MATH 475 Assignment 1

Mihail Anghelici 260928404

October 12, 2020

Question 1

First and foremost we know that

$$\text{Net Heat} = \text{External heat source} + \text{Heat through edges},$$

where the external heat source is

$$\int_V r(x,t)p\ dx$$

and the heat through the edges corresponds to

$$-\int_{\partial V} \boldsymbol{q}(x,t)\cdot n\ d\sigma.$$

Then the rate of change in energy ,defined by the thermal energy per unit mass $e(x,t)$,is equal to the the sum of the net flux and the external heat source;

$$\begin{aligned}\int_V \frac{\partial e(x,t)}{\partial t}p &= \int_V r(x,t)p\ dx - \int_{\partial V} \boldsymbol{q}(x,t)\cdot n\ d\sigma \\ &= \int_V r(x,t)p\ dx + \int_V \operatorname{div}\boldsymbol{q}(x,t)\ dx\end{aligned}$$

Since $\partial(e(x,t))/\partial t = cu_t$ then

$$\int_V cu_t p - \operatorname{div}\boldsymbol{q}(x,t)\ dx = \int_V r(x,t)p$$

Sicne $\operatorname{div}\boldsymbol{q}(x,t) = \operatorname{div}(\kappa(x)\nabla u)$ we obtain

$$\int_V cu_t p - \operatorname{div}(\kappa(x)\nabla u)\ dx = \int_V r(x,t)p$$

This holds $\forall\ V \subseteq \mathbb{R}^n$, i.e, it is pointwise ,therefore

$$\implies\ cu_t p - \operatorname{div}(\kappa(x)\nabla u) = r(x,t)p$$

We rearrange by dividing cp on both sides

$$\implies u_t - \frac{1}{cp}\mathrm{div}(\kappa(x)\nabla u) = \frac{r(x,t)}{c}$$

Let us define $k(x) = \kappa(x)/cp$, we can take a constant out of a div argument such that

$$\therefore u_t - \mathrm{div}(k(x)\nabla u) = r(x,t).$$

Question 2

Let $w = u - v$ solve

$$\begin{cases} w_t - \nabla\cdot(k(x)\nabla w) = 0 & \text{in } \Omega_T, \\ w(x,0) = 0 & \text{in } \Omega, \\ w(\sigma,t) = 0 & \text{in } \partial\Omega\times(0,T], \end{cases}$$

Then by the energy method,

$$\begin{aligned} E(t) &:= \int_\Omega w^2(x,t)\, dx \\ E'(t) &= \int_\Omega \frac{d}{dt} w^2(x,t)\, dx \\ &= 2\int_\Omega w w_t\, dx \end{aligned}$$

Using $w_t = \nabla\cdot(k(x)\nabla u)$,

$$\begin{aligned} E'(t) &= 2\int_\Omega w(\nabla\cdot(k(x)\nabla w))\, dx \\ &= 2\int_\Omega w(x)\mathrm{div}(k(x)\nabla w) \\ &= 2\int_\Omega w(x)\sum_i (k(x)w_{x_i})_{x_i} \\ &= 2\int_{\partial\Omega} w(x)k(x)\sum_i w_{x_i}\, d\sigma - 2\int_\Omega k(x)\sum_i (w_{x_i})^2\, dx \\ &= -2\int_\Omega k(x)\sum_i (w_{x_i})^2\, dx \\ E'(t) &= -2\int_\Omega k(x)\Delta(w) \end{aligned}$$

The last expression is ≤ 0 since $k(x) > 0$, therefore since $E(0) = 0$ it follows that $E(t) \le 0$, but by definition $E(t) \ge 0$, we conclude that

$$E'(t) = -2\int_\Omega k(x)\Delta w = 0 \implies w \equiv 0,$$

the solution is unique.

Question 3

We first note that $x + 1 + 2\sin^2(2\pi x) = x + 2 - \cos(4\pi x)$. We let $w(x,t) = u(x,t) + v(x,t)$ and set $v(x,t) = -(x+2)$. Then $w(x,t)$ solves

$$\begin{cases} w_t - 2w_{xx} = 0 \\ w(x,0) = -\cos(4\pi x) \\ w_x(0,t) = 0, \quad w_x(1,t) = 0. \end{cases}$$

The form of the solution is evidently $w(x,t) = e^{-2\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$, since it's the only one that matches the sinusoidal nature of the initial condition.

$$\begin{aligned} w_x(x,t) &= \lambda e^{-2\lambda^2 t}(B\cos\lambda x - A\sin\lambda x) \\ w_x(0,t) &= \lambda e^{-2\lambda^2 t}B = 0 \implies B = 0. \\ w_x(1,t) &= -^{-2\lambda^2 t}A\sin\lambda = 0 \implies \sin\lambda = 0 \implies \lambda = n\pi, \quad \text{for } n \in \mathbb{Z}. \end{aligned}$$

So we have that the solution is

$$w(x,t) = \sum_{n=0}^{N} e^{-2n^2\pi^2 t} A_n \cos(n\pi x).$$

We carry on with the initial condition ;

$$\begin{aligned} w(x,0) &= \sum_{n=0}^{N} A_n \cos(n\pi x) = -\cos(4\pi x) \\ w(x,0) &= 0 + 0 + 0 + 0 - 1\cos(4\pi x) = -\cos(4\pi x) \end{aligned}$$

Finally, having found the appropriate constants we write the general solution ;

$$w(x,t) = e^{-2(16)\pi^2 t}\cos(4\pi x) \implies u(x,t) = e^{-32\pi^2 t}\cos(4\pi x) + x + 2.$$

Question 4

Let $w = u - v$ solve

$$\begin{cases} w_t - kw_x x = 0 & \text{in } L_T := (0,L)\times(0,T] \\ w(x,0) = 0 & \text{in } (0,L], \\ w_x(0,t) - \alpha w(0,t) = 0 \quad , w_x(L,t) + \alpha w(L,t) = 0 & \text{in } \times (0,T] \end{cases}$$

Then by the energy method

$$\begin{aligned} E(t) &:= \int_\Omega w^2(x,t)\ dx \\ E'(t) &= \int_\Omega 2ww_t\ dx \\ &= \int_\Omega 2kww_xx\ dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} &= 2k\left[w(x,t)w_x(x,t)\Big|_0^L - \int_0^L w_x^2(x,t)\ dx\right] \\ &= 2k\left[(w(L,t)w_x(L,t) - w(0,t)w_x(0,t)) - \int_0^L w_x^2(x,t)\ dx\right] \\ &= 2k\left[-\alpha(w_x^2(L,t) - w_x^2(0,t)) - \int_0^L w_x^2(x,t)\ dx\right] \end{aligned}$$

The RHS ≤ 0 since $\alpha > 0, k > 0$ and $w_x^2 > 0$ with positive integral bounds given that $L \neq 0$. All of which implies that $E'(t) \leq 0 \implies E'(t) = 0$,then since $E(t) \geq 0$ by definition ,it follows that

$$\underbrace{2k\alpha(w_x^2(L,t) - w_x^2(0,t))}_{k>0,\alpha>0 \implies \geq 0} = \underbrace{-2k\int_0^L w_x^2(x,t)\ dx}_{k>0 \implies \leq 0}$$

$$\begin{aligned} &\therefore 2k\alpha(w_x^2(L,t) - w_x^2(0,t)) = -2k\int_0^L w_x^2(x,t)\ dx = 0 \\ &\implies w_x \equiv 0 \implies w \equiv 0 \end{aligned}$$

Question 5

Let the solution be of the form $u(x,t) = e^{-3\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$. Then,

$$\begin{aligned} u(0,t) = 0 &\implies e^{-3\lambda^2 t}A = 0 \implies A = 0. \\ u_x(x,t) &= \lambda e^{-3\lambda^2 t}(B\cos\lambda x - A\sin\lambda x) \\ \implies u_x(\pi,t) &=^{-3\lambda^2 t} B\cos\lambda\pi = 0 \implies \cos\lambda n = 0 \implies \lambda = \frac{n}{2} \quad \text{for } n \in \mathbb{Z}. \end{aligned}$$

So we have,

$$u(x,t) = e^{-3\lambda^2 t}B\sin\left(\frac{nx}{2}\right)$$

Extending the solution in the summation form

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{N} e^{-3\lambda^2 t} B_n \sin\left(\frac{nx}{2}\right) \\
u(x,0) &= \sum_{n=0}^{N} B_n \sin\left(\frac{nx}{2}\right) = 4\sin\frac{x}{2} - \frac{4}{3}\sin\frac{3x}{2} \\
&\implies B_1 = 0, B_2 = 5, B_3 = 0, B_4 = -\frac{4}{3}. \\
\therefore u(x,t) &= 4e^{-3t}\sin\left(\frac{x}{2}\right) - \frac{4}{3}e^{-12t}\sin\left(\frac{3x}{2}\right)
\end{aligned}
$$

Question 6

By the Weak Maximum Principle,

$$\min_{\partial_P \Omega_T} v \leq v(x,t) \leq \max_{\partial_P \Omega_T} v,$$

where $\partial_P \Omega_T$ represents the parabolic boundary of a space-time cylinder (i.e., the base and the sides). Now since $\partial_P \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times (0,T]$, it follows that

$$
\begin{aligned}
\min_{\partial_P \Omega_T} v &= \min\left(\min_{\Omega\times\{t=0\}} v, \min_{\partial\Omega\times(0,T]} v\right) \\
&= \min\left(\min_{\Omega\times\{t=0\}} v, \min\left(\min_{(0,T]}(1+\sqrt{3}t), \min_{(0,T]}(\pi^4 + e^t)\right)\right) \\
&= \min(1, \min(1, \pi^4)) \\
&= 1.
\end{aligned}
$$

$$\therefore v(x,t) \geq 1,$$

We conclude that $v(x,t)$ is a super solution. Then, we verify if the given function we're comparing to is a sub solution or a super solution to the PDE ; Let $w = x^4 + e^{-t}\sin(x)$ solve the PDE, then

$$
\begin{aligned}
w_t - w_{xx} &= (x^4 + e^{-5}\sin(x))_t - (x^4 + e^{-5}\sin(x))_{xx} \\
&= -e^{-t}\sin(x) - ((4x^3 + e^{-t}\cos(x))_x \\
&= -e^{-t}\sin(x) - 12x^2 + e^{-t}\sin(x) \\
&= -12x^2 \leq 0 \; \forall (x,t) \in (0,\pi) \times (0,T]
\end{aligned}
$$

We conclude that w as defined is a sub solution. Finally, by definition, super solutions are bigger than sub solutions so

$$v(x,t) \geq x^4 + e^{-5}\sin(x) \quad \forall (x,t) \in \pi_T.$$

Question 7

a)

Since u solves the heat equation, and u is the fundamental solution which is infinitely smooth ,we can take the derivative inside the integral. Indeed,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u\ dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\ dx \overset{u_t=u_{xx}}{=} \int_{-\infty}^{\infty} \frac{\partial u_x}{\partial x}\ dx = u_x(\infty) - u_x(-\infty)$$

Since $\lim\limits_{t\to-\infty} u_x = 0$ and $\lim\limits_{t\to\infty} u_x = 0$ then it follows that

$$E'(t) = 0.$$

Now since,

$$E'(t) = 0 \implies \int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty} u(x,0)\ dx$$

Since $g(x)$ is continuous on the specified domain and is integrable, and by theorem

$$\lim_{t\to 0} u(x,t) = g(x),$$

It follows that

$$\int_{-\infty}^{\infty} u(x,0)\ dx = \int_{-\infty}^{\infty} g(x)\ dx \implies \int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty} g(x)\ dx.$$

b)

$$\int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x-y,t)g(y)\ dy\ dx$$

By Fubinni's theorem, we may interchange the order of integration

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x-y,t)g(y)\ dx\ dy$$

Since $\int_{-\infty}^{\infty} \Gamma(x,t)\ dx = 1 \implies \int_{-\infty}^{\infty} \Gamma(x-y,t)\ dx = 1$, therefore

$$= \int_{-\infty}^{\infty} g(y)\ dy$$

Since y in this case is a dummy variable , we may let $y \to x$ such that

$$\int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty} g(x)\ dx.$$

Question 8

Fixing $t_0 \in (0,T]$, let

$$r(x,t) = t \sup_{\mathbb{R}\times(0,T]} f + \sup g = At + B$$
$$p(x,t) = -t \inf_{\mathbb{R}\times(0,T]} f - \inf g = -At - B.$$

Let $(u-r)$ and $(p-v)$ be two solutions. Plugging them in the given PDE we find trivially that for $(u-r)$ the PDE is $f - \sup f \leq 0$ therefore it is a sub solution and similarly for $(p-v)$ the PDE is $-f - \sup f \leq 0$ which is also a sub solution.Thus, we may apply the Global Maximum Principle . Let $w = u - r$, then

$$\begin{aligned} w(x,t) &\leq \sup_{\mathbb{R}^n} w(x,0) \\ u - r &\leq \sup_{\mathbb{R}^n} w(x,0) \\ u &\leq \underbrace{\sup_{\mathbb{R}^n}(g(x) - \sup g)}_{\leq 0} + r \\ \therefore u &\leq r \end{aligned}$$

Similarly letting $w = p - u$

$$\begin{aligned} w(x,t) &\leq \sup_{\mathbb{R}^n} w(x,0) \\ p - u &\leq \sup_{\mathbb{R}^n} w(x,0) \\ p &\leq \underbrace{\sup_{\mathbb{R}^n}(g(x) + \sup g)}_{\geq 0} + u \\ \therefore p &\leq u \end{aligned}$$

We conclude that $Pp \leq u \leq r$ which translates to

$$-t \inf_{\mathbb{R}\times(0,T]} f - \inf g \leq u(x,t) \leq t \sup_{\mathbb{R}\times(0,T]} f + \sup g,$$

as it should.

Question 9

$$|u(x,t)| = \left| \int_{-\infty}^{\infty} \Gamma(x-y,t) g(y)\, dy \right| \leq \int_{-\infty}^{\infty} |\Gamma(x-y,t)||g(y)|\, dy$$

Following the propreties of the absolute value and since the fundamental solution is always positive,

$$= \int_{-\infty}^{\infty} \Gamma(x-y,t)|g(y)|\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} |g(y)|\,dy$$

$(x-y)^2 \geq 0\,\forall x,y \implies -(x-y)^2/4kt$ is strictly decreasing. Moreover, $\max \exp\{-(x-y)^2/4kt\} = 1$ and $\min \exp\{-(x-y)^2/4kt\} = 0$, but it is never reached since it's an exponential function. We conclude that $0 < \exp\{-(x-y)^2/4kt\} \leq 1$, hence it follows that

$$\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} |g(y)|\,dy$$

Since g is bounded by C as defined , we conclude that

$$|u(x,t)| \leq \frac{C}{\sqrt{4\pi kt}} \geq 0$$
$$\implies u(x,t) \leq \frac{C}{\sqrt{4\pi kt}}.$$

It follows evidently that for each $x \in \mathbb{R}$

$$\lim_{t\to\infty} u(x,t) = \frac{C}{\infty} = 0,$$

as it should.

Question 10

a)

$$\begin{aligned}
p(x,t+\tau) &= \frac{1}{4}p(x+he_1,t) + \frac{1}{4}p(x-he_1,t) + \frac{1}{4}p(x+he_2,t) + \frac{1}{4}p(x-he_2,t) \\
&= p(x,t) + p_t(x,t)\tau + \mathcal{O}(\tau) \\
p(x+he_1,t) &= p(x,t) + p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2) \\
p(x-he_1,t) &= p(x,t) - p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2) \\
p(x+he_2,t) &= p(x,t) + p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2) \\
p(x-he_2,t) &= p(x,t) - p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
p(x,t) + p_t(x,t)\tau + \mathcal{O}(\tau) &= \frac{1}{4}\left(p(x,t) + p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2)\right)\\
&+ \frac{1}{4}\left(p(x,t) - p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2)\right)\\
&+ \frac{1}{4}\left(p(x,t) + p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2)\right)\\
&+ \frac{1}{4}\left(p(x,t) - p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2)\right)\\
\therefore\ p_t(x,t)\tau + \mathcal{O}(\tau) &= \frac{p_{xx}}{2}(x,t)(he_1)^2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_1|^2) + \mathcal{O}(|he_1|^2)
\end{aligned}$$

Dividing both sides by τ

$$p_t(x,t) + \frac{\mathcal{O}(\tau)}{\tau} = \frac{p_{xx}(x,t)}{2}\frac{((he_1)^2 + (he_2)^2)}{\tau} + \frac{\mathcal{O}(|he_1|^2 + |he_2|^2)}{\tau}$$

Let us suppose $\lim_{\tau\to 0}((he_1)^2 + (he_2)^2)/\tau \neq 0$, then $\tau \to 0$ should return something non trivial. Suppose $((he_1)^2 + (he_2)^2)/\tau = 2k > 0$. Then sending $\tau \to 0$ and $he_i \to 0$, which in return makes the remainder terms vanish results in

$$p_t(x,t) - kp_{xx}(x,t) = 0.$$

b)

$$\begin{aligned}
p(x,t+\tau) &= \frac{1}{8}p(x+he_1,t) + \frac{1}{8}p(x-he_1,t) + \frac{3}{8}p(x+he_2,t) + \frac{3}{8}p(x-he_2,t)\\
&= p(x,t) + p_t(x,t)\tau + \mathcal{O}(\tau)\\
p(x+he_1,t) &= p(x,t) + p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2)\\
p(x-he_1,t) &= p(x,t) - p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + \mathcal{O}(|he_1|^2)\\
p(x+he_2,t) &= p(x,t) + p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2)\\
p(x-he_2,t) &= p(x,t) - p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + \mathcal{O}(|he_2|^2)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
p(x,t) + p_t(x,t)\tau + O(\tau) &= \frac{1}{8}\left(p(x,t) + p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + O(|he_1|^2)\right)\\
&+ \frac{1}{8}\left(p(x,t) - p_x(x,t)he_1 + \frac{p_{xx}}{2}(x,t)(he_1)^2 + O(|he_1|^2)\right)\\
&+ \frac{3}{8}\left(p(x,t) + p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + O(|he_2|^2)\right)\\
&+ \frac{3}{8}\left(p(x,t) - p_x(x,t)he_2 + \frac{p_{xx}}{2}(x,t)(he_2)^2 + O(|he_2|^2)\right)\\
\therefore\ p_t(x,t)\tau + O(\tau) &= \frac{p_{xx}}{8}(x,t)(he_1)^2 + \frac{3p_{xx}}{8}(x,t)(he_2)^2 + O(|he_1|^2) + O(|he_1|^2) \qquad (1)
\end{aligned}
$$

Let

$$
A(x)D^2p \equiv \begin{pmatrix} \frac{1}{8}he_1 & 0 \\ 0 & \frac{3}{8}he_2 \end{pmatrix} p_{xx}(x,t),
$$

then dividing by τ on both sides of (1) we have

$$
p_t(x,t) + \frac{O}{\tau} - \mathrm{tr}A(x)\frac{D^2p}{\tau} = \frac{O(\tau)(|he_1|^2 + |he_2|^2)}{\tau}
$$

Let us suppose $\lim_{\tau\to 0}((he_1)^2 + (he_2)^2)/\tau \neq 0$, then $\tau \to 0$ should return something non trivial. Suppose $((he_1)^2 + (he_2)^2)/\tau = 2k > 0$. Then sending $\tau \to 0$ and $he_i \to 0$, which in return makes the remainder terms vanish results in

$$
p_t - \mathrm{tr}(A(x)D^2p) = 0.
$$

We note there is no drift because the random walk system as defined is stable. Indeed, the probabilities in each direction are symmetric such that there's no preferred direction for a given orientation.