MATH 475 Weekly Work 0

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Question 1

Let

$$u(x,t) = X(x)T(t) \tag{1}$$

a)

$$T'(t) = 0 \implies T(t) = \int 0 \; dt = c_1.$$

$$X''(x) = 0 \implies X'(x) = \int 0 \; dx = c_2 \implies X(x) = \int c_2 \; dx = c_2 x + c_3.$$

Replacing in (1) we have

$$u(x,t) = c_1(c_2 x + c_3).$$

Since c_1, c_2, c_3 are all arbitrary constants this is equivalent to $u(x,t) = Ax + B$ for $A, B \in \mathbb{R}$.

b)

$$T'(t) = -k\lambda^2 T(t) \implies \frac{T'(t)}{T(t)} = -k\lambda^2$$

$$\implies \int \frac{T'(t)}{T(t)} \; dt = \int -k\lambda^2 \; dt \implies \ln|T(t)| = -k\lambda^2 + c_1$$

$$\implies T(t) = e^{-kt\lambda^2 + c_1}$$

We set $e^{c_1} \equiv C$

$$\therefore T(t) = Ce^{-k\lambda^2 t}.$$

Next,

$$X''(x) = -\lambda^2 X(x) \implies \frac{X''(x)}{X(x)} = -\lambda^2$$

Solving the caracteristic equation yields

$$\implies X''(x) + \lambda^2 X(x) = 0 \implies X(x) = \pm i\lambda$$

Complex roots therefore we use the general solution :

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

Finally,

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= Ce^{-k\lambda^2 t}(A\cos\lambda x + B\sin\lambda x) \end{aligned}$$

C is an arbitrary constant which is absorbed by A and B, getting

$$u(x,t) = e^{-k\lambda^2 t}(A\cos\lambda x + B\sin\lambda x).$$

c)

First,

$$T'(t) = -\lambda^2 kT(t) \implies \ln|T(t)| = -\lambda^2 kt + c_1 \implies T(t) = Ce^{-\lambda^2 kt}$$

Then,

$$X''(x) = -\lambda^2 X(x)$$

Since $\pm i\lambda = \pm|\lambda|$ for λ immaginary we have

$$X(x) = \pm|\lambda|$$

Therefore the solution to the characteristic equation is

$$X(x) = Ae^{|\lambda|x} + Be^{-|\lambda|x}$$

Substituting everything back in (1) we get

$$u(x,t) = e^{-|\lambda|^2 kt}(Ae^{|\lambda|x} + Be^{-|\lambda|x}).$$

Question 2

a)

The only solution that can satisfy this PDE is evidently $u(x,t) = Ax + B$ since it's the only linear solution, hence the only solution that can satisfy the initial condition $u(x,0) = 3x$ on $(0, L)$. Solving :

$$u(0,t) = 0 \implies B = 0 \qquad , u(x,0) = Ax + B = 3x \implies A = 3.$$

Therefore the final solution is $u(x,t) = 3x$ since it also satisfies $u(L,t) = 3L$ by symmetry of the initial condition.

b)

The initial condition is of exponential nature so we dismiss the $\lambda^2 = 0$ linear case. Let us attempt the $\lambda^2 < 0$ case.

$$u(x,0) = Ae^{|\lambda|x} + B^{-|\lambda|x} = 2e^{3x} + 2e^{-3x}$$

The only solutions to this equality is $A = B = 2$ and $|\lambda| = 3$. We proceed further,

$$\begin{aligned} u_x &= |\lambda|e^{2|\lambda|^2 t}(Ae^{|\lambda|x} - Be^{-|\lambda|x}) \\ u_x(L,t) &= |\lambda|e^{2|\lambda|^2 t}(Ae^{|\lambda|L} - Be^{-|\lambda|L}) \\ &= 3e^{2|3|^2 t}(2e^{|3|L} - 2e^{-|3|L}) \\ &= 6e^{18t}(e^{3L} - e^{-3L}) \end{aligned}$$

We conclude the $\lambda^2 < 0$ case satisfies the given BVP since it also satisfies the boundary $u_x(0,t) = 0$ given that $(A = B)$, and the $\lambda^2 > 0$ case is dismissed since only one case can satisfy the BVP.

Question 3

a)

Since only the $\lambda^2 > 0$ satisfies the BVP, we check the boundary

$$u(0,t) = 0 \implies e^{-3\lambda^2 t}A = 0,$$

since $e^{-3\lambda^2 t} \neq 0 \ \forall \ t \in \mathcal{I}$ we conclude that $A = 0$.
Now verifying the right end boundary condition,

$$u(1,t) = 0 \implies e^{-3\lambda^2 t}B\sin(\lambda(1)) = 0$$

Expressing $\lambda = n\pi/1$ we get

$$u(1,t) = e^{-3n^2\pi^2 t}B\sin(n\pi(1))$$

Extending the spatial domain to x we obtain

$$u(x,t) = e^{-3\pi^2 n^2 t}B\sin(n\pi x).$$

b)

First, we note that the PDE as defined is linear (transport equation), therefore a linear combination of solutions is also a solution.$u(x,t) = Be^{-3n^2\pi^2 t}\sin(n\pi x)$ suggests there exists an infinite number of solutions since $n \in \mathbb{Z}$, thus

$$u(x,t) := \sum_{n=-N}^{N} B_n e^{-3n^2\pi^2 t}\sin(n\pi x) \tag{2}$$

is a solution to the given PDE.

We can truncate the sum from $n = -N \to 0$ since the $B_{n<0}$ are absorbed in the terms $B_{n>0}$. For illustrative purposes, let us consider the $n = -1 \to 1$ case :

$$\sum_{n=-1}^{1} = B_{-1}e^{\alpha}(\sin(-\pi x)) + B_0 0 + B_1 e^{\alpha}\sin(\pi x)$$

Using the identity $\sin(-\pi x) = -\sin(\pi x)$, this is equivalent to

$$= (B_1 - B_{-1})e^{\alpha}\sin(\pi x)$$

Since B_n are arbitrary numbers (2) can indeed be expressed as

$$u(x,t) := \sum_{n=0}^{N} B_n e^{-3n^2\pi^2 t}\sin(n\pi x).$$

c)

Using the general solution found in 3 b) ,

$$u(x,0) = \sum_{n=0}^{N} B_n \sin(n\pi x) = 5\sin(2\pi x) - 30\sin(n\pi x) \implies B_1 = 0\ , B_2 = 5\, B_3 = -30.$$

Therefore we may express the general solution to this particular PDE as

$$u(x,t) = 5e^{-3(\pi^2)2^2 t}\sin(2\pi x) - 30e^{-3\pi^2 3^2 t}\sin(3\pi x).$$