MATH 325 Ass 5.

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Question 1

We note that since $p(x) = 0 \implies p_n = 0 \;\forall\; n \geq 0$. Also, $q(x) = k^2t^2 \implies q_n = 0 \;\forall\; n \geq k^2t^2$. Using the recursive formula we have

$$
\begin{aligned}
a_{n+2} &= \frac{-1}{(n+2)(n+1)} \sum_{k\geq 0} (k+1)a_{k+1}p_{n-k} + a_k q_{n-k} \\
&= \frac{-1}{(n+2)(n+1)} \sum_{k\geq 0} a_k q_{n-k} = \frac{-a_n k^2 t^2}{(n+2)(n+1)} \quad \forall\; n \geq 0.
\end{aligned}
$$

$$
\left\Vert
\begin{aligned}
a_2 &= \frac{-a_0 k^2 t^k}{2!} \\
a_4 &= \frac{-a_0 k^4 t^4}{24} \\
a_6 &= \frac{-a_0 k^6 t^6}{(30)(12)(2)}
\end{aligned}
\quad\middle|\quad
\begin{aligned}
a_3 &= \frac{-a_1 k^2 t^k}{6} \\
a_5 &= \frac{a_1 k^4 t^4}{120} \\
a_7 &= \frac{-a_1 k^6 t^6}{(120)(60)}
\end{aligned}
\right\Vert
$$

This leads to the recursion relationship

$$
\textbf{EVEN} : a_n = a_{2j} = \frac{(-1)^j a_0 k^{2j} t^{2j}}{(2j)!}, \quad j \geq 1, \quad \textbf{ODD} : a_n = a_{2j+1} = \frac{(-1)^j a_1 k^{2j} t^{2j}}{(2j+1)!} \quad j \geq 1.
$$

$$
\begin{aligned}
\therefore y(x) &= a_0 \left(\sum_{j\geq 0} \frac{(-1)^j k^{2j} t^{2j} x^{2j}}{(2j)!} \right) + a_1 \left(\sum_{j\geq 0} \frac{(-1)^j k^{2j} t^{2j} x^{2j+1}}{(2j+1)!} \right) \\
&= a_0 \left(\sum_{j\geq 0} \frac{(-1)^j k^{2j} t^{2j} x^{2j}}{(2j)!} \right) + \frac{a_1}{kt} \left(\sum_{j\geq 0} \frac{(-1)^j k^{2j+1} t^{2j+1} x^{2j+1}}{(2j+1)!} \right) \\
&= a_0 \cos(ktx) + \frac{a_1}{kt} \sin(ktx)
\end{aligned}
$$

Using the initial conditions we replace the coefficients a_0 and a_1 yielding the final expression

$$
y(x) = y_0 \cos(ktx) + \frac{y_0'}{kt} \sin(ktx).
$$

Question 2

We first and foremost note that $p(x) = x \implies p_0 = x \implies p_n = 0 \ \forall \ n \geq x$. Moreover, $q(x) = 2 \implies q_n = 0 \ \forall \ n \geq 2 \implies q_0 = 2$. We'll solve the given IVP by using the differentiating method of the power series.

$$y = \sum_{k\geq 0} a_n x^n, \qquad xy' = \sum_{k\geq 1} n a_n x^n, \qquad y'' = \sum_{k\geq 0}(n+2)(n+1)a_{n+2}x^n.$$

$$\begin{aligned}
\therefore y(x) &= \sum_{k\geq 0}(n+2)(n+1)a_{n+2}x^n + \sum_{k\geq 1} n a_n x^n + 2\sum_{k\geq 0} a_n x^n = 0 \\
&= a_2(2)(1) + \sum_{k\geq 1}(n+2)(n+1)a_{n+2}x^n + \sum_{k\geq 1} n a_n x^n + 2a_0 + \sum_{k\geq 1} 2a_n x^n = 0 \\
&= 2(a_2 + a_0) + \sum_{k\geq 1} (((n+2)(n+1)a_{n+2} + n a_n + 2a_n)x^n) = 0
\end{aligned}$$

$\because$ linear independence,

$$a_2 + a_0 = 0 \implies a_2 = -a_0 \quad \text{and} \quad ((n+2)(n+1)a_{n+2} + na_n + 2a_n)x^n = 0$$
$$\therefore a_{n+2} = \frac{-a_n(n+2)}{(n+2)(n+1)} = \frac{-a_n}{n+1} \quad , n \geq 1.$$

$a_2 = \frac{-a_0}{1}$	$a_3 = \frac{-a_1}{2}$
$a_4 = \frac{a_0}{3}$	$a_5 = \frac{a_1}{8}$
$a_6 = \frac{a_0}{15}$	$a_7 = \frac{-a_1}{48}$

This leads to the recursion relationship

$$\textbf{EVEN} : a_n = a_{2k} = \frac{(-1)^k a_0 2^k k!}{(2k)!} \quad k \geq 1, \quad \textbf{ODD} : a_n = a_{2k+1} = \frac{(-1)^k a_1}{2^k k!} \quad k \geq 1.$$

Finally,

$$y(x) = a_0 \sum_{k\geq 0} \frac{(-1)^k 2^k k! x^{2k}}{(2k)!} + a_1 \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{2^k k!}$$

Using the initial conditions we may replace a_0 and a_1, giving

$$= -3 \sum_{k\geq 0} \frac{(-1)^k 2^k k! x^{2k}}{(2k)!} + 2 \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{2^k k!}.$$

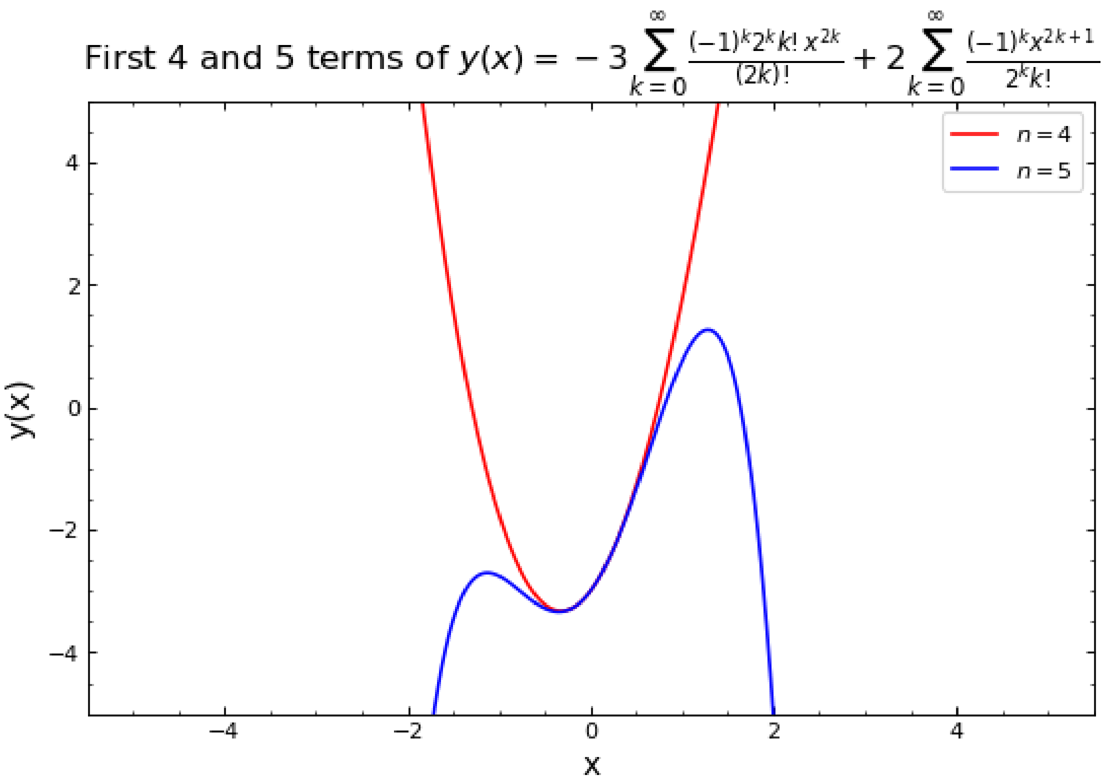

Figure 1: First 4 and 5 term approximation to the solution of Question 2.

Question 3

Due to confusion about the problem statement, I will provide two methods for finding the first three coefficients.

(METHOD 1) : The given IVP is exact thus

$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx$$
$$\sin^{-1}(y) = x + C,$$

given the initial condition $y(0) = 0$,

$$y = \sin(x+0) = \sin(x)$$
$$\therefore y(x) = \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \dots.$$

Therefore,using the initial condition $a_0 = 0$, the coefficients up to x^3 as requested are $\{0, 1, 0, -\frac{1}{6}\}$.

(METHOD 2) : Let $y(x)$ be analytic. Then it follows that

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$$

Since $a_0 = 0$,

$$\begin{aligned} &= a_1x + a_2x^2 + a_3x^3 + O(x^4) \\ \implies y'(x) &= a_1 + 2a_2x + 3a_3x^2 + O(x^3) \end{aligned}$$

Replacing in $y' = \sqrt{1-y^2}$ yields

$$\begin{aligned} y'(x) &= \sqrt{1-(a_1x + a_2x^2 + O(x^3))^2} \\ &= \sqrt{1-(a_1x^2 + 2a_1a_2x^3 + O(x^4))} \end{aligned}$$

We may now use the Taylor expansion for $\sqrt{1-x}$,

$$= 1 - \frac{1}{2}(a_1^2x^2 + 2a_1a_2x^3 + O(x^4))$$

Since we need the coefficients up to x^3 this reduces to

$$\begin{aligned} y'(x) &= 1 - \frac{1}{2}a_1^2x^2 + O(x^3) \\ \therefore a_1 + 2a_2x + 3a_3x^2 + O(x^3) &= 1 - \frac{1}{2}a_1^2x^2 + O(x^3) \\ \implies (a_1 - 1) + 2a_2x + \left(3a_3 + \frac{1}{2}a_1^2\right)x^2 &= 0 \end{aligned}$$

$\because$ linear independence,

$$\implies a_1 = 1 \quad ,a_2 = 0 \quad , a_3 = -\frac{1}{6}.$$

We conclude that the coefficients up to x^3 are $\{0, 1, 0, -\frac{1}{6}\}$.

Question 4

a)

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} - 2x\sum_{n=0}^{\infty} na_nx^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_nx^n - 2\sum_{n=0}^{\infty} na_nx^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \sum_{n=0}^{\infty} n(n-1)a_nx^n - 2\sum_{n=0}^{\infty} na_nx^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\implies \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n\right)x^n = 0$$

By linear independence,

$$\implies a_{n+2} = \frac{n(n-1)a_n + 2na_n - \alpha(\alpha+1)a_n}{(n+2)(n+1)}$$
$$= \frac{a_n((n-\alpha)(n+\alpha+1))}{(n+2)(n+1)} \quad n \geq 1.$$

$$\left| \begin{array}{l|l} a_2 = \dfrac{a_0(-\alpha)(\alpha+1)}{2} & a_3 = \dfrac{-a_1(\alpha-1)(\alpha+2)}{3!} \\ a_4 = \dfrac{a_0(\alpha+1)(\alpha-2)(\alpha+3)}{4} & a_5 = \dfrac{a_1(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)}{8} \\ a_6 = \dfrac{a_0-(\alpha+1)(\alpha-2)(\alpha+3)(\alpha-4)(\alpha+5)}{6!} & a_7 = \dfrac{-a_1(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)(\alpha-5)(\alpha+6)}{7!} \end{array} \right|$$

This leads to the recursion relationship

EVEN : $a_n = a_{2k} = \dfrac{(-1)^k a_0 \alpha(\alpha+2k-1)(\alpha+2k-3)\ldots(\alpha+1)(\alpha-2)\ldots(\alpha-2k+2)}{(2k)!}, \ k \geq 1$

ODD : $a_n = a_{2k+1} = \dfrac{(-1)^k a_1(\alpha+2k)(\alpha+2k-2)\ldots(\alpha+2)(\alpha-1)\ldots(\alpha-2k+1)}{(2k+1)!} \ ,k \geq 1.$

Therefore we write the series solution as

$$y(x) = y_0 y_1(x) + y_0' y_2(x),$$

where $\quad y_1(x) = 1 + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k a_0 \alpha(\alpha+2k-1)(\alpha+2n-3)\ldots(\alpha+1)(\alpha-2)\ldots(\alpha-2k+2)}{(2k)!}$

and $\quad y_2(x) = x + \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k a_1(\alpha+2k)(\alpha+2k-2)\ldots(\alpha+2)(\alpha-1)\ldots(\alpha-2k+1)}{(2k+1)!}$

b)

Letting $\alpha = 2k$, all the terms after the kth term are terminated since they contain the factor $\alpha - 2k$, therefore since the kth term has the highest power in x, that term will dictate the degree of the polynomial ,i.e., $2k$.

Similarly, if $\alpha = 2k+1$ then all the terms which contain $\alpha - (2k+1)$ will vanish ,leaving the kth term having the power of $2k+1$ such that $\deg(y_2(x)) = 2k+1$.

$$\left\| \begin{array}{ll|ll} y_1(x) = 1 & ,(\alpha=0) & y_2(x) = x & ,(\alpha=1) \\ y_1(x) = 1 - 3x^2 & ,(\alpha=2) & y_2(x) = x - \dfrac{5}{3}x^3 & ,(\alpha=3) \\ y_1(x) = 1 - 10x^2 + \dfrac{35}{3}x^4 & ,(\alpha=4) & y_2(x) = x - \dfrac{14}{3}x^3 + \dfrac{21}{5}x^5 & ,(\alpha=5) \end{array} \right\|$$

c)

$$
\begin{array}{llr}
\boldsymbol{(\alpha = 0)} \quad 1 \quad : & & P_0(1) = 1 \implies P_0(x) = 1\checkmark \\
\boldsymbol{(\alpha = 1)} \quad x \quad : & & P_1(1) = 1 \implies P_1(x) = x\checkmark \\
\boldsymbol{(\alpha = 2)} \quad 1 - 3x^2 \quad : & & P_2(1) = -2
\end{array}
$$

a(1-3) =1 $\implies a = -\dfrac{1}{2} \quad \therefore P_2(x) = \dfrac{3x^2 - 1}{2} \checkmark .$

$$\boldsymbol{(\alpha = 3)} \qquad x - \frac{5}{3}x^3 \quad : \qquad\qquad P_3(1) \neq 1$$

a$\left(1 - \dfrac{5}{3}\right) = 1 \implies a = -\dfrac{3}{2} \quad \therefore P_3(x) = \dfrac{5x^3 - 3x}{2} \checkmark .$

$$\boldsymbol{(\alpha = 4)} \qquad 1 - 10x^2 + \frac{35}{3}x^4 \qquad\qquad P_4(1) \neq 1$$

a$\left(1 - 10 + \dfrac{35}{3}\right) = 1 \implies a = \dfrac{3}{8} \quad \therefore P_4(x) = \dfrac{35x^4 - 30x^2 + 3}{8} \checkmark .$

$$\boldsymbol{(\alpha = 5)} \qquad x - \frac{14}{3}x^3 + \frac{21}{5}x^5 \qquad\qquad P_5(1) \neq 1$$

a$\left(1 - \dfrac{14}{3} + \dfrac{21}{5}\right) = 1 \implies a = \dfrac{15}{8} \quad \therefore P_5(x) = \dfrac{63x^5 - 70x^3 + 15x}{8} \checkmark .$

Question 5

$$y_{n+1} = y_n + h \begin{pmatrix} (y_1)_n(a - \alpha y_2) \\ (y_2)_n(-c + \gamma y_1). \end{pmatrix}$$

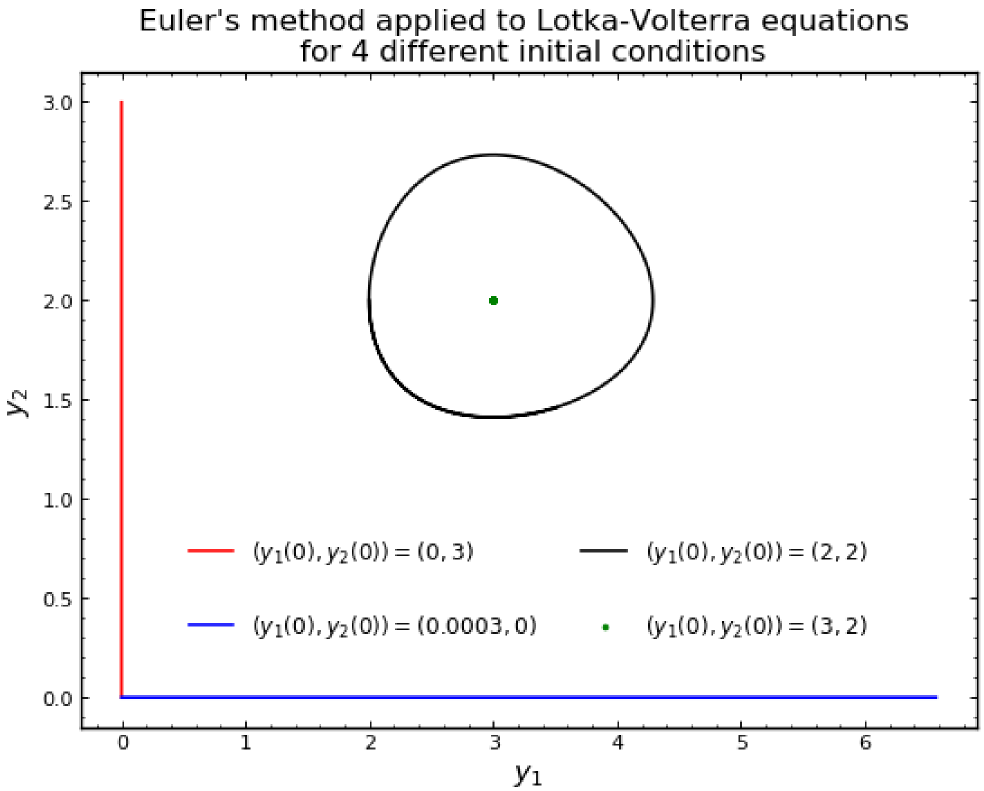

Figure 2: Euler's method applied to Lotka-Volterra biological model with $a, b, \alpha, \gamma = 1, 1/2, 3/4, 1/4$, respectively.

(a) In the absence of prey, the population of predators decrease exponentially since the interaction between the two populations is null and the death rate is negative. The number of predators start at 3 individuals as defined in the initial condition and converge to 0 as the time approaches infinity

(b) In the absence of predators, the population of prey increase exponentially since the interaction between the two populations is positive but small and the growth rate is positive. The number of prey start at 3/10000 individuals as defined in the initial condition and increase exponentially up to ≈ 6.7 given that 10000 cycle iterations are performed on the population, but would go indefinitely further given a larger number of iterations.

(c) IF there are initially 2 prey and 2 predators , the biological system will oscillate over time. The growth and death rate of the two species are inversely proportional to one other such that one species is striving while the other is not and the roles interchange consistently over time.

(d) If there are initially 3 prey and 2 predators, the biological system will remain constant over time. This means that the growth rate of prey and the death rate of the predators are both null since both species are ,within this model, mutually sustainable.