MATH 475 Weekly Work 2

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September 25, 2020

Question 1

a)

$M(T)$ is a non-decreasing function of T. Indeed let the maximum up until $t = T$ occur at the base of the space time cylinder, then if $u(x,t') < M(t)$ for $t' > T$, then $M(t)$ will not change. But if $u(x,t') > M(t) \implies M(t)$ increases. In other words, the maximum can increase if we extend the further the time domain over T, but it can not decrease if we extende the time domain backwards.

b)

By the Weak max principle,

$$\min_{\partial_P \Omega_T} u \leqslant u(x,t) \leqslant \max_{\partial_P \Omega_T} u,$$

therefore , since $\partial_P \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times (0,T]$,

$$\max_{\partial_P \Omega_T} u = \max \left(\max_{\Omega \times \{t=0\}}, \max_{\partial\Omega \times (0,T]} \right)$$

We set the Dirichlet boundary constant $\equiv k$,

$$= \max \left(\max_{\Omega \times \{t=0\}}, k \right) \tag{1}$$

Notice (1) is time-independent as requested .

Question 2

a)

Inserting the $(u - r)$ and $p - u$ in the given PDE we have

$$(u - r)_t - k\Delta(u - r) = f - A = f - \max_{\Omega_{t_0}} |f|$$

$$(p - u)_t - k\Delta(p - r) = -f - A = -f - \max_{\Omega_{t_0}} |f|$$

Since $|f(x,t)| \leqslant \max_{\Omega_{t_0}} |f|$, it follows that both $(u-r)$ and $(p-u)$ are subsolutions. We may apply the weak max principle.

$$\begin{aligned}
u - r &\leqslant \max_{\partial_P \Omega_T}(u-r) \\
u &\leqslant \max_{\partial_P \Omega_T}(u-r) + r \\
&= \max_{\partial_P \Omega_T}(u - At - B) + r \\
&= \max_{\partial_P \Omega_T}(u - At - \max_{\partial_P \Omega_{t_0}} |u|) + r
\end{aligned}$$

Since $u - \max_{\partial_P \Omega_{t_0}} |u| \leqslant 0$ and $A > 0$, $\implies \max_{\partial_P \Omega_T}(u - At - \max_{\partial_P \Omega_{t_0}} |u|) < 0$

$$\therefore u \leqslant r$$

Similarly for $(p-u)$,

$$\begin{aligned}
p - u &\leqslant \max_{\partial_P \Omega_T}(p-u) \\
&\leqslant \max_{\partial_P \Omega_T}(p-u) + u \\
&= \max_{\partial_P \Omega_T}(-At - B - u) \\
&= \max_{\partial_P \Omega_T}(-At - \max_{\partial_P \Omega_{t_0}} |u|) + u
\end{aligned}$$

Since $-\max_{\partial_P \Omega_{t_0}} |u| - u \leqslant 0$ and $A > 0$, $\implies \max_{\partial_P \Omega_T}(-At - \max_{\partial_P \Omega_{t_0}} |u| - u) < 0$

$$\therefore p \leqslant u$$

It follows that $p \leqslant u \leqslant r$ which in return implies that

$$-t_0 \max_{\overline{\Omega_{t_0}}} |f| - \max_{\partial_p \Omega_{t_0}} |u| \leqslant u(x_0, t_0) \leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f| + \max_{\partial_p \Omega_{t_0}} |u|.$$

b)

Let $u = v - w$. Then it follows that

$$u_t - k\Delta u = v_t - w_t - k\Delta v + k\Delta w = (v_t - k\Delta v) - (w_t - k\Delta w) = f_1 - f_2.$$

Following the identity proved in $2b$, this is equivalent to

$$\begin{aligned}
&u(x_0, t_0) \leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_1 - f_2| + \max_{\partial_P \Omega_{t_0}} |u| \\
\implies\ &v(x_0, t_0) - w(x_0, t_0) \leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_1 - f_2| + \max_{\partial_P \Omega_{t_0}} |v - w|. \qquad \text{(a)}
\end{aligned}$$

Now let $u = w - v$. Then it follows that

$$u_t - k\Delta u = w_t - v_t - k\Delta w + k\Delta v = (w_t - k\Delta w) - (v_t - k\Delta v) = f_2 - f_1.$$

Following the identity proved in $2b$, this is equivalent to

$$\begin{aligned} u(x_0, t_0) &\leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_2 - f_1| + \max_{\partial_P \Omega_{t_0}} |u| \\ \implies w(x_0, t_0) - v(x_0, t_0) &\leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_2 - f_1| + \max_{\partial_P \Omega_{t_0}} |w - v|. \end{aligned}$$

But we can rewrite this as follows

$$\begin{aligned} -(v(x_0, t_0) - w(x_0, t_0)) &\leqslant t_0 \max_{\overline{\Omega_{t_0}}} |-(f_1 - f_2)| + \max_{\partial_P \Omega_{t_0}} |-(v - w)| \\ \implies -(v(x_0, t_0) - w(x_0, t_0)) &\leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_1 - f_2| + \max_{\partial_P \Omega_{t_0}} |v - w| \qquad \text{(b)} \end{aligned}$$

Given (a) and (b), we conclue that

$$|v(x_0, t_0) - w(x_0, t_0)| \leqslant t_0 \max_{\overline{\Omega_{t_0}}} |f_1 - f_2| + \max_{\partial_P \Omega_{t_0}} |v - w|.$$

Question 3

a)

Let $u(x_1, t_1) = c$ for some value $c \in (0, 1) = \Omega_{t_1}$. Then by the strong max principle $u \equiv c$ in Ω_{t_1}. But, from the initial condition, $u(x, 0) = 4x(1 - x)$, we see that u is non-constant therefore we have a contraditction. It follows that $u \neq c$ in Ω_{t_1}. By the weak max principle, the minima and maxima occur on the boundaries. The minimum occurs on the boundary since $u(0, t) = u(1, t) = 0 \implies u > 0$. At the base the maximum occurs internaly at $x, t = (\frac{1}{2}, 0) < c \implies u < 1$. We conclude $0 < u < 1$ for the specified time and space domain.

b)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{d}{dt} E = \frac{d}{dt} \int_0^1 u^2(x, t) dx = 2k \left[u u_x \Big|_0^1 - \int_0^1 (u_x)^2 \, dx \right]$$

Since at the boundaries the function is 0, the first term vanishes

$$\frac{d}{dt} \int_0^1 u^2(x, t) \, dx = \underbrace{-2k \int_0^1 (u_x)^2 dx}_{\text{negative}}$$

$$\implies E(t) \text{ is strictly decreasing with respect to } t.$$

$$u(x,t) = e^{-2\lambda^2}(A\cos\lambda x + B\sin\lambda x)$$
$$u_x(x,t) = \lambda e^{-2\lambda^2 t}(B\cos\lambda x - A\sin\lambda x)$$

Boundary conditions to find a summation expression

$$u_x(0,t) = \lambda e^{-2\lambda^2 t}B = 1 \implies B = \frac{1}{\lambda e^{-2\lambda^2 t}}$$
$$u_x(1,t) = \lambda e^{-2\lambda^2 t}(B\cos x - A\sin\lambda x) = 1$$
$$\implies \quad \cos\lambda - \lambda e^{-2\lambda^2 t}A\sin\lambda = 1$$

Implementing $\sin^2(t)$,

$$\implies \quad -\lambda e^{-2\lambda^2 t}A\sin\lambda = 2\sin^2(\lambda/2)$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix}-i & 2\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}i\\ 2\end{pmatrix} = \frac{(-2i+2i)\hbar}{10} = 0$$