PHYS 350 Assignment 3

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Question 1

a)

The nucleus of Uranium-235 has 92 protons. The nuclear size is also the size at which two nucleons separate. Let us consider two spheres of 46 protons each. Then,

$$\frac{(46e)^2}{4\pi\epsilon_0(15\times 10^{-15})} = \frac{(46 \times 1.602\times 10^{-19})^2\times 8.98\times 10^9}{15\times 10^{-15}} = 3.25\times 10^{-11}\ \mathrm{J}.$$

b)

The total number of nucleons inside the Uranium is 235 and so

$$235m_p = 3.92\times 10^{-25}\ \mathrm{kg}.$$

Multiplying by c^2 we have the rest energy

$$235m_pc^2 = 3.54\times 10^{-8}\ \mathrm{J}.$$

Finally, the ratio (same for energy as for mass) is

$$(3.25\times 10^{-11})/(3.54\times 10^{-8}) \approxeq 0.001.$$

c)

The product of the efficiency ratio and c^2 is approximatively the energy stored in 1 kg, therefore

$$0.001c^2 = 9\times 10^{13}\ \mathrm{J}$$

Dividing by 3.6×10^6 to convert J $\to$ kW h, we get

$$1\ \mathrm{kg} = 25000000\ \mathrm{kW\,h}.$$

If it costs 0.1 $ to burn 1 kW h then

$$25000000\ \mathrm{kW\,h} = 2500000\$.$$

Question 2

No, if the charge is very close to the wall which is negatively charged on the outside, a charge of opposite sign will be induced on the wall such that a net force will be applied to the charge, such that it will be attracted to that wall.
The inner wall will slowly absorb the charge which has a net force towards it. Therefore, the charge configuration inside the conductor will slightly change and the final energy will be greater than the initial.

Question 3

We can treat the line as a cylinder and apply Gauss's Law,

$$E(2\pi rL) = \frac{\lambda L}{\epsilon_0} \implies E = \frac{\lambda}{2\pi\epsilon_0 r},$$

where r is the distance from the wire.
Let us consider the yz plane and set the reference point at $(0,0,0)$, the positions of $\lambda_\pm$ are respectively at $\pm a$ of the origin. Then we consider an arbitrary point at distance r and position (x,y,z) and let us consider this point in the $(+y,+z)$ quadrant. Since we're treating two positions, let $r = r_-$ for λ_- and $r = r_+$ for λ_+.

$$V = -\int_a^{r_+} \frac{\lambda}{2\pi\epsilon_0 r}\, dr = \frac{-\lambda}{2\pi\epsilon_0}\int_a^{r_+} \frac{1}{r}\, dr = \frac{-\lambda}{2\pi\epsilon_0} \ln\left|\frac{r_+}{a}\right|.$$

By symmetry, the potential of λ_- should also be

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left|\frac{r_-}{a}\right|$$

By the superposition principle, the potentials add up

$$V_T = \frac{\lambda}{2\pi\epsilon_0} \ln\left|\frac{r_-}{a}\right| + \frac{-\lambda}{2\pi\epsilon_0} \ln\left|\frac{r_+}{a}\right| = \frac{\lambda}{2\pi\epsilon_0} \ln\left|\frac{r_-}{r_+}\right|.$$

We now express r_+ and r_- as functions of y,z , these correspond to the norms ;

$$r_+ = \sqrt{(y-a)^2 + z^2} \qquad , r_- = \sqrt{(y+a)^2+z^2},$$

we get that

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left|\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right| = \frac{\lambda}{4\pi\epsilon_0} \ln\left|\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right|.$$

b)

$$V = V_0 = \frac{\lambda}{4\pi\epsilon_0} \ln\left|\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right| \implies \frac{4\pi\epsilon_0 V_0}{\lambda} = \ln\left|\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right|$$
$$\implies e^{4\pi\epsilon_0 V_0/\lambda} = \frac{(y+a)^2+z^2}{(y-a)^2+z^2}.$$

To simplify the algebra, let $k \equiv e^{4\pi\epsilon_0 V_0/\lambda}$. Then we carry on by looking for a circular relationship ;

$$(y+a)^2+z^2 = k[(y-a)^2+z^2]$$
$$y^2+2ya+a^2+z^2 = ky^2-2kya+ka^2+kz^2$$
$$\implies y^2(k-1)+a^2(k-1)+z^2(k-1)-2ay(k+1) = 0$$

We apply a division on both sides by $(k-1)$,

$$\therefore y^2+a^2+z^2-2ay\left(\frac{k+1}{k-1}\right) = 0 \tag{1}$$

The equation for a circle with radius r in a $2D$ plane is

$$(x-h)^2+(y-h)^2 = r^2$$

In this case the circles are centered around $y=\lambda$ and $z=0$ so then

$$(y-\lambda)^2+z^2 = r^2 \implies y^2+z^2+(\lambda^2-r^2)-2y\lambda = 0 \tag{2}$$

By comparing (1) and (2) we see that

$$\lambda = a\left(\frac{k+1}{k-1}\right) \qquad \text{and} \qquad a^2 = (\lambda^2-r^2),$$

so then

$$\begin{aligned} r = \sqrt{\lambda^2-a^2} &= \sqrt{a^2\left(\frac{k+1}{k-1}\right)^2 - a^2} \\ &= a^2\left(\frac{(k+1)^2}{(k-1)^2}-1\right) \\ &= a^2\left(\frac{k^2+2k+1-(k^2-2k+1)}{k^2-2k+1}\right) \\ &= \sqrt{a^2\frac{4k}{(k-1)^2}} = \frac{2a\sqrt{k}}{(k-1)}. \end{aligned}$$

Replacing the initial parameter ,

$$\lambda = a\frac{e^{4\pi\epsilon_0 V_0/\lambda}+1}{e^{4\pi\epsilon_0 V_0/\lambda}-1} \quad \text{and} \quad R = \frac{2a\sqrt{e^{4\pi\epsilon_0 V_0/\lambda}}}{e^{4\pi\epsilon_0 V_0/\lambda}-1}.$$

These correspond to circles at origin λ on the y axis with radius increasing towards the right and decreasing towards the left (for the λ_+) and the opposite for λ_-.

Question 4

a)

Any charge from a conductor reside on its surface. So the charges will be on the surfaces of the wires.

b)

The electric field will be maximal right over the surface of the wire, where the charges are located , since $E(r) \propto \frac{\hat{r}}{r}$ and $E_{\text{inside}} = 0$. Since the electric field of one wire doesn't affect the electric field of another we can apply Gauss's law on one wire and we get that

$$E = E_{\max} = \frac{\lambda}{2\pi\epsilon_0 R}.$$

c)

$$V(b) - V(a) = \int_D^{D-R} \frac{\lambda}{2\pi\epsilon_0 R}\, dR = \frac{\lambda}{2\pi\epsilon_0} \ln\left|\frac{D-R}{R}\right|.$$

By the principle of superposition we multiply by 2 the previous expression to account for the potential of the second wire

$$\therefore V = \frac{\lambda}{\pi\epsilon_0} \ln\left|\frac{D-R}{R}\right|.$$

d)

By substitution,

$$E_{\max}(2\pi\epsilon_0 R) = \lambda \implies V = 2RE_{\max} \ln\left|\frac{D-R}{R}\right|.$$

e)

We let $E_{\max} = 3 \times 10^6\ \mathrm{V\,m^{-1}}$, then

$$\frac{1}{2R\ln\left|\frac{D-R}{R}\right|} = 3 \times 10^6 \implies \frac{1}{6 \times 10^6} = R\ln\left|\frac{10-R}{R}\right|$$

Solving this numerically yields

$$R = 7.95 \times 10^{-9}\ \mathrm{m} \implies d = 1.59 \times 10^{-8}\ \mathrm{m}.$$

f)

$$V = 2RE_{\max}\ln\left|\frac{D-R}{R}\right| \implies 765 = 2\left(\frac{2.59}{2}\right)3\ln\left|\frac{D-\frac{2.59}{2}}{\frac{2.59}{2}}\right|$$
$$\implies \exp\left\{\frac{765}{3(2.59)}\right\}\left(\frac{2.59}{2}\right)+\left(\frac{2.59}{2}\right) = 7.43\times10^{42}\text{ mm}.$$

Question 5

a)

We use Poisson's equation

$$\nabla^2 V = \frac{-\rho}{\epsilon_0} \implies \Delta V = V_{xx} = \frac{-\rho}{\epsilon_0}.$$

b)

Since $W = qV$, by conservation of work-energy ,

$$W = qV = \frac{1}{2}mv_f^2 - \cancelto{0}{\frac{1}{2}mv_i^2} \implies v = \sqrt{\frac{2qV}{m}}.$$

c)

By definition the current is $I = dQ/dt$. For an infinitesimal charge along the plate,

$$dq = A\rho dx \xrightarrow{\times dt^{-1}} I = A\rho\frac{dx}{dt} = A\rho v.$$

d)

$$I = A\rho v, \qquad v = \sqrt{\frac{2qV}{m}}, \qquad V_{xx} = -\frac{\rho}{\epsilon_0}.$$

By substitution,

$$\frac{I}{Av} = \rho \implies V_{xx} = \frac{-I}{\epsilon_0 Av} = \frac{I}{\epsilon_0 A\sqrt{\frac{2qV}{m}}} \implies V_{xx} = \frac{I}{\epsilon_0 A\sqrt{\frac{2q}{m}}}V^{-1/2},$$

is the differential equation we're looking for.

e)

To solve this DE, we first let $k \equiv -I/\epsilon_0 A\sqrt{2q/m}$, we get

$$V_{xx} = kV^{-1/2},$$

this is a second order non linear differential equation. We will first reduce the order and then solve the separable equation. Let $\alpha = dV/dt$, we multiply both sides ;

$$\begin{aligned}
\alpha V_{xx} &= \alpha k V^{-1/2} \\
\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}x} &= \frac{\mathrm{d}V}{\mathrm{d}x} k V^{-1/2} \\
\int \alpha \ d\alpha &= k \int V^{-1/2} \ dV \\
\alpha^2 &= 4kV^{1/2} + C
\end{aligned}$$

At the cathode the field is 0 therefore $C \equiv 0$.

$$\begin{aligned}
\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 &= 4kV^{1/2} \\
\int \frac{dV}{V^{1/2}} &= 2\sqrt{k} \int dx \\
\frac{4}{3} V^{3/4} &= 2\sqrt{k}x + C
\end{aligned}$$

The potential at 0 is 0 therefore $C \equiv 0$, we get that

$$V^{3/4} = \frac{3}{2}\sqrt{k}x \implies V(x) = \underbrace{\left(\frac{3}{2}k^{1/2}\right)^{4/3}}_{:=V_0} x^{4/3}$$

It follows that

$$V(x) = V_0 x^{4/3}$$

If d is the farthest distance x can reach , this translates to

$$V(x) = V_0 \left(\frac{x}{d}\right)^{4/3}. \tag{3}$$

We now look for an expression for ρ and v as functions of x.

$$\begin{aligned}
\rho = -\epsilon_0(V_{xx}) = -\epsilon 0 V_0 \left(\frac{4}{3d}\left(\frac{x}{d}\right)^{1/3}\right)_x = -\epsilon_0 V_0 \left(\frac{4}{9d^2}\left(\frac{x}{d}\right)^{-2/3}\right) = \frac{-\epsilon_0 V_0 4x^{-2/3}}{9d^{4/3}}. \\
v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2q}{m}}\sqrt{V_0}\left(\frac{x}{d}\right)^{2/3} = \sqrt{\frac{2qV_0}{m}}\left(\frac{x}{d}\right)^{2/3}.
\end{aligned}$$

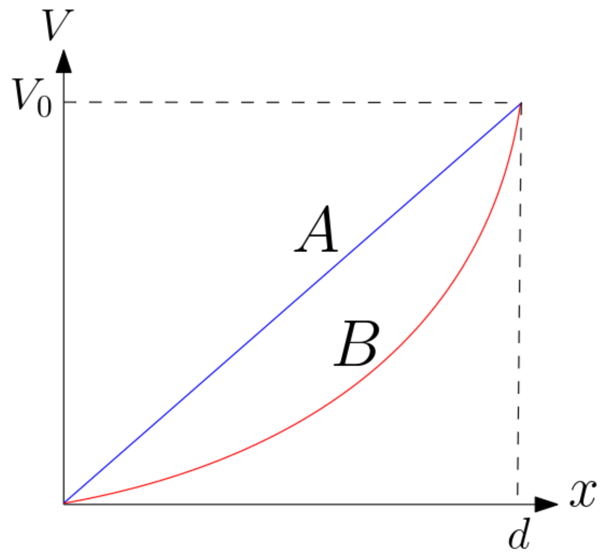

Figure 1: Approximative behaviour of (3) with (A) and without (B) space-charge.

f)

At $x = d$, $V(x) = V(d) = V_0$ therefore,

$$V(x) = \left(\frac{9}{4}k\right)^{2/3} x^{4/3} \implies V_0 = \left(\frac{9}{4}k\right)^{2/3} d^{4/3}$$
$$= \left(-\frac{9}{4}\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2q}}\right)^{2/3} d^{4/3}$$
$$V_0^3 = \frac{81}{16}\frac{I^2 m}{\epsilon_0^2 A^2 2q} d^4$$
$$\therefore I = \sqrt{\frac{V_0^3 16\epsilon_0^2 A^2 2q}{81 m d^4}}$$

Letting

$$\frac{16\epsilon_0^2 A^2 2q}{81 m d^4} \equiv K \qquad \implies I = K V_0^{3/2}.$$

Bonus

a)

Using Gauss's law for a spherical conductor,

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \qquad \text{and} \quad C = 4\pi\epsilon_0 r.$$

Since $Q = CV$, at a radius R (since $\vec{E} = \vec{0}$ inside the conductor),

$$E = \frac{CV}{4\pi\epsilon_0 r^2} = \frac{V}{r} = \frac{V}{R}.$$

We have that $R = 0.075$ m and $E_{\text{max}} = 3 \times 10^6 \text{ V m}^{-1}$ thus,

$$3 \times 10^6 = \frac{V}{0.075} \implies V = 225000 \text{ V}.$$

Moreover, the charge is found with

$$Q = CV \implies Q = (4\pi\epsilon_0)(0.075)(225000) = 1.87 \times 10^{-6} \text{ C}.$$

b)

Let us consider a disk of radius R and a point P on the x axis. Let the charge density be uniform across the disk ,then it follows that $dq = \sigma dA = \sigma 2\pi r\ dr$, such that

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{r^2+x^2}} = \frac{2\pi\sigma r}{4\pi\epsilon_0\sqrt{r^2+x^2}}\ dr$$

We integrate this expression with respect to the symmetry axis,

$$\begin{aligned} V = \int_0^R \frac{2\pi\sigma r}{4\pi\epsilon_0\sqrt{r^2+x^2}}\ dr &= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{2r}{\sqrt{r^2+x^2}}\ dr \\ &\overset{u=r^2+x^2}{=} \frac{\sigma}{4\epsilon_0}\int_0^R \frac{2}{2}\frac{1}{\sqrt{u}}\ du \\ &= \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+x^2}\Bigg|_0^R\right] = \frac{\sigma}{2\epsilon_0}(\sqrt{R^2+x^2}-x). \end{aligned}$$

Since $Q = \sigma\pi R^2$, we conclude that

$$V = \frac{Q}{2\pi\epsilon_0\pi R^2}(\sqrt{R^2+x^2}-x).$$

d)

Regardless of the mass of the plate it will either stay still given its symmetrical configuration along the sphere, or fly off if it is placed slightly unsymmetrically.