MATH 327 Assignment 4

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Question 1

First we deduce

$$\|A\|_F^2 = \sum_{i,j} |A_{ij}|^2 = \sum_i \left(\sum_j A_{ij}^T A_{ji} \right) = \sum_i (A^T A)_{ii} = \operatorname{tr}(A^T A) = \operatorname{tr}(AA^T).$$

So then,

$$\begin{aligned} \|A\|_F^2 = \sum_i \sum_j |a_{ij}|^2 &= \operatorname{tr}(AA^T) \\ &= \operatorname{tr}((U\Sigma V^T)(U\Sigma V^T)) \\ &= \operatorname{tr}((U\Sigma V^T)(\Sigma V^T)^T U^T) \\ &= \operatorname{tr}((U\Sigma V^T) V \Sigma^T U^T) \\ &= \operatorname{tr}(\Sigma\Sigma^T) = \sum_i (\sigma_i)^2 \end{aligned}$$

Thus indeed,

$$\|A\|_F = \sqrt{\sum_i (\sigma_i)^2}.$$

Question 2

(a)

For each row multiplication, there are $n-1$ additions and n multiplications, so for m rows in total that is $(n-1)m + nm = m(2n-1)$ operations in total.

(b)

For $v_i^T x$, we have $2n - 1$ operations. Multiplying by u_i from the left that is a total of $(2n - 1) + m$ operations. And finally, multiplying by the constant σ_i we conclude there are a total of $(2n - 1) + 2m$ operations.

(c)

We start by $\sigma_i u_i$ which encompasses m operations.Then, $(\sigma_i u_i)v_i^T$ is an outer product and represents $m + mn$ operations. Summing summing over k we get $(m + mn)(k - 1)$ operations with an additional $(k - 1)mn$ operations for summing up the matrices. Finally, from (a) we know that Ax requires $m(2n - 1)$ operations so we conclude that

$$A_k(x) \to (m + mn)(k - 1) + (k - 1)mn + m(2n - 1) \text{ operations.}$$

Evidently it is more efficient to compute Ax instead of $A_k x$. There is an additional $(k - 1)(m + mn) + (k - 1)m$ operations within the latter case so for $k, m, n \geq 1$ the Ax is more efficient.

Question 3

(a)

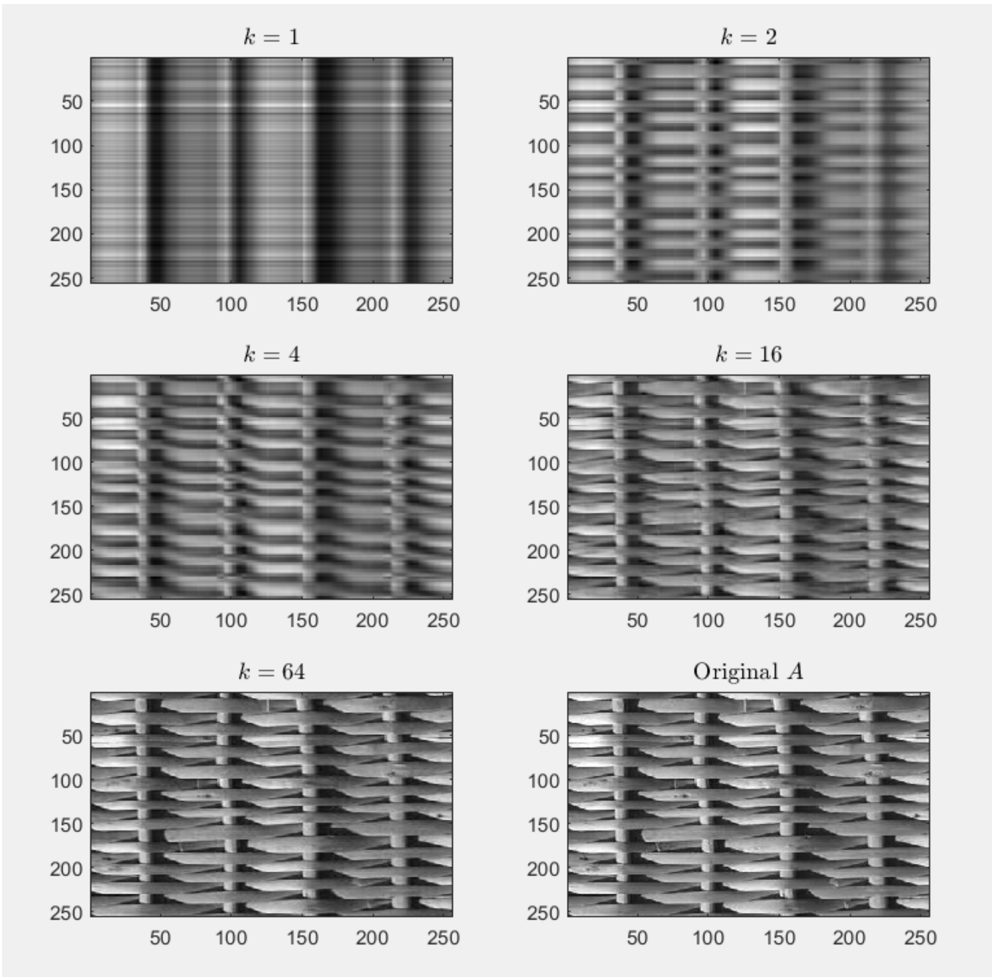

Figure 1: SVD decomposition for k approximations of $1, 2, 4, 16, 64$, compared to the original picture defined by A.

(b)

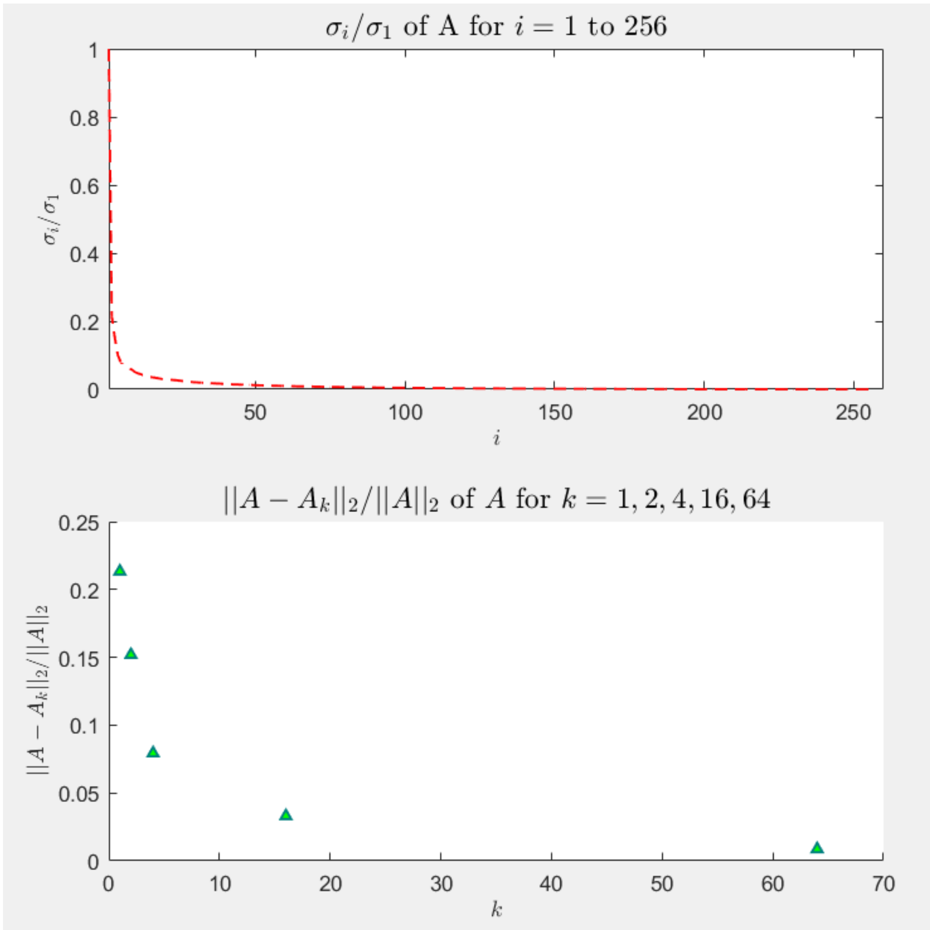

Figure 2

The largest value of k is 9 and is found using the following code snippet :

```
A_k_list = [];
[U,S,V] = svd(A);
i=1;
for k=1:64
        Ak = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
        A_k_list{i} = Ak;
        if (norm(A-A_k_list{i} ,2)/norm(A, 2) <=0.05)
                largest_k = k;
                break;
        end
        i=i+1;
end
```

Question 4

(a)

- $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_i \sigma_i(A) = 4.$
- $\|A\|_F = \sqrt{\sum_i \sigma_i^2} = \sqrt{30}.$
- Since $\|QA\| = \|A\|$ for $Q \perp$. Then $\|U\|_2 = 1$.
- By inspection of Σ, we deduce that $\text{rank}(A) = 4$.

(b)

$$\mathcal{R}(A) = \text{Span}\{u_1, \ldots, u_r\} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \dim(\mathcal{R}(A)) = 4$$

$$\mathcal{N}(A) = \text{Span}\{v_{r+1}, \ldots, v_n\} \rightarrow \text{ no basis since} \qquad \dim(\mathcal{N}(A)) = 0.$$

(c)

Let $U = [U_1 \ U_2]$ and $\Sigma = [\hat{\Sigma} \ 0]^T$ for $U_{m \times m}, U_{1,m \times n}, U_{2,m \times (m-n)}, \Sigma_{m \times n}$ and $\hat{\Sigma}_{n \times n}$. Then we state ;

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & -1/2 & 0 \\ -1/3 & 0 & -1/2 & 0 \\ -1/3 & 0 & -1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d)

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T U^T U \Sigma V^T = V(\hat{\Sigma})^2 V^T.$$

Here $\hat{\Sigma}$ is diagonal so $\sigma_i(A^T A) = \sigma_i^2(\hat{\Sigma}) = \{16, 9, 4, 1\}$. Moreover, the eigenvalues of $A^T A$ are the same as the singular values in the present case.

(e)

We use the definition ;

$$\frac{\|A-A_2\|_2}{\|A\|_2} = \frac{1}{\kappa_2(A)} \Longrightarrow \frac{\min\limits_{x\neq 0} \|Ax\|_2/\|x\|_2}{\max\limits_{x\neq 0} \|Ax\|_2/\|x\|_2}\|A\|_2 = \frac{\sigma_{\min}(A)}{\sigma_{\max}(A)}\sigma_{\max}(A) = \frac{1}{4}4 = 1.$$

Moreover,

$$A_2 = U(:,1:2)\Sigma(1:2,1:2)V(:,1:2)^T = \begin{pmatrix} 0 & 1 \\ 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Question 5

(a)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \to \text{Rank}(A) = 2.$$

(b)

It returns 1×10^{-15} which is actually the machine precision. This arises from the svd(A) function called. The answer therefore does not contradict that from (a).