PHYS 241 Final Exam

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Preliminary

My signature below certifies that I have not, nor will I, consult with any other person about the exam, or any other subject related to it

Question 1

(a)

We first and foremost write the KVL equations

$$\begin{array}{lr} \textbf{(KCL)} & I_1 + I_3 = I_2 \\ \textbf{(KVL1)} & V_1 - R_1 I_1 - R_2 I_2 = 0 \\ \textbf{(KVL2)} & V_2 - I_2(R_2+R_3) + I_1(R_3) = 0 \end{array} \implies \begin{cases} R_1 I_1 + R_2 I_2 & = \mathrm{V}_1 \\ -R_3 I_1 + (R_2+R_3) I_2 & = \mathrm{V}_2 \end{cases}$$

This leads to the linear equation system

$$\begin{pmatrix} R_1 & R_2 \\ -R_3 & (R_2+R_3) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \to \begin{pmatrix} 5\text{ k}\Omega & 1\text{ k}\Omega \\ -1\text{ k}\Omega & 2\text{ k}\Omega \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 12\text{ V} \\ 3\text{ V} \end{pmatrix}.$$

This system is trivially solved using *Cramer's Rule*

$$\begin{pmatrix} 5\text{ k}\Omega & 1\text{ k}\Omega \\ -1\text{ k}\Omega & 2\text{ k}\Omega \end{pmatrix}^{-1} \begin{pmatrix} 12\text{ V} \\ 3\text{ V} \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 2\text{ k}\Omega & -1\text{ k}\Omega \\ 1\text{ k}\Omega & 5\text{ k}\Omega \end{pmatrix} \begin{pmatrix} 12\text{ V} \\ 3\text{ V} \end{pmatrix} = \begin{pmatrix} 21/11 \\ 27/11 \end{pmatrix}.$$

Therefore, the values for I_1 and I_2 are respectively 1.909 mA and 2.455 mA.

To find the values of V_A and V_B the nodal method is applied

$$I_1 = \frac{V_1 - V_A}{R_2} \implies V_A = V_1 - I_1 R_1 = 12\text{ V} - 1.909 \times 10^{-3}\text{ A} \times 5000\ \Omega = 2.455\text{ V}.$$

The voltage at the point V_B is simply 0 V since that point is a ground.

(b)

Since voltage is conserved along a loop and since there is a ground placed right after the positive polarity of the f.e.m source, then V_B must necessarily be -12 V. Moreover, the direction of the currents is not altered by changing the ground's place in the present configuration so we may apply the nodal method for I_2 that was found in (a)

$$I_2 = \frac{V_A - V_B}{R_2} \implies V_A = I_2 R_2 + V_B = 2.455 \text{ mA} \times 1000\ \Omega - 12 \text{ V} = -9.455 \text{ V}.$$

Question 2

(a)

At $t = 0$ the capacitor is fully discharged,hence $Q(0) = 0$ C $\implies \Delta V_C(0) = 0$ V. Since R_2 is in parallel with C ,then it follows that $\Delta V_{R_2}(0) = 0$ V as well. Applying Kirchhoff's law along the first loop and isolating I yields

$$I(0) = \frac{V_0}{R_1}.$$

(b)

After a long time, the capacitor is *"fully"* charged such that no voltage is dropped across it. So we may effectively remove this component from the circuit ,compute the equivalent resistance from the resistors in series and apply Kirchhoff's law along the circuit to obtain

$$I(t)_{t\to\infty} = \frac{V_0}{R_1 + R_2}.$$

(c)

Three equations are needed

$$①.\ \ I = I_1 + I_2 \qquad ②.\ \ V_0 = I_1 R_1 + I_2 R_2 \qquad ③.\ \ R_2 I_2 = \frac{Q}{C}.$$

We first differentiate with respect to time ③ and isolate $\dot{I}_2$ yielding

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = \frac{I_1}{R_2 C}. \tag{1}$$

Then we substitute ① into ② giving

$$V_0 = R_1 I_1 + (R_1 + R_2) I_2. \tag{2}$$

Differentiating with respect to time Equation 2 and substitute Equation 1 inside

$$\begin{aligned} V_0 = R_1 I_1 + (R_1+R_2) I_2 \xrightarrow{d/dt} 0 &= R_1 \frac{\mathrm{d}I_1}{\mathrm{d}t} + (R_1+R_2)\frac{\mathrm{d}I_2}{\mathrm{d}t} \\ &= R_1 \frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{(R_1+R_2)}{R_2 C} I_1 \end{aligned}$$

Diving both sides by R_1 and letting $\tau \equiv R_1 R_2 C/(R_1+R_2)$ we have a first order linear differential equation

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{I_1}{\tau} = 0.$$

From ODEs , there exists an integrating factor $\mu(t) = e^{\int p(t)\ dt} = e^{\int 1/\tau\ dt} = e^{t/\tau + C} = e^{t/\tau} e^{C} = C_1 e^{t/\tau}$, where $C_1 \equiv e^C$. We let $C_1 = 1$ since only one integrating factor is needed and thus $\mu(t) = e^{t/\tau}$. Moreover, the general solution is given by $y(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\ dt$, i.e.,

$$I_1(t) = \frac{1}{e^{t/\tau}} \int e^{t/\tau}(0)\ dt = C_2 e^{-t/\tau}$$

The initial condition is the initial current therefore $C_2 \equiv I_1(0)$, finally

$$I_1(t) = I_1(0) e^{-t/\tau}. \tag{3}$$

We now integrate Equation 1 and substitute Equation 3 to have an expression for $I_2(t)$

$$I_2(t) = \int_0^t \frac{I_1}{CR_2}\ dt = \frac{I_1(0)}{CR_2} \int_0^t e^{-t/\tau}\ dt = \frac{I_1(0)}{CR_2} \tau \left(1 - e^{-t/\tau}\right).$$

Combining the previous result with Equation 3 provides a value for the current $I(t)$

$$\begin{aligned} I(t) = I_1(t) + I_2(t) &= I_1(0) e^{-t/\tau} + \frac{I_1(0)}{CR_2}\tau\left(1 - e^{-t/\tau}\right) \\ &= I_1(0)\left[e^{-t/\tau} + \frac{\tau\left(1 - e^{-t/\tau}\right)}{cR_2}\right] \end{aligned}$$

Using the result from (a) , $I_1(0)$ is the initial current in the circuit therefore

$$I(t) = \frac{V_0}{R_1}\left[e^{-t/\tau} + \frac{\tau\left(1 - e^{-t/\tau}\right)}{cR_2}\right] \qquad \text{, with} \quad \tau = \frac{R_1 R_2 C}{R_1 + R_2}.$$

Question 3

(a)

$$\frac{V'}{V} = \frac{Z_{R_2}}{Z_{R_1} + Z_{R_2}} = \frac{R_2}{R_1 + R_2}.$$

The amplitude is simply $R_2/(R_1+R_2)$ since this is not a complex number. The phase is 0 since there is no imaginary part and $\tan^{-1}(0) = 0$. Having found the essential terms , we write

$$V_{\text{out}} = \frac{V_0 \cos(\omega t) R_2}{(R_1 + R_2)}.$$

(b)

Let $R_T \equiv R_1 + R_2$ since the resistors are in series. Then

$$\frac{V'}{V} = \frac{Z_C}{Z_C + Z_{R_T}} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R_T} = \frac{1}{1 + j\omega C R_T} = \frac{1}{1 + j\omega\tau} \qquad \text{, with } \quad \tau = R_T C.$$

We compute the amplitude

$$\left|\frac{V'}{V}\right| = \left(\left(\frac{1}{1 + j\omega\tau}\right)\left(\frac{1}{1 - j\omega\tau}\right)\right)^{1/2} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}.$$

It follows that for the phase

$$\because \; \left(\frac{1}{1 + j\omega\tau}\right)\left(\frac{1 - j\omega\tau}{1 - j\omega\tau}\right) = \frac{1}{1 + (\omega\tau)^2} - j\frac{\omega\tau}{1 + (\omega\tau)^2}$$
$$\varphi(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}[V'/V]}{\mathrm{Re}[V'/V]}\right) = -\tan^{-1}\omega\tau.$$

Having found all the essential terms it follows that

$$V'(t) = V_{\text{out}} = \frac{V_0}{\sqrt{1 + (\omega\tau)^2}} \cos\big(\omega t - \tan^{-1}\omega\tau\big).$$

(c)

As $\omega \to 0$ the impedance of the capacitor is very high compared to the equivalent resistance, therefore little voltage is dropped across the resistors such that $V_0' \approx 0$. The current leads by $\pi/2$ meaning that the phase shift is $\pi/2$.

As $\omega \to \infty$ the capacitor's impedance vanishes and all voltage is dropped across the equivalent resistors such that $V_0' \approx V_0$ and thus the phase shift is ≈ 0.

Given these limits and the circuit's configuration , we can regard this circuit as a low-pass RC filter.

Question 4

(a)

Let us first compute a_0

$$\frac{a_0}{2} = \frac{1}{T}\left(\int_{-T/2}^{0} -V_0 \; dt + \int_{0}^{T/2} V_0 \; dt\right) = \frac{1}{T}\left(-\frac{T}{2} + \frac{T}{2}\right) = 0 \implies a_0 = 0.$$

The given square wave is an odd function therefore $a_n = 0 \; \forall \; n \in \mathbb{N}_+$.

$$\begin{aligned}
b_{n>0} &= \frac{1}{T}\left[\int_{-T/2}^{0} -V_0 \sin(\omega_n t)\; dt + \int_0^{T/2} V_0 \sin(\omega_n t)\; dt\right] \\
&= \frac{1}{T}\left[-\frac{V_0}{\omega_n}\left(-\cos(\omega_n t)\Big|_{-T/2}^{0}\right) + \frac{V_0}{\omega_n}\left(-\cos(\omega_n t)\Big|_{0}^{T/2}\right)\right] \\
&= \frac{1}{T}\left[\frac{-V_0}{\omega_n}\left(-1+\cos\left(\frac{-2n\pi T}{2T}\right)\right) + \frac{V_0}{\omega_n}\left(-\cos\left(\frac{2n\pi T}{2T}\right)+1\right)\right] \\
&= \frac{-V_0}{2n\pi}\cos(-\pi n) + \frac{V_0}{n\pi} - \frac{V_0}{2n\pi}\cos(n\pi)
\end{aligned}$$

Using the identity $\cos(-x) = \cos(x)$ and rearranging yields

$$\begin{aligned}
&= \frac{V_0}{n\pi}(1-\cos(n\pi)) \\
&= \frac{2V_0}{n\pi}\left(\frac{1-\cos(n\pi)}{2}\right) = \frac{2V_0}{n\pi}\sin^2\left(\frac{n\pi}{2}\right).
\end{aligned}$$

We note that

$$\frac{2V_0}{n\pi}\sin^2\left(\frac{n\pi}{2}\right) \quad \begin{cases} 1 & \text{, for } n \text{ odd} \\ 0 & \text{, for } n \text{ even} \end{cases}$$

Finally, we write the Fourier series for the given square wave

$$f(t) = \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left(\frac{2\pi(2n-1)}{T}t\right).$$

(b)

We first and foremost find an expression for the transfer function

$$H(\omega) = \frac{\left(\frac{1}{Z_L}+\frac{1}{Z_C}\right)^{-1}}{\left(\frac{1}{Z_L}+\frac{1}{Z_C}\right)^{-1}+Z_R} = \frac{\left(\frac{1}{j\omega L}+j\omega C\right)^{-1}}{\left(\frac{1}{j\omega L}+j\omega C\right)^{-1}+R} = \frac{1}{1+R\left(\frac{1}{j\omega L}+j\omega C\right)}$$

Multiplying by $j\omega L$ and diving by R whilst defining $\tau \equiv L/R$ with $LC = 1/\omega_0^2$ yields

$$H(\omega) = \frac{j\omega L}{j\omega L + R(1-(\omega/\omega_0)^2)} = \frac{j\omega\tau}{j\omega\tau + 1 - (\omega/\omega_0)^2}.$$

We carry on by finding the amplitude of $H(\omega)$

$$|H(\omega)| = \left(\left(\frac{j\omega\tau}{j\omega\tau+1-(\omega/\omega_0)^2}\right)\left(\frac{-j\omega\tau}{-j\omega\tau+1-(\omega/\omega_0)^2}\right)\right)^{1/2} = \frac{\omega\tau}{\sqrt{(1-(\omega/\omega_0)^2)^2+(\omega\tau)^2}}.$$

Finally,we look for the phase

$$\left(\frac{j\omega\tau}{j\omega\tau+1-(\omega/\omega_0)^2}\right)\left(\frac{1-(\omega/\omega_0)^2-j\omega\tau}{1-(\omega/\omega_0)^2-j\omega\tau}\right)=\frac{(\omega\tau)^2}{(1-(\omega/\omega_0)^2)^2+(\omega\tau)^2}+j\frac{(\omega\tau)(1-(\omega/\omega_0)^2)}{(1-(\omega/\omega_0)^2)^2+(\omega\tau)^2}$$
$$\therefore \varphi_n(\omega_n)=\tan^{-1}\left(\frac{(\omega\tau)(1-(\omega/\omega_0)^2)}{(\omega\tau)^2}\right)=\tan^{-1}\left(\frac{1-(\omega/\omega_0)^2}{\omega\tau}\right).$$

Using the result from (a) we write the V_{out} function

$$V_{\text{out}}=\sum_{n=1}^{\infty}\frac{2V_0}{\pi(2n-1)}\frac{\omega_n\tau}{\sqrt{(1-(\omega_n/\omega_0)^2)^2+(\omega_n\tau)^2}}\sin\left(\omega_n t+\tan^{-1}\left(\frac{1-(\omega_n/\omega_0)^2}{\omega_n\tau}\right)\right). \quad (4)$$

To see which frequencies go through , we look for $\omega_0\pm\Delta\omega$

$$\omega_0=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{(1\text{ mH})(2.8\text{ µF})}}=18898\text{ s}^{-1}.$$
$$\tau=\frac{L}{R}=\frac{1\text{ mH}}{3\text{ k}\Omega}=3.33\times10^{-7}\text{ s}.$$
$$Q=\frac{1}{\tau\omega_0}=\frac{1}{(3.33\times10^{-7}\text{ s})(18898\text{ s}^{-1})}=158.74$$
$$\Delta\omega=\frac{\omega_0}{Q}=\frac{18898\text{ s}^{-1}}{158.74}=119.05\text{ s}^{-1}.$$
$$\therefore \omega_0\pm\Delta\omega=(18898\pm119)\text{ s}^{-1}.$$

We now look for the frequency components which lie within this range

$$\omega_1=\frac{2\pi(1)}{T}=\cancel{6283.2\text{ s}^{-1}}\qquad \omega_2=\frac{2\pi(2)}{T}=\cancel{12566.4\text{ s}^{-1}}\qquad \omega_3=\frac{2\pi(3)}{T}=18849.6\text{ s}^{-1}\ \checkmark.$$

The amplitude can then be computed with

$$A_{\text{out}}=\frac{2V_0}{\pi(2(3)-1)}\frac{\omega_3\tau}{\sqrt{(1-(\omega_3/\omega_0)^2)^2+(\omega_3\tau)^2}}$$
$$=\frac{2(1)}{\pi(2(3)-1)}\frac{(18849.6)(3.33\times10^{-7})}{\sqrt{(1-(18849.6/18898)^2)^2+((18849.6)(3.33\times10^{-7}))^2}}=0.09869\text{ V}.$$

THe phase is also computed with

$$\varphi=-\frac{\pi}{2}+\tan^{-1}\left(\frac{1-\left(\frac{18849.6}{18898}\right)^2}{18849.6(3.33\times10^{-7})}\right)=-\frac{\pi}{2}+39^\circ\approxeq-\frac{\pi}{4}.$$

Question 5

Since $f(t) = t/T$ for $|t| \leq T/2$, let us compute C_k using the definition

$$\begin{aligned}
C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\omega_k t}\ dt &= \frac{1}{T^2}\int_{-T/2}^{T/2} te^{-j\omega_k t}\ dt \\
&= \frac{1}{T^2}\left[\left.\frac{-te^{-j\omega_k t}}{j\omega_k}\right|_{-T/2}^{T/2} + \frac{1}{j\omega_k}\int_{-T/2}^{T/2} e^{-j\omega_k t}\ dt\right] \\
&= \frac{1}{T^2}\left[\frac{1}{j\omega_k}\left(\frac{-Te^{\frac{-j\omega_k T}{2}}}{2} - \frac{Te^{\frac{j\omega_k T}{2}}}{2}\right) + \frac{2}{j\omega_k^2}\left(\frac{e^{\frac{j\omega_k T}{2}} - e^{-\frac{j\omega_k T}{2}}}{2j}\right)\right] \\
&= \frac{1}{T^2}\left[\frac{-T\cos\left(\frac{\omega_k T}{2}\right)}{j\omega_k} + \frac{2\sin\left(\frac{\omega_k T}{2}\right)}{j\omega_k^2}\right] \\
&= \frac{1}{T^2}\left[\frac{Ti\cos\left(\frac{\omega_k T}{2}\right)}{\omega_k} - \frac{2i\sin\left(\frac{\omega_k T}{2}\right)}{\omega_k^2}\right]
\end{aligned}$$

Using $\omega_k = 2\pi k/T$ the expresion reduces to

$$C_k = \frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}.$$

We now compute the Fourier transform

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty}\left(\frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}\right)e^{j\omega_k t} \\
\therefore\quad F(\omega) &= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left(\frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}\right)e^{j\omega_k t}e^{-j\omega t}\ dt \\
&= \sum_{k=-\infty}^{\infty}\left(\frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}\right)\int_{-\infty}^{\infty} e^{j(\omega_k-\omega)t}\ dt \\
&= \sum_{k=-\infty}^{\infty}\left(\frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}\right)\underbrace{\int_{-\infty}^{\infty} e^{-j(\omega-\omega_k)t}\ dt}_{\equiv\delta(\omega-\omega_k)} \\
F(\omega) &= \sum_{k=-\infty}^{\infty}\left(\frac{\pi i k\cos(\pi k) - i\sin(\pi k)}{2\pi^2 k^2}\right)\delta(\omega-\omega_k).
\end{aligned}$$