MATH 327 Assignment 3

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Question 1

(a)

$$\begin{aligned}\langle Qx, Qy\rangle &= (Qx)^T Qy \\ &= x^T Q^T Q y \\ &= x^T I y \\ &= x^T y \\ &= \langle x, y\rangle\end{aligned} \qquad \begin{aligned}\|Qx\|_2 &= \sqrt{\sum_{i=1}^{n}(Qx)_i^2} \\ &= \langle Qx, Qx\rangle \\ &= \langle x, x\rangle \\ &= \|x\|_2 \\ &\implies \|Qx\|_2 = \|x\|_2\end{aligned}$$

(b)

We note that $(1\ 0\ 0\ 0)^T$ is already in the desired form so we focus on $\hat{A} = (4\ 2\ 4)^T$. We want

$$\tilde{Q}\begin{pmatrix}4\\2\\4\end{pmatrix} = \begin{pmatrix}-6\\0\\0\end{pmatrix} \quad \text{, then } u = \begin{pmatrix}1\\1/5\\2/5\end{pmatrix} \implies \|u\|_2^2 = \frac{2\tau}{\tau + x_1} = \frac{6}{5} \implies \gamma = \frac{5}{3}.$$

Then we solve

$$\tilde{Q} = I - \gamma u u^T = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{5}{3}\begin{pmatrix}5\\1\\1\end{pmatrix}\begin{pmatrix}5&1&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{5}{3}\begin{pmatrix}25&5&5\\5&1&1\\5&1&1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{122}{3}&\frac{25}{3}&\frac{25}{3}\\\frac{25}{3}&\frac{2}{3}&\frac{5}{3}\\\frac{25}{3}&\frac{5}{3}&\frac{2}{3}\end{pmatrix} = \tilde{Q}^T$$

Then we know that

$$\hat{Q}_{2\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix} \qquad \text{and} \quad R = \begin{pmatrix} & \hat{R} & \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here the vector (x_2, x_3, x_4) is the first row of $\tilde{Q}^T$ since their product gives $-\tau$, so then we conclude that

$$\hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{122}{3} \\ 0 & \frac{25}{3} \\ 0 & \frac{25}{3} \end{pmatrix}, \qquad \hat{R} = \begin{pmatrix} 1 & 2 \\ 0 & -6 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ 1/5 \\ 2/5 \end{pmatrix}$$

(c)

In the rank deficient case we have $\hat{c} = \hat{Q}^T b$ so by theorem there exists a decomposition $A = QR$. We apply Q^T to the overdetermined system obtaining $Ax = b \Longrightarrow Rx = Q^T b = c$, here R is upper triangular so we can use b

$$\begin{pmatrix} R_{11} & R_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \Longrightarrow \|s\|_2 = \sqrt{\|c - R_{11}x_1 - R_{12}x_2\|_2^2 + \|d\|_2^2}.$$

With backward substitution we find x_1 then since $c - R_{11}x_1 - R_{12}x_2 = 0$ we get $y = (x_1 x_2)^T$ which is the solution that minimizes $\|b - Ax\|$ in $Ax = b$.

(d)

First, we know $Rx = Q^T b = c = (1 \ -2/3 \ 5/3 \ -5/3)^T = c$. So

$$\begin{pmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2/3 \\ 5/3 \\ -5/3 \end{pmatrix} \Longrightarrow \begin{cases} x_1 + 2x_2 & = 1 \\ -6x_2 & = -2/3 \end{cases} \Longrightarrow x = \begin{pmatrix} 7/9 \\ 1/9 \end{pmatrix}.$$

Then, we use $c - R_{11}x_1 - R_{12}x_2 = 0$;

$$\begin{pmatrix} 1 \\ -2/3 \end{pmatrix} - (1) \begin{pmatrix} 7/9 \\ 1/9 \end{pmatrix} - (2)x_2 = 0 \Longrightarrow x_2 = \left(\begin{pmatrix} -1 \\ 2/3 \end{pmatrix} + \begin{pmatrix} 7/9 \\ 1/9 \end{pmatrix} \right) \left(-\frac{1}{2} \right) = \begin{pmatrix} 1/9 \\ -7/18 \end{pmatrix}.$$

We conclude that

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7/9 \\ 1/9 \\ 1/9 \\ -7/18 \end{pmatrix}.$$

Question 2

(a)

The forward routine is

```
% Forward substitution
function [ y ] = fsub( R,b )
        y=nan(size(b));
        [m,n]=size(R);
        [mv,nv]=size(b);

        if m~=n
                disp('matrix not square')
                return
        elseif mv~=m || nv ~= 1
                disp('vector size not compatible with matrix')
                return
        end

        y(1)=b(1)/R(1,1);

        for i=2:n
                y(i)=b(i);
                for j=1:i-1
                        y(i) = y(i)-R(i,j)*y(j);
                end
        y(i)=y(i)/R(i,i);
        end
end
```

(b)

The coefficients found on Mathematica are

$$x_1 = \begin{pmatrix} 1.6580 \\ 3.3360 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 3.5523 \\ 0.6617 \\ 0.8914 \end{pmatrix}.$$

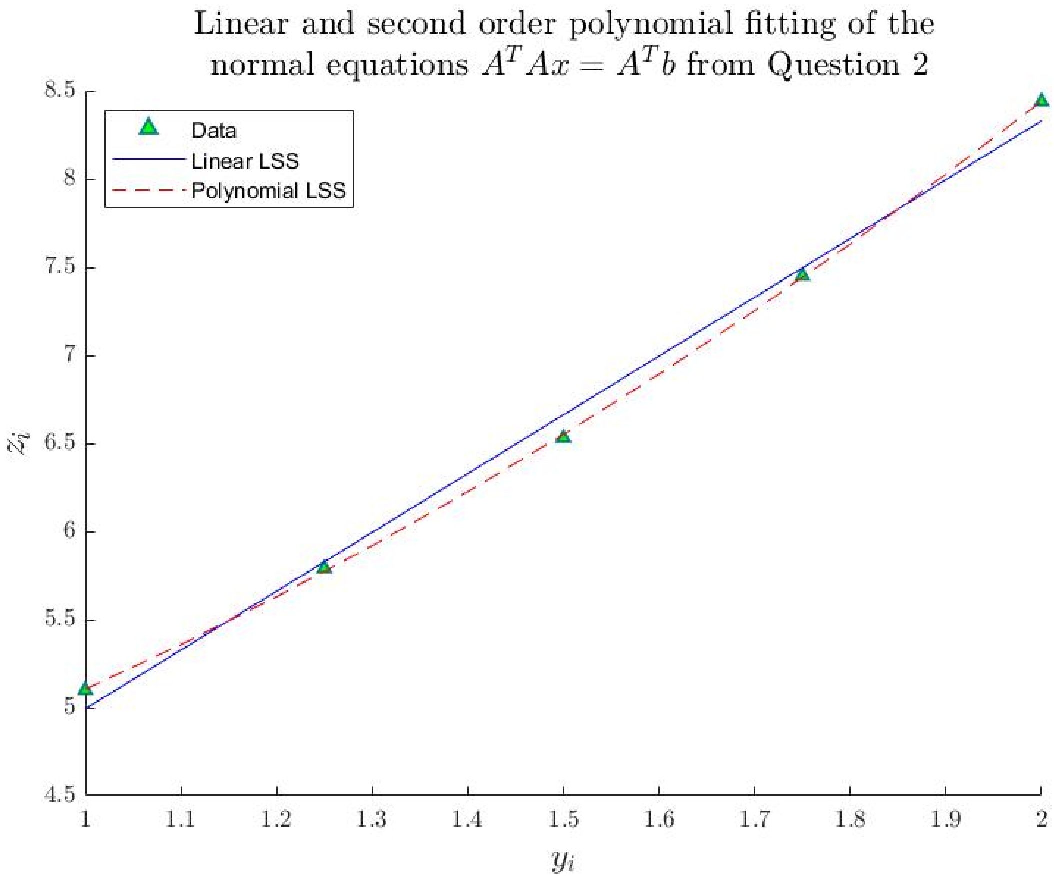

Figure 1

Question 3

(a)

Let $x \in \mathbb{R}^m$. Then

$$\|b - Ax\|_2 = \min_{\hat{x}\in\mathbb{R}^m} \|b - A\hat{x}\|_2 \iff b - Ax \in \mathcal{R}(A)^{\perp}.$$

Since $\mathcal{R}(A)^{\perp} = \mathcal{N}(A^T)$ by fact, then x solves the LLSP if and only if $b - Ax \in \mathcal{N}(A^T)$. By definition of the kernel, we write the equivalency

$$A^T(b - Ax) = 0 \Longrightarrow A^t b = A^T Ax \Longrightarrow A^T Ax = A^T b.$$

(b)

If A has full rank $(n \geq m)$ then $A^T A$ is positive definite. So $\exists (A^T A)^{-1} \Longrightarrow \exists!\ x \in \mathbb{R}^m$ that solves the LLSP. Indeed

$$x^T(A^T A)x = 0 \Longrightarrow (Ax)^T A^x = \langle Ax, Ax\rangle = 0 \Longrightarrow \mathcal{N}(A) = 0,$$

$\therefore A^T A$ is positive definite. In the rank deficient case, $(A^T A)^{-1}$ does not exist so the $x \in \mathbb{R}^m$ is not unique.

(c)

$$A^T A = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 9 & 18 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}.$$

We use Cholesky decomposition on $A^T A$ which is positive definite.

$$\begin{aligned} r_{11}^2 &= 9 & &\implies r_{11} = 3 \\ r_{12} r_{11} &= 9 & &\implies r_{12} = 3 \rightarrow R = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \\ r_{12}^2 + r_{22}^2 &= 18 & &\implies r_{22} = 3 \end{aligned}$$

We apply forward substitution then backward substitution

$$R^T y = b \rightarrow \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} y = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \implies \begin{cases} y_1 & = 9/3 = 3 \\ y_2 & = (0 - 3(3))/3 = -3 \end{cases} \qquad \therefore y = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$Rx = y \rightarrow \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} x = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \implies \begin{cases} x_2 & = -3/3 = -1 \\ x_1 & = (3 - (3)(-3))/3 = 2 \end{cases} \qquad \therefore x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Question 4

(a)

		Comparing Algorithms			
	m	*cgs.m*	*mgs.m*	*Householder*	*QR*
	2	1.6345e−16	1.6345e−16	7.2312e−16	4.7108e−16
0	4	3.9218e−14	5.7278e−15	2.8507e−16	3.8886e−16
	8	1.2777e−04	8.0473e−12	6.2207e−16	7.1078e−16
	16	8.2721e+00	1.2911e−04	1.3689e−15	1.1569e−15
	32	2.6692e+01	1.0000e+00	1.1423e−15	1.4161e−15

(b)

As m grows the values become less accurate across all cases. This inaccuracy grows particularly fast for the cgs case around $m = 8_+$. The *mgs* case's inaccuracy increases slower but increases nevertheless, while the *Householder triangularization* method seems to be very accurate and run on par with the MatLab built in *QR* method. For this reason it is evident that MatLab uses the Householder triangularization method with some slight adjustments.

(c)

Comparing Successive Algorithms		
Application	*cgs.m*	*mgs.m*
0		
1	2.6692e+01	1.0000e+00
2	2.1939e+01	4.4033e−11

Indeed the accuracy increases and it increases substantially faster for the *mgs.m* case, as it was expected.

(d)

Computationally, the safest method would be the QR built in method since it is an improved version of the Householder triangularization and it is the most accurate across all matrix sizes. While, by hand, the CGS method is sufficiently intuitive, easy, and accurate for low matrix size, so it is more appropriate.

Question 5

(a)

We first note that the rank of the matrix is 1 since each column is linearly dependent on u. By the outer product formulation theorem, we have

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T = \sigma_1 u_1 v1j^T = \hat{U}\hat{\Sigma}\hat{V}^T.$$

So then by the condensed SVD theorem we get

$$A = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \left(\sigma_1\right) \left(v_1 \dots v_n\right).$$

Where $\sigma_1 = 1$. But here $\hat{U}$ and $\hat{V}$ are not isometries as the theorem requires. So we normalize the vectors and conclude that

$$A = \underbrace{\frac{u}{\|u\|}}_{\hat{U}} \underbrace{(\|u\|\|v\|)}_{\hat{\Sigma}} \underbrace{\frac{v}{\|v\|}}_{\hat{V}^T}.$$

(b)

We first find V through A^TA

$$A^TA = \begin{pmatrix} 14 & 28 \\ 28 & 56 \end{pmatrix} \to \begin{array}{ll} \text{For } \lambda = 70: & \begin{pmatrix} -56 & 28 \\ 28 & -14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} := v_1 \\ \text{For } \lambda = 0: & \begin{pmatrix} 14 & 28 \\ 28 & 56 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} := v_2 \end{array}$$

The vectors need to be normalized so we have

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \Longrightarrow \hat{V}^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

Similarly, we find U through AA^T.

$$AA^T = \begin{pmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{pmatrix} \to \begin{array}{ll} \text{For } \lambda = 70: & \begin{pmatrix} -65 & 10 & 15 \\ 10 & -50 & 30 \\ 15 & 30 & -25 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} := u_1 \\ \text{For } \lambda = 0: & \begin{pmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \\ & := u_2 + u_3 \end{array}$$

The vectors need to be normalized so we have

$$U = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{10}} \\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{15}} & 0 \\ \frac{3}{\sqrt{14}} & 0 & -\frac{1}{\sqrt{10}} \end{pmatrix} \Longrightarrow \hat{U} = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}.$$

Also, from the eigenvalues of A^TA we get Σ

$$\Sigma = \begin{pmatrix} \sqrt{70} & 0 \\ 0 & \sqrt{0} \\ 0 & 0 \end{pmatrix} \Longrightarrow \hat{\Sigma} = \begin{pmatrix} \sqrt{70} \end{pmatrix}.$$

Finally we conclude the condensed SVD decomposition is

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}, \qquad \hat{\Sigma} = \begin{pmatrix} \sqrt{70} \end{pmatrix}, \qquad \hat{V}^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

$$
\begin{aligned}
& a, b, c, d, e, f, g, h, i \\
& \underbrace{a, b}_{\text{deck1}}, \quad \underbrace{c, d, e, f, g}_{\text{this deck}}, \quad \underbrace{h, i}_{\text{deck3}} \\
\to\ & \underbrace{a, b}_{\text{deck1}}, \quad \underbrace{h, i, c, d, e, f, g}_{\text{new deck}} \\
\to\ & \underbrace{h, i, c, d, e, f, g, a, b}_{\text{final deck}}
\end{aligned}
$$