# PHYS 241 Assignment 2.

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**Question1.**

**a)**

The relative permittivity $\epsilon_r$ is a textbook value given by 2.1. The vaccuum perimittiviy value is approximately $8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$ The capacitance $C$ is then given by

$$
\begin{aligned}
C &= \frac{\epsilon_0 \epsilon A}{d} \\
&= \frac{(0.01)^2(2.1)(8.85 \times 10^{-12})}{0.0001} \left[\frac{\mathrm{m^2\,F}}{\mathrm{m^2}}\right] \\
&= 1.8585 \times 10^{-11}\ \mathrm{F} = \boxed{18.585\ \mathrm{pF}}.
\end{aligned}
$$

**b)**

The circuit time constant for an $RC$ set up is given by

$$
\begin{aligned}
\tau = RC &= (1 \times 10^3)(1.8585 \times 10^{-11})\ \Omega\,\mathrm{F} \\
&= 1.8585 \times 10^{-8}\ \mathrm{s} = \boxed{0.018585\ \mathrm{\mu s}}.
\end{aligned}
$$

**c)**

The formula for a capacitor's charge who's discharging is given by

$$
\begin{aligned}
Q(t) &= Q_{\max}\left(1 - e^{\frac{-t}{\tau}}\right) \\
5\ \mathrm{C} &= 10\ \mathrm{C}(1 - e^{\frac{-t}{0.018585\ \mathrm{s}}}) \\
(-0.01858\ \mathrm{s})\left(\ln\left(\frac{1\ \mathrm{s}}{2\ \mathrm{s}}\right)\right) &\implies \boxed{t = 0.01288\ \mathrm{s}}.
\end{aligned}
$$

**d)**

The total energy dissipated across a resistor from a capacitor is given by

$$\begin{aligned}
U_C(t) &= \frac{1}{2}C(\Delta V_C(t))^2 \\
&= \frac{1}{2}C\left(\frac{Q}{C}\right)^2 \\
&= \frac{1}{2}\frac{Q^2}{C} \\
\implies U_C(0.01288\text{ s}) &= \frac{1}{2}\frac{(5\times10^{-6})^2}{(1.8585\times10^{-11})} \quad \left[\frac{\mathrm{C}^2}{\mathrm{F}}\right] \\
&= \boxed{0.6726\text{ J}}.
\end{aligned}$$

**Question 2.**

**a)**

The inductance is given by

$$\begin{aligned}
L = \mu_0 A\frac{N^2}{\ell} &= 4\pi\times10^{-7}(0.01)^2\pi\frac{(1000)^2}{0.05} \quad \left[\frac{\mathrm{H\,m}^2}{\mathrm{m}^2}\right] \\
&= 7.895\times10^{-3}\text{ H} = \boxed{7.895\text{ mH}}.
\end{aligned}$$

**b)**

The time constant for a LR circuit is given by

$$\tau = \frac{L}{R} = \frac{7.895\times10^{-3}\text{ H}}{1\times10^{3}\ \Omega} = 7.895\text{ s} = \boxed{7.896\times10^{6}\ \mu\text{s}}.$$

**c)**

The maximal value for the current in the given circuit is $I_{\max} = \frac{V_0}{R}$. Thus the time to reach the half value of $I_{\max}$ is

$$\frac{I_{\max}}{2} = I_{\max}\left(e^{\frac{-t_{1/2}}{\tau}}\right)$$

$$\ln\left(\frac{1}{2}\right)(-\tau) = t_{1/2}$$

$$\implies t_{1/2} = ln\left(\frac{1}{2}\right)(-7.896) = \boxed{5.472 \text{ s}}.$$

## Question 3.

Let $R_{\text{eq}} = R_1 + R_2$ since the two resistors are in series.

**a )**

$$V_0 = \Delta V_C + \Delta V_{R_{\text{eq}}} = \frac{Q}{C} + IR_{\text{eq}}$$

Let us apply a time derivative on both sides , since $I = I(t)$ and $Q = Q(t)$ this yields ,

$$0 = \frac{1}{C}\frac{dQ}{dt} + R_{\text{eq}}\frac{dI}{dt}.$$

Since $\frac{dQ}{dt} = I$ , we have

$$\frac{dI}{dt} = \frac{-I}{CR_{\text{eq}}}$$

$$\implies \frac{dI}{dt} + \frac{I}{\tau} = 0 \qquad \text{for} \quad \tau = C(R_1 + R_2).$$

**b)**

At $t = 0$, the capacitor is fully discharged, so only the resistor plays a role such that

$$I_0 = \frac{V_0}{R_{\text{eq}}}.$$

**c)**

Note that $\dfrac{dI}{dt} = \dfrac{-I}{CR_{\text{eq}}}$ is a separable equation. Therefore ,

$$\frac{dI}{dt} = \frac{-I}{CR_{\text{eq}}} \implies CR_{\text{eq}}\frac{I'}{I} = -1 \implies CR_{\text{eq}}\ln(I) = \int -1 \; dt$$
$$\implies CR_{\text{eq}}\ln(I) = -t + c \implies I - e^{\frac{-t+c}{CR_{\text{eq}}}} = e^{\frac{-t}{CR_{\text{eq}}}} e^{\frac{c}{CR_{\text{eq}}}}.$$

Define $A = e^{\frac{c}{CR_{\text{eq}}}}$ then we have the initial value problem

$$I = Ae^{\frac{-t}{CR_{\text{eq}}}} \qquad , I(0) = \frac{V_0}{R_{\text{eq}}}$$
$$\frac{V_0}{R_{\text{eq}}} = Ae^{\frac{0}{CR_{\text{eq}}}} \implies A = \frac{V_0}{R_{\text{eq}}}$$

thus the desired differential equation is $\quad I(t) = \dfrac{V_0}{R_{\text{eq}}} e^{\frac{-t}{\tau}} \quad$ for $\tau = C(R_1 + R_2)$.

**d)**

Since $I(t) = \dfrac{V_0}{R_{\text{eq}}} e^{\frac{-t}{\tau}}$, then $IR = V_0 e^{\frac{-t}{\tau}}$. Using conservation of voltage along the circuit along with the substitution for $IR$ we have

$$\Delta V_C(t) = \frac{1}{C}Q(t) = V_0 - IR = V_0 - V_0 e^{\frac{-t}{\tau}} = V_0(1 - e^{\frac{-t}{\tau}})$$
$$\text{Since} \;\; Q = C\Delta V_C, \;\; \text{we have } C\Delta V_C = CV_0(1 - e^{\frac{-t}{\tau}})$$
$$\implies Q(t) = CV_0(1 - e^{\frac{-t}{\tau}}).$$

**e)**

Since $\Delta V_C = V_0(1 - e^{\frac{-t}{\tau}})$, then since the capacitor is chargin and voltage drop is conserved throughout the circuit, symmetrically , we have

$$\Delta V_{R_2}(t) = \left(\frac{R_2}{R_1 + R_2}\right) V_0 e^{\frac{-t}{\tau}}$$

**f)**

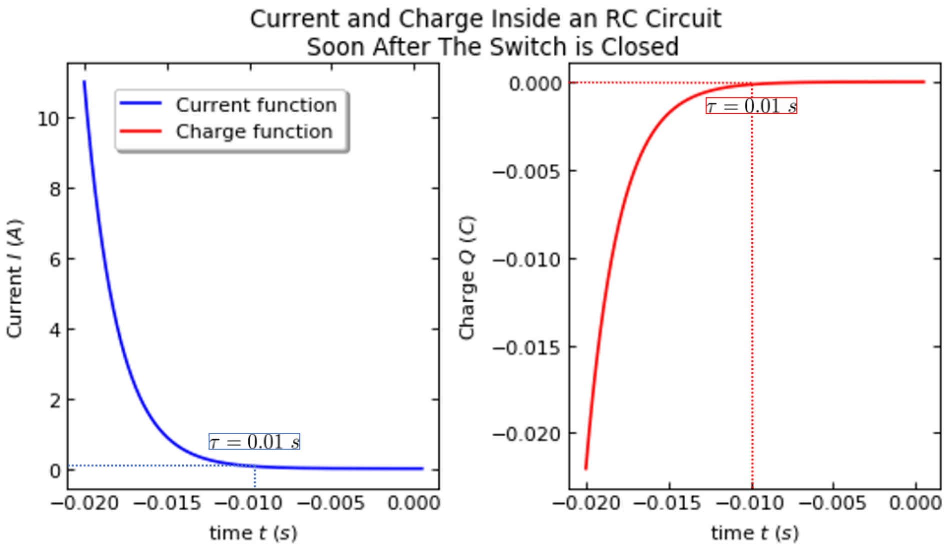


Figure 1: Current and Charge plots

## Question 4.

**a)**

The inductor acts like a short therefore the steady current is

$$I = \frac{V_0}{R_1}.$$

**b)**

Since the switch is opened no current is going through $R_1$ such that,

$$\frac{dI}{dt} = \frac{-R_2}{L} I \implies \frac{dI/dt}{I} = \frac{R_2}{L}$$
$$\ln(I) = \int \frac{-R_2}{L} dt = \frac{-R_2}{L} + C.$$
$$\implies I(t) = Ae^{\frac{-t}{\tau}} \qquad \text{for} \quad A = e^C.$$

$$\text{Since} I(0) = \frac{V_0}{R_1} \quad \implies I(t) = \frac{V_0}{R_1} e^{\frac{-t}{\tau}} \quad , \text{for} \quad \tau = \frac{L}{R_2}.$$

**c)**

The total energy at time $t$ formula is given by

$$U_L = \frac{1}{2} L (I(t))^2$$

Integrating from $0 \to t$ this formula whilst substituting the previous value found for $I(t)$ ,

$$\begin{aligned} U_L &= \frac{1}{2} L \left( \frac{V_0}{R_1} e^{\frac{-t}{\tau}} \right)^2 \\ &= \frac{V_0^2}{R_1^2} \frac{1}{2} L \int_0^t e^{\frac{-t}{\tau}} \ dt \\ &= \frac{V_0^2}{R_1^2} \frac{1}{2} L(\tau \left( 1 - e^{\frac{-t}{\tau}} \right)) \end{aligned}$$

$$\implies U_L = \frac{V_0^2}{R_1^2} \frac{1}{2} L(\tau \left( 1 - e^{\frac{-t}{\tau}} \right))$$

Taking the lim as $t \to \infty$ yields

$$\lim_{t \to \infty} U_L = \frac{V_0^2}{R_1^2} \frac{L\tau}{2}$$

**Question 5.**

Since potential difference is conserved in loops, we have

$$\begin{aligned} V_0 - IR_1 - L\frac{dI_1}{dt} &= V_0 - IR_1 - I_2R_2 \\ \implies L\frac{dI_1}{dt} &= I_2R_2. \end{aligned}$$

Three equations are necessary to solve this problem

1. $I = I_1 + I_2$ by the Mesh Law.
2. $L\frac{dI_1}{dt} = I_2R_2$.

3. $V_0 = R_1 I + L\frac{dI_1}{dt}$

$$\begin{aligned} V_0 &= R_1 I + L\frac{dI_1}{dt} \\ &= R_1 I_1 + R_1 I_2 + L\frac{dI1}{dt} \end{aligned}$$

Since $I_2 = \frac{L}{R_2}\frac{dI_1}{dt}$ , we have

$$\begin{aligned} &= R_1 I_1 + \frac{R_1}{R_2} L\frac{dI_1}{dt} + L\frac{dI_1}{dt} \\ &= R_1 I_1 + \left(\frac{R_1}{R_2} + 1\right) L\frac{dI_1}{dt} \\ \implies \frac{dI_1}{dt} &= \frac{V_0 - R_1 I_1}{L\left(1 + \frac{R_1}{R_2}\right)} \\ \frac{dI_1/dt}{V_0 - R_1 I_1} &= \frac{1}{L\left(1 + \frac{R_1}{R_2}\right)} \\ \ln|V_0 - R_1 I_1| &= -R_1\left(\frac{t}{L\left(1 + \frac{R_1}{R_2}\right)} + C\right) \end{aligned}$$

Let $A = e^{-R_1 C}$, we then have

$$V_0 - R_1 I_1 = Ae^{\frac{-R_1 t}{L(1+R_1/R_2)}}$$

Let $B = A/-R_1$ , we then have

$$I_1(t) = Be^{\frac{-tR_1}{(1+R_1/R_2)}} + \frac{V_0}{R_1}$$

Using the initial condition $I(0) = 0$ since initialy no current goes through,

$$\begin{aligned} I_1(0) &= Be^{\frac{-\infty R_1}{(1+R_1/R_2)}} + \frac{V_0}{R_1} = \frac{V_0}{R_1} \\ \implies I_1(t) &= \frac{V_0}{R_1}\left(1 + e^{\frac{-tR_1}{(1+R_1/R_2)}}\right) = \frac{V_0}{R_1}\left(1 + e^{\frac{-t}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}}\right) \xrightarrow{t\to\infty} \frac{V_0}{R_1} \end{aligned}$$

Which indeed agrees with the long time limit from Question 4.