PHYS 350 Assignment 4

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Question 1

a)

Since

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV,$$

we have that

$$C = \frac{2U}{V^2} = \frac{2(200)}{1000^2} = \frac{1}{2500}\text{ F}.$$

Similarly,

$$Q = \frac{2U}{V} = \frac{2(200)}{1000} = \frac{2}{5}\text{ C}.$$

b)

Air between the plates so we have natural dielectric

$$U = \frac{A}{2d}\epsilon_0 E^2 d^2 = \frac{1}{2}Ad\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 E^2(\text{Vol})$$

Letting $E = E_{\text{max}} = 3 \times 10^6\text{ V m}^{-1}$, then

$$\text{Vol} = \frac{2(200)}{\epsilon_0(3 \times 10^6)^2} = 5.02\text{ m}^3.$$

This volume is much larger than the volume that they use.

c)

Since $U = \epsilon_0 E^2/2$, then integrating over a sphere with respect to r in spherical coordinates yields

$$U = \frac{\pi}{3}\epsilon_0 E^2 r^3 \implies r = \sqrt[3]{\frac{3U}{\pi\epsilon_0 E^2}} = 1.32\text{ m} \implies d \approx 2.6\text{ m}.$$

Question 2

Let us consider the rods in the yz quadrant. Let a be the distance from the centre set at $(0, 0)$ for each rod. Then as it was found previously the equipotential lines are parametrized by

$$D = a\frac{e^{4\pi\epsilon_0 V_0/\lambda} + 1}{e^{4\pi\epsilon_0 V_0/\lambda} - 1} \qquad \text{and } R = 2a\frac{e^{2\pi V_0\epsilon_0/\lambda}}{e^{4\pi V_0\epsilon_0/\lambda} - 1}$$

We can perform the following operations , let $a \equiv 2\pi\epsilon_0 V_0/\lambda$

$$D = a\frac{e^{2a}+1}{e^{2a}-1} = a\frac{e^a e^a + 1}{e^a e^a - 1} = a\frac{e^a + \frac{1}{e^a}}{e^a - \frac{1}{e^a}} = a\frac{e^a + e^{-a}}{e^a - e^{-a}} = a\coth(a)$$
$$R = 2a\frac{e^a}{e^{2a}-1} = 2a\frac{1}{e^a - e^{-a}} = 2a\frac{1}{\sinh(a)} = a\,\mathrm{csch}(a)$$

and so we conclude that

$$D = a\coth(2\pi\epsilon_0 V_0/\lambda) \qquad \text{and } \; R = a\,\mathrm{csch}(2\pi\epsilon_0 V_0/\lambda).$$

Diving the two we can solve for λ;

$$\frac{D}{R} = \cosh(2\pi\epsilon_0 V_0/\lambda) \implies \lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(D/R)}.$$

We also concluded that the potential at any point in that quadrant is

$$V = \frac{\lambda}{4\pi\epsilon_0}\ln\left|\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right|,$$

with $a = \sqrt{D^2 - R^2}$.

Question 3

It was seen in the last assignment that the potential and maximal electric field for the log based solution,

$$E_{\max} = \frac{\lambda}{2\pi\epsilon_0 R}$$
$$V = \frac{\lambda}{\pi\epsilon_0}\ln\left|\frac{D-R}{R}\right| \implies \lambda = \frac{V_0\pi\epsilon_0}{\ln\left|\frac{D-R}{R}\right|}$$
$$\therefore E_{\max} = \frac{V_0\pi\epsilon_0}{\ln\left|\frac{D-R}{R}\right|}(2\pi\epsilon_0 R)^{-1} \tag{1}$$

For the new exact solution,

$$V = \frac{\lambda}{4\pi\epsilon_0}\ln\left|\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right|$$

$$\begin{aligned}
\implies E = -\nabla V &= \frac{-\lambda}{4\pi\epsilon_0}\left(\frac{2(y-a)}{(y-a)^2+z^2} - \frac{2(y+a)}{(y+a)^2+z^2}\right)\\
&= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{(y-a)}{(y-a)^2+z^2} - \frac{(y+a)}{(y+a)^2+z^2}\right)\\
&= \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(D/R)2\pi\epsilon_0}\left(\frac{(y-a)}{(y-a)^2+z^2} - \frac{(y+a)}{(y+a)^2+z^2}\right) \qquad (2)
\end{aligned}$$

Combining (1) and (2) and with $R = 0.025$ cm and $D = 10$ m and $V = V_0 = 765kV$ we get

$$\begin{aligned}
(1)\ :\qquad E_{\max} &= \frac{(765000)\pi(8.83\times 10^{-12})}{\ln\left|\frac{10-0.025}{0.025}\right|2\pi(8.83\times 10^{-12})0.025} = 2.55\times 10^{6}\ \mathrm{V\,m^{-1}}\\
(2)\ :\qquad E_{\max} &= \frac{2\pi(8.83\times 10^{-12})(765000)}{\cosh^{-1}(10/0.025)}\left(\frac{2}{5}\right) = 4.6\times 10^{5}\ \mathrm{V\,m^{-1}}
\end{aligned}$$

The two electric fields differ by about an order of magnitude, suggesting that the electric field or log based potential formulae derived on the last assignment are erroneous.

Remark. *No further improvements have been suggested by the TA whom I have contacted and assured me that the correction for this problem shall not be to harsh. Thence the answer is left as is.*

Question 4

The average of a potential along a sphere is given by

$$V_{\text{avg}} = \frac{1}{A}\int V\,da = \frac{1}{4\pi R^2}\int V\,da.$$

We show the equality as requested ;

$$V_{\text{avg}} = \frac{1}{4\pi R^2}\int V\,da$$

Since we're integrating over the surface of a sphere, we parameterize $V = V(R,\theta,\phi)$,

$$= \frac{1}{4\pi}\int V(R,\theta,\phi)R^2\sin\theta\,d\theta\,d\phi$$

By Leibnitz rule,

$$= \frac{1}{4\pi}\int \frac{\partial V}{\partial R}\sin\theta\,d\theta\,d\phi$$

From spherical coordinates we know that

$$\nabla V = \frac{\partial V}{\partial R}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \implies \hat{r}\cdot\nabla V = (\hat{r}\cdot\hat{r})\frac{\partial V}{\partial R} = \frac{\partial V}{\partial R}$$

$$\therefore \frac{\mathrm{d}V_{\text{avg}}}{\mathrm{d}R} = \frac{1}{4\pi}\int (\hat{r}\cdot\nabla V)\sin\theta \; d\theta \; d\phi$$

We add a factor of R^2 and rearrange

$$= \frac{1}{r\pi R^2}\int (\nabla V)\cdot\hat{r}R^2\sin\theta d\theta d\phi$$

$$\frac{\mathrm{d}V_{\text{avg}}}{\mathrm{d}R} = \frac{1}{4\pi R^2}\oint (\nabla V) d\boldsymbol{a} \quad \checkmark$$

Applying the divergence theorem , we take the gradient and integrate over the volume

$$= \frac{1}{4\pi R^2}\int \nabla\cdot(\nabla V)\; d\tau = \frac{1}{4\pi R^2}\int (\nabla^2 V)\; d\tau.$$

If V satisfies Laplace equation ,i.e., $\nabla^2 V = 0$ over the volume then

$$\frac{\mathrm{d}V_{avg}}{\mathrm{d}R} = 0,$$

Which suggests that regardless of the radius , the potential remains the same. So for a point P at which the ball is centered, it follows $V_{\text{avg}}(0) = V(P) = V_{\text{avg}}(0)$

Question 5

We know that

$$\int_{v} T\nabla^2 U + (\nabla T)\cdot(\nabla T)\; d\tau = \oint_S (T\nabla U)\cdot d\boldsymbol{a},$$

for $T = U = V_3$.Therefore ,

$$\int_{v} (V_3\nabla^2 V_3 + (\nabla V_3)\cdot(\nabla V_3))\; d\tau = \oint_S (V_3\nabla V_3)\cdot d\boldsymbol{a}$$

Since by construction $E_3 = E_1 - E_2 \implies V_3 = V_1 - V_2$. Moreover, we know that $\nabla E = -\rho/\epsilon_0$, and $\nabla V = -E$,thus

$$\int_{v} (V_3(\nabla^2 V_1 - \nabla^2 V_2) + {E_3}^2)\; d\tau = -\oint_S V_3 E_3\cdot d\boldsymbol{a}$$

$$\int_{v} V_3\left(\cancelto{0}{\frac{-\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0}}\right) + {E_3}^2\; d\tau = -\oint_S V_3 E_3\cdot d\boldsymbol{a}$$

$$\implies \int_{v} {E_3}^2\; d\tau = -\oint_S V_3 E_3\cdot d\boldsymbol{a}$$

Since V_3 is constant over all surfaces in ν and is 0 at ∞, we have that

$$\int_\nu {E_3}^2 \, d\tau = -V_3 \oint_S E_3 \cdot d\boldsymbol{a} = 0 \text{ since } \oint_S E \cdot d\boldsymbol{a} = 0 \text{ by definition}$$

$$\therefore \int_\nu {E_3}^2 \, d\tau = 0 \implies E_3 \equiv 0 \implies E_1 \equiv E_2.$$

Question 6

a)

We place a charge q' at the right of the centre with value $q' = -Rq/a$. We also place a second image charge q'' at the centre of the sphere such that the potential inside the sphere is no longer 0 but $V_0 = 4\pi\epsilon_0 q''/a$. Then since the sphere is neutral we have that $q' + q'' = 0$. It then follows by definition of the force for image charge configurations that The potential super posed is

$$V = \frac{q''}{4\pi e p_0 a} + \frac{q''}{4\pi\epsilon_0(a-b)} \implies E = -\nabla V = \frac{1}{4\pi\epsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right),$$

So then the force is $F = qE$,

$$F = \frac{1}{4\pi\epsilon_0} q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right).$$

Now since $q'' = -q'$ we can perform some rearrangement

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0} qq' \left(\frac{-1}{a^2} + \frac{1}{(a-b)^2}\right) \\ &= \frac{1}{4\pi\epsilon_0} qq' \left(\frac{-(a-b)^2 + a^2}{a^2(a-b)^2}\right) \\ &= \frac{1}{4\pi\epsilon_0} qq' \frac{(2a-b)b}{a^2(a-b)^2} \end{aligned}$$

Since $q' = -Rq/a$ and $b = R^2/a$ by definition, it follows that through substitution

$$\begin{aligned} &= \frac{1}{4\pi\epsilon_0} \frac{q\left(\frac{-R^2 q}{a}\right)(2a - R^2/a)R^2/a}{a^2(a - R^2/a)^2} \\ &= \frac{1}{4\pi\epsilon_0} q^2 \left(\frac{-R}{a}\right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^2} \end{aligned}$$

We drop the minus sign since the force is attractive so we care about the magnitude

$$F = \frac{1}{4\pi\epsilon_0} q^2 \left(\frac{R}{a}\right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^2}. \tag{3}$$

We now look for an expression for the leading order in a and R. Let $R \sim a$ since for the order it doesn't matter if one is larger or smaller than the other, then

$$
\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q^2 \cancelto{(1,\ \textit{"since they have the same order"})}{\left(\frac{R}{a}\right)^3} \frac{a^2\left(2 - \left(\frac{R}{a}\right)^2\right)}{a^4(1 - 2\left(\frac{R}{a}\right)^2 + \left(\frac{R}{a}\right)^4)} \\
&= \frac{1}{4\pi\epsilon_0} q^2 \frac{2 - \left(\frac{R}{a}\right)^2}{a^2\left(1 - 2\left(\frac{R}{a}\right)^2 + \cancelto{0}{O\left(\frac{R}{a}\right)}\right)} \\
&\approx \frac{1}{4\pi\epsilon_0} q^2 \frac{2 - \left(\frac{R}{a}\right)^2}{a^2\left(1 - 2\left(\frac{R}{a}\right)^2\right)},
\end{aligned}
$$

we see that the leading order between a and R is a with the second order magnitude. We conclude that the force for that leading order is approximatively

$$
F \approx \frac{1}{4\pi\epsilon_0} q^2 \frac{2 - \left(\frac{R}{a}\right)^2}{a^2\left(1 - \left(\frac{R}{a}\right)^2\right)}.
$$

b)

Since the charge is *placed* on the conductor and is brought all the way from infinity, the work would blow up to infinity as well since $V = 0$ on the surface of the conductor , since it's an equipotential.

c)

We use $E = E_{\text{max}} = 3 \times 10^6$ for an estimate since it's the ambient electric field around the ball given the corona discharge presence around the van der graph. For a spherical conductor,

$$
E = \frac{Q}{4\pi\epsilon_0 r^2} \implies Q = E4\pi\epsilon_0 R^2 = (3 \times 10^6)4\pi(8.83 \times 10^{-12})(0.05)^2 = 8.32 \times 10^{-7}\ \text{C}.
$$

Then, we use (3) to find the force but replace q with Q ;

$$
\begin{aligned}
F &= \frac{Q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^3} \\
&= \frac{(8.32 \times 10^{-7})^2}{4\pi(8.83 \times 10^{-12})}\left(\frac{0.05\ \text{m}}{0.1\ \text{m}}\right)^3 \frac{(2(0.1)^2 - 0.05^2)}{(0.1^2 - 0.05^2)^2} \\
&= 0.273\ \text{N}.
\end{aligned}
$$