PHYS 357 Assignment 2

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Problem 2.4

$$|+x\rangle \xrightarrow{S_y \text{ basis}} \begin{pmatrix} \langle +y|+x\rangle \\ \langle -y|+x\rangle \end{pmatrix} \quad , |-x\rangle \xrightarrow{S_y \text{ basis}} \begin{pmatrix} \langle +y|-x\rangle \\ \langle -y|-x\rangle \end{pmatrix} \tag{1}$$

Computing each braket,

$$\begin{aligned} \langle +y|+x\rangle &= \frac{1}{\sqrt{2}}\langle +y|+z\rangle + \frac{1}{\sqrt{2}}\langle +y|-z\rangle \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{-i}{\sqrt{2}}\right) = \frac{1}{2}(1-i)\,. \\ \langle -y|+x\rangle &= \frac{1}{\sqrt{2}}\langle -y|+z\rangle + \frac{1}{\sqrt{2}}\langle -y|-z\rangle \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2}(1+i) \end{aligned}$$

Following (1) and using the conjugate for the $|-x\rangle$ we conclude that

$$|+x\rangle = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \quad , |-x\rangle = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

Problem 2.5

$$|+z\rangle \xrightarrow{S_y \text{ basis}} \begin{pmatrix} \langle +y|+z\rangle \\ \langle -y|+z\rangle \end{pmatrix} \quad , |-z\rangle \xrightarrow{S_y \text{ basis}} \begin{pmatrix} \langle +y|-z\rangle \\ \langle -y|-z\rangle \end{pmatrix} \tag{2}$$

Computing each braket,

$$\begin{aligned}
\langle +y|+z\rangle &= \langle +z|+y\rangle^* = \left(\frac{1}{\sqrt{2}}\right)^* = \frac{1}{\sqrt{2}} \\
\langle +y|-z\rangle &= \langle -z|+y\rangle^* = \left(\frac{i}{\sqrt{2}}\right)^* = \frac{-i}{\sqrt{2}} \\
\langle -y|+z\rangle &= \langle +z|-y\rangle^* = \left(\frac{1}{\sqrt{2}}\right)^* = \frac{1}{\sqrt{2}} \\
\langle -y|-z\rangle &= \langle -z|-y\rangle^* = \left(\frac{-i}{\sqrt{2}}\right)^* = \frac{i}{\sqrt{2}}
\end{aligned}$$

We conclude that

$$|+z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad , |-z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\i\end{pmatrix}.$$

Finally,

$$\begin{aligned}
\hat{S}_z &= \frac{\hbar}{2}(|+z\rangle\langle +z| - |-z\rangle\langle -z|) \\
&= \left(\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}i & -i\end{pmatrix}\right) = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}
\end{aligned}$$

Therefore,

$$\langle S_z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i & 1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\hbar}{2}.$$

Problem 2.6

$$\begin{aligned}
|+z\rangle &= \frac{1}{\sqrt{2}}|+y\rangle + \frac{1}{\sqrt{2}}|-y\rangle \\
|+y\rangle &= \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle \quad |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle
\end{aligned}$$

We use the relationship

$$\hat{J}_y|\pm y\rangle = \pm\frac{\hbar}{2}|\pm y\rangle,$$

therefore,

$$\begin{aligned}
\hat{R}_y(\theta)|+z\rangle &= e^{\frac{-i}{\hbar}\hat{S}_y\theta}\left(\frac{|+y\rangle}{\sqrt{2}} + \frac{|-y\rangle}{\sqrt{2}}\right) \\
&= e^{\frac{i\theta}{2}}\frac{1}{\sqrt{2}}|+y\rangle + e^{\frac{-i\theta}{2}}\frac{1}{\sqrt{2}}|-y\rangle
\end{aligned}$$

We switch back in the z basis

$$
\begin{aligned}
&= \frac{e^{\frac{i\theta}{2}}\left|+z\right\rangle + ie^{\frac{i\theta}{2}}\left|-z\right\rangle}{2} + \frac{e^{\frac{-i\theta}{2}}\left|+z\right\rangle - ie^{\frac{-i\theta}{2}}\left|-z\right\rangle}{2} \\
&= \frac{\left(e^{\frac{i\theta}{2}} + e^{\frac{-i\theta}{2}}\right)}{2}\left|+z\right\rangle + \frac{i\left(e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}\right)}{2}\left|-z\right\rangle \\
&= \cos\left(\frac{\theta}{2}\right)\left|+z\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|-z\right\rangle .
\end{aligned}
$$

We evalaute at $\theta = \pi/2$,

$$
\begin{aligned}
\hat{R}_y\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{4}\right)\left|+z\right\rangle + \sin\left(\frac{\pi}{4}\right)\left|-z\right\rangle \\
&= \frac{1}{\sqrt{2}}\left|+z\right\rangle + \frac{1}{\sqrt{2}}\left|-z\right\rangle ,
\end{aligned}
$$

which is indeed $\left|+x\right\rangle$ in the z basis.

Problem 2.8

The Pauli matrix σ_x is already written in the z basis , so immediately,

$$
\langle\varphi|\sigma_x|\varphi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}-i & 2\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}i\\ 2\end{pmatrix} = 0
$$

Then it follows that

$$
\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \frac{\hbar}{2}.
$$

Question 5

a)

Let us compute with the Pauli matrices since they are both in the same basis

$$
\begin{aligned}
[\hat{S}_x, \hat{S}_z] = \hat{S}_x\hat{S}_z - \hat{S}_z\hat{S}_x &= \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\
&= \frac{\hbar^2}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = i\hbar\hat{S}_y \neq 0.,
\end{aligned}
$$

we conclude that the operators do not commute.

b)

Let us apply these operators to $|+x\rangle$ written in the z basis.

$$\hat{S}_x\hat{S}_z\,|+x\rangle = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\hbar^2}{4\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$\hat{S}_z\hat{S}_x\,|+x\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{-\hbar^2}{4\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}.$$