# MATH 327 Assignment 4

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## Question 1

### (a)

We find the upper triangular matrix $U$ and the three matrices $L_i^{-1}$.

$$
\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 0 & 3/5 & 0 & 1 \end{pmatrix} \\
\sim &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 1/2 & 1 \\ 0 & -3/2 & 7/4 & 1 \\ 0 & 3/5 & 0 & 1 \end{pmatrix} \\
\sim &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 3/10 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 17/8 & 7/4 \\ 0 & 0 & -3/20 & 7/10 \end{pmatrix} \\
\sim &\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L_1^{-1}}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 3/10 & 0 & 1 \end{pmatrix}}_{L_2^{-1}}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6/85 & 1 \end{pmatrix}}_{L_3^{-1}}\underbrace{\begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 17/8 & 7/4 \\ 0 & 0 & 0 & 14/17 \end{pmatrix}}_{U}
\end{aligned}
$$

It follows that $L_i = (L_i^{-1})^{-1}$

$$
L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 3/10 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3/4 & 1 & 0 \\ 0 & -3/10 & 0 & 1 \end{pmatrix}
$$

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6/85 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6/85 & 1 \end{pmatrix}$$

As a double-check,

$$L_1^{-1}L_2^{-1}L_3^{-1}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 3/10 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6/85 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 17/8 & 7/4 \\ 0 & 0 & 0 & 14/17 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 2 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 0 & 3/5 & 0 & 1 \end{pmatrix} = A\checkmark$$

**(b)**

$$L = L_1^{-1}L_2^{-1}L_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & -3/4 & 1 & 0 \\ 0 & -3/10 & -6/85 & 1 \end{pmatrix} \to \begin{aligned} y_1 &= b_1/a_{11} = 16/1 = 1 \\ y_2 &= (b_2 - a_{21}y_1)/a_{22} = (16+8)/1 = 24 \\ y_3 &= 30 \\ y_4 &= 1132/85 \end{aligned}$$

So $y = (1\ 24\ 30\ 1132/85)^T$.

**(c)**

$$U = L_3L_2L_1A = \begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 17/8 & 7/4 \\ 0 & 0 & 0 & 14/17 \end{pmatrix} \to \begin{aligned} z_4 &= y_4/a_{44} = (1132/85)(17/14) = 566/35 \\ z_3 &= (y_3 - a_{34}z_4)/a_{33} = 4/5 \\ z_2 &= 26/7 \\ z_1 &= -243/340 \end{aligned}$$

**(d)**

The solution $x$ to $Ax = b$ is the vector $z$ of $Uz = y$ ,

$$x = ({}^{566}/_{35}\ {}^{4}/_{5}\ {}^{26}/_{7}\ {}^{-243}/_{340})^T.$$

## Question 2

**(a)**

```
A = [25 4 0 1 ; 4 -15 -2 0 ; 0 -2 6 -1 ; 1 0 -1 3];

%%% a %%%

q0 = [1 1 1 1]';
norm(q0, 'inf')
q_list = [];
sigma_list = [];

for i=1:10
        if (norm(q0,'inf') ~= 1)
                q0 = q0/norm(q0, 'inf');
        end

        q_list{i} = (A*q0)/norm(A*q0,'inf');
        sigma_list{i} = norm(A*q0,'inf');
        q0 =q_list{i};
end
```

After 10 iterations over $q_0 = (1\ 1\ 1\ 1)^T$, we find

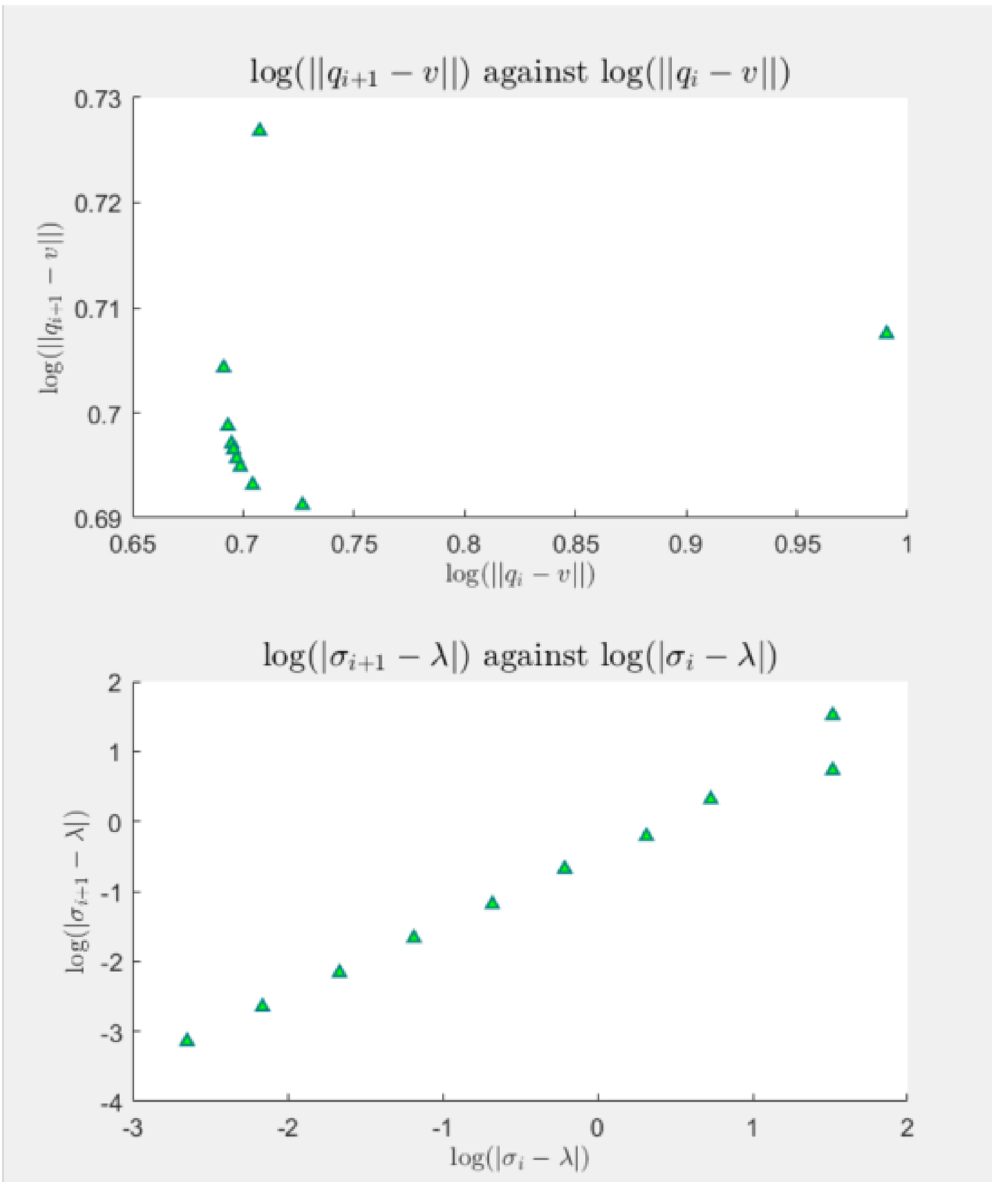
$$q_{10} = (1\ 0.106\ \ -0.012\ 0.0452)^T, \qquad \sigma_1 = 25.399$$

**(b)**

```
%%% b %%%

[V,D] = eig(A)
v = V(:,4)
lambda = D(4,4)

qy_list = [];
qx_list =[];
sigmay_list = [];
sigmax_list = [];
sigma_list = cell2mat(sigma_list);

sigma_list(5)
for i=1:10
        qy_list(i) = log(norm((q_list{i}-v),2));
        sigmay_list(i) = log(abs((sigma_list(i)-lambda)));
        if i==1
                q0 = [1 1 1 1]';
```

```
                    qx_list(i) = log(norm((q0-v),2));
                    sigmax_list(i) = log(abs(norm(A*q0, 'inf')
                       - lambda));
          else
                    qx_list(i) = log(norm((q_list{i-1}-v),2));
                    sigmax_list(i) = log(abs(sigma_list(i-1) -
                       lambda));
          end
end
```

Plotting the above arrays yields (code omitted for aesthetic purposes)



From the slope of $(ii)$, which is $\sim 4.8$, we deduce that the order of convergence is 5.

### (c)

As defined , the eigenvalues of $A - \rho I$ are $\lambda_i - \rho$. Moreover,

$$q_1 = \frac{Aq_0}{\|Aq_0\|_\infty} = q_{\text{list}}\{1\} \approx 30,$$

so given $\lambda_{\max} \approx \rho \Longrightarrow \lambda_1 - \rho = 30 - 23.99 \approx 6$.

## Question 3

### (a)

Let $\alpha = 5$ in $(A - \alpha I)^{-1}$, we use the $LU$ decomposition along with backward and forward substitution to get the matrix. We then apply the power method on $(A - \alpha I)^{-1}$ until we get the dominant eigenpair since

$$(A - \alpha I)^{-1} v = \frac{1}{\lambda - \alpha} v,$$

then the dominant eigenvalue found $\mu$, is equivalent to

$$\mu = \frac{1}{\lambda - \alpha} = \frac{1}{\lambda - 5},$$

since $(A - \alpha I)^{-1}$ and $A$ share the same eigenvector for that shift.

The algorithm converges when $|\lambda_{k-1} - 5| >> |\lambda_k - 5|$ and with ratio

$$\frac{|\lambda - 5|}{|\lambda_k - 5|},$$

where $\lambda_k - 5$ is the second smallest eigenvalue of $A - 5I$ in absolute value.

### (b)

By definition,

$$r(x) = \frac{x^T Ax}{x^T x} \in \mathbb{R}.$$

By theorem, a LLSP with normal equations is $A^T Az = A^T b$. Here we have,

$$r = \frac{x^T Ax}{x^T x} \Longrightarrow r x^T x = x^T Ax,$$

the symmetry between $r \leftrightarrow z$ , $x \leftrightarrow A$ and $Ax \leftrightarrow b$, suggests that $r x^T x = x^T Ax$ is a set of normal equations. Thence, $r$ solves its corresponding LLSP

$$r x^T x = x^T Ax \Longrightarrow \min_r \|Ax - xr\| \quad \forall x.$$

**(c)**

The difference is in that at each iteration we override the value of $\rho_j$ by $q_j^T A q_j / (q_j^T qj)$ (in the Rayleigh case) , while in the inverse power method iteration, the value $\rho$ remains unchanged and is defined once at the beginning of the iterations.

## Question 4

**(a)**

$$
\begin{aligned}
A &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \xrightarrow{\det(A-I\lambda)} \lambda^2 - 3\lambda + 2 = 0 && \Longrightarrow \lambda_1 = 1, \quad \lambda_2 = 2. \\
B &= \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \xrightarrow{\det(A-I\lambda)} \lambda^2 - 4\lambda = 0 && \Longrightarrow \lambda_1 = 4, \quad \lambda_2 = 0.
\end{aligned}
$$

**(b)**

The sub-routine *rayleigh˙quotient˙iteration.m* returns $\sim 2$ for the $11^{\text{th}}$ iteration on the array *rhos*, when replacing the sample-matrix by $A$ as defined in this problem. So one eigenvalue is $\lambda = 2$.

The sub-routine *rayleigh˙quotient˙iteration.m* returns $\sim 4$ for the $11^{\text{th}}$ iteration on the array *rhos*, when replacing the sample-matrix by $B$ as defined in this problem. So one eigenvalue is $\lambda = 4$.

**(c)**

The matrix $A$ converges to 2 after about 4 iterations with incremental precision increasing by about $\sim 1$ order of magnitude. Meanwhile, the matrix $B$ seems to bypass the convergence value of 4 after the second iteration and just approaches 0. This is due to the eigenvalue of 0 for the second matrix $B$ ; the convergence ratio $4/0$ is about the machine precision and so the algorithm is very imprecise and fails.

# Question 5

## (a)

We use the definition of convergence of a sequence and show it is equal to $\lambda_2/\lambda_1$. We also use the

$$\begin{aligned}
\lim_{j\to\infty} \frac{\left\| \frac{A^{j+1}q}{\lambda_1^{j+1}} - c_1 v_1 \right\|}{\left\| \frac{A^j q}{\lambda_1^j} - c_1 v_1 \right\|} &= \lim_{j\to\infty} \frac{\left\| \frac{\lambda_1^{j+1}\left(c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{j+1} v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{j+1} v_n\right)}{\lambda_1^{j+1}} - c_1 v_1 \right\|}{\left\| \frac{\lambda_1^{j}\left(c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{j} v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{j} v_n\right)}{\lambda_1^{j}} - c_1 v_1 \right\|} \\
&= \lim_{j\to\infty} \frac{\left\| c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{j+1} v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{j+1} v_n \right\|}{\left\| c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{j} v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{j} v_n \right\|} \\
&= \lim_{j\to\infty} \frac{\left\| \sum_{i=2}^n c_i \left(\frac{\lambda_2}{\lambda_1}\right)^{j+1} v_i \right\|}{\left\| \sum_{i=2}^n c_i \left(\frac{\lambda_2}{\lambda_1}\right)^{j} v_i \right\|} \\
&= \lim_{j\to\infty} \left\| \sum_{i=2}^n \frac{\lambda_2}{\lambda_1} \right\| \\
&= \frac{\lambda_2}{\lambda_1},
\end{aligned}$$

since $|\lambda_1| > |\lambda_2|$ then the convergence coefficient is bounded by 0 and 1 and so indeed the given sequence converges to $c_1 v_1$ with convergence ratio $\lambda_2/\lambda_1$.

## (b)

$$\begin{aligned}
q_{j+1} = \frac{Aq_j}{\sigma_{j+1}} \Longrightarrow \sigma_j &= \frac{Aq_{j-1}}{q_j} \\
\Longrightarrow \lim_{j\to\infty} \sigma_j &= \frac{\lim_{j\to\infty} Aq_{j-1}}{\lim_{j\to\infty} q_j} \\
&= \frac{\lim_{j\to\infty} Aq_{j-1}}{\alpha v_1}
\end{aligned}$$

Now since

$$q_j = \frac{A^j q}{\lambda_1^j} \Longrightarrow q_{j-1} = \frac{A^{j-1} q}{\lambda_1^{j-1}}$$

so then

$$\begin{aligned}
\lim_{j\to\infty} \sigma_j &= \frac{1}{\alpha v_1} \lim_{j\to\infty} A q_{j-1} \\
&= \frac{1}{\alpha v_1} \lim_{j\to\infty} \frac{A A^{j-1} q}{\lambda_1^{j-1}} \\
&= \frac{1}{\alpha v_1} \lim_{j\to\infty} \frac{A^j q}{\lambda_1^{j-1}} \\
&= \frac{1}{\alpha v_1} \lim_{j\to\infty} \frac{\left(\alpha \lambda^j v_1 + \cdots + c_n \lambda^j v_n\right)}{\lambda_1^{j-1}} \\
&= \frac{1}{\alpha v_1} \lim_{j\to\infty} \frac{\lambda_1^j \left(\alpha v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j v_2 \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^j v_n\right)}{\lambda_1^{j-1}}
\end{aligned}$$

Since $|\lambda_1| > |\lambda_i| \; \forall \; i > 1$, then all terms vanish as $j \to \infty$ ,except $\alpha v_1$

$$\begin{aligned}
&= \frac{1}{\alpha v_1} \lim_{j\to\infty} \frac{\lambda_1^j}{\lambda_1^{j-1}} (\alpha v_1) \\
&= \lambda
\end{aligned}$$