PHYS358 Assignment 8

Mihail Anghelici 260928404

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Question 1

a)

We normalize the state and thereby determine A

$$
\begin{aligned}
1 = \int_{-\infty}^{\infty} \mathrm{d}x \, \langle\psi|x\rangle \, \langle x|\psi\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2 \\
\implies 1 &= |A|^2 \int_{-R}^{R} \mathrm{d}x(R^2 - x^2) \\
&= |A|^2 \left(x\Big|_{-R}^{R} - 2R^2 \left[\frac{x^3}{r}\Big|_{-R}^{R} + \left[\frac{x^5}{5}\Big|_{-R}^{R} \right) \right.\right. \\
&= |A|^2 \frac{16}{5} R^5 \\
\therefore A &= \sqrt{\frac{5}{16R^5}}.
\end{aligned}
$$

b)

By definition,

$$
\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle = \int \mathrm{d}x \, \langle\psi|x\rangle \, x \, \langle x|\psi\rangle = \int \mathrm{d}x x|\psi(x)|^2.
$$

Thus we compute

$$
\begin{aligned}
\langle\hat{x}\rangle = \int \mathrm{d}x A^2 (R^2 - x^2)^2 &= \int \mathrm{d}x \frac{5}{16R^5}(R^4 - 2R^2x^2 + x^4) \\
&= \frac{5}{16} \frac{x}{R}\Big|_{-R}^{R} - \frac{2}{3R^3}\Big|_{-R}^{R} + \frac{x^5}{5r^5}\Big|_{-R}^{R} \\
&= \frac{1}{3}
\end{aligned}
$$

c)

We us the definition,

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-R}^{R} A(R^2 - x^2) e^{\frac{-ipx}{\hbar}} \mathrm{d}x,$$

solving this integral in Mathematica yields

$$\psi(p) = \sqrt{\frac{15\hbar^3}{2\pi R^5}} \frac{1}{p^3} \left(\hbar \sin\left(\frac{pR}{\hbar}\right) - pR\cos\left(\frac{pR}{\hbar}\right)\right).$$

So then by definition ,

$$\begin{aligned} |\psi\rangle &= \int \mathrm{d}P \, |p\rangle \langle p|\psi\rangle \\ &= \sqrt{\frac{15\hbar^3}{2\pi R^5}} \int_{-\infty}^{\infty} \mathrm{d}P \, \frac{1}{p^3} \left(\hbar \sin\left(\frac{pR}{\hbar}\right) - pR\cos\left(\frac{pR}{\hbar}\right)\right) |p\rangle \end{aligned}$$

d)

$$\begin{aligned} \langle \hat{p} \rangle = \langle \psi|p|p\rangle \langle p|\psi\rangle &= \int \mathrm{d}P p \left|\psi(p)^2\right| \\ &= \int_{-R}^{R} \mathrm{d}P \sqrt{\frac{15\hbar^3}{2\pi R^4}} \frac{1}{p^2} \left(\hbar \sin\left(\frac{pR}{\hbar}\right) - pR\cos\left(\frac{pR}{\hbar}\right)\right) \end{aligned}$$

Solving this integral in Mathematica yields

$$\langle \hat{p} \rangle = 0.$$

e)

Since we're studying a free-particle, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m},$$

so then by definition .

$$\psi(x,t) = \int \mathrm{d}P e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}\right)} \psi(p, t=0) = \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}\right)} \sqrt{\frac{15\hbar^3}{2\pi R^5}} \frac{1}{p^3} \left(\hbar \sin\left(\frac{pR}{\hbar}\right) - pR\cos\left(\frac{pR}{\hbar}\right)\right).$$

Question 2

a)

Using the normalization and splitting the absolute value,

$$\begin{aligned}
1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x &= |A|^2 \int_{-R}^{0} (R+x)^2 \mathrm{d}x + \int_{0}^{R} (R-x)^2 \mathrm{d}x \\
&= |A|^2 \left\{ \left[R^2 x \right|_{-R}^{0} + \left[\frac{2R}{2} x^2 \right|_{-R}^{0} + \left[\frac{1}{3} x^3 \right|_{-R}^{0} + \left[R^2 x \right|_{0}^{R} - \left[\frac{2R}{2} x^2 \right|_{0}^{R} + \left[\frac{1}{3} x^3 \right|_{0}^{R} \right\} \\
&= |A|^2 \frac{2}{3} R^3 \implies A = \sqrt{\frac{3}{2R^3}}.
\end{aligned}$$

b)

We apply the definition,

$$\begin{aligned}
\langle \hat{x} \rangle &= \int \mathrm{d}x |\psi(x)|^2 \\
&= |A|^2 \left(\int_{-R}^{0} \mathrm{d}x \; x(R+x)^2 + \int_{0}^{R} \mathrm{d}x \; x(R-x)^2 \right)
\end{aligned}$$

This is an integegral of an odd function over a symmetric interval, so this integrates to 0.

$$= 0.$$

c)

$$\begin{aligned}
\psi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-R}^{R} (R - |x|) e^{-\frac{ipx}{\hbar}} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-R}^{R} R e^{-\frac{ipx}{\hbar}} \mathrm{d}x - \int_{-R}^{0} (-x) e^{-\frac{ipx}{\hbar}} \mathrm{d}x - \int_{0}^{R} x e^{-\frac{ipx}{\hbar}} \mathrm{d}x
\end{aligned}$$

Solving this integral with Mathematica yields

$$= \sqrt{\frac{3\hbar^3}{4\pi R^3 p^4}} \left(1 - e^{\frac{ipR}{\hbar}}\right) \left(-e^{-\frac{ipR}{\hbar}}\right).$$

So then by definition ,

$$\begin{aligned}
|\psi\rangle &= \int \mathrm{d}P \, |p\rangle \langle p|\psi\rangle \\
&= \sqrt{\frac{3\hbar^3}{4\pi R^3}} \int_{-\infty}^{\infty} \mathrm{d}P \frac{1}{p^2} \left(1 - e^{\frac{ipR}{\hbar}}\right) \left(-e^{-\frac{ipR}{\hbar}}\right) |p\rangle .
\end{aligned}$$

for $\psi(p)$ as defined above.

d)

By definition ,

$$\langle \hat{p} \rangle = \int \mathrm{d}P \; p|\psi(p)|^2 = \int_{-R}^{R} \mathrm{d}P \; p\sqrt{\frac{3\hbar^3}{4\pi R^3 p^4}} \left(1 - e^{\frac{ipR}{\hbar}}\right)\left(-e^{-\frac{ipR}{\hbar}}\right),$$

solving this integral with Mathematica yields

$$\langle \hat{p} \rangle = 0.$$

e)

We apply the definition as before

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \sqrt{\frac{3\hbar^3}{4\pi R^3 p^4}} \left(1 - e^{\frac{ipR}{\hbar}}\right)\left(-e^{-\frac{ipR}{\hbar}}\right) e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}\right)} \mathrm{d}P.$$