PHYS241 Ass 5.

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April 14, 2020

Question 1

a)

$$\begin{aligned}
&\underline{KCL}: I_1 = I_2 + I_3 \\
&\underline{KVL1}: V_{01} - R_1 I_1 + V_{02} - I_3 R_2 = 0 \\
&\underline{KVL2}: V_{01} - R_1 I_1 - I_2 R_3 = 0
\end{aligned}$$

$$\begin{aligned}
&\therefore V_{01} - R_1(I_2 + I_3) + V_{02} - I_3 R_2 = 0 \\
&\quad V_{01} - R_1(I_2 + I_3) - I_2 R_3 = 0 \\
\implies &I_2[-R_1] + I_3[-R_1 - R_2] + V_{02} = -V_{01}
\end{aligned}$$

We need a third equation to replace V_{01} in previous equation

$$\begin{aligned}
&V_{01} = R_1(I_2 + I_3) + I_2 R_3 \\
\implies &-[R_1(I_2) + R_1 I_3 + I_2 R_3 + I_2(-R_1) + I_3(-R_1 - R_2)] = V_{02} \\
&\therefore I_2(-R_3) + I_3(R_2) = V_{02} \\
&\therefore I_2(R_1 + R_3) + I_3(R_1) = V_{01}
\end{aligned}$$

Converting the previous two equation's $I_3 \to I_1$ to accomodate for Question 1b yields

$$I_1(R_1) + I_2(R_3) = V_{01} \quad \text{and } I_1(R_2) + I_2(-R_3 - R_2) = V_{02}.$$

$$\begin{pmatrix} R_1 & R_3 \\ R_2 & (-R_3 - R_2) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_{01} \\ V_{02} \end{pmatrix}$$

b)

$$\begin{aligned}
&\underline{KVL1}: V_{01} - R_1 I_1 + V_{02} - (I_1 - I_2) R_2 = 0 \\
&\underline{KVL2}: -V_{02} - I_2 R_3 - (I_2 - I_1) R_2 = 0
\end{aligned}$$

Since $V_{02} = -I_2R_3 - I_2R_2 + I_1R_2$,

$$\begin{aligned} \implies I_1(R_1) + I_2(R_3) &= V_{01} \\ I_1(R_2) + I_2(-R_2 - R_3) &= V_{02}. \end{aligned}$$

The results agree with the answer in part a.

c)

$$\begin{pmatrix} R_1 & R_3 \\ R_2 & (-R_3 - R_2) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_{01} \\ V_{02} \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solving the above system of equations numerically yields $I_1 = 0.8$ mA , $I_2 = 0.2$ mA, which then implies that $I_3 = 0.6$ mA.

Question 2

$$\begin{aligned} \underline{KVL1} &: V_0 - I_1R_1 - (I_1 - I_2)R_2 - (I_1 - I_3)R_3 = 0 \\ \underline{KVL2} &: -I_2R_5 - (I_2 - I_3)R_4 - (I_2 - I_1)R_2 = 0 \\ \underline{KVL3} &: -(I_3 - I_2)R_4 - I_3R_6 - (I_3 - I_1)R_3 = 0 \end{aligned}$$

The common terms between KVL1 and KVL2 with KVL3 are respectively I_2R_2 and I_3R_3 so we isolate those to express KVL2 and KVL3 in terms of V_0

$$\begin{aligned} &\implies V_0 - I_1R_1 - I_1R_2 - (I_1 - I_3)R_3 = -I_2R_2 \\ &\text{and } V_0 - I_1R_1 - (I_1 - I_2)R_2 - I_1R_3 = -I_3R_3 \end{aligned}$$

Plugging those expressions in the equations for KVL_1 , KVL_2 and KVL_3 while simultaneously isolating for V_0 and canceling out terms yields the following system of equations

$$\begin{aligned} I_1(R_1 + R_2 + R_3) + I_2(-R_2) + I_3(-R_3) &= V_0 \\ I_1(R_1 + R_3) + I_2(R_5 + R_4) + I_3(-R_4 - R_3) &= V_0 \\ I_1(R_1 + R_2) = I_2(-R_4 - R_2) + I_3(R_4 + R_6) &= V_0 \end{aligned}$$

$$\therefore \begin{pmatrix} R_1 + R_2 + R_3 & -R_2 & -R_3 \\ R_1 + R_3 & R_5 + R_4 & -R_4 - R_3 \\ R_1 + R_2 & -R_4 - R_2 & R_4 + R_6 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_0 \\ V_0 \\ V_0 \end{pmatrix}.$$

Question 3

a)

We first and foremost compute the value of $a_0/2$.

$$\frac{a_0}{2} = 2\left(\frac{1}{T}\int_{-T/2}^{0} V_0 + \frac{2V_0 t}{T} dt\right) = \frac{2V_0}{T}\left(\int_{-T/2}^{0} dt + \int_{-T/2}^{0} \frac{2t}{T}\ dt\right)$$
$$= \frac{2V_0}{T}\left(\left(0\frac{T}{2}\right) + \left(\frac{0}{T} - \frac{(-T/2)^2}{T}\right)\right) = \frac{2V_0}{T}\left(\frac{T}{4}\right) = \frac{V_0}{2}.$$

Since the function is even $b_n = 0$ and so we now find an expression for a_n. By definition,

$$a_{n>0} = \frac{4V_0}{T}\int_{-T/2}^{0}\left(1 + \frac{2t}{T}\right)\cos(\omega_n t)\ dt$$

Using *Wolframalpha* the above integral redudces to

$$= \frac{4V_0 \sin^2\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}. = \frac{4V_0}{n^2\pi^2}\begin{cases} 0 & ,n = \text{ even} \\ 1 & ,n = \text{ odd.}\end{cases}$$

The final expression for the Fourier series of the given function is then

$$f(t) = \frac{V_0}{2} + \sum_{n=1}^{\infty}\frac{4V_0}{(2n-1)^2\pi^2}\cos\left(\frac{2\pi(2n-1)t}{T}\right).$$

b)

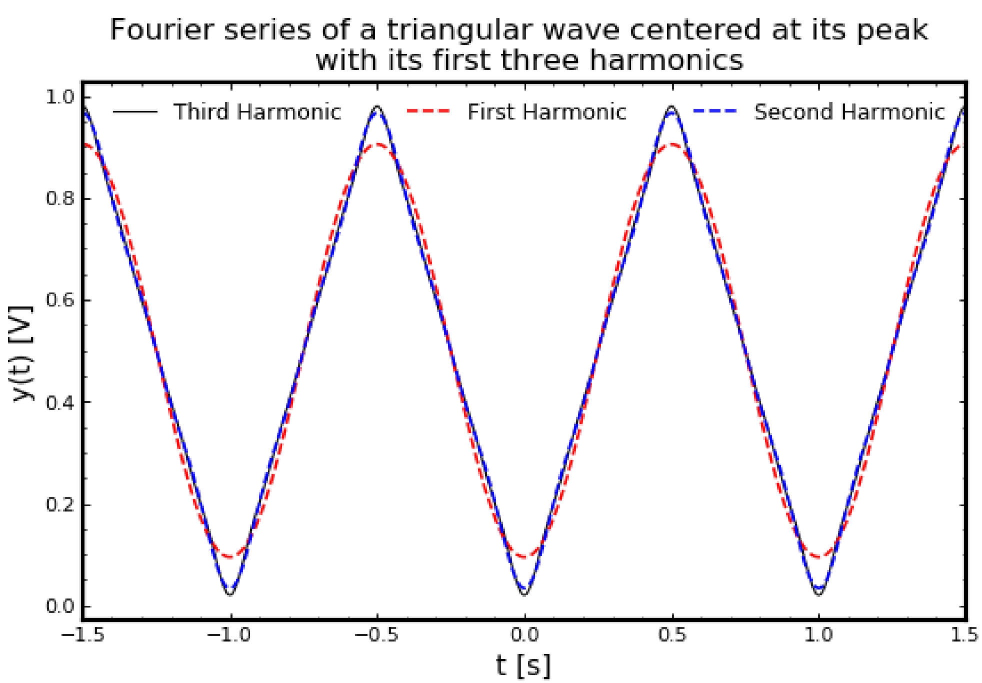

Figure 1: First three harmonics of the Fourier series in Question 2a, for $T = 1$ ms.

Question 4

a)

$$\begin{aligned}H(\omega) &= \frac{Z_R}{Z_C + Z_R} = \frac{R}{R + 1/j\omega C} = \frac{1}{1 + 1/j\tau\omega} \\ &= \frac{j\tau\omega}{j\tau\omega + 1} = \frac{j\tau\omega}{j\tau\omega + 1}\left(\frac{-j\omega\tau + 1}{-j\omega\tau + 1}\right) = \frac{(\omega\tau)^2 + j\omega\tau}{(\omega\tau)^2 + 1}.\end{aligned}$$

b)

We first compute b_n since this is an odd function.

$$b_{n>0} = \frac{4}{T}\int_0^{T/2} \frac{t}{T}\sin(\omega_n t)\ dt = \frac{4}{T^2}\int_0^{T/2} t\sin(\omega_n t)\ dt$$

Applying integration by parts with $u = t$ and $dv = \sin(\omega_n t)$ yields

$$= \frac{4}{T^2}\left(\frac{-1}{\omega_n}\, t\cos(\omega_n)\bigg|_0^{T/2}\right) = \frac{-T\cos(\pi n)}{2\omega_n} = \frac{-\cos(\pi n)}{\pi n} = -\frac{(-1)^2}{\pi n}.$$

The amplitude $|H(\omega)|$ is

$$|H(\omega)| = \sqrt{\mathrm{Re}(H(\omega))} = \frac{\omega_n\tau}{\sqrt{1 + (\omega_n\tau)^2}}.$$

The phase is immediately computed with

$$\varphi(\omega) = \tan^{-1}\left[\frac{\mathrm{Im}(H(\omega))}{\mathrm{Re}(H(\omega))}\right] = \tan^{-1}\frac{((\tau\omega/(\tau\omega)^2 + 1))}{(\tau\omega)^2/((\tau\omega)^2 + 1))} = \tan^{-1}\left(\frac{1}{\tau\omega}\right).$$

Recombining everything and following the definition of Fourier series we have

$$\begin{aligned}V_{\mathrm{out}}(t) &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi n}\frac{(\omega_n\tau)}{\sqrt{(\omega_n\tau)^2 + 1}}\sin\left(\omega_n t + \tan^{-1}\left(\frac{1}{\omega\tau}\right)\right) \\ V_{\mathrm{in}}(t) &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi n}\sin(\omega_n t).\end{aligned}$$

c)

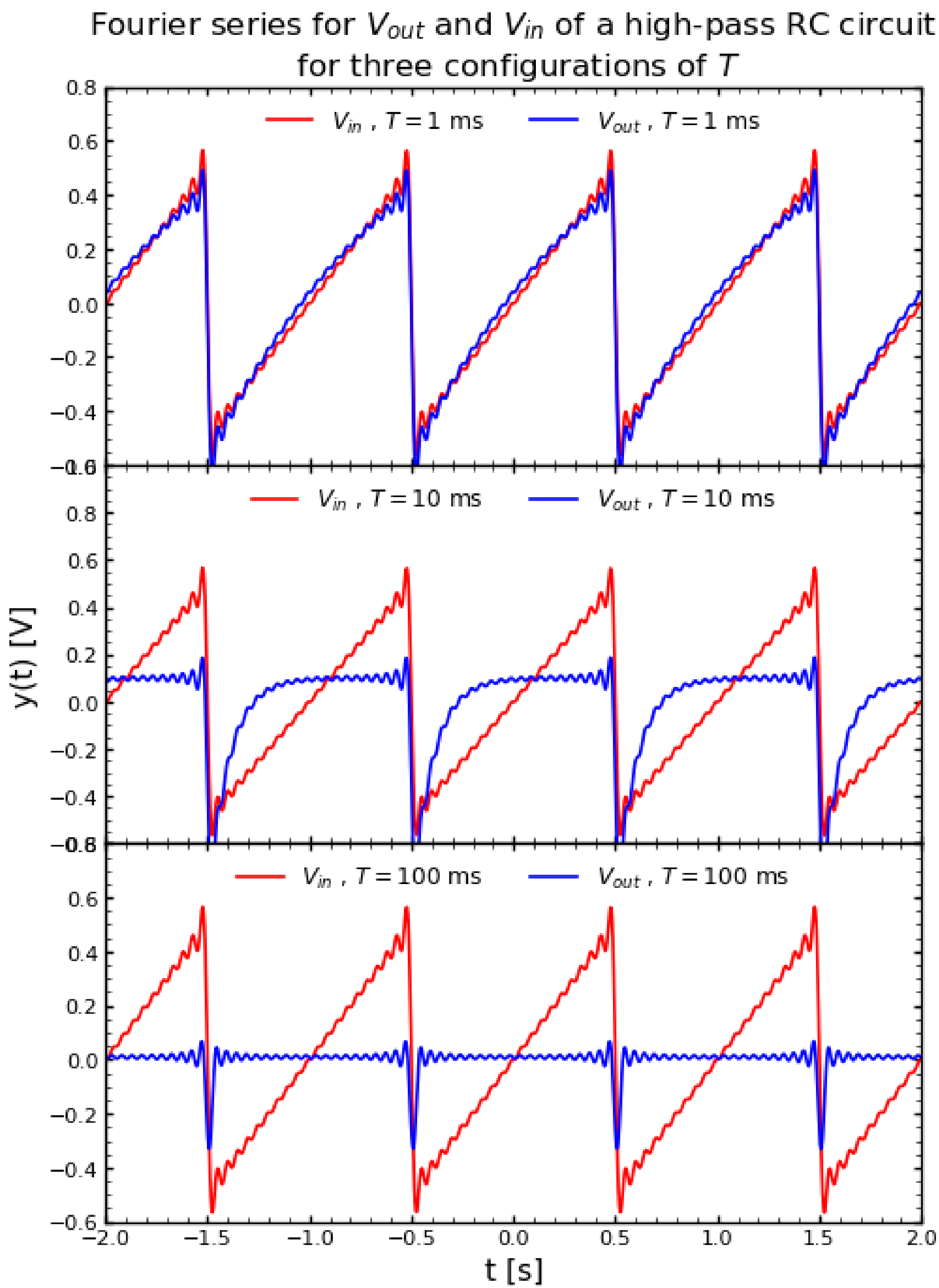

Figure 2: First 20 non vanishing terms of $V_{\text{in}}(t)$ and $V_{\text{out}}(t)$ for a fixed number of cycles plotted for three values of T.

V_{in} does not change as T increases since the number of cycles is re-scaled for each value of T. We also note that V_{out} diverges from V_{in} as T increases as this is a High-pass RC filter, so when the frequency decreases the capacitor charges faster to its full capacity, as perceived in Figure 2.

Question 5

a)

$$\begin{aligned} H(\omega) &= \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} \\ &= \frac{1}{j\omega CR - \omega^2 LC + 1} = \frac{1}{j\omega\tau - (\omega/\omega_0)^2 + 1} = \frac{1}{j\omega\tau - (\omega/\omega_0)^2 + 1}. \end{aligned}$$

b)

We first compute the amplitude $|H(\omega)|$

$$|H(\omega)| = \left(\left(\frac{1}{1 - \left(\frac{\omega_n}{\omega_0}\right)^2 + j\omega_n\tau} \right) \left(\frac{1}{1 - \left(\frac{\omega_n}{\omega_0}\right)^2 - j\omega_n\tau} \right) \right)^{-1/2} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega_n}{\omega_0}\right)^2\right) + (\omega_n\tau)^2}}.$$

Then, the phase is immediately computed as well

$$\left(\frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\omega\tau} \right) \left(\frac{1 - \left(\frac{\omega}{\omega_0}\right)^2 - j\omega\tau}{1 - \left(\frac{\omega}{\omega_0}\right)^2 - j\omega\tau} \right) \implies \tan^{-1}(\varphi(\omega)) = \frac{\omega\tau}{1 - \left(\frac{\omega}{\omega_0}\right)^2}.$$

We may write an expression for V_{out} ,

$$V_{\text{out}}(t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \frac{1}{\sqrt{\left(1 - \left(\frac{\omega_n}{\omega_0}\right)^2\right)^2 + (\omega_n\tau)^2}} \sin\left(\omega_n\tau - \tan^{-1}\left(\frac{\omega_n\tau}{1 - \left(\frac{\omega_n}{\omega_0}\right)^2} \right) \right). \quad (1)$$

To find the frequencies that will go through we use the definition of the quality factor.

$$Q = \frac{\omega_0 L}{R} = \left(\sqrt{\frac{L}{C}}\right)\frac{1}{R} = 5\sqrt{10}$$

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R} \implies \Delta\omega = \frac{R}{L} \implies \text{Range } = \omega_0 \pm \Delta\omega = 6424 \pm 400.$$

We may now look at which frequency mode pass through this range

$$\omega_1 = \frac{2\pi(1)}{T} = 6283\checkmark \qquad ,\omega_2 = \frac{2\pi(2)}{T} = \cancel{12588}.$$

Replacing $\omega_1 = 6424$ and the given values for T, L, C and R in Equation 1 , yields $V_{\text{out}} \approxeq 2.39$ V.

Question 6

a)

First and foremost, to simplify algebra , let $K = \pi/2a$. Then by definition of inverse transform,

$$\begin{aligned}
F(\omega) &= \int_{-a}^{a} f(t)e^{-i\omega t}\ dt \\
&= \frac{1}{2}\left(\int_{-a}^{a} e^{Kit}e^{-i\omega t}\ dt + \int_{-a}^{a} e^{-Kit}e^{-i\omega t}\ dt\right) \\
&= \frac{1}{2}\left(\int_{-a}^{a} e^{i(K-\omega)t}\ dt + \int_{-a}^{a} e^{i(-K-\omega)t}\ dt\right) \\
&= \frac{1}{2}\left(\frac{1}{i(K-\omega)}\ e^{i(K-\omega)t}\Big|_{-a}^{a} + \frac{1}{i(-K-\omega)}\ e^{i(-K-\omega)t}\Big|_{-a}^{a}\right) \\
&= \frac{1}{2}\left(\frac{1}{i(K-\omega)}\left(e^{i(K-\omega)a} - e^{-i(K-\omega)a}\right) + \frac{1}{i(-K-\omega)}\left(e^{i(-K-\omega)a} - e^{-i(-K-\omega)a}\right)\right) \\
&= \frac{\sin\left(\frac{\pi}{2} - \omega a\right)}{\left(\frac{\pi}{2a} - \omega\right)} + \frac{\sin\left(\frac{-\pi}{2} - \omega a\right)}{\left(\frac{-\pi}{2a} - \omega\right)}
\end{aligned}$$

Using the identities $\sin(\pi/2 \pm x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ yields the reduced expression

$$F(\omega) = \frac{\cos(\omega a)}{\left(\frac{\pi}{2a} - \omega\right)} - \frac{\cos(\omega a)}{\left(\frac{-\pi}{2a} - \omega\right)}.$$

b)

When compared to $F_2(\omega) = 2\sin(\omega a)/\omega$, we note that the parameter a has the same effect on both Fourier transforms in terms of frequency scaling, i.e., when varying that parameter the functions are compressed or stretched at the same rate.