PHYS356 Assignment 7

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November 9, 2020

Question 1

Since

$$\hat{H} = \frac{2A}{\hbar^2} S_{1z} S_{2z} + S_{1x} S_{2x} + S_{1y} S_{2y} + \omega_0 S_{1z},$$

and

$$S_{1x} S_{2x} + S_{1y} S_{2y} = \frac{1}{2} \left(S_{1+} S_{2-} + S_{1-} S_{2-} \right),$$

we have

$$\begin{aligned}
\hat{H} |1\rangle &= \left(\frac{2A}{\hbar 2} \frac{\hbar^2}{4} + \omega_0 \frac{\hbar}{2} \right) |+z, +z\rangle = \left(\frac{A}{2} + \omega_0 \frac{\hbar}{2} \right) |+z, +z\rangle \\
\hat{H} |2\rangle &= \frac{2A}{\hbar^2} \frac{\hbar}{2} \frac{-\hbar}{2} |+z, +z\rangle + \frac{A}{\hbar^2} \hbar^2 |-z, +z\rangle +_0 \frac{\hbar}{2} = \frac{-A}{2} |2\rangle + A |3\rangle + \omega_2 \frac{\hbar}{2} |2\rangle \\
\hat{H} |3\rangle &= \frac{-A}{2} |3\rangle + A |2\rangle - \omega_0 \frac{\hbar}{2} |3\rangle \\
\hat{H} |4\rangle &= \left(\frac{A}{2} - \omega_0 \frac{\hbar}{2} \right) |+z, +z\rangle
\end{aligned}$$

So the Hamiltonian is

$$\hat{H} = \begin{bmatrix} \frac{A+\hbar\omega_0}{2} & 0 & 0 & 0 \\ 0 & \frac{-A+\omega_0\hbar}{2} & 0 & 0 \\ 0 & 0 & \frac{-A-\omega_0\hbar}{2} & 0 \\ 0 & 0 & 0 & \frac{A-\hbar\omega_0}{2} \end{bmatrix}.$$

We look for the energy eigenvalues.

$$\det(\hat{H}) = \left(\frac{A + \hbar\omega_0}{2} - E \right) \left(\frac{-A + \hbar\omega_0}{2} - E \right) \left[\left(\frac{-A - \hbar\omega_0}{2} - E \right) - A^2 \right] \left(\frac{A - \hbar\omega_0}{2} - E \right) = 0,$$

from the extremities we clearly see we have

$$E_1 = \frac{A+\hbar\omega_0}{2}; \qquad E_4 = \frac{A-\hbar\omega_0}{2}.$$

For the remaining eigenvalues we solve the middle two factors

$$\left(\frac{-A+\hbar\omega_0}{2} - E\right)\left(\frac{-A-\hbar\omega_0}{2} - E\right) - A^2 = 0$$
$$\implies EA - \frac{\omega_0^2\hbar^2}{4} + E^2 - 3\frac{A^2}{4} = 0$$
$$\implies E_\pm = \frac{-A \pm \sqrt{A^2 - 4\left(-\frac{\omega_0^2\hbar^2}{4} - 3\frac{A^2}{4}\right)}}{2}$$
$$\therefore E_\pm = -\frac{A}{2} \pm \sqrt{A^2 + \left(\frac{\omega_0\hbar}{2}\right)^2}$$

Now we look for the two limits with the Taylor expansions. We first note that the Taylor expansions for a square root is

$$\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{2!}\left(\frac{-1}{4}\right) + \dots.$$

So then in our case,

Case 1 : $A >> \hbar\omega_0$.

$$E_\pm = -\frac{A}{2} \pm A\sqrt{1+\left(\frac{\omega_0\hbar}{2A}\right)^2} = -\frac{A}{2} \pm A\left(1+\left(\frac{\omega_0\hbar}{2A}\right)^2\frac{1}{2}\right) = -\frac{A}{2} \pm A \pm \frac{\omega_0^2\hbar^2}{8A}$$
$$\therefore E_2 = \frac{A}{2} + \frac{\omega_0^2\hbar^2}{8A}; \qquad E_3 = -\frac{3A}{2} - \frac{\omega_0^2\hbar^2}{8A}.$$

Case 2: $A << \hbar\omega_0$.

$$-\frac{A}{2} \pm \sqrt{A^2 + \left(\frac{\omega_0\hbar}{2}\right)^2} = -\frac{A}{2} + \frac{\omega_0\hbar}{2}\left(1+\left(\frac{2A}{\omega_0\hbar}\right)^2\frac{1}{2}\right)$$
$$\therefore E_2 = -\frac{A}{2} + \frac{\omega_0\hbar}{2} + \frac{A^2}{\omega_0\hbar}; \qquad E_3 = -\frac{A}{2} - \frac{\omega_0\hbar}{2} - \frac{A^2}{\hbar\omega_0}$$

Question 2

$$|{+n}\rangle = \cos\frac{\theta}{2}\,|{+z}\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|{-z}\rangle \tag{1}$$

$$|-n\rangle = \sin\frac{\theta}{2}|+z\rangle - e^{i\varphi}\cos\frac{\theta}{2}|-z\rangle \tag{2}$$

We need to isolate $|\pm z\rangle$ in terms of $|\pm n\rangle$. To do so, we will multiply (1) by $\cos\theta/2$ and add to (2) multiplied by $\sin\theta/2$, the opposite operation is also performed to obtain the opposite sign $|z\rangle$; this gives us

$$\cos\frac{\theta}{2}|+n\rangle + \sin\frac{\theta}{2}|-n\rangle = |+z\rangle\,.$$

Similarly,

$$\sin\frac{\theta}{2}|+n\rangle - \cos\frac{\theta}{2}|-n\rangle = e^{i\varphi}|-z\rangle\,.$$

Now we compute each bracket individually.

$$\begin{aligned}|+z,-z\rangle &= \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi}|+n,+n\rangle - \cos^2\frac{\theta}{2}e^{i\varphi}|+n,-n\rangle\\ &\quad + \sin^2\frac{\theta}{2}e^{-i\varphi}|-n,+n\rangle - \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\varphi}|-n,-n\rangle\end{aligned}$$

$|+n,+n\rangle$ and $|-n,-n\rangle$ have the same energy eigenvalue so they vanish here

$$= -\cos^2\frac{\theta}{2}e^{-i\varphi}|+n,-n\rangle + \sin^2\frac{\theta}{2}e^{-i\varphi}|-n,+n\rangle$$

Then by symmetry, we omit some trivial computations ;

$$|-z,+z\rangle = \sin^2\frac{\theta}{2}e^{-i\varphi}|+n,-n\rangle - \cos^2\frac{\theta}{2}e^{-i\varphi}|-n,+n\rangle$$

Thus,

$$\begin{aligned}|0,0\rangle &= \frac{1}{\sqrt{2}}\bigg(-\cos^2\frac{\theta}{2}e^{-i\varphi}|+n,-n\rangle + \sin^2\frac{\theta}{2}e^{-i\varphi}|-n,+n\rangle\\ &\qquad - \sin^2\frac{\theta}{2}e^{-i\varphi}|+n,-n\rangle + \cos^2\frac{\theta}{2}e^{-i\varphi}|-n,+n\rangle\bigg)\\ &= \frac{1}{\sqrt{2}}\left(-e^{-i\varphi}|+n,-n\rangle + e^{i\varphi}|-n,+n\rangle\right)\\ \therefore\ |0,0\rangle &= \frac{e^{-i\varphi}}{\sqrt{2}}\left(|-n,+n\rangle - |+n,-n\rangle\right).\end{aligned}$$

Question 3

First and foremost we compute $|1,1\rangle_x$, $|1,0\rangle_x$ and $|1,-1\rangle_x$.

$$|1,1\rangle_z = |+z,+z\rangle \implies |1,1\rangle_x = \frac{1}{\sqrt{2}}\left(|+x\rangle_1 + |-x\rangle_1\right)\frac{1}{\sqrt{2}}\left(|+x\rangle_2 + |-x\rangle_2\right) = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}^T$$

$$|1,0\rangle_z = \frac{1}{\sqrt{2}}|+z,-z\rangle + \frac{1}{\sqrt{2}}|-z,+z\rangle \implies |1,0\rangle_x = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|+x\rangle_1 + |-x\rangle_1\right)\right]\left[\frac{1}{\sqrt{2}}\left(|+x\rangle_2 - |-x\rangle_2\right)\right]$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}^T$$

$$|1,-1\rangle_z = |-z,-z\rangle \implies |1,-1\rangle_x = \frac{1}{\sqrt{2}}\left(|+x\rangle_1 - |-x\rangle_1\right)\frac{1}{\sqrt{2}}\left(|+x\rangle_2 - |-x\rangle_2\right) = \frac{1}{2}\begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix}^T$$

Following this, since the particle goes through an SG device oriented along a different axis, we need to compute $\hat{S}_x$. We use the tensor product and the fact that $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}$. That is,

$$\left.\begin{aligned}\hat{S}_{1x} &= \hat{S}_1 \otimes \mathbb{1} = \tfrac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \tfrac{\hbar}{2}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\\ \hat{S}_{2x} &= \mathbb{1} \otimes \hat{S}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \tfrac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \tfrac{\hbar}{2}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\end{aligned}\right\} \implies \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}.$$

Then we compute $|\psi\rangle$

$$|\psi\rangle = \hat{S}_x |+z,+z\rangle_z = \frac{\hbar}{2}\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\ 1\\ 1\\ 0\end{pmatrix}.$$

Finally we compute each probability, we also remove the $\hbar$ constant from $|\psi\rangle$ since we're looking for probabilities, which must sum up to 1.

$$|_x\langle 1,1|\psi\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}0\\ 1\\ 1\\ 0,\end{pmatrix}\right|^2 = \frac{1}{4}$$

$$|_x\langle 1,0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}\frac{1}{2}\begin{pmatrix}0\\ 1\\ 1\\ 0\end{pmatrix}\right|^2 = 0,$$

$$|_x\langle 1,-1|\psi\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}0\\ 1\\ 1\\ 0\end{pmatrix}\right|^2 = \frac{1}{4}.$$

Question 4

a)

Since $\hat{H} = -\vec{\mu} \cdot \vec{B}$ and for a two particle system we can sum the Hamiltonians given the $\hat{\vec{S}}$ dot product , then considering the charge of a positron being the negative of an electron with identical mass,

$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \frac{ge}{2mc} B_0 \hat{S}_{1z} - \frac{ge}{2mc} B_0 \hat{S}_{2z} = \left(\frac{ge}{2mc} B_0\right)(\hat{S}_{1z} - \hat{S}_{2z}).$$

b)

We need the time operator. That is

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\int_0^t \hat{H}(t')\mathrm{d}t'} = e^{-\frac{i}{\hbar}\hat{H}t}.$$

We look for the stationary energies of $(\hat{S}_{1z} - \hat{S}_{2z})$.

$$\omega_0(\hat{S}_{1z} - \hat{S}_{2z}) = \omega_0 S_{1z} \left|+z,-z\right\rangle - \omega_0 S_{2z} \left|+z,-z\right\rangle = \omega_0 \frac{\hbar}{2} \left|+z,-z\right\rangle - \omega_0 \frac{-\hbar}{2} \left|+z,-z\right\rangle = \omega_0 \hbar \left|+z,-z\right\rangle .$$

$$\omega_0(\hat{S}_{1z} - \hat{S}_{2z}) \left|-z,+z\right\rangle = \omega_0 S_{1z} \left|-z,+z\right\rangle - \omega_0 S_{2z} \left|-z,+z\right\rangle = -\omega_0 \frac{\hbar}{2} \left|-z,+z\right\rangle - \omega_0 \frac{\hbar}{2} \left|-z,+z\right\rangle = -\omega_0 \hbar \left|-z,+z\right\rangle ,$$

we conclude that the eigen values are $\pm\hbar\omega_0$.So then,

$$\left|\psi(t)\right\rangle = \hat{U}(t)\left|\psi(0)\right\rangle = e^{-\frac{i}{\hbar}Ht}\left|\psi(0)\right\rangle = \frac{e^{-i\omega_0 t}}{\sqrt{2}}\left|+z,-z\right\rangle - \frac{e^{i\omega_0 t}}{\sqrt{2}}\left|-z,+z\right\rangle$$

$$\implies \left|\psi(t)\right\rangle = \frac{e^{-i\omega_0 t}}{\sqrt{2}}\left(\left|+z,-z\right\rangle - e^{2i\omega_0 t}\left|+z,-z\right\rangle\right). \tag{3}$$

Then we verify that the system oscillates between states $|0,0\rangle$ and $|1,0\rangle$;

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\left|+z,-z\right\rangle - \left|-z,+z\right\rangle\right)$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(\left|+z,-z\right\rangle + \left|-z,+z\right\rangle\right)$$

We note that in (3), for $t = 2\pi n/\omega_0$ we obtain $|0,0\rangle$ and for $t = n\pi/\omega_0$ for $n \in \mathbb{N}_{\text{odd}}$ we get $|1,0\rangle$. So indeed the system oscillates between these two spin states.

c)

We essentially require $|1,1\rangle_x$. We can take $|1,1\rangle_z = \left|+z,+z\right\rangle$ and convert it to

$$|1,1\rangle_x = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}^T,$$

just as it was done in Question 3. Then the probability is

$$|_x \langle 1,1|\psi(t)\rangle|^2 = \left|\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\ e^{-i\omega_0 t}\\ -e^{i\omega_0 t}\\ 0\end{pmatrix}\right|^2 = \left|\frac{1}{2\sqrt{2}}\left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)\right|^2 = \frac{1}{2}\sin^2(\omega_0 t).$$

Question 5

If we extend a 2−particle system to a 3-particle system total angular momentum is conserved so we may equate the lowering operator between the 2 and 3 particle system such as

$$\begin{aligned}
\hat{S}_-\left|\frac{3}{2},\frac{3}{2}\right\rangle &= \left(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\right)\left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle\\
\hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{3}{2}\left(\frac{3}{2}-1\right)}\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left[\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right]\\
\implies \left|\frac{3}{2},\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)
\end{aligned}$$

Then we apply $\hat{S}_-$ again on both sides and distribute the lower operator on each ket. We also note that for all quadrulplets the lowering operator square root factor is always $\sqrt{1}$ so we omit this factor.

$$\begin{aligned}
\hat{S}_-\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \left(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\right)\frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\
\sqrt{3}\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle+\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle\right.\\
&\qquad\left.+\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle+\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\
\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle+\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)
\end{aligned}$$

We apply $\hat{S}_-$ on both sides one last time ;

$$\begin{aligned}
\hat{S}_-\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \left(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\right)\frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle+\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\
\sqrt{3}\hbar\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \frac{3}{\sqrt{3}}\left|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\\
\implies \left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \left|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle
\end{aligned}$$

Here we reached the maximal lower quadruplet state, by symmetry it is evident that the $\left|\frac{3}{2},\frac{3}{2}\right\rangle$ must be $\left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$.

In conclusion, we have

$$
\begin{aligned}
\left|\frac{3}{2},\frac{3}{2}\right\rangle &= \left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle,\\
\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\
\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle + \left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\
\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \left|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle.
\end{aligned}
$$