MATH 475 Weekly Work 3

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October 29, 2020

Question 1

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \Gamma_k(x-y,t)g(y)\ dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} e^{-y}\ dy \end{aligned}$$

Next we complete the square in the exponential of $\exp\left(\frac{-(x-y)^2}{4kt} - y\right)$ with respect to the y variable such that the denominator is a function of $4kt$ and there is a left-over term not involving y. From

$$\begin{aligned} (y+2kt-x)^2 &= \frac{y4kt + x^2 - 2xy + y^2}{4kt} - x + kt \\ \implies -(y+2kt-x)^2 &= \frac{-y4kt - x^2 + 2xy - y^2}{4kt} - x + kt \end{aligned}$$

So we add $-x+kt$, essentially the square is completed by

$$\frac{-(y+2kt-x)^2}{4kt} - x + kt.$$

We continue evaluating the integral by setting $p = \frac{-(y+2kt-x)^2}{4kt} - x + kt \implies dp = \frac{dy}{\sqrt{4kt}}$

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-p^2} e^{kt-x} dp \sqrt{4kt} \\ &= \frac{e^{kt-x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\ dp \end{aligned}$$

This is the gaussian function solved commonly in polar coordinates

$$u(x,t) = e^{kt-x}.$$

Question 2

Let $\bar{u}(x,t) \equiv u(x,t) - 1$ such that now the boundary condition is 0. Then

$$g_{\text{odd}} = \begin{cases} -1 & x \quad \geq 0 \\ 1 \quad x & < 0 \end{cases}$$

$$\begin{aligned}
\bar{u}(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g_{\text{odd}} \\
&= \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} (1) - \int_{\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} (1)
\end{aligned}$$

Let $-y = y'$ in the second integral ,then

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} \, dy + \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x+y')^2}{4kt}} \, dy'$$

Let $r = x - y/\sqrt{4kt} \implies dr = dy/\sqrt{4kt}$ and $r = x + y'/\sqrt{4kt} \implies dr = dy'/\sqrt{4kt}$

$$\begin{aligned}
&= \int_{-\infty}^{x/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-r^2} \, dr + \int_{-\infty}^{x/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-r^2} \, dr \\
&= 2F\left(\frac{x}{\sqrt{4kt}}\right).
\end{aligned}$$

Now since $\bar{u}(x,t) = u(x,t) - 1 \implies u(x,t) = \bar{u}(x,t) + 1 \equiv F'(y) + 1$. Then since $1 - F(y) = F'(y) \implies 2 - F(y) = F'(y) + 1$ we conclude

$$u(x,t) = 2 - 2F\left(\frac{x}{\sqrt{4kt}}\right).$$

Question 3

a)

$$\begin{aligned}
f'_{-}(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\
f'_{+}(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}
\end{aligned}$$

Letting the right derivative be $f(-h)$ since the function is even , equating we get

$$f(-h) = f(h) \implies f'(h) = 0 \quad \text{as } h \text{ approaches } 0,$$

hence $u_x(0,t) = 0$.We chose $g(y)$ to be even instead of odd since the derivative of an even function is an odd function. Let

$$g_{\text{even}} = \begin{cases} g(x) & x \quad \geq 0 \\ g(-x) & x \quad < 0 \end{cases}$$

Therefore,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g_{\text{even}}\ dy \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g(y)\ dy + \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g(-y)\ dy \end{aligned}$$

Let $y = -y$ in the second integral , we get

$$\begin{aligned} &= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g(y)\ dy - \int_{\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x+y')^2}{4kt}} g(y')\ dy' \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} g(y)\ dy + \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x+y')^2}{4kt}} g(y')\ dy' \end{aligned}$$

Letting $y = y'$, we end up

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}} \right) g(y)\ dy$$

b)

Since $g(x) = 1$ then we have

$$g_{\text{even}} = \begin{cases} 1 & x \quad \geq 0 \\ 1 & x \quad < 0 \end{cases}$$

Therefore,

$$u(x,t) = \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} \left(e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}} \right)\ dy$$

Letting $r = x - y/\sqrt{4kt} \implies dr = -dy/\sqrt{4kt}$ in the left integral and $r = x + y/\sqrt{4kt} \implies dr = dy/\sqrt{4kt}$ in the right integral we get

$$= -\int_{x/\sqrt{4kt}}^{-\infty} \frac{1}{\sqrt{\pi}} e^{-r^2}\ dr + \int_{x/\sqrt{4kt}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-r^2}\ dr$$

Letting $r \to -r$ in the second integral , we conclude

$$\begin{aligned} &= \int_{x/\sqrt{4kt}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-r^2}\ dr + \int_{x/\sqrt{4kt}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-r^2}\ dr \\ &= \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-r^2}\ dr. \end{aligned}$$