MATH 327 Assignment 1

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Question 1

a)

A possible connectivity path exists and is $1, 2, 3, 6, 5, 4, 1$.

b)

$$r_1 = \frac{r_2}{2} + \frac{r_5}{4} + \frac{r_4}{3} \qquad r_2 = \frac{r_4}{3} + \frac{r_5}{4} + \frac{r_6}{2} + \frac{r_1}{1}$$
$$r_3 = \frac{r_2}{2} \qquad r_4 = \frac{r_5}{4}$$
$$r_5 = \frac{r_4}{3} + \frac{r_6}{2} \qquad r_6 = \frac{r_3}{1} + \frac{r_5}{4}$$

Then, given the linear system $Ar = (r_1\ r_2\ r_3\ r_4\ r_5\ r_6)^T$,

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ 1 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \end{pmatrix}. \tag{1}$$

Each column indeed sum up to one and each row contains a non-zero entry suggesting that indeed this is a connectivity matrix representing a directional graph.

c)

Let $\varepsilon = 0.001$. Then considering the ∞ norm, the number of iterations are 13 and $r_f = (0.1963\ 0.3251\ 0.1621\ 0.0256\ 0.1025\ 0.01084)^T$, and the page rank is $2, 1, 6, 3, 5, 4$.The page rank is not surprising since 2 is the node which has the most outwards links and 4 is one which has the least.

d)

$$\widetilde{P} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{6} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{} & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{5} & 0 & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & \frac{1}{5} & \frac{1}{3} & 0 \end{pmatrix}. \tag{2}$$

Let $\varepsilon = 0.001$. Then considering the ∞ norm ,$r_f = (0.0340\, 0..0453\, 0.0255\, 0.01149\, 0.0233\, 0.0278\, 0.0597)^T$ and the page rank is $7, 2, 1, 6, 3, 5, 4$. Thence, yes, the order changes entirely.

Question2

a)

$$\begin{aligned} r_1 &= \frac{r_3}{1} & r_4 &= \frac{r_5}{1} \\ r_2 &= \frac{r_1}{1} & r_5 &= \frac{r_4}{2} \\ r_3 &= \frac{r_2}{1} + \frac{r_4}{2} \end{aligned} \qquad \longrightarrow P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}. \tag{3}$$

The Page Rank method does not converge. After multiple matrix power iterations, we notice that the r vector's 3 first entries oscillate while the other two converge to 0. This does make sense since the system is not connected but has a sub-path around the nodes $1 - 2 - 3$.

b)

$$\hat{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}. \tag{4}$$

The Page Rank is inconsistent with this system, the r vector converges to 0 as $\varepsilon \longrightarrow 0$. This system is different to the previous one in that the 2 node has no outward links and is hence isolated ; because of that, there does not exist a connectivity path in the matrix any more.

c)

Adding a sixth node ,

$$\widetilde{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix}. \tag{5}$$

The Page Rank for this connectivity matrix is 6, 3, 1, 2, 4, 5 with
$r_f = (0.1501\ 0.1392\ 0.1721\ 0.1154\ 0.1025\ 0.3207)^T$.

Question 3

a)

Let $v_1 := (0\ 0\ 0\ \ldots\ 1)^T \in \mathbb{R}^n$. This vector satisfies the given equivalence.

b)

Let $v_2 := (c\ c\ c\ \ldots\ c)^T \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then indeed,

$$n\|v_2\|_\infty = n \max_{i=1,\ldots,n} |x_i| = nc \quad \checkmark$$
$$\sqrt{n}\|v_2\|_2 = \sqrt{n}\sqrt{nc^2} = \sqrt{n}\sqrt{n}c = nc \quad \checkmark$$
$$\|v_2\|_1 = \sum_i^n |x_i| = nc \quad \checkmark$$

c)

(i) We show $\|v\|_\infty \leq \|v_2\|_2$.

$$\begin{aligned} \|v_2\|_2^2 = \sum_i x_i^2 = \sum_i |x_i|^2 \\ \geq \max_{i=1,\ldots,n} |x_i|^2 \\ = \|v\|_\infty^2 \end{aligned}$$
$$\therefore \|v\|_\infty^2 \leq \|v\|_2^2 \implies \|v\|_\infty \leq \|v\|_2 \checkmark.$$

(ii) We show $\|v\|_2 \le \|v\|_1$.

$$
\begin{aligned}
\|v\|_1^2 &= \left(\sum_i^n |x_i|\right)^2 \\
&= \sum_i^n |x_i|^2 + 2\sum_{i<j} |x_i||x_j| \\
&\ge \sum_i |x_i|^2 \\
&= \sum_i x_i^2 \\
&= \|v\|_2^2 \\
\therefore \|v\|_2^2 \le \|v\|_1^2 &\implies \|v\|_2 \le \|v\|_1 \checkmark.
\end{aligned}
$$

(iii) We show $\|v\|_1 \le n\|v\|_\infty$.

$$
\begin{aligned}
|x_i| \le \max_{i=1,\dots n} |x_i| \implies \|v\|_1 &= \sum_i^n |x_i| \\
&\le \sum_i^n \max_{i=1,\dots,n} |x_i| \\
&= \sum_i^n \|v\|_\infty \\
&= n\|v\|_\infty \\
\therefore \|v\|_1 &\le n\|v\|_\infty \checkmark.
\end{aligned}
$$

(iv) We show $\sqrt{n}\|v\|_2 \le n\|v\|_\infty$. We first note that $|x_i| \le \max_{i=1,\dots.n} |x_i|$.

$$
\begin{aligned}
\|v\|_2 &= \left(\sum_i^n |x_i|^2\right)^{1/2} \\
&\le \left(\sum_i^n \max_{i=1,\dots,n} |x_i|^2\right)^{1/2} \\
&= \left(\sum_i^n \|v\|_\infty^2\right)^{1/2} \\
&= \left(n\|v\|_\infty^2\right)^{1/2} \\
&= \sqrt{n}\|v\|_\infty \\
\therefore \|v\|_2 &\le \sqrt{n}\|v\|_\infty \checkmark.
\end{aligned}
$$

Question 4

a)

(i) $\|A\|_M$ is a maximum of absolute value so always positive. If A is the 0 matrix then $a_{ij} = 0 \; \forall i, j \mid \max_{i,j} a_{ij} = 0 \implies \|A\|_M = 0$. We conclude that positivity holds.

(ii)
$$\|\alpha A\|_M = \max_{i,j} \left|\alpha a_{ij}\right| = |\alpha| \max_{i,j} \left|a_{ij}\right| = |\alpha| \|A\|_M.$$

(iii)
$$\begin{aligned}\|A + B\|_M &= \max_{i,j} \left|a_{ij} + b_{ij}\right| \\ &\leq \max_{i,j} \left(\left|a_{ij}\right| + \left|b_{ij}\right|\right) \\ &\leq \max_{i,j} \left|a_{ij}\right| + \max_{i,j} \left|b_{ij}\right| \\ &= \|A\|_M + \|B\|_M\end{aligned}$$

b)

Let $r = m$ and define
$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{pmatrix} \longrightarrow \|A\|_M = 1, \qquad B = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{pmatrix} \longrightarrow \|B\|_M = 1.$$

Then it is evident that
$$AB = \begin{pmatrix} n & n & n & \dots & n \\ n & n & & & \\ n & & n & & \\ \vdots & & & \ddots & \\ n & & & & n \end{pmatrix} \longrightarrow \|B\|_M = n.$$

Since this is a square matrix, the case $n = m = 1$ for which the claim fails, is rejected since a matrix needs to be multidimensional by definition. Thence, it follows that $\|AB\|_M = n > \|A\|_M \|B\|_M$.

Question 5

a)

(i) If $A \neq 0$ then $\exists \, \hat{x} \in \mathbb{R}^n$ (non-zero) for which $A\hat{x} \neq 0$. The vector norm satisfies positivity so $\|\hat{x}\| > 0 \implies \|A\hat{x}\| > 0$.

(ii)

$$\begin{aligned}\|\alpha A\|_\infty &= \max_{\|x\|_\infty=1} \\ &= |\alpha| \max_{\|x\|_\infty=1} \|Ax\|_\infty \\ &= |\alpha|\|A\|_\infty\end{aligned}$$

(iii)

$$\begin{aligned}\|A+B\|_\infty &= \max_{\|x\|_\infty=1} \|(A+B)x\|_\infty \\ &= \max_{\|x\|_\infty=1} \|Ax+Bx\|_\infty \\ &\leq \max_{\|x\|_\infty=1} (\|Ax\|_\infty + \|Bx\|_\infty) \\ &\leq \max_{\|x\|_\infty=1} \|Ax\|_\infty + \max_{\|x\|_\infty=1} \|Bx\|_\infty \\ &= \|A\|_\infty + \|B\|_\infty\end{aligned}$$

b)

$$\begin{aligned}\|A\|_\infty &= \max_{\|x\|_\infty=1} \|Ax\|_\infty \\ &= \max_{\|x\|_\infty=1} \max_{1\leq i\leq n} \sum_{j=1}^n \left|a_{ij}x_j\right| \\ &= \max_{1\leq i\leq n} \max_{\|x\|_\infty=1} \sum_{j=1}^n \left|a_{ij}x_j\right| \\ &= \max_{1\leq i\leq n} \sum_{j=1}^n \left|a_{ij}\right|\end{aligned}$$

For the second equality, let r_i be the i-th row, a vector. Then

$$\|r_i\|_1 = \sum_{j=1}^n \left|r_{ij}\right| = \sum_{j=1}^n \left|a_{ij}\right|$$

$$\implies \max_{i=1,\dots,n} \sum_{j=1}^n \left|a_{ij}\right| = \max_{i=1,\dots,n} \sum_{j=1}^n \left|r_{ij}\right| = \max_{i=1,\dots,n} \|r_i\|_1$$

c)

(i) $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2$: Using the results proved in Problem 3,

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \implies \frac{\|A\|_\infty}{\sqrt{n}} \leq \|A\|_2.$$

(ii) $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$:

$$\text{Since , } \frac{1}{\sqrt{n}}\|v\|_2 \leq \|v\|_\infty \implies \|v\|_2 \leq \sqrt{n}\|v\|_\infty$$
$$\implies \|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sqrt{n} \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty$$
$$\therefore \|A\|_2 \leq \sqrt{n}\|A\|_\infty$$

Question 6

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

For x an eigenvector such that $Ax = \lambda x$, where $|\lambda| = 1$ and $\|x\| = 1$,

$$\begin{aligned} &\geq \max_{\|x\|=1} \|Ax\| \\ &= \max_{\|x\|=1} \|\lambda x\| \\ &= \rho(A) \max_{\|x\|=1} \|x\| \\ &= \rho(A) \end{aligned}$$
$$\therefore \|A\| \geq \rho(A)$$

Question 7

a)

(i)

$$\|I\|_1 = \max_{1\leq i\leq n} \sum_{i=1}^{n} \left|a_{ij}\right| = \max\{1, 1, \ldots, 1\} = 1.$$

(ii)

$$\|I\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} \left|a_{ij}\right| = \max\{1, 1, \ldots, 1\} = 1.$$

(iii) We use the property that the 2-norm is orthogonally invariant.

$$\|I\|_2 = \max_{\|x\|_2=1} \|Ix\|_2 = \max_{\|x\|_2=1} \|x\|_2 = 1.$$

(iv)

$$\|I\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} {a_{ij}}^2\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} {\delta_{ij}}^2\right)^{1/2} = \sqrt{n},$$

where δ_{ij} is the usual Kronecker delta function.

b)

(i)

$$\|A\|_1 = \max_{1\le i\le n} \sum_{i=1}^{n} \left|a_{ij}\right| = \max\{1, 102\} = 102.$$

(ii) By theorem, $\|A\|_2 = \max\limits_{1\le i\le n} \sigma_i(A)$, so we proceed

$$A^T A = \begin{pmatrix} 1 & 0 \\ 100 & 2 \end{pmatrix} \begin{pmatrix} 1 & 100 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 100 \\ 100 & 10004 \end{pmatrix}$$

$$\det\left(A^T A\right) = (1-\lambda)(10004-\lambda) - 10000 = 0 \implies \lambda_\pm = \frac{10005 \pm \sqrt{10005^2 - 4^2}}{2}$$

$$\therefore \|A\|_2 = \sqrt{\lambda_{\max}} = \sqrt{\frac{10005 \pm \sqrt{10005^2 - 4^2}}{2}} \approx 100.25$$

(iii)

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right| = \max\{101, 2\} = 101.$$

(iv)

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} {a_{ij}}^2 \right)^{1/2} = \sqrt{1^2 + 100^2 + 2^2} = \sqrt{10005} \approx 100.25$$

(v)

$$\rho(A) = \max_{1\le i\le n} |\lambda_i| = \max\{1, 2\} = 2.$$