PHYS356 Assignment 7

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November 16, 2020

Question 1

$$u(x,t) = X(x)T(t); \quad u_{tt} = X(x)T''(t); \quad u_{xx} = X''(x)T(t) \quad \implies X(x)T''(t) - kX''(x)T(t) = 0$$
$$\implies \frac{T''(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda^2.$$

In this problem we consider only the $\lambda > 0$ case since the initial conditions are trigonometric functions.

$$\begin{aligned} X''(x) &= -\lambda^2 X(x) \implies X(x) = A\sin(\lambda x) + B\cos(\lambda x) \\ T''(t) &= -\lambda^2 kT(t) \implies T(t) = C\sin\left(\sqrt{k}\lambda x\right) + D\cos\left(\sqrt{k}\lambda x\right). \end{aligned}$$

Thus, the general solution

$$u(x,t) = (A\sin(\lambda x) + B\cos(\lambda x))(C\sin\left(\sqrt{k}\lambda x\right) + D\cos\left(\sqrt{k}\lambda x\right)).$$

We only consider the $u(x,0)$ boundary condition since $u(\pi,t)$ does not provide valuable information ;

$$u(x,0) = B(C\sin\left(\sqrt{k}\lambda x\right) + D\cos\left(\sqrt{k}\lambda x\right)) = 0 \implies B = 0,$$

and so the general solution is refined to

$$u(x,t) = A\sin(\lambda x)(C\sin\left(\sqrt{k}\lambda x\right) + D\cos\left(\sqrt{k}\lambda x\right)).$$

We split the BVP in in 3 components, which is allowed by superposition ;

$$\begin{cases} v_{tt} - 4v_{xx} = 0, \\ v(0,t) = 0, \ v(\pi,t) = 0, \\ v(x,0) = 3\sin x, \\ v_t(x,0) = 0. \end{cases}, \quad \begin{cases} w_{tt} - 4w_{xx} = 0, \\ w(0,t) = 0, \ w(\pi,t) = 0, \\ w(x,0) = -\sin 4x, \\ w_t(x,0) = 0. \end{cases}, \quad \begin{cases} z_{tt} - 4z_{xx} = 0, \\ z(0,t) = 0, \ z(\pi,t) = 0, \\ v(x,0) = 0, \\ z_t(x,0) = \frac{1}{2}\sin 2x. \end{cases}$$

Solving for $v(x,t)$,

$$v(x,0) = A\sin(\lambda x)D = 3\sin x \implies A = 3, D = 1, C = 0, \lambda = 1,$$
$$\therefore v(x,t) = 3\sin(x)\cos(2t).$$

Similarly for $w(x,t)$,

$$w(x,0) = A\sin(\lambda x)D = -\sin 4x A = -1, D = 1, C = 0, \lambda = 4,$$
$$\therefore w(x,t) = -\sin(4x)\cos(8t).$$

Finally for $z(x,t)$,

$$z_t(x,0) = 2\lambda A\sin(\lambda x)C = \frac{1}{2}\sin 2x \implies A = \frac{1}{8}, C = 1, D = 0, \lambda = 2, \therefore z(x,t) = \frac{1}{8}\sin(2x)\sin(4t).$$

We conclude that the solution to the BVP is

$$u(x,t) = 3\sin(x)\cos(2t) - \sin(4x)\cos(8t) + \frac{1}{8}\sin(2x)\sin(4t).$$

Question 2

We split the BVP in 4 components

$$\begin{cases} v_{tt} - v_{xx} = 0, \\ v_x(0,t) = 0,\ v_x(1,t) = 0, \\ v(x,0) = 1, \\ v_t(x,0) = 0. \end{cases}, \quad \begin{cases} w_{tt} - w_{xx} = 0, \\ w_x(0,t) = 0,\ w_x(1,t) = 0, \\ w(x,0) = 8\cos 4\pi x, \\ w_t(x,0) = 0. \end{cases}, \quad \begin{cases} z_{tt} - z_{xx} = 0, \\ z_x(0,t) = 0,\ z_x(1,t) = 0, \\ z(x,0) = 0, \\ z_t(x,0) = 5. \end{cases}$$

$$\begin{cases} y_{tt} - z_{xx} = 0, \\ y_x(0,t) = 0,\ y_x(1,t) = 0, \\ y(x,0) = 0, \\ y_t(x,0) = 2\cos\pi x. \end{cases}$$

The $v(x,t)$ and $z(x,t)$ BVPs are $v(x,t) = 1$ and $z(x,t) = 5t$, to match the cases above. The given that linear functions solve the wave equation $v(x,t)$ and $z(x,t)$ are defined that way.

For the $w(x,t)$ and $y(x,t)$ we apply the same procedure outlined in Question 1 ;

$$w(x,t) = (A\sin\lambda x + B\cos\lambda x)(C\sin\sqrt{k}\lambda t + D\cos\sqrt{k}\lambda t)$$
$$w_x(0,t) = \lambda A(C\sin\sqrt{k}\lambda t + D\cos\sqrt{k}\lambda t) = 0 \implies A = 0$$
$$\therefore w(x,0) = B\cos(\lambda x)D = 8\cos 4\pi x \implies B = 8, D = 1, C = 0, \lambda = 4\pi$$
$$\therefore w(x,t) = 8\cos(4\pi x)\cos(4\pi t)$$

Similarly,

$$y_x(0,t) = \lambda A(C\sin\sqrt{k}\lambda t + D\cos\sqrt{k}\lambda t) = 0 \implies A = 0$$
$$y_t(x,t) = \lambda B\cos(\lambda x)(C\cos(\lambda t) - D\sin(\lambda t))$$
$$y_t(x,0) = \lambda B\cos(\lambda x)C = 2\cos\pi x \implies B = \frac{2}{\pi}, C = 1, D = 0, \lambda = \pi$$
$$\therefore\ y(x,t) = \frac{2}{\pi}\cos(\pi x)\sin(\pi t).$$

The final solution is

$$u(x,t) = 1 + 5t + 8\cos(4\pi x)\cos(4\pi t) + \frac{2}{\pi}\cos(\pi x)\sin(\pi t).$$

Question 3

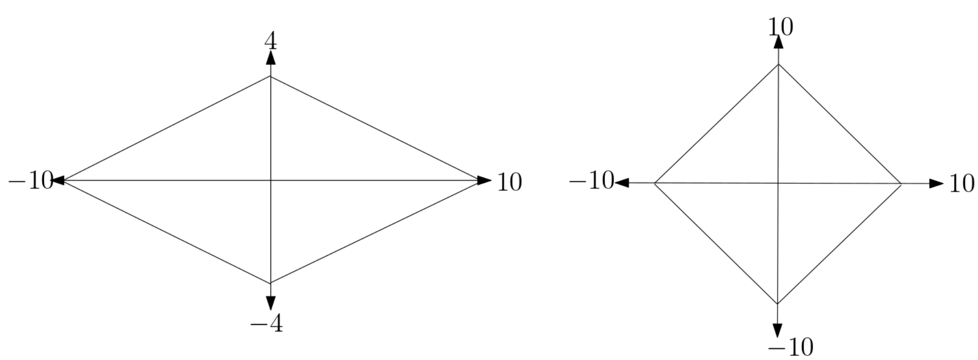

Figure 1: Region of influence and domain of dependence diagram
a) Diagram on a $x - t$ plane for $c = 2.5$ where $u \equiv 0$ outside the diamond shape.
b) Diagram on a $x - t$ plane for $c = 1$ where $u \equiv 0$ outside the diamond shape.

From Figure 1, we note that as c increases, the t_0 value on the t axis where the lines intersect decreases. For $u(0,4)$ to be non-zero, we require then that $t_0 \geq 4$. We conclude that

$$\text{For} \quad 0 < c \leq 2.5,\ u(0,4) = \frac{1}{2}\int_{-4c}^{4c} h(y)\mathrm{d}y.$$