PHYS241 Assignment 3.

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Question 1.

a)

$$V(t) = \Delta V_R + \Delta V_L = IR + \frac{dI}{dt}L.$$
$$V(t) = V_0\frac{t}{T} \implies \frac{V_0 t}{TL} = \frac{dI}{dt} + \frac{I}{\tau}$$

$$\begin{aligned}\frac{d}{dt}\left(Ie^{t/\tau}\right) &= \frac{dI}{dt}e^{t/\tau} + \frac{I}{\tau}e^{t/\tau}\\ &= e^{t/\tau}\left(\frac{dI}{dt} + \frac{I}{\tau}\right)\end{aligned}$$
$$\implies \frac{V_0 t e^{t/\tau}}{TL} = \frac{d}{dt}\left(Ie^{t/\tau}\right)$$

Now we may integrate to obtain I.

$$\int_0^t \frac{V_0 t}{TL}e^{t/\tau} = Ie^{t/\tau} - I(0)$$

Applying integration by parts with $u = t$ and $dv = e^{t/\tau}$ we get

$$\frac{V_0}{TL}\left(\tau t e^{t/\tau} - \int_0^t \tau e e^{t/\tau}\right) = \frac{\tau e^{t/\tau}V_0}{TL}(t-\tau)$$

The initial current $I(0) = I_0$ is obtained by the initial expression $V(t) = V_0\frac{t}{T} \implies I_0 = \frac{V_0 t}{RT}$. We may now formulate an expression for the current

$$I(t) = \frac{\tau e^{t/\tau}V_0(t-\tau)}{TLe^{t/\tau}} + \frac{V_0 t}{RTe^{t/\tau}}$$
$$I(t) = \frac{\tau V_0(t-\tau)}{TL} + \frac{V_0 t}{RTe^{t/\tau}}.$$

b)

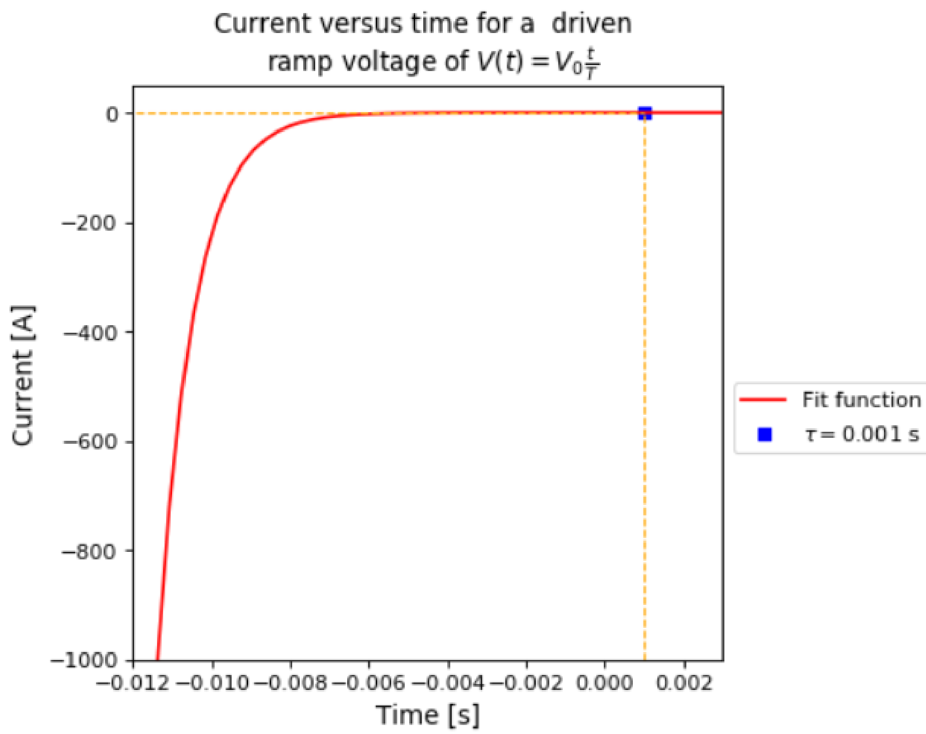

Figure 1: Visual representation of the current found in part a) along with indication for the transient time.

Question 2.

a)

$$\mathrm{Re}\{3 + j5\} = 3 \quad , \mathrm{Im}\{3 + j5\} = 5$$
$$|3 + 5j| = \sqrt{9 + 25} = \sqrt{34}$$
$$\tan \phi = \frac{5}{3} \implies \phi = \arctan\left(\frac{5}{3}\right).$$

b)

$$
\begin{gathered}
\mathrm{Re}\{2j\} = 0 \quad , \mathrm{Im}\{2j\} = 2 \\
|2j| = \sqrt{0+4} = 2 \\
\phi = \arctan\left(\frac{2}{0^+}\right) = \pi/2.
\end{gathered}
$$

c)

$$
\begin{gathered}
\frac{1}{2-3j} = \frac{2+3j}{2-3j} = \frac{2}{13} + \frac{3}{13}j. \\
\implies \mathrm{Re}\left\{\frac{1}{2-3j}\right\} = \frac{2}{13} \quad , \mathrm{Im}\left\{\frac{1}{2-3j}\right\} = \frac{3}{13}. \\
\left|\frac{1}{2-3j}\right| = \left|\frac{1}{2-3j}\frac{2+3j}{2-3j}\right| = \left|\frac{2+3j}{4+9}\right| = \frac{|2+3j|}{13} = \frac{1}{\sqrt{13}}. \\
\phi = \arctan\left(\frac{3}{2}\right)
\end{gathered}
$$

d)

$$
\begin{gathered}
3e^{j\frac{\pi}{4}} = |3|(\cos(\pi/4) + j\sin(\pi/4)) \\
\implies \mathrm{Re}\left\{3e^{j\frac{\pi}{4}}\right\} = 3\cos(\pi/4) \quad , \mathrm{Im}\left\{3e^{j\frac{\pi}{4}}\right\} = 3\sin(\pi/4). \\
\left|3e^{j\frac{\pi}{4}}\right| = 3 \\
\phi = \frac{\pi}{4}.
\end{gathered}
$$

Question 3.

$$
\begin{gathered}
\text{Assume } \ \widetilde{I}(t) = I_0 e^{j(\omega t+\phi_0)} \implies \widetilde{I}'(t) = I_0 j\omega e^{j(\omega t+\phi_0)} \\
\implies LI_0 j\omega e^{j(\omega t+\phi_0)} = \widetilde{V}_L \\
\implies Lj\omega\widetilde{I} = \widetilde{V}_L \implies \widetilde{I} = \frac{\widetilde{V}}{Lj\omega} \\
\therefore Z_C = Lj\omega.
\end{gathered}
$$

Question 4.

a)

The impedence for a capacitor and a resistor are respectively

$$Z_C = \frac{1}{j\omega C} \quad , Z_R = R.$$

We want the the driven sine wave to lower by one half i.e, $V'/V = 1/2$. We have

$$\frac{V'}{V} = \frac{1}{2} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega\tau}.$$

$$\begin{aligned}
\left|\frac{V_0'}{V_0}\right| &= \left(\frac{1}{1 + j\omega\tau}\right)\left(\frac{1}{1 - j\omega\tau}\right) = \frac{1}{1 + (\omega\tau)^2} \\
\implies \frac{V_0'}{V_0} &= \frac{1}{\sqrt{1 + (\omega\tau)^2}} \\
\therefore \frac{1}{2} &= \frac{1}{\sqrt{1 + (\omega\tau)^2}} \implies \tau = \frac{\sqrt{3}}{\omega} = \frac{\sqrt{3}T}{2\pi}.
\end{aligned}$$

b)

It was found in part a) that

$$\frac{V'}{V} = \frac{1}{1 + j\omega\tau}.$$

Let us convert this expression to extract a real and imaginary part,

$$\begin{aligned}
\frac{1}{1 + j\omega\tau} &= \frac{1}{1 + j\omega\tau}\left(\frac{1 - j\omega\tau}{1 - j\omega\tau}\right) = \frac{1 - j\omega\tau}{1 + (\tau\omega)^2} \\
\therefore \frac{1}{1 + j\omega\tau} &= \frac{1}{1 + (\omega\tau)^2} - j\frac{\omega\tau}{1 + (\omega\tau)^2}.
\end{aligned}$$

By definition ,

$$\phi = \arctan\frac{\mathrm{Im}\{\frac{V'}{V}\}}{\mathrm{Re}\{\frac{V'}{V}\}}$$

Substituting the previous result and whilst replacing ϕ by $-\pi/8$ we can solve for τ

$$\tan\left(\frac{-\pi}{9}\right)\left(\frac{1}{\omega}\right) = \tau.$$

Question 5.

a)

Let $V_0 \cos \omega t = V_0 e^{j\omega t}$.

$$\begin{aligned}\widetilde{I}\widetilde{I}^{\star} &= \left(\frac{V_0 e^{j\omega t}}{R + j\omega L}\right)\left(\frac{V_0 e^{-j\omega t}}{R - j\omega L}\right) = \frac{V_0^2}{R^2 - j^2(\omega L)^2} = \frac{V_0^2}{R^2 + (\omega L)^2}. \\ &= \frac{V_0^2/R^2}{1+(\omega\tau)^2} = \left(\frac{V_0^2}{R}\right)\frac{1}{1+(\omega\tau)^2} \implies \left|\widetilde{I}\right| = \frac{1}{\sqrt{1+(\omega\tau)^2}}.\end{aligned}$$

Let us find the offset associated with this complex current.

$$\frac{V_0 e^{j\omega t}}{R + j\omega L} = \frac{V_0}{R}\frac{e^{j\omega t}}{1 + j\omega\tau} = \frac{V_0}{R}\underbrace{\left(\frac{1}{1+j\omega\tau}\right)}_{=Z} e^{j\omega t}$$

$$\frac{1}{1+j\omega\tau}\left(\frac{1-j\omega\tau}{1+j\omega\tau}\right) = \frac{1-j\omega\tau}{1+(\omega\tau)^2} = \frac{1}{1+(\omega\tau)^2} - j\frac{\omega\tau}{1+(\omega\tau)^2}$$
$$\implies \phi = \arctan(\omega\tau)$$

Finally,

$$\widetilde{I} = \frac{V_0}{R}\frac{e^{j(\omega t+\phi)}}{\sqrt{1+(\tau\omega)^2}} \implies I(t) = \frac{V_0}{R}\frac{\cos(\omega t+\phi)}{\sqrt{1+(\omega\tau)^2}}$$

b)

$$\frac{\widetilde{V}'}{\widetilde{V}} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega\tau}{1+j\omega\tau}$$

Multiplying by the conjugate both sides we get

$$\frac{\widetilde{V}'}{\widetilde{V}} = \frac{(\omega\tau)^2}{1+(\omega\tau)^2} \implies \widetilde{V}' = \frac{V_0 e^{j\omega t}(\omega\tau)^2}{1+(\omega\tau)^2}.$$

Let us find the offset phase ,

$$\frac{\widetilde{V}'}{\widetilde{V}} = \frac{j\omega\tau}{1+j\omega\tau} = \frac{1}{1+\frac{1}{j\omega\tau}}$$

$$\implies \left(\frac{1}{1-\frac{j}{\omega\tau}}\right)\left(\frac{1+\frac{j}{\omega\tau}}{1+\frac{j}{\omega\tau}}\right) = \frac{(\omega\tau)^2}{1+(\omega\tau)^2} + j\frac{\omega\tau}{1+(\omega\tau)^2} \implies \phi = \frac{1}{\omega\tau}.$$

$$\text{Finally,} \quad \widetilde{V}' = \frac{V_0 e^{j\omega t}(\omega\tau)^2}{1+(\omega\tau)^2}$$

$$\therefore \frac{V_0 \cos\left(\omega\tau + \frac{1}{\omega\tau}\right)}{1+(\omega\tau)^2} = V'.$$

c)

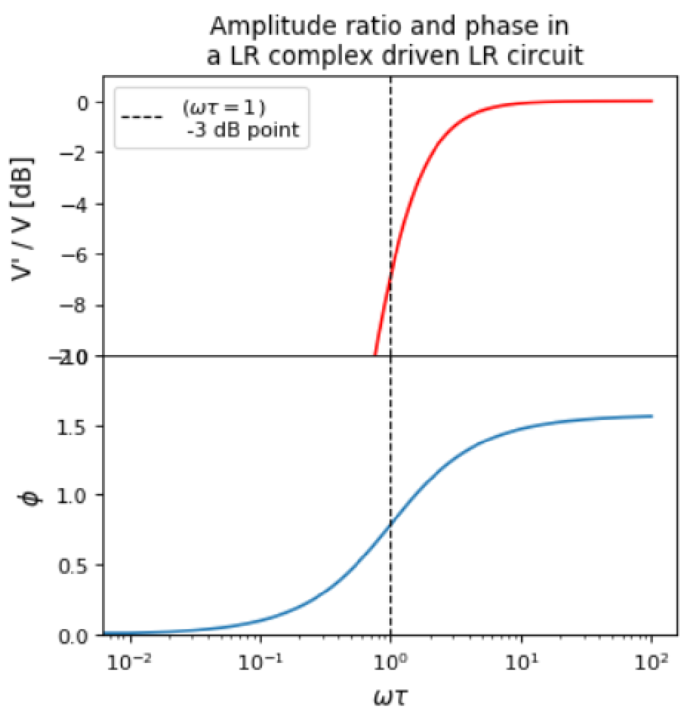

Figure 2: Visual representation of the ratio between the V' potential and V, along with the evolution of ϕ with respect to the independent variable $\omega\tau$.