MATH 314 Ass 4.

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Question 1

The orientation is positive therefore we may use the standard form of the definition , i.e.,

$$\int\limits_C P\ dx + Q\ dy = \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA.$$

Setting $P(x,y) = y^2 + x^3$ and $Q(x,y) = x^4$ we get

$$\begin{aligned}\int\limits_C y^2 + x^3 dx + x^4 dy &= \int_0^1 \int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dydx \\ &= \int_0^1 \int_0^1 (4x^3 y - 2y)\ dydx \\ &= \int_0^1 \left(4x^3 y\Big|_0^1 - y^2\Big|_0^1\right)\ dx = \int_0^1 4x^3 - 1\ dx = x^4\Big|_0^1 - x\Big|_0^1 = 0.\end{aligned}$$

Question 2

$$\text{Area}(D) = \int\limits_C x\ dy = -\int\limits_C y\ dx. \tag{1}$$

First assume (1) is true. Since the contour integral is linear,

$$\int\limits_C x\ dy + \int\limits_C y\ dx = 0 \implies \int\limits_C x\ dy + y\ dx = 0.$$

Converting the last integral using Green's theorem and show it is equal to 0 should imply that the RHS equality in (1) is correct.

$$\int\limits_C x\ dy + y\ dx = \int\limits_C Q\ dy + P\ dx \implies \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \iint\limits_D (1-1)\ dA = 0.$$

We now show that Area$(D) = \int\limits_C x\ dy$.

$$\int\limits_C x\ dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \iint\limits_D (1-0)\ dA = \iint\limits_D\ dA$$

By definition, $\iint\limits_D\ dA = \text{Area}(D)$, which completes the proof.

Question 3

$$\implies \vec{T}_u = (1,1,v) \quad , \vec{T}_v = (-1,1,u) \quad , \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & v \\ -1 & 1 & u \end{vmatrix} = (u-v, u+v, 2).$$

$$\implies \left\|\vec{T}_u \times \vec{T}_v\right\| = \sqrt{2u^2+2v^2+4} = \sqrt{2}\sqrt{u^2+v^2+2}$$

We may now use the surface area definition

$$\begin{aligned} \text{S.A} &= \iint\limits_D \left\|\vec{T}_u \times \vec{T}_v\right\|\ dudv \\ &= \sqrt{2}\iint\limits_D \sqrt{u^2+v^2+2}\ dudv \end{aligned}$$

Let us convert to polar coordinates , $u = r\cos\theta$ and $v = r\sin\theta$, yielding

$$\begin{aligned} &= \sqrt{2}\int_0^{2\pi}\int_0^1 \sqrt{r^2+2}r\ drd\theta \\ &\overset{u=r^2+2}{=} \frac{1}{\sqrt{2}}\int_0^{2\pi}\int_2^3 u^{1/2}\ dud\theta \\ &= \frac{1}{\sqrt{2}}\int_0^{2\pi} \frac{2}{3}\ u^{3/2}\Big|_2^3\ d\theta \\ &= \frac{\sqrt{2}}{3}\left(\int_0^{2\pi} 3^{3/2}\ d\theta - \int_0^{2\pi} 2^{3/2}\ d\theta\right) = \frac{\sqrt{2}}{3}2\pi(3^{3/2}-2^{3/2}). \end{aligned}$$

Question 4

$$\begin{aligned} \vec{T}_u &= (2\cos(u), -3\sin(u), 0) \\ \vec{T}_v &= (0,0,1) \end{aligned}$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos u & -3\sin u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-3\sin(u), 2\cos(u), 0).$$

Then we use the surface integral for vector fields definition

$$\begin{aligned}
\iint_S F \cdot ds &= \iint_D F(\phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v) dA \\
&= \int_0^{2\pi} \int_0^1 (2\sin u, 3\cos u, v) \cdot (-3\sin u, 2\cos u, 0)\ dvdu \\
&= \int_0^{2\pi} \int_0^1 6(\cos^2(u) - \sin^2(u))\ dvdu \\
&= \int_0^{2\pi} 6\cos(2u)\ du = 3\sin(2u)\Big|_0^{2\pi} = 0.
\end{aligned}$$

Question 5

First and foremost we set $x = \sqrt{1 - y^2 - z^2}$, $y = y$ and $z = z$.

$$\begin{aligned}
\implies \vec{T}_y &= \left(\frac{-y}{\sqrt{1-y^2-z^2}}, 1, 0 \right) \quad , \vec{T}_z = \left(\frac{-z}{\sqrt{1-y^2-z^2}}, 0, 1 \right) \\
\vec{T}_y \times \vec{T}_z &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-y}{\sqrt{1-y^2-z^2}} & 1 & 0 \\ \frac{-z}{\sqrt{1-y^2-z^2}} & 0 & 1 \end{vmatrix} = \left(1, \frac{-y}{\sqrt{1-y^2-z^2}}, \frac{z}{\sqrt{1-y^2-z^2}} \right),
\end{aligned}$$

$\therefore$ the vector $\vec{n}$ is pointing out of the surface. Since we're given that $x \geq 0$, $\implies 0 = \sqrt{1-y^2-z^2} \implies 1 = y^2 + z^2$. Thus, the boundary of the given shape at $x = 0$ is a circle of radius r parameterized as

$$c(t) = (0, \sin(t), \cos(t)) \qquad , 0 \leq t \leq 2\pi.$$

Finally, by Stoke's Theorem,

$$\int_S \mathrm{Curl} F \cdot ds = \int_0^{2\pi} F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\ dt$$

Since $c'(t) = (0, \cos(t), -\sin(t))$ this gives

$$\begin{aligned}
&= \int_0^{2\pi} (0, -\sin^3(t), 0) \cdot (0, \cos(t), -\sin(t))\ dt \\
&= \int_0^{2\pi} (0 - \sin^3(t)\cos(t) + 0)\ dt \\
&\overset{u=\sin(t)}{=} \frac{-1}{4} \sin^4(t)\Big|_0^{2\pi} = 0.
\end{aligned}$$

Question 6

The ellipse that results from the intersection of the cylinder and the plane is parameterized as

$$\begin{aligned} & C(t) = (\cos t, \sin t, 1 - \cos t - \sin t) \\ \implies & C'(t) = (-\sin t, \cos t, \sin t - \cos t). \end{aligned}$$

Then using the definition of line integral we get

$$\begin{aligned} \int_C -y^3 dx + x^3 dy - z dz &= \int_0^{2\pi} F(C(t)) \cdot C'(t)\ dt \\ &= \int_0^{2\pi} (\sin^3 t, \cos^3 t, -1 + \cos t + \sin t)\ dt \\ &\quad \cdot (-\sin t, \cos t, \sin t - \cos t)\ dt \\ &= \int_0^{2\pi} (1 - 2\sin^2 t \cos^2 t) - \sin t + \cos t - \cos(2t)\ dt \\ &= \int_0^{2\pi} 1\ dt - 2\int_0^{2\pi} \frac{1}{8}(1 - \cos(4t))\ dt - \int_0^{2\pi} \sin t + 0 - 0 \\ &= 2\pi - \frac{\pi}{2} - 0 = \frac{3\pi}{2}. \end{aligned}$$

Question 7

$$\text{div} F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = 3y^2 + 3x^2 + 3z^3 = 3(y^2 + x^2 + z^2).$$

Let us use spherical coordinates

$$x = \rho \sin\varphi \cos\theta \quad , y = \rho \sin\varphi \sin\theta \quad , z = \rho\cos\varphi.$$

We get ,

$$\begin{aligned} \int_S \vec{F} \cdot ds &= 3\int_0^{\pi} \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\varphi \sin^2\theta + \rho^2 \sin^2\varphi \rho \cos^2\theta + \rho^2\cos^2\varphi)\rho^2 \sin\varphi\ d\rho d\theta d\varphi \\ &= 3\int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^4 (\sin^3\varphi(\sin^2\theta + \cos^2\theta) + \cos^2\varphi \sin\varphi)\ d\rho d\theta d\varphi \\ &= \frac{3}{5}\int_0^{\pi} \int_0^{2\pi} \sin^3\varphi + \sin\varphi\cos^2\varphi\ d\theta d\varphi \\ &= \frac{3}{5}\int_0^{\pi} \int_0^{2\pi} \sin\varphi\ d\theta d\varphi \\ &= \frac{6\pi}{5}\int_0^{\pi} \sin\varphi\ d\varphi = \frac{12\pi}{5}. \end{aligned}$$