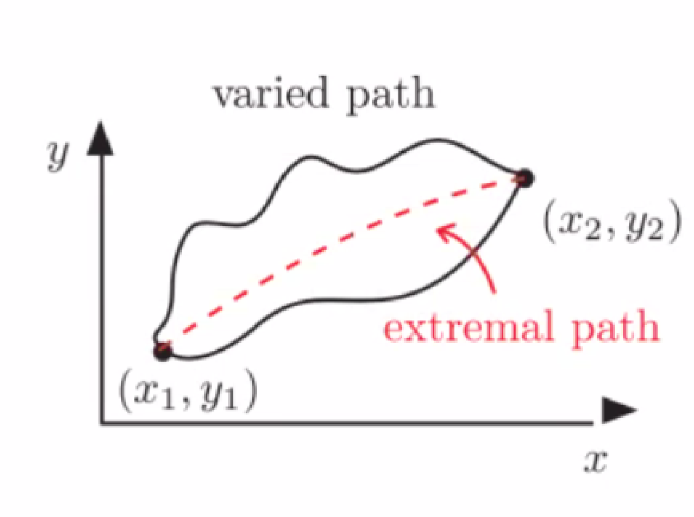
Lecture 1

Variational Calculus

We are interested in extremization. We wish to find the minimum and maximum of integrals of functions *functionals*.

Statement of the problem



We consider this function to be a path. Consider a function $F(y, y', x)$ defined on a path $y = y(x)$ between two points (x_1, y_1) and (x_2, y_2). Where $y' \equiv dy/dx$. There could be many paths. In fact we could argue there is an infinite number of paths that join these two plots.

We define the mathematical form as the line integral I and call it a *functional*, defined as

$$I = \int_{x_1}^{x_2} F(y, y', x)\, \mathrm{d}x$$

the line integral.

Q We know I is an integral. When is it minimum and when is it maximum (extremize a line integral) ?

This is *variational calculus*.

The problem is to find *the path* $y(x)$ such that the line integral I of the function above between x_1 and x_2 is *extremal*. We can show that small variations of I, noted δI, when the function $y(x)$ is varied arbitrarily from $y(x) \to \delta y(x)$ is

$$\delta I = \int_{x_a}^{x_b} \left[\frac{\partial}{\partial y} F(y, y', x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y(x) \mathrm{d}x$$

so it is equal to 0 only if this integrand is 0.

So $\delta I = 0$ (stationary point) whenever

$$\frac{\partial}{\partial y} F(y, y', x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which is known as the Euler equation in the context of mathematics.

Proof

Assume $y(x,\alpha) = y(x,0) + \alpha\eta(x)$ such that the function $\eta(x)$ is the varied path.

$$I(\alpha) = \int_{x_1}^{x_2} F(y(x,\alpha), y'(x,\alpha), x)\mathrm{d}x$$

We define the variation

$$\delta I \equiv \left(\frac{\partial I}{\partial \alpha}\right)_0 \delta\alpha$$

There is a stationary point if $\delta I = 0$, so

$$\delta I \equiv \left(\frac{\partial I}{\partial \alpha}\right)\delta\alpha = \int_{x_1}^{x_2} \left\{\frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right\}$$

The y' is annoying. We do integration by path

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\delta\alpha\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{\partial^2 y}{\partial\alpha\partial x}\delta\alpha\mathrm{d}x$$

Integrating by parts $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$ and using the fact that all varied curves must go through (x_1, y_1) and (x_2, y_2) we get

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x\delta\alpha$$

Then since $\partial y/\partial\alpha$ is arbitrary, we obtain

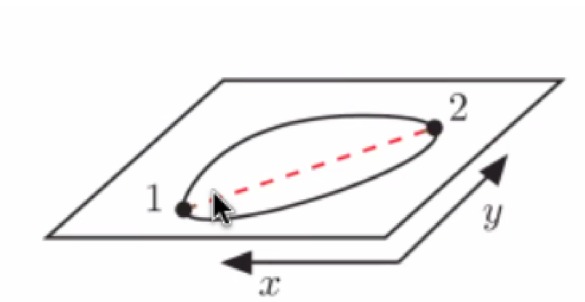
$$\delta I = 0 \iff \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which is known in mathematics as Euler's equation.

Remark We will interchange Euler's-Lagrange equation

Note $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial x}$ are partial derivatives while $\frac{\mathrm{d}}{\mathrm{d}x}$ is a total derivative.

[Example What is the shortest distance between two points and a plane



Recall an arc length is given by $\mathrm{d}x^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \implies \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$. The total length is

$$I = \int_1^2 \mathrm{d}s = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x$$

Is this a functional ? Yes. Does it depend on y ? No. Is it an explicit function of x ? Yes. It is a function of y'.

So here , $F(y, y', x)$ is only a function of y' : $F = \sqrt{1 + y'^2}$. We know that

$$\delta I = 0 \iff \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y;}\right) = 0$$
$$\implies \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

and we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \implies \frac{y'}{\sqrt{1+y'^2}} = \text{constant} \equiv c$$

which can only be true if y' is also a constant. Let $y' \equiv a$ and related to c throught $a = c/\sqrt{1-c^2} \implies y' = c/\sqrt{1-c^2}$, hence

$$\implies \left(\frac{c}{\sqrt{1-c^2}}\right)x + b = y(x)$$

c, b are determined with conditions 1 and 2

The Brachistocrone Problem

Famous problem in the 1600 that challenged mathematicians for years.

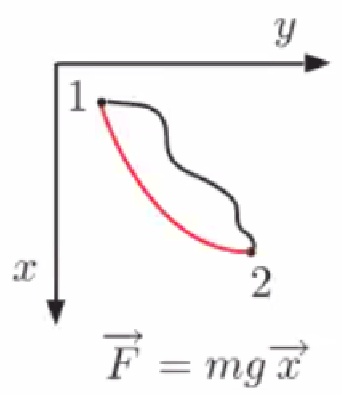


Figure 12: Brachistochrone

> Finf the trajectory between two points x_1, y_1 and x_2, y_2 for a free falling body, i.e. for a free body moving in a constant force field $F = -mg$, that will yield the least amount of time spent ? It's not a straights line, not an arc (As Galileo proposed)

The time transit between two points is

$$t_{12} = \int_1^2 \frac{\mathrm{d}s}{v(s)},$$

where $v(s)$ is the velocity along the path. We use conservation of energy ; $1/2mv^2 \implies v = \sqrt{2gx}$. Initial condition : $E_1 = 0$ and $E_2 = 1/2mv^2 - mgx = 0$.

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \implies \mathrm{d}s = \sqrt{1+y'^2}\mathrm{d}x$$

Lecture 2

After some work we find the integral

$$y=\int\frac{x\mathrm{d}x}{\sqrt{2\alpha x-x^2}}$$

We use the change of variables $x=\alpha(1-\cos\theta)$ such that $\mathrm{d}x=\alpha\ sin\theta\mathrm{d}\theta$, such that

$$y=\int\frac{\alpha^2(1-\cos\theta)\sin\theta\mathrm{d}\theta}{\sqrt{2\alpha^2(1-\cos\theta)-\alpha^2-\alpha^2\cos^2\theta+2\alpha^2\cos\theta}}=\int\frac{\alpha^2(1-\cos\theta)\sin\theta\mathrm{d}\theta}{\sqrt{\alpha^2(1-\cos^2\theta)}}$$

so then we get

$$y=\int\alpha(1-\cos\theta)\mathrm{d}\theta$$

with solution

$$y=\alpha(\theta-\sin\theta)+\mathrm{constant}$$

This theta parametrization describes a cycloid passing through the origin

$$y(\theta)=\alpha(\theta-\sin\theta),\qquad x(\theta)=\alpha(1-\cos\theta)$$

In a $x-y$ plsnse $y(\theta)$ and $x(\theta)$ describes the path which minimizes the travel time t_{12}

> the cycloid shape wins !

Recap

we developed an equation that can extranaze a function $F=(y,y',x)$. Next we consdider some special cases and then we consider the concept of constraints.

Case where $\frac{\partial F}{\partial x}=0$: Euler's second form

So if F does not depend on x we can use a shortcut

For any function $F(y,y';x)$, the total derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$ is

$$\frac{\mathrm{d}F}{\mathrm{d}x}=\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}y''+\frac{\partial F}{\partial x}$$

we note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right)=y''\frac{\partial F}{\partial y'}+y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}$$

Then syvstituting back ,

$$=\frac{\mathrm{d}F}{\mathrm{d}x}-\frac{\partial F}{\partial x}-y'\frac{\partial F}{\partial y}+y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y}$$

where the two last terms are

$$y'\left\{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}-\frac{\partial F}{\partial y}\right\}\equiv 0$$

it's equal 0 because it's the *Euler's equation* from before. Whence we get :

$$\boxed{\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{ - y' \frac{\partial F}{\partial y'} + F \right\} = 0}$$

Q Why is this important ? Why does teacher like this ?

What is the total derivative of something if the RHS is 0 ? It means the inside is equal to a constant. Here we still have the partial on the left though. But if $\frac{\partial F}{\partial x} = 0$, then we don't have the term on the left ! So setting the left term to 0, we get

$$F(y, y') - y' \frac{\partial F(y, y')}{\partial y'} = \text{constant} \tag{1}$$

Midterm Show shortest distance in a $x - y$ plane is a straight line by using the second Euler form

Remark (1) is Actually the Hamiltonian in QM for time independent.

In this section we won't be too formal since it's not too imporant. So far we did physics in 1 dimension? We said $F(y, y'; x)$, here we have only one y.

With 50 particles we would ahve $y_1, y_2, \ldots, y_{50}$. Let us consider a function $F = F(y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x); x)$. We can generalize and show that in this case the "functional" I$ becomes extremal for

$$\delta I = \left(\frac{\delta I}{\delta \alpha}\right) \delta\alpha = \int_{x_1}^{x_2} \sum_i \left\{ \frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y_i'} \right\} \eta_i(x) \mathrm{d}x \delta\alpha$$

Which is only $\delta I = 0$ when

$$\boxed{\frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y_i'} = 0 \qquad i = 1, 2, \ldots, n}$$

Imeagine now the function F is a function of $F = f(y, y', y'', x)$ i.e., if the function to be considered is a function of second order derivative in $y(x) \to y''$ then the Euler equation is written as : (*we will not use the equation*)

Euler Equation with Constraint

Problem We want to extremize a function $F = F(y, y', x)$ under a particular constraint function $g(y, x) = 0$

Example Consider a surface S described by $g(y, x) = 0$. We want to clculate the shortest path between twopoints on the surface with the constrain the the path satisfy the equation of the surface $g(y, x) = 0$.

We can show that for the general case of a function $F = F(y_i, y_i', x)$ where we have $i = 1, 2, \ldots, n$ independent variables and $g_j\{y_i; x\} = 0, j = 1, 2, \ldots, m$ ie., m equations of constraints the set of equations that will minimize the functional

$$I = \int F\{y_i, y_i'; x\} \mathrm{d}x$$

is

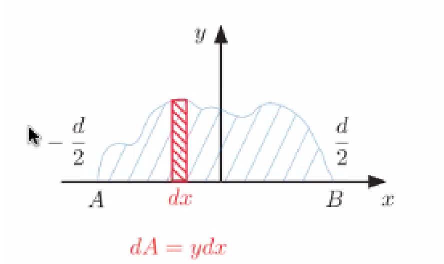
$$\frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y_i'} + \sum_j \lambda_j \frac{\partial g_j}{\partial y_i} = 0$$

$g_j(y_i; x) = 0$ where $i = 1 \to n$ and $j = 1 \to m$. The λ_j coefficients are **Lagrange multipliers** introduced to take the equation of constraints into account. We thus have n equations with a total of $m+n$ unknownws. But we also have m equations of constraint. In other words, the system of equations is *solvable*

Midterm Average in midterm is prertty high, not a speed contest

Lecture 3

Problem III : An isoperimetric problem



Q What will be the shape of the string which will yield the largest area?

We want to minimize $I = \int_{-d/2}^{d/2} y\mathrm{d}x$ with a fixed length $K = \int_{-d/2}^{d/2} \sqrt{1+y'^2}\mathrm{d}x$.

Since $J = I + \lambda K \implies \delta J = 0$ and so : ...

This gives

$$\frac{\partial h}{\partial y'} = \frac{\lambda y'}{\sqrt{1+y'^2} \implies y + \lambda\sqrt{1+y'^2}} - -$$

(review this at home for the assigment actually)

Lagrangian and Hamiltonian Dynamics

Q Which coordinates to use ?

- A pendulum swinging ? Only 1 degree of freedom θ so use *polar coordinates*.
- A ball on a sphere ? Only 2 degree of freedom so use *spherical coordinates*.
- billard on a pool with no slip condition ? 2 degrees of freedom use $x - y$
- cilynder that's rolled on an incline no slip condition? 1 degree of freedom
- pendullum with spring on bottom (2 degree of freedom)

  If we add a joint such that it splits the pendulum in half, then still 2 degree of freedom
- 3 particles with 2 springs :

Remark Midterm do not solve the differential equation ??

Lagrange Equations

Generalized Coordinates

Let's consider the position of a dot in space, which we shall denote with the raidus vector $\vec{r} = r(x, y, z)$. We denote the first derivative of $\vec{r}$ with respect to time $\dot{\vec{r}}$. It's descrived by Newton's second law

$$F = \frac{\mathrm{d}m\vec{r}}{\mathrm{d}t}$$

Let us consider a system with N points in space. We need a total of $3N$ coordinates. In general, we need a number of coordinates corresponding to the l degrees-of-freedom of the system to completely position it. The number of l of degree of freedom is

$$l = 3N - m,$$

where l are the independent coordinates required to position the system.

Let us denote the set of coordinates $\{q_1, q_2, \ldots, q_l\}$ be a set that completely characterize the position of a system with l degrees of freedom. We refer to that set as the *generalized coordinates*, and their first derivatives $\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_l\}$ the *generalized velocities*of the system.

Goal Mathematically determine the accelartion at all time t $\{\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_l\}$. For the deterministic time evolution one needs to derive the **equation of motion** for the system

Momenta and Energies

Using the generalized coordinates q_i's we define the **Kinetic energy** T of a system by

$$T = T(\dot{q}_i) = \frac{m}{2} \sum_{\alpha} \dot{x}_{\alpha}^2(q_i)$$

We define the mathematical function describing the enrgy *difference* $T - V$ of a system as the **Lagrangian** of a mechanical system, which we denote by $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$:

$$\mathcal{L} = T - V$$

Similary, we can define the generalized (canonical) p_i momenta by making use of the generalized coordinates

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

which *does not necessarily represent the linar moomentum* of a system of particles.

Example In cartesian coordinates $\vec{r} = r(x, y, z)$,

$$\begin{aligned} q_1 &= x \quad q_2 &= y \quad q_3 = z \\ \dot{q}_1 &= \dot{x} \quad \dot{q}_2 &= \dot{y} \quad \dot{q}_3 = \dot{z} \end{aligned}$$

and

$$\begin{aligned} T &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ p_i &= \frac{\partial T}{\partial \dot{q}_i} = m\{\dot{x}, \dot{y}, \dot{z}\} \end{aligned}$$

i.e, in this case, the generalized momenta are the linear momenta of the system.

Example Same example as above by in cylindrical coordiantes (ρ, θ, z)

$$\begin{aligned} T = T(\rho, \theta, z) &= \frac{1}{2} m \sum_i \dot{x}_i^2 \\ &= \frac{1}{2} \{\rho^2 \dot{\theta}^2 \cos^2\theta + \dot{\rho}^2 \sin^2\theta + 2\rho\dot{\rho} \dots\} \end{aligned}$$

Lecture 4

Recap We sa wa simple pendulum.

Q

- particle rolling on a sphere (how many degrees of freedom) ? $\rightarrow 2$,
- for $3 \rightarrow 6$, $n \rightarrow 3n$.
- particle rolling on a table $\rightarrow 1$ dof.

> today is a nice lecure. We will derive the lagrange equation from a principle.

Recall The Lagrangian , $L \equiv T - V$, with $T = T(q, \dot{q}, t)$ and $V = (q, t)$.

It's conterintuitive to form a function with $T - V$. Say $L = (q, \dot{q}, t)$ and $F = F(y, y', x)$. The goal was to figure $y(x)$. WIth L the goal was to find $q(t)$, the position at a time. That is the goal *Position at all times*.

> classical mechanics is deterministic, *up to some point*.

$$\begin{cases} F = F(y, y', x) \\ y(x) \end{cases}$$

> insert picture

Q How many coordinates do we need to locate that thing ?

Dropping a particle $\rightarrow$ choses <u>one</u> path.

3.1.3 Hamilton's Principle

We wish to "derive" the $q_i(t)$ (trajectory of a particle). We know we can form $\{q_i\}, \{\dot{q}_i\}, \{\ddot{q}_i\}$. How do we do this link from speed to acceleration though ? We're not sure yet.

Consider a mechanical system :

$$\begin{aligned} &L = L(q_1, q_2, \dots, q_l, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_l, t) \\ &L = 3N - m \end{aligned}$$

We define the action S of the system as the line integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t$$

Note that S here is a "functional".

claim the path is the extremal path.

Definition (*Principle*) : Statement proved experimentally or mathematically, that is true at posteriori.

Definition (*Hamilton's Principle Of Least Action*) : "of all possible paths along which a dynamical system may move from one point to another in configuration space, the path realized is the one that minimize the action of the system." In other words,

$$\implies \boxed{\delta S = 0}, \qquad S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t \tag{1}$$

If we think deeply,

$$F(y, y', t) \quad I = \int F(y, y', x) \mathrm{d}x \qquad \delta I = 0$$

Q What sets of equations will set condition (1) true ? Euler equation.

Consider a variation $q(t) + \delta q(t)$, where δq is a small variation. So we can write

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) \mathrm{d}t - \int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t$$

$$\delta S = \delta \left[\int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t \right] = 0$$

$$\delta S = \int_{t_1}^{t_2} \left[\underbrace{\left(\frac{\partial L}{\partial q} \right) \delta q}_{1} + \underbrace{\left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q}}_{2} \right] \mathrm{d}t$$

We have 2 terms here. We don't like the $\dot{q}$ term of course.

Second term :

$$\delta \dot{q} = \frac{\mathrm{d}\delta q}{\mathrm{d}t} \quad u = \frac{\partial L}{\partial \dot{q}} \quad \mathrm{d}u = ? \quad \mathrm{d}v = \delta \dot{q} \mathrm{d}t$$

Then by parts, the second term is

$$\underbrace{\frac{\partial L}{\partial \dot{q}} \delta q \Bigg|_{t_1}^{t_2}}_{\star} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \mathrm{d}t$$

Q what is $\star$ equal to ? It's zero because there's no variation there.

Therefore, if we now combine term 1 and term 2, we obtain

$$\delta S = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q} \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \mathrm{d}t = 0$$

Since $\delta q \neq 0$

$$\boxed{\iff \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0}$$

> Deriving $F = ma$
>
> $$T = \frac{m \ddot{q}}{2} - V(q) \qquad \frac{\partial L}{\partial q} = \frac{\partial V}{\partial q} \qquad \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$
>
> $$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) = m \ddot{q}$$
>
> $$\frac{\partial V}{\partial q} - m \ddot{q} = 0 \implies m \ddot{q} = - \frac{\partial V}{\partial q} \implies ma = F$$

Lecture 5

Recall We defined the *action* (S).

$$S=\int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t \qquad \delta s=0 \implies \frac{\partial L}{\partial \dot{q}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)=0, \quad L=T-V$$

Recall The principle of least action $\delta S=0 \implies (\ldots)$.

Remark This will fail if we consider a particle with a deBrouigle wavelength.

Q Cat at 1m/s, what will happen if cat crosses door with $\hbar=1$ instead 10^{-34} ? 2 doors eparated by $1m$ which one will it pick ? IT will go though both at the same time.

> In classical world we don't have the deBrouigle wavelength, because of *decoherence* which makes it lose quantum properties.

In classical mechanics $q(t)$ is deterministic.

Recipe for Solving Lagrange

1. : Choose generalized coordinates
   - Note In an exam these will be given or hinted
   - Remark In lagrangian mechanics we need to imemdiately think of generalized coordinates
2. : Express T and V with $\{q_i\}$
3. : $L=T-V$
4. : Solve the Lagrange equations.
5. : Obtain Differential equation (*which we can solve, or not*).

Example (*1*) : Simple pendulum

Given an $x:\rightarrow$, $y:\downarrow$ plane with a pendulum in the down right direction. Let L be fixed.

$$\begin{aligned} \vec{r} &= (l\sin]theta, -l\cos\theta) \\ \dot{x} &= l\cos\theta\dot{\theta} \qquad\qquad\qquad \text{with } V=mgy=-mgl\cos\theta \\ \dot{y} &= l\sin\theta \end{aligned}$$

Then , we get

$$\begin{aligned} L=T-V &= \frac{m}{2}(l\cos^2\theta\dot{\theta}^2+l^2\sin^\theta\dot{\theta}^2)+mgl\cos\theta \\ &= \frac{ml^2\dot{\theta}^2}{2}+mgl\cos\theta \end{aligned}$$

Then we do Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

We say

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=ml^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta}=-mgl\sin\theta \\ \implies &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=ml^2\ddot{\theta}+mgl\sin\theta=0\boxed{\implies \ddot{\theta}+\frac{g}{l}\sin\theta=0} \end{aligned}$$

This is non-linear differential equation. If we plot position - momentum we see that this is a *chaotic* equation.

Example (*2*) : Consider a pouley with m_1 to the left and m_2 to the right with inertia $I = 0$.

```
          _ _ _    <-- I=0
        /       \
 ------|----o----|--------
|       \ _ _ _ /        |
|        |     |         |
| y_1 -->|     |         |
|      m_1     |    <-- y_2
               |
             m_2
```

We have that $y_1 + y_2 = L \implies 1$ variable "y", with $y = L - y_2$. It follows that

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}^2$$
$$V = -m_1 g y - m_2 g(L - y)$$
$$L = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + (m_1 - m_2)gy - \underbrace{m_2 g L}_{\text{cancel ,const.}}$$

Let us perform

$$\frac{\partial L}{\partial y} = (m_1 - m_2)g - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y} = (m_1 + m_2)\ddot{y}$$
$$(m_1 + m_2)\ddot{y} - (m_1 - m_2)g = 0$$
$$\boxed{\ddot{y} = \frac{(m_1 - m_2)g}{m_1 + m_2}}$$

Remark We didn't gp throught force and tension !!

Q Are we able to solve the previous differential equation ? Yes. Solution is what ? IF we set $m_1 = m_2$ nothing happens.

Remark Always check the limits when getting an asnwer to verify that the algebra was correct.

Example (*3*) : Perl falling on a parabolic roating wire. POint particle with mass m. Let $z = c\rho^2$ (rotating at a frequency wt). Gravity mgz.

$$x = \rho\cos\omega t \qquad y = \rho\sin\omega t \qquad \omega t \equiv \text{constant} \qquad z = c\rho^2$$
$$\dot{x} = \dot{\rho}\cos\omega t - \rho\sin\omega t\omega$$
$$\dot{y} = \dot{\rho}\sin\omega t + \rho\cos\omega t\omega$$

Therefore,

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{\rho}^2 + \omega^2\rho^2 + 2c\rho\dot{\rho}^2)$$

We write teh lagrangian

$$
\begin{aligned}
L = L(\rho, \dot\rho) &= \frac{m}{2}(\dot\rho^2 + \omega^2\rho^2 + 4e^2\rho^2\dot\rho^2) - mge\rho^2 \\
&= \frac{m}{2}(2\dot\rho + 8c^2\rho^2\dot\rho^2) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\rho} &= \frac{m}{2}(2\ddot\rho + 16c^2\rho^2\dot\rho^2 + 8c^2\rho^2\ddot\rho) && Arrangineverythingweget \\
\frac{\partial L}{\partial \rho} &= m(4c^2\rho\dot\rho^2 + \rho\omega^2 - 2gc\rho)
\end{aligned}
$$

$$
m\ddot\rho(1 + 4c^2\rho^2) + m\ddot\rho(4c^2\rho) + m(2gc - w^2)\rho = 0
$$

With $c \to 0$ and $\ddot\rho - \omega^2\rho = 0$, we get hyperbolic stuff.

Q Where is Newton in the previous derivation ?

\section*{Question 1}

Let

$$
L = \int_0^D \sqrt{1 + y'^2}\mathrm{d}x \qquad \text{and} \qquad E = \int_0^D \mu g y\sqrt{1 + y'^2}\mathrm{d}x
$$

Then we let $h(y, y') = E + \lambda L$,obtaining

$$
h(y, y') = E + \lambda L = (\mu g y + \lambda)\int_0^D \sqrt{1 + y'^2}\mathrm{d}x
$$

Here since $F \neq$ function(x), then we use the formula

$$
h - y'\frac{\partial h}{\partial y'} \equiv \text{constant},
$$

where $h = F + \lambda g$, for $F = \mu g y\sqrt{1 + y'^2}$ and $g = \sqrt{1 + y'^2}$. Then,

$$
\begin{aligned}
\frac{\partial h}{\partial y'} &= \frac{\partial}{\partial y'}\left((\mu g y + \lambda)\sqrt{1 + y'^2}\right) \\
&= (\mu g y + \lambda)\frac{y'}{\sqrt{1 + y'^2}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
h - y'\frac{\partial h}{\partial y'} &= \mu g y\sqrt{1 + y'^2} + \lambda\sqrt{1 + y'^2} - (\mu g y + \lambda)\frac{y'^2}{\sqrt{1 + y'^2}} \\
&= (\mu g y + \lambda)\left(\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right) \\
&\equiv \text{constant}
\end{aligned}
$$

Let this constant be $-y_0$. Then solving for y' we get

$$
y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{(\mu g y + \lambda)^2}{y_0^2} - 1}
$$

Which is an ordinary differential equation which we can solve by separation of variables

$$\int \mathrm{d}x = \int \frac{\mathrm{d}y}{\sqrt{\frac{(\mu g y+\lambda)^2}{y_0^2} - 1}}$$
$$x + x_0 = \int \frac{\mathrm{d}y}{\sqrt{\frac{(\mu g y+\lambda)^2}{y_0^2} - 1}}$$

We perform a change of variable with $(\mu g y + \lambda)/y_0 = \cosh\alpha$, then $\mathrm{d}y = \mathrm{d}\alpha y_0 \sinh\alpha / mg$,

$$\begin{aligned} x + x_0 &= \frac{y_0}{mg} \int \mathrm{d}\alpha \frac{\sinh\alpha}{\sqrt{\cosh^2\alpha - 1}} \\ &= \frac{y_0}{mg}\alpha \\ &= \frac{y_0}{mg}\cosh^{-1}\left(\frac{mgy+\lambda}{y_0}\right) \end{aligned}$$

Rearranging the last expression yields

$$y(x) = \frac{y_0}{mg}\cosh\left(\frac{mg}{y_0}(x+x_0)\right) - \frac{\lambda}{y_0},$$

which represents the shape of the given cable, where y_0 can be found using the initial conditions $y(0) = 0$ and $y(D) = 0$, x_0 is determined to be 0 using the identity $\cosh(\alpha) = \cosh(-\alpha)$, and λ can be found using the given constraint $L = \int \mathrm{d}s$.

\section*{Question 2}

\subsection*{(a)}

We wish to minimize E given

$$I = \int \mathrm{d}E = \int \underbrace{K(s)s'^2 \mathrm{d}s}_{F(t,s,s')}$$

Here since $F \neq$ function(t), then we use the simplification

$$\begin{aligned} F - y'\frac{\partial F}{\partial y'} &\equiv \text{constant} \\ K(s)s'^2 - s'\frac{\partial}{\partial s'}\left(K(s)s'^2\right) &= c \\ K(s)s'^2 - s'K(s)s's'' &= c \\ K(s)s'^2 s'' &= \underbrace{K(s)s'^2 - c}_{\equiv c'} \\ K(s)s'^2 s'' &= c' \\ s'' &= \frac{c'}{K(s)s'^2} \end{aligned}$$

We perform separation of variables on the previous ordinary differential equation

$$\begin{aligned} \int \mathrm{d}s' &= \int \mathrm{d}t \left(\frac{c'}{K(s)s'^2}\right) \\ s' &= \frac{tc'}{K(s)s'^2} - c'' \end{aligned}$$

Here both t and c'' get absorbed in c' , redefine $c' \to c$, and we are left off with

$$s'^3 = \frac{c}{K(s)} \implies v(s) = \frac{c}{K(s)^{1/3}}$$

\subsection*{(b)}

Given $\mathrm{d}t = \mathrm{d}s/v(s)$, then

$$\begin{aligned}
\mathrm{d}t = \mathrm{d}s\frac{K(s)^{1/3}}{c} &\implies \int_0^T \mathrm{d}t = \frac{1}{c}\int_0^L \mathrm{d}sK(s)^{1/3} \\
&\implies T = \frac{1}{c}\int_0^L \mathrm{d}sK(s)^{1/3} \\
&\implies C = \frac{\int_0^L \mathrm{d}sK(s)^{1/3}}{T}
\end{aligned}$$

\subsection*{(c)}

We substitute $v(s)$ with the new variables

$$\begin{aligned}
v(s) = \frac{\int_0^L \mathrm{d}sK(s)^{1/3}}{TK(s)^{1/3}} \implies E_{\text{opt}} = \int \mathrm{d}E &= \int_0^L \mathrm{d}sK(s)v^(s) \\
&= \int_0^L \mathrm{d}sK(s)\left(\frac{\int_0^L \mathrm{d}sK(s)^{1/3}}{TK(s)^{1/3}}\right)^2 \\
&= \int_0^L \mathrm{d}sK(s)\frac{L^2}{T^2}\frac{\langle K(s)^{1/3}\rangle^2}{K(s)^{2/3}} \\
&= \frac{L^2}{T^2}\int_0^L \mathrm{d}sK(s)^{1/3}\langle K(s)^{1/3}\rangle^2 \\
&= \frac{L^3}{T^2}\langle K(s)^{1/3}\rangle^3 \\
&= c^3T
\end{aligned}$$

\subsection*{(d)}

If v is constant, i.e., $v \neq \mathrm{func(s)} \mid v = v_0 = L/T$, then

$$E = \int_0^L K(s)v_0^2\mathrm{d}s = \frac{L^2}{T^2}\int_0^L \mathrm{d}sK(s)$$

Then if $\langle K(s)^{1/3}\rangle^3 \leq \langle K(s)\rangle$ then it follows trivially that $E \geq E_{\text{opt}}$, i.e., the energy consumption is lower whenever the speed is varied as a function of time, instead of being fixed at the average speed L/T.

\section*{Question 3}

We use the cylindrical coordinates

$$x = \rho\cos\theta, \quad y = \rho\cos\theta, \quad z = z$$

Let $\rho = 1$, i.e., the radius is 1, then

$$\mathrm{d}x^2 = (-\sin\theta)^2\mathrm{d}\theta^2, \quad \mathrm{d}y^2 = \cos^\theta\mathrm{d}\theta^2, \quad \mathrm{d}z^2 = \mathrm{d}z^2$$

From the above we can express the length integral

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \mathrm{d}z^2 \implies \mathrm{d}s = \sqrt{1 + z'^2}\mathrm{d}\theta$$

From which it follows that

$$I = \int \mathrm{d}s = \int \sqrt{1+z'^2}\mathrm{d}\theta = F(\theta, z, z').$$

We use Euler-Lagrange second form to find z',

$$\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial F}{\partial z'} = 0$$

$$\begin{aligned} F \neq \mathrm{func}(z) &\implies \frac{\partial F}{\partial z'} = c \\ &\implies \frac{\partial}{\partial z}\sqrt{1+z'^2} = c \\ &\implies \frac{z'}{\sqrt{1+z'^2}} = c \end{aligned}$$

Solving for z' yields

$$z' = \frac{c}{\sqrt{1-c^2}}$$

\section*{Question 4}

Let F be $n(y)\sqrt{1+x'^2}$. We use Euler-Lagrange second form

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial F}{\partial x'} = 0 \overset{F\neq \mathrm{func(x)}}{\implies} \frac{\partial F}{\partial x'} = y_0$$

We look for an expression for x'

$$\begin{aligned} \frac{\partial}{\partial x'}\left(n(y)\sqrt{1+x'^2}\right) &= y_0 \\ n(y)\frac{x'}{\sqrt{1+x'^2}} &= y_0 \\ n(y)^2x'^2 &= y_0^2 + x'^2y_0^2 \\ x' = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y_0}{\sqrt{n(y)^2 - y_0^2}} \end{aligned}$$

We use separation of variables to solve the previous ordinary differential equation

$$\begin{aligned} \int \mathrm{d}x &= y_0 \int \frac{\mathrm{d}y}{\sqrt{n(y)^2-y_0^2}} \\ x + x_0 &= y_0 \arcsin\left(\frac{y_0}{n(y)}\right) \end{aligned}$$

We wish to express this as $y(x)$ so we rearrange the above expression

$$\begin{aligned} \sin\left(\frac{x+x_0}{y_0}\right) = \frac{y_0}{n(y)} &\implies n(y) = y_0 \sin\left(\frac{x+x_0}{y_0}\right) \\ &\implies n_0\left(1-\frac{y^2}{a^2}\right) = y_0\sin\left(\frac{x+x_0}{y_0}\right) \\ &\implies \left[1-\left(\frac{y_0}{n_0}\sin\left(\frac{x+x_0}{y_0}\right)\right)\right]a^2 = y^2 \end{aligned}$$

We conclude,

$$\therefore y(x) = a\sqrt{1 - \frac{y_0}{n_0}\sin\left(\frac{x+x_0}{y_0}\right)}$$

Lecture 6

Recall

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\mathrm{d}t \qquad , L \equiv T - V \qquad , \delta S = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

3.2 D'Alembert's Principle and Constraints

3.2.1 Constraints

Let there be a system of N particles with m constraints such that $L = 3N - m$ (numbers of degrees of freedom)

Example A parabola x^2 shape with radius e , $z = c\rho^2$. $z - c\rho^2 = 0$.

In general, one could write

$$z_i - g(x_i, y_i) = 0$$

$g(x_i, y_i) \equiv$ equation of constraint $\{x_i, y_i\} \to \{z_i\}$.

In general, we say that a constraint is holonomic if the function f_p that describes the can be written as

$$f_p(\vec{r_1}, \vec{r_2}, \vec{r_3}, \ldots, t) = 0 \qquad , p = 1, 2, \ldots, m$$

Recall In general, we have $L = 3N - m$ degrees of freedom, which means $q_1, q_2, \ldots, q_L$ (coordinates).

3.2.2 D'Alembert's Principle of Virtual Work

> Based on the concept of a virtual displacement of the system
>
> $\equiv$ arbitrary change of coordinates leading to a infinitesimal displacement δr_j at an instantaneous time t.

This is to be distinct (not the same thing) of a *real* displacement $\mathrm{d}r_j$ in a time interval $\mathrm{d}t$.

A real displacement $\mathrm{d}r_j$ has to be in a time $\mathrm{d}t$. It must follow some laws of physics. Here we define a virtual displacement, where we instantaneously displace the coordinates of the object (teleportation ?) $\implies$ not physics in a sense

Let's call $\vec{F}_j$ the force on a particle j at equilibrium ($\vec{F} = 0$). Let us decompose $\vec{F}_j$ at equilibrium :

$$\vec{F}_j = \vec{F}_j^{\text{app}} + \vec{F}_j^{\text{cont}} = \vec{0}$$

where *cont.* is *constraint*.

We can generalize to a system

$$\sum_j^N F_j = \vec{0} \qquad \therefore \sum_j^N \vec{F}_j \cdot \delta \underbrace{\vec{r}_j}_{\text{virtual}} = 0$$

$$\implies \sum_j^N \vec{F}_j^{\text{app}} \cdot \delta\vec{r}_j + \sum_j^N \vec{F}_j^{\text{cont}} \cdot \delta\vec{r}_j = 0$$

```
|                 |
 \   (F_j^cont)   /
   \      |      /
     \    |    /
       - - - -----> delta r_j
```

holonomic constraint, the force of constraint is always $\perp$ to $\delta\vec{r}_j$, tangential,

$$\boxed{\vec{F}_j^{\text{cont}} \cdot \delta\vec{r}_j = 0} \equiv \text{virtual work}$$

We say that a holomic system is in equilibrium if the virtual work of the applied forces is also zero :

$$\boxed{\delta W = \sum_j^N \vec{F}_j^{\text{app}} \cdot \delta\vec{r}_j = 0} \quad \underline{\text{static}}$$

Dynamical System :

$$\vec{F}_j = \vec{F}_j^{\text{app}} + \vec{F}_j^{\text{cont}} = \dot{p}_j,$$

Therefore, we can generalize the D'Alembert principle with dynamics

$$\sum_j^N \left[\vec{F}_j^{\text{app}} - \dot{p}_j\right] \cdot \delta\vec{r}_j = 0 \quad \stackrel{??}{\longleftrightarrow} \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} = 0$$

3.2.3 Lagrange Equation From Virtual Work

Consider a *radius vector* : $\vec{r}_j = \vec{r}_j(q_1, q_2, \ldots, q_l)$, $L = 3N - m$, $j = 1, \ldots, N$. With out loss of generality,

Error: Missing argument for \vec

Then,

- Real

$$\dot{\vec{r}}_j = \sum_{k=1}^{l} \frac{\partial \vec{r}_j}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_j}{\partial t}$$

- Virtual

$$\delta\vec{r}_j = \sum_{k=1}^{l} \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k$$

So then

$$\sum_j \left[\underbrace{\vec{F}_j^{\text{app}}}_{(1)} - \underbrace{\dot{p}_k}_{(2)}\right] \cdot \delta q_k = 0$$

Term (1) :

$$\sum_j \vec{F}_j^{\text{app}} \cdot \delta\vec{r}_j = \sum_{j,k} \vec{F}_j^{\text{app}} \cdot \frac{\partial \vec{r}_j}{\partial q_k}\delta q_k$$
$$\equiv \sum_{k=1}^{l} Q_k^{\text{app}}\delta q_k,$$

where

$$Q_k^{\text{app}} \equiv \sum_j \vec{F}_j^{\text{app}} \cdot \frac{\partial \vec{r}_j}{\partial q_k}$$

We have the Q_k^{app} as the *"generalized applied forces of the system"*. Meaning that for each degrees of freedom k we can define a q_k . We transformed a basis in generalized coordinates and we can now this Q applied on the generalized forces.

> next lecture we will show that
>
> $$(1) + (2) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k^{\text{app}}.$$

Lecture 7

D'Alembert's principle of virtual work (dynamical)

$$\sum_j (\underbrace{\vec{F}_j^{\text{app}}}_{(1)} - \underbrace{\dot{\vec{p}}_j}_{(2)}) \cdot \delta\vec{r}_j = 0 \qquad \delta\vec{r}_j = \sum_{k=1}^{l}\frac{\partial \vec{r}_j}{\partial q_k}\delta q_k$$

Generalized applied forces from (1) :

$$\sum_j \vec{F}_j^{\text{app}} \cdot \delta\vec{r}_j = \sum_{j,k} \vec{F}_j^{\text{app}} \cdot \frac{\partial \vec{r}_j}{\partial q_k}\delta q_k = \sum_{k=1}^{l} Q_k^{\text{app}}\delta q_k \ ; \ Q_k^{\text{app}} \equiv \sum_j \vec{F}_j^{\text{app}} \cdot \frac{\partial \vec{r}_j}{\partial q_k}$$

The second term :

$$\sum_j^N \cdot \dot{P}_j \cdot \delta\vec{r}_j = \sum_j m_j \ddot{\vec{r}}_j \cdot \vec{r}_j = \sum_{j,k} m_j \ddot{\vec{r}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k}\delta q_k$$
$$= \sum_K \left[\sum_j \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(m_j \dot{r}_j \frac{\partial \vec{r}_j}{\partial q_k}\right) - m_j \dot{r}_j \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \delta r_j}{\partial \delta q_k}\right\}\right]\delta q_k$$
$$= \sum_k \left[\sum_j \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(m_j \dot{\vec{r}}_j \cdot \frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_k}\right) - m_J \dot{r}_j \left(\frac{\partial \dot{r}_j}{\partial q_k}\right)\right\}\right]\delta q_k$$
$$= \sum_k \left[\sum_j \left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial q_k}\left(\frac{m_j \dot{\vec{r}}_j^2}{2}\right) - \frac{\partial}{\partial q_k}\left(\frac{m_j \dot{\vec{r}}_j^2}{2}\right)\right\}\right]$$
$$= \sum_k^l \left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right]\delta q_k$$

From

$$\sum_j \left(\vec{F}_j^{\text{app}} - \dot{p}_j\right) \cdot \delta\vec{r}_j = \sum_{k=1}^{l} \left[Q_k^{\text{app}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} + \frac{\partial T}{\partial q_k}\right] 0$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k^{\text{app}}, \qquad k = 1, 2, \ldots, l$$

For Conservative Forces :

$$\vec{F}_j^{\text{app}} - \vec{\nabla}_j V(\vec{r}_1, \vec{r}_2 m \ldots, \vec{r}_n)$$

For Generalized Forces :

$$Q_k^{\text{app}} = \sum_j F_j^{\text{app}} \cdot \frac{\partial \vec{r}_j}{\partial q_k} = -na\vec{b}la(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n) \cdot \frac{\partial \vec{r}_1}{\partial q_k}$$
$$= -\frac{\partial V}{\partial q_k}$$

So then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

Q Is this Lagrange ?

Let us rewrite this

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}(T - V) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0$$

When $V \neq V(t)$ and $V \neq V(\dot{q})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

3.2.4 Lagrange Multipliers and Constraints

- Least action $\longleftrightarrow$ Aesthetically beautiful
- D'Alembert $\longleftrightarrow$ more "powerfulness" $Q_k^{\text{app}}\ ;\ Q_k^{\text{cont}}$

Lecture 8

3.2.4 Lagrange multipliers and constraints

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k^{\text{app}},$$

where $Q_k^{\text{app}} = \sum_j \vec{F}_j^{\text{app}}(\partial\vec{r}_j/\partial q_k)$. Consider

$$f_p(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = 0 \equiv f_p(x_1, x_2, \ldots, x_N) = 0,$$

$x_i \equiv$ cartesian coordinates. Let us make a small variation in the constraint

$$\delta f_p = \sum_i^{3N} \frac{\partial f_r}{\partial x_i}\delta x_i = 0 \implies \sum_{p=1}^{m} \lambda_p \sum_i^{3N} \frac{\partial f_p}{\partial x_i}\delta x_i = 0$$

D'Alambert's Principle

$$\sum_{i=1}^{3N}\left(F_i^{\text{app}}-\dot{P}_i\right)\cdot\delta x_i=0,$$

here $i\equiv$ not generalized.

$$\sum_{i=1}^{3N}\left\{F_i^{\text{app}}-\dot{P}_i+\sum_{p=1}^{m}\lambda_p\frac{\partial f_p}{\partial x_i}\right\}\cdot x_i=0$$

> Note that here we added a zero. This is not that intuitive.

Interpretation a la Newton

$$\underbrace{\overset{(1)}{F_i^{\text{app}}}}+\underbrace{\overbrace{\sum_{p=1}^{m}\lambda_p\frac{\partial f_p}{\partial x_i}}^{(2)}}_{F_i^{\text{const}}}=\overbrace{\dot{P}_i}^{(3)}$$

$$F_i^{\text{app}}+F_i^{\text{const}}=\dot{P}_i$$

1. For (1) and (2)) :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k^{\text{app}}.$$

Let us define the generalized forces of constraints :

$$Q_k^{\text{const}}=\sum_{i=1}^{3N}\sum_{p=1}^{m}\lambda_p\frac{\partial f_p}{\partial x_i}\frac{\partial x_i}{\partial q_k}=\sum_{p=1}^{m}\lambda_p\frac{\partial f_p}{\partial q_k}$$

Therefore ,

2. (1) (2) (3) :

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k^{\text{app}}+Q_k^{\text{const}}},\qquad k=1,2,\ldots,3N$$

Note generalized coordinates may or may note be independent. The above gives us $3m$ equations and we have m equations of constraint. So we have a total of $3N+m$. In the above we have $3N$ unknowns and m unknowns , so the system is solvable.

Recall

$$f_p\longrightarrow f_p=1,\ldots,m\qquad\text{we have } m\ \lambda_p\ \text{ unknowns.}$$

Example

Consider an incline right triangle with angle α. Along with a disk of radius R rolling over it, where y is the distance, parallel to the hypothenuse from the top up till the center of the disk. Let the mass be M and $I=mR^2/2$.

Let us work out the kinetic energy.

$$\begin{aligned}\text{Kinetic Energy }&:\frac{1}{2}m\dot{y}^2+\frac{1}{2}I\dot{\theta}^2\\ \text{Potential Energy }&:V=mg(l-y)\sin\alpha\end{aligned}$$

So then ,

$$L=\frac{1}{2}m\dot{y}^2+\frac{1}{4}MR^2\dot{\theta}^2+Mg(y-l)\sin\alpha \tag{1}$$

The disk is rolling without slipping. Which means mathematically that $f(y,\theta)=y-R\theta=0$ or in other words ,$y=R\theta$

Choice :

- we could feed in $y=R\theta$ in (1) and get 1 Lagrange Equation
- Keep both $\longrightarrow$ get 2 Lagrange equations.

Let us write down the set of Lagrange equations.

Note We need only one lambda since we have only one constraint. *(There's also the normal force pushing on the incline but we ignore this force for simplicity)*.

1. In y :

$$\frac{\partial L}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}}+\lambda\frac{\partial f}{\partial y}=0 \qquad , f(y,\theta)=y-R\theta.$$

2. In θ :

$$\frac{\partial L}{\partial \theta}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}}+\lambda\frac{\partial f}{\partial \theta}=0$$

Here

$$\begin{aligned}\frac{\partial L}{\partial \dot{y}}&=m\ddot{y}\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}}&=\frac{1}{2}mR^2\ddot{\theta}\\ \frac{\partial L}{\partial y}&=mg\sin\alpha\\ \frac{\partial L}{\partial \theta}&=0\end{aligned}$$

Sanity Check

y,θ,λ that's three equationss, $2L-1$ constants.

- $y: mg\sin\alpha-m\ddot{y}+\lambda=0$
- $\theta: -1/2mR^2\ddot{\theta}-\lambda R=0$

$$y=R\theta \implies \ddot{y}=R\ddot{\theta}$$

If we plug this in then we get

$$-\frac{1}{2}mR\ddot{y}-\lambda R=0 \implies \lambda=-\frac{1}{2}m\ddot{y}$$

plug this in y , and get

$$\begin{aligned}mg\sin\alpha-m\ddot{y}-\frac{1}{2}m\ddot{y}=0 &\implies \ddot{y}=\frac{2g}{3}\sin\alpha\\ &\implies \ddot{\theta}=\frac{2g\sin\alpha}{3R}\end{aligned}$$

Plug this again ang we get

$$\lambda = -\frac{mg}{3}\sin\alpha$$

Q How much to roll without slipping ? How much *Reaction* must be provided by the incline ?

Recall $Q_k^{\text{const}} = \lambda \partial f_k / \partial Q_k$

Since we have $f(R,\theta) = y - R\theta$, then we consider both variables :

- Along y :

$$\begin{aligned} Q_y &= \lambda \frac{\partial f}{\partial y} = \lambda = -\frac{mg}{3}\sin\alpha \qquad \longleftarrow \text{ Force} \\ Q_\theta &= \lambda \frac{\partial f}{\partial \theta} = -\lambda R = \frac{mgR}{3}\sin\alpha \qquad \longleftarrow \text{ Torque} \end{aligned}$$

These two Qs is what needs to "happen" to roll without slipping

Lecture 3

3.3 Symmetry and Conservation Theorem

[Theorem] (*Norther's theorem*) "it's from symmetries that arises physical conservation laws".

Property Inertial frame	Properties of Lagrangian	Conserved quantity
Time uniformity	$\frac{\mathrm{d}L}{\mathrm{d}t} = 0$	Energy
Space Homogeneity	$L'(\vec{r}') = L(\vec{r}); \vec{r}' = \vec{r} + \delta\vec{r}$	Linear Momentum
Space isotropy	$L'(\vec{r}') = L(\vec{r})$ $\vec{r}' = \vec{r} + \delta\vec{r} \quad \delta\vec{r} + \delta\vec{\theta} \wedge \vec{r}$	Angular Momentum

3.3.1 Time translational Invaried : Conservation Energy

$$L = T - V,$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \frac{\partial L}{\partial q_i} \cdot \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}$$

Lagrange tells us :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_i}\right) = \frac{\partial L}{\partial q_i}$$

By substitution we get

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_i \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} \end{aligned}$$

Note that in both lines we have total derivatives so we can put them together ?

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right\} + \frac{\partial L}{\partial t} = 0$$

Now we say

$$F=F(y,y',x)$$

$$\frac{\partial F}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}x}\left\{F-y'\frac{\partial F}{\partial y'}\right\}=0$$

$$F\neq F(x)\implies F-y'\frac{\partial F}{\partial y'}\equiv \text{constant!}$$

if $L\neq L(t)$ then $\partial L/\partial t=0$ and so

$$\sum_i \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L=E$$

Q What is E ?

$L=T(q,\dot{q})-V(q)$. T is a quadratic function of generalized coordinate $\dot{q}$.

Theorem (*Euler's theorem*) :

$$\sum_i \dot{q}_i\frac{\partial L}{\partial \dot{q}}=\sum_i \dot{q}_i\frac{\partial T}{\partial \dot{q}_i}=2T$$
$$\therefore E=2T-T+V=T+V$$

3.3.2 Space Translational Invariance

1. Cartesian $\vec{r}$:

   Let us do the transformation $\vec{r}_\alpha=\vec{r}+\epsilon$, where $\vec{\epsilon}\ll\vec{r}_\alpha$, and α stands for all degrees of freedom.

   Let us do a variation on L .

$$\delta L=\sum_\alpha\frac{\partial L}{\partial\vec{r}}\delta\vec{r}_\alpha=\vec{\epsilon}\frac{\partial L}{\partial\vec{r}_\alpha}.$$

   Since $\vec{\epsilon}\neq 0$, then

$$\implies\sum_\alpha\frac{\partial L}{\partial\vec{r}_\alpha}=0,$$

   because $\delta L=0$.

   So the Lagrange equation gives us

$$\sum_\alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{r}}_\alpha}\right)-\underbrace{\frac{\partial L}{\partial\vec{r}_\alpha}}_{=0}=0$$

$$\implies\sum_\alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{r}}_\alpha}\right)=0\quad\boxed{\frac{\mathrm{d}\vec{P}_\alpha}{\mathrm{d}t}}=\vec{e}$$

   > L translationally invariant (homogeneous) $\implies$ linear momentum conservation.

   Recall that $L(x)=L(x+\delta(x))$ in 1 dimension. Since $L=\frac{1}{2}m\dot{x}^2$, does this L depend on x ? No, so the lagrangian conserves linear momentum. Now take $L=\frac{1}{2}m\dot{x}^2-V(x)=\frac{1}{2}m\dot{x}^2-k/x$, then here the Lagrangian momentum is not the same. This means that space must be homogeneous.

Recall (*Hamiltonian*) :

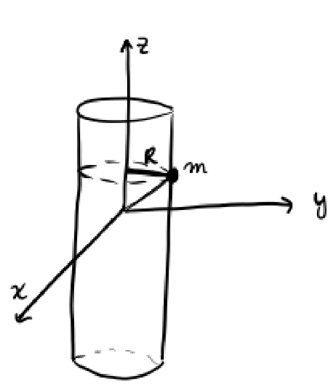
$$H(p,q,t) = \sum_i p_i \dot{q}_i - \mathcal{L} \qquad \text{Hamilton}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \qquad \text{Hamilton's canonical Equation}$$

Example



COnsider a particle on a cylinder. Then $\vec{F} = -k\vec{r}$. We have

$$x^2 + y^2 = R^2$$

$$\vec{r} = \begin{cases} x = R\cos\theta & \dot{x} = -R\sin\dot{\theta} \\ y = R\sin\theta & \dot{y} = R\cos\dot{\theta} \\ z = z & \dot{z} = \dot{z} \end{cases}$$

It follows that

$$T = \frac{m}{2}(R^2\dot{\theta}^2 + \dot{z}^2)$$

$$V = \frac{1}{2}k(x^2 + y^2 + z^2\frac{1}{2}k(R^2 + z^2)$$

$$\mathcal{L} = T - V = \frac{m}{2}(R^2\dot{\theta}^2 + z^2) - \frac{1}{2}k(R^2 + z^2)$$

We compute the relative momentums generalized coordinates

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

We apply Hamilton's equation

$$\begin{aligned} H(p,q,t) &= \sum_i p_i \dot{q}_i - \mathcal{L} \\ &= p_z\dot{z} + p_\theta\dot{\theta} - \frac{m}{2}(R^2\dot{\theta}^2 + z^2) - \frac{1}{2}k(R^2 + z^2) \\ &= m\dot{z}^2 + mR^2\dot{\theta}^2 - \frac{m}{2}(R^2\dot{\theta}^2 + \dot{z}^2) + \frac{k}{2}(R^2 + z^2) \\ &= \frac{m}{2}(R^2\dot{\theta}^2 + \dot{z}^2) + \frac{k}{2}(R^2 + z^2) \end{aligned}$$

Futher, by definition of p_θ, p_z,

$$H(p,q,t) = \underbrace{\frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m}}_{T} + \underbrace{\frac{k}{2}(R^2+z^2)}_{V}$$

Is this an instance of *conserved* ? $\rightarrow$ Yes. Is it energy conserved $(T+V)$? Yes. Thus,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \\ \dot{p}_z = -\frac{\partial H}{\partial z} = -kz \end{cases} \quad \begin{array}{l} p_\theta \equiv \text{constant} \\ \frac{\mathrm{d}p_z}{\mathrm{d}t} = -kz \end{array}$$

$$\dot{q}_i = -\frac{\partial H}{\partial p_i} \quad \begin{cases} \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \\ \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \end{cases} \quad \begin{array}{l} p_\theta mR^2\dot{\theta} \\ p_z = m\dot{z} \end{array}$$

$$\therefore \frac{\mathrm{d}p_z}{\mathrm{d}t} = -kz = m\ddot{z} \implies \underbrace{\boxed{\ddot{z} + \frac{k}{m}z = 0}}_{\text{Harmonic oscillator}}$$

We have the correspondence

$$\begin{array}{lcr} \text{quantum} & \longleftrightarrow & \text{Classical} \\ H(p,q) & & H, \mathcal{L} \\ \textit{Schrodinger} & & \textit{Lagrange, Hamilton} \\ [H,\hat{\alpha}] = 0 \;\; [\hat{p},\hat{q}] = i\hbar = 0 & & \{p_i, q_i\} \text{ Poisson Bracket} \\ \text{Heisenberg} & & \{H, q_i\} \end{array}$$

3.4.3 Poisson Brackets

$$\text{Hamilton's formalism} \implies \text{useful} \implies \begin{cases} \text{constant of motion} \\ \text{conserved ...} \\ \text{First intergral of motion} \end{cases}$$

Consider a function $f = f(p,q,t)$.

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \sum_i \left(\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right) & \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \right) & \dot{q}_i = -\frac{\partial H}{\partial p_i} \end{aligned}$$

Then

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{H, f\}}$$

where

$$\{H, f\} = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} = \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$$

If $f \neq f(t)$ and $\{f, H\} = 0$, then $\mathrm{d}f/\mathrm{d}t = 0$.

$$H = \frac{p^2}{2m} + V(q)$$

$$\{H, p\} = \frac{\partial H}{\partial p}\cancel{\frac{\partial p}{\partial q}} - \frac{\partial p}{\partial p}\cancel{\frac{\partial H}{\partial q}} = 0 \quad \text{since } -\frac{\partial V(q)}{\partial q} \neq 0$$

We have the correspondence

$$\text{Poisson Bracket} \longrightarrow \text{Lie group } (\textit{Lie algebra})$$

Bracket Properties

1. $\{p, q\} = -\{g, f\}$
2. $\{f, c\} = 0$ where $c \equiv$ constant
3. $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$
4. $\{f_1 f_2, g\} = f_1\{f_i, g\} + f_2\{f_1, g\}$
5. $\frac{\partial}{\partial t}\{f, g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\}$
6. $\{f, q_i\} = \frac{\partial f}{\partial p_i}$
7. $\{f, p_i\} = -\frac{\partial f}{\partial q_i}$
8. $\{q_i, q_k\} = \{p_i, p_k\} = 0, \quad \{p_i, q_k\} = \delta_{ik}$
9. $\{f, \{g, h\}\} + \{g, \{f, h\}\} + \{h, \{f, g\}\} = 0$ Jacobian Identity

[Theorem] (*Poisson Theorem*) :

If g, h are constant of motion, $\{f, H\} = \{g, H\} = 0$, then $\{f, g\} = 0$

Lecture 12

Given $f = f(p, q, t)$, we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{H, f\} \implies \frac{\mathrm{d}\langle A\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle [A, H]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$$

$$\{H, f\} \equiv \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$\{H, f\} = 0 \text{ if } f \neq f(t) \implies \frac{\mathrm{d}f}{\mathrm{d}t} = 0$$

Where $\{H, f\}$ the Poisson bracket is a linear operator

3.4.4 Action and Hamiltonian : Hamilton's modified least action principle

Consider the 2D plan with t against q, with the points (t_1, q_1) and (t_2, q_2) and consider the extremal path connecting them. Then

Recall

$$\delta S = \frac{\partial L}{\partial \dot q}\delta q\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}\right)\delta q \mathrm{d}t$$

Consider now

$$S \equiv \int_{t_1}^{t_2} \mathcal{L}(q, \dot q, t)\mathrm{d}t, \qquad \mathcal{L} = \sum_i p_i \dot q_i - H$$

So that now we write the modified Hamilton's Principle as

$$\delta \int_{t_1}^{t_2}\left(\sum_i p_i \dot q_i - H\right)\mathrm{d}t = 0$$

Now we need to carry in the δ variation, so the above will be equal to

$$= \int_{t_1}^{t_2}\sum_i\left(p_i\delta\dot q_i + \dot q_i \delta p_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)\mathrm{d}t = 0$$

Not that two terms have the same δp_i and two others do not. So what we do is

$$\delta \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i \implies \int_{t_1}^{t_2}\sum_i p_i \delta \dot{q}_i = -\int_{t_1}^{t_2}\sum_i \dot{p}_i \delta q_i \mathrm{d}t$$

Remark In the above we used $uv - \int v\mathrm{d}u$.

Now we got rid of the *dot*, this is great because now we get

$$\implies \int_{t_1}^{t_2}\sum_i \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right]\mathrm{d}t = 0$$

where $\delta p_i, \delta q_i$ are independent variations so $\neq 0$

Q What do we conclude ?

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

Hamilton's Canonical Equations!

4. Small Oscillations

```
|      k_1   x_1    k_2   x_2
|----....---o---....---o---...----          "n" of them
|          m_1 = M     m_2 = M              ---->\hat{x}
```

Definition (*Small Oscillations*) : "periodic motion of the system in the vicinity of its equilibrium position"

Given a spring in $\hat{x}$ being pulled by a force F, which is attached to a wall, we have the decomposition

$$F(x) = F_0 + x\left(\frac{\partial F}{\partial x}\right)_0 + \frac{1}{2!}\left(\frac{\partial^2 F}{\partial x^2}\right)_0 + \dots$$

We call a Hook's Spring if we neglect all the other terms, so we have

$$F_{\text{Hook}}(x) = x\left(\frac{\partial F}{\partial x}\right)_0 \equiv -kx \qquad \text{Hook's Law}$$

Further the potential is

$$V(x) = V(x_0) + (x - x_0)\frac{\partial V}{\partial x}\bigg|_{x_0} + \underbrace{\frac{1}{2!}\left(\frac{\partial^2 V}{\partial x^2}\right)\bigg|_{x_0}}_{\text{The first nonzero term}} + \dots$$

Q What is $\frac{\partial V}{\partial x}\big|_x \equiv 0$ of equilibrium.

Conditions

- $$\left(\frac{\partial V}{\partial x}\right)\bigg|_{x_0} = 0 \implies \text{equilibrium}$$
- $$\text{Stable} \quad \frac{\partial V}{\partial x}\bigg|_{x_0} > 0$$

Let us *expand* "q" in 1 D

$$V(q) - V(q_0) \approxeq \frac{k}{2}(q-q_0)^2 \qquad k > 0 \equiv \left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0}$$

This is the harmonic potential.

$$q - q_0 \equiv x \qquad V(x) = \frac{kx^2}{2} \qquad \mathcal{L} = \frac{m\ddot{x}^2}{2} - \frac{kx^2}{2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial \mathcal{L}}{\partial x} = -kx$$
$$\boxed{\ddot{x} + \omega_0^2 x = 0} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

And the solution, as we know it, is $A\cos(\omega_0 x + \alpha)$

```
|      k   x_1    k_12   x_2        k      |
|----....---o---....---o---...----...|
|          m_1 = M     m_2 = M             |
```

Q How many degrees of freedom do we want here ? 3

Lecture 14

We want to solve algebraic equations :

$$\sum_j (A_{ij} - \omega^2 M_{ij})a_{ij} = 0 \qquad q_j(t) = a_j e^{i(\omega t - \delta)}$$

We need to find the determinant of

$$\det\left|A_{ij} - \omega^2 M_{ij}\right| = 0$$

We can then solve for the eigenstate and eigenvectors.

We have Secular Equations (also known as characteristic equations) of degree N in ω^2

Remarks

1. The $A_{ij} = A_{ji}$ and $M_{ij} = M_{ji}$.
2. ω_r is the *eigenfrequencies*.
3. If several ω_r are equal (we have several motions that have the same frequencies) then we say there are *degeneracies*, meaning more than one degenerate oscillatory mode with same ω_r.

   That should not surprise us because to lift the degeneracy of a spin we put it in a magnet. That's basically the same thing here.
4. N eigenfrequencies $\implies$ we can construct N eigenvectors $\vec{a}_r$,each of which associated with an eigenfrequency.

   Denote a_{jr} be the j^{th} component or the r^{th} eigenvector.

Q What would be if we run the experiment and we nave N oscillators and we walk out of equilibrium. What will be the more general solution ? $\longrightarrow$ A sum of all of them a superposition (*sum of normal modes*).

The most general solution is gonna be

$$q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)} \qquad \text{Only the Real part}$$

Q why do we go through complex solution ? For easiness, because it's nice when it's in the exponent (exponents are easy). In final we can be asked to put numbers and we can just add exponents up.

We can show that the amplitude of a_{jr} of $q_j(t)$ forms an orthogonal basis (*From linear algebra*), where

$$\sum_{ij} M_{ij} a_{jr} a_{is} = 0 \qquad \text{, for } r = s$$

Definition (*Orthonormal*) : A orthogonal that is normalized to 1.

We can form an orthonormal basis

$$\begin{aligned} q_j(t) &= \sum_r \alpha_r a_{jr} e^{i(\omega_r t - \delta_r)} \qquad \text{mecanic} \\ &= \sum_r \beta_r a_{jr} e^{i(\omega_r t)} \\ &= \sum a_{jr} \eta_r(t), \end{aligned}$$

where $\eta_r(t) \equiv \beta_r e^{i\omega_r t}$. Here η_s is called the Normal coordinates.

Recipe to Find The Normal Modes and Frequencies

1. You have to choose wisely the generalized coordinates $\longrightarrow$ Then form $T, V \to \mathcal{L}$
2. Form the matrices M_{ij} and A_{ij} as $N \times N$ arrays.

$$A_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 \qquad T = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j.$$

3. Solve $\left|A_{ij} - \omega^2 M_{ij}\right| = 0 \implies \omega_r$.
4. For each ω_r : $a_{ir} : a_{2r} : \cdots : a_{nr}$ do

$$\sum_i (A_{ij} - \omega_r^2 M_{ij}) a_{jr} = c$$

5. If you wish normalize.

Example

```
|      k    m_1 k_12 m_2   k     |
|---......---......---....---|
|         ->x_1      -> x_2      |
```

here we have

$$V = \frac{kx_1^2}{2} + \frac{1}{2}kx_2^2 + \frac{1}{2}k_{12}(x_1 - x_2)^2$$

Then calculate the matrices

$$A_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \qquad A_{ij} = \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}$$

Note It has to be a harmonic potential because we're doing the general theory of small oscillations, so every thing we write down is good if it's harmonic (first nonzero term in the oscillation).

Then we form the kinetic energy (*easy*)

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$
$$M_{ij} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \qquad \text{because } M_{ij} = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j$$

Then we compute the determinant

$$\begin{vmatrix} k + k_{12} - M\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - \omega^2 M \end{vmatrix} = 0$$

And the second equation

$$(k + k_{12} - M\omega^2)^2 - k_{12}^2 = 0$$
$$k + k_{12} - M\omega^2 = \pm k_{12}$$

Then solving for ω we get

$$\omega = \sqrt{\frac{k + k_{12} \pm k_{12}}{m}} \qquad \implies \omega_1 = \sqrt{\frac{k + 2k_{12}}{m}},\ \omega_2 = \sqrt{\frac{k}{M}}.$$

Note Final Exam One problem on Small Oscillation. Three questions so we can guess them (Lagrange, Hamilton, etc).

---

a.

Given the Lagrangian

$$\mathcal{L} = T - V = \frac{m\dot{\theta}^2}{2} + mg\cos\theta,$$

then we have the canonical variables $p_\theta = \partial\mathcal{L}/\partial\dot{\theta} = m\dot{\theta}$ and $q = \theta$.

b.

We apply the definition of the Hamiltonian

$$\begin{aligned} H(p, q, t) &= p_\theta\dot{\theta} - \theta, \dot{\theta}, t \\ &= m\dot{\theta}^2 - \left(\frac{m\dot{\theta}^2}{2} + mg\cos\theta\right) \\ &= m\dot{\theta}^2 - mg\cos\theta \\ &= \frac{p_\theta^2}{m} - mg\cos\theta \end{aligned}$$

c.

By definition, given $\partial\mathcal{L}/\partial t = 0$, then

$$E = \dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L},$$

let us verify this.

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = m\dot{\theta}^2 - mg\cos\theta = H.$$

Here $H = E$, therefore we conclude that the energy is conserved.

d.

$$\begin{aligned}
\{H, \theta\} &= \left(\frac{\partial H}{\partial p_\theta}\frac{\partial \theta}{\partial \theta} - \frac{\partial \theta}{\partial p_\theta}\frac{\partial H}{\partial \theta}\right) = \frac{2}{m}p_\theta \\
\{H, p_\theta\} &= \left(\frac{\partial H}{\partial p_\theta}\frac{\partial p_\theta}{\partial \theta} - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial H}{\partial \theta}\right) = mg\sin\theta
\end{aligned}$$

Here we observe that

$$\{H, \theta\} = \frac{2}{m}p_\theta = \frac{\partial H}{\partial p_\theta} \qquad \text{and} \qquad \{H, p_\theta\} = mg\sin\theta = \frac{\partial H}{\partial \theta},$$

so we conclude that

Lecture 15

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5. Central Force Motion

5.1 The two-body Problem (general)

2 masses m_1 , m_2 subjected to a potential $V(\vec{r})$ where $|\vec{r}| = |\vec{r}_2 - \vec{r}_1|$. This is a very special potential, it's a potential $V(r) \sim 1/r$ depends only on the distance between the two masses.

Q Give an example of such potential that you know , Coulombe for instance.

The Central potential has a lot of important properties. Let us define $\vec{r}$ as the distance between the two masses and $\vec{R}_{\text{CM}}$ the center of mass coordinate. We wish to write the Lagrange

$$\begin{aligned}
\mathcal{L} &= T(\dot{\vec{R}}_{\text{CM}}, \dot{\vec{r}}) - V(\vec{r}, \vec{R}_{\text{CM}}) \\
\vec{R}_{\text{CM}} &\equiv \vec{0} \qquad m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}
\end{aligned}$$

Now we write

$$\begin{aligned}
\vec{r}_1 &= \frac{m_2}{m_1 + m_2}\vec{r} \qquad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r} \\
\mathcal{L} &= \frac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\vec{r}}_2\right|^2 - V(r) \\
\mathcal{L} &= \frac{1}{2}\mu|\dot{r}|^2 - V(r) \qquad , \mu \equiv \frac{m_1 m_2}{m_1 + m_2},
\end{aligned}$$

where we define μ as the *reduced mass*.

Note Everything depends *only* on $\vec{r}$!

First integral of motion and Kepler's law.

Consider $V = V(r)$ and $r \equiv \left|\vec{r_2} - \vec{r_1}\right|$, this is called the Central potential.

- It has spherical symmetry.
- It is invariant under rotation .
- The angular coordinate is cyclic.
- It conserve $\vec{\mathcal{L}} = \vec{r} \cdot \vec{p}$, i.e, $\vec{\mathcal{L}} =$ constant $\forall\, \hat{m}$.
- $\vec{r}$ always perpendicular to $\vec{\mathcal{L}}$,
- Two particle in a plane

We have the result

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Let us look at the canonical momentum

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

So then this means that

$$\frac{\mathrm{d}p_\theta}{\mathrm{d}t} = 0 \qquad \mu r^2 \theta \equiv l = \text{constant} = \left|\vec{L}\right|$$

> Say "p_θ is a first integral of motion"

Consider an angle of $\delta\theta$ for the vectors $r(t_1)$ and $r(t_2)$ then

$$\delta r = r\delta\theta$$
$$\delta A = \frac{1}{2} r r \delta\theta = \frac{1}{2} r^2 \delta\theta$$

Then

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \delta\dot{\theta} = \frac{l}{2\mu} = \text{constant},}$$

also known as *Kepler's second law*.

Area spanned per unit time by a particle moving in a central potential must be constant

5.1 Equations of Motion for the Orbits

We know wish to solve for the "radial" equation of the orbits.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \implies \mu\ddot{r} - \mu r\dot{\theta} = -\frac{\mathrm{d}V}{\mathrm{d}r} \equiv f(r)$$

And then we obtain the important equation

$$\boxed{\mu\ddot{r} - \frac{l^2}{\mu r^3} = f(r)},$$

where recall that $l = \mu r^2 \dot{\theta}$.

Let us now consider the energy

$$\begin{aligned} H = E &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) \end{aligned}$$

What is the time derivative of the Energy here ? $\mathrm{d}E/\mathrm{d}t$

Error: Missing argument for \dot

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\ddot{r} + V + \frac{l^2}{2\mu r^2}\right) = 0$$

And also we induce from this that

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + V + \frac{l^2}{2\mu r^2} = \text{constant}}$$

Remark Pretty much every thing we' rederiving is a constant LOL

We write

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{\frac{2}{\mu}(E - V) - \frac{l^2}{\mu^2 r^2}}$$

5.3 Classification of Orbits

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E - V) - \frac{l^2}{\mu^2 r^2}}, \quad , \dot{r} = 0 \quad E - V - \frac{l^2}{2\mu r^2} = 0$$

In general there are two types of orbits

1. Closed : Motion in periodic
2. Open : Motion is non-periodic

Consider

$$\begin{gathered} \dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - V\left(\frac{1}{r}\right)\right) - \frac{l^2}{\mu^2 r^2}} \\ V_{\text{eff}} - V(r) + \frac{l^2}{2\mu r^2} \\ \dot{r} = \pm\sqrt{\frac{2}{\mu}(E - V_{\text{eff}})} \end{gathered}$$

In Kepler law we have

$$V_{\text{eff}} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

Assignment 5

Q1

Using Lagrange formulation we find the corresponding kinetic and potential energy of the system

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k_{12}(x_2 - x_1)^2 + \frac{1}{2}kx_2^2 = \frac{1}{2}(k + k_{12})x_1^2 + \frac{1}{2}(k + k_{12})x_2^2 - k_{12}x_1x_2$$
$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2.$$

Then we find the corresponding A matrix entries through

$$A_{ij} = \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_0 \qquad \implies A_{11} = \left.\frac{\partial^2 V}{\partial x_1^2}\right|_0 = k + k_{12}$$
$$A_{12} = A_{21} = \left.\frac{\partial^2 V}{\partial x_1 \partial x_2}\right|_0 = -k_{12}$$
$$A_{22} = \left.\frac{\partial^2 V}{\partial x_2^2}\right|_0 = k + k_{12}$$

Further, One can show that $T = \sum_{ij} M_{ij}\dot{q}_i\dot{q}_j/2$ then it must be that $m_{11} = m_{22} = M$ and $m_{12} = m_{21} = 0$. From this we obtain the M matrix and we state the A matrix as well

$$A = \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix} \qquad M = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

The eigenfrequencies are found using $\left|A_{ij} - \omega^2 M_{ij}\right| = 0$ so then

$$\begin{vmatrix} k + k_{12} - M\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - M\omega^2 \end{vmatrix} = 0 \implies (k + k_{12} - M\omega^2)^2 - k_{12}^2 = 0$$
$$\implies \omega = \sqrt{(k + k_{12} \pm k_{12})/M}$$
$$\implies \omega_1 = \sqrt{(k + 2k_{12})/M} \ , \omega_2 = \sqrt{k/M}$$

We then find the eigenvectors by substituting ω_1 and ω_2 in $\sum_j (A_{ij} - \omega_r^2 M_{ij})a_{jr} = 0$.

$$(A_{11} - \omega_1^2 M_{11})a_{11} + (A_{21} - \omega_1^2 M_{21})a_{21} = 0 \overset{\omega_1}{\implies} a_{11} = -a_{21}$$
$$(A_{11} - \omega_2^2 M_{11})a_{12} + (A_{12} - \omega_2^2 M_{12})a_{22} = 0 \overset{\omega_2}{\implies} a_{12} = a_{22}$$

From these two relationships we conclude that the corresponding two eigenmodes of the system are

$$\vec{a}_1 = (-1 \quad 1)^T , \qquad \vec{a}_2 = (1 \quad 1)^T.$$

Further, the first normal mode is found by letting $\vec{a}_2$ be trivial so then $x_1 = -x_2$ and vice versa and this happens at frequency $\omega = \sqrt{(k + 2k_{12})/M}$. Similarly, the second normal mode is found by letting $\vec{a}_1$ be trivial so then $x_1 = x_2$ and this happens at frequency $\omega = \sqrt{k/M}$.

The most general solution is found by first finding the normal basis vectors η_r. Given $\vec{a}_r$ as defined above, solving the system of equations $x_1 = a_{11}\eta_1 + a_{22}\eta_2$ and $x_2 = -a_{11}\eta_1 + a_{22}\eta_2$, we obtain

$$\eta_1 = \frac{1}{2a_{11}}(x_1 - x_2), \quad \eta_2 = \frac{1}{2a_2}(x_1 + x_2) \implies \underset{\sim}{q} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a_{11}}(x_1 - x_2) \\ \frac{1}{a_{22}}(x_1 + x_2) \end{pmatrix}.$$

If $m_1 \neq m_2$ then $T = (m_1\dot{x}_1^2 + m_2\dot{x}_2^2)/2$ and since by definition $T = \sum_{ij} M_{ij}\dot{q}_i\dot{q}_j/2$, it follows that $m_{12} = m_{21} = 0$and $m_{11} = m_1$ with $m_{22} = m_2$. Then, in this system, the A matrix remains the same as that in the $m_1 = m_2$ case, so we solve immediately for the eigenfrequencies using $\left|A_{ij} - \omega^2 M_{ij}\right| = 0$. That is,

$$\begin{vmatrix} k + k_{12} - m_1\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - m_2\omega^2 \end{vmatrix} = 0 \implies (k + k_{12} - m_1\omega^2)(k + k_{12} - m_2\omega^2) - k_{12}^2 = 0$$

$$\implies \omega = \sqrt{(k + k_{12} \pm k_{12})(m_1 + m_2)/(m_1 m_2)}$$

$$\implies \omega_1 = \sqrt{(k + 2k_{12})(m_1 + m_2)/(m_1 m_2)}$$

$$\implies \omega_2 = \sqrt{k(m_1 + m_2)/(m_1 m_2)},$$

where in the second line we have used the *reduces mass* formulation.

Given $x = l\sin\theta$ and $y = l - l\cos\theta$, then $\dot{v}_i^2 = l^2\dot{\theta}_i^2$ and so we can state the kinetic energy immediately

$$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2).$$

Further, since the length of the string is l and imposes a constraint on the height of a given mass, then the potential of the system is

$$V = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2) - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2,$$

where for the restorive force of the spring we assume that the two pendulums are separated by a distance of l.

Consider the series expansion of $\cos\theta$

$$\cos\theta \approxeq 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots,$$

then considering a second order approximation, we let $\cos\theta = 1 - \theta^2/2$, such that $1 - \cos\theta_i = \theta_i^2/2$. The potential then becomes

$$V = mgl\left(\frac{\theta_1^2}{2}\right) + mgl\left(\frac{\theta_2^2}{2}\right) - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2 = \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2.$$

Given T as defined above, and since $T = \sum_{ij} m_{ij}\dot{q}_i\dot{q}_j/2$, it follows that $m_{12} = m_{21} = 0$ and $m_{11} = m_{22} = ml^2$. Further, since

$$A_{11} = A_{22} = \left.\frac{\partial^2 V}{\partial\theta_1^2}\right|_1 = mgl - kl^2, \quad A_{12} = A_{21} = \left.\frac{\partial^2 V}{\partial\theta_1\partial\theta_2}\right|_1 = -kl^2,$$

then we may state the A and M matrices

$$A = \begin{pmatrix} mgl - kl^2 & -kl^2 \\ -kl^2 & mgl - kl^2 \end{pmatrix}, \qquad M = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}.$$

Then we solve $\left|A_{ij} - \omega^2 M_{ij}\right| = 0$ to find the eigenfrequencies of the system

$$\begin{vmatrix} mgl - kl^2 - \omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl - kl^2 - \omega^2 ml^2 \end{vmatrix} = 0 \implies (mgl - kl^2 - \omega^2 ml^2)^2 - kl^2 = 0$$

$$\implies \omega = \sqrt{(\mp kl^2 + mgl - kl^2)/ml^2}$$

$$\implies \omega_1 = \sqrt{\frac{g}{l}}, \quad \omega_2 = \sqrt{\frac{g}{l} - \frac{2k}{m}}$$

Further, we find the eigenmodes by first computing the eigenvectors associated with the eigenfrequencies ω_1 and ω_2, by substituting back in the determinant matrix

$$\left(mgl - kl^2 - \frac{g}{l}ml^2\right)a_{11} + (-kl^2)a_{21} = 0 \implies a_{11} = -a_{21}$$

$$(-kl^2)a_{12} + \left(mgl - kl^2 - \left(\frac{g}{l} - \frac{2k}{m}\right)ml^2\right)a_{22} = 0 \implies a_{12} = a_{22}$$

So the eigenmodes are the same as in Question 1 part (a) but happening at frequencies ω_1 and ω_2 defined above.

Lecture 18

Consider the Lorentz $\vec{v} = v\hat{x}$.

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x' \\ y \\ z \\ ct \end{pmatrix}$$

And with

$$\beta \equiv \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} \implies \gamma^2 - \beta^2\gamma^2 = 1$$

6.1.3 Momenta and Energies

In special relativity we still have

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\vec{v}) \qquad , \vec{p} = m\gamma\vec{v}$$

The kinetic energy by consuming work can be calculated

$$\Delta T = \Delta W = \int_1^2 \vec{F} \cdot \mathrm{d}l = m\int_0^v v\mathrm{d}(\gamma v)$$

Relativistic Kinetic energy is then , given $T = (\gamma - 1)mc^2$, $[\mathcal{L} = T - V]$?

Well since

$$T = \gamma mc^2 - mc^2 = E - E_0,$$

where $E = \gamma mc^2$ the *relativistic energy* and $E_0 = mc^2$ the *rest energy*.

Q How do we know that $E = mc^2$?

Not just an equivalence postulate, it's a fact, it's seem all the time in nuclear fission.

Recall

$$E^2 = p^2c^2 + m^2c^4 = p^2c^2 + E_0^2$$

6.3 Relativistic Lagrangian

Q $\mathcal{L} = T - V = (\gamma - 1)mc^2 - V \quad [??]$

$$\begin{aligned} p_i &= \frac{\partial \mathcal{L}}{\partial \dot{q}_i} && \text{canonical momentum} \\ p = \frac{\partial \mathcal{L}}{\partial v} &\implies m\gamma\vec{v} && \text{well defined} \end{aligned}$$

With $mv_i/\sqrt{1-\beta^2}$.

Lecture 19 (Review Section)

We saw that

$$l\left(\eta, \frac{\mathrm{d}\eta}{\mathrm{d}x}, \frac{\mathrm{d}\eta}{\mathrm{d}t}, x, t\right) = \frac{\mu}{2}\dot{\eta}^2 - \frac{y}{2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2$$

There's a new dependency so we have to invoke the least action principle again

$$\delta I = \delta \int_{t_1}^{t_2} \int l \mathrm{d}r^3 \mathrm{d}t = 0$$

We have developed a new formalism , now l depends on y'.

7.4 Schrodinger's Field

We treat $\psi(\vec{r}, t)$ as a field of probabilities. We can think of it as a probability *flop*. And, we also consider its complex conjugate $\psi^\dagger(\vec{r}, t)$. Turns out we can write

$$l(\psi, \dot{\psi}, \psi^\star, \dot{\psi}^\star, \vec{\nabla}\psi, \vec{\nabla}\psi^\star, \vec{r}, t) = \frac{1}{2}i\hbar(\psi^\star, \dot{\psi} - \dot{\psi}^\star\psi) - \frac{\hbar^2}{2m}\vec{\nabla}\psi^\star\vec{\nabla}\psi - \psi^\dagger \vee \psi$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l}{\partial \psi^\star} + \vec{\nabla}\frac{\partial l}{\partial \vec{\nabla}\psi^\star} - \frac{\partial l}{\partial \psi^\star} = 0$$

The solution of this, *like magic* is

$$\implies i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi - \vee\psi,$$

Remark this is the Schrodinger's equation. turns out we can derive it from the variational principles.

"Baby QFT"

In Quantum mechanics we use the variables which we map to observables.

$$\vec{r}, \vec{p} \longrightarrow \hat{Q}, \hat{P} \to [R_i, P_j] = i\hbar\delta_{ij} \ \text{(canonical quantization of QM)}$$

We have a field $\psi(\vec{x}, t)$, and we consider a canonically conjugated field which has a bidirectional map to $\Pi(\vec{x}, t)$, where we define

$$\Pi(\vec{x}, t) = \frac{\partial l}{\partial \dot{\psi}}.$$

Let us *quantize it*,

$$[\psi_{op}(\vec{x}, t), \Pi_{op}(\vec{x}', t)] = \frac{i\hbar}{2}[\psi_{op}(\vec{x}, t), \psi^{\dagger}(x', t)] = i\hbar\delta(\vec{x} - \vec{x}').$$

Essentially the point is that when we did $\mathcal{L} = T - V$, this was a discrete map to lagrangian density l. Now we can map this Lagrangian density to $l(\psi, \psi^{\star})$, which we then quantize to QFT.