McGill

Advanced Calculus

MATH 314

CLASS NOTES FOR MATH 314

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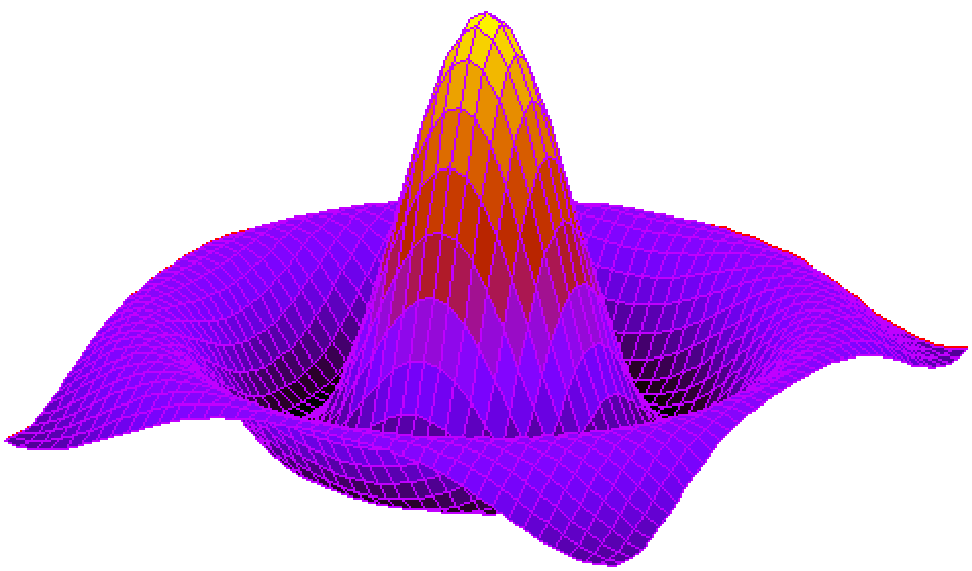

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Contents

1	Basics of Scalar and Vector Valued Functions	3
2	Limits and Continuity	5
	2.1 Limit Review	8
3	Differentiation	10
4	Properties of Derivatives	17
	4.1 Chain Rule	18
5	Taylor Series	20
6	Implicit Functions	28
	6.1 The Implicit Function Theorem	28
	6.2 The Inverse Function Theorem	31
7	Double Integrals and Triple Integrals	33
	7.1 Properties of Integrable Functions	35
	7.2 Integration Over More General Domain	37
	7.3 Triple Integrals	37
	7.4 Change of Variables Coordinates	40
	7.5 Cylindrical Coordinates	42
	7.6 Spherical Coordinates	43
8	Line Integrals and Path Integrals	46
	8.1 Divergence and Curl	47
9	Line Integrals and Force Fields	50
	9.1 Path Independence	51
	9.2 Green's Theorem	54
	9.3 Vector Form of Green's Theorem	61
10	Parametric Surfaces	62
	10.1 Surface Area	64
	10.2 Surface Integrals of Vector Fields	69
	10.3 Stoke's Theorem	71
	10.4 Divergence Theorem	75

Disclaimer and Preliminary Information

The following lecture notes are based on the class presented by *Dr. Geoffrey McGregor* and are typesetted by *Mihail Anghelici*. Any errors - mathematical or otherwise - originate from the author. Most mistakes arise from the author's miss-typing on the keyboard or from miss-writing on the notebook while re-transcribing the board during class sessions.

These notes do not entirely cover what was presented in the lectures, but since the author is a self-proclaimed amateur speed-writer, it is fair to assume that most essential information is covered in this document.

For further reference, the manuscript was type-setted using **TeXstudio** LaTeXeditor and the figures were created using **IPE** editor.

I ask kindly that these notes be not sold or posted on websites such as *Course-Hero*. These types of websites are inherently counter productive in terms of academic learning and consequently, in my opinion, should avoided.

"Mathematics takes us into the region of absolute necessity, to which not only the actual word, but every possible word, must conform."

—Bertrand Russell

1 Basics of Scalar and Vector Valued Functions

Let f be a function whose domain A, is a subset of $\mathbb{R}^n$ with range contained in $\mathbb{R}^n$ and image in $\mathbb{R}^m$,written $f : \mathbb{R}^n \to \mathbb{R}^m$ and more specifically, $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ .Therefore for each $\vec{x} = (x_1, x_2, \ldots, x_n) \in A$, f assigns a value $f(\vec{x})$, an m-tuple in $\mathbb{R}^m$. Such f functions are called vector-valued if $m > 1$. If $m = 1$ then f is called scalar-valued.

Example 1.1.

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \to f : \mathbb{R}^3 \backslash \{\vec{0}\}$$

Is a scalar-valued function.

Example 1.2.

$$f(x_1, x_2, x_3, x_4, x_5) = \left(x_1 x_2 x_3 x_4, \sqrt{x_1^2 + x_5^2}\right)$$

$$\to f : \mathbb{R}^5 \to \mathbb{R}^2$$

Is a vector-valued function.

Why bother studying such functions from $\mathbb{R}^n \to \mathbb{R}^m$?

Example 1.3 (Scalar-valued example). Let T measure the temperature at each point in this room at time t. If $T : \mathbb{R}^n \to \mathbb{R}^m$, what is the n and m? Each point in this room has (x, y, z) , but also time !

$$T : \mathbb{R}^4 \to \mathbb{R} \qquad [T(x, y, z, t)]$$

extra info : Since $t > 0$, we have more precisely $T : \mathbb{R}^3 \times \mathbb{R}^+$.

Example 1.4 (Vector-valued example). Let V represent the velocity at time t at each point in a body of water, what is the m and n ?
Domain : V is a function of x, y, z and t , thus $V(x, y, z, t) \implies$ domain $\subset \mathbb{R}^4$, therefore $V : \mathbb{R}^4 \to \mathbb{R}^3$ Since the velocity vectors has three components.

For functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$, we can discuss its graph . When $f : U \in \mathbb{R} \to \mathbb{R}$ (Calc 1), the graph is a subset of $\mathbb{R}^2$ consisting of all points $(x, f(x)) \in \mathbb{R}^2$ where $x \in U$, Figure 1a illustrates this situation and Figure 1b portrays this in a generalized fashion.

$$\text{Graph of f : } \{(x, f(x)) \in \mathbb{R}^2 \mid x \in U\}$$

reading this out loud : "set of all .. such that ..".

Definition 1.1. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The graph of f is the subset of $\mathbb{R}^{n+1}$ consisting of all points*

$$(x_1, x_2, \ldots .x_n, f(x_1, \ldots f(x_n)) \in \mathbb{R}^{n+1})$$

where $(x_1, \ldots x_n) \in U$ (the domain).

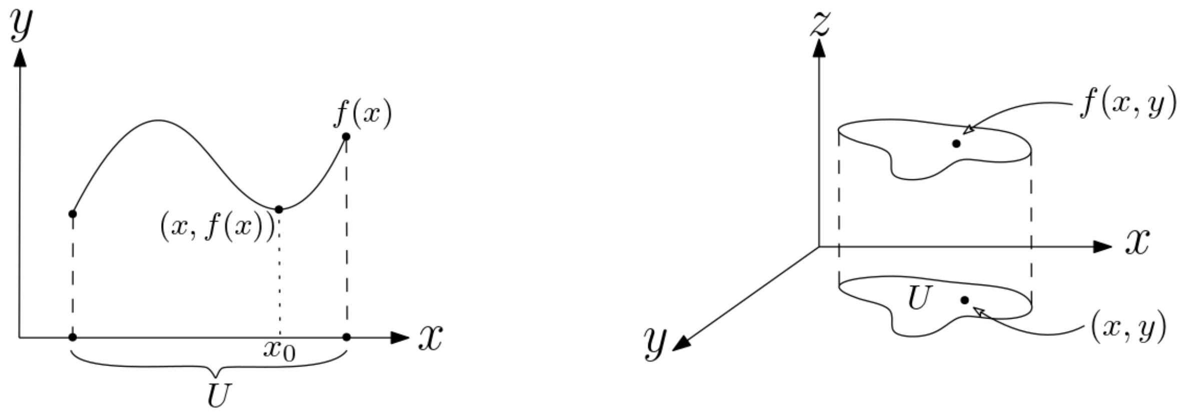


Figure 1: **a)** Two dimensional interpretation, $f : U \subset \mathbb{R} \to \mathbb{R}$. **b)** Three dimensional interpretation, can be generalized to $f : U \subset \mathbb{R}^n \to \mathbb{R}$.

Definition 1.2. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and let $c \in \mathbb{R}$ .Then the level set of value c is defined to be the point $\vec{x} \in U$ which $f(\vec{x}) = c$.*

$$\begin{aligned} &\textit{If } \; n = 2, \quad \textit{we say level curves} \\ &\textit{If } \; n = 3, \quad \textit{we say level surfaces} \end{aligned}$$

Example 1.5. Describe the level curves of $f(x, y) = x^2 + y^2$.
Recall that level curves live in $\mathbb{R}^2$ (in domain)
Let $c = 1$,$\to$ $x^2 + y^2 = 1^2$ The level curves are circles of radius $\sqrt{c}$.

Example 1.6. Describe the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$
Let $c = 1$,$\to$ $x^2 + y^2 + z^2 = 1$ (...). The level curves are spheres of radius $\sqrt{c}$. Note these are surfaces living in $\mathbb{R}^3$ space as illustrated in Figure 2a.

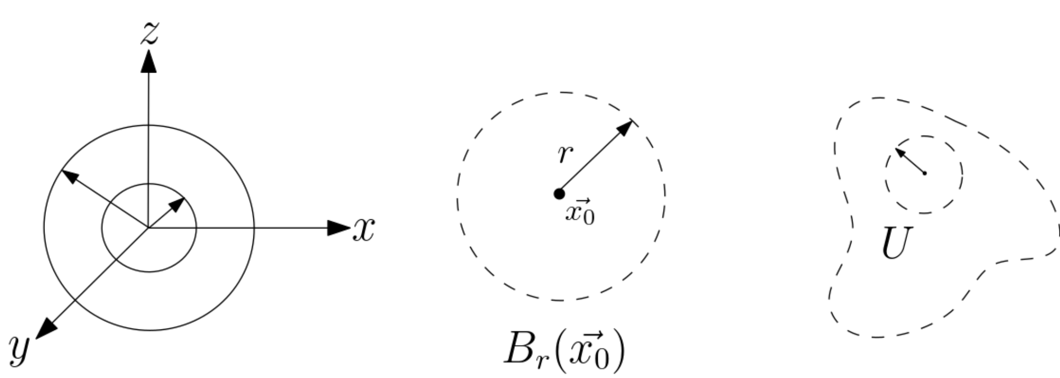


Figure 2: **a)** Level curves in $\mathbb{R}^3$. **b)** Open ball of radius r centered at $\vec{x_0}$. **c)** Illustration of an open-set.

2 Limits and Continuity

Let $\vec{x_0} \in \mathbb{R}^n$ and let r be a positive real number. The open ball of radius r centered at $\vec{x_0}$, as illustrated in Figure 2b, is defined as the set of all vectors $\vec{x}$ such that

$$\{\|\vec{x} - \vec{x_0}\| < r\}_{\text{distance between } \vec{x} \text{ and } \vec{x_0}}$$

$$\underbrace{B_r(\vec{x_0})}_{\text{Ball centered at } r} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{x_0}\| < r\} \tag{2.0.1}$$

If $\|\vec{x} - \vec{x_0}\| = r$ the $\vec{y} \notin B_r(\vec{x_0})$ i.e, the dotted line is not in the set.

Definition 2.1. (Open set) *Let $U \in \mathbb{R}^n$ we call U an open set if for every point $\vec{x_0} \in U$, there exists a $r > 0$ such that $B_r(\vec{x_0}) \subset U$.*

An attempt for a visual interpretation of Definition (2.1) is provided in Figure 2c.

Example 2.1. Prove that $(0, 1) \times (0, 1)$ is an open set, where

$$(0, 1) \times (0, 1) = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), y \in (0, 1)\}$$

. Take a point $(x_0, y_0) \in S \implies x_0 \in (0, 1)$ and $y_0 \in (0, 1)$

$$\begin{aligned} \text{Let } r_1 &= \min(x_0, 1 - x_0) \\ r_2 &= \min(y_0, 1 - y_0) \\ r &= \frac{\min(r_1, r_2)}{2} \end{aligned}$$

By this construction,

$$\begin{matrix} (x_0 - r, x_0 + r) \in (0, 1) \\ (y_0 - r, y_0 + r) \in (0, 1) \end{matrix} \rightarrow \underbrace{(x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r)}_{\text{is an open square of radius } r}$$

Moreover recalling (2.0.1), we can construct

$$\underbrace{B_r(x_0, y_0)}_{\text{open circle of radius } r} \subset (x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r)$$

Conclusion : S as defined is open.
"Idea is to show that there is a circle in there"

Definition 2.2. (Neighborhood) *We say a neighborhood of $\vec{x} \in \mathbb{R}^n$ is any open set containing $\vec{x}$.*

Definition 2.3. (Boundary point) *Let $A \subset \mathbb{R}^n$. A point $\vec{x} \in \mathbb{R}^n$ is called a boundary point of A if every neighborhood of $\vec{x}$ contains points in A and not in A.*

Definition 2.4. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is open. Let $\vec{x_0} \in A$ or a boundary point of A , and let N be a neighborhood of $\vec{b} \in \mathbb{R}^m$, we say f is eventually in N as $\vec{x} \to \vec{x_0}$ if there exists a neighborhood U of $\vec{x_0}$ such that*
if $\vec{x} \neq \vec{x_0}$, $\vec{x} \in U$ and $\vec{x} \in A$ (in the domain) $\implies f(\vec{x}) = N$. We say $f \to \vec{b}$ as $\vec{x} \to \vec{x_0}$

$$\text{i.e,} \quad \lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = \vec{b} \tag{2.0.2}$$

this happens when given any neighborhood of N of $\vec{b}$ f is eventually in N as $\vec{x} \to \vec{x_0}$ (approaches $\vec{x_0}$)
$\implies$ it may be that $\vec{x} \to \vec{x_0}$ that the values of $f(\vec{x})$ do not get close to any particular number.In this case the limit DNE.

Let $A = (\alpha, \beta)$ an open interval.
-Pick a random point x_0, b
-Now pick any neighbourhood of b
-If i put any x in the set U then $f(x) \in N$.
-$U \subset A$

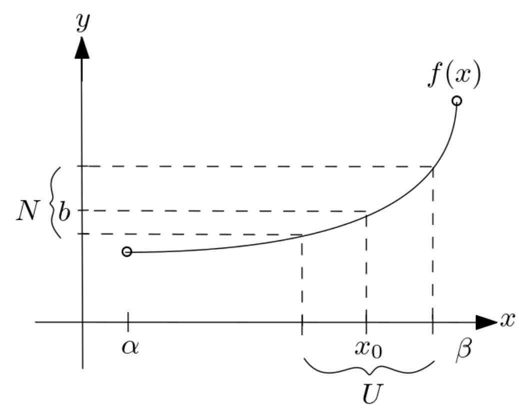


Figure 3: 1-D interpretation

Example 2.2. Use the idea of Definition (2.0.2) to show

$$\lim_{x \to 0} f(x) \quad \text{DNE}$$

where

$$f = \begin{cases} 1 & ,x > 0 \\ 0 & ,x = 0 \\ -1 & ,x < 0 \end{cases} \quad \to$$ Limit from the right $\neq$ limit from the left so DNE ,from calc.1

Take any open-set U containing 0, for example : $U = (-r, r), r > 0$ .Then for any r , if $x < 0$ and $x \in U \implies f(x) = -1$. If $x > 0$ and $x \in U \implies f(x) = 1$. Therefore , the elements of U aren't approaching a particular value for any U containing 0, thus DNE. Indeed, if it's elements are not approaching any single value then DNE.

Definition 2.5. (Continuous) *Let $f : A \in \mathbb{R}^n \to \mathbb{R}^m$ and let $\vec{x_0} \in A$. We say f is continuous at $\vec{x_0}$ if and only if*

$$\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = f(\vec{x_0}) \tag{2.0.3}$$

Saying f is continuous means continuous at all points in its domain (not just $\vec{x_0}$).

Theorem 2.1. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}$.*

(i) If f is continuous at $\vec{x_0}$, then so is the function $cf = cf(x)$ (i.e scalar multiplication doesn't affect continuity).

(ii) If f, g are continuous at $\vec{x_0}$, then so is $(f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x})$ is continuous at $\vec{x_0}$

(iii) If $f(\vec{x} = (f_1(\vec{x}) + f_2(\vec{x}), \dots, f_m(\vec{x})))$ then f is continuous at $\vec{x_0}$ if and only if each real-valued function $f_1(\vec{x}) \quad i = 1, \dots, m$ is continuous at $\vec{x_0}$

(iv) If $m = 1$ and f, g real valued and

$$\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = a, \quad \lim_{\vec{x} \to \vec{x_0}} g(\vec{x}) = b \quad \Longrightarrow \quad \lim_{\vec{x} \to \vec{x_0}} f(\vec{x})g(\vec{x}) = ab$$

Example 2.3. Show that $f(x, y) = \left(x^2 y, \dfrac{y + x^3}{1 + x^2}\right)$ is continuous using Theorem 2.1.

Let $f_1(x, y) = x^2 y$ and let $f_2(x, y) = \dfrac{y + x^3}{1 + x^2}$, then by (iii) ,we need to show that f_1 and f_2 are continuous.

$$f_1(x, y) = x^2 y \qquad , x^2 \text{ and } y \text{ are both polynomials} \implies \text{continuous.}$$

Also, by (iv), the product is also continuous

$$f_2(x, y) = \frac{y + x^3}{1 + x^2} = \underbrace{(y + x^3)}_{\text{sum of cont. functions} \checkmark} \left(\frac{1}{1 + x^2}\right).$$

For $1/(1 + x^2)$ the denominator is never 0 so by Theorem 2.1, is continuous.
Conclusion: by (iv) , $f_2(x, y)$ is continuous $\implies$ f is continuous as well by (iii).

Theorem 2.2. *Let $g : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : B \subset \mathbb{R}^m \to \mathbb{R}^p$.*
Suppose $g(A) \subset B$ (i.e B much larger than the range of g so that $f \circ g$ is defined on A.)
If g is continuous at $\vec{x_0} \in A$ and f is continuous at $\vec{y_0} = g(\vec{x_0})$, then $f \circ g$ is continuous at $\vec{x_0}$.

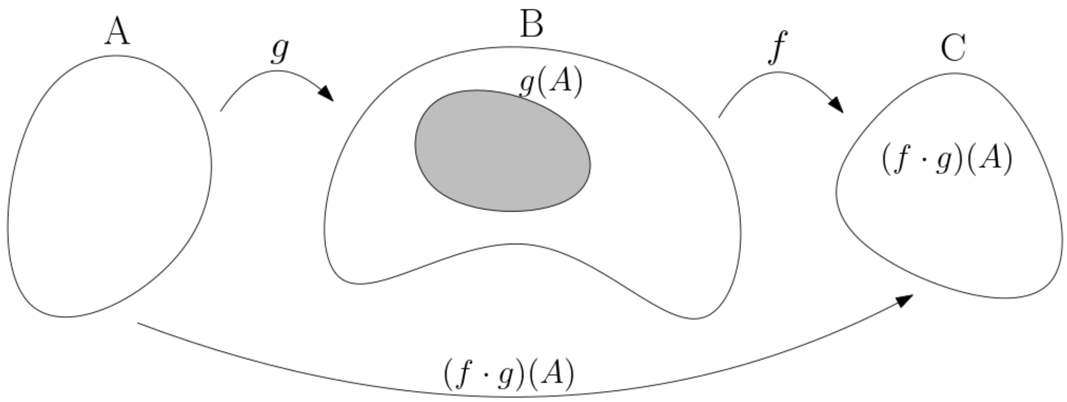


Figure 4: Visual picture for comprehension.

Example 2.4. Show that $f(x, y, z) = (xy + z)^2 + (\cos(y))^3$ is continuous.
We first and foremost define $f_1 = (xy + z)^2$ and $f_2 = (\cos(y))^3$.
Say $f_1(x, y, z) = F_1(G_1(x, y, z)), \quad G_1 : \mathbb{R}^3 \to \mathbb{R},\ F_1 : \mathbb{R} \to \mathbb{R}^3$.
So $G_1(x, y, z) = xy + z$ is continuous (polynomial and product) by (ii) and (iv).
$f_1(x) = x^2$ is continuous because it is also a polynomial , therefore f_1 is continuous by Theorem 2.1.

Similarly, we define $f_2(x, y, z) = F_2(G_2(x, y, z)),\ G_2 : \mathbb{R}^3 \to \mathbb{R},\ \ F_2 : \mathbb{R} \to \mathbb{R}$.
Where $G_2(x, y, z) = \cos(y)$, which is obviously continuous. $F_2(x) = x^3$ is also continuous so by same analysis f is continuous as well.

2.1 Limit Review

Example 2.5. Compute

$$\lim_{(x,y)\to(0,0)} \frac{x^4 \arctan\left(\dfrac{1}{x^2+y^2}\right)}{x^2+y^2}.$$

Let us show that it exists (intuitively) using polar coordinates.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &= \lim_{r\to 0} \frac{r^4(\cos\theta)^4 \arctan\left(\dfrac{1}{r^2}\right)}{r^2} \\ &= \lim_{r\to 0} r^2(\cos\theta)^4 \underbrace{\arctan\left(\frac{1}{r^2}\right)}_{=\pi/2} \\ &= \frac{\pi}{2} \underbrace{\lim_{r\to 0} r^2(\cos\theta)^4}_{\text{Squeeze it !}} \end{aligned}$$

$$0 \le (\cos\theta)^4 \le 1 \implies 0 \le \lim_{r\to 0} r^2(\cos\theta)^4 \le \underbrace{\lim_{r\to 0} r^2}_{=0}$$

Finally we conclude that the limit ,as defined, converges to 0.

Example 2.6. Compute

$$\lim_{(x,y)\to(0,0)} \frac{x^6y^3}{x^{12}+x^6}$$

Step 1 : Show that limit exists when approaching $(0,0)$ along any line. Set $y = mx$.

$$\lim_{x\to 0} \frac{x^6(mx)^3}{x^{12}+(mx)^6} = m^3 \lim_{x\to 0} \frac{x^9}{x^{12}+x^6m^6} = m^3 \lim_{x\to 0} \frac{x^3}{x^6+m^6}$$

The numerator goes to 0 and the denominator goes to $0 + m^6$, but if $y = 0$ then lim $= 0$.
Therefore, the limit equals zero when approaching any line since we arbitrarily chose $mx \quad , m \in \mathbb{R}$.
Step 2 : Let $y = x^2$

$$\lim_{x\to 0} \frac{x^6(x^2)^3}{x^{12}+(x^2)^6} = \frac{x^{12}}{x^{12}+x^{12}} = \frac{1}{2}$$

Different result $1/2 \neq 0 \implies$ limit DNE.

3 Differentiation

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ (real-valued)

$$\text{Then } \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \quad \begin{cases} \text{the partials of } f \text{ with respect to its first, second} \\ \text{and } n\text{'th variables , are all real-valued functions} \\ \text{of } n \text{ variables.} \end{cases}$$

i.e, $\frac{\partial f}{\partial x_i} : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \dots, n$ which at point $\vec{x} = (\vec{x_1}, \dots, \vec{x_n})$ are defined by

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_{j+h}, \dots, x_n) - f(x_1, \dots, x_n)}{h}.$$

Note that this may be written shorter,

$$\lim_{h \to 0} \frac{f(\vec{x} + \vec{e}_j h) - f(\vec{x})}{h} \quad \text{where, } \vec{e}_j = (0, 0, \dots, \underbrace{1}_{j^{\text{th}} \text{ component}}, \dots, 0).$$

Extra information : $\vec{e}_j$ may be seen a standard basis vector.

Example 3.1.

$$f(x_1, x_2, x_3, x_4, x_5) = x_1^2 x_3 + x_4 \sin(x_1 x_2)$$

is continuous.

$$\frac{\partial f}{\partial x_1} = 2x_1 x_3 + x_4 \cos(x_1 x_2) x_2$$

is also continuous and works for the other x_i as well.

Example 3.2. Let $f(x, y) = x^{1/3} x^{1/3}$. Compute $f_x(0, 0)$.

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \quad \text{the right answer} \checkmark$$

Note that if we would had computed $f_x(x, y) = \frac{1}{3} x^{-2/3} y^{1/3}$, at $(0, 0)$ it cannot be evaluated.
$\to$ So we conclude that simply having partials exist at a point is not enough to have differentiability.

We ended last time by studying the function

$$\left.\begin{aligned} f(x, y) &= x^{1/3} y^{1/3} \\ f_x(0, 0) &= 0, \quad f_y(0, 0) = 0 \end{aligned}\right\} \quad \text{,but we don't expect } f(x, y) \text{ to be differentiable at } (0, 0).$$

In Calculus 1, we say f is differentiable at x_0 if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{exists.}$$

Therefore we said that that

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} \quad \text{, for some } f'(x_0) \in \mathbb{R}, \tag{3.0.1}$$

so f is differentiable at x_0 if such a number $f'(x_0)$ exists to satisfy the limit.

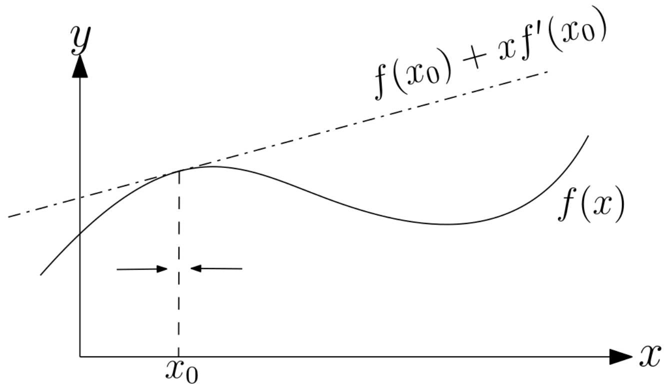


Figure 5: Tangent line to a function following (3.0.1).

Example 3.3. Using the definition of differentiability of real-valued function, show that for $f(x) = x^3$ that $f'(x) = 3x^2$.

$$\begin{aligned}
&\lim_{h\to 0} \frac{(x_0+h)^3 - x_0^2 - (3x_0^2)h}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + 3x_0h + 3x_0h + h^3) - x_0^3 - (3x_0^2)h}{h} \\
&= \lim_{h\to 0} \frac{x_0h^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x_0h + h^2 = 0
\end{aligned}$$

Conclusion : f as defined , is indeed differentiable.

Now what about $f : \mathbb{R}^2 \to \mathbb{R}$ functions?

Definition 3.1. *We say $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_o, y_0)$ exist and*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \frac{f_x(x_0,y_0) - (f_x)(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}{\|(x,y) - (x_0,y_0)\|}. \tag{3.0.2}$$

Recall that the tangent plane to a function $z = f(x, y)$ at some point $P(x_0, y_0, z_0)$ is

$$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{3.0.3}$$

Note. *The definition of a tangent plane* (3.0.3) *requires* $f_x(x, y)$ *and* $f_y(x, y)$ *to be continuous at* (x_0, y_0)*, however our definition of differentiability doesn't require this.*

Example 3.4. Using the definitions of partial derivatives and differentiability , show that $f(x, y) = x^2 + y^4 + xy$ is differentiable at $(x, y) = (1, 0)$.
If we compute f_x and so on, we assume continuity but,

$$\begin{aligned}
f_x(1,0) &= \lim_{h\to 0} \frac{f(1+h,0) - f(1,0)}{h} && = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{1+2h}{+} h^2 - 1h && = \lim_{h\to 0} 2 + h = 2 \\
f_y(1,0) &= \lim_{h\to 0} \frac{f(1,h) - f(1,0)}{h} && = \lim_{h\to 0} \frac{1 + h^4 + h - 1}{h} \\
&= \lim_{h\to 0} \frac{h^4 + h}{h} = 1
\end{aligned}$$

Now we compute the final limit :

$$\begin{aligned}
&= \lim_{(x,y)\to(1,0)} \frac{f(x,y) - f(1,0) - (f_x(1,0)(x-1) + f_y(1,0)y)}{\|(x,y)\| - (1,0)} \\
&= \lim_{(x,y)\to(1,0)} \frac{(x^2 + y^4 + xy) - (1) - (2(x-1) + y)}{\underbrace{\sqrt{(x-1)^2 + y^2}}_{\text{Euclidian Norm !}}} \\
&= \lim_{(x,y)\to(1,0)} \frac{x^2 + 2x + 1 + y^4 - y + xy}{\sqrt{(x-1)^2 + y^2}} \\
&= \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 + y^4 - y + xy}{\underbrace{\sqrt{(x-1)^2 + y^2}}_{\text{Relabel } x\to x-1}}
\end{aligned}$$

Thus, the limit now goes to $(0, 0)$;

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 + y^4 - y + (x+1)y}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2 + y^4 + xy}{\sqrt{x^2 + y^2}}$$

Now we transform to polar coordinates - Set $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}
&= \lim_{r\to 0} \frac{r^2\cos^2\theta + r^4\sin^4\theta + r^2\cos\theta\sin\theta}{r} \\
&= \lim_{r\to 0} r\cos^2\theta + r^3\sin^4\theta + r\cos\theta\sin\theta = \boxed{0}
\end{aligned}$$

Conclusion : f as defined, is differentiable at $(1,0)$. Note that regardless of the value for θ , r dominates and goes to zero. Therefore, just like mx in Example 2.6 where we wanted the limit to come from "all sides", here all directions (θ) point to zero.

Example 3.5. Show that $f(x,y) = x^{1/3}y^{1/3}$ is differentiable at $(0,0)$ (question from last class).
Recall : We found that $f_x(0,0) = 0 = f_y(0,0)$.

$$\begin{aligned}
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (f_x(0,0)x + f_y(0,0)y)}{\|(x,y)\| - (0,0)} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^{1/3}y^{1/3} - 0 - 0 - 0}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^{1/3}y^{1/3}}{\sqrt{x^2+y^2}}
\end{aligned}$$

Note that because the numerator power is $2/3$ we expect the expression to blow up to ∞.
Moreover, note that this example works with both polar and $y = mx$ methods.
Let us try setting $y = mx$.

$$\begin{aligned}
\implies &= \lim_{x\to 0} \frac{x^{1/3}(mx)^{1/3}}{\sqrt{x^2+(mx)^2}} = \lim_{x\to 0} \frac{m^{1/3}x^{2/3}}{|x|\sqrt{1+m^2}} \\
&= \lim_{x\to 0} \underbrace{\frac{1}{|x|^{1/3}}}_{\text{goes to } \infty} \underbrace{\frac{m^{1/3}}{\sqrt{1+m^2}}}_{} \implies \text{goes to } \pm\infty \\
&\qquad \text{if } m < 0 \implies -\infty \\
&\qquad\quad m > 0 \implies +\infty
\end{aligned}$$

Conclusion : f as defined, is differentiable at $(0,0)$.

Note. *We can rewrite the differentiability limit as*

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \overbrace{[f_x(x_0,y_0) f_y(x_0,y_0)]}^{\text{define } = Df(x_0,y_0)} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}{\|(x,y) - (x_0,y_0)\|}. \tag{3.0.4}$$

Df here is called the derivative of f , but also here $Df = \nabla f$ (gradient). More generally, we can write

$$\text{for}, \; f : \mathbb{R}^n \to \mathbb{R} \quad , \lim_{\vec{x}\to\vec{x_0}} \frac{f(\vec{x}) - f(\vec{x_0}) - \nabla f(\vec{x} - \vec{x_0})}{\|\vec{x} - \vec{x_0}\|}. \tag{3.0.5}$$

Definition 3.2. *Let U be an open set of $\mathbb{R}^m$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ we say f is differentiable* *at $\vec{x_0} \in U$ if the partials of f exist and if*

$$\lim_{\vec{x}\to\vec{x_0}} \frac{\|f(\vec{x}) - f(\vec{x_0}) - T(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|} = 0, \tag{3.0.6}$$

for some matrix T (Linear transformation).

Theorem 3.1. *$T = Df(\vec{x_0})$ is the partial derivatives matrix of $f(\vec{x_0}) = (f_1(\vec{x_0}), f_2(\vec{x_0}), \ldots, f_m(\vec{x_0}))$,*

$$T = Df(\vec{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \vdots \quad \vdots \\ \frac{\partial f_m}{\partial x_1} \cdots \frac{\partial f_m}{\partial x_n} \end{bmatrix}. \tag{3.0.7}$$

Proof. Suppose this limit is true i.e., it is equal to 0.
Recall from linear algebra that ($\|\vec{x}\| = 0 \implies \vec{x} = 0$) which implies each component goes to 0 as well. Let us rewrite (3.0.6):

$$\lim_{\vec{x}\to\vec{x_0}} \frac{|f_i(\vec{x_0}) - f_i(\vec{x_0}) - (T(\vec{x} - \vec{x_0})_i)|}{\|\vec{x} - \vec{x_0}\|} = 0 \quad , \forall i = 1, \ldots, m$$

Set $\vec{x_0} + \vec{h} = \vec{x}$,

$$\lim_{\vec{h}\to\vec{0}} \frac{\left|f_i(\vec{x_0} + \vec{h}) - f_i(\vec{x_0}) - (T(\vec{h}))_i\right|}{\left\|\vec{h}\right\|} = 0$$

This limit holds along any path $\vec{h} \to \vec{0}$. Now set $\vec{h} = (0, 0, \ldots, a, \ldots, 0) = \underbrace{a\vec{e}_j}_{\text{standard basis vector}}$,

$$\begin{aligned} &\lim_{a\to 0} \frac{\left|f_i(\vec{x_0} + a\vec{e}_j) - f_i(\vec{x_0}) - (Ta\vec{e}_j)\right|}{|a|} = \lim_{a\to 0} \frac{\left|f_i(\vec{x_0} + a\vec{e}_j) - f_i(\vec{x_0}) - a(T\vec{e}_j)_i\right|}{|a|} \\ &= \lim_{a\to 0} \left|\frac{f_i(\vec{x_0} + a\vec{e}_j) - f_i(\vec{x_0}) - a(T\vec{e}_j)_i}{a}\right| = \lim_{a\to 0} \left|\frac{f_i(\vec{x_0} + a\vec{e}_j) - f_i(\vec{x_0})}{a} - (T\vec{e}_j)_i\right| \\ &= \lim_{a\to 0} \frac{f_i(\vec{x} + a\vec{e}_j) - f_i(\vec{x_0})}{a} = (T\vec{e}_j)_i \\ &\quad \implies \frac{\partial f_i}{\partial x_j} = (T\vec{e})_i = (i, j)^{\text{th}} \text{ component of } T. \end{aligned}$$

Conclusion : T is the matrix of partials also known as the Jacobian of f. □

Last time we introduced the notation of differentiability for functions $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, where f is differentiable at $\vec{x_0} \in \mathbb{R}^n$ provided that all partials exist and recalling (3.0.6)

$$\lim_{\vec{x} \to \vec{x_0}} \frac{\|f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|} = 0.$$

Where Df is the Jacobian matrix of f, with $(Df)_{ij} = \dfrac{\partial f_i}{\partial x_j}$.

Theorem 3.2. (Continouity for real-valued functions) *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and let f be differentiable at $\vec{x_0} \in \mathbb{R}^n$, then f is continuous at $\vec{x_0}$.*

Theorem 3.3. (Differentiability for real-valued functions) *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose all partials $\partial f_i / \partial x_j$ exist and are continuous in a neighborhood of a point $\vec{x_0} \in U$. Then f is differentiable at $\vec{x_0}$.*

Note. $\boxed{\textit{Continuous partials} \implies \textit{Differentiability} \implies \textit{Partials exist.}}$

This is useful because if conditions are met, we don't have to worry about using the long limit formula.

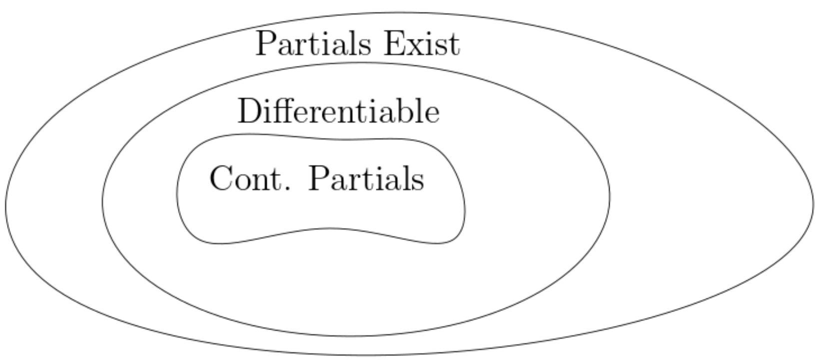


Figure 6: Hierarchical inclusion of differentiability ,continuity and the partials associated. Note that within the differentiable set, functions have varying tangent plane.

Example 3.6. Using the Theorems 3.2 and 3.3, show that $f(x, y) = \dfrac{xy}{(x^2 + y^2)^2}$ is differentiable at all $(x, y) \neq (0, 0)$.

$$f_x = \frac{y(x^2 + y^2)^2 - xy(2(x^2 + y^2)2x)}{(x^2 + y^2)^4}, \quad f_y = \frac{x(x^2 + y^2)^2 - xy(2(x^2 + y^2)2y)}{(x^2 + y^2)^4}.$$

Both the numerator and denominator are continuous $\implies$ f_x and f_y are continuous whenever $(x^2 + y^2) \neq 0$. Therefore, they're continuous away from $(0, 0)$. Thus, we conclude that, since f_x and f_y are continuous $\forall(x, y) \neq (0, 0) \implies f$ is differentiable at all $(x, y) \neq (0, 0)$.

Example 3.7. Using the derivative , find a function $f(x, y)$ and approximate $(0.99)^3 + (2.01)^3 - 6(0.99)(2.01)$.

Let $f(x, y) = x^3 + y^3 - 6xy$, which is obviously differentiable $\forall(x, y) \in \mathbb{R}^2$.
Points in a neighborhood of (x, y) are well approximated by the tangent plane of f. Let $x = 1$,$y = 2$, $\Delta x = 0.01$ and $\Delta y = 0.01$.Then ,

$$f(x - \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)(-\Delta x) + f_y(x, y)(+\Delta y)$$

$$\begin{aligned} f_x(x, y) = 3x^2 - 6y &\implies f_x(1, 2) = 3 - 12 = \boxed{-9} \\ f_y(x, y) = 3y^2 - 6x &\implies f_y(1, 2) = 12 - 6 = \boxed{6} \\ &\implies f(1, 2) = 1 + 8 - 12 = \boxed{-3}. \end{aligned}$$

Now put the $\Delta x, \Delta y$ results in the initial expression, get

$$\implies f(0.99, 2.01) \approxeq -3 + (-9)(-0.01) + (6)(0.01) \approxeq -2.85.$$

4 Properties of Derivatives

Suppose $z = f(u(x), v(x), w(x))$, where $u, v, w : \mathbb{R} \to \mathbb{R}$. Then ,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}. \tag{4.0.1}$$

This is the classic chain rule as discussed in Calculus 3. Here we will generalize the concepts of differentiation to functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

1. **The Constant Multiple Rule**
   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x_0} \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. Then letting $h(\vec{x}) = cf(\vec{x})$,

   $$Dh(\vec{x_0}) = cDf(\vec{x_0}). \tag{4.0.2}$$

2. **The Sum Rule**
   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$,which are both differentiable at $\vec{x_0}$. Then, $h(\vec{x}) + g(\vec{x})$ is also differentiable with

   $$Dh(\vec{x_0}) = Df(\vec{x_0}) + Dg(\vec{x_0}). \tag{4.0.3}$$

   Proof.
   $$\lim_{\vec{x}\to\vec{x_0}} \frac{\|h(\vec{x}) - h(\vec{x_0}) - (Df(\vec{x_0}) + Dg(\vec{x_0}))(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|}$$

   Using $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$,
   $$= \lim_{\vec{x}\to\vec{x_0}} \frac{\|f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) + g(\vec{x_0}) - Dg(\vec{x_0})(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|}$$

   Using the Triangel Inequality : $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$,
   $$\leq \lim_{\vec{x}\to\vec{x_0}} \left[\frac{\|f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|} + \frac{\|g(\vec{x}) - g(\vec{x_0}) - Dg(\vec{x_0})(\vec{x} - \vec{x_0})\|}{\|\vec{x} - \vec{x_0}\|} \right]$$
   $$\leq 0 \implies Dh(\vec{x_0}) = Df(\vec{x_0}) + Dg(\vec{x_0}).$$

   □

3. **Product Rule**
   Let $F : U \subset \mathbb{R}^n \to \mathbb{R},\ g : U \subset \mathbb{R}^n \to R$, both differentiable at $\vec{x_0}$. Let $h(\vec{x}) = f(\vec{x})g(\vec{x})$ $\implies h : U \subset \mathbb{R}^n \to \mathbb{R}$ and h is differentiable

   $$\underbrace{Dh(\vec{x_0})}_{\text{vector}} = \underbrace{g(\vec{x_0})Df(\vec{x_0}) + f(\vec{x_0})Dg(\vec{x_0})}_{\text{vector}}. \tag{4.0.4}$$

4. **Quotient Rule**
Let f and g as above. If $g \neq 0$ in a neighborhood of $\vec{x_0}$, then h is differentiable ($h(\vec{x}) = f(\vec{x})/g(\vec{x})$) at $\vec{x_0}$ with

$$Dh(\vec{x_0}) = \frac{g(\vec{x_0})Df(\vec{x_0}) - f(\vec{x_0})Dg(\vec{x_0})}{(g(\vec{x_0}))^2}. \tag{4.0.5}$$

Example 4.1.

$$\text{Let} \quad \frac{h(\vec{x})f(\vec{x})}{g(\vec{x})}, \text{ where } \begin{cases} f(\vec{x}) & = xy\cos(z) \\ g(\vec{x}) & = x^2 + y^2 \end{cases}.$$

Compute Dh.

$$\begin{aligned} h_x(x,y,z) &= \frac{y\cos(z)(x^2+y^2) - xy\cos(z)(2x)}{(x^2+y^2)^2} \\ h_y(x,y,z) &= \frac{x\cos(z)(x^2+z^2)}{(x^2+z^2)^2} = \frac{x\cos(z)}{(x^2+y^2)} \\ h_z(x,y,z) &= \frac{-xy\sin(z)(x^2+y^2) - xy\cos(z)(2x)}{(x^2+y^2)^2} \end{aligned}$$

$$Dh = \begin{bmatrix} h_x(x,y,z) \\ h_y(x,y,z) \\ h_z(x,y,z) \end{bmatrix} \text{ or } \begin{aligned} Df &= (y\cos(z), x\cos(z), -xy\sin(z)) \\ Dg &= (2x, 0, 2z) \\ Dh &= \frac{(x^2+y^2)(y\cos(z)), x\cos(z) - xy\sin(z) - xy\cos(z)(2x,0,2z)}{(x^2+y^2)^2}. \end{aligned}$$

4.1 Chain Rule

Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$, be given such that g maps $U \to V$ (to ensure $f \circ g$) is defined. Suppose g is differentiable at $\vec{x_0}$, with

$$D(f \circ g)(\vec{x_0}) = \underbrace{Df(\vec{y_0}) \cdot Dg(\vec{x_0})}_{\text{Product of matrices}}.$$

$$\begin{aligned} \text{Take } h = f \circ g \quad &, g = (g_1(\vec{x}), g_2(\vec{x}), \ldots .g_n(\vec{x})) \\ &= (f_1(g_1, \ldots, g_m), f_2(g_1, \ldots, g_m), \ldots, f_p(g_1, \ldots, g_m)) \end{aligned}$$

$$\implies \left(\frac{\partial h}{\partial x_1}\right)_1 = \left(\frac{\partial h}{\partial x}\right)_{1,1} = \frac{\partial f_1}{\partial g_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial g_2}\frac{\partial g_2}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial g_m}\frac{\partial g_m}{\partial x_1}$$

$$\begin{aligned}
Dh &= Df(\vec{y_0})Dg(\vec{x_0}), \quad \text{where} \quad \vec{g} = g(\vec{x_0}) \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} & \cdots & \frac{\partial f_1}{\partial g_m} \\ \vdots & & \ddots & \\ \frac{\partial f_p}{\partial g_1} & & \cdots & \frac{\partial f_p}{\partial g_m} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial g_m}{\partial x_1} & & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \\
\implies (Dh)_{1,1} &= \frac{\partial f_1}{\partial g_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial g_2}\frac{\partial g_2}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial g_m}\frac{\partial g_m}{\partial x_1}
\end{aligned}$$

These are the same.

Example 4.2. Let $f(x, y) = (\cos(y)+x^2, e^{xy})$, $g(u, v) = (e^{u^2}, v-\sin(a))$. Compute $D(f\circ g)(0, 0)$.

$$\begin{aligned}
D(f \circ g)(0,0) &= Df(g(0,0)) \cdot Dg(0,0) \\
Dg(0,0) &= \begin{bmatrix} 2ue^{u^2} & 0 \\ \cos(u) & 1 \end{bmatrix}\Bigg|_{(u,v)=(0,0)} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \\
Df(g(0,0)) &= \begin{bmatrix} 2x & -\sin(y) \\ y & xe^{xy} \end{bmatrix}\Bigg|_{g(0,0)=(1,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
\implies D(f \circ g)(0,0) &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.
\end{aligned}$$

Last class we introduced several rules of differentiation including product rule, quotient rule and chain rule.

Example 4.3. Consider $f(x, y, z) = x^2 + y\sin(xz) - z^3$ where $x(r, s, t) = rst$, $y(r, s, t) = se^t$, $z(r, s, t) = \cos(rs)$.Find a formula for $h(r, s, t) = f(x, y, z)$ and compute $\dfrac{\partial h}{\partial r}$ and verify with the chain rule on f.

$$h(r,s,t) = (rst)^2 + se^t \sin(rst\cos(rs)) - \cos^3(rs)$$

$$\therefore \frac{\partial H}{\partial r} = 2(rst)st + se^t(\cos(rst\cos(rs)) \cdot (st\cos(rs) - rs^2t\sin(rs)) - 3\cos^2(rs)\cdot(-\sin(rs)\cdot s))$$

Verify with the chain rule : $(\ldots)$ trivial.

5 Taylor Series

In the next few pages, we will explore and derive formulae for Taylor expansions in for vector-valued functions. If only the results are of interest to the reader , fast forward to $p.23$.
Thinking of functions $f : \mathbb{R} \to \mathbb{R}$, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x - x_0)^2}{2} + \underbrace{\theta((x - x_0)^3)}_{Notation...}. \tag{5.0.1}$$

Provided f is analytic, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x - x_0)^n}{n!}. \tag{5.0.2}$$

When we truncate, $f(x) \approx \sum_{n=0}^{k} \dfrac{f^n(x - x_0)^n}{n!}$, we make an error. The larger our choice of k, the better the approximation. We would like to generalize this idea of function $f : \mathbb{R}^n \to \mathbb{R}$.
We want to find an approximation of $f(\vec{x_0} + t\vec{h})$ in terms of a multi-nominal $\vec{x_0} \in \mathbb{R}^n$,
$\vec{h} \in \mathbb{R}^n$ and $\left\|\vec{h}\right\| = 1$.

$$f_{\vec{u}} = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{u}) - f(\vec{x_0})}{t}, \quad \text{where} \quad \|\vec{u}\| = 1.$$

Also , when f is differentiable, we get :

$$f_{\vec{u}}(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \vec{u}.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable , recalling 3.0.6

$$\lim_{\vec{x} \to \vec{x_0}} \frac{f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0})}{\|\vec{x} - \vec{x_0}\|} = 0$$

$$\text{Let } \; f(\vec{x}) = f(\vec{x_0}) + Df(\vec{x_0})(\vec{x} - \vec{x_0}) + \underbrace{R(\vec{x}, \vec{x_0})}_{\text{Remainder}}$$

$$\text{Note that differentiability} \implies \lim_{\vec{x} - \vec{x_0}} \frac{R(\vec{x}, \vec{x_0})}{\|\vec{x} - \vec{x_0}\|} = 0.$$

Can we find an equation for R ? Suppose f has continuous second order partial derivatives. Let $\vec{x} = \vec{x_0} + t\vec{h}$, for $\vec{h} \in \mathbb{R}^n$ (so now it's bigger than 1). Then this implies that

$$\begin{aligned} \frac{d}{dx} f(\vec{x_0} + t\vec{h}) &= Df(\vec{x} + t\vec{h}) \cdot \vec{h}, \quad \text{by the chain rule ,} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x} + t\vec{h}) h_i. \end{aligned}$$

Integrating both sides with respect to t from $t = 0 \to 1$, we get

$$\int_0^1 \frac{d}{dt} f(\vec{x_0} + t\vec{h})dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \ dt$$
$$\implies f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) = \sum_{i=1}^{n} \int_0^1 \frac{\partial f}{\partial x_i} h_i \ dt.$$

Doing integration by parts,

$$\int_0^1 u \frac{dv}{dt} \ dt = uv\Big|_0^1 - \int_0^1 \frac{du}{dt} v dt$$

Take v anything , take for instance $v(t) = t - 1$, $u(t) = \dfrac{\partial f}{\partial x_i}(\vec{x_0} + t\vec{h})h_i$

$$\begin{aligned} \text{so} \quad f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t-1)\Big|_0^1 \\ &\quad - \sum_{i=1}^{n} \int_0^1 \frac{d}{dt}\left(\frac{\partial f}{\partial x_i} h_i\right)(t-1) \ dt \\ \implies f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \end{aligned}$$

Let's figure it out ;

$$\underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial x_i}\right) h_i}_{\text{Some stuff}} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0} + t\vec{h}) h_i h_j$$

$$\therefore f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i + \underbrace{\sum_{i=0}^{n} \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0} + t\vec{h}) h_i h_j}_{R_1(\vec{h}, \vec{x_0}) \ldots \text{Eq. } 1}$$

$$\therefore f(\vec{x_0} + \vec{h}) = f(\vec{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i + R_1(\vec{h}, \vec{x_0}).$$

Where $R_1(\vec{h}, \vec{x_0})$ is given by Eq. 1
Can we find more terms this way ? Integrate $R_1(\vec{h}, \vec{x_0})$ by parts again.

$$\begin{aligned} \therefore R_1(\vec{h}, \vec{x_0}) &= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0} + \vec{h}) \left(\frac{-(1-t)^2}{2}\right)\Bigg|_0^1 \\ &\quad + \sum_{i,j=1}^{n} \int_0^1 \frac{(1-t)^2}{2} \frac{d}{dt}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0} + t\vec{h})\right) h_i h_j \ dt. \end{aligned}$$

Why $t-1$ initially chosen ? If we don't, we do not get an expansion for x_0.

$$= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0})h_i h_j + \underbrace{\sum_{i,j,k=1}^{n} \Big|_0^1 \frac{(1-t)^2}{2}\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\vec{x_0}+t\vec{h})h_i h_j h_k \; dt}_{R_2(\vec{h},\vec{x_0})}.$$

$$\therefore\ f(\vec{x_0}+\vec{h}) = f(\vec{x_0}) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{x_0})h_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0})h_i h_j + R_2(\vec{h},\vec{x_0})$$

We can derive a more familiar form of the remainder. Consider $\int_a^b h(t)g(t)dt$, where h, g are continuous on $[a,b]$ closed bounded. $g(t) \geq 0 \neq 0 \implies \int_a^b g(t)dt > 0$,
By continuity of $h(t)$ on $[a,b] \implies h$ attains its minimum m and max M in $[a,b]$. We then have that,

$$h(t_m) = m \leq h(t) \leq h(t_M) = M.$$

$$\text{Since} \quad g(t) > 0 \implies m\int_a^b g(t)dt \leq \int_a^b h(t)g(t)dt \leq M\int_a^b g(t)dt$$

$$\implies m \leq \underbrace{\frac{\int_a^b h(t)g(t)dt}{\int_a^b g(t)dt}}_{\text{Denote } = Q} \leq M.$$

By intermediate value theorem,

$$h(c) = 0 \ \text{ for some } \ c \in [a,b] \implies h(c)\int_a^b g(t)\; dt = \int_a^b h(t)g(t)\; dt \ \text{, for some } \ c \in [a,b].$$

$$R_1(\vec{h},\vec{x_0}) = \sum_{i,j=1}^{n}\int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)h_i h_j \; dt,$$

$$\text{Take} \quad g(t) = (1-t), \quad h(t) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)h_i h_j$$

$$\implies R_1(\vec{h},\vec{x_0}) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(c_{ij})h_i h_j.$$

For some C_{ij} on the line between $\vec{x_0}$ and $\vec{x_0}+\vec{h}$,

$$\text{Similarly,} \quad R_2(\vec{h},\vec{x_0}) = \frac{1}{3!}\sum_{i,j,k=1}^{n}\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(c_{ijk})h_i h_j h_k$$

For some C_{ijk} on the line between $\vec{x_0}$ and $\vec{x_0} + t\vec{h}$.

These are called the *Lagrange form of the remainder*.

Example 5.1. Find the second order Taylor expansion of $f(x, y) = e^{x^2+y}$.

$$f(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + f_{xx}\frac{(0, 0)}{2}x^2 + f_{xy}(0, 0)xy + f_{yy}\frac{(0, 0)}{2}y^2 + R_2((x, y), \vec{0})$$

$$\left\| \begin{array}{l|l} f(0,0) = 1 & f_{xx}(0,0) = 2 \\ f_x(0,0) = 2xe^{x^2+y} = 0 & f_{xy}(0,0) = 0 \\ f_y(0,0) = 1 & f_{yy}(0,0) = 0 \end{array} \right\|$$

$$e^{x^2+1} = 1 + y + x^2 + \frac{y^2}{2} + R_2((x, y), \vec{0}).$$

Note that,

$$\begin{aligned} e^{x^2+y} = e^{x^2}e^y &= (1 + x^2 + \ldots)(1 + y + \frac{y^2}{2} + \ldots) \\ &= (1 + y + x^2 + \frac{y^2}{2} + \ldots). \end{aligned}$$

Last time we introduced Taylor's theorem function $f : \mathbb{R}^n \to \mathbb{R}$.
Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has continuous partials of third order, then

$$f(\vec{x_0} + \vec{h}) = f(\vec{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)h_i h_j + R_2(\vec{h}, \vec{x_2}), \tag{5.0.3}$$

where

$$\underbrace{R_2(\vec{h}, \vec{x_0})}_{\text{Lagrange form}} = \frac{1}{3!}\sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(c_{ijk})h_i h_j h_k, \tag{5.0.4}$$

for which c_{ijk} is somewhere on the line joining $\vec{x_0}$ and $\vec{x_0} + \vec{h}$.

• Note that these equations are very tedious , especially to write any form of generality. For this reason, we introduce multi-index notation.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an *n-tuple* (non-zero), we say $\alpha \leq k$, for some non-negative integer k if $\alpha_i \leq k \; \forall \, i = 1, \ldots, n$. We write ,

$$|\alpha| = \alpha_1 + 2 + \cdots + \alpha_n = \sum_{i=1}^{n} \alpha_i \qquad \to \text{sum of } \alpha. \tag{5.0.5}$$

$$\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n! \qquad \to \text{product of functions} \tag{5.0.6}$$

For example : $(2, 2, 3)! = 2!2!3! = 24$.

By $\vec{h}^{\alpha}$, for $\vec{h} \in \mathbb{R}^n$, we mean $\vec{h}^{\alpha} = h_1^{\alpha_1} h_2^{\alpha_2} \ldots h_n^{\alpha_n}$, and

$$\partial^{\alpha} f(\vec{x}) = \underbrace{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} (f\vec{x})}_{\alpha_i \text{ partials in } x_i} . \tag{5.0.7}$$

$\therefore$ The Taylor series of k^{th} order of $f : \mathbb{R}^n \to \mathbb{R}$ can be written as

$$f(\vec{x_0} + \vec{h}) = \underbrace{\sum_{|\alpha| \le k} \frac{\partial^{\alpha} f(\vec{x})}{\alpha!} h^{\alpha}}_{\text{Series}} + \underbrace{R_k(\vec{h}, \vec{x_0})}_{\text{Remainder}} \tag{5.0.8}$$

$$\text{, where } R_k(\vec{h}, \vec{x_0}) = \sum_{|\alpha| = k+1} \frac{\partial^{\alpha} f(c_{\alpha})}{\alpha!} h^{\alpha}, \tag{5.0.9}$$

for some c_{α} on the line joining $\vec{x_0}$ to $\vec{x_0} + \vec{h}$.

Example 5.2. Write out

$$\sum_{|\alpha| \le 2} \frac{\partial^{\alpha} f(\vec{x_0})}{\alpha!} h^{\alpha} \quad \text{, where } f : \mathbb{R}^2 \to \mathbb{R}.$$

$$|\alpha| = 0 \implies \vec{\alpha} = \langle 0, 0 \rangle.$$

Since $\sum = 0$ and all tuple terms are non-zero,

$$\begin{aligned} |\alpha| = 1 &\implies \vec{\alpha} = \langle 1, 0 \rangle, \vec{\alpha} = \langle 0, 1 \rangle, \\ |\alpha| = 2 &\implies \vec{\alpha} = \langle 2, 0 \rangle, \vec{\alpha} = \langle 1, 1 \rangle, \vec{\alpha} = \langle 0, 2 \rangle. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sum_{|\alpha| \le 2} \frac{\partial^{\alpha} f(\vec{x_0}) h^{\alpha}}{\alpha!} &= \sum_{|\alpha| = 0} \frac{\partial^{\alpha} f(\vec{x_0}) h^{\alpha}}{\alpha!} + \sum_{|\alpha| = 1} \frac{\partial^{\alpha} f(\vec{x_0}) h^{\alpha}}{\alpha!} + \ldots \\ &= f(\vec{x_0}) \\ \text{Then , } \sum_{|\alpha| = 1} \frac{\partial^{\langle} f(\vec{x_0}) h^{\alpha}}{\alpha!} &= \frac{\partial f}{\partial x} h^{\alpha} + \frac{\partial f}{\partial y} h^{\alpha}. \end{aligned}$$

If $\vec{h} = (h_1, h_2)$ is a directional vector we have,

$$\begin{aligned} &= \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 \\ \text{And so , } \sum_{|\alpha| = 2} \frac{\partial^{\alpha} f(x_0) \vec{h}^{\alpha}}{\alpha!} &= \frac{f_{xx}(\vec{x_0}) h_1^2}{2} + \frac{f_{xy}(\vec{x_0}) h_1 h_2}{1} + \frac{f_{yy}(\vec{x_0}) h_2^2}{2} \end{aligned}$$

If $\vec{x_0} = (0,0)$, $\vec{h} = (x, y)$, we get

$$f(x,y) \approxeq f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{f_{xx}(0,0)x^2}{2} + f_{xy}xy + \frac{f_{yy}(0,0)y^2}{2}.$$

Note. *We can only express the remainder of f in this manner if f has enough continuous derivatives and if the line joining $\vec{x_0}$ to $\vec{x_0} + \vec{h}$ is in the domain of f. (Because (c_{ijk}) is in the derivative)*

The remainder requires us to evaluate $\partial^\alpha f(c_\alpha)$, where c_α is on the line joining $\vec{x_0}$ to $\vec{x_0} + \vec{h}$. Thus, it needs to be in the domain of f (Figure 7). To be able to draw a line from any $\vec{x_0}$ to any $\vec{x_0} + \vec{h}$ in the domain, and have the line also be in the domain of $f \implies \text{dom}(f)$ needs to be *convex*.

There is another way of viewing terms in the Taylor series expansion

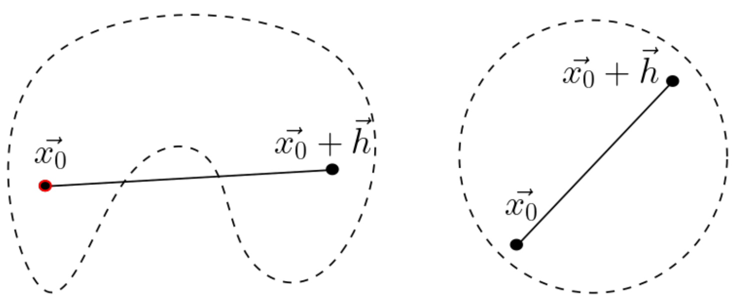


Figure 7: a) Not in the domain (hole) **b)** In the domain (no hole)

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i = Df(\vec{x_0})(\vec{x} - \vec{x_0}) = \underbrace{\nabla f(\vec{x_0}) \cdot (\vec{x} - \vec{x_0})}_{\text{First term of Taylor's.}}$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n, \ \ \nabla f = \left(\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m}\right) \implies D(\nabla f) \text{ is a matrix given by}$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \to$$

This is called the *Hessian* matrix of f. Note that this matrix is symmetric.

Note. $D(\nabla f) = H$ *the derivative of the gradient.*

Next we show for a matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & & \dots & a_{nn} \end{bmatrix} \quad \text{that} \quad \sum_{i,j=1}^{n} a_{ij} h_i h_j = [h_1, \dots, h_n] A \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = \vec{h}^T A \vec{h}$$

$$\implies A\vec{h} = \begin{bmatrix} a_{11}h_1 + & \dots & +a_{1n}h_n \\ a_{21}h_1 + & \dots & +a_{2n}h_n \\ \vdots & \ddots & \vdots \\ a_{n1}h_1 + & \dots & +a_{nn}h_n \end{bmatrix} \implies \begin{aligned} h^T(A\vec{h}) = {} & (h_1 a_{11} h_1 + h_1 a_{12} h_2 + \cdots + h_1 a_{1n} h_n) + \\ & (h_2 a_{21} h_1 + h_2 a_{22} h_2 + \cdots + h_2 a_{2n} h_n) + \\ & (h_n a_{n1} h_1 + h_n a_{n2} h_2 + \cdots + h_n a_{nn} h_n) \\ = {} & \sum_{(i,j)=1}^{n} a_{ij} h_i h_j \end{aligned}$$

The second term of the Taylor expansion is

$$\sum_{(i,j)=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0}) h_i h_j \quad \text{therefore,} \quad = \vec{h}^T H(f(\vec{x_0}))\vec{h}.$$

Thus, the other way to write Taylor expansion is

$$f(\vec{x_0} + \vec{h}) = f(\vec{x_0}) + \nabla f(\vec{x_0}) \cdot \vec{h} + \vec{h}^T H(\vec{x_0})\vec{h} + R_2(\vec{h}, \vec{x_0}). \tag{5.0.10}$$

• The Hessian is very useful in optimization.

Recall f has a critical point at $\vec{x_0}^\star$ if $\nabla f(\vec{x_0}^\star) = \vec{0}$

$$\implies f(\vec{x_0}^\star + \vec{h}) = f(\vec{x_0}^\star) + \vec{h}^T H(\vec{x_0}^\star)\vec{h} + R_2(\vec{h}, \vec{x_0}^\star)$$

and for $\left\|\vec{h}\right\|$ small $\implies \left|R_2(\vec{h}, \vec{x_0})\right| << \vec{h}^T H(\vec{x_0}^\star)\vec{h}$ since $\left|R_2(\vec{h}, \vec{x_0}^\star)\right| \cong \left|\vec{h}\right|^3$,
while $\left|h^T H h\right| \cong \left|\vec{h}\right|^2$ Therefore, in a neighborhood of $\vec{x}^\star$, $h^T H(\vec{x_0}^\star)\vec{h}$ dictates the behavior of f.
A symmetric $n \times n$ matrix M is called positive definite if $h^T M h > 0 \quad \forall \vec{h} \neq 0$.
Similarly, if $h^T M h < 0$, for $\vec{h} \neq 0$, then M is called *Negative Definite*.

Theorem 5.1. *A symmetric $n \times n$ matrix M is positive definite if and only if all eigenvalues of M are positive (If all negative then M is negative definite).*

Following the previous theorem and recalling (5.0.10), letting $x_0^\star$ be a critical point of f , then

$$f(\vec{x_o^\star} + \vec{h}) - f(\vec{x_0}^\star) = h^T H(\vec{x_0}^\star) h + \underbrace{R_2(\vec{h}, \vec{x_0})}_{\text{negligable}}. \tag{5.0.11}$$

Therefore, if $H(\vec{x_0}^\star)$ is positive definite then $f(\vec{x_0}^\star + \vec{h}) - f(\vec{x_0}^\star) > 0 \quad \text{for } \left\|\vec{h}\right\| \text{ small},$ $\vec{x_0}^\star$, is a local minimum.	Conversely, if $H(\vec{x_0}^\star)$ is negative definite then $f(\vec{x_0}^\star + \vec{h}) - f(\vec{x_0}^\star) < 0 \quad \text{for } \left\|\vec{h}\right\| \text{ small},$ $\vec{x_0}^\star$, is a local maximum.

Example 5.3. Find second order Taylor of $f(x,y) = \log\left(x^2 + y^2 + 1\right)$ about $(0,0)$. Then show $(0,0)$ is a local min.

$$\begin{aligned} f(0,0) &= 0 & f_{xx}(0,0) &= 2 \\ f_x(0,0) &= 0 & f_{xy}(0,0) &= 0 \\ f_y(0,0) &= 0 & f_{yy}(0,0) &= 2 \end{aligned}$$

$$\text{Thus ,recalling (5.0.11)} \quad f(x,y) = [x \ y] \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{Hessien}} \begin{bmatrix} x \\ y \end{bmatrix} + R_2(x,y,\vec{0})$$

$$\det(H - \lambda I) = 0 \implies (2-\lambda)^2 = 0 \implies \lambda = 2 > 0.$$

Therefore, positive eigenvalues of H meaning that it's a local min.

6 Implicit Functions

6.1 The Implicit Function Theorem

Consider a function $F(x,y)=0$, where F has continuous first order partials. Suppose $(a,b)\in\mathbb{R}^2$ such that $F(a,b)=0$. Can the equation $F(x,y)=0$ be solved as a function of x in a neighborhood of (a,b) ? I.e., does there exist a function $y(x)$ defined on $I=(a-h,a+h)\ni x$ such that $F(x,y(x))=0\ \forall\, x\in I$. If such a $y(x)$ exists , then we can find its derivative

$$\frac{d}{dx}F(x,y(x))=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}=0. \tag{6.1.1}$$

$$\implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}=\frac{-\partial F/\partial x}{\partial F/\partial y} \text{ provided } \frac{\partial F}{\partial y}\neq 0. \tag{6.1.2}$$

From Ordinary Differential Equations, if $dy/dx=G(x,y)$ with $y(a)=b$, then provided G and dG/dy are continuous in a neighborhood of (a,b) then $\exists!\ y(x)$ in a neighborhood with $y(a)=b$. However, if we have that $|G(x,y)|\leq M$ on some neighborhood of (a,b) then we get the same result.

$$\therefore \text{ For } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\partial F/\partial x}{\partial F/\partial y}\ \text{, we have } \left|\frac{\partial F}{\partial x}\right|\leq M \quad \text{on some neighborhood.}$$

$$\text{and } \left|\frac{\partial F}{\partial y}\right|\leq B \text{ in a neighborhood of } (a,b).$$

$$\implies \left|\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}\right|\leq\frac{M}{B} \quad \text{on a neighborhood of (a,b)} \quad \therefore \exists! \text{ solution.}$$

Note. $\exists!$ *represents a notation for "Exists a unique..."*

Example 6.1. Consider $x^2+y^2-1=0$, where can we solve for $y(x)$? Here, recalling (6.1.1), $\frac{d}{dx}F(x,y(x))=0$

$$\implies \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}=0 \implies 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y} \text{ provided } y\neq 0.$$

The solution to the ODE is $y(x)=\pm\sqrt{1-x^2}$. Note the plus minus and note that these are the horizontal extremities of a regular circle scaled by -1.

Theorem 6.1. *Suppose that $F:\mathbb{R}^{n+1}\to\mathbb{R}$ has cont. partials denoting points in $\mathbb{R}^{n+1}$ as $(\vec{x},z)$ where $\vec{x}\in\mathbb{R}^n$ and $z\in\mathbb{R}$, assuming that $F(\vec{x}_0,z_0)=0$ and $\frac{\partial F}{\partial z}\neq 0$. Then. there is an open ball containing $\vec{x_0}$ and a neighborhood V of z_0 such that $\exists$ a unique function $z=g(\vec{x})$ defined for $\vec{x}\in U$ and $z\in V$ that satisfies $F(\vec{x},g(\vec{x}))=0$. Finally, $z=g(\vec{x})$ is differentiable* *with*

$$Dg(\vec{x}) = -\left.\frac{DxF(\vec{x}, z)}{{}^{\partial F}/_{\partial z}(\vec{x}, z)}\right|_{z=g(\vec{x})}. \tag{6.1.3}$$

Notation $D\vec{x}F = \left(\dfrac{\partial F}{\partial x_1}, \dots, \dfrac{\partial F}{\partial x_n}\right) \implies$ *"Gradient in x components".*

$$\therefore \frac{\partial g}{\partial x_i} = -\frac{{}^{\partial F}/_{\partial x_i}}{{}^{\partial F}/_{\partial z}} \quad , i = 1, \dots, n. \tag{6.1.4}$$

Example 6.2. Near what points may the surface $x^3 + 3y^2 + 8xz^3 - ex^3y = 1$ be respresented as graph of a differentiable function $z = g(x, y)$?

$$\frac{\partial F}{\partial z} = 16xz - 9z^2y = z(16x - 9zy)$$

$\therefore$ provided $z \neq 0$ and $16x \neq 9zy$, we can find $z = g(x, y)$ in a neighborhood.

Now suppose we have m equations , can we solve these in terms of m variables ? Let $F_i : \mathbb{R}^{n+m} \to \mathbb{R}$ for $i = 1, \dots, m$, then

$$\left.\begin{aligned} \text{Let } \mathrm{F}_1(x_1, \dots, x_n, z_1, \dots, z_m) &= 0 \\ &\vdots \\ \mathrm{F}_m(x_1, \dots, x_n, z_1, \dots, z_m) &= 0 \end{aligned}\right\} \text{Can we find } z_1(\vec{x}), z_2(\vec{x}), \dots, z_m(\vec{x}) \text{ in a neighborhood ?}$$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial x_j} + \frac{\partial F_i}{\partial z_1}\frac{\partial z_1}{\partial x_j} + \frac{\partial F_i}{\partial z_2}\frac{\partial z_2}{\partial x_j} + \dots + \frac{\partial F_i}{\partial z_m}\frac{\partial z_m}{\partial x_j} = 0$$

$$\implies \left(\frac{\partial F_i}{\partial z_1}, \dots, \frac{\partial F_i}{\partial z_m}\right) \cdot \left(\frac{\partial z_1}{\partial x_j}, \frac{\partial z_2}{\partial x_j}, \dots, \frac{\partial z_m}{\partial x_j}\right) = -\frac{\partial F_i}{\partial x_j}.$$

Apply d/dx_j for all F_i, we get

$$J = \begin{bmatrix} \frac{\partial F_i}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & \dots & \frac{\partial F_1}{\partial z_m} \\ \vdots & & \ddots & \\ \frac{\partial F_m}{\partial z_1} & \dots & \dots & \frac{\partial F_m}{\partial z_m} \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial x_j} \\ \vdots \\ \frac{\partial z_m}{\partial x_j} \end{bmatrix} = \begin{bmatrix} -{}^{\partial F_1}/_{\partial x_j} \\ \vdots \\ -{}^{F_m}/_{x_j} \end{bmatrix}$$

Recall that $A\vec{x} = B$ can be solved if $\det(A) \neq 0$, $\therefore Jx = B$ can be solved if $\det(J) \neq 0$. If we had done d/dx_k we still get $Jx = B$. Therefore the system can be solved for $z_1(\vec{x}), z_2(\vec{x}), \dots, z_m(\vec{x})$ provided

$$\det\begin{vmatrix} \frac{\partial F_1}{\partial z_1} & \dots & \frac{\partial F_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \dots & \frac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0. \tag{6.1.5}$$

Last time we introduced the implicit function theorem. Let $F(\vec{x}, z) = 0$ with $F : \mathbb{R}^{n+1} \to \mathbb{R}$ who has continuous partial derivatives. Then, if $\frac{\partial F}{\partial x} \neq 0$ (with $F(\vec{x_0}, z_0) = 0$) then there exists $z = g(\vec{x})$ such that $F(\vec{x}, g(\vec{x})) = 0$ in a neighborhood of $\vec{x_0}$ and

$$\left.\frac{\partial z}{\partial x_i}\right|_{(\vec{x_0}, z_0)} = \frac{-\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial z}}.$$

If we have a system of m equations , we showed that it leads to a system of matrix equation. The system can be solved uniquely provided JF has a non-zero determinant (6.1.5) Moreover, this connection remained regardless of which partial we took.

Example 6.3. Consider the system $x^3(y^3 + z^3) = 0$ with $(x-y)^3 - z^2 - 7 = 0$. Show that there exists a neighborhood of $(x, y, z) = (1, -1, 1)$ such that the curve of intersection of the two surfaces can be described by $y = f(x)$ and $z = g(x)$.

$$\begin{aligned} F_1(x, y, z) &= 0 = x^3(y^3 + z^3) \\ f_2(x, y, z) &= 0 = (x-y)^3 - z^2 - 7 \end{aligned}$$

The question implicitly asks us to show that in a neighborhood of $(1, -1, 1)$ we can find $f(x), g(x)$ such that

$$F_1(x, f(x), g(x)) = F_2(x, f(x), g(x)) = 0.$$

Note that here the y and z are the z_1, z_2 from the theorem described previously, and x is our $\vec{x}$. We therefore have that

$$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial x} \\ \frac{\partial z}{\partial x} \end{bmatrix} = \left.\begin{bmatrix} -\,{}^{\partial F_1}\!/_{\partial x} \\ -\,{}^{\partial F_2}\!/_{\partial x} \end{bmatrix}\right|_{(1,-1,1)}$$

$$JF = \left.\begin{bmatrix} 3x^3y^3 & 3x^3z^2 \\ -3(x-y)^2 & -2z \end{bmatrix}\right|_{(1,-1,1)} = \begin{bmatrix} 3 & 3 \\ -24 & -2 \end{bmatrix} \implies \left.\det(JF)\right|_{(1,-1,1)} = 66 \neq 0.$$

We conclude that in a neighborhood of $(1, -1, 1)$ we can find $f(x), g(x)$ such that $F_i(x, f(x), g(x)) = 0$.

Example 6.4. Consider the system of equations $x^2 + y^2 - 1 = 0$ and $x^2 + y^2 + z^2 - 1 = 0$. Use the implicit function theorem to find the curve of intersection in terms of $y(x)$, then show that we can find $F_i(x, y(x), z(x)) = 0$ with $z(x) = 0$.

Note that graphically, this system corresponds in $\mathbb{R}^3$ to a semi-sphere inside a cylinder. Set $F_1(x, y, z) = x^2 + y^2 - 1$ and $F_2(x, y, z) = x^2 + y^2 + z^2 - 1$. We get that

$$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial x} \\ \frac{\partial z}{\partial x} \end{bmatrix} = \begin{bmatrix} -{}^{\partial F_1}\!/_{\partial x} \\ -{}^{\partial F_2}\!/_{\partial x} \end{bmatrix} \implies \begin{bmatrix} 2y & 0 \\ 2y & 2z \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -2x \\ -2x \end{bmatrix}$$

Now we use Cramer's Rule stating that when solving $A\vec{x} = \vec{b}$ we have $x_i = \det(A_i)/\det(A)$ where A_i is A with i$^{\text{th}}$ column replaced by $\vec{b}$.

$$\frac{dz}{dx} = \frac{\det\begin{vmatrix} 2y & -2x \\ 2y & -2x \end{vmatrix}}{\det\begin{vmatrix} 2y & 0 \\ 2y & 2z \end{vmatrix}} = 0, \implies z(x) = 0 \quad \frac{dy}{dx} = \frac{\det\begin{vmatrix} -2x & 0 \\ -2x & 2z \end{vmatrix}}{\det\begin{vmatrix} 2y & 0 \\ 2y & 2z \end{vmatrix}} = \frac{-4xz}{4yz} = \frac{-x}{y}$$

We conclude that $dy/dx = -x/y$ implying that provided $y \neq 0$, we have $y(x) = \pm\sqrt{1 - x^2}$.

6.2 The Inverse Function Theorem

Here we try to solve equations of the form

$$\begin{aligned} y_1(\vec{x}) &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ y_n(\vec{x}) &= f_n(x_1, \ldots, x_n) \end{aligned}$$

For $x_1, \ldots, x_n$ as functions of $y_1, \ldots y_n$ in a neighborhood of $\vec{x_0}$. Setting ,

$$\begin{aligned} F_1(\vec{x}, y_1) &= f_1(\vec{x}) - y_1 = 0 \\ &\vdots \\ F_n(\vec{x}, y_n) &= f_n(\vec{x}) - y_n = 0 \end{aligned}$$

We thing o y_i's as x_i's from before. We want $(x_1(\vec{y}), x_2(\vec{y}), \ldots, x_n(\vec{y}))$, such that

$$\underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_{1n}} \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}\Bigg|_{(x_0, y_0)}}_{\text{We get the same matrix regardless of the partials chosen}} \begin{bmatrix} \frac{\partial x_i}{\partial y_j} \\ \vdots \\ \frac{\partial x_n}{\partial y_j} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_i}{\partial y_j} \\ \vdots \\ -\frac{\partial F_n}{\partial y_j} \end{bmatrix}\Bigg|_{(x_0, y_0)}$$

Therefore the system of equations is locally invertible around $\vec{x_0}$ provided

$$\underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{JF(\vec{x_0})} \neq 0.$$

- The notation for this matrix is $\frac{\partial(f_1, \ldots f_n)}{\partial(x_1, \ldots, x_n)}\Big|_{\vec{x}=\vec{x_0}} = \det(Jf(\vec{x_0}))$
- This is also commonly called the *Jacobian determinant of* f.

We conclude that provided $J(f(\vec{x_0})) \neq 0 \implies \exists x_1(\vec{y}), \ldots, x_n(\vec{y})$ with $x_i(\vec{y}) = y_i(\vec{i})$ for $i = 1, \ldots, n$.

Example 6.5. Let $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$. Show that we can find local inverse where $r \neq 0$.

$$\begin{aligned} f_1(r,\theta) - x &= 0 \\ f_2(r,\theta) - y &= 0 \end{aligned} \implies Jf = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\begin{aligned} &\therefore \det(Jf) = r\cos^2\theta + r\sin^2\theta = r \neq 0 \;\text{ if }\; r \neq 0. \\ &\therefore \text{We can find } \; r(x,y) \; \text{ and } \; \theta(x,y) \; \text{ locally.} \end{aligned}$$

Example 6.6. Consider $u(x,y) = \dfrac{x^4 + y^4}{x}$, $v(x,y) = \sin(x) + \cos(y)$. Determine which points we can find $x(u,v)$ and $y(u,v)$ locally.

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{3x^4 - y^4}{x} & \frac{4y^3}{x} \\ \cos(x) & -\sin y \end{bmatrix}$$

So $\det(Jf) = -\left(\dfrac{3x^4 - y^4}{x^2}\right)\sin(y) - \frac{4y^3}{x}\cos(x)$.
Therefore we can find local inverses $x(u,v)$ and $y(u,v)$ when $\det(Jf) \neq 0$.

Note. *Finding a global inverse is not easy. For example let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *whose components are all polynomials and suppose* $\det(Jf) = 1$ *everywhere. Is* f *globally invertible ? This is actually an open problem !*

7 Double Integrals and Triple Integrals

To begin we consider functions $f : \mathbb{R}^2 \to \mathbb{R}$ where the domain of integration is a rectangle $R = [a, b] \times [c, d]$, this means $x \in [a, b]$ and $y \in [c, d]$.

Taking $f(x, y) \geq 0$ on R , the volume enclosed by the graph of f and the four planes $x = a, x = b, y = c, y = d$ is called the double integral of f over R and is denoted

$$\int_R f, \quad \int_R f(x,y)\, dA, \quad \int_R f(x,y)\, dxdy, \quad \iint_R f(x,y)\, dxdy, \tag{7.0.1}$$

all these expressions are equivalent.

Definition 7.1. (Double Integral) *Just like the definition from Calc.2, we portioned $[a, b]$ into n sub intervals with $n + 1$ points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and for any $c_i \in [x_i, x_{i+1}]$*

$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) \approx \int_a^b f\, dx.$$

We compute volumes in a very similar way. To start, we use Cavalier's principle. Suppose a solid body has cross sectional area at x given by $A(x)$ as portrayed in Figure 8b.

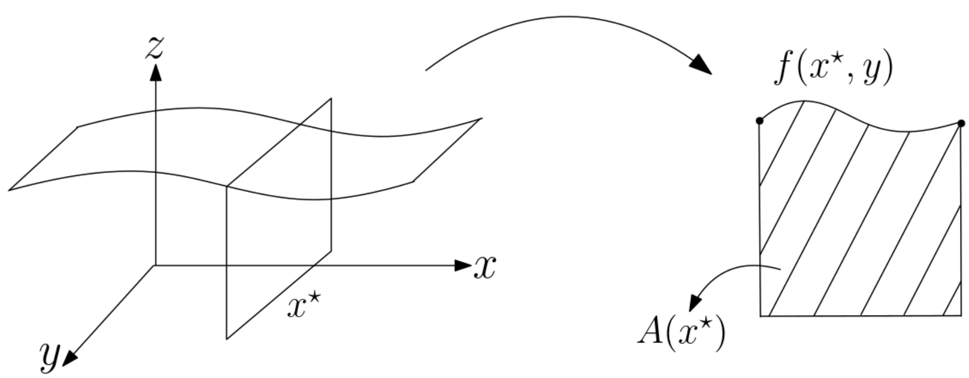


Figure 8: a) Three dimensional interpretation of a cross-sectional intersection with a body. **b)** Cross-sectional area at an arbitrary $x^\star$.

His principle states that the volume of the body is $\int_a^b A(x)\, dx$. To see this we portion $[a, b]$ into $a = x_0 < x_1 < \cdots < x_n = b$, which in return implies

$$\int_a^b A(x)\, dx \approxeq \sum_{i=0}^{n-1} \underbrace{A(c_i)(x_{i+1} - x_i)}_{\text{olume of each slab (Figure)}}.$$

$A(x_i)$ gives the cross-sectional area at some $c_i \in [x_i, x_{i+1}]$. Therefore adding each slab between x_i

and x_{i+1} yields the approximate volume. Moreover, using (7.0.1) ,

$$\text{If volume } = \int_a^b A(x)\ dx = \int_a^b \int_c^d f(x,y)\ dydx \implies A(x) = \int_c^d f(x,y)\ dy.$$

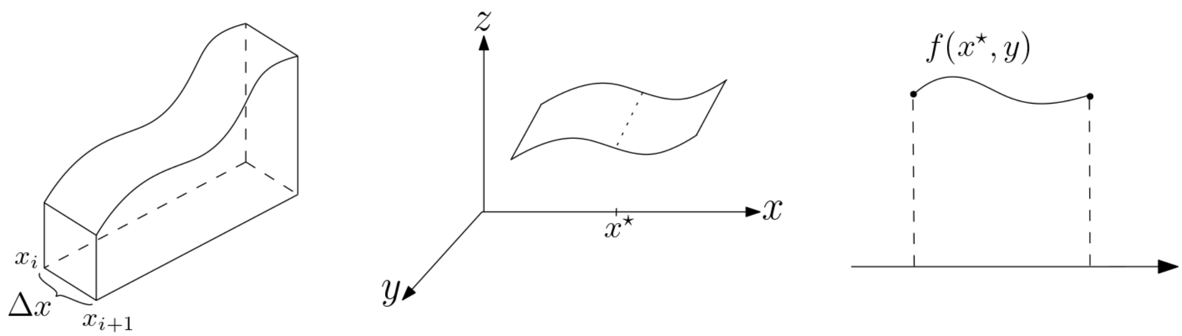


Figure 9: **a)**A slab of Reinmann sum partition. **b)** Setting an arbitrary $x^\star$ on a given plane **c)** The curve of the sliced function at $x^\star$.

But does this make sense ? Fix a $x^\star$, as shown in Figure ,then slicing at $x^\star$ we get the curve as illustrated in Figure. Clearly, $\int_c^d f(x^\star, y)\ dy = A(x^\star)$. Similarly, we can think of the volume as

$$\int_c^d A(y)\ dy \quad \text{,where } A(y) = \int_a^b f(x,y)\ dx.$$

To summarize, we have the following definition

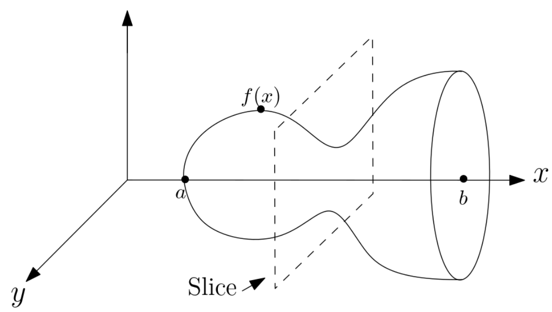
Definition 7.2. (Cavalier's Principle) *Suppose a body has cross sectional area at x given by $A(x)$, then the volume of the body is $\int_a^b A(x)\ dx$. Moreover ,*

$$\textit{If volume } = \int_a^b A(x)\ dx = \int_a^b \int_c^d f(x,y)\ dydx \implies A(x) = \int_c^d f(x,y)\ dy. \tag{7.0.2}$$

Similarly, we can think of the volume as

$$\int_c^d A(y)\ dy, \qquad \textit{where } A(y) = \int_a^b f(x,y)\ dx.$$

Example 7.1. Use Cavalier's principle to compute the volume of a function $f(x) \geq 0$ defined for $x \in [a, b]$ rotated about the x-axis.



If we look at this through yz we in fact see a circle, therefore using (7.0.2)

$$A(x) = \pi(f(x))^2 \implies V = \int_a^b \pi(f(x))^2 \, dx. \tag{7.0.3}$$

Definition 7.3. *If the sequence S_n converges as $n \to \infty$ and the limit is the same regardless of the c_{jk} chosen then f is called integrable over R, with*

$$\lim_{n\to\infty} \sum_{j,k=0}^{n-1} f(c_{jk})\Delta A = \int_R f.$$

Theorem 7.1. *Let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be bounded on R and suppose the set of points where f is discontinuous comprises a finite union of graphs of continuous functions. In this situation , f is integrable on R.*

7.1 Properties of Integrable Functions

(i) **Homogeneity** $\int_R cf(x,y)\, dA = c \int_R f(x,y)\, dA,\ c \in \mathbb{R}.$

(ii) **Monotonicity** If $f(x,y) \geq g(x,y)$, then $\int_R f(x,y)\, dA \geq \int_R g(x,y)\, dA.$

(iii) **Additivity** If R_i for $i = 1, \ldots, n$ are disjoint rectangles, then f is integrable for each R_i. Then if $Q = \bigcup_{i=1} R_i = R_1 \cup R_2 \cup \cdots \cup R_n$, then

$$\sum_{i=1}^{m} \int_{R_1} f\, dA = \int_Q f\, dA, \tag{7.1.1}$$

i.e., integrating rectangles individually on all is the same.

Theorem 7.2. (Fubini's Theorem) *Let f be a continuous function with domain $R = [a, b] \times [c, d]$, then*

$$\int_a^b \int_c^d f(x, y)\, dxdy = \int_c^d \int_a^b f(x, y)\, dydx. \tag{7.1.2}$$

Proof. Partition $[c, d]$ $\quad c = y_0 < y_1 < \cdots < y_n = d.$

$$\text{Let } F(x) = \int_c^d f(x, y)\, dy. \implies F(x) = \sum_{k=0}^{n-1} \int_{y_k}^{y_{k+1}} f(x, y)\, dy \quad \text{(by additivity)}$$

Recall: The mean value theorem for integration,

$$\int_a^b h(t)g(t)\, dt = h(c) \int_a^b g(t)\, dt \quad , c \in [a, b].$$

$$\begin{aligned} \implies \int_{y_k}^{y_{k+1}} f(x, y)\, dy &= f(x, y_k(x)) \int_{y_k}^{y_{k+1}} dy \\ &= f(x, y_k(x))(y_{k+1} - y_k) \quad , y_{k(x)} \in [x_k, x_{k+1}] \end{aligned}$$

$$\therefore f(x) = \sum_{k=0}^{n-1} f(x, y_k(x))(y_{k+1} - y_k)$$

$$\text{use } \therefore \int_a^b F(x)\, dx = \lim_{n\to\infty} \sum_{j=0}^{n-1} F(p_j)(x_{j+1} - x_j),$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $p_j \in [x_k, x_{k+1}]$.

$$\text{set } c_{jk} = (p_j, y_k(p_j)) \implies (p_j) = \sum_{k=0}^{n-1} f(c_{jk})(y_{k+1} - y_k)$$

$$\begin{aligned} \therefore \int_a^b \int_c^d f(x, y)\, dydx = \int_a^b F(x)\, dx &= \lim_{n\to\infty} \sum_{j=0}^{n-1} F(p_j)(x_{k+1} - x_k) \\ &= \lim_{n\to\infty} \underbrace{\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(c_{jk})(y_{k+1} - y_k)(x_{k+1} - x_k)}_{\text{Rienmann sum for } \int_R f\, dA.} \\ &= \int_R f(x, y)\, dA. \end{aligned}$$

$\square$

Example 7.2. Use Fubini's theorem (7.2) to show that

$$\int_0^1 \int_0^2 x^2 + y \, dydx = \int_0^2 \int_0^1 x^2 + y \, dxdy.$$

$$\int_0^1 x^2 y + \frac{y^2}{2}\bigg|_0^2 \, dx = \int_0^1 2x^2 + 2x \, dx = \frac{2x^3}{3} + 2x\bigg|_0^1 = \frac{8}{3}.$$
$$\int_0^2 \int_0^1 x^2 + y \, dxdy = \int_0^2 \frac{y}{3} + \frac{y^2}{2}\bigg|_0^2 = \frac{8}{3}.$$

7.2 Integration Over More General Domain

Here we consider domain of integration where $x \in [a, b], \phi_1(x) \leq y \leq \phi_2(x)$. Or similarly, $y \in [c, d], \phi_1(y) \leq x \leq \phi_2(y)$. Suppose a region $D \subseteq \mathbb{R}$ defined by the previous domain of integration, can we say that

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dydx = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) \, dxdy?$$

To do this we use the more general form of Fubini's theorem.

Theorem 7.3. (Fubini's Theorem Generalized) *Let f be a bounded function with domain $R = [a, b] \times [c, d]$ and suppose the discontinuity of f from a finite union of graphs of continuous functions.*

$$\text{Then if } \int_c^d f(x, y) \, dy \; \exists \, \forall \, x \in [a, b] \text{ and } \int_a^b f(x, y) \, dx \; \exists \, \forall \, y \in [c, d],$$
$$\text{then } \int_a^b \int_c^d f(x, y) \, dydx = \int_c^d \int_a^b f(x, y) \, dxdy. \tag{7.2.1}$$

7.3 Triple Integrals

Given a continuous function $f : B \longrightarrow \mathbb{R}$ where B is a rectangular parallelepiped in $\mathbb{R}^3$. We define the integral of f over B as

$$\lim_{n\to\infty} S_n \qquad \text{,where } S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(c_{ijk}) \Delta V,$$

for which c_{ijk} are some points in B_{ijk}.

Definition 7.4. *Let f be a bounded function of 3 variables defined on B. If $\lim_{n\to\infty} S_n$ exists and is the same for any choice of c_{ijk}, then f is integrable with*

$$\lim_{n\to\infty} S_n = \int_B f\ dV = \iiint_B f = \iiint f(x,y,z)\ dxdydz. \tag{7.3.1}$$

Moreover, from Fubini's theorem (7.3) *since f is continuous and $B = [a,b] \times [c,d] \times [u,v]$ then,*

$$\int_B f\ dV = \int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_d^c \int_a^b \int_u^v f\ dzdxdy = \ldots \text{ all combinations.} \tag{7.3.2}$$

We can discuss integrability and switching order of integration on the domain $\omega := x \in [a,b], \phi_1(x) \leq y \leq \phi_2(x), \phi_1(x,y) \leq z \leq \phi_2(x,y)$, by placing ω in a larger parallelepiped B and using same ideas as in $\mathbb{R}^2$. The volume of ω is given by $\iiint_\omega 1\ dV = \text{Volume}(\omega)$.

Using the ω described above we have

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} 1\ dV = \underbrace{\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \phi_2(x,y) - \phi_1(x,y)\ dydx}_{\text{Double integral, i.e., volume below } \phi_2 \text{ and above } \phi_1}.$$

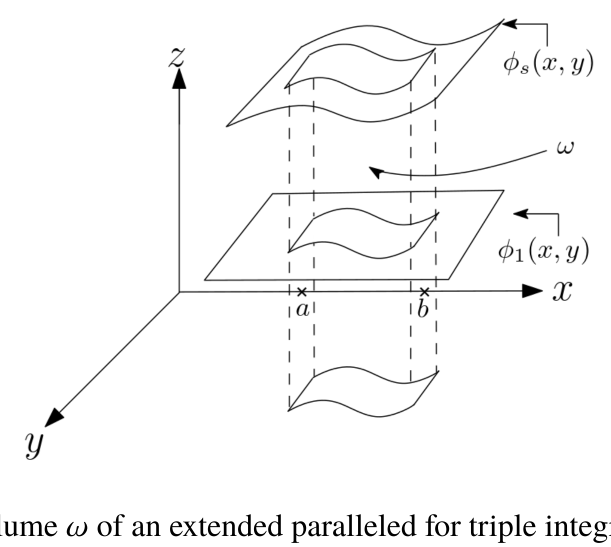


Figure 10: Volume ω of an extended paralleled for triple integral integration.

Example 7.3. Compute the volume of the unit sphere.
Sketch the graphs and deduce that we must set $\phi_2(x,y) = -\sqrt{1-x^2-y^2}$, $\phi_1(x,y) = \sqrt{1-x^2-y^2}$

, $\phi_1(x) = -\sqrt{1-x^2}$ and finally $\phi_2(x) = \sqrt{1-x^2}$. So we have

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dzdydx = 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dydx.$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{(1-x^2)-y^2}\, dy, \quad \text{let } u = \sqrt{1-x^2}, \quad \int_{-a}^{a} \sqrt{a^2-y^2}\, dy$$

let $y = \cos\theta$ and $dy/d\theta = -\sin\theta$, so then

$$\int \sqrt{a^2-x^2}\, dy = -a\int \sin\theta\sqrt{a^2-x^2\cos^2\theta}\, \underset{\cdot}{\theta} = -a^2\int \sin^2\theta$$

$$= -a^2\int \frac{1-\cos(2\theta)}{2}\, d\theta = \frac{-a^2\theta}{2} + \frac{\sin(2\theta)a^2}{4} + C.$$

Using $\sin(2\theta) = 2\sin\theta\cos\theta$ we get

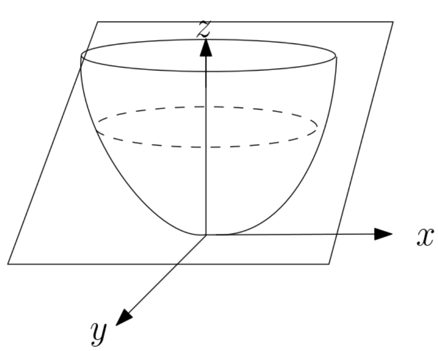
$$= -a^2\left(\frac{\arccos(y/a)}{2}\right) + \frac{a}{2}y\sin\theta + C$$

$\sin\theta = \frac{\sqrt{a^2-y^2}}{2}$ so we may write the result

$$= \frac{-a^2}{2}\arccos(1) + \frac{a^2}{2}\arccos(-1) = \frac{a^2}{2}\pi$$

$$\therefore \text{ the volume is : } \int_{-1}^{1}\frac{(1-x^2)\pi}{2}\, dx = \frac{4\pi}{3}.$$

Example 7.4. Find an integral for the volume of the region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 3 - 2y$



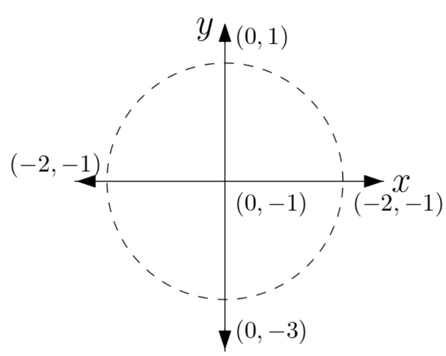


Figure 11: Figure associated with the current example. **a)** The given plane intersect the paraboloid. **b)** An elevated view of the intersection along with coordinates are provided.

From the figure we have to deduce that z is bounded as $x^2 + y^2 \leq z \leq 3 - 2y$. Now we need either $x \in [a, b], \phi_1(x) \leq y \leq \phi_2(x)$ or $y \in [c, d], \phi_1(y) \leq x \leq \phi_2(y)$. The surfaces intersect along the curve given by $x^2 + y^2 = 3 - 2y \implies x^2 + y^2 + 2y = 3$. This is in fact a circle of radius 2 shifted down since $x^2 + y^2 + 2y + 1 - 1 = 3 \equiv x^2 + (y + 1)^2 = 4$!
And so we have $-\sqrt{4 - (y + 1)^2} \leq x \leq \sqrt{4 - (y + 1)^2}, y \in [-3, 1]$.

$$\therefore \text{ Volume is given by } \int_{-3}^{1} \int_{-\sqrt{4-(y+1)^2}}^{\sqrt{4-(y+1)^2}} \int_{x^2+y^2}^{3-2y} dzdxdy.$$

7.4 Change of Variables Coordinates

Theorem 7.4. *Let A be a 2×2 matrix and suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\vec{x}) = A\vec{x}$, then T maps parallelograms to parallelograms.*

Example 7.5. Consider $T(x, y) = ((x + y)/2, (x - y)/2)$

$$T(x, y) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

T maps paral. to paral.

How do we apply this to integration ? Consider $\iint\limits_D f(x, y)\, dA$ and let $T : (u, v) \to (x(u, v), y(u, v))$.

- The point in the xy-plane can be given by points in the uv-plane through T.
- $T(u, v) = (x(u, v), y(u, v))$.

Near a point (u_0, v_0) we have,

$$T(u, v) \approx T(u_0, v_0) + DT(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix},$$

where $DT(u_0, v_0)$ is the Jacobian. Near (u_0, v_0),

$$T(u, v) \approx T(u_0, v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \Bigg|_{(u_0, v_0)} \begin{bmatrix} (u - u_0) \\ (v - v_0) \end{bmatrix}. \tag{7.4.1}$$

Note that this is a linear transformation with $T = A\vec{x} \implies T$ sends parallelograms to parallelograms. Through some algebraic manipulations that we will omit in this manuscript, we obtain the following important result

$$\iint\limits_D f(x, y)\, dxdy = \iint\limits_S f(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv, \tag{7.4.2}$$

where S is the domain D in the uv coordinate system, also called *the description of D*.

Example 7.6.

$$\text{Solve } \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy.$$

Set $x = r\cos\theta$ and $y = r\sin\theta$. Recalling (7.4.2)

$$\implies \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}\right| = r$$

$$\implies = \iint_S r\, dr d\theta \text{ where } S \text{ is the unit circle in polar coordinates}$$

$$= \int_0^{2\pi}\int_0^1 r\, dr d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^1 d\theta = \pi.$$

Example 7.7.

$$\text{Compute } \iint_R e^{\frac{x+y}{x-y}}\, dA,$$

where R is the trapezoid with vertices $(1,0), (2,0), (0,-1), (0,-2)$. Try $u = x + y$ with $v = x - y$. $x = u - y \implies v = (u - y) - y \implies 2y = u - v \implies y = \frac{u-v}{2}$ and $x = \frac{u+v}{2}$

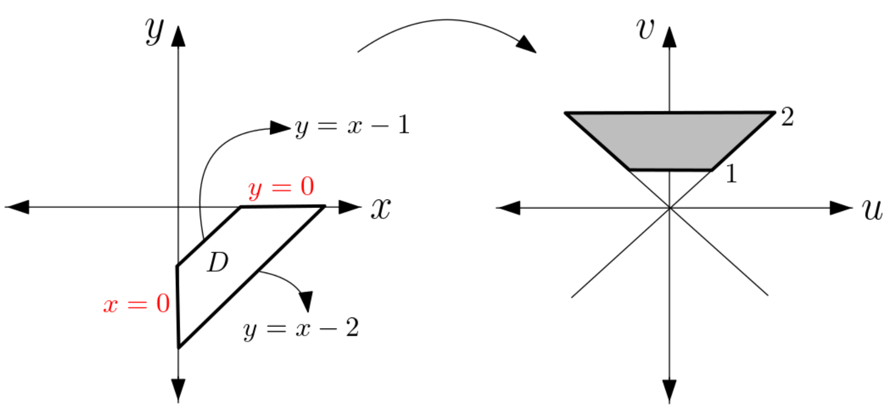


Figure 12: Figure for the current example.**a)** The original trapezoid. **b)** The transformed trapezoid.

From a) and b), we have that $y = 0 \implies u = v$ and $x = 0 \implies u = -v$. From $u = x + y$ and $v = x - y$ we get $y = x - 1 \implies v = 1$ and $y = x - 2 \implies v = 2$.
Now how do we integrate over b)?

$$\int_1^2\int_{-v}^{v} e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \to \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{vmatrix}1/2 & 1/2 \\ 1/2 & -1/2\end{vmatrix}\right| = \frac{1}{2}$$

$$\implies \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\, dudv = \frac{1}{2}\int_1^2 ve^{u/v}\Big|_{-v}^{v}\, dv = \frac{1}{2}\int_1^2 v\left(e^{v/v} - e^{-v/v}\right)\, dv$$
$$= \frac{1}{2}\left(e^1 - e^{-1}\right)\int_1^2 v\, dv = \frac{1}{2}(e^1 - e^{-1})(2 - \frac{1}{2}) = \frac{3}{4}(e - e^{-1}).$$

Last time we introduced change of coordinates for double integrals

$$\iint_D f(x,y)\, dxdy = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where S is the description of D in the uv-plane.
The same thing holds for triple integrals, when integrating functions $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose $f(x,y,z) = f(x(u,v,w), y(u,v,w), z(u,v,w))$ (change of coordinates), then we have

$$T(u,v,w) \approx T(u_0,v_0,w_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}\Bigg|_{(u_0,v_0,w_0)} \begin{bmatrix} u - u_0 \\ v - v_0 \\ w - w_0 \end{bmatrix}. \tag{7.4.3}$$

Using the same argument as before we get,

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_S f(y,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \tag{7.4.4}$$

7.5 Cylindrical Coordinates

The following transformations are done in the cylindrical coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \implies \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Example 7.8. Find the volume enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 3 - 2y$, same example as seen before.

$$\int_1^3 \int_{-\sqrt{3-2y-y^2}}^{\sqrt{3-2y-y^2}} \int_{x^2+y^2}^{3-2y} dzdxdy,$$

Let us translate this into cylindrical coordinates. Recall that we said the intersection circle is a shifted by -1 , with radius 2 and is overall described by $x^2 + (y+1)^2 = 4$. Now let us set $x = r\cos\theta$, $y = -1 + r\sin\theta$ and $z = z$. This is a *shifted cylinder*.

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2\theta + (r\sin\theta - 1)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta + 2r\sin\theta + 1 = r^2 - 2r\sin\theta + 1. \end{aligned}$$

Also, $3-2y = 3-2(r\sin\theta - 1) = 5 - 2r\sin\theta$

Having transformed the given structures , we may solve the triple integral

$$= \int_0^{2\pi}\int_0^2\int_{r^2-2r\sin\theta+1}^{5-2r\sin\theta} r\; dzdrd\theta = \int_0^{2\pi}\int_0^2 ((t-2r\sin\theta)-(r^2-2r\sin\theta+1))r\; drd\theta$$
$$= \int_0^{2\pi}\int_0^2 (4-r^2)r\; drd\theta = (\ldots) = 8\pi.$$

Example 7.9. Find the volume of the region below $z = x+2$ and between $x^2+y^2 = 1$ and $x^2+y^2 = 4$. Sketching these constraints we get a region defined between a circle of radius 1 a circle of radius 2. We may immediately solve the problem ,

$$\int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+z} r\; dzdrd\theta = \int_0^{2\pi}\int_1^2 r^2\cos\theta+2r\; drd\theta = \int_0^{2\pi} \frac{r^3}{3}\cos\theta + r^3\Big|_1^2 d\theta = (\ldots) = 6\pi.$$

7.6 Spherical Coordinates

For spherical coordinates we have the following transformations

- $x = \rho\sin\varphi\cos\theta$
- $y = \rho\sin\varphi\sin\theta$
- $z = \rho\cos\varphi$.

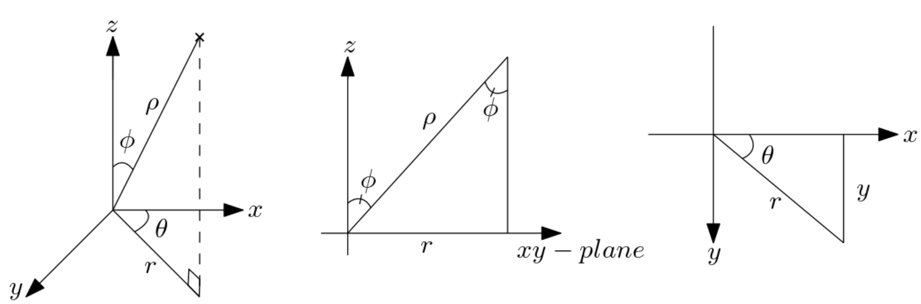


Figure 13: Three perspectives of the spherical coordinates. Important notes include that ρ is the distance from the origin, θ is the angle between the positive x-axis and the vector $(x, y, 0)$, $(0 \leq \theta \leq 2\pi)$. Finally, ϕ is the angle between the positive z-axis and the vector (x, y, z), $(0 \leq \phi \leq \pi)$.

Let us determine the corresponding Jacobian to spherical coordinates transformation.

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$\begin{aligned} &= \rho^2\cos\varphi(-\sin\varphi\cos\phi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta) - \rho^2\sin\varphi(\sin^2\varphi\cos^2\theta + \sin\varphi\sin^2\theta) \\ &= -\rho^2\cos^2\varphi\sin\varphi - \rho^2\sin^3\varphi = \boxed{-\rho^2\sin\varphi}. \end{aligned}$$

Example 7.10. Compute the volume of the sphere of radius r in spherical coordinates. The squelet of a spherical integral is

$$\int_\varphi\int_\theta\int_\rho \rho^2\sin\varphi\ d\varphi d\theta d\rho.$$

So we have

$$\begin{aligned} \int_0^\pi\int_0^{2\pi}\int_0^r \rho^2\sin\varphi\ d\varphi d\theta d\rho &= \int_0^\pi\int_0^{2\pi} \frac{\rho^3}{3}\sin\varphi\Big|_0^r\ d\theta d\varphi \\ &= \int_0^\pi\int_0^{2\pi}\frac{r^2}{3}\sin\varphi\ d\theta d\varphi \\ &= \int_0^\pi\frac{2\pi r^3}{3}\sin\varphi\ d\varphi = (\dots) = \frac{4\pi r^3}{3}. \end{aligned}$$

Example 7.11. Find the volume of the region above the cone $z = \sqrt{x^2+y^2}$ and below the unit sphere. Sketching these constraints gives us an upside down cone , with a semi-sphere sitting on top of it. Which coordinates should we use ?

Note. *For problems involving spheres and especially cones, spherical coordinates are appropriate.*

Let us find (φ,θ,ρ). We're looking for the intersection of the cone and the unit sphere which implies that $\rho = 1$. Express the given functions as $z^2 = x^2+y^2$ (cone) and $z^2 = 1-x^2-y^2$ ((Sphere)).

$$\text{Equate these : } \to x^2+y^2 = 1-x^2-y^2 \implies z(x^2+y^2) = 1.$$

Now we go to the spherical coordinates,

$$\left.\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}\right\} \implies \begin{aligned} x^2+y^2 &= \rho^2\sin\varphi\cos^2\theta + \rho^2\sin\varphi\sin^2\theta \\ &= \rho^2\sin^2\varphi(\cos^2\theta = \sin^2\theta) \\ &= \rho^2\sin^2\theta. \end{aligned}$$

$$\begin{aligned} &\implies z\rho^2\sin^2\varphi = 1, \quad \text{with} \quad \rho^2 = 1 \\ &\implies z\sin^2\varphi = 1 \implies \sin^2\varphi = \frac{1}{2} \implies \sin\varphi = \frac{1}{\sqrt{2}} \quad \therefore \varphi = \frac{\pi}{4}. \end{aligned}$$

So we have $\varphi \in [0, \frac{\pi}{4}]$, $\theta \in [0, 2\pi]$, $\rho \in [0, 1]$

$$\therefore \text{ Volume is } \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi \; d\rho d\theta d\varphi = (...).$$

Calculations for the above integral are omitted since they're trivial.
Note that the problem here is done, since we're only asking to find the volume - the problem could be more complicated if we were asked to integrate over or under a function.

The density of a region $R \subseteq \mathbb{R}^3$ is given by $\rho(x, y, z)$, then the mass of R is given by

$$m(R) = \iiint\limits_R \rho(x, y, z) \; dV, \tag{7.6.1}$$

with the center of mass defined as $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{\iiint\limits_R x\rho(x, y, z) \; dV}{m(R)}, \quad \bar{y} = \frac{\iiint\limits_R y\rho(x, y, z) \; dV}{m(R)}, \quad \bar{z} = (\ldots).$$

Example 7.12. Find the center of mass for the previous example (Cone/sphere) for $\rho = 1$(density).
We know by rotational symmetry that $\bar{x} = \bar{y} = 0$, we therefore just need

$$\bar{z} = \frac{\iiint\limits_R z \; dV}{m(R)} = \frac{\int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^3 \cos\varphi \sin\varphi \; d\rho d\theta d\varphi}{m(R)} = (...) = \frac{\frac{\pi}{2} \int_0^{\pi/4} \cos\varphi \sin\varphi \; d\varphi}{m(R)}.$$

$$\cos\varphi \sin\varphi = \frac{\sin(2\varphi)}{2} \implies \bar{z} = \frac{\frac{\pi}{4} \int_0^{\pi/4} \sin(2\varphi) \; d\varphi}{m(R)} = \frac{\frac{\pi}{8}\left[-\cos(2(\pi/4)) + \cos(0)\right]}{\frac{2\pi}{3}\left[1 - \frac{1}{1\sqrt{2}}\right]} = \frac{3}{16(1 - 1/\sqrt{2})}.$$

$$\therefore \text{Center of mass is } \left(0, 0, \frac{3}{16\left(1 - \frac{1}{\sqrt{2}}\right)}\right).$$

8 Line Integrals and Path Integrals

Definition 8.1. (Path Integral) *The path integral , or integral of $f(x, y, z)$ along the curve $\vec{\sigma}$, is defined when $\vec{\sigma} : [a, b] \to \mathbb{R}^3$ is continuous and differentiable on $[a, b]$ and when $f(\vec{\sigma}(t)) : t \to f(x(t), y(t), z(t))$ is continuous on $[a, b]$. The integral is defined as*

$$\int_{\vec{\sigma}} f\, ds = \int_a^b f(x(t), y(t), z(t))\|\vec{\sigma}'(t)\|\, dt = \int_a^b f(\vec{\sigma}(t))\|\vec{\sigma}(t)\|\, dt. \tag{8.0.1}$$

Note. *If $f := 1$, then this gives the arclength of $\vec{\sigma}$ between a and b.*

Example 8.1. Evaluate $\int_{\vec{\sigma}}$, where σ consists of σ_1 which is the arc of a parabola "$(y = x^2)$" from $(0, 0)$ to $(1, 1)$. And, $\sqrt{2}$ is a function $(1, 1) \to (1, 2)$.

$$\implies \int_\sigma 2x\, dx = \int_{\sigma_1} 2x\, dx + \int_{\sigma_2} 2x\, dx.$$

Let us parameterize the functions and express what we're looking for . We want ,

$$\begin{aligned} &\sigma_1 : t \to \mathbb{R}^2 \\ &\vec{\sigma_1}(t) : \langle t, t^2\rangle \quad t \in [0, 1] \\ &\vec{\sigma_2}(t) : \langle 1, t\rangle \quad t \in [1, 2] \end{aligned}$$

Let us find the first integral and second,

$$\int_{\vec{\sigma_1}} 2x\, dx = \int_a^b f(x(t), y(t))\left\|\vec{\sigma_1}'(t)\right\| dt \quad \begin{cases} x(t) = t \quad , y(t) = t^2 \\ \left\|\vec{\sigma_1}'(t)\right\| = \sqrt{(x'(t))^2 + (y'(t))^2} \end{cases}$$

$$\therefore \int_{\vec{\sigma_1}} 2x\, dx = \int_0^1 2t\sqrt{1 + 4t^2}\, dt \overset{u=1+4t^2}{=} \frac{1}{4}\int_1^5 \sqrt{u}\, du = (\ldots) = \left(\frac{5\sqrt{5} - 1}{6}\right).$$

For $\displaystyle\int_{\vec{\sigma_2}} 2x\, dx = \int_1^2 2\sqrt{1}\, dt = 2.$ $\quad\therefore$ Path integral is $\boxed{\dfrac{5\sqrt{5} - 1}{6} + 2}$.

Definition 8.2. (Vector Field) *A vector field F on $\mathbb{R}^n$ is a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\vec{x} \in U$ a vector $\vec{F}(\vec{x})$.*
For example, oxygen in a room , or any other quantity that is subjected to some forces inside a dimensional space.

Definition 8.3. (Flow line) *If $\vec{F}$ is a vector field, a flow line for $\vec{F}$ is a path $\vec{\sigma}(t)$ such that $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$.*
This can be extended to

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\sigma(\vec{x},t) &= F(\vec{\sigma}(\vec{x}(t))) \\ \sigma(\vec{x},0) &= \vec{x} \end{aligned}\right\} \quad \textit{The flowline with initial condition at } \vec{x} \textit{ when } t = 0.$$

8.1 Divergence and Curl

Let $\vec{F}$ be continuously differentiable with

$$\vec{F}(\vec{x}) = \langle F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x})\rangle = F_1(\vec{x})\vec{i} + F_2)(\vec{x})\vec{j} + F_3(\vec{x})\vec{k}. \tag{8.1.1}$$

$$\text{The Curl of } F = \nabla \times \vec{F} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{8.1.2}$$

$$\implies \text{Curl}(\vec{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}. \tag{8.1.3}$$

The curl describes the rotation of a vector field. The curl of $\vec{F}$ of $\vec{x}$ yields a vector pointing along the axis of rotation where the length $\|\nabla \times F\|$ describes the magnitude of the rotation.

Theorem 8.1. *If f is twice continuously differentiable (C^2) then $\nabla \times (\nabla f) = 0$, (where f is scalar-valued)*

Definition 8.4. (Conservative) *A vector field is called conservative if $F = \nabla f$, for some real-valued function f, implying that the curl of any conservative vector field is* 0.

Example 8.2. Determine if $f(x,y,z) = y^2z^3\vec{i} + 3xyz^3\vec{j} + 3x^2z^2\vec{k}$ is conservative. Using (8.1.3),

$$\begin{aligned} \text{curl}(F) &= (f_{3y} - f_{2z})\vec{i} + (f_{1z} - f_{3x})\vec{j} + (f_{2x} - f_{1y})\vec{k} \\ &= (6xyz^2 - 6xyz^2)\vec{i} + (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2yz^2)\vec{k} \\ &= \vec{0} \quad \therefore F \text{ is conservative (by Definition 8.4)} \end{aligned}$$

Definition 8.5. (Divergence) *The divergence of a continuously differentiable vector field F is*

$$\nabla \cdot \vec{F} = \nabla \cdot \langle F_1, F_2, F_3\rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{8.1.4}$$

Example 8.3. Compute the $\mathrm{Div}(\nabla f)$, where $f : \mathbb{R}^3 \to \mathbb{R}$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$= \frac{\partial f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f = \Delta f$$

Note. $\nabla^2 = \Delta$ *is called the Laplace operator. For example, the heat equation* $\partial/\partial t(u(x,t) = \Delta u)$.

What does div $\vec{F}$ represent? Recall that the flow line $\vec{\sigma}$ starting at point $\vec{x}$ flowing under a vector field $\vec{F}$ is defined by

$$\begin{cases} \frac{\partial}{\partial t}\vec{\sigma}(\vec{x},t) = F(\vec{\sigma}(\vec{x},t)) & (\star) \\ \vec{\sigma}(\vec{x},0) = \vec{x} = (x_1,x_2,x_3) \end{cases}$$

where $\sigma : \mathbb{R}^{3+1} \to \mathbb{R}$
and $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $F(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x}))$.

Let us differentiate $(\star)$ in $\vec{x}$, we get

$$\underbrace{D\vec{x}\frac{\partial}{\partial t}\vec{\sigma}(\vec{x},t)}_{\text{Matrix}} = \underbrace{D\vec{x}F(\vec{\sigma}(\vec{x},t))}_{\text{Matrix}}\ \underbrace{D\vec{x}\vec{\sigma}(\vec{x},t)}_{\text{Matrix}} \quad \text{(by chain rule).}$$

Now consider the 3 vectors $\vec{v_1} = \epsilon\vec{i},\ \vec{v_2} = \epsilon\vec{j},\ \vec{v_3} = \epsilon\vec{k}$ for some $\epsilon > 0$.

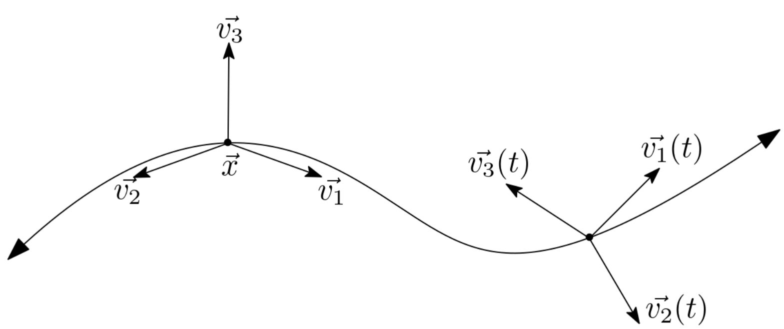


Figure 14: The direction of sub-vectors change over time.

The volume of the parallelepiped $\vec{v_1}, \vec{v_2}, \vec{v_3}$ is

$$\vec{v_1} \cdot (\vec{v_2} \times \vec{v_3}) = (\epsilon, 0, 0)\det\begin{vmatrix} i & j & k \\ 0 & \epsilon & 0 \\ \epsilon & 0 & \epsilon \end{vmatrix} = \epsilon^3.$$

Note that $\vec{v_1} \cdot (\vec{v_2} \times \vec{v_3}) = \vec{v_3} \cdot (\vec{v_1} \times \vec{v_2}) = \vec{v_2} \cdot (\vec{v_3} \times \vec{v_1})$.

We want an approximation of $\vec{v}_1(t) \cdot (\vec{v}_3(t) \times \vec{v}_1(t))$, so let us compute the first order Taylor expansion of σ centered at $\vec{x}$

$$\vec{\sigma}(z,t) \approxeq \sigma(\vec{x},t) + \underbrace{D\vec{x}\sigma(\vec{x},t)}_{\text{Jacobian}}(\vec{z}-\vec{x}) \implies \vec{\sigma}(z,t) - \sigma(\vec{x},t) = D\vec{x}\sigma(x,t)(\vec{z}-\vec{x}).$$

At $t = 0$, we have $\vec{z} - \vec{x} = D\vec{x}(\vec{x})(\vec{z} - \vec{x})$,and when $\vec{z} = \vec{x} + \epsilon\vec{i} = \vec{x} + \vec{v}_1$

$$\implies \vec{v}_1 = D\vec{x}(\vec{x})\vec{v}_1 = \vec{v}_1$$

We define $\vec{v}_1(t) = \sigma(\vec{x} + \epsilon\vec{i}, t) - \sigma(\vec{x}, t)$,

$$\begin{aligned}
\implies\ & v_1(t) = D\vec{x}(\sigma(\vec{x},t))\vec{v}_1 \\
& v_2(t) = D\vec{x}(\sigma(\vec{x},t))\vec{v}_2 \\
& v_3(t) = D\vec{x}(\sigma(\vec{x},t))\vec{v}_3 \\
\implies\ & \frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \frac{\partial}{\partial t}Dx(\sigma(\vec{x},t))v_i = \left(Dx\frac{\partial}{\partial t}\sigma(\vec{x},t)\right)\vec{v}_i \\
\implies\ & \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}v_i(t)}_{\text{Vector}} = \underbrace{DxF(\sigma(\vec{x},t))}_{\text{Matrix}}\underbrace{Dx\sigma(\vec{x},t)\vec{v}_i}_{\text{Matrix}} \\
\implies\ & \left.\frac{\mathrm{d}}{\mathrm{d}t}v_i(t)\right|_{t=0} = D\vec{x}F(\vec{x})\vec{v}_i
\end{aligned}$$

The volume of the parallelepiped at time t is then

$$\begin{aligned}
v(t) &= v_i(t) \cdot (v_2(t) \times v_3(t)) \\
v'(t) &= \frac{\mathrm{d}v_1(t)}{\mathrm{d}t} + v_1(t)\left(\frac{\mathrm{d}v_2(t)}{\mathrm{d}t} \times v_3(t)\right) + v_1(t)\left(v_2(t) \times \frac{\mathrm{d}v_3(t)}{\mathrm{d}t}\right) \\
&= (DxF(\vec{x})\epsilon\vec{i})(\epsilon^2\vec{i}) + \epsilon\vec{i}(DxF(\vec{x})\epsilon\vec{j} \times \epsilon\vec{k}) + \epsilon\vec{i} \cdot (\epsilon\vec{j} \times DxF(x)\epsilon\vec{k}) \\
&= DxF(\vec{x})\epsilon\vec{i} \cdot \epsilon^2\vec{i} + DxF(\vec{x})\epsilon\vec{j} \cdot (\epsilon\vec{k} \times \epsilon\vec{i}) + DxF(\vec{x})\epsilon\vec{k}(\epsilon\vec{i} \times \epsilon\vec{j}) \\
&= DxF(\vec{x})\epsilon\vec{i} \cdot \epsilon^2\vec{i} + DxF(\vec{x})\epsilon\vec{j} \cdot (\epsilon^2\vec{j}) + DxF(\vec{x})\epsilon\vec{k} \cdot (\epsilon^2\vec{k}) \\
Dx(f(\vec{x}))\epsilon\vec{i} &= \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_1} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_1} \end{bmatrix} \underbrace{\begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix}}_{\epsilon_i} \\
&= \left(\epsilon\frac{\partial F_1}{\partial x_1}, \epsilon\frac{\partial F_2}{\partial x_1}, \epsilon\frac{\partial F_3}{\partial x_1}\right) = Dx(F(\vec{x}))\epsilon\vec{i} \cdot \epsilon^2\vec{i} = \epsilon^3\frac{\partial F_1}{\partial x_1} \\
&\implies \left.\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right|_{t=0} = \epsilon^3\mathrm{div}(F(\vec{x})) = V(0)\mathrm{div}F(\vec{x})
\end{aligned}$$

So div essentialy represents a volume changing.

Note. *The divergence of a vector field can be viewed as a volume changing.*

9 Line Integrals and Force Fields

Suppose a particle moves along a path σ while being acted on by $\vec{F}$. We are interested in the work done on the particle as it traces out the path σ.

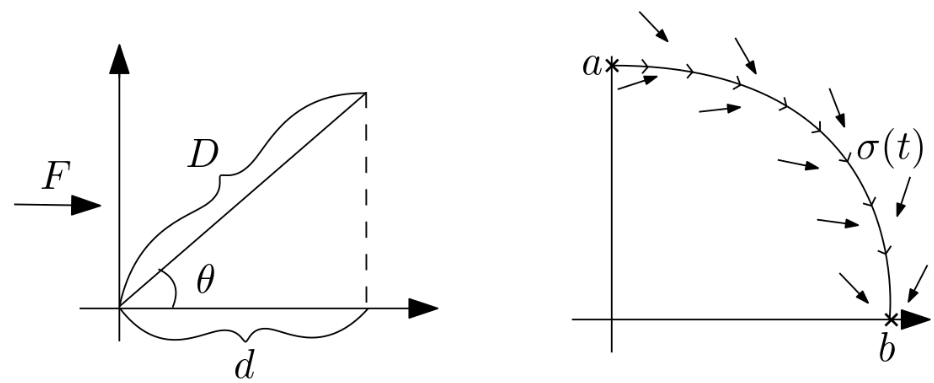


Figure 15: Force acting on a particle. **a)** A force F is applied on a particle along a straight line.**b)** We see a force of two dimensions and the particle moves along a varying path $\sigma(t)$.

From the above figure in a), the work done on the particle is along a straight line ,thus $W = FD\cos\theta = Fd$.. In the second case b), Since $\sigma(t+\Delta t) - \sigma(t) \approxeq \sigma'(t)\Delta t$, the work done by the vector field at time t is

$$\|F(\sigma(t))\|\|\sigma'(t)\Delta t\|\cos\theta = F(\sigma(t)) \cdot \sigma'(t)\Delta t, \tag{9.0.1}$$

here we projected F on the x-axis.

Now, partioning $a = t_0 < t_1 < t_2 < \cdots < t_n < t_n = b$,the work done is $\sum_{i=0}^{n} F(\sigma(t_i)) \cdot \sigma'(t)\Delta t$.

Definition 9.1. (Integral along a path) *Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is continuous on the C^1 path $\sigma : [a,b] \to \mathbb{R}^3$. Then we define $\int_\sigma F \cdot ds$, the integral of F along σ by*

$$\int_\sigma F \cdot ds = \int_a^b f(\sigma(t)) \cdot \sigma'(t)\, dt. \tag{9.0.2}$$

Note. *Take $F = (F_1, F_2, F_3), \sigma = (x(t), y(t), z(t))$*

$$\implies \int_\sigma F \cdot ds = \int_a^b F \cdot \sigma'(t)\, dt = \int_a^b \left(F_1\frac{\mathrm{d}x}{\mathrm{d}t} + F_2\frac{\mathrm{d}y}{\mathrm{d}t} + F_3\frac{\mathrm{d}z}{\mathrm{d}t}\right) dt, \tag{9.0.3}$$

also frequently written as $F_1\, dx + F_2\, dy + F_3\, dz$. (the differential form).

Example 9.1. Ket $\sigma(t) = (\cos^3(t), \sin^2(t), t)$, $t \in [0, 7\pi/2]$. Compute $\int_\sigma \sin(z)\, dx + \cos(z)\, dy - (xy)^{1/3}\, dz$.

$$= \int_0^{7\pi/2} \sin(t) 3\cos^2(t)(-\sin(t)) + \cos(t)(3\sin^2(t))\cos(t) - \cos(t)\sin(t)\, dt$$
$$= \frac{-1}{2}\int_0^{7\pi/2} \sin(2t)\Big|_0^{\frac{7\pi}{2}} = \frac{-1}{2}.$$

Theorem 9.1. (Fundamental Line Integral Theorem) *Let $\sigma : [a,b] \to \mathbb{R}^3$ be C^1 and suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is also C^1, then*

$$\int_\sigma \nabla f\, ds = \int_a^b \nabla f \cdot \sigma'(t)\, dt \cdot f(\sigma(b) - f(\sigma(a))) \tag{9.0.4}$$

Theorem 9.2. (Conservative) *If σ is closed , C^1 curve and if F is conservative, then*

$$\int_\sigma F \cdot ds = 0. \tag{9.0.5}$$

Equivalently,

$$\sigma(b) = \sigma(a) \implies \int_\sigma F \cdot ds = \int_a^b \nabla f \cdot \sigma'(t)\, dt = \int \sigma(b) - \sigma(a) = 0. \tag{9.0.6}$$

Note. *F conservative means $F = \nabla f$.*

9.1 Path Independence

The line integral of a vector field F is called path independent if for any two curves σ_1, σ_2, where $\sigma_1(a) = \sigma_2(a)$ and $\sigma_1(b) = \sigma_2(b)$ (i,e., they start and end at common points) , then $\int_{\sigma_1} F \cdot ds = \int_{\sigma_2} F \cdot ds$.

Theorem 9.3. (Path Independence)

$$\int_\sigma F \cdot ds \text{ is path independent} \iff \int_c F \cdot ds = 0\ \forall \text{closed curve } c.$$

Proof. ($\Longrightarrow$) Assume F is path independent. Take any close curve $c : [a, b] \to \mathbb{R}^3,\ c(a) = c(b)$.

Let $a < \beta < b$
Let $\sigma_1 : [a, \beta] \to \mathbb{R}^3$, where $\sigma_1(t) = c(t),\ t \in [a, \beta]$.
Let $\sigma_2 : [\beta, b] \to \mathbb{R}^3$, where $\sigma_2(t) = c(t),\ t \in [\beta, b]$.
Let $-\sigma_2$ be the opposite direction mapping $c(b) \to c(\beta)$.

Then, by path independence it follows

$$\int_{\sigma_1} F \cdot ds = \int_{-\sigma_2} F \cdot ds \implies \int_{\sigma_1} F \cdot ds - \int_{-\sigma_2} F \cdot ds = 0.$$

Using that $\int_{\sigma} F \cdot ds = -\int_{-\sigma} F \cdot ds$,

$$\begin{aligned} &= \int_{\sigma_1} F \cdot ds + \int_{\sigma_2} F \cdot ds = 0 \\ &= \int_{c} F \cdot ds = 0. \end{aligned}$$

($\Longleftarrow$) Suppose $\int_c F \cdot ds = 0$ on any closed curve. Suppose we have two paths $\sigma_1[a, b] \to \mathbb{R}^3$ and $\sigma_2[a, b] \to R^3$, where $\sigma_1(a) = \sigma_2(a)$ and $\sigma_1(b) = \sigma_2(b)$. Taking σ_1 from $a \to b$ and $-\sigma_2$ from $b \to a$ gives a closed curve ;

$$\implies \int_{\sigma_1} F \cdot ds + \int_{-\sigma_2} F \cdot ds = 0 \implies \int_{\sigma_1} F \cdot ds = -\int_{-\sigma_2} F \cdot ds = \int_{\sigma_2} F \cdot ds,$$

$\therefore$ Path independent. $\square$

How to determine if a vector field is conservative ? Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $F = (P, Q)$ where P and Q have continuous partials. If $\nabla f = F \implies P_y = Q_x$. If $\nabla f = (P, Q) = f_x = P,\ f_y = 0$, then by Clairot's theorem we have $f_{xy} = f_{yx} = P_y = Q_x$.

We conclude that if F is conservative with $F = (P, Q)$ where P, Q have continuous partials then $P_y = Q_x$. Now a question arises, if $F = (P, Q)$ and $P_y = Q_x$ does there exist a function f such that $f_x = P$ and $f_y = Q$? This is only true for certain domains, which leads us to the next important definitions.

Definition 9.2. (Simple Closed Curve) *A simple closed curve is a curve $\sigma : [a, b] \to \mathbb{R}^n$ which does not intersect itself outside of its endpoints which implies that $\sigma(t_i) = \sigma(t_j)$ for $a \le t_i \le t_j \le b \implies t_i = a,\ t_j = b$.*

Definition 9.3. ((Path) Connected) *A domain D is called (path) connected if any two points in D can be joined by a path that lies in D.*

Definition 9.4. (Simply Connected) *A simply connected region D is a connected region such that every simple closed curve in D encloses points only contained in D.*

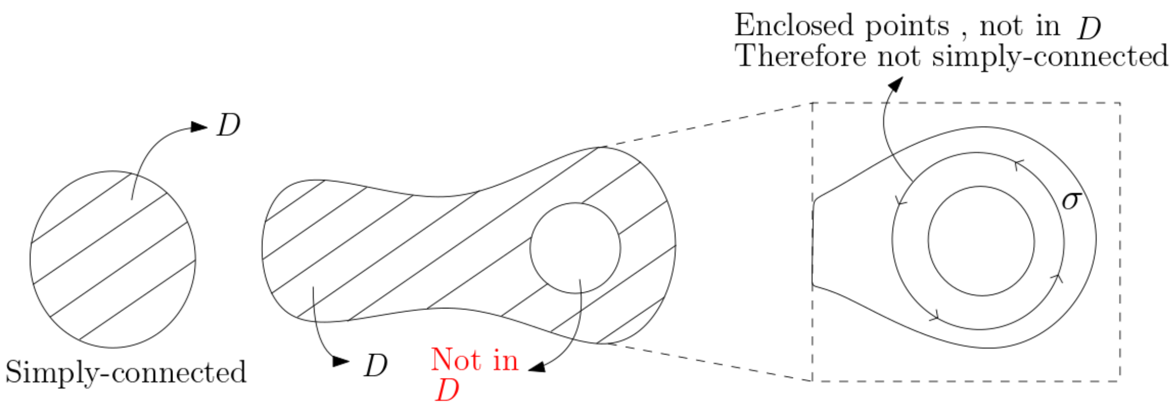


Figure 16: Illustration of simply-connected and not simply-connected domains, following the previous definitions.

Theorem 9.4. *If $F = (P, Q)$ is a vector field on a open simply connected region D and P, Q have continuous partials with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then F is conservative ($F = \nabla f$ where $f \in C^1$).*

Theorem 9.5. *Let $F = (F_1, F_2, F_3)$ with partials of F_i continuous in D. If curl$\vec{F} = \vec{0}$ in D and D is simply connected, then F is conservative ($F = \nabla f, f$ is C^1).*

Example 9.2. Let $F + (3 + 2xy, x^2 - 3y^2)$, determine if F is conservative. If so, find f such that $\nabla f = F$.

- **Step 1:** Check $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$.
- **Step 2:** Is the domain simply connected ?

$\frac{\partial}{\partial y} = 2x$, $\frac{\partial}{\partial x} = 3x$ $\quad\therefore$ same. Moreover, F has polynominal components therefore no singularities. Which then implies that the domain is simply connected and consequently F is conservative.
Now how do we find f such that $\nabla f = F$? Conservative implies that $F = (f_x, f_y)$. We use the procedure for Exact Equations from ODEs ;

$$\Longrightarrow \frac{\partial}{\partial x} f(x, y) = 3 + 2xy \Longrightarrow f(x, y) = \int 3 + 2xy dx + g(y) \Longrightarrow f(x, y) = 3x + x^2 y + g(y).$$

We use $f_y = x^2 - 3y^2$ and get ,

$$\frac{\partial}{\partial y} f(x,y) = x^2 + g'(y) = x - 2y^2 \implies g'(y) = -3y^2 \implies g(y) = -y^3.$$

$$\therefore f(x,y) = 3x + x^2 y - y^3.$$

Example 9.3. Show that $\int_C 2xe^{-y}\, dx + (2y - x^2 e^{-y})dy$ is path independent.

First we show that $F = (2xe^{-y}, 2y - x^2 e^{-y})$ is conservative

$$\frac{\partial}{\partial y} = -2xe^{-y} \quad , \frac{\partial}{\partial x} = -2xe^{-y} \ \checkmark.$$

And, since both components are sums of products of polynomials and exponential this implies there are no singularities and consequently F is conservative.

9.2 Green's Theorem

We say a simple closed curve has positive orientation if the closed curve is traced on counter-clockwise.

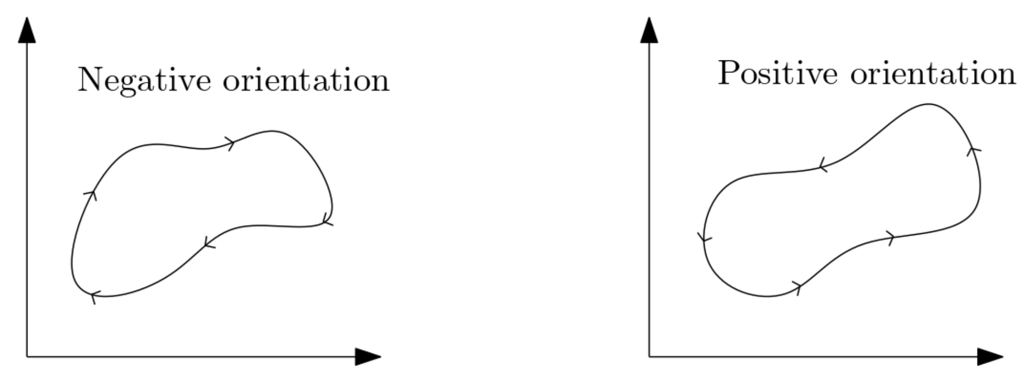


Figure 17: Positive and negative orientation of a closed curve.

Theorem 9.6. (Green's Theorem) *Let C be a positively oriented piece wise smooth , simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partials on an open region containing D, then*

$$\int_C P dx + Q dy = \iint_D \frac{\mathrm{d}Q}{\mathrm{d}x} - \frac{\mathrm{d}P}{\mathrm{d}y}\, dA. \tag{9.2.1}$$

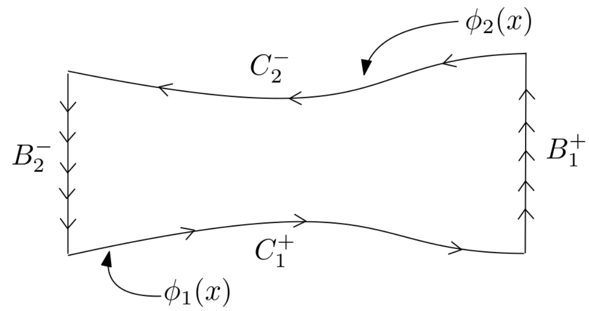


Figure 18: A type 1 domain D for, figure for illustrative purpose and to facilitate the proof's notation.

Proof. We start the proof by assuming D is of type 1 ,

$$D := x \in [a,b],\ \varphi_1(x) \le y \le \varphi_2(x) \qquad ,C^+ = C_1^+ \cup B_1^+ \cup C_2^- \cup B_2^-,$$

$$\text{Show that } \int\limits_{C^+} P dx = \iint\limits_D \frac{\partial P}{\partial y} dA.$$

$$\iint\limits_D \frac{\partial P}{\partial y} dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\ dy dx = \int_a^b P(x,\varphi_2(x)) - P(x,\varphi_1(x))\ dx$$

Notice that

$$\int_a^b P(x,\varphi_2(x)) dx = \int\limits_{-C_2^-} P\ dx = -\int\limits_{C_2^-} P\ dx.$$

Also notice that similarly

$$-\int_a^b P(x,\varphi_1(x))\ dx = -\int\limits_{C_1^+} P\ dx.$$

Since x is constant on B_1^+ and B_2^+ , $\implies \int\limits_{B_1^+} P dx = 0 = \int\limits_{B_2^-} P dx.$

Putting this all tohether we get,

$$\iint\limits_D \frac{\partial P}{\partial y} dA = -\int\limits_{C_1^+} P dx - \int\limits_{C_2^-} P dx - \int\limits_{B_1^+} P dx - \int\limits_{B_2^-} P dx = -\int\limits_{C^+} P dx.$$

$\square$

Theorem 9.7. *If σ is a simple closed curve which bounds domain D , where $\sigma = \partial D$ (boundary of D), then*

$$\text{Area of } D \text{ is } \frac{1}{2}\int\limits_{\partial D} xdy - ydx. \tag{9.2.2}$$

Proof. Area(D)= $\frac{1}{2}\int\limits_{\partial D} -ydx + xdy$. By the Green's theorem this imply that $P(x, y) = -y$ and $Q(x, y) = x$.

$$\implies \frac{1}{2}\int\limits_{\partial D} -ydx + xdy = \frac{1}{2}\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint\limits_{D} dA \ \checkmark.$$

□

Now what if the region is not of type 3 ? The idea is to take the region, divide it in two sub regions and apply our machinery to those sub-regions.

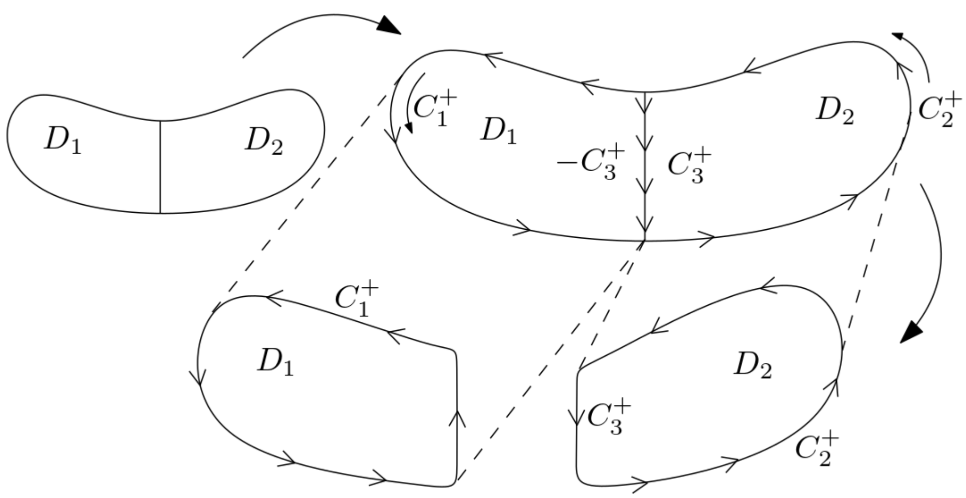


Figure 19: How to handle regions that are not type 3.

Following Figure 19 , let us show rigorously that Green's theorem holds for these types of regions.

$$\iint\limits_{D_1} \frac{\partial Q}{\partial c} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_1^+ \cup -C_3^+} Pdx - Qdy \quad , \iint\limits_{D_2} \frac{\partial Q}{\partial c} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_2^+ \cup C_3^+} Pdx - Qdy.$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_{D_1} \frac{\partial Q}{\partial c} - \frac{\partial P}{\partial y}\, dA + \iint_{D_2} \frac{\partial Q}{\partial c} - \frac{\partial P}{\partial y}\, dA$$

$$= \int_{C_1^+} Pdx + Qdy - \int_{C_3^+} Pdx + Qdy + \int_{C_2^+} Pdx + Qdy \int_{C_3^+} Pdx + Qdy$$

$$= \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{C_1^+ \cup C_2^+} Pdx + Qdy = \int_{C^+} Pdx + qdy.$$

$\therefore$ Green's theorem holds on the union of type 3 regions.

Example 9.4. Verify Green's theorem in the unit disc where $P(x, y) = x$, $Q(x, y) = xy$.
Recall Green's theorem states that

$$\int_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

Note that this corresponds to a circle of path $\sigma(t) = (\cos(t), \sin(t)) \equiv (x(t), y(t))$, with domain D inside. Let us first verify the left hand side of the above formula then the RHS.

$$\int_\sigma Pdx + Qdy = \int_0^{2\pi} P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$$

$$= \int_0^{2\pi} \cos(t)(-\sin(t)) + \cos^2(t)\sin(t)\, dt$$

$$\overset{u=\cos(t)}{=} \int_1^1 u - u^2\, du = 0.$$

Now for the RHS,

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D y\, dA = \int_0^{2\pi}\int_0^1 r^2 \sin\theta\, dr d\theta = \int_0^{2\pi} \frac{\sin\theta}{3}\, d\theta = 0.$$

Green's theorem is verified.

On the previous class we proved Green's theorem on type 3 domains i.e.,

$$\left.\begin{array}{ll} D := & x \in [a, b], \varphi_1(x) \le y \le \varphi_2(x) \\ \text{or} & y \in [c, d], \varphi_1(y) \le x \le \varphi_2(y) \end{array}\right\} \text{ Equivalent ways to write.}$$

What if the domain is not of type 3 ?

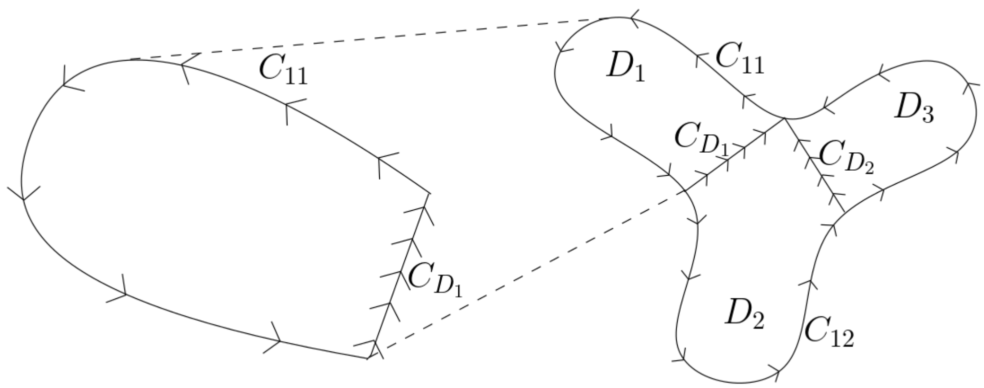


Figure 20: Other example of non type 3 domain.

Following Figure 20, let us suppose D_1, D_2, D_3 are type 3. Then we want to do,

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint\limits_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint\limits_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint\limits_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\text{so , } \iint\limits_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{11}} Pdx + Qdy + \int\limits_{C_{D_1}} Pdx + Qdy$$

$$\iint\limits_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \underbrace{\int\limits_{C_{12}} Pdx + Qdy + \int\limits_{C_{D_2}} Pdx + Qdy \overbrace{- \int\limits_{C_{D_1}} Pdx + Qdy}^{\int_{-C_{D_1}} = -\int_{C_{D_1}}}}_{\text{all these components to complete one loop}}$$

$$\iint\limits_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{13}} Pdx + Qdy - \int\limits_{C_{D_2}} Pdx + Qdy.$$

Recollecting , many terms cancel out so then we have

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{11}} Pdx + Qdy + \int\limits_{C_{12}} Pdx + Qdy + \int\limits_{C_{13}} Pdx + Qdy.$$

Letting $c := C_{11} \cup C_{12} \cup C_{12} = \partial D$ we get

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C = \partial D} Pdx + Qdy.$$

Now what about regions enclosed by 2 closed curves ?

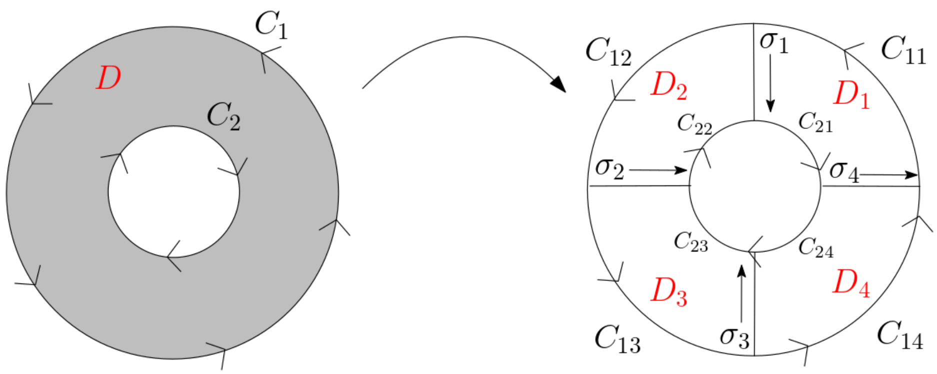


Figure 21: Example of transforming non-type 3 domain enclosed by two curves ,into a type 3 domain.

Following Figure 21, note that in the LHS picture , C_1 is positively oriented while C_2 is negatively oriented , as stated in Figure 17. We can immediately apply Green's theorem, first of , let $C_1 = C_{11} \cup C_{12} \cup C_{13} \cup C_{14}$ and let $C_2 = C_{21} \cup C_{22} \cup C_{23} \cup C_{24}$. We then have, by definition

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint\limits_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint\limits_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$
$$+ \iint\limits_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint\limits_{D_4} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

Using Figure 20, we can express each integral on the RHS as follows ,

$$\iint\limits_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{11} \cup \sigma_1 \cup C_{21} \cup \sigma_1} P dx + Q dy \quad , \iint\limits_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{12} \cup \sigma_2 \cup C_{22} \cup -\sigma_1} P dx + Q dy$$
$$\iint\limits_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{13} \cup \sigma_3 \cup C_{23} \cup -\sigma_2} P dx + Q dy \quad , \iint\limits_{D_4} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_{14} \cup -\sigma_4 \cup C_{24} \cup -\sigma_3} P dx + Q dy.$$

Adding this all up, all the σ_i cancel and we are left with

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_{C_1} P dx + Q dy + \int\limits_{C_2} P dx + Q dy.$$

Example 9.5. Let $F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, show that $\int\limits_{\sigma} F \cdot ds = 2\pi$, for any simple closed curve σ enclosing (wraps around it) the point $(0, 0)$. Assume σ is positively oriented.

The strategy for this type of problem is as follows , first , find a circle of radius a centered at $(0,0)$ which is contained in the region enclosed by σ. Then solve for $c(t) = (a\cos(t), a\sin(t))$, for some $a > 0$. Replacing in Figure 21a C_1 by σ and C_2 by C whilst allowing a non-circular shape and allowing a negative orientation for C, we have our corresponding domain. Given F as defined and since the region enclosed by C is not in the domain we can say that F has continuous partials on D, therefore,

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int\limits_\sigma P dx + Q dy - \int\limits_C P dx + Q dy,$$

here we're subtracting since C and σ have opposite directions. Note that we chose the directions of C and σ purely arbitrarily , as longs as we stay consistent with the signs in the previous equation everything should remain consistent as well.

Now let's go back to our function F,

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \implies P = \frac{-y}{x^2+y^2},\ Q = \frac{x}{x^2+y^2}$$

$$\text{Therefore },\frac{\partial Q}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\text{And similarly, } \frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \text{ by Green's theorem, } \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint\limits_D \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) - \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) dA = 0.$$

This result then implies that

$$\int\limits_D F \cdot ds = \int\limits_C F \cdot ds.$$

We may now compute the path integral,

$$\begin{aligned}
\int\limits_\sigma F \cdot ds &= \int_0^{2\pi} F(a\cos(t), a\sin(t)) \cdot \underbrace{(-a\sin(t), a\cos(t))}_{\sigma'(t) \text{ from the path integral definition}}\, dt \\
&= \int_0^{2\pi} \frac{-a\sin(t)}{a^2\cos^2(t)+a^2\sin^2(t)}(-a\sin(t)) + \frac{a\cos(t)}{a^2\cos^2(t)+a^2\sin^2(t)}(a\cos(t))\, dt \\
&= \int_0^{2\pi} dt = 2\pi.
\end{aligned}$$

Theorem 9.8. *Let $F = (P, Q)$ be a vector field on a simply connected domain D where P and Q have continuous partials on D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D. Then, $\int\limits_C F \cdot ds = 0$ for any simple closed curve c contained in D.*

Proof. Follows from Green's theorem ! $\square$

9.3 Vector Form of Green's Theorem

Let D be a simply connected region of $\mathbb{R}^2$ which can be portioned into type 3 domains. Let $\partial D = C$, positively oriented. Letting F be C^1 vector field on D , we have

$$\int\limits_{\partial D} F \cdot ds = \int\limits_{C} F \cdot ds = \iint\limits_{D} (\text{Curl}F) \cdot \vec{k}\, dA = \iint\limits_{D} (\nabla \times F) \cdot \vec{k}\, dA, \tag{9.3.1}$$

where $\vec{k}$ is the usual unit vector $(0, 0, 1)$.

Taking $F = (P, Q, 0)$ and recalling (8.1.2)

$$\implies \text{Curl } F = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\therefore\ (\text{Curl}F) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad \text{Standard form of Green's theorem}$$

Theorem 9.9. **(Divergence Theorem in $\mathbb{R}^2$)** *Let $D \subseteq \mathbb{R}^2$ be simply connected and a union of type 3 domain. Let $\sigma(t) : [a, b] \to \mathbb{R}^2$ be a positively oriented parametrization of the boundary ∂D, where $\sigma(t) = (x(t), y(t))$.*
Let $\vec{n}$ denote the unit outward normal of ∂D, given by

$$\vec{n} = \frac{(y'(t) - x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}. \tag{9.3.2}$$

Then, if F is C^1 vector field on D we have,

$$\int\limits_{\partial D = \sigma} F \cdot \vec{n}\, ds = \iint\limits_{D} DivF\ dA. \tag{9.3.3}$$

Remark. *$\vec{n}$ is a vector perpendicular to $\sigma'(t)$ which implies that $\vec{n} \cdot \sigma'(t) = 0$.*

Note. *$\vec{F} \cdot \vec{n}$ is a scalar, i.e., a real-valued function. Therefore, $\int\limits_{\sigma} F \cdot \vec{n}$ is a path integral.*

Let $F(x, y) = (P(x, y), Q(x, y))$ then

$$\int\limits_{\sigma} \vec{F} \cdot \vec{n}\, ds = \int_a^b \left(\frac{P(x(t), y(t))y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} - \frac{Q(x(t), y(t))x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \right) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int\limits_{\sigma} P dy - Q dx$$

Observe the previous equation, usual Green's theorem is $\int_{\sigma} Pdx + Pdy$.

$$= \int_{\sigma} (-Q)dx + Pdy = \iint_{D} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_{D} \mathrm{Div}F\, dA.$$

What will we cover in the next two remanding weeks ? Green's theorem is finished and line integrals , path integrals , curl, div ... Now we need to do *surface integrals* (integrals over surface in 3D). *Stoke's theorem (generalization of Green's in 3D)* and *divergence theorem*.

10 Parametric Surfaces

So far we have looked at surfaces given by the graph of a function $z = f(x, y)$, where $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$, but also level surfaces like $k = f(x, y, z)$.

Definition 10.1. (Parametric surface) *A parameterized surface also known as a parametric surface is a function $\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. This surface, denoted S corresponding to the function ϕ is its image $S = \phi(D)$.*
We can write $\phi(u, v) = (x(u, v), y(u, v), z(u, v))$. We say S is a differentiable surface if ϕ is differentiable $\implies$ $x(u, v), y(u, v)$ and $z(u, v)$ are differentiable.

Think of these as an extension of parametric curves $\vec{r}(t) = (x(t), y(t), z(t))$, to surfaces. So if we're going to vary u and v everywhere in the domain this is going to create a surface in $\mathbb{R}^3$.

Example 10.1. Identify the parametric surface

$$x = u\cos(v), y = u\sin(v), z = u \quad u \geq 0.$$

Note there's both a cos and a sin with u in front.

$$\begin{aligned} \implies x^2 + y^2 &= u^2(\cos^2(v) + \sin^2(v)) \\ &= u^2 \\ \implies z &= \sqrt{x^2 + y^2} \end{aligned}$$

Therefore this is a cone. Hence, $(u\cos(v), u\sin(v), u)$ parameterizes a cone.

Definition 10.2. (Smoothness (*Informal*)) *We say a surface is smooth is there exists a tangent plane at any point of that surface.*

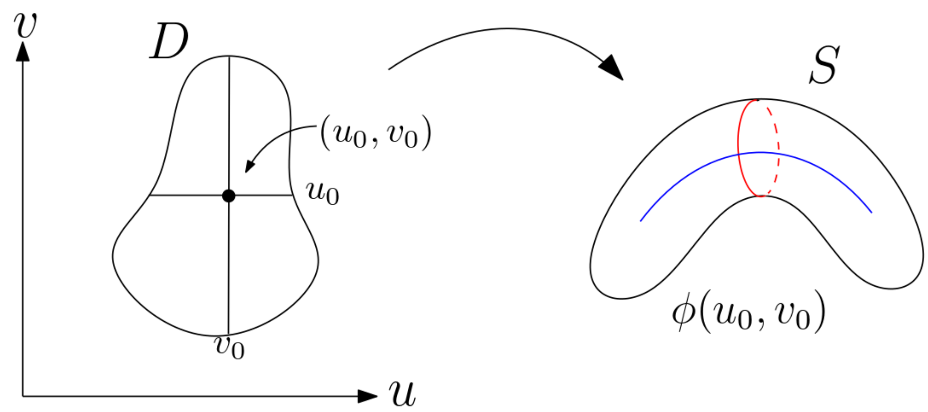


Figure 22: **a)** Two dimensional surface living in $\mathbb{R}^2$. **b)** The surface S living in $\mathbb{R}^3$; Line projected while v_0 is fixed, Line projected while u_0 is fixed.

Following Figure 22, suppose we fix $u = u_0$ and let v vary in D. This implies $\varphi(u_0, v) = (x(u_0, v), y(u_0, v), z(u_0, v))$ is a parametric curbe on S. The vector tangent to the curve $\varphi(u_0, v)$ at $\varphi(u_0, v_0)$ is

$$\vec{T}_v := \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} := \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

Similarly, fixing v_0 and letting u vary in D , we obtain the tangent vector at $\varphi(u_0, v_0)$

$$\vec{T}_u := \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}.$$

Using $\vec{T}_u$ and $\vec{T}_v$ we can form the tangent plane to S at $\varphi(u_0, v_0)$ by computing the normal vector

$$\vec{n} = \vec{T}_u \times \vec{T}_v.$$

Putting this all together we have the following definition

> **Definition 10.3.** **(Smoothness (*Formal*))** *A parameterized surface is smooth at a point (u_0, v_0) if $\vec{T}_u \times \vec{T}_v \neq \vec{0}$ at (u_0, v_0).*
>
> *Moreover, we say S is smooth if $\vec{T}_u \times \vec{T}_v \neq \vec{0}$ anywhere in D (or all points $\varphi(u, v) \in S$).*

Having defined smoothness let us analyse the previous example (**Example 10.1**).

Example. *We had,*

$$x = u\cos(v), y = u\sin(v), z = u \quad u \geq 0.$$

- *Is the surface differentiable ? The surface is indeed differentiable because x, y, z are all differentiable functions. Recalling Section 3, product of polynomial functions is differentiable given the components are differentiable.*

- *Is the surface smooth ?*

$$\vec{T}_u = (\cos(v), \sin(v), 1), \qquad \vec{T}_v = (-u\sin(v), u\cos(v), 0)$$

$$\vec{T}_u \times \vec{T}_v = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \cos(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$
$$= (-u\cos(v), -u\sin(v), u\cos^2(v) + u\sin^2(v)).$$

Notice that when $u = 0$, we have $\vec{T}_u \times \vec{T}_v = \vec{0}$, $\therefore$ the surface is not smooth. Indeed, the point at the sharp end of a cone has no tangent plane there !

Example 10.2. Find the tangent plane to the surface

$$x = u^2, y = v^2, z = u + 2v \qquad \text{at } (1, 1, 3).$$

We compute the corresponding tangent vectors then evaluate them at the given points.

$$\vec{T}_u = (2u, 0, 1), \qquad \vec{T}_v = (0, 2v, 2)$$
$$\implies \text{ at } (1, 1, 3), \quad \vec{T}_u = (2, 0, 1), \qquad \vec{T}_v = (0, 2, 2).$$
$$\therefore \vec{n} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2, -4, 4)$$

Using the definition of a tangent plane from Calculus 3, the tangent plane has the equation

$$Z_{x,y,z} = -2(x - 1) - 4(y - 1) + 4(z - 3) = 0.$$

10.1 Surface Area

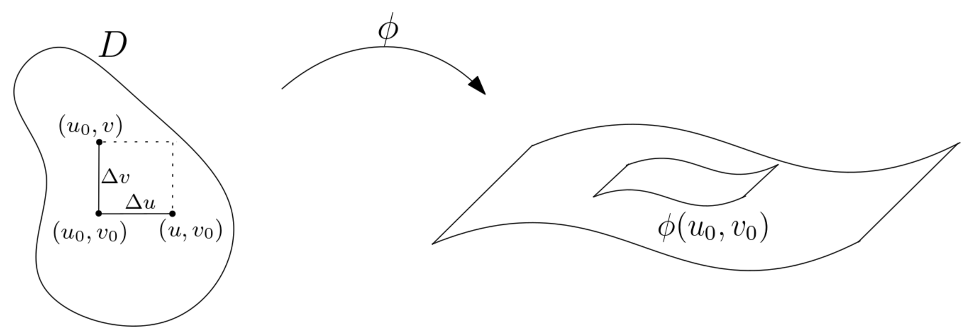


Figure 23: TODO

$$\varphi(u, v_0) \approxeq \varphi(u_0, v_0) + \vec{T}_u \Delta u \implies \varphi(u, v_0) - \varphi(u_0, v_0) \approxeq \vec{T}_u \Delta u.$$
$$\text{Similarly, } \varphi(u_0, v) - \varphi(u_0, v_0) \approxeq \vec{T}_v \Delta v.$$

We want to find are of the parallelogram formed by $\vec{T}_v \Delta v$ and $\vec{T}_u \Delta u$.

$$\text{Area} = \left\| \vec{T}_u \Delta u \times \vec{T}_v \Delta v \right\| = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} \Delta u & \frac{\partial y}{\partial u} \Delta u & \frac{\partial z}{\partial u} \Delta u \\ \frac{\partial x}{\partial v} \Delta v & \frac{\partial y}{\partial v} \Delta v & \frac{\partial z}{\partial v} \Delta v \end{vmatrix} = \left\| \vec{T}_u \times \vec{T}_v \right\| \Delta u \Delta v.$$

Definition 10.4. (Surface Area) *We define the surface area of a parameterized surface by*

$$\iint_D \left\| \vec{T}_u \times \vec{T}_v \right\| dudv, \tag{10.1.1}$$

where $\left\| \vec{T}_u \times \vec{T}_v \right\|$ *is the standard Euclidian norm of the vector* $\vec{T}_u \times \vec{T}_v$.

Example 10.3. Compute the surface area of a sphere of raidus a.
Here we'll use spherical coordinates. Let

$$x = a \sin \varphi \cos \theta, \quad y = a \sin \varphi \sin \theta, \quad z = a \cos \varphi.$$

The bounds of integration are then defined from

$$D := \{(\theta, \varphi) | 0 \le \theta \le 2\pi, \;\; 0 \le \varphi \le \pi\}$$

Following Definition 10.4, we must compute $\vec{T}_\theta, \vec{T}_\varphi, \vec{T}_\theta \times \vec{T}_\varphi$ and $\left\| \vec{T}_\theta \times \vec{T}_\varphi \right\|$.

$$\begin{aligned}
\vec{T}_\varphi &= (a \cos \varphi \cos \theta . a \cos \varphi \sin \theta, -a \sin \varphi) \\
\vec{T}_\theta &= (-a \sin \varphi \sin \theta, a \sin \varphi \cos \theta, 0) \\
\vec{T}_\varphi \times \vec{T}_\theta &= \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a \cos \varphi \cos \theta & a \cos \varphi \sin \theta & -a \sin \varphi \\ -a \sin \varphi \sin \theta & a \sin \theta \cos \theta & 0 \end{vmatrix} \\
&= (a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \sin \varphi \cos \varphi) \\
\left\| \vec{T}_\varphi \times \vec{T}_\theta \right\| &= \sqrt{a^4 \sin^4 \varphi \cos^2 \theta + a^4 \sin^4 \varphi \sin^2 \theta + a^4 \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{a^4 \sin^4 \varphi + a^4 \sin^2 \varphi \cos^2 \varphi} \\
&= a^4 \sin^4 \varphi + a^4 \sin^2 \varphi (1 - \cos^2 \varphi) = \left| a^2 \sin \varphi \right|.
\end{aligned}$$

Finally,

$$\begin{aligned}\text{Surface area } &= \int_0^{\pi}\int_0^{2\pi} \left|a^2 \sin\varphi\right| d\theta d\varphi \qquad \sin\varphi \geq 0 \text{ on } [0, 2\pi] \\ &= 2\pi a^2 \int_0^{\pi} \sin\varphi \, d\varphi = 4\pi a^2.\end{aligned}$$

Definition 10.5. (Surface area of the graph of a function $f(x, y)$) *Letting $x = u \quad y = v, \quad z = f(u, v) \implies \vec{T}_u = \left(1, 0, \frac{\partial f}{\partial u}\right), \vec{T}_v = \left(0, 1, \frac{\partial f}{\partial v}\right)$. Computing the determinant $\vec{T}_u \times \vec{T}_v$ yields $\left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$. Thus,*

$$\text{Surface area is } \iint_D \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1} \, dudv. \tag{10.1.2}$$

Example 10.4. Find the surface are of a cone in the region enclosed by $y = x$, $y = x^2$. Following Definition 10.5 , we must first calculate the required terms and then the answer is immediate.

$$f(x, y) = \sqrt{x^2 + y^2} \ , \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

$$\begin{aligned}\therefore \text{ Surface area } &= \int_0^1 \int_x^{x^2} \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} \, dydx \\ &= \sqrt{2} \int_0^1 \int_x^{x^2} dydx = \sqrt{2}x - x^2 \, dx = \sqrt{2}\left[\frac{1}{2} - \frac{1}{3}\right] \frac{\sqrt{2}}{6}.\end{aligned}$$

Definition 10.6. (Surface integrals) *If $f(x, y, z)$ is a real-valued function defined on S (a surface), we define the integral fo f over S to be*

$$\int_S f(x, y, z) \, ds = \iint_D f(\varphi(u, v)) \left\|\vec{T}_u \times \vec{T}_v\right\| dudv, \tag{10.1.3}$$

where S is given by $\varphi(D)$.

Remark. *In Definition 10.6, if $f = 1$ we recover the surface area formula from Definition 10.4.*

Example 10.5. Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = \theta$,
where $D = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$. Let $f(x, y, z) = \sqrt{x^2 + y^2 + 1}$, compute $\int_S f \, ds$.

As before, we first compute the necessary terms

$$\vec{T}_r = (\cos\theta, \sin\theta, 0) \quad \vec{T}_\theta = (-r\sin\theta, r\cos\theta, 1)$$

$$\implies \vec{T}_r \times \vec{T}_\theta = \det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r) \;\therefore\; \left\|\vec{T}_u \times \vec{T}_\theta\right\| = \sqrt{r^2+1}.$$

Let us now evaluate f on our surface,

$$f(x(r,\theta), y(r,\theta), z(r,\theta)) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \sqrt{r^2+1}$$

$$\therefore \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\sqrt{r^2+1}\; dr d\theta = \int_0^{2\pi}\int_0^1 r^2+1\; dr d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}.$$

Example 10.6. Compute $\iint\limits_S z^2\, ds$, where S is the hemisphere defined by $z = \sqrt{a^2 - x^2 - y^2}$.

We already know that spherical coordinates parameterizes the sphere, so let us use them. Set

$$x = a\sin\varphi\cos\theta, \quad y = a\sin\varphi\sin\theta, \quad z = a\cos\varphi \;\; , D = \{(\varphi,\theta) \mid 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi\}.$$

As seen in Example 10.3 , $\left\|\vec{T}_\varphi \times \vec{T}_\theta\right\| = \left|a^2\sin\varphi\right|$. Since yet again $\sin\varphi > 0$ in our present region, we have $\left\|\vec{T}_\varphi \times \vec{T}_\theta\right\| = a^2\sin\varphi$.

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\pi} a^2\cos^2\varphi a^2\sin\varphi\; d\theta d\varphi &= a^4\int_0^{\pi/2}\int_0^{\pi}\cos^2\varphi\sin\varphi\; d\theta d\varphi \\ &= 2\pi a^4\int_0^{\pi/2}\cos^2\varphi\sin\varphi\; d\varphi.\end{aligned}$$

This is a common integral seen in Calc 2. Let $u = \cos\varphi \implies du = -\sin\varphi\, d\varphi$, thus

$$= -2\pi a^4\int_1^0 u^2\; du = \frac{2\pi}{3}a^4.$$

Before moving to surface integrals of vector fields, we first discuss orientable surfaces.

Definition 10.7. (Oriented surface) *An oriented surface is a two-sided surface with one side specified as the outside or positive side ; we call the other side the inside or negative side. At each point $(x, y, z) \in S$ there are two unit normal vectors $\vec{n}_1, \vec{n}_2$, where $\vec{n}_1 = -\vec{n}_2$. Each of these normals can be associated with one side of the surface.*

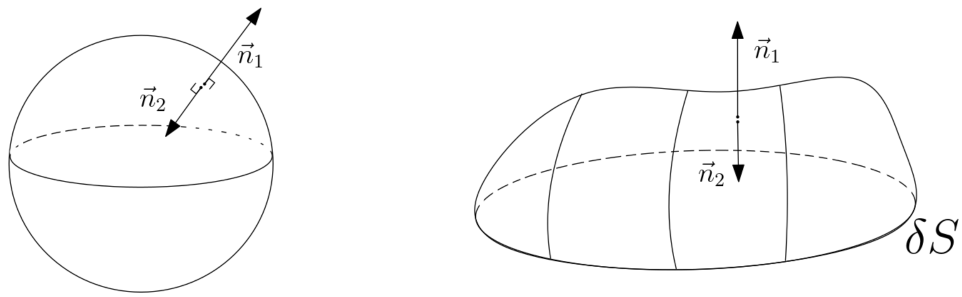


Figure 24: **a)** Closed surface with $\vec{n}_1$ pointing outwards and $\vec{n}_2$ pointing inwards. **b)** $\vec{n}_1$ always points outwards and $\vec{n}_2$ always points inwards. To have $\vec{n}_1$ point inwards it would have to cross the boundary (δS).

Following Definition 10.7, a natural question is what a non-oriented surface would look like ?The Möbious strip is a classical example and is illustrated in the following figure.

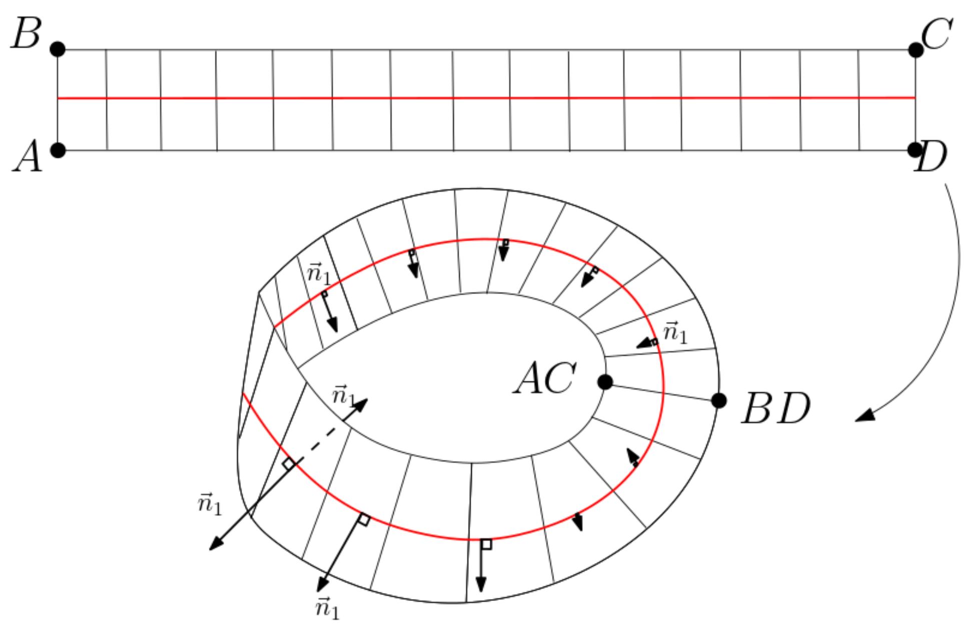


Figure 25: **a)** A rectangular object. **b)** Twist and attach the extremities of the rectangular object to obtain the Möbius strip.

Since the Mobius strip in Figure 25 is a one-sided surface, it is not orientable.
For a surface given by $z = g(x, y)$, we use the orientation provided by the normal vector to the tangent plane. As we did before, for

$$x = u,\ y = v,\ z = g(u, v)\ \ \vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{\left\|\vec{T}_u \times \vec{T}_v\right\|} = \frac{\left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}.$$

Since $\vec{k}$ component is positive, this gives us the upward orientation. Therefore for a parametric surface

$$\vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{\left\|\vec{T}_u \times \vec{T}_v\right\|}. \tag{10.1.4}$$

Example 10.7. Fing the outward pointing orientation for the sphere of radius one.

$$x = a \sin\varphi \cos\theta, \quad y = a \sin\varphi \sin\theta, \quad z = a\cos\varphi$$

$$\vec{n} = \frac{\vec{T}_\varphi \times \vec{T}_\theta}{\left\|\vec{T}_\varphi \times \vec{T}_\theta\right\|} = \frac{(a^2 \sin^2\varphi \cos\theta, a^2 \sin^2\varphi \sin\theta, a^2 \sin\varphi\cos\varphi)}{a^2 \sin\varphi}$$
$$= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Indeed, as a double check

- When $\varphi = 0$ (Top of sphere) $\implies \vec{n} = (0, 0, 1)$
- When $\varphi = \pi$ (Bottom of sphere) $\implies \vec{n} = (0, 0, -1)$
- When $\varphi = \pi/2$ (Equatior) $\implies \vec{n} = (\cos\theta, \sin\theta, 0)$

$\therefore$ The surface points outward at every point using the above formula, as expected.

10.2 Surface Integrals of Vector Fields

Definition 10.8. (Surface integral over a vector field) *If $\vec{F}$ is a continuous vector field defined on a oriented surface S with unit normal $\vec{n}$, then the surface integral of $\vec{F}$ over S is*

$$\iint\limits_S F \cdot ds = \iint\limits_S \vec{F} \cdot \vec{n}\, ds. \tag{10.2.1}$$

This is commonly called the flux integral across S.

By substituting Equation 10.1.1 in the previous definition we have

$$\iint\limits_S \vec{F} \cdot ds = \iint\limits_S \vec{F} \cdot \left|\frac{\vec{T}_u \times \vec{T}_v}{\left\|\vec{T}_u \times \vec{T}_v\right\|}\right| ds = \iint\limits_D F \cdot \left|\frac{\vec{T}_u \times \vec{T}_v}{\left\|\vec{T}_u \times \vec{T}_v\right\|}\right| \left\|\vec{T}_u \times \vec{T}_v\right\| dA$$
$$\implies \iint\limits_S \vec{F} \cdot ds = \iint\limits_D F \cdot (\vec{T}_u \times \vec{T}_v)\, du dv. \tag{10.2.2}$$

Remark. *If we think of $\vec{F}$ as the velocity field of a fluid, then $F \cdot (\vec{T}_u \times \vec{T}_v)$ is a measure of the net quantity of fluid flowing outward across the surface per unit time.*

Note. *Essentially the dot product in Equation 10.2.1 is the projection of the vector field onto the unit normal*

Example 10.8. Calculate the total flux of $F = (x, y, z)$ outward through the cylinder $x^2 + y^2 \leq a^2$ with $-h \leq z \leq h$. The cylinder has three sides (top, bottom and side) so there are three different outward vectors. The top and bottom vectors are respectively $\vec{n}_{\text{top}} = (0, 0, 1)$ and $\vec{n}_{\text{bot}} = (0, 0, -1)$. To find the side vector we let

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = z,$$

where $\theta \in [0, 2\pi]$ and $z \in [-h, h]$. Then we compute the outward vector using (10.1.4).

$$\vec{T}_\theta = \langle -a\sin\theta, a\cos\theta, 0\rangle$$

$$\vec{T}_z = \langle 0, 0, 1\rangle \implies \vec{T}_\theta \times \vec{T}_z = \det\begin{vmatrix} i & j & k \\ -a\cos\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos\theta, a\sin\theta, 0).$$

Therefore, $\vec{n}_{\text{side}} = (\cos\theta, \sin\theta, 0)$ as we expected.

Now we compute each individual side flux using (10.2.2).
(**Top**) Letting $(a\cos\theta, a\sin\theta, h)$, we have

$$\vec{T}_a = (\cos\theta, \sin\theta, 0) \quad \vec{T}_\theta = (-a\sin\theta, a\cos\theta, 0) \implies \vec{T}_a \times \vec{T}_\theta = (0, 0, a).$$

$$\therefore \iint_{\text{Top}} F \cdot (\vec{T}_a \times \vec{T}_\theta)\, dA = \int_0^{2\pi} \int_0^a (x(a, \theta), y(a, \theta), z(a, \theta)) \cdot (0, 0, a)\, dad\theta$$

$$= h \int_0^{2\pi} \int_0^a a\, dad\theta = \frac{a^2}{2} h \int_0^{2\pi} = a^2 h\pi.$$

(**Bottom**) Here The parameterization is very similar,

$$(a\cos\theta, a\sin\theta, -h) \implies \vec{n} = (0, 0, a),$$

same vector again but this time inward.

$$\therefore \vec{n} = 0\left(\frac{\vec{T}_a \times \vec{T}_\theta}{\left\|\vec{T}_a \times \vec{T}_\theta\right\|}\right) = \int_0^{2\pi} \int_0^a (a\cos\theta, a\sin\theta, -h)(0, 0, -a)$$

$$= \int_0^{2\pi} \int_0^a ah\, dad\theta = a^2 h\pi.$$

(**Side**)

$$\iint_{\text{Side}} \vec{F} \cdot \vec{n}\, ds = \int_0^{2\pi} \int_{-h}^h (a\cos\theta, a\sin\theta, z) \cdot (a\cos\theta, a\sin\theta, 0)\, dzd\theta$$

$$= \int_0^{2\pi} \int_{-h}^h a^2\cos^2\theta + a^2\sin^2\theta\, dzd\theta = (\ldots) = 4\pi ha^2.$$

We conclude that the total flux of the given object is $6\pi ha^2$.

10.3 Stoke's Theorem

Stoke's theorem relates the integral around a simple closed curve $C \in \mathbb{R}^3$ to an integral over a surface S for which C is the boundary.

Note. *Very similar to Green's theorem because integrating over a curve was related to integrating over a domain D.*

Recall. The surface integral from previous classes (10.1.3)

$$\int_S F \cdot ds = \int F \cdot \vec{n} = \int_S F \cdot (\vec{T}_u \times \vec{T}_v)\, ds.$$

If S is given by the surface associated with a function $f(x, y)$ then if $F = (F_1, F_2, F_3)$ (is a vector field) then

$$\int_S F \cdot ds = \int_D \left(F_1 \left(-\frac{\partial z}{\partial x} \right) + F_2 \left(-\frac{\partial z}{\partial y} \right) + F_3 \right)\, dxdy, \tag{10.3.1}$$

where $z = f(x, y)$.
Recall. For Green's theorem we proved the result for type 3 domains for which

$$D = x \in [a, b] \text{ and } \varphi_1(x) \le y \le \varphi_2(x)$$
$$y \in [c, d] \text{ and } \varphi_1(y) \le x \le \varphi_2(y).$$

We proved Green's theorem for this byt then we extended the result for a much larger class of domains (i.e., we partitioned into smaller domains and took the union between them Figure 19 ; Figure 20). We prove Stoke's theorem on domains where Green's theorem applies.

Note. *In this class, wherever Green's theorem applies Stoke's will apply as well even though it can be extended further.*

Before invoking the formal definition we must clarify some concepts such as Orientation of Boundary of S (δS).
Suppose that $\sigma : [a, b] \longrightarrow \mathbb{R}^2$, where $\sigma(t) = (x(t), y(t))$ is a parameterization of δD. We define the parameterization of δS as

$$\eta : t \longrightarrow (x(t), y(t), f(x(t), y(t))). \tag{10.3.2}$$

Theorem 10.1. (Stoke's Theorem) *Let S be an oriented surface (surface with defined sides* 10.7*) defined by a C^2 function $z = f(x, y)$, where $(x, y) \in D$ and let D be a C^1 vector field on S. Then, if δS denotes the oriented boundart curve of S as defined above then we have*

$$\int_S curlF \cdot dS = \int_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds \tag{10.3.3}$$

If $\eta : [a,b] \to \mathbb{R}^3$, where $\eta = (x(t), y(t), f(x(t), y(t)))$ is an orientation preserving parameterization of the simple closed curve δS , then

$$\int\limits_{\partial S} F \cdot ds = \int\limits_{\eta} F_1 dx + F_2 dy + F_3 dz = \int_a^b \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t} \right) dt. \tag{10.3.4}$$

Proof. Recalling (8.1.3) , the curl of F is defined as

$$\begin{aligned} \mathrm{Curl}(\vec{F}) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k} \\ \implies \int\limits_S \mathrm{Curl} F \cdot ds &= \int\limits_D \mathrm{Curl} F \cdot (\vec{T}_u \times \vec{T}_v)\, dA, \end{aligned}$$

where $\vec{T}_u \times \vec{T}_v = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right)$.

$$\therefore \int\limits_S \mathrm{Curl} F \cdot ds = \int\limits_D \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \left(-\frac{\partial z}{\partial x} \right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \left(-\frac{\partial z}{\partial y} \right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \quad (\star).$$

Recalling that z is a function of x and y implies that

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} \\ \implies \int\limits_{\partial S} F \cdot ds &= \int_a^b \left(F_1 + F_3 \frac{\partial z}{\partial x} \right) \frac{\mathrm{d}x}{\mathrm{d}t} + \left(F_2 + F_3 \frac{\partial z}{\partial y} \right) \frac{\mathrm{d}y}{\mathrm{d}t}\, dt \end{aligned}$$

$$= \int_a^b \underbrace{\left(F_1 + F_3 \frac{\partial z}{\partial x} \right)}_{P} dx + \underbrace{\left(F_2 + F_3 \frac{\partial z}{\partial y} \right)}_{Q} dy \implies \int\limits_C P dx + Q dy = \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA,$$

where C is the boundary of D.

Now recall that $F = \langle F_1(x, y, z(x, y)), F_2(x, y, z(x, y)), F_3(x, y, z(x, y)) \rangle$, thus

$$\begin{aligned} \frac{\partial Q}{\partial x} &= \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} \frac{\partial z}{\partial x} \right) + \left(\frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial z} \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial y} + \left(F_3 \frac{\partial^2 z}{\partial x \partial y} \right) \\ \frac{\partial P}{\partial y} &= \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial y} \right) + \left(\frac{\partial F_3}{\partial y} + \frac{\partial F_3}{\partial z} \frac{\partial z}{\partial y} \right) \frac{\partial z}{\partial x} + \left(F_3 \frac{\partial^2 z}{\partial x \partial y} \right) \\ \implies \int\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA &= \int\limits_D \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y} \right) \frac{\partial z}{\partial x} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = (\star). \end{aligned}$$

□

Example 10.9. Use Stoke's theorem to evaluate the integral

$$\int_C -y^3 dx + x^3 dy - z^3 dz,$$

where C is the intersection of $x^2 + y^2 = 1$(cylinder) and the plane $x + y + z = 1$.
Note that the surface we are integrating is the plane on the domain $x^2 + y^2 \leq 1$. Stoke's theorem says that

$$\int_{\partial S} F \cdot ds = \int_D \overbrace{\text{Curl}F}^{\text{Vector field}} \cdot \underbrace{(\vec{T}_u \times \vec{T}_v)}_{\text{Given by the plane}} \; dA.$$

So first and foremost we compute the curl.

$$\text{Curl}F = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^2 & z^2 \end{vmatrix} = 0\vec{i} - 0\vec{j} + (3x^2 + 3y^2)\vec{k} \implies \text{Curl } F = (0, 0, 3(x^2 + y^2)).$$

Now for $\vec{T}_u \times \vec{T}_v$ we have the surface $z(x, y) = 1 - x - y$

$$\therefore \int_D \text{Curl } F \cdot (\vec{T}_u \times \vec{T}_v) \; dA = \int_S \text{Curl } F \cdot \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) \; dA = \int_D 3x^2 + 3y^2 \; dA$$

In polar coordinates , $D = \{(r, \theta) | 0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi\}$ therefore

$$= 3 \int_0^{2\pi} \int_0^1 r^2 r \; dr d\theta = \frac{3\pi}{2}.$$

Stoke's theorem also holds on parameterized surfaces given by $\Phi : D \subset \mathbb{R}^2 \to S$. The orientation along δS is induced by the orientation of th surface given by $\vec{n}$

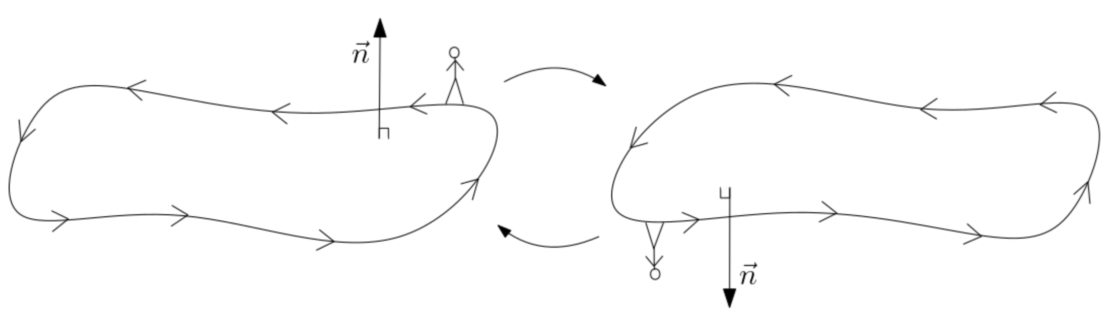


Figure 26: Both sub-figures are equivalent ways of determining if a given surface has an orientation along its boundary. Standing up in the same direction as the normal vector and walking along the surface being at our left hand , implies that surface's boundary is oriented.

Note. *If we're given a parameterized surface the first thing we do is compute and outward normal.*

Example 10.10. Use Stoke's theorem to evaluate $\int_S \text{Curl } F \cdot ds$, where $F = (z^2-1)\vec{i}+(z+xy^3)\vec{j}+6\vec{k}$ and S is given by $x = 6-4y^2-4z^2$ for $x \geq -2$. We first verify what is the orientation of the surface. Our parametrization here is $x = 6-4y^2-4z^2$, $y = y$ and $z = z$. Therefore

$$\vec{T}_y = (-8y, 1, 0) \quad , \vec{T}_z = (-8z, 0, 1) \quad , \vec{T}_y \times \vec{T}_z = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8y & 1 & 0 \\ -8z & 0 & 1 \end{vmatrix} = (1, 8y, 8z),$$

which means the normal vector is pointing out of the surface. Now since $-2 = 6-4y^2-4z^2$, $\implies 2 = y^2+z^2$. So at $x = 2$ the given surface has as boundary a circle of radius two , which can be trivially parameterized as

$$C(t) = (2, \sqrt{2}\sin(t), \sqrt{2}\cos(t)) \qquad , 0 \leq t \leq 2\pi.$$

Note that we knew a paraboloid has for its base a circle so it make sense to have trigonometric functions in the parameterization. This parameterization also represents the orientation of our boundary. Now by Stoke's Theorem (10.1), we know that

$$\int_S \text{Curl } F \cdot ds = \int_0^{2\pi} F \cdot ds.$$

Therefore since $\vec{F} = (z^2-1)\vec{i} + (z+xy^3)\vec{j} + 6\vec{k}$ and $C'(t) = (0, \sqrt{2}\cos(t), -\sqrt{2}\sin(t))$, we have

$$\begin{aligned} \int_0^{2\pi} F \cdot ds &= \int_0^{2\pi} F(C(t)) \cdot C'(t)\, dt \\ &= \int_0^{2\pi} 0 + (\sqrt{(2)}\cos(t) - 2(\sqrt{(2)}\sin(t))^3)\sqrt{2}\cos(t) - 6\sqrt{2}\sin(t)\, dt \\ &= \int_0^{2\pi} 2\cos^2(t) - 8\sin^3(t)\cos(t) - 6\sqrt{2}\sin(t)\, dt \\ &= \left(t + \frac{\sin(2t)}{2} - 2\sin^4(t) + 6\sqrt{2}\cos(t)\right)\Bigg|_0^{2\pi} = 2\pi. \end{aligned}$$

10.4 Divergence Theorem

The third of the big 3 theorems (Green's , Stokes, Divergence) is the Divergence Theorem which relates the integral of the divergence of a vector field over a solid region with a flux integral over the boundary surface(s). For example a sphere has one boundary surface while cylinders have multiple. Here we will consider solid regions E which we'll name *Simple solid regions*, where

$$\begin{aligned}E :=& \{(x,y,z) | (x,y) \in D_1, u_1(x,y) \leq z \leq u_2(x,y)\} \\ \text{or } & \{(x,y,z) | (x,z) \in D_2, v_1(x,z) \leq y \leq v_2(x,z)\} \\ \text{or } & \{(x,y,z) | (y,z) \in D_3, w_1(y,z) \leq x \leq w_2(y,z)\}.\end{aligned}$$

Note that these can be extended to more general solids.

Theorem 10.2. (The Divergence Theorem) *Let E be a simple solid region and let S be the boundary surface of E , given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partials on an open set containing E. Then,*

$$\iint_S F \cdot ds = \iiint_E divF\ dV. \tag{10.4.1}$$

Therefore the flux of F across the boundary equals the integral over the solid region of the divergence of F.

Note. *Basically, if one wants to know how much air is changing inside a ball, one can either look at how much air goes in and out by the boundary (LHS of* (10.4.1)*) or by looking at how much air is inside the ball and how is that quantity changing over time (RHS of* (10.4.1)*).*

Proof. Let $F = (P, Q, R) \implies \text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. Therefore,

$$\iiint_E \text{div}F\ dV = \iint_E \frac{\partial P}{\partial x}\ dV + \iint_E \frac{\partial Q}{\partial y}\ dV + \iint_E \frac{\partial R}{\partial z}\ dV$$

$$\text{and } \iint_S F \cdot ds = \iint_S F \cdot \vec{n}\ ds = \iint_S P\vec{i} \cdot \vec{n} ds + \iint_S Q\vec{i} \cdot \vec{n}\ ds + \iint_S R\vec{k} \cdot \vec{n}\ ds.$$

We prove that

$$\iint_S R\vec{k} \cdot \vec{n}\ ds = \iiint_E \frac{\partial P}{\partial z}\ dV.$$

Since E is a simple solid region this implies $E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$.Thus,

$$\iiint_R \frac{\partial R}{\partial z}\ dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}\ dz\right] dA = \iint_D R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))\ dA.$$

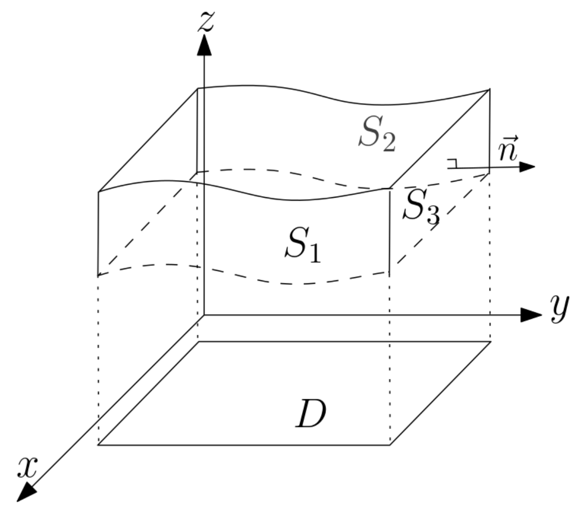


Figure 27: Decomposition of $\iint_S R\vec{k}\cdot\vec{n}ds$ into three regions ; S_1 is the bottom ; S_2 is the top ; S_3 is the sides.

We know S_1 is given by $z = u_1(x, y)$ and S_2 is given by $z = u_2(x, y)$ (since the object is simple solid region). What about S_3?

$$\iint_S R\vec{k}\cdot\vec{n}\,ds = \iint_{S_1} R\vec{k}\cdot\vec{n}\,ds + \iint_{S_2} R\vec{k}\cdot\vec{n}\,ds + \iint_{S_3} R\vec{k}\cdot\vec{n}\,ds$$

S_3 are are all vertical $\implies$ $\vec{n}$ = horizontal $\implies$ $\vec{k}\cdot\vec{n} = \vec{0}$, thus

$$\iint_{S_3} R\vec{k}\cdot\vec{n}\,ds = \vec{0}.$$

Therefore,

$$\begin{aligned}
\iint_{S_1} R\vec{k}\cdot\vec{n}\,ds &= \iint_{S_1} \frac{-R}{\sqrt{\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 + 1}}\,ds \\
&= \iint_D \frac{-R(x, y, u_1(x, y))}{\sqrt{\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 + 1}} \left(\sqrt{\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 + 1}\right) dA \\
&= \iint_{S_1} R\vec{k}\cdot\vec{n}\,ds = \iint_D -R(x, y, u_1(x, y))\,dA.
\end{aligned}$$

Similarly,

$$\iint_{S_2} R\vec{k}\cdot\vec{n}\,ds = \iint_D R(x, y, u_2(x, y))\,dA.$$

$$\therefore \iint\limits_S R\vec{k} \cdot \vec{n}\, ds = \iint\limits_D R(x, y, u(x, y)) - R(x, y, u_1(x, y))dA = \iiint\limits_E \frac{\partial R}{\partial z}\, dV.$$

□

Example 10.11. Consider $\vec{F} = (2x, y^2, z^2)$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Evaluate $\iint_S \vec{F} \cdot ds$ by using the divergence theorem. Recalling (10.2),

$$\iint\limits_S \vec{F} \cdot ds = \iiint\limits_E \mathrm{div} F\, dV.$$

In this case, $\mathrm{div}F = 2+2y+2z = 2(1+y+z)$. E is the unit ball so we will use spherical coordinates

$$\begin{aligned}\iiint\limits_E \mathrm{div}F\, dV &= 2\int_0^{\pi}\int_0^{2\pi}\int_0^{1}(1 + \rho\sin\varphi\sin\theta + \rho\cos\varphi)\rho^2\sin\varphi\, d\rho d\theta d\varphi \\ &= \frac{8\pi}{3} + \int_0^{\pi}\int_0^{2\pi}\int_0^{1}\rho^3\sin^2\varphi\sin\theta + \rho^3\sin\varphi\cos\varphi\, d\rho d\theta d\varphi\end{aligned}$$

Trick for the above integral, the RHS has no values of θ so it is equal to 0. The LHS has $\sin\theta \implies \cos\theta\big|_0^{2\pi} = 0 \implies$ all LHS is equal to 0.

$$\therefore \iiint\limits_E \mathrm{div}F\, dV = \frac{8\pi}{3}.$$

Example 10.12. Evaluate $\iint\limits_{\partial w} x^2 + y + z\, ds$, where w is the unit ball $x^2 + y^2 + z^2 \leq 1$. Recalling (10.2),

$$\iint\limits_{\partial w} \vec{F} \cdot ds = \iiint\limits_w \mathrm{div}F\, dV.$$

Notice this problem is special in that we're already given $\vec{F} \cdot ds$, therefore we must find $\vec{F}$ to apply (10.4.1).Let $\vec{n}$ be the unit outward normal for the sphere $\implies \vec{n} = (x, y, z)$. Since $\vec{F} \cdot ds = x^2 + y + z$,

$$\implies \vec{F} \cdot \vec{n} = F_1 x + F_2 y + F_3 z = x^2 + y + z \implies \vec{F} = (x, 1, 1) \implies \mathrm{div}F = 1.$$

Finally, we apply the definition

$$\iint\limits_{\partial w} x^2 + y + z\, ds = \iiint\limits_{\text{unit ball}} 1\, dV = \frac{4\pi}{3}.$$

Example 10.13. Evaluate $\iint\limits_S \vec{F} \cdot ds$, where $F(x, y, z) = (xy^2, x^2y, y)$ and S is the surface of the cylinder $x^2 + y^2 = 1$ bounded by $z = 1$ and $z = -1$.

Here we must include $x^2 + y^2 \leq 1$ on $z = 1$ and $z = -1$, giving

$$\begin{aligned}
\iint_S \vec{F} \cdot ds &= \iiint_E \mathrm{div} F \; dV \\
&= \int_{-1}^{1} \left[\iint_{x^2+y^2\leq 1} \mathrm{div} F \; dA \right] dz = \int_{-1}^{1} \left[\iint_{x^2+y^2\leq 1} x^2 + y^2 \; dA \right] dz
\end{aligned}$$

Let us use cylidrical coordinates to solve this, recalling (7.5) we get

$$= \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{1} r^2 r \; dr d\theta dz = (\dots) = \pi.$$