PHYS241 Assignment 4.

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April 4, 2020

Question 1.

a)

The conductor has an induced f.e.m with current in opposite direction therefore voltage conservation across the circuit gives

$$\frac{Q}{C} + L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = 0 \xrightarrow{d/dt} \frac{1}{C}I + L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

Dividing by L and since dQ/dt in this set-up equals I ,

$$\therefore \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \Gamma\frac{\mathrm{d}I}{\mathrm{d}t} + \omega_0^2 I = 0.$$

Homogeneous second order differential equations, let us find the roots to have a solution.

$$\implies \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2} = \frac{-\Gamma}{2} \pm \sqrt{\left(\frac{-\Gamma}{2}\right)^2 - \omega_0^2}$$

Since we're only interested in the undamped case , the solutions are of complex form with $I(t) = -\frac{\Gamma}{2} \pm j\omega_f$. So we have the general solution,

$$\begin{aligned} I(t) &= c_1 e^{-\left(\frac{\Gamma}{2} - j\omega_f\right)t} + c_2 e^{\left(\frac{-\Gamma}{2} + j\omega_f\right)t}, \\ &= e^{\frac{-\Gamma}{2}t}\left(A\cos\omega_f t + B\sin\omega_f t\right). \end{aligned}$$

Let us find the initial conditions A and B.

$$\text{The current in the circuit is initially 0} \implies A = 0.$$

$$\text{Relating voltage conservation to current gives } \frac{Q}{C} + L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = 0$$

At $t = 0$ the current is 0 therefore and $Q(0) = CV_0$,

$$\frac{Q}{C} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \implies -V_0 = L\frac{\mathrm{d}I}{\mathrm{d}t} \implies B = \frac{-V_0}{L\omega_f}.$$

The general solution is therefore

$$I(t) = \frac{-V_0}{\omega_f L} e^{\frac{-\Gamma}{2}t}\sin\omega_f t.$$

b)

Since $Q(t) = \int_0^t I(t')\, dt'$ and $\Delta V_C(t) = Q(t)/C$, it follows that

$$\begin{aligned}\Delta V_C(t) &= \frac{1}{C}\int_0^t I(t')\, dt'.\\ &= \left(\frac{1}{C}\right)\int_0^t \frac{-V_0}{\omega_f L} e^{\frac{-\Gamma}{2}} \sin \omega_f t \; dt\end{aligned}$$

Applying integral by parts twice with $u = e^{\frac{-\Gamma}{2}t}$ and $dv = \sin \omega_f t$, we obtain

$$\begin{aligned}&= \left(\frac{V_0}{C\omega_f L}\right)\left(\frac{\frac{e^{\frac{-\Gamma}{2}t}}{\omega_f}\left(-\cos \omega_f - \frac{\Gamma \sin \omega_f t}{2\omega_f}\right)}{\left(1+\frac{\Gamma^2}{4\omega_f^2}\right)}\right)\\ &= \left(\frac{-V_0}{LC}\right)\frac{4e^{\frac{-\Gamma}{2}t}\left(-\cos \omega_f t - \frac{\Gamma \sin \omega_f t}{2\omega_f}\right)}{(4\omega_f^2+\Gamma^2)}.\end{aligned}$$

c)

Let us verify that the voltage drop across the capacitor converges to 0 as $t \to \infty$.

$$\begin{aligned}&\lim_{t\to\infty}\left(\frac{-V_0}{LC}\right)\frac{4e^{\frac{-\Gamma}{2}t}\left(-\cos \omega_f t - \frac{\Gamma \sin \omega_f t}{2\omega_f}\right)}{(4\omega_f^2+\Gamma^2)}\\ &= \left(\frac{-V_0}{(LC)(4\omega_f^2+\Gamma^2)}\right)\lim_{t\to\infty} 4e^{\frac{-\Gamma}{2}t}\left(-\cos \omega_f t - \frac{\Gamma \sin \omega_f t}{2\omega_f}\right) = 0.\end{aligned}$$

Since indeed $e^{-\Gamma t/2}$ converges to 0 while the waves just oscillate.

$$\therefore \checkmark$$

Question 2.

Let us take the derivative with respect to x and set equal to zero to find the maximal value.

$$\left|\frac{\widetilde{\Delta V_C}}{\widetilde{V}}\right| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + (\omega\tau)^2}}$$

Taking the derivative and setting equal to 0 we get

$$2\tau^2\omega - \frac{4\omega\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}{\omega_0^2} = 0$$

$$\implies \omega_0^4\tau^2\omega - 2\omega(\omega_0^2-\omega^2) = 0 \implies \omega(\omega_0^4\tau^2 - 2(\omega_0^2-\omega^2)) = 0$$

$\omega \neq 0$ for the largest value

$$\omega_0^4\tau^2 - 2(\omega_0^2-\omega^2) = 0 \implies 2\omega^2 = 2\omega_0^2 - \omega_0^4\tau^2$$
$$\therefore \omega = \omega_0 \pm \sqrt{1-\frac{\omega_0^2\tau^2}{2}}.$$

Question 3.

a)

Let us find the equivalent impedence,

$$\frac{1}{Z_2} = \left(\frac{1}{j\omega L} + j\omega C\right)^{-1} \implies Z_2 = \frac{j\omega L}{1-\omega^2 LC},$$
$$\therefore Z_{\text{eq}} = R + \frac{j\omega L}{1-\omega^2 LC}.$$

We may now find simplify an expression for the current,

$$\widetilde{I} = \frac{V_0 e^{j\omega t}}{R + \frac{j\omega L}{1-\omega^2 LC}} = \left(\frac{V_0}{R}\right)\frac{j\omega\tau}{j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}e^{j\omega t}$$

Let us find the amplitude of the previous expression

$$\left|\widetilde{I}\right|^2 = \left(\frac{V_0}{R}\right)^2\left(\frac{j\omega\tau}{j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}\right)\left(\frac{-j\omega\tau}{-j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}\right) = \left(\frac{V_0}{R}\right)^2\frac{(\omega\tau)^2}{(\omega\tau)^2 + \left(\frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}\right)^2}$$
$$\implies \left|\widetilde{I}\right| = \left(\frac{V_0}{R}\right)\frac{\omega\tau}{\sqrt{(\omega\tau)^2 + \left(\frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}\right)^2}}.$$

Let us now find the phase offset of the complex current,

$$\widetilde{I} = \frac{\widetilde{V}}{R}\frac{j\omega\tau}{j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}} = \frac{\widetilde{V}}{R}\left(\frac{j\omega\tau}{j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}\right)\left(\frac{-j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}{-j\omega\tau - \frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}\right)$$
$$\implies Z_{\text{Numerator}} = (\omega\tau)^2 - j\omega\tau\left(\frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}\right) \implies \phi = \tan^{-1}\left(\frac{\omega\tau\left(\frac{-\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}\right)}{(\omega\tau)^2}\right)$$
$$\therefore \phi = \tan^{-1}\left(\frac{\frac{-\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}{\omega\tau}\right)$$

Finally, we have an expression for the current inside the circuit

$$I = \left(\frac{V_0}{R}\right) \frac{\omega\tau}{\sqrt{(\omega\tau)^2 + \left(\frac{\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}\right)^2}} e^{j(\omega t+\phi)} \quad \text{, where } \phi = \tan^{-1}\left(\frac{\frac{-\omega^2/\omega_0^2}{1-\omega^2/\omega_0^2}}{\omega\tau}\right).$$

b)

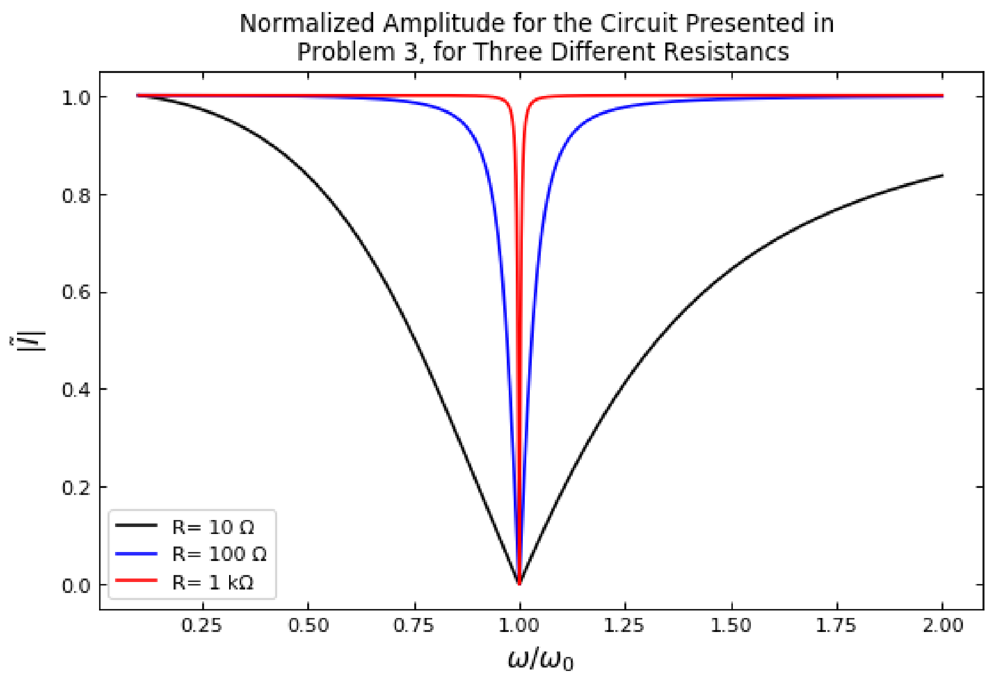

Figure 1: Plot of the current's amplitude found in Question 2, for three resistor configurations R. We note that increasing the value for the resistance in return increases the narrowness around the resonance point.

Question 4.

a)

Let us find the equivalent impedance in sub portion of the rightward side of the given circuit.

$$Z_1 = j\omega L + r \qquad Z_2 = \frac{1}{j\omega C}$$
$$Z_{||} = \left(\frac{1}{j\omega L} + j\omega C\right)^{-1}$$
$$\therefore Z_{||} = \frac{(j\omega L + r)}{1 + (j\omega L + r) j\omega C} = \frac{j\omega L + r}{1 - \frac{\omega^2}{\omega_0^2} + j\omega C r}$$

b)

$$Z_{\text{eq}} = R + \frac{j\omega L + r}{1 - \frac{\omega^2}{\omega_0^2} + j\omega C r}.$$

We may now find an expression for the current,

$$\widetilde{I} = \frac{V_0 e^{j\omega t}}{R + \frac{j\omega L + r}{1 - \omega^2/\omega_0^2 + j\omega C r}} = \left(\frac{V_0}{R}\right) \frac{j\omega\tau}{j\omega\tau + \frac{-\omega^2/\omega_0^2 + j\omega C r}{1 - \omega^2/\omega_0^2 + j\omega C r}}.$$

Now we find the amplitude of the previous expression

$$\left|\widetilde{I}\right|^2 = \left(\frac{V_0}{R}\right)^2 \left(\frac{j\omega\tau}{j\omega\tau + \frac{-\omega^2/\omega_0^2 + j\omega C r}{1 - \omega^2/\omega_0^2 + j\omega C r}}\right) \left(\frac{-j\omega\tau}{-j\omega\tau + \frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}}\right)$$
$$= \left(\frac{V_0}{R}\right)^2 \frac{(\omega\tau)^2}{(\omega\tau)^2 + \frac{(\omega C r)^2 + \left(\frac{\omega}{\omega_0}\right)^4}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + (\omega C r)^2}} \quad \therefore \left|\widetilde{I}\right| = \left(\frac{V_0}{R}\right) \frac{\omega\tau}{\sqrt{(\omega\tau)^2 + \frac{(\omega C r)^2 + \left(\frac{\omega}{\omega_0}\right)^4}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + (\omega C r)^2}}}.$$

The current's amplitude is found and the problem is done, let us nevertheless find a complete expression for the current, we find the offset

$$\widetilde{I} = \frac{\widetilde{V}}{R} \left(\frac{j\omega\tau}{j\omega\tau + \frac{-\omega^2/\omega_0^2 + j\omega C r}{1 - \omega^2/\omega_0^2 + j\omega C r}}\right) \left(\frac{-j\omega\tau + \frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}}{-j\omega\tau + \frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}}\right)$$
$$\implies Z_{\text{Numerator}} = (\omega\tau)^2 + j\omega\tau \left(\frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}\right)$$
$$\implies \phi = \tan^{-1}\left(\frac{\omega\tau\left(\frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}\right)}{(\omega\tau)^2}\right) \quad \therefore \phi = \tan^{-1}\left(\frac{\frac{-\omega^2/\omega_0^2 - j\omega C r}{1 - \omega^2/\omega_0^2 - j\omega C r}}{\omega\tau}\right).$$

Finally, we have an expression for the current inside the cirucit ,

$$I = \left(\frac{V_0}{R}\right) \frac{\omega\tau}{\sqrt{(\omega\tau)^2 + + \frac{(\omega Cr)^2 + \left(\frac{\omega}{\omega_0}\right)^4}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + (\omega Cr)^2}}} e^{j(\omega t + \phi)}, \quad \text{where } \phi = \tan^{-1}\left(\frac{\frac{-\omega^2/\omega_0^2 - j\omega Cr}{1 - \omega^2/\omega_0^2 - j\omega Cr}}{\omega\tau}\right).$$

c)

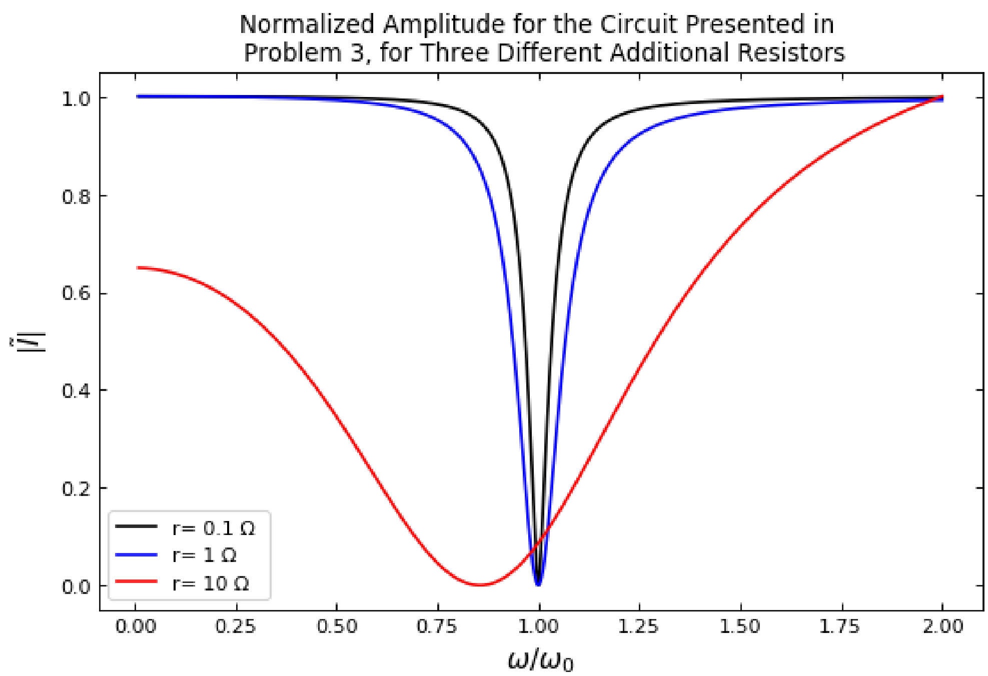

Figure 2: Plot of the current's amplitude found in Question 3, for three additional resistor configurations r. We note that increasing the value for the resistance in return decreases the narrowness at the resonance point. Moreover, there exists a critical value of r for which the corresponding plot starts shifting progressively to the left.