MATH475 Homework 3

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Question 1

a)

We use the general method of characteristics to find the solution $u(x,t)$. We note that $(\alpha x u)_x = \alpha u + \alpha x u_x$, so then

$$\frac{dx}{d\tau} = \alpha x; \quad \frac{dt}{d\tau} = 1; \quad \frac{dz}{d\tau} = -\alpha z,$$

with initial conditions

$$x(s,0) = s; \quad t(s,0) = 0; \qquad z(s,0) = u_0(s).$$

Thus, solving each ODE with the respective initial condition yields

$$\begin{aligned} z &= Ce^{-\alpha\tau} \overset{z(s,0)=u_0(s)}{\Longrightarrow} C = u_0(s) \implies z = u_0(s)e^{-\alpha\tau}, \\ t &= \tau + C \overset{t(s,0)=0}{\Longrightarrow} C = 0 \implies t = \tau, \\ x &= Ce^{\alpha\tau} \overset{x(s,0)=s}{\Longrightarrow} C = s \implies x = se^{\alpha\tau}. \end{aligned}$$

Converting with $z(s,\tau) = u(x,t)$ we get

$$u(x,t) = u_0\left(\frac{x}{e^{\alpha t}}\right)e^{-\alpha t}.$$

b)

Qualitatively, the velocity function $v(x,t) = \alpha x$ represents the flux of the substance studied. In space, it varies linearly with respect to x so then the velocity grows linearly moving away from the origin

c)

$$\lim_{t\to 0} u(0,t) = u_0\left(\frac{0}{e^{\alpha t}}\right)e^{-\alpha t} = u_0(0)e^{-\infty},$$

which converges to 0 unless $u_0(0) = \infty$; since this is unrealistic we must conclude that $\lim_{t\to\infty} u(0,t) \to 0$.
In this limit the material vanishes ,which is in accordance with our remark in (b) where we said that the velocity increases away from the origin linearly, for which it implies $\exp(-\alpha t) \xrightarrow[t\to\infty]{} 0$.

d)

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} u_0\left(\frac{x}{e^{\alpha t}}\right)e^{-\alpha t}\mathrm{d}x$$

Let $\eta = x/e^{\alpha t} \implies e^{\alpha t}\mathrm{d}\eta = \mathrm{d}x$ so then

$$= \int_{-\infty}^{\infty} u_0(\eta)\mathrm{d}\eta,$$

this is an integral over $\mathbb{R}$ where u_0 is integrable, the integral is improper so the final answer will not involve the parameter t in the case where it diverges, and will be a constant , also independent of t , in the case where it converges.

Question 2

$$au_x + bu_y = \nabla u \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

which implies that u is constant along lines of the form $bx - ay = c \ \forall\, c \in \mathbb{R}$. Therefore the solution is

$$u(x,t) = g(bx - ay), \qquad \text{for } g \text{ arbitrary and } \ g : \mathbb{R} \to \mathbb{R}.$$

$$\begin{aligned} \because u(1,2) = u(3,6) &\implies g(b-2a) = g(3b-6a), \\ &\overset{g \text{ is arbitrary}}{\implies} b - 2a = 3b - 6a \\ &\therefore \frac{b}{a} = 2 \end{aligned}$$

Question 3

We use the method of characteristics.

$$\frac{dx}{d\tau} = z; \quad \frac{dt}{d\tau} = 1; \quad \frac{dz}{d\tau} = 0,$$

with initial conditions

$$x(s,0) = 1; \quad t(s,0) = 1; \qquad z(s,0) = 5.$$

Thus, solving each ODE with the respective initial condition yields

$$\begin{aligned} z = C &\overset{z(s,0)=5}{\Longrightarrow} C = 5 \implies z = 5, \\ t = \tau + C &\overset{t(s,0)=1}{\Longrightarrow} C = 1 \implies t = \tau + 1, \\ x = z\tau + C &\overset{x(s,0)=1}{\Longrightarrow} C = 1 \implies x = z\tau + 1. \end{aligned}$$

Combining the results above we have the equations

$$x(s,\tau) = 5t - 4; \quad t(s,\tau) = \tau + 1; \quad z(s,\tau) = 5.$$

The solution to Burger's equation is constant on the characteristic lines. Given x as defined above, we note that only the couples $(6,2)$ and $(11,3)$ are on this line.

Question 4

a)

We have

$$uu_x + uu_y = \frac{1}{2},$$

for which we'll use the general method of characteristics.

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}\tau} = z; && \frac{\mathrm{d}y}{\mathrm{d}\tau} = z; && \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}, \\ &x(s,0) = s; && y(s,0) = 2s; && z(s,0) = 1. \end{aligned}$$

Solving each ODE with their respective initial condition yields

$$x = 2\tau + s; \qquad y = 2\tau + 2s; \qquad z = \frac{\tau}{2} + 1.$$

We cancel the s variable and substitute $\tau(x,y)$ in $z(s,\tau)$,

$$2x - y = 2\tau \implies x - \frac{y}{2} = \tau \to z = \frac{x}{2} - \frac{y}{4} + 1 \implies z = \frac{2x - y + 4}{4}.$$

We conclude that the solution is

$$z(s,\tau) = u(x,y) = \frac{2x - y + 4}{4}.$$

b)

If $u(x,x) = 1$ then we get the characteristics

$$x = 2\tau + s; \qquad u = 2\tau + s; \qquad z = \frac{\tau}{2} + 1.$$

There exists a solution $u(x,y)$ provided that $\det D\Phi(s_0,\tau_0) \neq 0$. In our case,

$$\det \begin{vmatrix} \frac{\partial x}{\partial s}(s,0) & \frac{\partial y}{\partial s}(s,0) \\ \frac{\partial x}{\partial \tau}(s,0) & \frac{\partial y}{\partial \tau}(s,0) \end{vmatrix} = \det \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0,$$

therefore the inverse function theorem fails for this particular initial condition ,thence no solution exists.

Question 5

From the notes and weekly worksheet , we know that the implicit solution to the Burger's equation given is

$$u(x,t) = u_0(x - ut).$$

Using the problem statement,

$$\begin{aligned} u(x,t) = (x-ut)^2 = x^2 - 2xut + u^2t^2 \implies u^2t^2 - (2xt+1)u + x^2 = 0 \\ u(x,t) &= \frac{2xt + 1 \pm \sqrt{(2xt+1)^2 - 4t^2u^2}}{2t^2} \\ &= \frac{2xt + 1 \pm \sqrt{4xt+1}}{2t^2}. \end{aligned}$$

We verify $u(x,0)$ for the positive root ; Since this is a product of two functions which both limit's do not converge to 0 so we apply the product rule ;

$$\lim_{t\to 0} \frac{2xt + 1 + \sqrt{4xt+1}}{2t^2} = \lim_{t\to 0} \frac{1}{t^2} \frac{1}{2} \lim_{t\to 0}(2xt + 1 + \sqrt{4xt+1}) = \lim_{t\to 0} \frac{1}{t^2} = \infty,$$

the positive root solution diverges at the initial condition which does not match the given function. We verify the negative root ; We can not apply the product rule here, we try rationalizing the root

$$\lim_{t\to 0} \frac{2xt + 1 - \sqrt{4xt+1}}{2t^2} = \lim_{t\to 0} \frac{2xt + 1 - \sqrt{4xt+1}}{2t^2} \frac{2xt + 1 + \sqrt{4xt+1}}{2xt + 1 + \sqrt{4xt+1}} = \lim_{t\to 0} \frac{4x^2}{1 + 2tx + \sqrt{4tx+1}} = x^2,$$

the positive solution indeed matches the initial condition. We conclude that

$$u(x,t) = \begin{cases} \frac{2xt+1-\sqrt{4xt+1}}{2t^2} & \text{for } x \geq ut, \\ 0 & \text{for } x < ut. \end{cases}$$