MATH 475 Weekly Work 0

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September 6, 2020

Question 1

(i)

$$-\Delta u = -(u_{x_1 x_1} + \cdots + u_{x_n x_n}) = -\operatorname{tr}(D^2 u).$$

Since $L[u] = -\operatorname{tr}(A(x)D^2 u)$, and $G(Du, u, x) = 0$ we have

$$-\operatorname{tr}(A(x)D^2 u) = 0.$$

From which it follows that

$$-\operatorname{tr}(D^2 u) = -\operatorname{tr}(A(x)D^2) \implies A(x) = I_n.$$

(ii)

$$\begin{aligned} u_{tt} - \Delta u = u_{tt} - u_{x_1 x_1} - \cdots - u_{x_n x_n} &= -\operatorname{tr}(A(x,t)D^2_{x,t} u) \\ \implies u_{x_1 x_1} + \ldots u_{x_n x_n} - u_{tt} &= \operatorname{tr}(A(x,t)D^2_{x,t} u). \end{aligned}$$

It is evident that the matrix $A(x,t)$ corresponds to

$$A(x,t) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & -1 \end{bmatrix}$$

(iii)

Since

$$F(D^2 u, Du, u, x) = L[u] + G(Du, u, x),$$

we set $G(Du, u, x) = u_t$. Therefore,

$$\begin{aligned} u_t - \Delta u &= u_t - (u_{x_1x_1} + \cdots + u_{x_nx_n}) = 0 \\ \implies -\mathrm{tr}(A(x,t)D^2_{x,t}u) + u_t &= u_t + (u_{x_1x_1} + \cdots + u_{x_nx_n}). \end{aligned}$$

It follows evidently, that the matrix $A(x,t)$ corresponds to

$$A(x,t) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix}$$

Question 2

(i)

From $L[u] = -\mathrm{tr}(A(x,y)D^2u)$ we deduce $A(x,y)$ to be ,

$$au_{xx} + 2bu_{xy} + cu_{yy} = \mathrm{tr}\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}\right).$$

We then look for the eigenvalues of A

$$\begin{aligned} (a-\lambda)(c-\lambda) - b^2 = 0 &\implies ac - \lambda(a+c) + \lambda^2 - b^2 = 0 \\ &\implies \lambda^2 - \lambda(c+a) + ac - b^2 = 0 \\ &\implies \lambda = \frac{(c+a) \pm \sqrt{(c+a)^2 + 4(b^2 - ac)}}{2} \end{aligned}$$

Assuming $|4(b^2 - ac)| < (c+a)^2$ to preserve a real part for λ_i, then if $(b^2 - ac) < 0 \implies 4(b^2 - ac) < 0$. We find that both solutions for λ are non-zero and of positive sign since $c + a > \Delta\ \forall a, b, c \in \mathbb{R}$.

(ii)

Similarly as in (i) if $(b^2 - ac) > 0 \implies 4(b^2 - ac) > 0$. Thus $\Delta > 0\ \forall a, b, c \in \mathbb{R}$. Moreover, $\Delta > (c+a)\ \forall a, b, c \in \mathbb{R}$ such that $\lambda_1 > 0$ and $\lambda_2 < 0$, both being nonzero since $\Delta > 0$.

(iii)

Similarly as in (i), if $(b^2 - ac) = 0 \implies 4(b^2 - ac) = 0$. Therefore, $\Delta = (c+a)$ from which it is evident that one eigenvalue is positive or negative while the other is zero.

Question 3

We assume $A(x,y)$ is not symmetric, thus $a_{12} \neq a_{21}$. Conversely, $\widetilde{A}(x,y)$ is symmetric therefore $\widetilde{a}_{12} = \widetilde{a}_{21}$. D^2u is known so if we expand the trace of the matrix product for both equations we find

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{21}u_{yx} + a_{22}u_{yy} = \widetilde{a}_{11}u_{xx} + 2\widetilde{a}_{12}u_{xy} + \widetilde{a}_{22}u_{yy}. \tag{1}$$

Since the hessian matrix is symmetric ($u \in C^2$) then $u_{xy} = u_{yx}$ such that $a_{12}u_{xy} + a_{21}u_{yx} = (a_{12} + a_{21})u_{xy}$. Thus, matching the coefficients of (1) we get an expression for $\widetilde{A}(x,y)$:

$$\widetilde{A}(x,y) = \begin{bmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} \end{bmatrix}.$$