PHYS 350 Assignment 4

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Question 0

To express the potential everywhere using only sines and cosines then the A and B terms in

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \tag{1}$$

should vanish. This condition is met if and only if $V \longrightarrow 0$ as $x \longrightarrow \infty$ and with $V \longrightarrow 0$ as $x \longrightarrow -\infty$, which is possible in a setting where no potential barrier is set in the x direction such that the potential vanishes in both directions

Question 1

a)

Using the general form of separation of variables (1), from $V(x,0) = 0 \implies$

$$V(x,0) = 0 \implies \underbrace{(Ae^{kx} + Be^{-kx})}_{\neq 0}(0 + D) = 0 \implies D = 0,$$
$$V(0,y) = 0 \implies A = -B\ quad \text{ since } C = 0 \text{ solution is rejected(trivial general solution)}$$
$$V(x,a) = 0 \implies \sin ky = \frac{n\pi}{a}$$

Therefore,

$$\begin{aligned}
V(x,y) &= A(e^{kx} - e^{-kx})C\sin(ky) \\
&= A(e^{n\pi x/a} - e^{-n\pi x/a})\sin\left(\frac{n\pi y}{a}\right) \\
&= \sum_{n=1}^{\infty} A_n(e^{n\pi x/a} - e^{-n\pi x/a})\sin\left(\frac{n\pi y}{a}\right) \\
&= 2\sum_{n=1}^{\infty} \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)
\end{aligned}$$

b)

We now determine the coefficients A_n using the boundary condition $V_0(y) = V_0$.We use Fourier's trick

$$\int_0^a V_0(y)\sin\left(\frac{n'ny}{a}\right)\,\mathrm{d}y = \left(\frac{a}{2}\right)A_n\sinh\left(\frac{n\pi b}{a}\right)$$
$$\implies A_n = \frac{2}{\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)\,\mathrm{d}y$$
$$\implies \int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right) = \begin{cases} 0 & n \text{ even} \\ \frac{2V_0}{n\pi} & n \text{ odd} \end{cases}$$

We conclude that

$$A_n = \begin{cases} \frac{4}{\sinh\left(\frac{n\pi b}{a}\right)}\left(\frac{V_0}{n\pi}\right) & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

The potential everywhere given the boundary condition is then

$$V(x,y) = \sum_{\text{odd}} \frac{4V_0}{n\pi}\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi y}{a}\right). \tag{2}$$

Remark. *We note that if the setup was a square instead of a rectangle the* sinh *arguments in the* $V(x,y)$ *expression including the boundary condition* (2) *will cancel out such that the potential in the centre would be*

$$V(x,y) = \sum_{odd} \frac{4V_0}{n\pi}\sin\left(\frac{n\pi y}{a}\right).$$

Question 2

The potential around the sphere is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r < R)$$
$$V(r,\theta) = \sum_{l=0}^{\infty} A_l' r^{-(l+1)} P_l(\cos\theta) \qquad (r \geq R),$$

Then at the boundary $r = R$ it follows that

$$V(R,\theta) = \sum A_l R^l P_l(\cos\theta) = \sum A_l' R^{-(l+1)} P_l(\cos\theta) \implies A_l' = A_l R^{2l+1}.$$

There exists a relationship between the surface charge and the potential difference at the boundary ;

$$\hat{n}\left(E_{\text{out}} - E_{\text{in}}\right) = \frac{\sigma}{\epsilon_0} = \left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right) = -\frac{\sigma(\theta)}{\epsilon_0}.$$

Combining the latter relationships we get

$$\sum (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\epsilon_0}. \tag{3}$$

We use Fourier's Trick for $\sum A_l R^l P_l(\cos\theta) = V_0(\theta)$

$$\int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta = \begin{cases} 0 & \text{if } \ l' \neq l \\ \frac{2}{2l+1} & \text{if } \ l' = l \end{cases}$$

So after integrating anf multiplying by $P_l(\cos\theta)\sin\theta$ the above then rearranging we obtain

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta \, \mathrm{d}\theta$$

Then by susbsitution ,

$$\frac{\sigma(\theta)}{\epsilon_0} = \sum (2l+1) R^{l-1} \frac{2l+1}{2R^l} P_l(\cos\theta) \underbrace{\int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta}_{\equiv C_l}$$

So finally we conclude

$$\sigma = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos\theta)$$
$$\text{where } \ C_l = \int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta \, \mathrm{d}\theta.$$

Question 3

We know that

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r < R)$$
$$V(r,\theta) = \sum_{l=0}^{\infty} A_l' r^{-(l+1)} P_l(\cos\theta) \qquad (r \geq R),$$

So we have the relationship

$$A_l' = A_l R^{2l+1} \implies \sum (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\epsilon_0} \tag{4}$$

We use Fubini's trick ,

$$
\begin{aligned}
(2l+1)A_{l'}R^{l'-1}\int_0^\pi P_{l'}(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \sigma_0(\theta)\epsilon_0\int_0^\pi P_{l'}(\cos\theta)\sin\theta \\
\implies 2A_lR^{l-1} &= \frac{\sigma}{\epsilon_0}\int_0^\pi P_{l'}(\cos\theta)\sin\theta\,\mathrm{d}\theta \\
\implies A_l &= \frac{1}{2\epsilon_0R^{l-1}}\int_0^\pi \sigma_0(\theta)P_l(\cos\theta)\sin\theta\mathrm{d}\theta
\end{aligned}
$$

Then since

$$
\sigma_0(\theta)=\sigma_0 \quad \text{for } \theta \in (0,\pi 2) \text{ and } \sigma_0(\theta)=-\sigma_0 \quad \text{for } \theta \in (\pi,\pi/2),
$$

this is equivalent to

$$
\begin{aligned}
A_l &= \frac{\sigma_0}{2\epsilon_0R^{l-1}}\left(\int_0^{\pi/2}P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta - \int_{\pi/2}^{\pi}P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta\right) \\
&\overset{x=\cos(\theta)}{=} \frac{\sigma_0}{2\epsilon_0R^{l-1}}\left(-\int_1^0 P_l(x)\,\mathrm{d}x + \int_0^{-1}P_l(x)\,\mathrm{d}x\right)
\end{aligned}
$$

We use the properties of integral bounds $\int_a^b = -\int_b^a$, we get

$$
= \frac{\sigma_0}{2\epsilon_0R^{l-1}}\left(\int_0^1 P_l(x)\,\mathrm{d}x - \int_{-1}^0 P_l(x)\,\mathrm{d}x\right)
$$

We use the general property of Legendre polynomials that $P_l(-x)=(-1)^nP_l(x)$,to exploit this poperty we perform a change of variable $x\to -x$ in the second integral. We get

$$
\begin{aligned}
&= \frac{\sigma_0}{2\epsilon_0R^{l-1}}\left(\int_0^1 P_l(x)\,\mathrm{d}x - \int_1^0 P_l(-x)\,\mathrm{d}(-x)\right) \\
&= \frac{\sigma_0}{2\epsilon_0R^{l-1}}\int_0^1 P_l(x)(1-(-1)^n)\,\mathrm{d}x \\
\therefore A_l &= \begin{cases} 0 & \text{for } l \text{ even} \\ \dfrac{\sigma_0}{2\epsilon_0R^{l-1}}\displaystyle\int_0^1 P_l(x)\,\mathrm{d}x & \text{for } l \text{ odd.}\end{cases} \qquad (5)
\end{aligned}
$$

We look for A_l for $l\in[0,6]$. We note following (5) that A_l for l even are all 0. Therefore, using the Legendre polynomials expressions (source: Wikipedia) we have

$$
A_1 = \frac{\sigma_0}{\epsilon_0}\int_0^1 P_1(x)\,\mathrm{d}x = \frac{\sigma_0}{\epsilon_0}\int_0^1 x\,\mathrm{d}x = \frac{\sigma_0}{2\epsilon_0}
$$

$$A_3 = \frac{\sigma_0}{\epsilon_0 R^2}\int_0^1 P_3(x)\mathrm{d}x = \int_0^1 \frac{5x^3-3x}{2}\,\mathrm{d}x = \frac{-\sigma_0}{8\epsilon_0 R^2}$$
$$A_5 = \frac{\sigma_0}{\epsilon_0 R^4}\int_0^1 P_5\,\mathrm{d}x = \int_0^1 \frac{63x^5-70x^3+15x}{8}\,\mathrm{d}x = \frac{\sigma_0}{16\epsilon_0 R^4}$$

We also need the $B_l(A_l')$ coefficients for $l \in [0,6]$. We use the relationship (4), which essentially amounts to multiplying the A_l coefficients by R^{2l+1}

$$B_1 = A_1 R^3 = \frac{\sigma_0}{2\epsilon_0}R^3 \qquad , B_3 = A_3 R^7 = \frac{-\sigma_0}{8\epsilon_0}R^5 \qquad , B_5 = A_5 R^{11} = \frac{\sigma_0}{16\epsilon_0}R^7.$$

Question 4

We know that
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} \overset{\text{cylindrical}}{=} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) = 0. \tag{6}$$

We look for solution $V(r,\phi) = R(r)\Phi(\phi)$ so from (6) ,

$$\implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\left(\frac{\partial^2 \Phi}{\partial \phi^2}\right) = 0.$$

Let us divide the above vy Φ and multiply by r^2 for convenience.

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\left(\frac{\partial^2 \Phi}{\partial \phi^2}\right),$$

since the above is true point wise for functions of different variables then we get

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \pm\lambda \qquad \text{and} \quad \frac{1}{\Phi}\left(\frac{\partial^2 \Phi}{\partial \phi^2}\right) = -\lambda^2$$

For first solution in the RHS, $\Phi_{\phi\phi} + \lambda^2\Phi = 0$, solving the characteristic equation yields $k^2 + \lambda^2 = 0 \implies k = \pm i\lambda$. Complex roots so the general solution is

$$\Phi = A\cos(\lambda\phi) + B\cos(\lambda\phi). \tag{7}$$

To solve the RHS PDE, we first use product rule

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda^2 \overset{P.R}{\implies} r^2 R_{rr} + rR_r - \lambda^2 R = 0.$$

This is a Cauchy-Euler equation. To solve we let

$$s = \ln(r) \implies R(r) = \varphi(\ln(s)) = \varphi(s),$$

so then

$$
\begin{gathered}
R_r = \varphi'(s)\frac{1}{r} \qquad \text{and} \quad R_{rr} = \frac{1}{r^2}(\varphi''(s) + \varphi'(s)) \\
\implies \left[\frac{r^2}{r^2}\varphi''(s) + \frac{r^2}{r^2}\,\varphi'(s)\right] - \frac{\varphi'(s)}{r} - \lambda^2\varphi(s) = 0 \\
\implies \varphi''(s) - \lambda^2\varphi(s) = 0
\end{gathered}
$$

Solving the characteristic equation we get $k = \pm\lambda$. Similar roots so the general solution takes the form

$$
\varphi(s) = Ae^{\lambda s} + Be^{-\lambda s} \implies R(r) = Ar^{\lambda} + Br^{-\lambda}. \tag{8}
$$

We may extract another solution related to (7) if we set the initial PDE equal to $\lambda^2 = 0$, we obtain

$$
\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \implies \Phi(\phi) = A\phi + B.
$$

Moreover we can extract another solution linked to (8) if we set the initial PDE equal to $\lambda^2 = 0$, we obtain

$$
\int \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \int 0 \implies rR_r = C \implies C\ln(r) + D = R(r).
$$

Our final 4 solutions are

$$
\begin{cases}
\Phi(\phi) = A\cos(\lambda\phi) + B\cos(\lambda\phi) \\
\Phi(\phi) = A\phi + B \\
R(r) = Ar^{\lambda} + Br^{-\lambda} \\
R(r) = C\ln(r) + D
\end{cases} \tag{9}
$$

Question 5

a)

We can not use separation of variables because at $\xrightarrow{x\to\infty} \neq \infty \neq 0$, such that the exponentials can't be used. And we know from Question 0 that such case is not possible.

b)

Let $V = V_1 + V_2$ where V_1 is the potential everywhere from a setup where we have V_s on the strip and $V = 0$ on both plates. Also, V_2 is the potential everywhere from a setup where we have $V_s = 0$ on the strip, $V = V_0$ on the top plate and $V = 0$ on the bottom plate.
For V_1, given the symmetrical nature of the problem it follows that

$$
V_s = -V_s + V_0 \implies V_s = \frac{V_0}{2},
$$

the solution is then of the following form along with its boundary conditions

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \quad \text{with} \quad \begin{cases} V = 0 & \text{when } \; y = 0, \\ V = 0 & \text{when } \; y = a, \\ V = \frac{V_0(y)}{2} & \text{when } \; x = 0, \\ V \to 0 & \text{as } \; x \to \infty. \end{cases}.$$

c)

Given the boundary conditions we conclude that the solution is of the form

$$V(x,y) = Ce^{-kx}\sin ky \qquad \text{with } k = n\pi/a, \quad \text{for } \; n \in N_+$$
$$\implies V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a).$$

We then use Fourier's trick, the same procedure as outlined in previous questions, we get

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = \frac{V_0(y)}{2}$$
$$C_n = \frac{V_0}{a}\int_0^a \sin(n\pi y/a)\,\mathrm{d}y = \begin{cases} 0, & \text{if } \; n \; \text{ even} \\ \frac{2V_0}{n\pi} & \text{if } \; n \; \text{ if odd.} \end{cases}$$
$$\therefore V(x,y) = \frac{2V_0}{\pi}\sum_{\text{odd}} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a).$$

Then we find the potential from V_2. The potential from an infinite sheet of charge is given by

$$V_2 = -\frac{\sigma y}{2\epsilon_0},$$

where σ is the surface density charge. We conclude that the potential everywhere for the whole setup is

$$V(x,y) = V_1 + V_2 = \frac{2V_0}{\pi}\sum_{\text{odd}} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a) - \frac{\sigma y}{2\epsilon_0}.$$

Question 6

We use almost the same procedure as outlined in Question 2 The potential around the sphere is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad\qquad (r < R)$$
$$V(r,\theta) = \sum_{l=0}^{\infty} A_l' r^{-(l+1)} P_l(\cos\theta) \qquad (r \geq R),$$

Then at the boundary $r = R$ it follows that

$$V(R,\theta) = \sum A_l R^l P_l(\cos\theta) = \sum A_l' R^{-(l+1)} P_l(\cos\theta) \implies A_l' = A_l R^{2l+1},$$

which is our first relationship.
We require that the electric fields be continuous around both sides, specifically in the hole , hence we set

$$\frac{\partial V_{\text{out}}}{\partial r} = \frac{\partial V_{\text{in}}}{\partial r}.$$

Given the condition at the boundary $r = R$, we conclude that The coefficients are then given by

$$C_l = \int_0^{\theta_0} V_0(\theta) P_l \cos(\theta) \sin\theta \, \mathrm{d}\theta.$$