PHYS 350 Assignment 1

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September 16, 2020

Question 1

a)

To show this relationship we will use two lemmas :

- $\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = 1 + 1 + 1 = 3.$
- $\delta_{ij}\delta_{jk} = \delta_{ik}.$

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} &= \begin{vmatrix} \delta_{ik} & \delta_{il} & \delta_{im} \\ \delta_{jk} & \delta_{jl} & \delta_{jm} \\ \delta_{kk} & \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= 3(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) - \delta_{jk}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) + \delta_{ik}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \\
&= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} - \delta_{jm}\delta_{il} + \delta_{jl}\delta_{im} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} \\
&= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.
\end{aligned}$$

b)

$$\begin{aligned}
\vec{A} \times (\vec{B} \times \vec{C}) &= \epsilon_{ijk}A_j(\epsilon_{klm}B_lC_m) \\
&= \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m \\
&= \delta_{il}\delta_{jm}(A_jB_lC_m) - \delta_{im}\delta_{jl}(A_jB_lC_m) \\
&= A_jB_iC_j - A_jB_jC_i \\
&= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}).
\end{aligned}$$

Question 2

a)

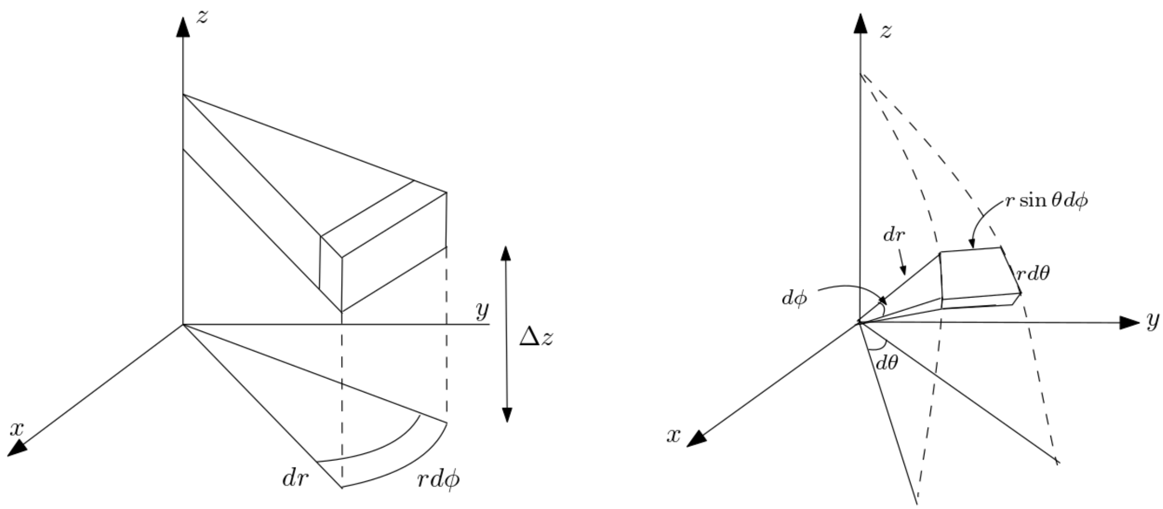

Figure 1: (a) Infinitesimal slab of a cylinder. **(b)** Infinitesimal slab of a sphere

In the r direction we have

- $f_r r d\phi dz$
- $f_{r+\Delta r} r d\phi dz = (f_r r + dr\frac{\partial(f_r r)}{\partial\partial r})d\phi dz$

We note that r is an implicit variable therefore it is embedded in the argument of $\partial(f_r r)$. The net force is then

$$\text{Net } = (f_r r + dr\frac{\partial(f_r r)}{\partial r})d\phi dz - f_r r d\phi dz = \frac{\partial(f_r r)}{\partial r} dr d\phi dz.$$

In the ϕ direction :

- $f_\phi dr dz$
- $f_{\phi+\Delta\phi} dr dz = (f_\phi + d\phi\frac{\partial f_\phi}{\partial\phi})dr dz$

Net force is then

$$\text{Net } = (f_\phi + d\phi\frac{\partial f_\phi}{\partial\phi})dr dz - f_\phi dr dz = \frac{\partial f_\phi}{\partial\phi} dr d\phi dz.$$

In the z direction :

- $f_z r dr \phi$

- $f_{z+\Delta z} r dr d\phi = (f_z + dz\frac{\partial f_z}{\partial z}) r dr d\phi$

The net force is then

$$\text{Net } = (f_z + dz\frac{\partial f_z}{\partial z}) r dr d\phi - f_z r dr \phi = \frac{\partial f_z}{\partial z} r dr d\phi dz.$$

Finally adding up all the results and dividing by r to account for the extra r in the z direction we have

$$\vec{\nabla} \cdot \vec{f} = \left(\left(\frac{1}{r}\right)\frac{\partial f_r r}{\partial r} + \left(\frac{1}{r}\right)\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}\right) dr d\phi dz.$$

b)

We proceed similarly for the spherical coordinates. First, in the r direction

- $f_r r^2 \sin\theta d\phi d\theta$
- $f_{r+\Delta r} r^2 \sin\theta d\phi d\theta = (f_r r^2 + dr\frac{\partial f_r r^2}{\partial r}) d\phi d\theta$

We proceed similarly as in the r direction of Question 2a in which we embedded the r value in the argument of the partial derivative , since it's an implicit value. The net force is then

$$\text{Net } = (f_r r^2 + dr\frac{\partial f_r r^2}{\partial r}) d\phi d\theta - f_r r^2 \sin\theta d\phi d\theta = \frac{\partial f_r r^2}{\partial r} dr d\phi d\theta.$$

In the θ direction :

- $f_\theta r \sin\theta d\phi dr$
- $f_{\theta+\Delta\theta} r \sin\theta d\phi dr = (f_r \sin\theta + d\theta\frac{\partial f_\theta \sin\theta}{\partial \theta}) r d\phi dr$

The net force is then

$$\text{Net } = (f_r \sin\theta + d\theta\frac{\partial f_\theta \sin\theta}{\partial \theta}) r d\phi dr - f_\theta r \sin\theta d\phi dr = \frac{\partial f_\theta \sin\theta}{\partial \theta} r d\theta dr d\phi.$$

In the ϕ direction :

- $f_\phi r dr d\theta$
- $f_{\phi+\Delta\phi} r dr d\theta = (f_\phi + d\phi\frac{\partial f_\phi}{\partial \phi}) r dr d\theta$

The net force is

$$\text{Net } = (f_\phi + d\phi\frac{\partial f_\phi}{\partial \phi}) r dr d\theta - f_\phi r dr d\theta = \frac{\partial f_\phi}{\partial \phi} r dr d\theta d\phi.$$

Adding up everything and diving by $r^2 \sin\theta$ to clean up the expression and match the arguments with the divided elements we obtain

$$\vec{\nabla} \cdot \vec{f} = \left(\left(\frac{1}{r^2}\right)\frac{\partial (f_r r^2)}{\partial r} + \left(\frac{1}{r\sin\theta}\right)\frac{\partial f_\theta \sin\theta}{\partial \theta} + \left(\frac{1}{r\sin\theta}\right)\frac{\partial f_\phi}{\partial \phi}\right) dr d\theta d\phi. \quad (1)$$

Question 3

To solve this, we use the result found previously (1).

$$\begin{aligned}
\mathrm{div} F = \vec{\nabla} \cdot \vec{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^{2+\epsilon}}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^{-\epsilon}) \\
&= \left(\frac{-\epsilon}{r^2}\right)\frac{1}{r^{\epsilon+1}} = -\frac{\epsilon}{r^{\epsilon+3}} \quad , \text{for} \quad r \in \mathbb{R} \setminus \{0\}.
\end{aligned}$$

Question 4

$q_1 = q_2 \equiv q$ and $m_p = m_p \equiv m$ In (MKS),

$$\frac{F_e}{F_g} = \frac{\frac{kq^2}{r^2}}{\frac{Gm^2}{r^2}} = \frac{kq^2}{Gm^2} = \frac{(8.987 \times 10^9)(1.602 \times 10^{-19})^2}{(1,672 \times 10^{-27})^2 (6.67 \times 10^{-11})} \frac{\mathrm{N}}{\mathrm{N}} \approx 1.237 \times 10^{36}.$$

It is to be noted that the choice of the unit system (MKS, CGS, English) does not matter in this context since the fraction is dimensionless.

Question 5

a)

We find the minimal δt by setting the uncertainty principle equal.

$$\delta t = \frac{\hbar}{2mc^2}.$$

The maximal rest mass is found in the same way while setting $c\delta t = d = 100000$ ly.

$$\delta E \delta t \leq \frac{\hbar}{2} \to d = c\delta t \implies d = \frac{\hbar}{2mc} \implies m = \frac{\hbar}{200000 \text{ ly c}}.$$

b)

If the photon doesn't have a mass then $E \sim \frac{1}{4\pi\epsilon_0 r^2}$. Alternatively, if the photon does have a mass, then the electric field has a bonus decay $E \sim \frac{q}{4\pi e p_0 r^2} e^{-r/r_0}$.
We set $r_0 = 100000$ ly $= 100000(9.46 \times 10^{15})$ m and set $r = 10000$ km $= 10^7$ m. We take the fraction of the two electric fields

$$\frac{\frac{q}{4\pi e p_0 r^2} e^{-r/r_0}}{\frac{1}{4\pi\epsilon_0 r^2}} = e^{-r/r_0} = \exp\left\{-\frac{10^7 \text{ m}}{100000(9.46 \times 10^{15}) \text{ m}}\right\} = e^{-1.05 \times 10^{-14}} \approx 0.999.$$

Bonus

GitHub username : PhysicsKush
URL to the project code :
`https://github.com/physicskush/PHYS350_Ass1/blob/master/PHYS350_Ass1_bonus.ipynb`