Notes For MATH223

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1 Vector Spaces

Propreties of Vectors Spaces

1. The element 0_v of A4 is unique ,it's the zero vector
2. $\forall u \in V$ the vector $-u$ from A5 is also unique.
3. $\alpha 0_v = 0_v$
4. $0u = 0_v \qquad \forall\ u \in V$
5. $\forall u, v, w \in V \qquad u + v = u + w \implies v = w$
6. $\alpha \in \mathbb{R}, u\ \in V$
$\alpha u = 0_v \iff (\alpha = 0)$ or $(u = 0_v)$

1.1 Subspaces

Let $(V, +, \cdot)$ be a vector space and let W be a subset of V.

Definition 1.1. W is said to be a subspace of V if W equipped with the two operations is a vector space.

Remark We do not need to verify all 10 axioms on W . It is enough to verify that

1. W is not empty
2. W is closed under addition i.e whenever $u, v \in W, u + v \in W$
3. W is closed under scalar multiplication
   i.e whenever $u \in W, \alpha \in \mathbb{R}, \alpha u \in W$

Proposition 1.1. Let V be a vector space and let W be a non-empty subset of V. W is a subpace of V iff whenever $u, v \in W \alpha, \beta \in \mathbb{R}, \alpha u + \beta v \in W$

Example
Let $\mathscr{P}_n$ be the set of all polynominals defined on $\mathbb{R}$ with degree $\leq n$. $\mathscr{P}_n$ is a subspace of $\mathbb{F}(\mathbb{R})$.
Let $n = 2$.

$$\begin{aligned} \text{take } f(x) &= x^2 \\ g(x) &= -x^2 + 1 \\ (f+g)(x) &= 1 \quad \rightarrow \deg(0) \end{aligned}$$

So it's not closed under addition nor scalar multiplication.

Example
Let $V = \{M \in \mathscr{M}_{n\times n} |$ such that $\quad A = A^T \rightarrow$ set of all symmetric matrices $\}$

$$\begin{aligned} &\textbf{i. } \underset{\sim}{O} \in V \qquad \text{Verify emptyness} \\ &\textbf{ii, } \text{Let } A, B \in V \quad , (A+B)^T = A^T + B^T = A + B \\ &\textbf{iii, } \text{Let } A \in V,\ \alpha \in \mathbb{R} \qquad , \alpha A^T = \alpha A \end{aligned}$$

So V as defined is a vector space of $\mathscr{M}_{n\times n}$.

2 Spanning Sets

Definition 2.1. Linear Combination Let V be a vector space.
Let $u_1, u_2, \ldots, n$ be vectors in V.
A linear combination of $u_1, u_2, \ldots, n$ is any vector $u \in V$ of the form

$$u = \sum_{i=1}^{n} \alpha_i u_i$$
$$\implies \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \qquad \text{where } \alpha_1, \alpha_2, \ldots, \alpha_2 \in \mathbb{R}$$

Definition 2.2. Spanning Set
Let S be a non empty subset of a vector space V. Span$\{S\}$ is the subset of V defined as Span$(\mathcal{S})$.

$$\text{Span}(\mathcal{S}) = \{u = \sum_{i=1}^{n} \alpha_i u_i | n \in \mathbb{N},$$
$$u_1, u_2, \ldots, n \in \mathcal{S}, \quad \alpha_1, \alpha_2, \ldots, \alpha_2 \in \mathbb{R}\}$$

Remark $\mathcal{S} \subseteq \text{Span}(\mathcal{S})$

Proposition 2.1. Let V be a vector space and $\mathcal{S}$ be a non empty subset of V. Then , Span($\mathcal{S}$)is the smallest vector subspace of V which contains $\mathcal{S}$

Propreties of spanning sets

1. $\mathcal{S} = \text{Span}(\mathcal{S})$ if an only if $\mathcal{S}$ is a subspace

2. $\text{Span}((\mathcal{S})) = (\mathcal{S})$same span of span of span...

Example
$\mathcal{P}_2$ as a subspace of $\mathbb{F}(\mathbb{R})$
Let $f \in \mathcal{P}_2$, $\exists$ $a, b, c \in \mathbb{R}$ such that $f(x) = ax^2 + bx + c$

Exercise Let $W = \{p \in \mathcal{P}_3 | p(1) = 0\}$
Prove that W is a subspace of $\mathcal{P}_3$

$$p(x) = (x-1)[ax^2 + bx + c]$$
$$W = \text{Span}\{x-1, x(x-1), x^2(x-1)\}$$

That's how a cubic polynominal vanishes at x =1

Example Let us find a spanning set $\mathcal{M}_{(2\times2)}$.

$M \in \mathcal{M}_{(2\times2)}, M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Let

$$E_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\mathcal{M}_{(2\times2)} = Span\{E_1, E_2, E_3, E_4\}$

Note

1. For a $(n \times n)$ matrix we have a spanning set of $(m \times n)$ matrices.
2. A spanning set is not unique, we can come up with an infinite number of them.

3 Linearly Independent Subsets

Let ($\mathcal{S}$)be a non-empty subset of a vector space V.

Definition 3.1. $\mathcal{S} = \{u_1, u_2, \ldots, n\}$ is said to be linearly independent (LI) iff whenever

$$\sum_{i=1}^{n} \alpha_i u_i = 0 \quad \text{where} \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Remark

If $\mathcal{S} = \{u_1, u_2, \ldots, n\}$ is not linearly independent we say that $\mathcal{S}$ is linearly dependent.

Propreties of Linear Independce

V is a vector space

1. $\{0_V\}$ is linearly dependent (rank = 0)
   Moreover any subsets of $\mathcal{S}$ of V which contains 0_V is linearly dependent.

2. $\{u_n\}$ is linearly independent iff $u \neq 0_V$

3. Let $\mathcal{S} = \{u_1, u_2, \ldots, n\}$ be linearly independent.
   Let $u \in V$. $\mathcal{S} \cup \{u\} = \{u_1, u_2, \ldots, n, u\}$

4 Basis and Dimension

Lemma 4.1. Let V be a vector space and $\mathcal{S} = \{u_1, u_2, \ldots, n\}$ be a spanning set of V and $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ a linearly independent set of vectors in V, then

$$\boxed{m \leq n}$$

Definition 4.1. Exchange Lemma In the setting above, we can replace m vectors u_i in the subset $\mathcal{S}$ by the v_i (all of them) and the resulting set will still be a spanning set of V.

Definition 4.2. Let V be a vector space. A subset $\mathcal{B} = u_1, u_2, \ldots, n$ is called a basis of V if $\mathcal{B}$ is both a spanning set of V and linearly independent.

Proposition 4.1. Let V be a vector space. Then all bases of V have the same number of vectors, called dimension of V and denoted $\dim V$.

Example
Let $A_{(n\times m)}$, $\quad$ $\text{Null}(A) = \{x \in \mathcal{B}^m \quad \text{such that} \quad AX = Q\}$
Then Null(A) is a subspace of $\mathcal{B}^m$
and $\dim(\text{Null}(A)) = m - \text{rank}(A)$
Remark
Dim(V) where $V = \{0_V\}$ is 0.

Proposition 4.2. Let V be a finite dimensional vector space and E, F be two subspaces of V.

1. If $E \subseteq F \rightarrow \dim(E) \leq \dim(F)$
2. If $E \subseteq F$ and the $\dim(E) = \dim(F)$ then $E = F$
3. If $\dim(E) = k$ then
   (a) Every spanning set of E has at least k elements
   (b) Every linearly independent subsets of E has at most k elements.
   (c) Every spanning set of E which has exactly k elements is a basis of E.
   (d) Every linearly independent subsets of E which contains exactly k elements is a basis of E.

Remark
If you know the dimension of a subspace you can then know a lot of things , it's a critical information !

5 Sum and Direct Sum of Subspaces

Definition 5.1. Let E and F be 2 subspaces of a vector space. The sum of E and F , $E + F$ is the subset defined as

$$E + F = \{u + v \quad \text{where } u \in E, v \in F\}$$

Proposition 5.1. $E + F$ is a vector subspace of V.

Definition 5.2. IF F and E are 2 subspaces of V such that $E \cap F = \{0_V\}$ then $E + F$ is called the direct sum of E and F and is denoted $E \oplus F$

Proposition 5.2. Let E and F be 2 subspaces of a vector space V. If $E \cap F = \{0_V\}$, then every vector w in $E \oplus F$ can be written uniquely as $w = u + v$, where $u \in E$ and $v \in F$

Proposition 5.3. Let $S = \{u_1, u_2, \ldots, n\}$ and $F = \{v_1, v_2, \ldots, v_m\}$ be subsets of a vector space V.
Define $E = \text{span}\{S\}$ and $F = \text{Span}\{F\}$ then ,

1. $S \cup F$ is a spanning set of $E + F$
2. If S and F are linearly independent , then
   $S \cup F$ is linearly independent iff $E \cap F = \{0_V\}$

Corollary 5.0.1. If E and F are 2 finite dimensional subspaces of V and $S = u_1, u_2, \ldots, n$, $F = \{v_1, v_2, \ldots, v_m\}$ are the respective bases of E and F If $E \cap F = \{0_V\}$ then $S \cup F$ is a basis of $E \oplus F$ i.e :

$$\dim(E \oplus F) = \dim(E) + \dim(F)$$

Lemma 5.1. Let V be a vector space (finite dimensional) and E be a subspace of V. There exists a subspace E_1 of V such that $V = E \oplus E_1$.

Proposition 5.4. If E and F are finite dimensional subspaces of a vector space V , then

$$\dim(E + F) = \dim(E) + \dim(F) - \dim(E \cap F)$$

Example
$V = \mathscr{P}_3$

$$E = \{p \in \mathscr{P}_3 | p'(1) = 0\} \to \dim = 3$$
$$F = \{p \in \mathscr{P}_3 | p(0) = 0\} \to \dim = 3$$

True or false? $V = E + F$

Let us find the dimension of $E \cap F$.

Let $p \in E \cap F$

$$\text{then} \quad p \in F \quad \text{i.e} \quad p(x) = ax^3 + bx^2 + cx$$
$$\text{also} \quad p \in E \quad \text{i.e} \quad p'(1) = 3a + 2b + c = 0$$
$$\implies c = -3a - 2b$$
$$p(x) = a(x^3 - 3x) + b(x^2 - 2x) \qquad , a, b \in \mathbb{R}$$

Therefore

$$E \cap F = \text{Span}\{x^3 - 3x, x^2 - 2x\}$$
$$\text{So} \quad \{x^3 - 3x, x^2 - 2x\} \text{ is L.I, therefore a basis of } E \cap F$$

Since $\dim(E \cap F) = 2, \quad$ thus $\dim(E + F) = 3 + 3 - 2 = 4.$

6 Coordinates of a Vector Relative to a Basis

Proposition 6.1. Let V be a vector space such that $\dim(V) = n$
Let $\mathcal{B} = u_1, u_2, \ldots, n$ be a basis of V.

For every vector $u \in V$, there exists a unique $(n \times 1)$ matrix $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Such that , $u = \sum_{i=1}^{n} x_i u_i$

Definition 6.1. Given $\dim(V) = n$ and $\mathcal{B} = u_1, u_2, \ldots, n$ a basis of V , for $u \in V$,the unique vector

$\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ such that , $u = \sum_{i=1}^{n} x_i u_i$, is called the coordinate vector of u relative to $\mathcal{B}$ and denoted $\underbrace{[u]_{\mathcal{B}}}_{n\times 1}$

Proposition 6.2.

$$\begin{aligned} \text{Let} \quad & \mathcal{B} = \{u_1, u_2, \ldots, n\} \\ & \mathcal{S} = \{v_1, v_2, \ldots, v_n\} \quad \text{2 bases of same vector space} \end{aligned}$$

There exists a unique matrix $(n \times n)$ denoted $\mathcal{P}_{\mathcal{B},\mathcal{S}}$ and called the transition matrix from $\mathcal{B}$ to $\mathcal{S}$ such that $\forall u \in V$

$$\boxed{[w]_{\mathcal{S}} = \mathcal{P}_{\mathcal{B},\mathcal{S}}[w]_{\mathcal{B}}}$$

Corollary 6.0.1. Given $\mathcal{S}$ and $\mathcal{B}$,

$$\mathcal{P}_{\mathcal{B},\mathcal{S}}\mathcal{P}_{\mathcal{S},\mathcal{B}} = I_n$$

7 Linear Transformations

Definition 7.1. Let V and W be two vector spaces
A function $T : V \to W$ (from V to W) is called a linear transformation iff whenever $u_1, u_2 \in V$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

Propreties of Linear Transformations

1. If $T : V \to W$ is a linear transformation then $T(0_V) = 0_W$
2. If T_1 and T_2 are two linear transformations from V into W and $\alpha_1, \alpha_2 \in \mathbb{R}$,then

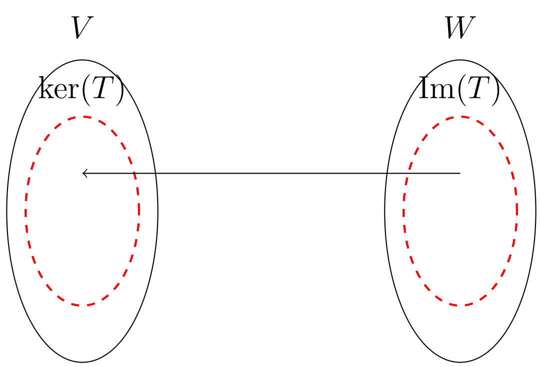
   $$(\alpha_1 T_1 + \alpha_2 T_2)(v) = \alpha_1 T_1(v) + \alpha_2 T_2(v)$$

   Then this is a linear transformation from V into W

Definition 7.2. Given $T : V \to W$ a linear transformation, the set $\{u \in V | T(u) = 0_W\}$ is a vector subspace of V called the Kernel of T and is denoted $\ker(T)$

$$\begin{aligned} \text{Ker}(T) =& \{u \in V | T(u) = 0_W\} \\ =& T^{-1}(\{0_W\}) \end{aligned}$$

Picture



Definition 7.3. One-to-One $T : V \to W$ a linear transformation is said to be one-to-one (or injective) if whenever

$$T(u_1) = T(u_2) \implies u_1 = u_2$$

i.e , we can't have same vector with different output.

Proposition 7.1. $T : V \to W$, a linear transformation is one-to-one iff

$$\ker(T) = \{0_V\}$$

i.e , only one element mapped to 0_W is 0_V.

Proposition 7.2. Let $T : V \to W$ be a one-to-one transformation
Let $\mathcal{B} = \{u_1, u_2, \ldots, n\}$ be a linearly independent subset of V ,then

$$T(\mathcal{B}) = \{T(u_1), T(u_2), \ldots, T(u_n)\} \quad \text{is L.I as well}$$

So a one-to-one transformation carries linear independence in W

Definition 7.4. Image of a Linear Transformation $T : V \to W$ is a linear transformation. The image of T denoted $\mathrm{Im}(T)$ is the range of T (a subset of W).

$$\mathrm{Im}(T) = \{w \in W | w = T(u) \quad \text{for some } u \in V\}$$

Proposition 7.3. $T : V \to W$ a linear transformation, $\mathrm{Im}(T)$ is a vector subspace of W.

Definition 7.5. Image of Linear Transformations $T : V \to W$ a linear transformation then,

$$\mathrm{Im}(T) = \{w \in W | \exists u \in V \quad \text{and} \quad w = T(u)\}$$

$\mathrm{Im}(T)$ is the range of T.

Definition 7.6. $T : V \to W$, a linear transformation, is said to be onto (surjective) if

$$\mathrm{Im}(T) = W$$

Propreties of surjective trasformations

Let $T : V \to W$ be a linear transformation and $\mathcal{S} = \{u_1, u_2, \ldots, n\}$ a spanning set of V then $\{T(u_1), \ldots, T(u_n)\}$ is a spanning set of $\mathrm{Im}(T)$

Note If we're asked to find a basis of $\mathrm{Im}(T)$ the previous proprety is very usefull.

Definition 7.7. .A linear trnasformation $T : V \to W$ is said to be an **isomorphism** if T is one-to-one and onto.

Proposition 7.4. .Let $T : V \to W$ be an isomorphism. Assume that dim(V) is finite.
Then,

$$\dim(V) = \dim(W)$$

Indeed since T is defined to be an isomorphism, its range has to be W itself, therefore the number of vectors that span each subspaces is the same and consequently dimension is preserved.
Under these circumstances, we say that isomorphism carries not only span, but the basis as well.

8 Composition of Linear Transformation

Let $V1, V2, V3$ and T_1, T_2 be defined as:

$$T_1 : V_1 \to V_2$$
$$T_2 : V_2 \to V_3$$

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3$$

Where T_i are linear transformations.

Proposition 8.1. The function $T_2 \circ T_1 : V_1 \to V_3$ is a linear transformation.

Proposition 8.2. Let $T : V \to W$ be an isomorphism. The inverse of T_1 denoted $T^{-1} : W \to V$ is also a linear transformation.

Note By definition of T^{-1}, given $w \in W$, $T^{-1}(w)$ is the unique vector in V such that $T(T^{-1}(w)) = w$.

Remark

$$\begin{aligned} &1) T \circ T^{-1} = Id_W \qquad \text{Identity L.T of W} \\ &2) T^{-1} \circ T = Id_V \end{aligned}$$

$$V \xrightarrow{T} W \xrightarrow{T^{-1}} V$$

Where Id_W and Id_V are the identity transformations on V and W respectively.

$$\begin{aligned} &Id_V : V \to W \\ &Id_V(u) = u \qquad \forall u \in V \end{aligned}$$

Theorem 8.1. *Rank Nullity Theorem* $T : V \to W$ is a linear transformation and $\dim(V)$ is finite,

$$\boxed{\text{dim(V)} = \text{dim(Ker(T))} + \text{dim(Im(T))}}$$

Definition 8.1. $T : R^n \to R^n$ What makes this linear transformation? then there $\exists$ a unique standard matrix of T

For example we can take

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + 2x \\ z - 3y \\ x + y \end{bmatrix} \longmapsto \qquad [T] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Definition 8.2. $\widetilde{T} = i_{\mathscr{B}} \circ i_{\mathscr{B}}^{-1}$
$\widetilde{T}$ is a linear transformation from $R^n \to R^n$ so there $\exists$ a unique matrix $(n \times n)$ $[\widetilde{T}]$ such that $\widetilde{T}(X) = [\widetilde{T}]X \quad \forall\ X \in \mathbb{R}^n, \forall u \in V$

$$\boxed{[T(u)]_{\mathscr{B}} = [\widetilde{T}][u]_{\mathscr{B}}}$$

Proposition 8.3. If $T : V \to W$ is a linear transformation and $\mathscr{B} = \{u_1, ..., u_n\}$ is a basis of V, then $\exists$ a unique matrix $(n \times n)$ matrix denoted $[T]_{\mathscr{B}}$ called the representation of T relative to $\mathscr{B}$ such that

$$[T(u)]_{\mathscr{B}} = [T]_{\mathscr{B}}[u]_{\mathscr{B}} \qquad \forall u \in V$$

Remark $[T]_{\mathscr{B}}$ is the standard matrix of $\widetilde{T}$ where $\widetilde{T} = i_{\mathscr{B}} \circ T \circ i_{\mathscr{B}}^{-1}$
How to find $[T]_{\mathscr{B}}$ in practice ?

$$\text{set } u = u_i \text{ in the result of proposition 1.3}$$

$$[u_i]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \longleftarrow \text{ ith position}$$

$$[T]_{\mathscr{B}}[u_i]_{\mathscr{B}} = \text{ ith column of } [T(u_i)]_{\mathscr{B}}$$

9 Matrix Representation of Linear Transformations

$T : V \to W$ Assume dim(V) = n and $\mathscr{B} = \{u_1, .., u_n\}$ a basis of V.
then we get the following result ; $\exists$ a unique matrix $(n \times n)$ denoted $(\exists!)$ of T , relative to $\mathscr{B}$. Such that $\forall u$

Remark The ith column of $[T]_{\mathscr{B}}$ is equal to $[T(u_i)]_{\mathscr{B}}$ where u_i is the ith vector of the (ordered) basis $\mathscr{B}$

Picture:

$$\begin{array}{ccc} & V \xrightarrow{T} V & \\ i_{\mathscr{B}}^{-1}\uparrow & & \downarrow i_{\mathscr{B}} \\ & \mathbb{R}^n \xrightarrow{\widetilde{T}} \mathbb{R}^n & \end{array}$$

Definition 9.1. Similar Matrices. Two $(n \times n)$ matrices A and B are said to be similar if $\exists$ a $(n \times n)$ invertible matrix P such that,

$$A = PBP^{-1}$$

Note that diagonalizable matrices are similar to diagonal matrices.

Proposition 9.1. Assume dim(V) = n , and $\mathscr{B}$ and $\mathscr{S}$ are two bases of V. If $T : V \to V$ is a linear transformation then, $[T]_{\mathscr{B}}$ and $[T]_{\mathscr{S}}$ are similar. More precisely,

$$[T]_{\mathcal{S}} = P_{\mathcal{B},\mathcal{S}}[T]_{\mathcal{B}}P^{-1}_{\mathcal{B},\mathcal{S}}$$

Remark $T : V \to V$, dim(V) = n , Question: is there a basis $\mathcal{S}$ of V such that $[T]_{\mathcal{S}}$ is diagonal ?
Starting with a basis $\mathcal{B}$ of V, if such a basis exists then

$$[T]_{\mathcal{B}} = P_{\mathcal{S},\mathcal{B}} \underbrace{[T]_{\mathcal{S}}}_{\text{diagonal}} P^{-1}_{\mathcal{S},\mathcal{B}}$$

(i.e) $[T]_{\mathcal{B}}$ is diagonalizable but not necessarily diagonal.
In summary, such a basis $\mathcal{S}$ exists iff $[T]_{\mathcal{B}}$ is diagonalizable for any other basis $\mathcal{B}$.
Note that the ith column of P is an eigenvector coresponding to λ_i
From this reasoning we conclude that if $P_{\mathcal{B},\mathcal{S}} = P$, then $[T]_{\mathcal{S}} = D$, moreover $P_{\mathcal{S},\mathcal{B}} = P^{-1}$

Remark The identity map Id_V is not necessarely the usual identity matrix. Indeed, let $I : \mathbb{R}^2 \to \mathbb{R}^2$

$$\text{by definition I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if we change the basis from $\vec{i}, \vec{j}$ to something else, then Id_V changes correspondingly.

10 Composition of Linear Transformations

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3$$

T_1 and T_2 are linear transformations.
Let $\mathcal{B}_i$ be a baiss of V_i where $i = 1, 2, 3$. Then what is the matrix representation of $[T_2 \circ T_1]_{\mathcal{B}_1,\mathcal{B}_3}$
Let $u \in V_1$

$$\begin{aligned}[T_2 \circ T_1(u)]_{\mathcal{B}_3} &= [T_2(T_1(u))]_{\mathcal{B}_3} \\ &= [T_2]_{\mathcal{B}_2,\mathcal{B}_3}[T_1(u)]_{\mathcal{B}_2} \\ &= [T_2]_{\mathcal{B}_2,\mathcal{B}_3}[T_1]_{\mathcal{B}_1,\mathcal{B}_3}[u]_{\mathcal{B}_1}\end{aligned}$$

Therefore,

$$[T_2 \circ T_1]_{\mathcal{B}_1,\mathcal{B}_3} = [T_2]_{\mathcal{B}_2,\mathcal{B}_3}[T_1]_{\mathcal{B}_1,\mathcal{B}_2}$$

Application
Let V and W be 2 vector spaces such that $T : V \to W$ is an isomoprhism.
Set $n = \dim(V) = \dim(W)$
Let $\mathcal{B}$ and $\mathcal{S}$ be bases of V and W respectively, then

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{T^{-1}} & V \\ \mathcal{B} & & \mathcal{S} & & \mathcal{B} \end{array}$$

Using the formula deribved before,

$$\begin{aligned} &[Id_V]|_{\mathcal{B},\mathcal{B}} = [T^{-1}]_{\mathcal{S},\mathcal{B}}[T]_{\mathcal{B},\mathcal{S}} \\ &I_n = [T^{-1}]_{\mathcal{S},\mathcal{B}}[T]_{\mathcal{B},\mathcal{S}} \end{aligned}$$

This result is telling us that if T is an isomprhism between $V \to W$ then T representation is invertible with inverse map.

Proposition 10.1. Let $T : V \to W$ be an isomorphism, $\mathcal{B}, \mathcal{S}$ are two basis of V and W respectively,
then $[T]_{\mathcal{B},\mathcal{S}}$ is invertible, moreover its inverse is :

$$([T]_{\mathcal{B},\mathcal{S}})^{-1} = [T^{-1}]_{\mathcal{S},\mathcal{B}}$$

"Therefore it's a matter of invertibility of matrix for isomorphism"

11 Finding $Ker(T)$ and $Im(T)$ using a matrix representation of T

Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ and
$\mathcal{B} = \{x^2 + 2x, x^2 + 2, 1 + x, 1 - x\}$ and $\mathcal{S} = \{x^2, x, 1\}$ basis of $\mathcal{P}_3$ and $\mathcal{P}_2$ respectively.
Note $[T]_{\mathcal{B},\mathcal{S}}$ is a 3×4 matrix not the opposite !

Picture:

$$\begin{array}{ccc} \mathcal{P}_3 & \xrightarrow{T} & \mathcal{P}_2 \\ i_{\mathcal{B}}\downarrow & & \downarrow i_{\mathcal{S}} \\ \mathbb{R}^4 & \xrightarrow{\widetilde{T}} & \mathbb{R}^3 \end{array}$$

From the above picture, the kernel of T is the column space of $\mathbb{R}^4$ and its image is the row space in $\mathbb{R}^3$, which are both isomorphic to $\mathscr{P}_3$ and $\mathscr{P}_2$ respectively through $i_{\mathscr{B}}$ and $i_{\mathscr{S}}$.

12 Inner Products

Recall Dot product in $\mathbb{R}^3$ is defined as

$$\vec{u_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \ \vec{u_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \qquad \vec{u_1} \cdot \vec{u_2} = a_1a_2 + b_1b_2 + c_1c_2$$

Propreties

1. $\vec{u_1} \cdot \vec{u_2} = \vec{u_2} \cdot \vec{u_1}$ (Symmetry)
2. $(\vec{u_1} + \vec{u_2}) \cdot \vec{u_3} = \vec{u_1} \cdot \vec{u_3} + \vec{u_2} \cdot \vec{u_3}$ (Distributivity)
3. $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$ (Dot product is positive definitive)

Definition 12.1. Let V be a vector space and consider a map from $V \times V$ into $\mathbb{R}$ which assigns to every pair $(u, v) u \in V, v \in V$ a number denoted $\langle u, v\rangle$ The above map is called "inner product" on V if the following proprieties hold.

1. $\langle u, v\rangle = \langle v, u\rangle$ (Symmetry)
2. $\langle \alpha_1 u_1 + \alpha_2 u_2, v\rangle = \alpha_1\langle u_1, v\rangle + \alpha_2\langle u_2, v\rangle$
   $\forall u_1, u_2 \in V, v \in V, \alpha_1, \alpha_2 \in \mathbb{R}$
3. $\forall u \in V, \qquad \langle u, v\rangle \geq 0$ and, $\langle u, v\rangle = 0 \iff u = 0_v$

Note In the above propositions, in number 2 α doesn't get distributed to each element since

$$\langle \alpha u, \alpha v\rangle = \alpha\langle u, \alpha v\rangle = \alpha^2\langle u, v\rangle \quad \to \mathit{bilinearity}!$$

Example Let $V = \mathbb{R}^2$. Define $\langle u, v\rangle = u^T Av$, where A is (2×2).
Under what conditions is A in the above map an inner product in $\mathbb{R}^2$?

$$\begin{aligned}
\text{Proprety 1. (Symmetry)}\langle u, v\rangle = \langle v, u\rangle \forall u, v \in \mathbb{R}^2 \\
\langle u, v\rangle \quad \text{is by definition } u^T Av \\
\langle v, u\rangle \quad \text{is by definition } v^T Au \\
\rightarrow \langle u, v\rangle \quad \text{is also} &= (u^T Av)^T \\
&= v^T A^T u
\end{aligned}$$

$$\text{So} \quad v^T A^T u = v^T Au \quad \iff A = A^T \quad \text{(Symmetry)}$$

We conclude that we must have $A = A^T$, i.e A myst be a symmetryc matrix.

$$\begin{aligned}
\text{Proprety 2. Let} \quad u_1, u_2, v \in \mathbb{R}^2, \alpha_1, \alpha_2 \in \mathbb{R} \\
\langle \alpha_1 u_1 + \alpha_2 u_2, v\rangle &= (\alpha_1 u_1 + \alpha_2 u_2)^T Av \\
&= \alpha_1 u_1^T Av + \alpha_2 u_2^T Av \\
&= \alpha_1 \langle u_1, v\rangle + \alpha_2 \langle u_2, v\rangle
\end{aligned}$$

In summary, if $A(2 \times 2)$ symmetric matrix, the map $\langle u, v\rangle = u^T Av\rangle$ satisfies propreties 1 and 2 of the definition of the inner product.

Definition 12.2. Let A be a $n \times n$ symmetric matrix. A is said to be positive definite if

$$\begin{aligned}
&1. \forall u \in \mathbb{R}^n, u^T Au \geq 0 \quad \text{and} \\
&2. u^T Au = 0 \quad \iff u = 0_v
\end{aligned}$$

Proposition 12.1. Let A be a positive definite matrix, the formula

$$\langle u, v\rangle = u^T Av \text{ defines an inner product in } \mathbb{R}^n$$

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$
then $\qquad \langle u, v\rangle = u^T Av \quad$ satisfy prop.1 and .2. Question : is A positive definite ?

$$\text{Take} \quad \langle u,u\rangle = u^T A u \quad \xrightarrow{so} u = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{and then , } \langle u,u\rangle = 5x^2 + 2xy + 5y^2$$

Method 1 :

$$\begin{aligned}
&\langle u,u\rangle = 4(x^2+y^2) + (x+y)^2 \geq 0 \\
&\langle u,u\rangle = 0 \implies x^2+y^2 = 0 \text{ and} \\
&\qquad\qquad\qquad\quad x+y=0 \\
&\text{so then} \quad x=y=0
\end{aligned}$$

Therefore A is positive definite and $\langle u,v\rangle = uA^Tv$ is an inner product in $\mathbb{R}^2$

Method 2 : Through diagonalization

$$\begin{aligned}
&p_A(\lambda) = (\lambda-4)(\lambda-6) \\
&A = PDP^{-1} \quad \text{where } D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \\
&P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

$$\langle u,u\rangle = u^T A u = \frac{1}{2} u^T \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{Q^T} \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{Q} u$$

$$\begin{aligned}
\text{Set} \quad u = \begin{bmatrix} x \\ y \end{bmatrix} \qquad & \text{then } \langle u,u\rangle = u^T A u \\
&= \frac{1}{2} u^T \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&\rightarrow= \frac{1}{2}(Qu)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} (Qu)
\end{aligned}$$

$$\rightarrow Qu = \begin{bmatrix} x-y \\ x+y \end{bmatrix} \quad \text{So that}$$

$$\langle u,u\rangle = \frac{1}{2}(4(x-y)^2 + 6(x+y)^2)$$

So in that new coordinate system it's 4 times the first coordinate and 6 times the second coordinate of P ,indeed the column vector of the matrix P defined in the above example are two new basis vectors.

Facts If A is a $n \times n$ symmetric matrix then,

1. A is diagonalizable

2. $A = PDP^{-1}$

then matrix P can be chosen such that

$$P^{-1} = P^T$$

so that $A = PDP^T \rightarrow$ orthogonal diagonalization

Remark : Given 2×2 symmetric matrix, when is A positive definite ?

When eigenvalues are larger than 0.

$$A = PDP^T, \qquad \text{set} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$P = \begin{bmatrix} \underbrace{u_1}_{2 \times 1} & \underbrace{u_2}_{2 \times 1} \end{bmatrix}$$

Proposition 12.2. In general, a symmegric $n \times n$ matrix A is positive definite if and only if all eigenvalues of A are positive

13 Examples of Inner Products

Example Let $V = \mathscr{P}_n$

$$\text{set } \langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

1. $\langle f, g\rangle = \langle g, f\rangle$ Symmetry proprety holds

2. $\langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle$

$$\begin{aligned}
&\int_0^1 (\alpha_1 f_1(t) + \alpha_2 f_2(t))g(t)dt \\
&= \alpha_1 \int_0^1 f_1(t)g(t)dt + \alpha_2 \int_0^1 f_2(t)g(t)dt \\
&= \alpha_1\langle f_1, g\rangle + \alpha_2\langle f_2, g\rangle \quad \rightarrow \text{ holds!}
\end{aligned}$$

3. $\langle f, f\rangle =$

$$\int_0^1 (f(t))^2 dt \geq 0 \qquad \text{by propreties of integrals}$$

Now when is this equal to zero ?

$$\int_0^1 (f(t))^2 dt = 0 \iff f(t) = 0 \quad \forall t \in [0,1]$$

Note about previous example The integral is 0 on $[0,1]$ which implies that all of this function on $\mathbb{R}^n$ is equal to zero as well since we're in $\mathscr{P}_n$ which has at most n roots , but on interval $[0,1]$ there are ∞ roots !

Example Let $V = \mathscr{M}_{2\times 2}$ where $\langle A, B\rangle = \operatorname{tr}\big(AB^T\big)$

1. $\langle A, B\rangle = \operatorname{tr}\big(AB^T\big)$

$$\begin{aligned}
\operatorname{tr}\big(AB^T\big)^T &= \operatorname{tr}\big(BA^T\big) \\
&= \langle B, A\rangle
\end{aligned}$$

2. $\langle \alpha_1 A_1 + \alpha_2 A_2, B\rangle$

$$\begin{aligned}
&= \operatorname{tr}\big((\alpha_1 A_1 + \alpha_2 A_2)B^T\big) \\
&= \operatorname{tr}\big(\alpha_1 A_1 B^T\big) + \operatorname{tr}\big(\alpha_2 A_2 B^T\big) \\
&= \alpha_1\langle A_1, B\rangle + \alpha_2\langle A_2, B\rangle
\end{aligned}$$

3. Set $A = (a_{ij}), \quad \langle A, A\rangle = \operatorname{tr}\big(AA^T\big) = \sum_{i,j} a_{ij} \geq 0$

so $\langle A, A\rangle = 0 \rightarrow a_{ij} = 0 \quad \forall i, j$

$$A = \underbrace{\underset{\sim}{O}}_{n \times n}$$

Definition 13.1. Norm of a vector
We will generalize the definition of norm of a vector in $\mathbb{R}^3$ which is $\sqrt{x_1^2 + x_2^2 + x_3^2}$.
Since this is true in $\mathbb{R}, \mathbb{R}^2$
Let V be equipped with an inner product. The norm of $u (u \in V)$ is denoted $\|u\|$ and defined as

$$\|u\| = \sqrt{\langle u, u \rangle}$$

Example

$$V = \mathscr{P}_2 \qquad \langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$$

$$p(t) = t \qquad \implies \|p\| = \frac{\sqrt{3}}{3}$$

Definition 13.2. Orthogonal vectors
Let V be a vector space equipped with an inner product $\langle , \rangle$
Two vectors $u, v \in V$ are said to be orthogonal if

$$\langle u, v \rangle = 0$$

Note Orthogonality is implicitly with respect to a dot product

Propreties

1. **Cauchy-Schwarz Inequality**
   $|\langle u, v \rangle| \leq \sqrt{\langle u, u \rangle}\sqrt{\langle v, v \rangle}$
   this equality holds iff u and v are //. i.e
   $u = \alpha v$ or
   $v = \beta u$
2. **Pythagorean Equality**
   if $u \perp v \iff \langle u, v \rangle = 0$ then
   $$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$
   else
   $$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle$$

Note For the previous proprieties, note that the discriminant in the quadratic formula is basically Cauchy-Schwarz's Inequality ,because if quadratic function is larger than 0 then there are no roots.

Definition 13.3. Let V be a vector space equipped with an inner product.

1. A set $\mathcal{S} = u_1, u_2, \dots, n$ is called an orthogonal set if $\langle u_i, u_j \rangle = 0$ whenever $i \neq j$

2. A proper orthogonal set is an orthogonal set which does not contain the zero vector 0_V

3. An orthonormal set is an orthogonal set $\mathcal{S} = u_1, u_2, \dots, n$ such that

$$\begin{aligned} \|u_i\| = 1 \qquad \forall i = 1, 2 \dots, n \\ \langle u_i, u_i \rangle = 1 \qquad \forall i = 1, 2 \dots, n \end{aligned}$$

Example$V = \mathcal{R}^3$, equipped with an inner product.

$$\mathcal{S}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{is not an orthogonal set , therefore not orthonormal as well}$$

Find x such that it's an orthogonal set

$$\mathcal{S}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ x \\ 0 \end{bmatrix} \qquad x = -2 \quad \text{proper orthogonal set, but not orthonormal}$$

Turn this into an orthonormal set

$$\mathcal{S}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{this is an orthonormal set}$$

Note An orthonormal set is a proper orthogonal set. Every orthonormal vector should have length one, but length of zero vector is zero.

Proposition 13.1. V is a vector space equipped with an inner product. Then every finite proper orthogonal set of V is linearly independent.

Note last proposition "You can increase dimension only when you have a bigger orthogonal set"

Corollary 13.0.1. Assume $\dim(V) = n$

1. Every proper orthogonal set with n vectors is a basis of V called an orthogonal basis of V.
2. Every orthonormal set of V with n vectors is a basis of V called orthonormal basis.

Note $\{x^2, x, 1\} \implies x \perp 1, x \perp x^2$ but x^2 is not perpendicular to 1.

Proposition 13.2. Let V be a vbector space equipped with an inner product.
Let E be a subspace of V such that $\dim(E) = n$.
Let $\mathcal{B} = u_1, u_2, \ldots, n$ be an orthogonal basis of E.

1. (Bersels Inequality)

$$\forall u \in V, \quad \text{the vector}$$
$$w = u - \sum_{i=1}^{n} \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i \quad \text{is orthogonal to every vector in } E$$
$$\text{Moreover,} \quad \|u\|^2 \geq \sum_{i=1}^{n} \frac{(\langle u, u_i \rangle)^2}{\langle u_i, u_i \rangle}$$

2. (Fourier's Indentity)

$$\text{Whenever } u \in E \text{ then,}$$
$$u = \sum_{i=1}^{n} \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i$$

3. (Perseval Equality)

$$\text{Whenever } u \in E \text{ then,}$$
$$\|u\|^2 = \sum_{i=1}^{n} \frac{(\langle u, u_i \rangle)^2}{\langle u_i, u_i \rangle}$$

Remark Given $u \in V$ $\exists!$ $v \in E$ such that $w = u - v \perp E$.

Definition 13.4. Let E be a finite dimensional subspace of a vector space V (which is equipped with an inner product). Let $u \in V$
The unique vector $v \in E$ such that $w = u - v \in E$ is called the orthogonal projection of u onto E, and is denoted $proj_{\mathbf{E}}\mathbf{u}$.
Moreoever, if $u_1, u_2, \ldots, n$ is an orthogonal basis of E then ,

$$proj_{\mathbf{E}}\mathbf{u} = \sum_{i=1}^{n} \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i$$

Note about polynominal projections If we're asked to project $\mathscr{P}_2$ onto $\mathscr{P}_3$ then since $\mathscr{P}_2 \in \mathscr{P}_3$ the projection is itself.

Proposition 13.3. Finding a $\perp$ basis. Gram=Schmidt Algorithm.
$\dim(E) = n$, E is a subspace of a vector space V *that is equipped with an inner product).
Let $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ an arbitrary basis of E (so not necessarely orthogonal)
Set

$$\begin{aligned} E_1 &= \text{Span}\{v_1\} \\ E_2 &= \text{Span}\{v_1, v_2\} \\ &\vdots \\ E_k &= \text{Span}\{v_1, v_2, \ldots, v_k\} \end{aligned}$$

Note $E_1 \subseteq E_2 \subseteq E_3 \cdots \subseteq E_n \to$ Nested subspaces.
Define

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - proj_{\mathbf{E_1}}\mathbf{v_2} \\ u_3 &= v_3 - proj_{\mathbf{E_2}}\mathbf{v_3} \\ &\vdots \\ u_k &= v_k - proj_{\mathbf{E_{k-1}}}\mathbf{v_k} \end{aligned}$$

And so we just jumped from an arbitrary basis to a $\perp$ basis, indeed $u_1 \perp u_2, u_3 \perp (u_1$ and $u_2)$ Finally,

$$E_k = \text{Span}\{u_1, u_2, \ldots, u_k\} \quad \text{where} \quad \{u1, \ldots, u_k\} \text{ is a basis of } E_k$$

Example

$V = \mathscr{P}_2$, $\mathscr{B} = \{p_1, p_2, p_3\}$ the canonical basis of $\mathscr{P}_2$. Define the usual inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\ dt$$

Find an orthogonal basis.
Let us use the Gram-Schmidt algorithm to find a $\perp$ basis.

$$\begin{aligned}
q_1 &= p_1 \\
q_2 &= p_2 - \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1 \to q_2(x) = x - 0 = x \\
q_3 &= p_3 - \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 - \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2
\end{aligned}$$

$$\text{replacing the values in the given integral yields :}$$
$$q_3(x) = x^2 - \frac{2/3}{2}(1) - 0 = x^2 - 1/3$$

Thus,

$$\{q_1, q_2, q_3\} \quad \text{is an orthogonal basis of } \mathscr{P}_2$$

14 Orthogonal Sets

Let V be a vector space equipped with an inner product. Let $\mathscr{S}$ be a subset of V.

Definition 14.1. The orthogonal set to $\mathscr{S}$ denoted

$$\mathscr{S}^{\perp} = \{v \in V | \langle u, v\rangle = 0\ \forall u\ \in\ \mathscr{S}\}$$

Example

What is orthogonal to 0_V and V ?

$$\begin{aligned}
\{0_V\}^{\perp} &= V \\
\{V\}^{\perp} &= \{0_V\}
\end{aligned}$$

Note

The bigger the set, the smaller the orthogonal compliment

Proposition 14.1. Propreties of orthogonal sets.

1. $\mathcal{S}^\perp$ is a subspace of V

2. Assume $\mathcal{S}$ is finite (can be infinite as well) ,then

$$\mathcal{S}^\perp = (\text{Span}\{\mathcal{S}\})^\perp$$

So it's convenient to check for spanning set.

Proposition 14.2. Let V be a vector space equipped with an inner product. Let E be a subspace of V such that $\dim(E) = n$, then

1. $E \oplus E^\perp = V$

2. $(E^\perp)^\perp = E$

For that reason , $E^\perp$ is called the orthogonal compliment of E.

Important Example
$V = \mathcal{P}_2, \quad \mathcal{B} = \{x^2, x, 1\} = \{p_1, p_2, p_3\}$

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\ dt \quad \text{Canonical isomorphism induced by } V$$

$$i_{\mathcal{B}} : V \to \mathbb{R}^3 \ \ \forall\ x, y\ \in\ \mathbb{R}^3, i_{\mathcal{B}}^{-1}(x), i_{\mathcal{B}}^{-1}(y)$$

$$\text{Ex :} \quad \underset{\sim}{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad i_{\mathcal{B}}^{-1}(x) = p = 2x^2 + x \quad \text{"Rewriting polynominal"}$$

go from $\mathbb{R}^3$ into the set of polynominals

Claim :

$$\langle x, y \rangle_1 - \langle i_{\mathcal{B}}^{-1}(x), i_{\mathcal{B}}^{-1}(y) \rangle \quad \text{is an inner product of } \mathbb{R}^3$$

We have to prove all propreties :

1. $\langle x,y\rangle = \langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(y)\rangle$ both are $\in V$ so we can use symmetry
   $\langle x,y\rangle = \langle i_{\mathscr{B}}^{-1}(y), i_{\mathscr{B}}^{-1}(x)\rangle = \langle y,x\rangle.$
2. Let $x, y_1, y_2 \in \mathbb{R}^3$, $\alpha_1, \alpha_2 \in \mathbb{R}$
   $$\begin{aligned}\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle &= \langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(\alpha_1 y u_1 + \alpha_2 y_2)\rangle \\ &= \langle i_{\mathscr{B}}^{-1}(x), \alpha_1 i_{\mathscr{B}}^{-1}(y_1) + \alpha_2 i_{\mathscr{B}}^{-1}(y_2)\rangle \\ &= \alpha_1\langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(y_1)\rangle + \alpha_2\langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(y_2)\rangle \\ &= \alpha_1\langle x, y_1\rangle + \alpha_2\langle x, y_2\rangle.\end{aligned}$$
3. Let $x \in \mathbb{R}^3$
   $$\langle x,x\rangle_1 = \langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(x)\rangle \geq 0$$
   And , $\langle x,x\rangle_1 = 0 \iff i_{\mathscr{B}}^{-1}(x) = 0 \implies x = 0$

So it is an inner product, the inner product is defined as :

$\langle x,y\rangle = x^T A y$ For some positive definite matrix A, but what is A ?

$$\langle i_{\mathscr{B}}^{-1}(x), i_{\mathscr{B}}^{-1}(y)\rangle = x^T A y$$

If $p, q \in \mathscr{P}_2$, set $x = [p]_{\mathscr{B}}, y = [q]_{\mathscr{B}}$ the coordinate vectors

$$\implies \langle p,q\rangle = [p]_{\mathscr{B}}^T A [q]_{\mathscr{B}}$$

The inner product on $\mathbb{R}^3$ can be represented through coordinate system. Therefore, we can find the transition matrix this way !

Our basis is $\mathscr{B} = \{p_1, p_2, p_3\}$

$$\implies [p]_{\mathscr{B}}^T A [q]_{\mathscr{B}} = \int_{-1}^{1} p_i(t)p_j(t)$$

So the matrix A , after computation is :

$$A = \begin{bmatrix} 2/5 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2 \end{bmatrix}$$ For instance , a_{11} was found by doing

$$a_{11} = \int_{-1}^{1} x^2 x^2 \, dx = \frac{2}{5}$$

Note
If we're given $T : \mathbb{R}^n \to \mathbb{R}$ and suppose $T \neq 0$, can we say something about Ker(T) ?

$$\dim(\mathrm{Ker}T) = n - 1 \qquad \text{because } \dim(\mathrm{Im}(T)) = 1 \text{ , by Rank-N-theorem}$$

Note

1. Orthogonality implies linear independence ,but linear independence doesn't imply orthogonality.
2. If A is positive definite then A is a symmetric matrix. If A is a symmetric matrix , then A is not necessarely positive definite.