MATH 475 Weekly Work 5

Mihail Anghelici 260928404

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Question 1

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned} x_r &= \cos\theta, & x_\theta &= -r\sin\theta \\ y_r &= \sin\theta, & y_\theta &= r\cos\theta \end{aligned}$$

We compute u_{rr} then $u_{\theta\theta}$. We use the chain rule since $r = r(x, y)$ and $\theta = \theta(x, y)$.

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta \\ u_{rr} &= u_{xr}\cos\theta + u_{yr}\sin\theta \\ &= (u_{xx}x_r + u_{xy}y_r)\cos\theta + (u_{yx}x_r + u_{yy}y_r)\sin\theta \end{aligned}$$

u is harmonic therefore $u \in C^2$ and we apply Clairot's theorem

$$\therefore\ u_{rr} = u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$

Similarly for $u_{\theta\theta}$,

$$u_\theta = u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

For the double derivative we apply the product rule and chain rule

$$\begin{aligned} u_{\theta\theta} &= -u_x r\cos\theta - r\sin\theta(u_{xx}x_\theta + u_{xy}y_\theta) = u_y r\sin\theta + r\cos\theta(u_{yx}x_\theta + u_{yy}y_\theta) \\ &= -r\underbrace{(u_x\cos\theta + u_y\sin\theta)}_{u_r} + u_{xx}r^2\sin^2\theta - 2u_{xy}r^2\sin\theta\cos\theta + u_{yy}r^2\cos^2\theta \end{aligned}$$

$$\implies \frac{u_{\theta\theta}}{r^2} = -\frac{u_r}{r} + u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta$$

We finally add these two expressions ,

$$\begin{aligned} u_{rr} + \frac{u_{\theta\theta}}{r^2} &= -\frac{u_r}{r} + u_{xx} + u_{yy} \\ \implies \Delta_{r,\theta}U &= U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} \qquad \checkmark. \end{aligned}$$

Question 2

Corollary 1. If u is a $C^2(\Omega)$ harmonic function on a domain Ω which is $C(\bar{\Omega})$, and the values of u on the boundary are bounded between m and M, then the values of u everywhere are bounded between m and M. (*Ref : R. Choksi, p.411*)

Following the corollary 1, since u is harmonic, the boundary $g = \sin^2\theta$ is evidently bounded by $0 \to 1$, hence it follows that $u(x, y) = U(r, \theta)$ is bounded by $0 \to 1$, so we conclude

$$0 \le U(r, \theta) \le 1.$$

Alternatively, since the function is harmonic, it satisfies the MVP such that we can apply the maximum principles. The maximum of $u(x)$ occurs at the boundary, which is bounded by 1. Similarly, since u is harmonic then so is $-u$ such that $\max(-u) = -\min(u) \implies -\max(-u) = \min(u)$. The minimum occurs at the boundary which is 0 for $\sin^2(\theta)$. We arrive at the same conclusion.

Next we look for the mean value of $u(\boldsymbol{x})$, since we're integrating in polar coordinates over the circumference of a circle it follows that,

$$\begin{aligned} u(\boldsymbol{x}) &= \frac{1}{|\partial B(x,R)|}\int_{\partial B(x,R)} u(y)\ dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sin^2(\theta)\ d\theta \\ &= \frac{1}{2\pi}\left(\pi - \cancelto{0}{\frac{\sin 2\theta}{4}\Big|_0^{2\pi}}\right) \\ &= \frac{1}{2}. \end{aligned}$$

Since u is harmonic then it follows that it satisfies the Mean Value Property, thence,

$$u(0,0) = U(0,0) = \frac{1}{2}.$$

Question 3

Let $U(r, \theta) \equiv R(r)\Theta(\theta)$ and

$$-\left(R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)\right) = 0. \tag{1}$$

Rearranging (1) we get

$$\frac{R''(r)r^2 - rR'(r)}{R(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)},$$

for which it follows that point wise,

$$R''(r)r^2 - rR'(r) = -\lambda^2 R(r) \qquad \text{, and} \quad \Theta''(\theta) = -\lambda^2\Theta(\theta). \tag{2}$$

We verify the three cases for λ^2 :
$\boldsymbol{\lambda^2 < 0}$ **:**

$$\Theta''(\theta) - \lambda^2\Theta(\theta) = 0$$

The caracteristic equation has repeated roots ($\lambda^2 = k^2$) so the general solution is

$$\Theta(\theta) = Ae^{\lambda\theta} + Be^{-\lambda\theta}$$

Since Θ is periodic, then $\Theta(0) = \Theta(2\pi)$,

$$\Theta(0) = A + B \neq \Theta(2\pi) = Ae^{2\pi} + Be^{-2\pi}$$

We reject the $\lambda^2 < 0$ solution.

$\boldsymbol{\lambda^2 = 0}$ **:**

$$\Theta''(\theta) = 0 \implies \int \Theta'(\theta) = A \implies \Theta(\theta) = Ax + B$$

Since Θ is periodic, then $\Theta(0) = \Theta(2\pi)$. In this case we clearly do not have periodicity , so the λ^2 case is also dismissed.

$\boldsymbol{\lambda^2 > 0}$ **:**

$$\Theta''(\theta) + \lambda^2\Theta(\theta) = 0$$

The caracteristic equation has complex roots ($\lambda^2 = -k^2$) so the general solution is

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

Since Θ is periodic, then $\Theta(0) = \Theta(2\pi)$,

$$\Theta(0) = B = \Theta(2\pi) = B \quad \checkmark.$$

We look for $R(r)$. We perform a change of variables in (2). Let $s = \ln(r) \implies R(r) = \varphi(\ln(s)) = \varphi(s)$.Then,

$$R'(r) = \varphi'(s)\frac{1}{r} \qquad \text{and} \quad R''(r) = \frac{1}{r^2}(\varphi''(s) + \varphi'(s))$$

$$\overset{(2)}{\Longrightarrow} \left[\frac{r^2}{r^2}\varphi''(s) + \frac{r^2}{r^2}\varphi'(s)\right] - \frac{r}{r}\varphi'(s) - \lambda^2\varphi(s) = 0$$
$$\Longrightarrow \varphi''(s) - \lambda^2\varphi(s) = 0$$

Solving the caracteristic equation we get similar roots $k_1 = \lambda$, $k_2 = -\lambda$, it follows that the general solution is

$$\varphi(s) = Ae^{\lambda s} + Be^{-\lambda s}$$

Converting back to the initial variable dependence,

$$R(r) = Ar^{\lambda} + Br^{-\lambda}$$

We note that $R(r)$ needs to be bounded so we exclude the $r^{-\lambda}$ term by setting $B = 0$, then

$$R(r) = Ar^{\lambda}.$$

Recombining the solutions ,

$$U(r,\theta) = R(r)\Theta(\theta) = r^{\lambda}(A\cos(\lambda\theta) + B\sin(\lambda\theta))$$

This represents infinite solutions sicne $\Theta(\theta)$ is periodic , so we can extend with

$$= r^{\lambda}(A_\lambda\cos(\lambda\theta) + B_\lambda\sin(\lambda\theta)) \qquad \text{for } \lambda \in \mathbb{N}.$$

The general solution is then

$$U(r,\theta) = \sum_{\lambda=0}^{N} r^{\lambda}(A_\lambda\cos(\lambda\theta) + B_\lambda\sin(\lambda\theta))$$

We use the boundary condition and the half angle identity,

$$U(1,\theta) = \frac{1}{2} - \frac{\cos(2\theta)}{2} = \sum_{n=0}^{N} A_\lambda\cos(\lambda\theta) + B_\lambda\cos(\lambda\theta) \implies A_0 = \frac{1}{2}, A_1 = 0, A_2 = -\frac{1}{2} \text{ and } B_n = 0\,\forall n.$$

Then it follows that

$$U(r,\theta) = \frac{1}{2} - \frac{r^2}{2}\cos(2\theta), \qquad \therefore U(0,0) = \frac{1}{2} \quad \checkmark.$$

Note. *It is to be noted that the procedure outlined in question* 3 *is far more computationally heavy when compared to that of question* 2*. Suggesting that the Mean Value Property combined with the maximum principles are strong tools.*