MATH 325 Assignment 2.

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Question 1.

a)

$$\begin{aligned}
&a_0\frac{d^n(e^{\lambda t})}{dt^n}+a_1\frac{d^{n-1}(e^{\lambda t})}{dt^{n-1}}+\cdots+a_{n-1}\frac{d(e^{\lambda t})}{dt}+a_ne^{\lambda t}=0\\
&a_0\lambda^n(e^{\lambda t})+a_1\lambda^{n-1}(e^{\lambda t})+\cdots+a_{n-1}\lambda e^{\lambda t}+a_ne^{\lambda t}=0\\
&e^{\lambda t}(a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n)=0
\end{aligned}$$

$e^{\lambda t}>0 \ \ \forall\ \lambda\in\mathbb{C},\mathbb{R}$ and $t\in\mathbb{R}$, so then

$$\implies a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n=0.$$

b)

From our results in a), we can set the given ODE as

$$2\lambda^4-\lambda^3-9\lambda^2+4\lambda^4+4=0$$

Using the rational root theorem, we find $\lambda_1=-1/2,\lambda_2=1,\lambda_3=2,\lambda_4=-2$. Therefore the general solution is

$$y(t)=c_1e^{\frac{-t}{2}}+c_2e^t+c_3e^{2t}+c_4e^{2t}$$

With the initial conditions we have

$$\begin{aligned}
y(0)=-2 &\implies c_1+c_2+c_3+c_4=-2\\
y'(0)=0 &\implies -c_1/2+c_2+2c_3+-2c_4=0\\
y''(0)=-2 &\implies c_1/4+c_2+4c_3+4c_4=-2\\
y'''(0)=0 &\implies -c_1/8+c_2+8c_3+-8c_4=0
\end{aligned}$$

This can be transformed in a matrix and solved for c_i

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -2 \\ -1/2 & 1 & 2 & -2 & 0 \\ 1/4 & 1 & 4 & 4 & -2 \\ -1/8 & 1 & 8 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & -2 \\ 0 & -3 & -5 & 3 & 2 \\ 0 & 0 & 17 & -15 & -2 \\ 0 & 0 & 0 & 19 & -2 \end{bmatrix} \implies \begin{aligned} c_1 &= \frac{-56}{19} \\ c_2 &= \frac{24}{19} \\ c_3 &= \frac{-4}{19} \\ c_4 &= \frac{-2}{19} \end{aligned}$$

So finally we can write the unique solution :

$$y(t) = \frac{-56}{19}e^{\frac{-t}{2}} + \frac{24}{19}e^{t} - \frac{4}{19}e^{2t} - \frac{2}{19}e^{-2t}.$$

c)

We have $\lambda^4 - 1 = 0$. This can be rewritten as

$$\underbrace{(\lambda - 1)}_{\lambda=1}\underbrace{(\lambda + 1)}_{\lambda=-1}\underbrace{(\lambda^2 + 1)}_{\lambda=\pm i} = 0$$

So we have the general solution

$$y(t) = c_1e^t + c_2e^{-t} + c_3e^{it} + c_4e^{-it}$$

By regrouping the real and imaginary parts we get

$$y(t) = c_1e^t + c_2e^{-t} + (c_3 + c_4)\cos(t) + (c_3 - c_4)\sin(t)$$
$$\text{Letting } (c_3 + c_4) = c_3 \text{ and } (c_3 - c_4) = c_4 \text{ we get}$$
$$y(t) = c_1e^t + c_2e^{-t} + c_3\cos(t) + c_4\sin(t).$$

Similarly as in b), we construct a matrix representing the constant values for the given initial conditions and we solve it

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= 0 \\ c_3 &= \frac{-1}{2} \\ c_4 &= \frac{-1}{2} \end{aligned}$$

So we have the final answer unique solution

$$y(t) = \frac{1}{2}e^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t).$$

d)

We have $\lambda^4 + 2\lambda^2 + 1 = 0$. This can be rewritten as

$$(\lambda^2 + 1)^2 = 0 \implies \lambda = \pm i \quad , m = 2.$$

Therefore the solutions are $e^{it}, te^{it}, e^{-it}, te^{-it}$.. Taking the real parts and the imaginary parts and recombining everything whilst defining arbitrary constants like in c) we get

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t)$$

. By taking the matrix corresponding to the different initial conditions and successively applying product rule to take the derivatives ,and solving it for c_i, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 & 1 \end{bmatrix} \implies \begin{aligned} c_1 &= 0 \\ c_2 &= \frac{1}{3} \\ c_3 &= \frac{-1}{3} \\ c_4 &= \frac{1}{2} \end{aligned}$$

So the final answer unique solution is

$$y(t) = \frac{\sin(t)}{3} - \frac{t\cos(t)}{3} + \frac{t\sin(t)}{2}.$$

Question 2.

a)

If $x = \ln(t) \implies x = x(t)$ such that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right).$$

$$\frac{d^2t}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\left(\frac{1}{t}\right)\right) = \frac{d^2y}{dtdx}\left(\frac{1}{t}\right) + \frac{dy}{dx}\left(\frac{1}{t}\right)'$$

Since $\frac{dy}{dt} = \frac{dy}{dx}\frac{1}{t}$ we then have

$$\frac{d^2y}{dtdx}\left(\frac{1}{t}\right) = \frac{d^2y}{dx^2}\left(\frac{1}{t}\right)^2 - \frac{dy}{dx}\left(\frac{1}{t^2}\right).$$

b)

Substituting the results back in the initial expression and rearranging yields

$$\begin{aligned}
t^2\left(\frac{d^2y}{dx^2}\frac{1}{t^2} - \frac{dy}{dx}\frac{1}{t^2}\right) + \alpha t\left(\frac{dy}{dx}\left(\frac{1}{t}\right)\right) + \beta y = 0 \\
\implies \frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y = 0 \\
\implies \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0 \\
\therefore y''(x) + (\alpha - 1)y'(x) + \beta y(x) = 0.
\end{aligned}$$

c)

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\left(\frac{1}{t}\right) \\
\frac{d^2y}{dt^2} &= \frac{d^2y}{dx^2}\frac{1}{t^2} - \frac{dy}{dx}\frac{1}{t^2}
\end{aligned}$$

Substituting back we get

$$\begin{aligned}
&\frac{d^2y}{dt^2} - \frac{dy}{dx} + 7\frac{dy}{dx} + 10y = 0 \\
&\implies y''(x) + 6y'(x) + 10y(x) = 0 \\
&\implies \lambda^2 + 6\lambda + 10 = 0 \implies (\lambda + 3)^2 = -1
\end{aligned}$$

The roots are $\lambda_1 = -3+i$ and $\lambda_2 = -3-i$. By taking the real and imaginary part while assigning arbitrary constants , we get the general solution

$$y(t) = c_1e^{-3t}\cos(t) + c_2e^{-3t}\sin(t).$$

Question 3.

a)

$$\begin{aligned}
&y_1(t) = t \implies y_2(t) = v(t)t \\
&y_2'(t) = v'(t)t + v(t) \\
&y_2''(t) = v''(t)t + 2v'(t)
\end{aligned}$$

Substituting these in the original expression and regrouping terms we have

$$\begin{gathered}
v''(t)[t^3] + v'(t)[2t^2 - t^3 - 2t^2] + v(t)[-t^2 - 2t + t^2 + 2t] = 0 \\
\implies v''(t)[t^3] + v'(t)[-t^3] = 0 \\
\implies \int -1 \ dt = -\int \frac{v''}{v'} \ dv \\
\implies t = \ln(v') + C \implies v = e^{t-C}
\end{gathered}$$

Finally, we substitute in the original expression for $y_1(t)$ and obtain the general solution

$$y(t) = c_1 y_1 + c_2 y_1 e^{t-C} = c_1 t + c_2 t e^t.$$

b)

$$\begin{aligned}
&y_2'(x) = v'(x)e^x + v(x)e^x \\
&y_2''(x) = v''(x)e^x + v'(x)e^x + e^x v(x) = 0
\end{aligned}$$

Substituting in the original expression and regrouping terms we get

$$\begin{gathered}
v''[e^x(x-1)] + v'[2e^x(x-1) - xe^x] + v[xe^x - e^x - xe^x + e^x] = 0 \\
\implies v''[e^x(x-1)] + v'[2e^x(x-1) - xe^x] = 0 \\
\implies \int \frac{v''}{v'} \ dv = \int \frac{e^x(x-1)}{e^x(x-2)} \ dx \\
\implies \ln(v') = \int 1 \ dx + \int \frac{1}{x-2} \\
\ln(v') = x + \ln(x-2) \implies e^x e^{\ln|x-2|} = v' \\
\implies v = \int e^x(x-2) \ dx
\end{gathered}$$

Solving this integral by parts with two substitutions yields

$$v = xe^x - 3e^x.$$

So we have that $y_2(x) - (xe^x - 3e^x)e^x \implies y_2(x) = xe^{2x} - 3e^{2x}$. Finally, the general solution is

$$y(t) = c_1 e^{2x} + c_2 e^{2x}(x-3).$$

Question 4.

a)

We have that $y'' - y' + \frac{y}{4} = 4e^{t/2}$. We first find the solutions to the homogenous equation

$$\lambda^2 - \lambda + 1/4 = 0 \quad \implies \left(\lambda - \frac{1}{2}\right)^2 = 0 \implies \lambda = \frac{1}{2}, m = 2.$$

So we have that $y_h(t) = c_1 e^{t/2} + c_2 te^{t/2}$. Following the proceedure we have

$$\begin{aligned} y(t) &= u_1 e^{t/2} + u_2 te^{t/2} \\ y'(t) &= u_1' e^{t/2} + \frac{u_1 e^{t/2}}{2} + u_2' te^{t/2} + u_2(e^{t/2} + \frac{te^{t/2}}{2}) \end{aligned}$$

We require $u_1' e^{t/2} + u_2' te^{t/2} = 0$, yielding

$$\begin{aligned} \implies y'(t) &= \frac{u_1 e^{t/2}}{2} + u_2 e^{t/2} + \frac{u_2 e^{t/2}}{2} \\ y''(t) &= \frac{u_1' e^{t/2}}{2} + \frac{u_1 e^{t/2}}{4} + u_2' e^{t/2} + \frac{u_2 e^{t/2}}{2} + \frac{u_2' te^{t/2}}{2} + \frac{u_2 te^{t/2}}{4}. \end{aligned}$$

Substituting these results in the initial ODE whilst also regrouping for u_1 and u_2 yields

$$\underbrace{u_1\left(\frac{-1}{4}\right)}_{=0} + \underbrace{u_2\left(\frac{-1}{2} - \frac{t}{4}\right)}_{=0} + \frac{u_1'}{2} + u_2' + \frac{tu_2'}{2} = 4$$
$$\implies u_1' + 2u_2' + u_2'(t) = 8$$

Forming a system of equation by adding the required equation : $u_1' + u_2' t = 0$

$$\begin{pmatrix} 1 & (2+t) \\ 1 & t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \to W(t) = t - (2+t) = -2 \neq 0$$

$$u_1' = \frac{8t}{-2} = -4t, \qquad u_2' = \frac{-8}{-2} = 4$$
$$\therefore \int u_1' \, dt = -4 \int t \, dt = -2t^2$$
$$\text{similarly,} \quad \int u_2' \, dt = 4 \int dt = 4t.$$

Substituting to find the particular solution gives

$$\begin{aligned} y(t) &= u_1 y_1(t) + u_2 y_2(t) \\ &= -2t^2 e^{t/2} + 4t^2 e^{t/2} \end{aligned}$$

Finally the general solution is

$$y(t) = c_1 e^{t/2} + c_2 e^{t/2} - 2t^2 e^{t/2} + 4t^2 e^{t/2}.$$

b)

We have $y'' + 4y = 0 \implies \lambda^2 + 4 = 0$. The solutions are $\lambda = \pm 2i$. Taking the real part and imaginary part we find that the homogenous solution is $y_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$.

$$y'(t) = u_1' \cos(2t) - 2u_1 \sin(2t) + u_2' \sin(2t) + 2u_2 \cos(2t).$$

We require that $u_1' \cos(2t) + u_2' \sin(2t)$,yielding :

$$\begin{aligned} y'(t) &= 2 - u_1 \sin(2t) + 2u_2 \cos(2t) \\ y''(t) &= 2 - u_1' \sin(2t) - 2u_1 \cos(2t) + 2u_2' \cos(2t) - 2u_2 \sin(2t) \end{aligned}$$

Substituting these results in the original expression and canceling the u_1 and u_2 terms we get the following system of equations

$$\begin{cases} 2u_2' \cos(2t) - 2u_1' \sin(2t) = t^2 + 3e^t \\ u_1' \cos(2t) + u_2' \sin(2t) = 0 \end{cases}$$
$$\implies \begin{pmatrix} -2\sin(2t) & 2\cos(2t) \\ \cos(2t) & \sin(2t) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t^3 + 3e^t \\ 0 \end{pmatrix}$$

$$\implies W(t) = -\sin^2(2t) - 2\cos^2(2t) = -2 \neq 0$$

Therefore we have have the solutions for the u_i':

$$u_1' = \int \frac{(t^2+3e^t)\sin(2t)}{-2}\,dt = \frac{-1}{2}\left(\int t^2\sin(2t) + 3\int e^t\sin(2t)\right)$$

$$u_2' = \int \frac{-(t^2+3e^t)\cos(2t)}{-2}\,dt = \frac{1}{2}\left(\int t^2\cos(2t) + 3\int e^t\cos(2t)\right)$$

We have that

$$y_p = \cos(2t)\frac{-1}{2}\left(\int t^2\sin(2t) + 3\int e^t\sin(2t)\right) + \sin(2t)\frac{1}{2}\left(\int t^2\cos(2t) + 3\int e^t\cos(2t)\right)$$

$$= \frac{1}{4}\cos^2(2t)t^2 - \frac{1}{4}\sin(2t)\cos(2t)t - \frac{1}{8}\cos^2(2t) + \frac{3}{5}e^t\cos^2(2t) - \frac{3}{10}\cos(2t)\sin(2t)e^t$$

$$+\frac{1}{4}\sin^2(2t)t^2 + \frac{1}{4}\cos(2t)\sin(2t)t - \frac{1}{8}\sin^2(2t) + \frac{3}{5}e^t\sin^2(2t) + \frac{3}{10}\sin(2t)\cos(2t)e^t$$

$$\implies y_p(t) = \frac{1}{4}t^2 + \frac{3}{5}e^t - \frac{1}{8}.$$

Finally we have the general solution

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}t^2 + \frac{3}{5}e^t - \frac{1}{8}.$$

Now solving for the unique solution :

$$y(0) = 0 \implies c_1 + \frac{3}{5} - \frac{1}{8} = 0 \implies c_1 = \frac{3}{40}.$$

$$y'(0) = 0 \implies c_2 + \frac{3}{5} = 0 \implies c_2 = -\frac{3}{5}$$

The unique solution to the IVP is

$$y(t) = \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{5} + \frac{1}{4}t^2 + \frac{3}{5}e^t - \frac{1}{8}.$$

Question 5.

a)

$$\text{Tank 1.}\quad C_{\text{In}} : 1.5 + \frac{1.5x_2(t)}{20}, C_{\text{Out}} : \frac{3x_1(t)}{30}$$

$$\text{Tank 2.}\quad C_{\text{In}} : 3 + \frac{3x_1(t)}{30}, C_{\text{Out}} : \frac{2.5x_2(t)}{20}$$

These two systems of equations can be represented as a matrix linear system
:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

b)

Let us first solve the $y' = Ay$ system.

$$p(\lambda) = \left(\frac{-1}{10} - \lambda\right)\left(\frac{-1}{8} - \lambda\right) - \frac{3}{400} = 0 \implies \lambda_1 = \frac{-1}{5}, \lambda_2 = \frac{-1}{40}.$$

Let us find the corresponding eigenvectors to these eigenvalues.

$$\text{For } \lambda_1, \rightarrow \begin{pmatrix} (\frac{-1}{10} + \frac{1}{5}) & \frac{3}{40} \\ \frac{1}{10} & (\frac{-1}{8} + \frac{1}{5}) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a = \frac{-3b}{4} \implies u_1 = \begin{pmatrix} -3/4 \\ 1 \end{pmatrix}$$

$$\text{For } \lambda_2, \rightarrow \begin{pmatrix} \frac{-3}{40} & \frac{3}{40} \\ \frac{1}{10} & \frac{-1}{10} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a = b \implies u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The corresponding matrix solution $Y(t)$ is then

$$\begin{pmatrix} \frac{-3}{4} e^{-t/5} & e^{-t/40} \\ e^{-t/5} & e^{-t/40} \end{pmatrix}$$

$y(t) = Y(t)c + Y(t)\int Y^{-1}(t) r(t)\, dt$, let us find $Y^{-1}(t)$.

$$\det(Y(t)) = \frac{-3}{4} e^{-t/5} e^{-t/40} - e^{-t/40} e^{-t/5} = \frac{-7}{4} e^{-9/40}$$

$$\therefore Y^{-1}(t) = \frac{1}{\det(Y(t))} \begin{pmatrix} \frac{-3}{4} e^{-t/5} & e^{-t/40} \\ e^{-t/5} & e^{-t/40} \end{pmatrix} = \begin{pmatrix} \frac{-4}{7} e^{-t/5} & \frac{4}{7} e^{-t/5} \\ \frac{4}{7} e^{t/40} & \frac{3}{7} e^{t/40} \end{pmatrix}$$

$$\implies \int Y^{-1}(t) r(t)\, dt = \begin{pmatrix} \frac{-4}{7} e^{-t/5} & \frac{4}{7} e^{-t/5} \\ \frac{4}{7} e^{t/40} & \frac{3}{7} e^{t/40} \end{pmatrix} \begin{pmatrix} 3/2 \\ 3 \end{pmatrix} = \begin{pmatrix} \int \frac{-6}{7} e^{-t/5} + \int \frac{12}{7} e^{-t/5} \\ \int \frac{6}{7} e^{t/40} + \int \frac{9}{7} e^{t/40} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t/5} \frac{-30}{7} \\ e^{t/40} \frac{600}{7} \end{pmatrix}$$

Multiplying by $Y(t)$ yields

$$= \begin{pmatrix} \frac{90}{28} e^{-2t/5} + \frac{600}{7} \\ e^{-2t/5} \frac{-30}{7} + \frac{600}{7} \end{pmatrix}$$

Now let us find c_1 and c_2 for the homogenous solution using $c = P^{-1}y_0$, where P is the eigenvector matrix and y_0 is the initial conditions on the tanks , i.e., 25 for Tank 1 and 15 for Tank 2.

$$c = P^{-1}Y_0 = \frac{-4}{7}\begin{pmatrix} 1 & -1 \\ -1 & -3/4 \end{pmatrix}\begin{pmatrix} 25 \\ 15 \end{pmatrix} = \begin{pmatrix} -4/7 & 4/7 \\ 4/7 & 3/7 \end{pmatrix}\begin{pmatrix} 25 \\ 15 \end{pmatrix} = \begin{pmatrix} -40/7 \\ 145/7 \end{pmatrix}$$

$$\therefore y_h(t) = Y(t)c = \frac{-40}{7}e^{-t/5}\begin{pmatrix} -3/4 \\ 1 \end{pmatrix} + \frac{145}{7}e^{-t/40}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally the solution to the IVP is

$$y(t) = e^{-t/5}\begin{pmatrix} 30/7 \\ -40/7 \end{pmatrix} + e^{-t/40}\begin{pmatrix} 145/7 \\ 145/7 \end{pmatrix} + e^{-2t/5}\begin{pmatrix} 90/28 \\ -30/7 \end{pmatrix} + \frac{600}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Question 6.

a)

$$\begin{aligned} A = \lambda I + N \implies e^{At} &= e^{t(\lambda I + N)} = e^{\lambda I t}e^{Nt} \\ &= e^{\lambda t}\sum_{n=0}^{k-1}\frac{t^k}{k!}N^k \end{aligned}$$

Since N is nilpotent we may truncate the previous expression up to k.

$$e^{At} = e^{\lambda t}\sum_{r=0}^{k-1}\frac{N^r t^r}{(r)!} = e^{\lambda t}\left(I + \sum_{r=1}^{k-1}\frac{N^r t^r}{r!}\right)$$

b)

Let N be the upper triangular matrix ,which doesn't contain the diagonal ,of the given matrix. N is nilpotent up to $k = 4$. From the characteristic polynomial of the given matrix we find that

$$p(\lambda) = (2 - \lambda)^4 = 0 \implies \lambda = -2, \quad m = 4.$$

Therefore, using the result from a we get that

$$e^{At} = e^{-2t}\begin{pmatrix} 1 & t & t + \frac{t^2}{2} & t + \frac{t^2}{2} + \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using the initial condition $Y(0) = y_0$ we may solve for the c_i constants.

$$e^{0t}\begin{pmatrix} 1 & 0 & 0+\frac{0^2}{2} & 0+\frac{0^2}{2}+\frac{0^3}{6} \\ 0 & 1 & 0 & \frac{0^2}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \implies \begin{matrix} c_1 = 1 \\ c_2 = 2 \\ c_3 = 3 \\ c_4 = 4 \end{matrix}$$

So finally, the solution to the IVP is

$$y(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2e^{-2t}\begin{pmatrix} t \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3e^{-2t}\begin{pmatrix} t+\frac{t^2}{2} \\ t \\ 1 \\ 0 \end{pmatrix} + 4e^{-2t}\begin{pmatrix} t+\frac{t^2}{2}+\frac{t^3}{6} \\ t+\frac{t^2}{2} \\ t \\ 1 \end{pmatrix}$$

Question 7.

By lemma we know that if A is diagonalizable then $\exists P$ such that $A = PDP^{-1}$, where D is a specific diagonal matrix. Moreover, by other lemma, we know that $A^k = PD^kP^{-1}$. Indeed if f is an analytic function, then $f(A) = Pf(D)P^{-1}$. e is analytic

$$\therefore e^A = Pe^DP^{-1}$$

.

$$e^D = \sum_{k=0}^{\infty}\frac{t^k}{k!}D^k = \begin{pmatrix} \sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda_n^k \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} = \Lambda$$

Thus $e^A = P\Lambda P^{-1}$ where $P = [u_1, \ldots, u_n]$ the eigenvector matrix.

Question 8.

Solving the characteristic polynomial for A yields $\lambda_1 = 10$ and $\lambda_2 = -2$. The eigenvectors are immediately computed

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{resepctively}$$

Let Λ be the diagonal matrix of e^A for which the diagonal elements correspond to the exponentials of the eigenvalues of A. We can then construct the matrix P of the eigenvectors of A. We can then invert the matrix P easily since it's 2×2

$$P = \begin{pmatrix} e^{10} & -e^{-2} \\ e^{10} & e^{-2} \end{pmatrix} \qquad , P^{-1} \begin{pmatrix} \frac{1}{2e^{10}} & \frac{1}{2e^{10}} \\ \frac{-e^2}{2} & \frac{e^2}{2} \end{pmatrix}$$

Finally, multiplying all these matrices together gives the desired expression for e^A.

$$e^A = \begin{pmatrix} e^{10} & -e^{-2} \\ e^{10} & e^{-2} \end{pmatrix} \begin{pmatrix} e^{10} & 0 \\ 0 & e^{-2} \end{pmatrix} \begin{pmatrix} \frac{1}{2e^{10}} & \frac{1}{2e^{10}} \\ \frac{-e^2}{2} & \frac{e^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{e^{12}+1}{2e} & \frac{e^{12}-1}{2e^2} \\ \frac{e^{10}-e^2}{2} & \frac{e^{10}+e^2}{2} \end{pmatrix}$$