PHYS356 Assignment 3

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Question 1

In electromagnetism for a wave travelling in the z direction we can think of x and y transmission polarization axis as a basis. For electromagnetic radiation there is a concept involving right/left circularity polarization which can be described in the basis as

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \qquad , |L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle).$$

The matrix to change basis from $x \to x'$ and $y \to y'$, representing a transmission axis at an angle is

$$S = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{1}$$

So essentially after transformation

$$|R'\rangle = \frac{1}{\sqrt{2}}(|x'\rangle + i\,|y'\rangle) \qquad , |L'\rangle = \frac{1}{\sqrt{2}}(|x'\rangle - i\,|y'\rangle).$$

A relationship between $|R\rangle$ and $|R\rangle'$ exists :

$$|R'\rangle = \frac{1}{\sqrt{2}}(|x'\rangle + i\,|y\rangle')$$

Using (1),

$$\begin{aligned} &= \frac{(\cos\phi - i\sin\phi)}{\sqrt{2}}(|x\rangle + y\,|y\rangle) \\ &= e^{-i\phi}\,|R\rangle\,, \end{aligned} \tag{2}$$

we see that the relationship is a rotation ,therefore we can apply the definition of rotations in a quantum mechanical frame

$$|R'\rangle = \hat{R}_z(\phi)\,|R\rangle = e^{-i\hat{J}_z\phi/\hbar}\,|R\rangle\,,$$

Following (2) , this implies that $\hat{J}_z\,|R\rangle = \hbar R$, and similarly for $\hat{J}_z\,|L\rangle = -\hbar L$, we conclude that the two eigenvalues of the generator of rotations which correspond to the angular momentum of the photon is $\pm\hbar$.

Question 2

We first find the matrix representation $\hat{S}_z$

$$\hat{S}_z \xrightarrow{R,L \text{ basis}}= \begin{pmatrix} \langle R|S_z|R\rangle & \langle R|S_z|L\rangle \\ \langle L|S_z|R\rangle & \langle L|S_z|L\rangle \end{pmatrix} = \begin{pmatrix} \hbar & 0 \\ 0 & \hbar \end{pmatrix}.$$

Now we transform the vectors

$$|x\rangle \xrightarrow{R,L \text{ basis}}= \begin{pmatrix} \langle R|x\rangle \\ \langle L|x\rangle \end{pmatrix}, \qquad |y\rangle \xrightarrow{R,L \text{ basis}}= \begin{pmatrix} \langle R|y\rangle \\ \langle L|y\rangle \end{pmatrix}.$$

We compute each element;

$$\begin{aligned} \langle R|x\rangle &= \langle x|R\rangle^* = \left(\frac{1}{\sqrt{2}}\right)^* = \frac{1}{\sqrt{2}} \\ \langle L|x\rangle &= \langle x|L\rangle^* = \left(\frac{1}{\sqrt{2}}\right)^* = \frac{1}{\sqrt{2}} \\ \langle R|y\rangle &= \langle y|R\rangle^* = \left(\frac{i}{\sqrt{2}}\right)^* = \frac{-i}{\sqrt{2}} \\ \langle L|y\rangle &= \langle y|L\rangle^* = \left(\frac{-i}{\sqrt{2}}\right)^* = \frac{i}{\sqrt{2}} \end{aligned}$$

Finally ,

$$|x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ i \end{pmatrix}.$$

Question 3

The spin of a particle essentially dictates the direction of its angular momentum , i.e., the direction of the propagation. If the spin for a photon is 0, then the photon is stationary which is impossible since photons have no rest frame.

Question 4

a)

First and foremost we note that $|\psi\rangle$ is normalized. Therefore,

$$|\langle y|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

b)

We first perform a change of basis with $S^\dagger$ as defined in (1)

$$\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\phi + \frac{i}{\sqrt{2}}\sin\phi \\ -\frac{\sqrt{2}}{3}\sin\phi + \frac{i}{\sqrt{3}}\cos\phi \end{pmatrix},$$

thus,

$$|\phi\rangle = \left(\sqrt{\frac{2}{3}}\cos\phi + \frac{i}{\sqrt{3}}\sin\phi\right)|x'\rangle + \left(-\sqrt{\frac{2}{3}}\sin\phi + \frac{i}{\sqrt{3}}\cos\phi\right)|y'\rangle\,.$$

The initial state is normalized ,hence it follows that

$$|\langle y'|\psi\rangle|^2 = \left|-\sqrt{\frac{2}{3}}\sin\phi + \frac{i}{\sqrt{3}}\cos\phi\right|^2 = \frac{2}{3} - \frac{2}{3}\cos^2\phi + \frac{1}{3}\cos^2\phi = \frac{2}{3} - \frac{\cos^2\phi}{3}.$$

c)

We use the probabilities associated with right/left circularities.

$$\begin{aligned} |R\rangle &= \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \\ \langle R| &= \frac{1}{\sqrt{2}}(\langle x| - i\,\langle y|) \\ \implies |\langle R|\psi\rangle|^2 &= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} - \frac{i}{\sqrt{2}}\frac{i}{\sqrt{3}}\right|^2 = \frac{1}{2} + \frac{\sqrt{2}}{3}. \end{aligned}$$

Since $|R\rangle$ and $|L\rangle$ for a normalized basis, it follows that

$$|\langle L|\psi\rangle|^2 = 1 - |\langle R|\psi\rangle|^2 = \frac{1}{2} - \frac{\sqrt{2}}{3}.$$

The latter correspond to normalized probability constants therefore since $\pm\hbar$ is the angular momentum corresponding to right/left circularity states, it follows that the net torque is

$$\text{Net torque } = N\hbar|\langle R|\psi\rangle|^2 - N\hbar|\langle L|\psi\rangle|^2 = N\hbar\frac{2\sqrt{2}}{3},$$

which is a value greater that 0, implying that the disk's rotation is clockwise as with respect to $z_- \to z_+$ orientation.

d)

The amplitudes would change :

$$|\langle y|\psi'\rangle|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

Moreover,

$$\begin{aligned}
|\langle y'|\psi'\rangle|^2 &= \left|-\sqrt{\frac{2}{3}}\sin\phi + \frac{1}{\sqrt{3}}\cos\phi\right|^2 \\
&= \frac{2}{3} - \frac{2}{3}\cos^2\phi - \frac{2\sqrt{2}}{3}\sin\phi\cos\phi + \frac{1}{3}\cos^2\phi \\
&= \frac{2}{3} - \frac{\cos^2\phi}{3} - \frac{\sqrt{2}}{3}\sin(2\phi).
\end{aligned}$$

The net torque should change as well

$$\begin{aligned}
|\langle R|\psi'\rangle|^2 &= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{2} \\
\implies |\langle L|\psi'\rangle|^2 &= \frac{1}{2},
\end{aligned}$$

We conclude the net torque is 0 since both probabilities are equal , so the disk will not rotate.

Question 5

a)

Let $a = (1\ 0\ 0)^T,\ b = (0\ 1\ 0)^T$ and $c = (0\ 0\ 1)^T$. Then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Solving the system of equations yields

$$\hat{T} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix found is indeed unitary since multiplied by its conjugate transpose it is equal to the identity,

$$\hat{T}\hat{T}^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark.$$

b)

We first find the eigenvalues with

$$p(\lambda) = \det(\hat{T} - I\lambda) = 0 \implies \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \implies (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\therefore \lambda_1 = 1, \qquad \lambda_2 = \frac{1}{2}(-1 + i\sqrt{3}), \qquad \lambda_3 = \frac{1}{2}(-1 - i\sqrt{3}).$$

The corresponding eigenvectors to these eigenvalues are found computing the kernel. That is by solving $(\hat{T} - I\lambda_i)\vec{v_i} = \vec{0}$, for $\vec{v_i}$, we obtain

$$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{v_2} = \begin{pmatrix} -1 - i\sqrt{3}) \\ -1 + i\sqrt{3} \\ 2 \end{pmatrix}, \qquad \vec{v_3} = \begin{pmatrix} -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \\ 2 \end{pmatrix}.$$

It follows that for a state we have the corresponding eigenstates in braket notation

$$|\psi\rangle = \frac{1}{\sqrt{3}}|v_1\rangle + \frac{1}{\sqrt{3}}|v_2\rangle + \frac{1}{\sqrt{3}}|v_3\rangle.$$

The probabilities are the same for each state so the probability to find in b state is given by $|\langle v_i|\psi\rangle|^2 = 1/3$.

c)

Let $\hat{T} \equiv \hat{R}_z(\varphi)$. A full circle is 2π therefore each rotation si $\varphi = 2\pi/3$. Let J_z be diagonal in the eigenbasis with α_i as diagonal entries.Then it follows that

$$\hat{R}_z\left(\frac{2\pi}{3}\right) = e^{\frac{-i\hat{J}_z 2\pi}{3\hbar}} \to e^{\frac{-i2\pi\alpha_1}{3\hbar}}|v_1\rangle = \lambda_1|v_1\rangle \implies e^{\frac{-i2\pi\alpha_1}{3\hbar}} = 1 \implies \alpha_1 = 0$$

$$\to e^{\frac{-i2\pi\alpha_2}{3\hbar}}|v_2\rangle = \lambda_2|v_2\rangle \implies e^{\frac{-i2\pi\alpha_2}{3\hbar}} = \frac{1}{2}(-1 + i\sqrt{3}) \xrightarrow{\text{Euler's identity}} \alpha_2 = -\hbar$$

$$\to e^{\frac{-i2\pi\alpha_2}{3\hbar}}|v_3\rangle = \lambda_3|v_3\rangle \implies e^{\frac{-i2\pi\alpha_3}{3\hbar}} = \frac{1}{2}(-1 - i\sqrt{3}) \xrightarrow{\text{Euler's identity}} \alpha_3 = \hbar,$$

We conclude that $\hat{J}_z$ in the eigenbasis is

$$\hat{J}_z = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, we convert to a, b, c basis

$$\begin{aligned}
\hat{J}_z \xrightarrow{a,b,c \text{ basis}} & \begin{pmatrix} \langle a|v_1\rangle & \langle a|v_2\rangle & \langle a|v_3\rangle \\ \langle b|v_1\rangle & \langle b|v_2\rangle & \langle b|v_3\rangle \\ \langle c|v_1\rangle & \langle c|v_2\rangle & \langle c|v_3\rangle \end{pmatrix} \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle v_1|a\rangle & \langle v_1|b\rangle & \langle v_1|c\rangle \\ \langle v_2|a\rangle & \langle v_2|b\rangle & \langle v_2|c\rangle \\ \langle v_3|a\rangle & \langle v_3|b\rangle & \langle v_3|c\rangle \end{pmatrix} \\
= \frac{1}{\sqrt{3}} & \begin{pmatrix} 1 & (-1-i\sqrt{3}) & (-1+i\sqrt{3}) \\ 1 & (-1+i\sqrt{3}) & (-1-i\sqrt{3}) \\ 1 & 2 & 2 \end{pmatrix} \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ (-1+i\sqrt{3}) & (-1-i\sqrt{3}) & 2 \\ (-1-i\sqrt{3}) & (-1+i\sqrt{3}) & 2 \end{pmatrix} \\
= \frac{\hbar}{\sqrt{3}} & \begin{pmatrix} 0 & -i4 & i4 \\ i4 & 0 & -i4 \\ -i4 & i4 & 0 \end{pmatrix},
\end{aligned}$$

The last matrix has the same eigenvectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ as $\hat{T}$ like defined in $5b$, thence they have the same eigenstates.