Assignment 3 MATH223

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1 Problem 1

We need to find a linear transformation such that the following equality holds

$$T(x+y) = T(x) + T(y) \qquad \forall x, y \in \mathbb{R}$$

Let $T : ln(x)$,indeed ln's propriety of addition makes the transformation linear

$$\begin{aligned} \ln x + y &= \ln xy \\ &= \ln x + \ln y \qquad \forall x, y \in \mathbb{R} \end{aligned}$$

We now prove that T is bijective. Let u and w $\in$V, then

$$\begin{aligned} \ln(u) &= \ln(w) \qquad \text{,Act e on both sides} \\ e^{\ln(u)} &= e^{\ln(w)} \rightarrow u = w \end{aligned}$$

Then we can prove that T is also surjective Let u and w be $\in$ V and k $\in \mathbb{R}$, then

$$\ln(u)^k = k \ln(u) \quad \text{by proprieties of ln(x)}$$

since k is an arbitrary constant, $\forall v \in \mathbb{R}\, \exists u \in$ U such that $T(u) = v$. T as defined is then surjective and injective ,consequently it's isomorphic.

2 Problem 2

2.1 a)

Let a $\in$ Im(T - I) then $\exists$ v $\in$ V such that

$$\begin{aligned} &(T - I)(v) = a \\ &\text{since} \quad (T^2 - I) = (T - I)(T + I) \\ &\text{then} \quad (T + I)(a) = (T - I)(T + I)(v) \\ &\qquad\qquad\qquad\quad = (T^2 - I)(v) \\ &\qquad\qquad\qquad\quad = 0 \times v = 0 \end{aligned}$$

Therefore a $\in$ Ker(T + I) and consequently, Im(T - I) $\subseteq$ Ker(T + I)
Similarly,

Let Let b ∈ Im(T + I) then ∃ y ∈ V such that

$$\begin{aligned}
&(T+I)(y) = b \\
&\text{since} \quad (T^2 - I) = (T-I)(T+I) \\
&\text{then} \quad (T-I)(b) = (T-I)(T+I)(y) \\
&\qquad\qquad\qquad = (T^2 - I)(y) \\
&\qquad\qquad\qquad = 0 \times y = 0
\end{aligned}$$

Therefore b ∈ Ker(T - I) and consequently, Im(T + I) ⊆ Ker(T - I)

2.2 b)

Let u ∈ Ker(T-I) ∩ Ker(T+I) , then by definition of the intersection ,
u ∈ Ker(T-I) and u ∈ Ker(T+I).

$$\begin{aligned}
&(T-I)(u) = 0 && (T+I)(u) = 0 \\
&T(u) - I(u) = 0 && (T)(u) + I(u) = 0 \\
&T(u) = I(u) && T(u) = -I(u) \\
&= u && = -(u) \qquad \text{,By definition of Id(u)}
\end{aligned}$$

Thus if u=-(u) → u=0 and so Ker(T-I) ∩ Ker(T+I) = 0.
Then since dim(V) = dim(Ker(T-I)) + dim(Ker(T+I)), V = Ker(T-I) ⊕ Ker(T+I)

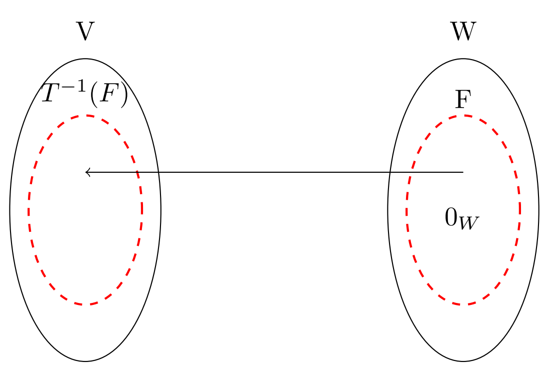
2.3 c)

Since (T^2-I)=0, that implies that (T-I)(T+I)=0 which has solutions 1 and -1, resulting in two eigenvectors for T. Then the two eigenvalues are respectively -1 and 1, giving

$$[T]_{\mathbb{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 Problem 3

3.1 a)



Let $x, y \in T^{-1}(F)$ then $\exists u, v \in V$ such that

$$T(x) = u \qquad T(y) = v$$

$$\begin{aligned}
&\text{Since } T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)\\
&\qquad\qquad\qquad\quad = \alpha u+\beta v \qquad \forall \alpha,\beta\in\mathbb{R} \;\;\&\;\; u,v\in V\\
&\text{then } (\alpha x+\beta y)\in T^{-1}(F)\\
&\text{since } \alpha x+\beta y\in V\\
&\text{then } T^{-1}(F)\subseteq V\\
&\text{Moreover, } T^{-1}(F) \text{ is a subspace of } V
\end{aligned}$$

For the second part, Since $T^{-1}(F)$ is a subspace of V and T : V $\rightarrow$ W, then

$$\dim(T^{-1}(F)) = \dim(\ker(T)\cap T^{-1}(F)) + \dim(\mathrm{Im(T)}\cap T(T^{-1}(F))$$

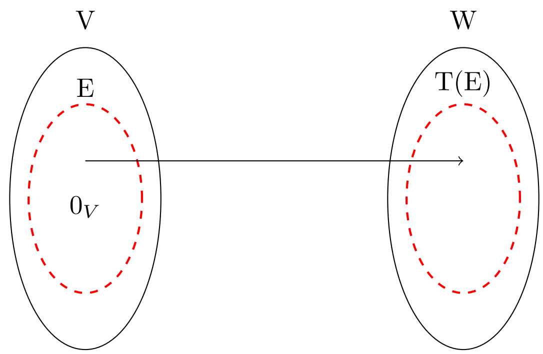
Since Ker(T) $\subseteq T^{-1}(F)$, then

$$\mathrm{Ker}(T)\cap T^{-1}(F) = \mathrm{Ker}(T)$$

Moreover, $T(T^{-1}(F)) = I_{dv}(F)$ = F by definition of the identity map. So we have shown that

$$\dim(T^{-1}(F)) = \dim(\mathrm{Ker}(T)) + \dim(F\cap \mathrm{Im}(T))$$

3.2 b)



Let $x,y\in T(E)$ then $\exists u,v\in W$ such that

$$T(x)=u \qquad T(y)=v$$

$$\begin{aligned}
&\text{Since } T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)\\
&\qquad\qquad\qquad\quad = \alpha u+\beta v \qquad \forall \alpha,\beta\in\mathbb{R} \;\;\&\;\; u,v\in\\
&\text{then } (\alpha x+\beta y)\in T(E)\\
&\text{since } \alpha x+\beta y\in W\\
&\text{then } T(E)\subseteq W\\
&\text{Moreover, } T(E) \text{ is a subspace of } W
\end{aligned}$$

For the second part, we'll use the rank-nullity theorem

$$\dim(E) = \dim(\mathrm{Ker}(\widetilde{T})) + \dim(\mathrm{Im}(\widetilde{T}))$$

Where $\dim(\mathrm{Ker}(\widetilde{T})) = \dim(\mathrm{Ker}(T)\cap T(T^{-1}(\mathrm{E})) = \dim(\mathrm{Ker}(T)\cap I_{dv}(E))$ =$\dim(\mathrm{Ker}(T)\cap \mathrm{E})$ By definition of the identity map.
Moreover, $\dim(\mathrm{Im}(\widetilde{T}))$ = Im(T) $\cap$ T(E).
Since Im(T) = W , W $\cap$ T(E) = T(E) , by definition of intersection.
Thus, we obtain the following final result

$$\dim(E) = \dim(\mathrm{Ker}(T)\cap E) + \dim(T(E))$$

4 Problem 4

4.1 a)

Let $\mathcal{B} = \{x^2$,x,1$\}$ be a basis of $\mathcal{P}_2$. Then

$$\begin{aligned}
[T(p_1)]_{\mathcal{B}} &\rightarrow T(p_1)(x) = x2(x) + 2x = 2x^2 + 2x \\
[T(p_2)]_{\mathcal{B}} &\rightarrow T(p_1)(x) = x(1) + 1 = x + 1 \\
[T(p_3)]_{\mathcal{B}} &\rightarrow T(p_3)(x) = x(0) + 0 = 0
\end{aligned}$$

So then

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

4.2 b)

Let $\mathcal{S} = \{x^2 - x$,x+1,x-1$\}$ be a basis of $\mathcal{P}_2$. Then

$$\begin{aligned}
[T(p_1)]_{\mathcal{B}} &\rightarrow T(p_1)(x) = x(2x - 1) + 2x - 1 = 2x^2 + x - 1 \\
[T(p_2)]_{\mathcal{B}} &\rightarrow T(p_1)(x) = x(1 + 0) + 1 + 0 = x + 1 \\
[T(p_3)]_{\mathcal{B}} &\rightarrow T(p_3)(x) = x(1 + 0) + 1 = x + 1
\end{aligned}$$

So then

$$[T]_{\mathcal{S}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

4.3 c)

Let $\mathcal{S} = \{x^2$,x,1$\}$ be a basis of $\mathcal{P}_2$. Then let us diagonalize this matrix. The characteristic polynominal is

$$\begin{aligned}
p(\lambda) &= \det([T]_{\mathcal{S}} - \lambda I_3) \\
&= \det \begin{vmatrix} 2-\lambda & 0 & 0 \\ 2 & 1-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} \\
&= (2-\lambda)((1-\lambda)(-\lambda)) = 0 \\
&= -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda - 1)(\lambda - 2)
\end{aligned}$$

Multiplicity is one and the corresponding eigenvalues are 0,1 and 2.The resulting diagonal matrix is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We now compute the corresponding eigenvectors.
For $\lambda = 0$, $([T]_{\mathcal{B}} - 0I_3)$ yields

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced echelon form gives $a = b = 0$ and c can take any value, therefore the corresponding eigenvector for v_1 is $[0, 0, 1]$
For $\lambda = 1$, $([T]_{\mathscr{B}} - 1I_3)$ yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced echelon form gives $a = 0$ and $b = c$, therefore the corresponding eigenvector for v_2 is $[0, 1, 1]$
For $\lambda = 2$, $([T]_{\mathscr{B}} - 2I_3)$ yields

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced echelon form gives $a = c$ and $b = 2c$, therefore the corresponding eigenvector for v_3 is $[1, 2, 1]$
In summary, the eigenvectors are

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Given $\mathscr{S} = \{x^2$,x,1$\}$ and P found above, we compute $\mathscr{B} = \{1$,$x^2 + x$,$x^2 + 2x + 1\}$ is a basis of $\mathscr{P}_2$ such that $[T]_{\mathscr{B}}$ is diagonal.

5 Problem 5

Let $\mathscr{S} = \{$u,v,w$\}$ be a basis of V. Then, since
T(u)=v+w, T(v)=u+w and T(w)=u+v,

$$[T]_{\mathscr{S}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynominal of this matrix is

$$\begin{aligned} p(\lambda) &= \det \begin{vmatrix} 0 - \lambda & 1 & 1 \\ 1 & 0 - \lambda & 1 \\ 1 & 1 & 0 - \lambda \end{vmatrix} \\ &= -\lambda(\lambda^2 - 1) + \lambda + 1 + 1 + \lambda \\ &= -\lambda^3 + 3\lambda + 2 = 0 \end{aligned}$$

This equation has solutions λ = -1 and λ = 2. Since dim(V) = 3 , we have 2 distinct eigenvalues but we need 3 for diagonalization. Therefore, we can't find a basis $\mathscr{B}$ such that $[T]_{\mathscr{B}}$ is diagonalizable.

6 Problem 6

6.1 a)

Let V = $\mathbb{R}$, S : V $\rightarrow$ V and T : V $\rightarrow$ V be both isomorphic.
Let S be defined as $f_1(x) = x$ and T as $f_2(x) = -x$. Both have their corresponding inverse $1/x$ and $-1/x$

respectively ,hence they're surjective.

$$f_1(x+x) = f_1(2x) = 2x \qquad\qquad f_1(x) + f_1(x) = x + x = 2x$$
$$f_2(x+x) = f_2(2x) = -2x \qquad\qquad f_2(x) + f_2(x) = -x + -x = -2x$$

Both have elements uniquely mapped, thus they're injective as well. Since S and T are bijective, they're indeed isomorphisms.
Now we verify if their addition is isomorphic between V as well.

$$\begin{aligned}(f_1 + f_2)(x+x) &= f_1(x) + f_2(x) \qquad \text{By proprieties of linear transformations}\\ &= x - x\\ &= 0\end{aligned}$$

Therefore, S+T is not an isomorphism from V into V since it's range is not $\mathbb{R}$ .And so the statement is <u>FALSE</u>

6.2 b)

Let V = R , then dim(V) = n = 1. We'll show that the statement is false , i.e rank(S) + rank(T) $\neq$ 1.
Let S : V$\to$V and T : V $\to$ V be two linear transformations defined as:

$$S(u) = u' \quad \forall u \in \mathbb{R}$$
$$T(u) = 0 \quad \forall u \in \mathbb{R}$$

$$\text{then, } (S \circ T)(u) = S(T(u)) = S(0) = 0' = 0$$
$$\text{thus, the condition } S \circ T = 0 \quad \text{is satisfied}$$

Since the rank of a linear transformation is by definition the dimension of it's image,

$$\text{rank}(S) = \text{rank}(T) = \dim(\text{Im}(S)) = \dim(\text{Im}(T)) = 0$$

Consequently, rank(S) + rank(T) = 0 + 0 $\ngeq$ 1. And so the statement is <u>FALSE</u>