Q1) From Schrodinger's equation,

a) $$\frac{-\hbar^2}{2m}\nabla^2\psi + V(x)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$\Rightarrow \nabla^2\psi = \left(V(x)\psi - i\hbar\frac{\partial}{\partial t}\psi\right)\frac{2m}{\hbar^2}$$

$\because$ A complex wa function can be expresed

$\therefore$ Sy $\psi = u + iv$, we know that $u_x = v_t$ and $u_t = -v_x$

$$\Rightarrow \nabla^2\psi^* = \left(V(x)\psi^* + i\hbar\frac{\partial}{\partial t}\psi^*\right)\frac{2m}{\hbar^2}.$$

oops

So we proceed,

$$J = \left(\frac{-i\hbar}{2m}\right)\left(\frac{2m}{\hbar^2}\right)\Big\{\psi^*\left(V(x)\psi - i\hbar\frac{\partial}{\partial t}\psi\right) - \psi\left(V(x)\psi^* + i\hbar\frac{\partial}{\partial t}\psi^*\right)\Big\}$$

$$= \frac{-i}{\hbar}\Big\{\psi^* V(x)\psi - \psi V(x)\psi^* - \psi^* i\hbar\frac{\partial}{\partial t}\psi - \psi i\hbar\frac{\partial}{\partial t}\psi^*\Big\}$$

$$= \frac{-i}{\hbar}\Big\{-\psi^* i\hbar\frac{\partial}{\partial t}\psi - \psi i\hbar\frac{\partial}{\partial t}\psi^*\Big\}$$

$$= -\Big\{\psi^*\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^*\Big\}$$

$$= -\frac{\partial}{\partial t}(\psi^*\psi) = \frac{-\partial}{\partial t}|\psi|^2 = \frac{-\partial}{\partial t}\rho$$

$$\nabla\cdot J = \nabla\cdot\Big\{\frac{-i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\Big\}$$

$$= \frac{-i\hbar}{2m}\left(\nabla\psi^*\nabla\psi + \psi^*\nabla^2\psi - \nabla\psi\nabla\psi^* - \psi\nabla^2\psi^*\right)$$

$$= \frac{-i\hbar}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi\right)$$

$$\Rightarrow \frac{-\partial}{\partial t}\rho = \nabla\cdot J \Rightarrow \frac{\partial}{\partial t}\rho + \nabla\cdot J = 0.$$

5) The Hamiltonian for a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m},$$

the time evolution operator yields,

$$e^{\frac{-ip^2}{2m}\frac{t}{\hbar}} \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}\left(px - \frac{p^2 t}{2m}\right)}$$

So we calculate,

$$J_x = \frac{-i\hbar}{2m}\left\{\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right)$$

$$= \frac{-i\hbar}{2m}\left\{\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^2 e^{\frac{-i}{\hbar}\left(px - \frac{p^2 t}{2m}\right)}\left(\frac{ip}{\hbar}\right) e^{\frac{i}{\hbar}\left(px - \frac{p^2 t}{2m}\right)}\right.$$

$$\left. - \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^2 e^{\frac{i}{\hbar}\left(px - \frac{p^2 t}{2m}\right)}\left(\frac{-ip}{\hbar}\right) e^{\frac{-i}{\hbar}\left(px - \frac{p^2 t}{2m}\right)}\right\}$$

$$= \left(\frac{-i\hbar}{2m}\right)\left(\frac{1}{2\pi\hbar}\right)\left(\frac{ip}{\hbar} + \frac{ip}{\hbar}\right)$$

$$\boxed{J_x = \frac{p}{2\pi m\hbar}}$$

Moreover, since $\rho = |\psi|^2 = \psi^*\psi$

$$= \frac{1}{2\pi\hbar} \rightarrow \text{evidently.}$$

Q6.3)

a) We use the property of the translation operator:

$$|\psi'\rangle = \hat{T}(\delta x)|\psi\rangle \Rightarrow \psi'(x) = \psi(x - \delta x)$$

$$\Rightarrow (\psi'(x))^* = \psi^*(x - \delta x)$$

Then by definition,

$$\langle x\rangle' = \langle\psi'|\hat{x}|\psi'\rangle, \text{ inserting } \int dx|x\rangle\langle x|$$

$$= \int dx\, \psi'^*(x)\, x\, \psi'(x)$$

$$= \int dx\, \psi^*(x - \delta x)\, x\, \psi(x - \delta x)$$

Let the substitution $x \mapsto x + \delta x \rightarrow dx = dx$

$\therefore \langle x \rangle = \int dx\, \psi^*(x)\,(x+\delta x)\,\psi(x)$
$= \int dx\, \psi^*(x)\, x\, \psi(x) + \int dx\, \delta x\, \psi^*(x)\psi(x)$

since $\int dx\, \psi^* \psi = 1 \Rightarrow \langle x \rangle' = \langle x \rangle + \delta x$

5) By definition, $\hat{T}(\delta x) = e^{-\frac{i\hat{p}\cdot\delta x}{\hbar}}$. We then proceed,
$\langle p_x \rangle = \langle \psi' | p_x | \psi' \rangle$, here we insert $\int dx\, |x\rangle\langle x|$

$$= \int dx\, \psi^*(x)\, e^{\frac{ip\delta x}{\hbar}}\, p_x\, \psi(x)\, e^{-\frac{ip\delta x}{\hbar}}$$
$$= \int dx\, \psi^*(x)\, e^{\frac{ip\delta x}{\hbar}} (-i\hbar) \frac{\partial}{\partial x}\left(\psi(x)\, e^{-\frac{ip\delta x}{\hbar}}\right)$$
$$= \int dx\, \psi^*(x)\, e^{\frac{ip\delta x}{\hbar}} (-i\hbar) \left\{ \frac{\partial}{\partial x}\psi(x)\, e^{-\frac{ip\delta x}{\hbar}} + \psi(x) \underbrace{\frac{\partial}{\partial x}\left(-\frac{ip\delta x}{\hbar}\right)}_{\to 0} e^{-\frac{ip\delta x}{\hbar}} \right\}$$
$$= \int dx\, \psi^*(x)\, e^{\frac{ip\delta x}{\hbar}} (-i\hbar) \frac{\partial}{\partial x}\psi(x)\, e^{-\frac{ip\delta x}{\hbar}}$$
$$= \int dx\, \psi^*(x)\, p_x\, \psi(x)$$
$$= \langle p_x \rangle$$

Q6,5)
a) From normalization, $1 = \langle \psi | \psi \rangle = \int_{-P/2}^{P/2} dP\, |\psi|^2$
$$= |N|^2 \int_{-P/2}^{P/2} dP$$
$$\Rightarrow 1 = |N|^2 P \Rightarrow N = 1/\sqrt{P}$$

b) By definition, $\psi(x) = \int dP\, \psi(p)\, e^{\frac{ipx}{\hbar}} \frac{1}{\sqrt{2\pi\hbar}}$
$$= \int_{-P/2}^{P/2} dP\, \frac{1}{\sqrt{P}}\, e^{\frac{ipx}{\hbar}} \frac{1}{\sqrt{2\pi\hbar}},$$

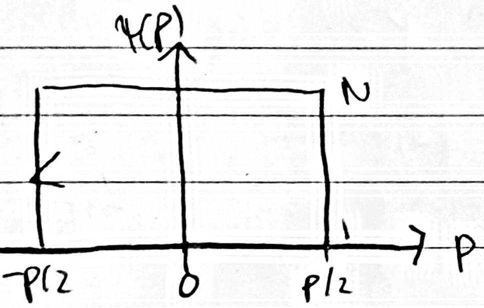
here we treat $1/\sqrt{P}$ as a constant since it's the normalization. So then,

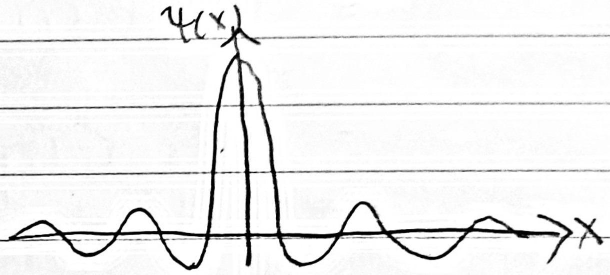
$$= \frac{1}{\sqrt{P}}\frac{1}{\sqrt{2\pi\hbar}}\int_{-P/2}^{P/2} e^{\frac{ipx}{\hbar}}\,dp$$

$$= \frac{1}{\sqrt{P2\pi\hbar}}\frac{\hbar}{ix}\left(e^{\frac{iPx}{2\hbar}} - e^{-\frac{iPx}{2\hbar}}\right)$$

$$\Rightarrow \psi(x) = \frac{2\hbar}{x}\sin\left(\frac{Px}{2\hbar}\right)\frac{1}{\sqrt{P}}\frac{1}{\sqrt{2\pi\hbar}} = \sqrt{\frac{P}{2\pi\hbar}}\;\frac{\sin(Px/2\hbar)}{Px/2\hbar}$$

c) $\psi(p)$ is just a square whell function, $\psi(x)$ has the form of a Dirichlet integral so





$$\Rightarrow \Delta p_x = \frac{1}{\sqrt{P}}, \quad \Delta x = \sqrt{\frac{P}{2\pi\hbar}}$$

$$\Rightarrow \Delta x \Delta p_x = 1/\sqrt{2\pi\hbar}$$, which is independent of P.

Q6.6)

a) if $\psi \in$ Real $\Rightarrow \psi(x) = \psi^*(x)$

$\langle p_x \rangle = \langle \psi | p_x | \psi \rangle$, here we insert $\int dx\, |x\rangle\langle x$

$$= \int dx\, \langle \psi | p_x | x \rangle \langle x | \psi \rangle$$

$$= \int dx\, \psi^*(x)\; p_x\, \psi(x)$$

$$= \int dx\, \psi^* (-i\hbar) \frac{\partial}{\partial x} \psi(x)$$

$$= \underbrace{(-i\hbar) \underbrace{\int dx\, \psi\, \partial_x \psi}_{\in \mathbb{R}}}_{\in \mathbb{C}}$$

$$\Rightarrow \langle p_x \rangle \in \mathbb{C} \Rightarrow \mathrm{Re}\{\langle p_x \rangle\} = 0.$$

b) Given $\psi'(x) \longmapsto e^{\frac{ip_0 x}{\hbar}} \psi(x)$,

$\langle x \rangle' = \langle \psi' | x | \psi' \rangle$

$= \langle e^{-ip_0 x/\hbar} \psi | x | e^{ip_0 x/\hbar} \psi \rangle$

$= \langle \psi | e^{-ip_0 x/\hbar} x e^{ip_0 x/\hbar} | \psi \rangle$

since $[e^{i\alpha x}, x] = 0 \ \forall \alpha$, then

$= \langle \psi | x | \psi \rangle$

$= \langle x \rangle$

$\langle p_x \rangle = \langle \psi' | p_x | \psi' \rangle$, we insert $\int dx |x\rangle\langle x|$

$= \int dx\, \psi^*(x) e^{-ip_0 x/\hbar} p_x \psi(x) e^{ip_0 x/\hbar}$

$= \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} (-i\hbar) \frac{\partial}{\partial x} \left(\psi(x) e^{\frac{ip_0 x}{\hbar}} \right)$

$= \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} (-i\hbar) \left\{ \frac{\partial}{\partial x} \psi(x) e^{\frac{ip_0 x}{\hbar}} + \left(\frac{ip_0}{\hbar}\right) \psi(x) e^{\frac{ip_0 x}{\hbar}} \right\}$

$= \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} (-i\hbar) \frac{\partial}{\partial x} \psi(x) e^{\frac{ip_0 x}{\hbar}}$

$+ \int dx\, \psi^*(x) e^{-ip_0 x/\hbar} p_0 \psi(x) e^{\frac{ip_0 x}{\hbar}}$

$= \int dx\, \psi^*(x) (-i\hbar) \frac{\partial}{\partial x} \psi(x) + p_0 \int dx\, \psi^*(x) \psi(x)$

$= \int dx\, \psi^*(x)\, p_x \psi(x) + p_0$

$= \langle p \rangle + \langle p_0 \rangle$.