PHYS350 Assignment 6

Mihail Anghelici 260928404

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Question 1

a)

The charge inside the ${C_{60}}^{-}$ ion is negative so we use the method of images with $q' + q'' = q$. We then proceed with

$$F = qE \implies f = \frac{q}{4\pi\epsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right)$$

Since $q'' = q - q'$,

$$\begin{aligned}
&= \frac{q}{4\pi\epsilon_0}\left(\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2}\right)\\
&= \frac{q^2}{4\pi\epsilon_0} + \underbrace{\frac{qq'}{4\pi\epsilon_0}\left(\frac{-1}{a^2} + \frac{1}{(a-b)^2}\right)}_{\text{HW 4}}\\
&= \frac{q^2}{4\pi\epsilon_0 a^2} + \frac{q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}\\
&= \frac{q^2}{4\pi\epsilon_0}\left(a - R^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}\right).
\end{aligned}$$

We look for r at $F = 0$. At that force, $a(a^2 - R^2)^2 = R^3(2a^2 - R^2)$. Solving numerically we get

$$R_+ = \frac{1}{2}(\sqrt{5} - 1)a; \qquad R_- - \frac{1}{2}(1 + \sqrt{5})a.$$

We chose the positive root since the distance is a positive quantity in this setting.

$$\implies R = \frac{1}{2}(\sqrt{5}-1)a \implies a - \frac{2}{\sqrt{5}-1} = \frac{1}{2}(1+\sqrt{5}) \implies a \approxeq 5.663 \text{ Å}.$$

b)

Let $a = \gamma R$. By definition

$$W = -\int_\infty^a F \, \mathrm{d}a = \underbrace{\int_\gamma^\infty \frac{1}{4\pi\epsilon_0 R}\left(\frac{1}{\gamma^2}\right)\mathrm{d}\gamma}_{:=I_1} - \underbrace{\int_\gamma^\infty \frac{2(\gamma^2-1)}{\gamma^3(\gamma^2-1)^2}\mathrm{d}\gamma}_{:=I_2}.$$

We solve the two integrals ,I_1 is trivial. For I_2, let $u = \gamma^2$,

$$I_2 = -\frac{1}{2}\int \frac{2u-1}{(u-1)^2u^2}\mathrm{d}u = \frac{1}{2}\int \frac{1}{u^2}\mathrm{d}u - \frac{1}{2}\int \frac{1}{(u-1)^2}\mathrm{d}u = -\frac{1}{2u} - \frac{1}{2}\int \frac{1}{s^2}\mathrm{d}s = \frac{1}{2(u-1)u}.$$

$$\therefore I_2 = \frac{1}{2\gamma^2(\gamma^2-1)}.$$

We conclude

$$W = \frac{q^2}{4\pi\epsilon_0 R}\left(-\frac{1}{\gamma} + \frac{1}{2\gamma^2(\gamma^2-1)}\right) = \frac{q^2}{4\pi\epsilon_0 R}\frac{1-2\gamma^3+2\gamma}{2\gamma^2(\gamma^2-1)}.$$

Substituting $\gamma = (1+\sqrt{5}/2)$ and replacing with the appropriate constants,

$$W = \frac{q^2}{8\pi\epsilon_0 R} = \frac{(1.6\times10^{-19})^2}{8\pi(8.85\times10^{-12})(5.66\times10^{-10})^2} = 2.03\times10^{-14}\text{ J} = 1.27\text{ eV}.$$

Question 2

We show that $3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} = 2\cos\theta\hat{r} + \sin\theta\hat{\theta}$ then we are done. In spherical coordinates with $\varphi \equiv 0$ and $\vec{p} \parallel \hat{z}$, we have that

$$(\vec{p}\cdot\hat{r})\hat{r} = p\cos\theta \qquad \text{and} \qquad (\vec{p}\cdot\hat{r})\hat{\theta} = -p\sin\theta,$$

so then

$$\vec{p}(r,\theta) = p\cos\theta - p\sin\theta \implies 3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} = 3p\cos\theta - p\cos\theta + p\sin\theta = 2p\cos\theta + p\sin\theta \ \checkmark.$$

Question 3

a)

The density volume charge is $\rho(r) = Ar$. Then by Gauss's law,

$$E(4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0} = \oint E\cdot\mathrm{d}a.$$

We find Q_{enc} ;

$$Q_{\text{enc}} = \int_V \rho(r)\mathrm{d}\tau = \int_V Ar\mathrm{d}\tau = \int_0^r Ar(4\pi r^2)\mathrm{d}r \implies \frac{4Ar^4\pi}{4\epsilon_0} = 4\pi r^2 E \implies E = \frac{Ar^2}{4\epsilon_0}.$$

Since the dipole moment is $p = ed$, we need to find d . Let $d \equiv r$, then

$$E = \frac{Ad^2}{4\epsilon_0} \implies d = \sqrt{\frac{4\epsilon_0 E}{A}} = \implies p = 2e\sqrt{\frac{E\epsilon_0}{A}} \;\therefore p \propto \sqrt{E}.$$

b)

Since $E \propto r$ and $E(0) \neq 0 \implies \rho(r) \propto E \implies \rho(r) \propto r$ and $\rho(0) \neq 0$.

Question 4

By symmetry, each dipole induces an electric field of the same strength. So we let $\vec{p} = \vec{p}_1$ and let the electric field be due to the second dipole. Then,

$$\begin{aligned} E_{\text{dip},2} = \frac{1}{4\pi\epsilon_0 r^3}(3(\vec{p}_2 \cdot \hat{r})\hat{r} - \vec{p}_1) \implies -\vec{p}_1 \cdot E_2 &= -\vec{p}_1 \cdot (3(\vec{p}_2 \cdot \hat{r})) + \vec{p}_1 \cdot \vec{p}_2 \\ &= (-\vec{p}_1 \cdot \hat{r})3(\vec{p}_2 \cdot \hat{r}) + \vec{p}_1 \cdot \vec{p}_2 \\ &= -3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r}) + \vec{p}_1 \cdot \vec{p}_2. \end{aligned}$$

Question 5

Given $\sigma_b = P \cdot \hat{n}$ and $\rho_b = -\nabla \cdot P$, the total charge on the dielectric is

$$\begin{aligned} Q_{\text{tot}} = \oint_S \sigma_b \mathrm{d}a + \int_V \rho_b \mathrm{d}\tau &= \oint_S (P \cdot \hat{n})\mathrm{d}a - \int_V (\nabla \cdot P)\mathrm{d}\tau \\ &= \oint_S P \cdot \mathrm{d}\boldsymbol{a} - \int_V \nabla \cdot P\mathrm{d}\tau, \end{aligned}$$

by the divergence theorem the last two integrals are equal so $Q_{\text{tot}} = 0$; the total bound charge vanishes.

Question 6

We use

$$E_{\text{dip}}(\vec{r}, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Sicne the distance is fixed we set $\hat{r} = 0$ giving us

$$= \frac{p}{4\pi\epsilon_0 r^3}(\sin\theta\hat{\theta})$$

Since $0 \leq \sin\theta \leq 1$, it follows that

$$\max E_{\text{dip}}(\vec{r}, \theta) = \frac{p}{4\pi\epsilon_0 r^3}.$$

Since the dipole moment is on the $\hat{z}$ axis and $\sin(0) = 1$ we conclude that this $E_{\max}$ is parallel transposed on the $\vec{P} \implies$ same orientation .

b)

We have that

$$U = \frac{1}{4\pi\epsilon_0 r^3}(\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r}))$$

$$= \frac{1}{4\pi\epsilon_0 r^3}(|p_1||p_2|\cos\theta_{p_1p_2} - 3\cancel{(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}) \quad 0,\ \text{since no } r\text{dependence}$$

It follows that when $\cos\theta_{p_1p_2} = 0$ or 1 we have the minmimum and maximum interaction energy respectively

$$\implies \max U = \frac{p_1p_2}{4\pi\epsilon_0 r^3}; \qquad \min U = 0,$$

these values physically represent two dipoles oriented along the same axis and oriented perpendicularly ,respectively.

c)

Given that one mole of proton is $\sim$ 1 g, then $m_p = 1/N_A = 1\ 6 \times 10^{23}$. Then we have that for 18 amu, $18/(6 \times 10^{23}) = 3 \times 10^{-26}$ kg. We then compute the normal vector

$$\hat{n} = \frac{\rho}{m_{H_2O}} = \frac{1000\ \text{kg}\,\text{m}^{-3}}{3 \times 10^{26}\ \text{kg}} = 3 \times 10^{28}\ \text{m}^3$$

. Then we use

$$U \sim \vec{p} \cdot \vec{E} = \frac{p^2}{4\pi\epsilon_0 d^3} = \frac{p^2}{4\pi\epsilon_0 (n^{(-1/3)})^3} = \frac{(1.85 \times 3.33 \times 10^{-30})^2}{4\pi \times 8.85 \times 10^{-12}(3.22 \times 10^{-10})^3)} \approxeq 1.01 \times 10^{-20}\ \text{J}.$$

For steam, we use $pV = NkT$ for which

$$pV = NkT \implies (10^5\ \text{Pa})d^3 = (1)k(373.15\ \text{K}) \implies d = 3.62 \times 10^{-9},$$

we then use the same interaction energy formula ;

$$U = \frac{p^2}{4\pi\epsilon_0 r^3} = \frac{(1.85 \times 3.33 \times 10^{-30})^2}{4\pi \times 8.85 \times 10^{-12}(3.62 \times 10^{-9})^3} \approxeq 7.2 \times 10^{-24}\ \text{J}.$$

Finally, we know that kT actually represents the energy in J such that for liquid water

$$kT = \text{Energy} = 1.01 \times 10^{(} - 20) \xrightarrow{k=1.38\times10^{-23}} T = 732.6\ \text{K}.$$

And for steam,

$$kT = \text{Energy} = 7.2 \times 10^{-24} \xrightarrow{k=1.38\times10^{-23}} T = 0.51\ \text{K}.$$

the temperatures should be larger than these values to exceed the dipole interaction energy, respectively.