MATH 325 Assignment 1.

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1 Question 1.

Due to the initial condition $y(1) = 1$ then the compact cylinder is given by

$$D_{\alpha,\delta} = \{(y,t) \in \mathbb{R}^2 : \|y - y_0\| \leq \alpha \text{ and } |t - t_0| \leq \delta\} \subset D \times (a,b).$$
$$D_{\alpha,\delta} = [1-\alpha, 1+\alpha] \times [1-\delta, 1+\delta]$$

Due to the configuration of the given funciton and since $\alpha, \delta > 0$, the function f is maximized for $\alpha + 1$ and $\delta - 1$,thus

$$\epsilon = \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right)$$
$$M_{\alpha,\delta} = \sup_{(y,t)\in[1-\alpha,1+\alpha]\times[1-\delta,1+\delta]} \left|(y+1)^2 + \frac{1}{1-t}\right|$$
$$\implies \epsilon = \min\left(\delta, \frac{\alpha}{(1+\alpha)^2 + \frac{1}{1-\delta}}\right)$$

This is an optimization problem with 2 variables to be found. Let us find the maximum for the prevous equation's right hand side :

$$\begin{aligned}
\frac{d}{d\alpha}\left(\frac{\alpha}{(1+\alpha)^2 + \frac{1}{1-\delta}}\right) &= \frac{\left((\alpha+1)^2 + \frac{1}{1-\delta}\right) - 2(\alpha+1)\alpha}{\left((\alpha+1)^2 + \frac{1}{1-\delta}\right)^2} = 0 \\
&= \left((\alpha+1)^2 + \frac{1}{1-\delta}\right) - 2(\alpha+1)\alpha = 0 \\
&= (\alpha+1)^2 + \frac{1}{1-\delta} - 2(\alpha+1)\alpha = 0 \\
&= \alpha^2 + \frac{1}{1-\delta} = 0 \implies \delta = 1 + \frac{1}{1-\alpha^2} + 1. \\
&\implies \text{or} \quad \alpha = \sqrt{1 - \frac{1}{\delta - 1}}
\end{aligned}$$

Since δ is an increasing function and $\alpha/((\alpha+1)^2+\frac{1}{1-\delta})$ is a decreasing function , the local minimum between them occurs at their interesection point, thus

$$\frac{\alpha}{(\alpha+1)^2+\frac{1}{1-\delta}} = \delta$$
$$\implies (\alpha+1)^2 + \frac{1}{1-\delta} = \alpha$$

rearranging this equation we have a quadratic equation in $\alpha+1$:

$$(\alpha+1)^2 - (\alpha+1) + \left(\frac{1}{1-\delta} + 1\right) = 0$$
$$\implies \alpha+1 = \frac{1+\sqrt{1-\frac{4\delta^2}{1-\delta} - 4\delta}}{2\delta}$$

Since $\alpha = \sqrt{1-\frac{1}{\delta-1}}$, we have

$$\frac{1+\sqrt{1-\frac{4\delta^2}{1-\delta} - 4\delta}}{2\delta} = \sqrt{1-\frac{1}{\delta-1}} + 1$$
$$1+\sqrt{\frac{5\delta-1}{\delta-1}} = \frac{2\delta\sqrt{\delta-2}}{\sqrt{\delta-1}} + 2\delta$$
$$\frac{5\delta-1}{\delta-1} = \left(2\delta\frac{\sqrt{\delta-2}}{\sqrt{\delta-1}} + 2\delta - 1\right)^2$$
$$= 3\delta^2\frac{\delta-2}{\delta-1} + 4\delta^2\frac{\delta-2}{\delta-1} - 2\delta\frac{\sqrt{\delta-2}}{\sqrt{\delta-1}}$$
$$+ 4\delta^2\frac{\sqrt{\delta-2}}{\sqrt{\delta-1}} + 4\delta^2 - 2\delta$$
$$- 2\delta\frac{\sqrt{\delta-2}}{\sqrt{\delta-1}} - 2\delta + 1$$
$$\left(\frac{\left(\frac{5\delta-1}{\delta-1}\right) - \frac{4\delta^2(\delta-2)}{\delta-1} - 4\delta^2 + 4\delta - 1}{8\delta^2 - 4\delta}\right)^2 = \frac{\delta-2}{\delta-1}$$

After further expansion and rearrangement we obtain

$$\frac{(2\delta(2-\delta))^2}{(\delta-1)(2\delta-1)^2} = \delta - 2$$
$$-4\delta^2(2-\delta) = (\delta-1)(2\delta-1)^2$$
$$4\delta + \delta - 1 = 0 \implies \delta = \frac{1}{5}.$$
$$\implies \alpha = \sqrt{1 - \frac{1}{\frac{1}{5}-1}} = \frac{3}{2}.$$

2 Question 2.

2.1 a)

Let $\mu(t) \neq 0$

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t)$$

By the chain rule the left hand side becomes

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t).$$

Integrating both sides and isolating $y(t)$ gives the general soluton :

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)q(t)\ dt + C$$

We need to find the integrating factor $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$
$$\implies \frac{d}{dt}\ln|\mu(t)| = p(t)$$
$$|\mu(t)| = Ce^{\int p(t)\ dt}$$

So finally, combining these results we have the solution :

$$y(t) = \frac{1}{e^{\int p(t)\ dt}}(\underbrace{\int \mu(t)q(t)\ dt + C)}_{\int 0\ dt = C}$$

So then finally if we let $2C$ be an undefined constant we have

$$y(t) = Ce^{-\int p(t)\ dt}$$

2.2 b)

Let us first and foremost find $y'(t)$

$$\begin{aligned}
y'(t) = \frac{d}{dt}y(t) &= \frac{d}{dt}C(t)e^{-\int p(t)\ dt} \\
&= \frac{C'(t)}{e^{\int p(t)\ dt}} + C(t)e^{-\int p(t)\ dt}(-p(t)) \\
&= \frac{1}{e^{\int p(t)\ dt}}(C'(t) - C(t)p(t))
\end{aligned}$$

Substituting $y'(t)$ and $y(t)$ in the non-homogeneous yields

$$\frac{1}{e^{\int p(t)\ dt}}(C'(t) - C(t)p(t)) + p(t)C(t)e^{-\int p(t)\ dt} = q(t)$$

Some terms cancel out and we obtain

$$C'(t) = q(t)e^{\int p(t)\ dt}$$

2.3 c)

Integrating the final expression found in 2. c), we get

$$C(t) = \int \left(q(t)e^{\int p(t)\ dt} + C\right)$$

Substituting in the expression for non-homogenous equation we get :

$$y(t) = \frac{\int \left(q(t)e^{\int p(t)\ dt} + C\right)}{e^{\int p(t)\ dt}}$$

3 Question 3.

Since this is a linear first order ODE the non-homogenous solution is

$$y(t) = C(t)e^{\int 2\ dt}$$

Differentiating yields

$$y'(t) = C'(t)e^{\int 2\ dt} + C(t)(2)e^{\int 2\ dt}$$

Substituting these results in the original expression produces

$$C'(t)e^{\int 2\ dt} + 2C(t)e^{\int 2\ dt} - 2C(t)e^{\int 2\ dt} = t^2e^{2t}$$

Canceling some terms and rearranging yields

$$C'(t) = t^2 \implies C(t) = \int t^2\ dt$$

And thus,

$$\begin{aligned} y(t) &= \left(\int t^2\ dt\right)e^{\int 2\ dt} \\ &= \left(\frac{t^3}{3} + C\right)e^{2t}. \end{aligned}$$

4 Question 4.

4.1 a)

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{x(y/x-4)}{x(1-y/x)} = \frac{(y/x)-4}{1-(y/x)}.$$

4.2 b)

Since $y(x) = xv(x)$, and $y = v/x$ then by the chain rule ,

$$\frac{dv}{dx} = \frac{dy/dx}{x} - \frac{y(x)}{x^2} \implies \frac{dy}{dx} = x\frac{dv}{dx} + v.$$

4.3 c)

Since $dy/dx = xdv/dx$ and $(y/x) = v$ then we have

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4-v+v^2}{1-v}.$$

$$\implies x\frac{dv}{dx} = \frac{v^2-4}{1-v}.$$

4.4 d)

Since the ODE is separable , then

$$\int \frac{1-v}{v^2-4}\, dv = \int \frac{1}{x}\, dx$$

The left hand side integral can be evaluated using partial fraction decomposition

$$\frac{A}{v-2} + \frac{B}{v+2} = 1 - v$$
$$\implies \begin{cases} (A+B) & = 1 \\ (2A-2B) & = -1 \end{cases} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & -3 \end{bmatrix}$$

So then $A = -1/4$ and $B = -3/4$.

$$\begin{aligned} \frac{-3}{4}\int \frac{1}{v+2}dv - \frac{1}{4}\int \frac{1}{v-2}dv &= \int \frac{1}{x}dx \\ -3\ln|v+2| - \ln|v-2| &= 4\ln|x| + 4C \\ \frac{1}{e^{\ln|v-2|}e^{\ln|(v+2)^3|}} &= e^{\ln|x^4|}e^{4C} \\ \frac{1}{(v-2)(v+2)^3} &= x^4e^{4C} \end{aligned}$$

4.5 e)

Since $v = y/x$, replacing in the previous equation yields

$$\begin{aligned} \frac{1}{\left(\frac{y-2x}{x}\right)\left(\frac{y+2x}{x}\right)^3} &= x^4e^{4C} \\ \frac{1}{(y-2x)(y+2x)^3} &= e^{4C} \end{aligned}$$

Let $C' \equiv e^{4C}$,then the implicit solution is given by

$$\frac{1}{(y_2x)(y+2x)^3} = C'.$$

5 Question 5.

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Since the ODE can be rewritten as fraction of x and y it implies it is homogeneous. Let $v = y/x$ and $y(x) = xv(x)$, then by the chain rule ,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Therefore we have

$$x\frac{dv}{dx} + v = 1 + +\frac{y}{x} + \left(\frac{y}{x}\right)^2 = 1 + v + v^2 = 1 + v^2.$$

$$\int \frac{1}{1+v^2}\, dv = \int \frac{1}{x}\, dx$$
$$\arctan(v) = \frac{-1}{x^2} + C.$$

replaceing $v = y/x$,

$$\arctan\left(\frac{y}{x}\right) = \frac{-1}{x^2} + C$$
$$y(x) = x\tan\left(\frac{-1}{x^2} + C\right).$$

6 Question 6.

6.1 a)

$y'(t) =$ rate in - rate out. The rate in is the concentration multiplied by the flow i.e., rate in $= \frac{1}{2}(1 + \frac{1}{2}\sin(t))$ gal/min. Moreover,

$$\text{Rate out } = 2\left(\frac{y(t)}{100}\right).$$

So then we have the IVP :

$$y'(t) + \frac{y(t)}{50} = -2\frac{(2+\sin(t)}{4} \quad , y(0) = 50.$$

This is a linear ODE therefore the integrating factor is given by

$$\mu(t) = e^{\int \frac{1}{50}\, dt}.$$

Multiplying both sides of the IVP and using the chain rule we obtain

$$\frac{d}{dt}\left(e^{t/50}y(t)\right) = e^{t/50}\frac{(-2-\sin(t))}{4}$$

$$\begin{aligned} e^{t/50}y(t) &= \frac{1}{4}\left(\int -2e^{t/50}\, dt - \int e^{t/50}\sin(t)\right) \\ &= \frac{1}{4}\left(\left(-100e^{t/50}+C\right) - \int e^{t/50}\sin(t)\right). \end{aligned}$$

The left hand side integral is evaluated using integration by parts twice :

$$\begin{aligned} \int e^{t/50}\sin(t) &= \sin(t)e^{t/50} - 50\int e^{t/50}\cos(t) \\ &= \sin(t)e^{t/50} - 50(50e^{t/50}\cos(t) + 50\int e^{t/50}\sin(t))\int e^{t/50}\sin(t) \\ &+ 2500\int e^{t/50}\sin(t) = \sin(t)e^{t/50} - 2500e^{t/50}\cos(t) \\ \int e^{t/50}\sin(t) &= \frac{\sin(t)e^{t/50} - 2500e^{t/50}\cos(t)}{2501} + C \end{aligned}$$

So then we have

$$\begin{aligned} e^{t/50}y(t) &= -25e^{t/50} - e^{t/50}\frac{(\sin(t) - 2500\cos(t)}{2501} + C \\ y(t) &= -25 - \frac{(\sin(t) - 2500\cos(t)}{2501} + \frac{C}{e^{t/50}}. \end{aligned}$$

$y(0) = 50$ so we may find C :

$$\implies 50 + 25 - \left(\frac{2500}{4(2501)}\right) \approxeq 74.75.$$

Therefore the amount of salt at any time is

$$y(t) = -25 - \frac{(\sin(t) - 2500\cos(t))}{4(2501)} + \frac{75.75}{e^{t/50}}.$$

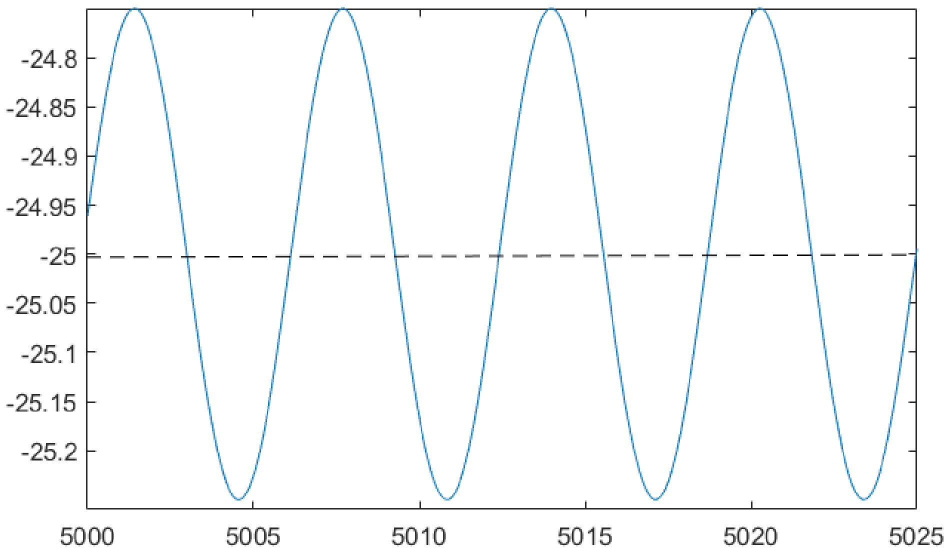

Figure 1: Solution Plot

6.2 b)

6.3 c)

$$y(t) = \underbrace{-25}_{\text{Point}} - \frac{(\sin(t) - 2500\cos(t))}{4(2501)} + \frac{75.75}{e^{t/50}}.$$

Evidently, the wave is oscillating about -25. Moreover, by taking the limit,

$$\lim_{t\to\infty} -25 - \frac{\sin(t)}{4(2500)} - \frac{2500}{2(2500)} + \frac{75.75}{e^{t/50}} = \underbrace{-25}_{\text{Off-set}} - \frac{\sin(t)}{4(2500)} - \frac{2500}{2(2500)}\cos(t)$$

The coefficient of $\sin(t)$ doesn't contributy much compared to that of $\cos(t)$ thus the amplitude is given by $\frac{2500}{4(2500)} = 0.25$

7 Question 7.

Let us assume that M, N, and their respective partials M_y and N_x are all continous on some simply connected domain $D \subset \mathbb{R}^2$. Then following this assumption, $\mu(xy)M(x,y) + \mu(xy)N(x,y)y' = 0$ is exact if and only if

$$\frac{\partial}{\partial y}(\mu(xy)M(x,y)) = \frac{\partial}{\partial x}(\mu(xy)N(x,y))$$

that is

$$y\mu'(xy)M(x,y) + \mu(xy)\frac{\partial M}{\partial y} = x\mu'(xy)N(x,y) + \mu(xy)\frac{\partial N}{\partial x}.$$

Rearranging the previous equation yields

$$\mu'(xy)(xM - yM) = \mu(xy)\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$
$$\implies \frac{\mu'(xy)}{\mu(xy)} = \frac{N_x - M_y}{xM - yN}.$$

Thus an integrating factor exists if the right hand side satisfies the initial equality $h(xy)$

$$\int \frac{d}{dt}\ln|\mu(xy)| = \int h(xy)\ dt$$
$$\implies \mu(xy) = Ce^{\int h(xy)\ dt}.$$

8 Question 8.

$$\underbrace{\left(3x + \frac{6}{y}\right)}_{M} + \underbrace{\left(\frac{x^2}{y} + \frac{3y}{x}\right)}_{N}\frac{dy}{dx} = 0$$

$$M_y = \frac{-6}{y^2} \neq N_x = \frac{2x}{y} - \frac{3y}{x^2} \implies \text{Not exact.}$$

Let us see if we can find an integrating factor as a function of x or y alone :

$$\frac{M_y - N_x}{N} = \frac{\frac{-6}{y^2} - \frac{2x}{y} + \frac{3y}{x^2}}{\frac{x^2}{y} + \frac{3y}{x}} \neq f(x)$$
$$\frac{N_x - M_y}{M} = \frac{\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}}{3x + \frac{6}{y}} \neq f(y).$$

So then let's try to find an integrating factor from the PDE :

$$\begin{aligned}
\frac{N_x - M_y}{xM - yN} &= \frac{\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}}{\left(3x^2 + \frac{6x}{y}\right) - \left(x^2 + \frac{3y^2}{x}\right)} \\
&= \frac{\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}}{2x^2 + \frac{6x}{y} - \frac{3y^2}{x}} = \frac{\frac{2x^3y}{x^2y^2} - \frac{3y^3}{x^2y^2} + \frac{6x^2}{x^2y^2}}{\frac{2x^3y}{xy} + \frac{6x^2}{xy} - \frac{3y^3}{xy}} \\
&= \frac{2x^3y - 3y^3 + 6x^2}{2x^3y + 6x^2 - 3y^3}\left(\frac{1}{xy}\right) = \frac{1}{xy}.
\end{aligned}$$

Thus there exists an integrating factor μ that depends on xy such that $\mu = \mu(x, y)$.
Let $t = xy$, then

$$\mu(x, y) = e^{\int \frac{1}{t}\ dt} = e^{\ln(t)} = t = xy.$$

Multiplying M and N by xy from the initial given expression we see that

$$\begin{aligned}
xy\left(3x + \frac{6}{y}\right) + xy\left(\frac{x^2}{y} + \frac{3y}{x}\right)\frac{dy}{dx} &= 0. \\
3x^2y + 6x + \left(x^3 + 3y^2\right)\frac{dy}{dx} &= 0 \\
\text{then,} \quad M_y = 3x^2 = N_x \implies \text{Exact.}
\end{aligned}$$

Therefore $\exists \phi(x, y)$.

$$\begin{aligned}
\phi(x, y) &= \int \left(3x + \frac{6}{y}\right) \\
&= \frac{3x^2}{2} + \frac{6x}{y} + h(y) \\
\phi_y(x, y) &= \frac{d}{dy}\left(\frac{3x^2}{2} + \frac{6x}{y} + h(y)\right) \\
&= \frac{-6x}{y^2} + h'(y) \implies h'(y) = 0 \implies h(y) = 0.
\end{aligned}$$

The general solution is given by

$$\phi(x, y) = \frac{3x^2}{2} + \frac{6x}{y} = c, \quad c \in \mathbb{R}.$$