PHYS 350 Assignment 4

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Question 1

a)

i)

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$$

We use the Lemma $[\hat{A}, [\hat{B}, \hat{C}]] = ABC - ACB - BCA + CBA$, we get

$$\begin{aligned} &= ABC - ACB - BCA + CBA + BCA - BAC - CAB \\ &+ ACB + CAB - CBA - ABC + BAC \\ &= 0 \qquad \checkmark. \end{aligned}$$

ii)

We use the dagger properties,

$$\begin{aligned} (\hat{A}, \hat{B})^\dagger &= (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger \\ &= (B^\dagger A^\dagger) - (B^\dagger A^\dagger) = (BA)^\dagger - (BA)^\dagger = (BA - AB)^\dagger = [\hat{B}^\dagger, \hat{A}^\dagger] \end{aligned}$$

b)

First we show $[\hat{A}, \hat{B}^n] = n\hat{B}^{n-1}[\hat{A}, \hat{B}]$.

$$\begin{aligned} [\hat{A}, \hat{B}^n] &= \hat{A}\hat{B}^n - \hat{B}^n\hat{A} \\ &= [\hat{A}, \hat{B}]\hat{B}^{n-1} + \hat{B}[\hat{A}, \hat{B}^{n-1}] \\ &= [\hat{A}, \hat{B}]\hat{B}^{n-1} + \hat{B}[\hat{A}, \hat{B}](n-1)\hat{B}^{n-2} \\ &= [\hat{A}, \hat{B}](\hat{B}^{n-1} + (n-1)\hat{B}^{n-1}) \\ &= [\hat{A}, \hat{B}]n\hat{B}^{n-1} \end{aligned}$$

$[\hat{A},\hat{B}]n = n[\hat{A},\hat{B}]$ so then finally,

$$\implies [\hat{A},\hat{B}^n] = n[\hat{A},\hat{B}]\hat{B}^{n-1}. \tag{1}$$

Then we show the claim.

$$\begin{aligned}
[\hat{A},F(\hat{B})] &= \hat{A}F(\hat{B}) - F(\hat{B})\hat{A} \\
&= \hat{A}\left(I - \left(\frac{i}{\hbar}\hat{B}\right) + \frac{1}{4}\left(\frac{i}{\hbar}\hat{B}\right)^2 - \frac{1}{24}\left(\frac{i}{\hbar}\hat{B}\right)^3 + \dots\right) \\
&\quad - \left(I - \left(\frac{i}{\hbar}\hat{B}\right) + \frac{1}{4}\left(\frac{i}{\hbar}\hat{B}\right)^2 - \frac{1}{24}\left(\frac{i}{\hbar}\hat{B}\right)^3 + \dots\right)\hat{A} \\
&= \left(\hat{A} - \left(\frac{i}{\hbar}\right)\hat{A}\hat{B} + \frac{1}{4}\left(\frac{i}{\hbar}\right)^2\hat{A}\hat{B}^2 - \frac{1}{24}\left(\frac{i}{\hbar}\right)^3\hat{A}\hat{B}^3 + \dots\right) \\
&\quad - \left(\hat{A} - \left(\frac{i}{\hbar}\right)\hat{B}\hat{A} + \frac{1}{4}\left(\frac{i}{\hbar}\right)^2\hat{B}^2\hat{A} - \frac{1}{24}\left(\frac{i}{\hbar}\right)^3\hat{B}^3\hat{A} + \dots\right) \\
&= \frac{-i}{\hbar}[\hat{A},\hat{B}] + \frac{1}{4}\left(\frac{i}{\hbar}\right)^2[\hat{A},\hat{B}^2] - \frac{1}{24}\left(\frac{i}{\hbar}\right)^3[\hat{A},\hat{B}^3] + \dots
\end{aligned}$$

We use the relationship $[\hat{A},\hat{B}^n] = cn\hat{B}^{n-1}$ proved in $(1) which means that the commutator between [,$
$^n]$ is the derivative with respect to $\hat{B}$ acting on $\hat{B}^n$.

$$\begin{aligned}
&= c\left(\frac{-i}{\hbar} + \frac{1}{4}\left(\frac{i}{\hbar}\right)^2 2\hat{B} - \frac{1}{24}\left(\frac{i}{\hbar}\right)^3 3\hat{B}^2 + \dots\right) \\
&= cF'(\hat{B})
\end{aligned}$$

For $F'(\hat{B})$ defined as $F'(\hat{B}) = \sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}\right)\left(\frac{-i}{\hbar}\hat{B}\right)^{n-1}$, i.e., the derivative applied to an exponential with a matrix argument.

c)

First of all we note that since $[\hat{A},\hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$, then We perform the exact proceedure outlined in $1b$.

$$\begin{aligned}
[\hat{A},F(\hat{B})] &= \hat{A}F(\hat{B}) - F(\hat{B})\hat{A} \\
&= \hat{A}\left(I - \left(\frac{i}{\hbar}\hat{B}\right) + \frac{1}{4}\left(\frac{i}{\hbar}\hat{B}\right)^2 - \frac{1}{24}\left(\frac{i}{\hbar}\hat{B}\right)^3 + \dots\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(I-\left(\frac{i}{\hbar}\hat{B}\right)+\frac{1}{4}\left(\frac{i}{\hbar}\hat{B}\right)^2-\frac{1}{24}\left(\frac{i}{\hbar}\hat{B}\right)^3+\dots\right)\hat{A}\\
&=\left(\hat{A}-\left(\frac{i}{\hbar}\right)\hat{A}\hat{B}+\frac{1}{4}\left(\frac{i}{\hbar}\right)^2\hat{A}\hat{B}^2-\frac{1}{24}\left(\frac{i}{\hbar}\right)^3\hat{A}\hat{B}^3+\dots\right)\\
&-\left(\hat{A}-\left(\frac{i}{\hbar}\right)\hat{B}\hat{A}+\frac{1}{4}\left(\frac{i}{\hbar}\right)^2\hat{B}^2\hat{A}-\frac{1}{24}\left(\frac{i}{\hbar}\right)^3\hat{B}^3\hat{A}+\dots\right)\\
&=\frac{-i}{\hbar}[\hat{A},\hat{B}]+\frac{1}{4}\left(\frac{i}{\hbar}\right)^2[\hat{A},\hat{B}^2]-\frac{1}{24}\left(\frac{i}{\hbar}\right)^3[\hat{A},\hat{B}^3]+\dots
\end{aligned}
$$

We use the relationship $[\hat{A},\hat{B}^n]=n\hat{C}\hat{B}^{n-1}$. Indeed, since $\hat{A}$ and $\hat{B}$ commutes with $[\hat{A},\hat{B}]$, then it follows that $[\hat{A},\hat{B}]=AB-BA \implies AB=BA$. Therefore, $[\hat{A},\hat{B}^n]=[\hat{A},\hat{B}]\hat{B}^{n-1}-\hat{B}^{n-1}[\hat{A},\hat{B}] \implies [\hat{A},\hat{B}]\hat{B}^{n-1}=\hat{B}^{n-1}[\hat{A},\hat{B}]$. We conclude that $[\hat{A},\hat{B}^n]=n\hat{C}\hat{B}^{n-1}$ holds for non-integer commutator.

$$
\begin{aligned}
&=\hat{C}\left(\frac{-i}{\hbar}+\frac{1}{4}\left(\frac{i}{\hbar}\right)^2 2\hat{B}-\frac{1}{24}\left(\frac{i}{\hbar}\right)^3 3\hat{B}^2+\dots\right)\\
&=\hat{C}F'(\hat{B})
\end{aligned}
$$

For $F'(\hat{B})$ defined as $F'(\hat{B})=\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}\right)\left(\frac{-i}{\hbar}\hat{B}\right)^{n-1}$, i.e., the derivative applied to an exponential with a matrix argument.

d)

Let $\hat{\beta}(x)\equiv e^{x\hat{A}}e^{x\hat{B}}$. Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\hat{\beta}(x)\right)&=\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x\hat{A}}e^{x\hat{B}}\right)\\
&=\hat{A}e^{x\hat{A}}e^{x\hat{B}}+e^{x\hat{A}}\hat{B}e^{x\hat{B}}\\
&=\hat{A}e^{x\hat{A}}e^{x\hat{B}}+[e^{x\hat{A}},\hat{B}]e^{x\hat{B}}+Be^{x\hat{A}}e^{x\hat{B}}\\
&=(\hat{A}+\hat{B})e^{x\hat{A}}e^{x\hat{B}}+[e^{x\hat{A}},\hat{B}]e^{x\hat{B}}
\end{aligned}
$$

Let $F(x\hat{A})\equiv e^{x\hat{A}}$, then as found before $[F'(x\hat{A}),\hat{B}]=xF(x\hat{A})[\hat{A},\hat{B}]$, it follows that

$$
\begin{aligned}
&=(\hat{A}+\hat{B})e^{x\hat{A}}e^{x\hat{B}}+\left(xe^{x\hat{A}}[\hat{A},\hat{B}]\right)e^{x\hat{B}}\\
&=\left(\hat{A}+\hat{B}+x[\hat{A},\hat{B}]\right)\hat{\beta}(x)
\end{aligned}
$$

$$\implies \int \frac{\hat{\beta}'(x)}{\hat{\beta}(x)}\, dx = \int \hat{A} + \hat{B} + x[\hat{A}, \hat{B}]\, dx$$
$$Ce^{\ln \hat{\beta}(x)} = e^{\hat{A}x + \hat{B}x + \frac{x^2}{2}[\hat{A},\hat{B}]}$$

We set $x \equiv 1$, then C vanishes since it's an integration constant with respect to x, we're left off with the requested claim

$$\hat{\beta}(1) = e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \qquad \checkmark.$$

Question 2

• For the first part , let $|\varphi\rangle = |+z\rangle$. Then

$$|+z\rangle \xrightarrow{x \text{ basis}} \begin{pmatrix} \langle +x|+z\rangle \\ \langle -x|+z\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

therefore

$$\langle \varphi | S_x | \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}.$$

We conclude that $\Delta S_x = 0$. Similarly for $\langle S_y \rangle$,

$$|+z\rangle \xrightarrow{y \text{ basis}} \begin{pmatrix} \langle +y|+z\rangle \\ \langle -y|+z\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

therefore

$$\langle \varphi | S_x | \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

We conclude that $\Delta S_y = \hbar/2$. We expect $\langle S_z \rangle = 0$.

$$\langle \varphi | S_z | \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \qquad \checkmark.$$

• Similarly for eigenstates of $\hat{S}_x$. Let $|\varphi\rangle = |+x\rangle$.Then

$$\langle \varphi | S_x | \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2},$$

we conclude that $\Delta S_x = 0$. Then similarly,

$$|+x\rangle \xrightarrow{y \text{ basis}}= \begin{pmatrix} \langle +y|+x\rangle \\ \langle +y|-x\rangle \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}.$$

Thus,

$$\langle\varphi|S_y|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1+i & 1-i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}10 & -1\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1-i\\ 1+i\end{pmatrix} = \frac{\hbar}{2},$$

we conclude that $\Delta S_y = 0$. We finally expect $\langle S_z\rangle = 0$

$$\langle\varphi|S_z|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = 0 \qquad \checkmark.$$

Question 3

By definition the eigenbasis of S_z is $(1\ 0\ 0)^T, (0\ 1\ 0)^T$ and $(0\ 0\ 1)^T$ since z is the default basis direction. Then we convert the transformation matrix

$$\hat{S}_x = \frac{\hat{S}_+ - \hat{S}_-}{2} \overset{Notes}{=} \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

By definition, the eigenvalues must be (ignoring the scaling constants)

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad \lambda_3 = -1.$$

We look for the eigenvectors

$$\lambda_1 = 1 \implies \hbar\begin{pmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \implies \begin{cases} a - c = 0\\ b - \sqrt{2}c = 0\end{cases} \implies \boldsymbol{v_1} = \begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

$$\lambda_2 = 0 \implies \hbar\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \implies \begin{cases} a + c = 0\\ b = 0\end{cases} \implies \boldsymbol{v_2} = \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

$$\lambda_3 = -1 \implies \hbar\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \implies \begin{cases} c - a = 0\\ b + \sqrt{2}c = 0\end{cases} \implies \boldsymbol{v_1} = \begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

Finally, we may express the eigenstates normalized in Ket notation.

$$\begin{aligned}
|1,1\rangle_x &= \frac{1}{2}|1,1\rangle_z + \frac{1}{\sqrt{2}}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_z\\
|1,0\rangle_x &= -\frac{1}{\sqrt{2}}|1,1\rangle_z + \frac{1}{\sqrt{2}}|1,-1\rangle_z\\
|1,-1\rangle_x &= \frac{1}{2}|1,1\rangle_z - \frac{1}{\sqrt{2}}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_x
\end{aligned}$$

Question 4

Following the results from Question 3,

$$\left|{}_x\langle 1,0|1,1\rangle_z\right|^2 = \left|{}_z\langle 1,1|1,0\rangle_x{}^*\right|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Question 5

First and foremost,

$$\hat{J}_x \overset{Notes}{=} \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

For a spin $3/2$ particle, by definition the eigenvalues are (ignoring the scaling constants)

$$\lambda_1 = 3, \qquad \lambda_2 = 1, \qquad \lambda_3 = -1, \qquad \lambda_4 = -3.$$

We look for the eigenvectors

$$\lambda_1 = 3 \implies \hbar\begin{pmatrix} -3 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -3 & 2 & 0 \\ 0 & 2 & -3 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \left\{ \begin{array}{l} d = \frac{c}{\sqrt{3}} \\ b = c \\ a = \frac{b}{\sqrt{3}} \end{array} \right. \implies \boldsymbol{v_1} = \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \implies \hbar\begin{pmatrix} -1 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -1 & 2 & 0 \\ 0 & 2 & -1 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \left\{ \begin{array}{l} d = \sqrt{3}c \\ b = -c \\ a = \sqrt{3}b \end{array} \right. \implies \boldsymbol{v_2} = \begin{pmatrix} -\sqrt{3} \\ -1 \\ 1 \\ \sqrt{3} \end{pmatrix}$$

$$\lambda_3 = -1 \implies \hbar\begin{pmatrix} 1 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 1 & 2 & 0 \\ 0 & 2 & 1 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \left\{ \begin{array}{l} d = -\sqrt{3}c \\ b = c \\ a = -\sqrt{3}b \end{array} \right. \implies \boldsymbol{v_3} = \begin{pmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{pmatrix}$$

$$\lambda_4 = -3 \implies \hbar\begin{pmatrix} 3 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 3 & 2 & 0 \\ 0 & 2 & 3 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \left\{ \begin{array}{l} d = -\frac{c}{\sqrt{3}} \\ b = -c \\ a = -\frac{b}{\sqrt{3}} \end{array} \right. \implies \boldsymbol{v_1} = \begin{pmatrix} -1 \\ \sqrt{3} \\ -\sqrt{3} \\ 1 \end{pmatrix}$$

We write the Ket notation for the state $S_x = \hbar/2$, with the proper normalization factor

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle_x = \frac{1}{\sqrt{8}}\left(-\sqrt{3}\left|\frac{3}{2},\frac{3}{2}\right\rangle_z - \left|\frac{3}{2},\frac{1}{2}\right\rangle_z + \left|\frac{3}{2},-\frac{1}{2}\right\rangle_z + \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle_z\right).$$

It then follows that

$$\begin{aligned}
P_{z=\frac{3\hbar}{2}} &= \left|{}_z\left\langle\frac{3}{2},\frac{3}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle_x\right|^2 = \left|-\sqrt{\frac{3}{8}}\right|^2 = \frac{3}{8}\\
P_{z=\frac{\hbar}{2}} &= \left|{}_z\left\langle\frac{3}{2},\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle_z\right|^2 = \left|-\frac{1}{\sqrt{8}}\right|^2 = \frac{1}{8}\\
P_{z=-\frac{\hbar}{2}} &= \left|{}_z\left\langle\frac{3}{2},-\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle_x\right|^2 = \left|\frac{1}{\sqrt{8}}\right|^2 = \frac{1}{8}\\
P_{z=-\frac{3\hbar}{2}} &= \left|{}_z\left\langle\frac{3}{2},-\frac{3}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle_x\right|^2 = \left|\sqrt{\frac{3}{8}}\right|^2 = \frac{3}{8}.
\end{aligned}$$

$$\frac{3}{2} \geq 1\frac{1}{}$$