $v \equiv (u-w)$ and take the integral ;

$$\begin{aligned}\int_{\Omega} -\Delta u(u-w)\ \mathrm{d}x &= -\int_{\partial\Omega} g(u-w)\ \mathrm{d}\sigma + \int_{\Omega} \nabla u \cdot \nabla(u-w)\mathrm{d}x = 0 \\ &= -\int_{\partial\Omega} gu\mathrm{d}\sigma + \int_{\partial\Omega} gw\ \mathrm{d}\sigma + \int_{\Omega} |\nabla u|^2\ \mathrm{d}x - \int_{\Omega} \nabla u \nabla w\ \mathrm{d}x = 0\end{aligned}$$

$$\implies \int_{\Omega} |\nabla u|^2\ \mathrm{d}x - \int_{\partial\Omega} gu\ \mathrm{d}\sigma = \int_{\Omega} \nabla u \nabla w\ \mathrm{d}x - \int_{\partial\Omega} gw\ \mathrm{d}\sigma$$

We use the identity $\nabla u \nabla w \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2$, getting

$$\begin{aligned}\int_{\Omega} |\nabla u|^2\ \mathrm{d}x - \int_{\partial\Omega} gu\ \mathrm{d}\sigma &\leq \int_{\partial\Omega} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\int_{\Omega} |\nabla w|^2 - \int_{\partial\Omega} gw\ \mathrm{d}\sigma \\ \implies \frac{1}{2}\int_{\Omega} |\nabla u|^2\ \mathrm{d}x - \int_{\partial\Omega} gu\ \mathrm{d}\sigma &\leq \frac{1}{2}\int_{\Omega} |\nabla w|^2\ \mathrm{d}x - \int_{\partial\Omega} gw\ \mathrm{d}\sigma\end{aligned}$$

$$\because \int_{\partial\Omega} gw\ \mathrm{d}\sigma - \int_{\partial\Omega} gu\ \mathrm{d}\sigma = \int_{\partial\Omega} g \overbrace{(u-w)}^{v \xrightarrow{\text{boundary}} 0} \mathrm{d}\sigma = 0$$

$$\begin{aligned}\therefore \frac{1}{2}\int_{\Omega} |\nabla u|^2\ \mathrm{d}x &\leq \frac{1}{2}\int_{\Omega} |\nabla w|^2\ \mathrm{d}x \\ \implies E[u] &\leq E[w]\end{aligned}$$

Question 2

We will prove the claim by contradiction. First we note that u is harmonic so it respects the MVP, for which it follows that we can apply the maximum principles. Let us assume $\exists x_0 \in \Omega$ such that $u(x_0) \leq 0$ then by the Strong Maximum Principle this $\implies \exists M \in \Omega$ for which

$$\min_{\overline{\Omega}} u = M,$$

since $u = g$ on the boundary, and $g \geq 0 \; \forall x \in \partial\Omega$. Since u is constant in Ω ($\max_{\overline{\Omega}} u = \max_{\Omega} u$), then we conclude that $u(x) \equiv M \leq 0 \; \forall x \in \Omega$. This is a contradiction since $\exists x \in \partial\Omega$ with $g(x) > 0 \implies u > 0$. We conclude that $u > 0 \; \forall x \in \Omega$.

Question 3

Corollary 1.0.1. If u is a $C^2(\Omega)$ harmonic function on a domain Ω which is $C(\bar{\Omega})$, and the values of u on the boundary are bounded between m and M, then the values of u everywhere are bounded between m and M. (*Ref : R. Choksi, p.411*)

Let us define $\overline{\Omega} := (x_1, x_2) \times (-1, 1) \subseteq \overline{Q}$, for arbitrary $(x_1 \neq x_2) \in \mathbb{R}$. Then, as defined, $u \equiv 0$ on the boundary of $\overline{\Omega}$, i.e.,

$$\max_{\partial\Omega} u = \min_{\partial\Omega} \equiv 0.$$

Following Corollary 1 , $u \equiv 0$ everywhere in $\overline{\Omega}$.
Since $u(x, y)$ is periodic in x and since x_1 and x_2 are chosen arbitrarily, we can chose $x_1' = x_1 + 2$ such that $u \equiv 0 \in \overline{\Omega}'$, where $\overline{\Omega}' := (x_1', x_2) \times (-1, 1) \subseteq \overline{Q}$, for arbitrary $x_2 \in \mathbb{R}$. We can repeat this process over any arbitrary x_1 and x_2 with difference ε between them such that for all possible domain subsets of $\overline{Q}$ which include its y boundary $u \equiv 0$ in them. We conclude finally that $u \equiv 0$ in $\overline{Q}$.

Question 4

Let $u(x)$ be subharmonic such that $-\Delta u(x) \leq 0$ on the bounded domain Ω.
Let $v(x) := u(x) + \varepsilon|\boldsymbol{x}|^2$ for $\varepsilon > 0$. Then by the second derivative test, u achieves an interior maximum if and only if

$$u_{x_i x_i} \leq 0 \implies \Delta u \leq 0.$$

But by construction ,

$$\begin{aligned}
\Delta v(x) &= \Delta(u(x) + \varepsilon|\boldsymbol{x}|^2) \\
&= \underbrace{\Delta u(x)}_{\geq 0,\, \equiv K} + \Delta\varepsilon|\boldsymbol{x}|^2 \\
&\geq K + \varepsilon \underbrace{\Delta|\boldsymbol{x}|^2}_{\text{Non null and positive perturbation}}
\end{aligned}$$

$$
\begin{aligned}
&> 0 \\
\implies &\Delta v(x) > 0.
\end{aligned}
$$

We conclude that v does not attain an interior maximum ,which implies max v is on the boundary. Let x_0 pe the point at which the maximum occurs, then it follows that

$$
\begin{aligned}
u(x) \le v(x) &\le v(x_0) \\
&= u(x_0) + \varepsilon |x_0|^2 \\
&= \max_{\partial\Omega} u + \varepsilon |x_0|^2 \\
\implies u(x) &\le \max_{\partial\Omega} u + \underbrace{\varepsilon |x_0|^2}_{\text{a constant}} \\
\implies \lim_{\varepsilon \to 0} u(x) &\le \lim_{\varepsilon \to 0} \left(\max_{\partial\Omega} u + \varepsilon |x_0|^2 \right) \\
u(x) &\le \max_{\partial\Omega} u.
\end{aligned}
$$

Then by the Weak maximum principle,

$$
\max_{\partial\Omega} u = \max_{\overline{\Omega}} u \implies u(x) \le \max_{\partial\Omega} u = \max_{\overline{\Omega}} u.
$$

Question 5

Let

$$
M := \max_{B_1(0)} |f|,
$$

and similarly to Question 4 let us consider $\varphi(x) := u(x) + M \dfrac{|x|^2}{2n}$. Then .

$$
\begin{aligned}
-\Delta \varphi(x) &= -\Delta \left(u(x) + M \frac{|x|^2}{2n} \right) \\
&= f - M = f - \max_{B_1(0)} |f| \le 0
\end{aligned}
$$

So we conclude that $\varphi(x)$ is a sub-solution. By construction, it follows that

$$
\max_{B_1(0)} u(x) \le \max_{B_1(0)} \varphi(x)
$$

Applying the Weak Maximum Principle, since u as defined is harmonic, thence satisfies the MVP and hence we may apply the Weak maximum principle,

$$\begin{aligned} &= \max_{\partial B_1(0)} \varphi(x) \\ &= \max_{\partial B_1(0)} g + M \max_{\partial B_1(0)} \frac{|x|^2}{2n} \\ &= \max_{\partial B_1(0)} g + \max_{\overline{B_1(0)}} |f| \max_{\partial B_1(0)} \frac{|x|^2}{2n} \end{aligned}$$

Let $C := \max_{\partial B_1(0)} \dfrac{|x|^2}{2n}$, for x a point on the boundary, then

$$= \max_{\partial B_1(0)} g + C \max_{\overline{B_1(0)}} |f|$$

Since $\max |g| \geq \max g$, it follows that

$$\max_{\overline{B_1(0)}} u(x) \leq \max_{\partial B_1(0)} |g| + C \max_{\overline{B_1(0)}} |f|$$

Question 6

There is a discontinuity at $\boldsymbol{x} = \boldsymbol{y}$ for $x, y \in \Omega$. Particularly, this discontinuity occurs in the term $\Phi(x - y)$ in $G(x, y)$, with

$$\Phi(x-y) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \quad \text{which} \quad \xrightarrow[\boldsymbol{x}\to\boldsymbol{y}]{} \infty. \tag{1}$$

We cut the domain Ω to Ω_ε to remove the discontinuity at the centre. Then $\Omega_\varepsilon := \Omega\backslash\{B(x, \varepsilon)\}$. This new domain has 2 boundaries, namely $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B(x, \varepsilon)$. Since $\Omega_\varepsilon \subseteq \Omega$ and Ω is a bounded and connected domain then it follows that Ω_ε is as well ; so let us apply the weak maximum principle on $u(y) = G(x, y)$ on Ω_ε

$$\begin{aligned} \max_{\Omega_\varepsilon} u(y) = \max_{\Omega_\varepsilon} G(x,y) &= \max_{\partial\Omega_\varepsilon} G(x,y) \\ &= \max\left(\max_{\partial\Omega} G(x,y), \max_{\partial B(x,\varepsilon)} G(x,y)\right) \end{aligned}$$

Since by definition $G(x, \sigma) = 0$ for $\sigma = y \in \partial B$, then we have

$$= \max\left(0, \max_{\partial B(x,\varepsilon)} G(x,y)\right)$$

We send $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} \max_{\Omega_\varepsilon} G(x,y) = \lim_{\varepsilon\to 0} \max(0, \underbrace{\max_{\partial B(x,\varepsilon)} G(x,y)}_{\to\infty \text{ by } (1)})$$

$\lim_{\varepsilon \to 0} \Omega_\varepsilon$ is precisely Ω therefore,

$$\max_{\Omega} G(x, y) = \max(0, \infty),$$

We conclude that $G(x, y)$ is bounded by 0 and ∞, implying that the values of $G(x, y)$ in the domain are positive.