MATH 325 Ass 4.

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Question 1

a)

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p \ dt$$

Let $r = st \implies dt = dr/s$ such that

$$\begin{aligned} &= \int_0^\infty e^{-r} \left(\frac{r}{s}\right)^p \\ &= \int_0^\infty \frac{e^{-r} r^p}{s^p} \frac{dr}{s} = \frac{1}{s^{p+1}} \int_0^\infty e^{-r} r^p \ dr \end{aligned}$$

Since $\Gamma(p+1) \equiv \int_0^\infty e^{-r} r^p dr$,

$$\implies \mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0.$$

b)

Let us prove the claim using mathematical induction. Let us assume that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$.

- Base case $(n = 0)$:

$$\mathcal{L}[t^0] = \mathcal{L}[1] = \frac{1}{s} = \frac{0!}{s^{0+1}} \qquad , s > 0 \quad \checkmark.$$

- Inductive step $(n = n + 1)$:

$$\mathcal{L}[t^{n+1}] = \int_0^\infty t^{n+1} e^{-st} \ dt$$

Applying integration by parts with $u = t^{n+1}$ and $dv = e^{-st}$ yields

$$\begin{aligned}
&= \frac{e^{-st}}{-s} t^{n+1}\Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}(n+1)t^n \; dt \\
&= (0-0) + \left(\frac{n+1}{s}\right)\int_0^\infty e^{-st}t^n \; dt = \left(\frac{n+1}{s}\right)\mathcal{L}[t^n] \\
\mathcal{L}[t^{n+1}] &= \frac{n+1}{s}\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}} \qquad , s > 0 \quad \checkmark .
\end{aligned}$$

We conclude that the initial assumption is true since both cases are satisfied.

c)

$$\mathcal{L}[t^{-1/2}] = \int_0^\infty e^{-st}t^{-1/2} \; dt.$$

Letting $u = st \implies du/s = dt$, therefore

$$= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^{-1/2}\frac{du}{s} = \frac{1}{\sqrt{s}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \; du. \quad , s > 0$$

Letting $\omega = \sqrt{u} \implies \omega^2 = u$ and $du = 2\omega d\omega$ such that

$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-\omega^2} \; d\omega \quad s > 0.$$

We'll use Fubini's theorem to solve the above integrsal ;

$$\frac{2}{\sqrt{s}}\int_0^\infty e^{-\omega^2} \; d\omega = \frac{2}{\sqrt{s}}\left(\frac{1}{2}\right)\int_{-\infty}^\infty e^{-\omega^2} \; d\omega = \frac{1}{\sqrt{s}}\left(\iint_{\mathbb{R}^2} e^{-(x+y)^2} \; dA\right)^{-1/2} .$$

Using polar coordinates ,

$$= \frac{1}{\sqrt{s}}\left(\int_0^\infty \int_0^{2\pi} e^{-r^2} r \; dr d\theta\right)^{-1/2} .$$

Applying integration by parts once yields

$$= \frac{1}{\sqrt{s}}\left(-\pi \; e^{-w^2}\Big|_0^\infty\right)^{-1/2} = \frac{1}{\sqrt{s}}\sqrt{\pi} \quad , s > 0$$

$$\therefore \mathcal{L}[t^{-1/2}] = \sqrt{\frac{\pi}{s}} \quad , s > 0.$$

d)

$$\mathcal{L}[\sqrt{t}] = \int_0^\infty e^{-st} t^{1/2} \ dt$$

Letting $u = st \implies dt = du/s$,

$$= \int_0^\infty \frac{e^{-u} u^{1/2}}{s^{3/2}} \ du = \frac{1}{s^{3/2}} \mathcal{L}[u^{1/2}] \quad , s > 0.$$

Since $\mathcal{L}[u^{1/2}] = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \ du = \int_0^\infty e^{-w^2} \ dw = \sqrt{\pi}/2$,

$$\mathcal{L}[\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} \qquad , s > 0.$$

Question 2

a)

$$\begin{aligned}
\mathcal{L}[\sin t] &= \mathcal{L}[t] - \frac{\mathcal{L}[t^3]}{3!} + \frac{\mathcal{L}[t^5]}{5!} - \dots \\
&= \frac{1}{s^2} - \left(\frac{3!}{3!}\right)\left(\frac{1}{s^4}\right) + \left(\frac{5!}{5!}\right)\left(\frac{1}{s^6}\right) - \dots \\
&= \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \dots \\
&= \left(\frac{1}{s^2}\right) \sum_{n=0}^\infty \left(\frac{-1}{s^2}\right)^n = \frac{1}{s^2}\left(\frac{1}{1+\frac{1}{s^2}}\right) = \frac{1}{s^2+1}, \quad s > 1.
\end{aligned}$$

b)

$$(\sin(t))\left(\frac{1}{t}\right) = \sin(t)t^{-1} = \sum_{n=0}^\infty \frac{(-1)^n t^{2n+1}}{(2n+1)!}(t^{-1}) = \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{(2n+1)!} \equiv f$$

$$\begin{aligned}
\mathcal{L}[f] &= \mathcal{L}[1] - \frac{\mathcal{L}[t^2]}{3!} + \frac{\mathcal{L}[t^4]}{5!} - \dots \\
&= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \dots = \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{s}\right)^{2n+1}}{2n+1} = \arctan(1/s) \quad , s > 0
\end{aligned}$$

Question 3

a)

Let us first compute the Laplace transform of $f(t)$.

$$\mathcal{L}[f(t)] = \int_0^{10} e^{-st}(1)\ dt + \int_{10}^{\infty} e^{-st}(0)\ dt.$$

Since there's no area under $\int_0^\infty 0\ dt$,

$$= \left.\frac{e^{-st}}{-s}\right|_0^{10} + 0 = \frac{-e^{-10s}}{s} + \frac{1}{s} = \frac{1-e^{-10s}}{s} \quad , s > 0.$$

We may now solve the IVP,

$$\begin{aligned}
\mathcal{L}[y'' + 3y' + 2] &= \mathcal{L}[f] \\
s^2Y(s) + 3sY(s) + 2Y(s) &= \frac{1-e^{-10s}}{s} \\
\implies Y(s) &= \frac{1-e^{-10s}}{s(s^2+3s+2)} \\
&= \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)} \\
y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] - \left(\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]\right) \\
&= 1 * \left(\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right) \\
&\quad - \left(u_{10}(t) * \left(\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right)\right) \\
&= \int_0^{\tau} (e^{-\tau} - e^{-2\tau})\ d\tau - \left(g(t-10)\int_0^{\tau} (e^{-\tau} - e^{-2\tau})\ d\tau\right)
\end{aligned}$$

Solving the LHS and RHS integral yields

$$y(t) = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} - \left(\frac{1}{2} - e^{-(t-10)} + \frac{e^{-2(t-10)}}{2}\right) u_{10}(t)$$

b)

$$\begin{aligned}
\mathcal{L}[y'' + y' + \frac{5y}{4}] &= \mathcal{L}[t - (t - \pi/2)u_{\pi/2}(t)] \\
Y(s)\left(s^2 + s + \frac{5}{4}\right) &= \frac{1}{s^2} - \frac{e^{\frac{-\pi s}{2}}}{s^2} \\
Y(s) &= \frac{1 - e^{\frac{-\pi s}{2}}}{s^2\left(s^2 + s + \frac{5}{4}\right)} \\
&= \frac{1}{s^2\left(\left(s + \frac{1}{2}\right)^2 + 1\right)} - \frac{e^{\frac{-\pi s}{2}}}{s^2\left(\left(s + \frac{1}{2}\right)^2 + 1\right)} \\
y(s) &= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{1}{\left(\left(s + \frac{1}{2}\right)^2 + 1\right)}\right] \\
&- \left(\mathcal{L}^{-1}\left[e^{\frac{-\pi s}{2}}\right] * \mathcal{L}^{-1}\left[\frac{1}{\left(\left(s + \frac{1}{2}\right)^2 + 1\right)}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\right) \\
&= t * e^{\frac{-t}{2}}\sin(t) - \left(u_{\pi/2}g(t - \pi/2)\right) \\
&= \int_0^t \tau e^{-\tau/2}\sin(\tau) - \left(u_{\pi/2}g(t - \pi/2)\right)
\end{aligned}$$

We find $g(t-\pi/2)$ by solving the LHS integral. Let us apply integration by parts with $u = \tau$ and $dv = e^{-\tau/2}\sin(t)$. We must apply another integration by parts to find v such that

$$\int_0^t e^{-\tau/2}\sin(t) = \frac{-2}{5}e^{-\tau/2}(\sin(\tau) + 2\cos(\tau))$$

then we apply the initial integration by parts,

$$y(t) = (4 + 2t)\frac{e^{\frac{t}{2}}\sin(t)}{5} + \frac{4t}{5}\cos(t) - \left(u_{\pi/2}g(t - \pi/2)\right)$$

We now express $g(t - \pi/2)$ given the previous result,

$$\begin{aligned}
&= (4 + 2t)\frac{e^{\frac{t}{2}}\sin(t)}{5} + \frac{4t}{5}\cos(t) - \left(4 + 2\left(t - \frac{\pi}{2}\right)\right)\frac{e^{\frac{\left(t - \frac{\pi}{2}\right)}{2}}\sin\left(\left(t - \frac{\pi}{2}\right)\right)}{5}u_{\pi/2}(t) \\
&\quad - \frac{4\left(t - \frac{\pi}{2}\right)}{5}\cos\left(t - \frac{\pi}{2}\right)u_{\pi/2}(t)
\end{aligned}$$

Question 4

a)

Let us multiply by $(s-r_k)$ on both sides yielding

$$\frac{A_1}{s-r_1}(s-r_k)+\cdots+\frac{A_n}{s-r_n}(s-r_k)=\frac{P(s)}{Q(s)}(s-r_k)$$

Since $Q'(s)=(s-r_k)Q(s)$,

$$=\frac{P(s)}{Q'(s)}$$

Taking the limit as s approaches r_k,

$$\lim_{s\to r_k}(s-r_k)\frac{P(s)}{Q(s)}=\frac{P(r_k)}{Q'(r_k)}=0+0+\cdots+\frac{A_k}{r_k-r_k}(r_k-r_k)+0+\cdots+0=A_k$$

b)

$$\begin{aligned}\mathcal{L}^{-1}\left[F(s)\right]=\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right]&=A_1\mathcal{L}^{-1}\left[\frac{1}{s-r_1}\right]+\cdots+A_n\mathcal{L}^{-1}\left[\frac{1}{s-r_n}\right]\\&=A_1e^{r_1t}+\cdots+A_ne^{r_nt}=\sum_{k=1}^{n}A_ke^{r_kt}=\sum_{k=0}^{n}\frac{P(r_k)}{Q'(r_k)}e^{r_kt}.\end{aligned}$$

Question 5

a)

$$\begin{aligned}\mathcal{L}[y''+ty]&=\mathcal{L}[0]\\ s\mathcal{L}[y']-y'(0)-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[y]&=0\\ s^2Y(s)-s-Y'(s)&=0\implies Y'(s)-s^2Y(s)=-s\end{aligned}$$

b)

$$\begin{aligned}\mathcal{L}[(1-t^2)y''-2ty'+\alpha(\alpha+1)y]&=\mathcal{L}[0]\\ \mathcal{L}[y'']-\mathcal{L}[t^2y'']-2\mathcal{L}[ty']+\alpha(\alpha+1)\mathcal{L}[y]&=0\\ s^2Y(s)-1-\frac{\mathrm{d}^2(s^2Y(s)-1)}{\mathrm{d}s^2}+2\frac{\mathrm{d}(sY(s))}{\mathrm{d}s}+\alpha(\alpha+1)Y(s)&=0\end{aligned}$$

Using the product rule on both terms and recollecting everything yields

$$\begin{aligned}s^2Y(s)-1-s^2Y''(s)-Y'(s)4s-Y(s)2+2Y(s)+2sY'(s)+\alpha(\alpha+1)Y(s)&=0\\ \therefore s^2Y''(s)+2sY'(s)-(s^2+\alpha(\alpha+1))Y(s)&=-1\end{aligned}$$

Question 6

$$
\begin{aligned}
\mathcal{L}[y'] + \mathcal{L}[t] &= \frac{1}{2}\mathcal{L}[t^2]\mathcal{L}[y] \\
sY(s) - 1 + \frac{1}{s^2} &= \frac{Y(s)}{s^3} \\
\therefore Y(s) &= \frac{1-\frac{1}{s^2}}{s-\frac{1}{s^3}} = \frac{s(s^2-1)}{s^4-1} = \frac{s}{s^2+1} \quad , s>0 \\
y(t) &= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos(t).
\end{aligned}
$$

Question 7

a)

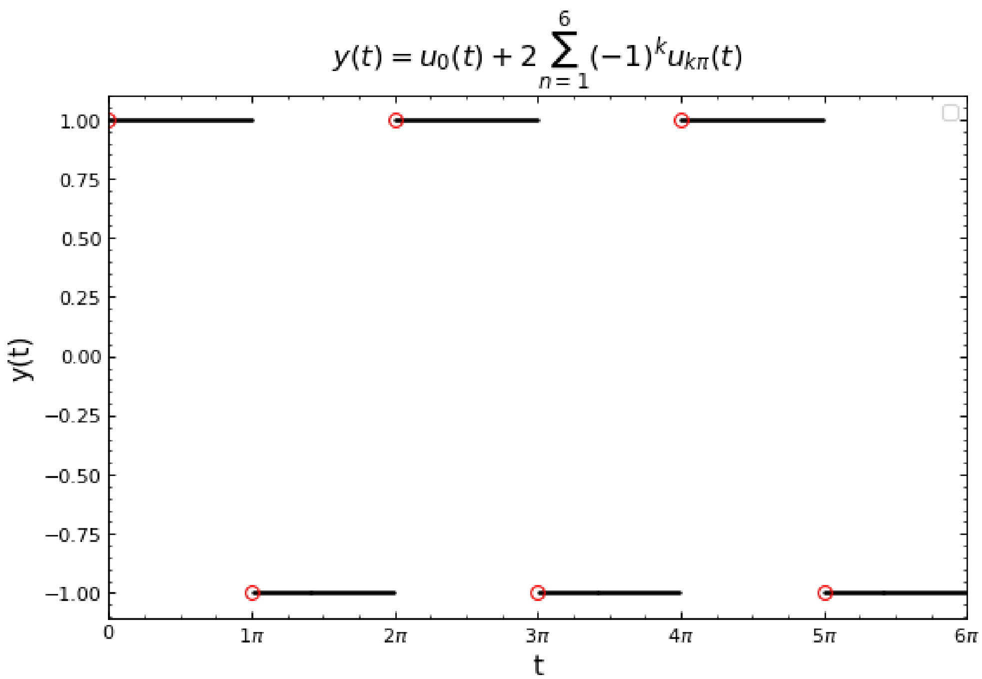

Figure 1: Graph of $f(t)$ for $0 \le t \le 6\pi$.

b)

$$\begin{aligned}
\mathcal{L}[y''] + \mathcal{L}[y] &= \mathcal{L}[u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)] \\
s^2Y(s) + Y(s) &= \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \mathcal{L}[u_{k\pi}(t)] \\
Y(s)(s^2+1) &= \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \frac{e^{-k\pi s}}{s} \\
Y(s) &= \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k \frac{e^{-k\pi s}}{s(s^2+1)} \quad , s > 0.
\end{aligned}$$

Using partial fraction decomposition on $1/(s(s^2+1))$ yields

$$\begin{aligned}
&= \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \\
\therefore y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \left[\frac{s}{s^2+1}\right] + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}\mathcal{L}^{-1}[g(t-k\pi)] \\
&= 1 - \cos(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)(1-\cos(t-k\pi)).
\end{aligned}$$

c)

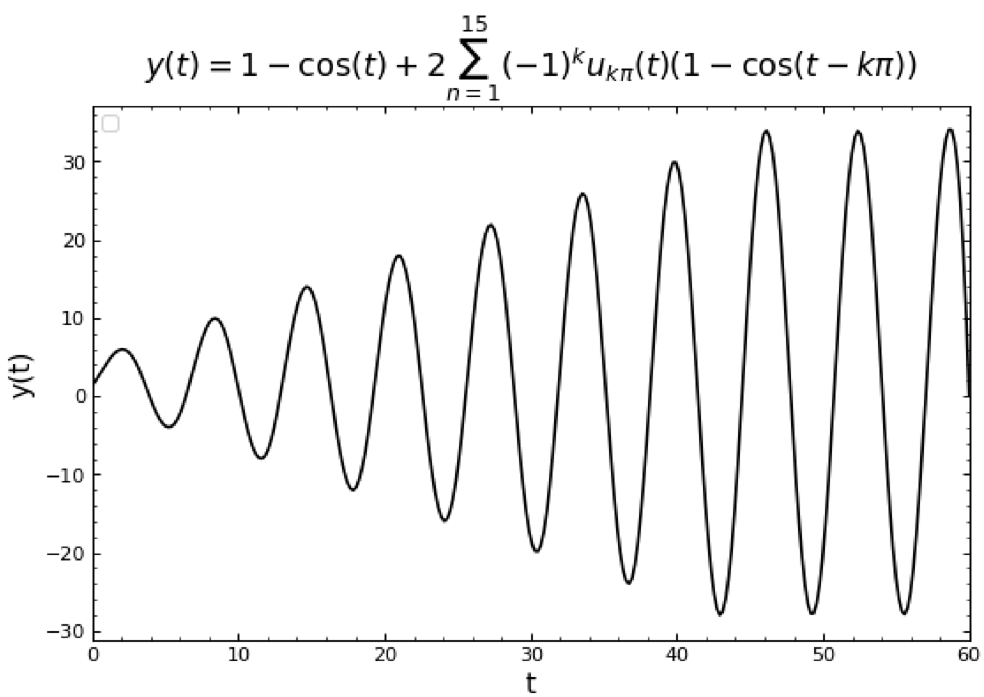

Figure 2: Graph of the solution $y(t)$ for $t \in [0, 60]$.

From Figure 2 we note that the amplitude increases steadily and scales with $y_A = 2y_{0,A}$. This amplitude change is due to the unit step function factor.The amplitude stops increasing at $t = 15\pi$ since the summation is defined up to $n = 15$, hence no additional terms are added to the $y(t_{n\geq 15})$.Following this previous remark, we note the solution reaches a steady state at approximately at $t = 50$. Moreover, the function is sinusoidal because of the cosine terms inside the equation.

d)

As n increases,only the critical point at which the amplitude stops increasing increases linearly with n, such that the function's aspect remains identical $\forall n$. As n approaches infinity , $y(t)$ diverges as the sum is alternating series with $b_{n+1} > b_n,\ \forall n\ \in \mathbb{N}$, but nevertheless the steady state solution does not change.

Question 8

$$
\begin{aligned}
\mathcal{L}[2y'' + y' + 4y] &= \mathcal{L}[\delta(t - \pi/6)\sin(t)] \\
2\mathcal{L}[y''] + \mathcal{L}[y'] + 4\mathcal{L}[y] &= \sin(-\pi/6)\mathcal{L}[\delta(t - \pi/6)] \\
Y(s) &= \frac{\sin(-\pi/6)e^{\frac{-\pi s}{6}}}{2\left(s^2 + \frac{s}{2} + 2\right)} \\
&= \left(-\frac{1}{4}\right)\left(\left(\frac{1}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}}\right) + \left(\frac{e^{\frac{-s\pi}{6}}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}}\right)\right) \\
y(t) &= \left(-\frac{1}{4}\right)\left(\frac{8}{15}\mathcal{L}^{-1}\left[\frac{\frac{15}{8}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}}\right] + \frac{8}{15}\left(\mathcal{L}^{-1}\left[e^{\frac{-\pi s}{6}}\right] * \mathcal{L}^{-1}\left[\frac{\frac{15}{8}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}}\right]\right)\right) \\
&= \left(-\frac{1}{4}\right)\left(\frac{8}{15}\right)\left(e^{\dfrac{-t}{4}}\sin\left(\frac{15t}{8}\right) + \left(e^{\dfrac{-\left(t-\dfrac{\pi}{6}\right)}{4}}\sin\left(\frac{15(t-\frac{\pi}{6})}{8}\right)\right)u_{\pi/6}(t)\right)
\end{aligned}
$$