MATH 327 Assignment 2

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Question 1

The given matrices are in fact Hermitian. We will use Sylvester's criterion to verify positive definiteness.

a)

$$A := \begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{pmatrix} \qquad \begin{aligned} \det(A_{11}) &= 4 > 0 \\ \det(A_{22}) &= 4 > 0 \\ \det(A_{33}) &= 64 > 0 \end{aligned}$$

A as defined is positive definite. Then we can apply the algorithm given

$$A = \begin{pmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \longrightarrow \begin{cases} 4 = {r_{11}}^2 & \implies r_{11} = 2 \\ 2 = r_{11}r_{12} & \implies r_{12} = 1 \\ 6 = r_{12}r_{13} & \implies r_{13} = 3 \\ 2 = r_{12}^2 + {r_{22}}^2 & \implies r_{22} = 1 \\ 5 = r_{12}r_{13} + r_{22}r_{23} & \implies r_{23} = 2 \\ 29 = {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 & \implies r_{33} = 4 \end{cases}$$

$$\therefore R = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix},$$

R is upper triangular the algorithm is successful.

b)

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \qquad \begin{aligned} \det(A_{11}) &= 1 > 0 \\ \det(A_{22}) &= 1 > 0 \\ \det(A_{33}) &= -1 < 0 \end{aligned}$$

A as defined is not positive definite. We expect the algorithm to fail.

$$A = \begin{pmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \longrightarrow \begin{cases} 1 = {r_{11}}^2 & \implies r_{11} = 1 \\ 1 = r_{11}r_{12} & \implies r_{12} = 1 \\ 1 = r_{12}r_{13} & \implies r_{13} = 1 \\ 2 = r_{12}^2 + {r_{22}}^2 & \implies r_{22} = 1 \\ 2 = r_{12}r_{13} + r_{22}r_{23} & \implies r_{23} = 1 \\ 1 = {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 & \implies r_{33} = \sqrt{-1} \\ & \qquad\quad \not> 0 \end{cases},$$

the algorithm fails, a contradiction arises in the last step.

Question 2

a)

We show that $A^{-1} = I + \alpha uv^T$ is the inverse of A if their product is the identity. That is,

$$\begin{aligned} AA^{-1} = I &\implies (I + uv^T)(I + \alpha uv^T) = I \\ &\implies I^2 + \alpha uv^T + uv^T + uv^T \alpha uv^T = I \\ &\implies \alpha uv^T + uv^T + uv^T \alpha uv^T = 0 \end{aligned}$$

We use the fact that by matrix associativity, $uv^T \alpha uv^T = \alpha(uv^T uv^T) = \alpha u(v^T u)v^T$. Then following from matrix commutative proerty , since $v^T u$ is a scalar, this reduces to $\alpha(v^T u)uv^T$. So we write,

$$\implies (\alpha + 1 + \alpha v^T u)uv^T = 0$$

If $uv^T = O$ then $A = I$ which doesn't hold. So it must be that $(\alpha + 1 + \alpha v^T u) = 0$ such that

$$\alpha(1 + v^T u) + 1 = 0 \implies -\frac{1}{1 + v^T u} = \alpha,$$

which $\exists$ for $v^T u \neq -1$.

b)

If A is singular $det(A) = 0$. So let us verify,

$$\det(A) = \det\big(I + uv^T\big)$$

we use the property $\det(A+B) = \det(A) + \det(B) + \det(A)\,\mathrm{tr}\big(A^{-1}B\big)$, giving

$$\begin{aligned} &= \det(I) + \det\big(uv^T\big) + \det(I)\,\mathrm{tr}\big(I^{-1}uv^T\big) \\ &= 1 + \det\big(uv^T\big) + \mathrm{tr}\big(uv^T\big) \end{aligned}$$

The inner product is the trace of the outer product [source : Optimal Control and Estimation , *p.26*].

$$= 1 + \det\big(uv^T\big) + v^T u$$

For $\det\big(uv^T\big)$, every row of b is scalar multiple of a , such that the determinant of uv^T is automatically 0. Finally,

$$= 1 + v^T u = 0 \implies v^T u = -1,$$

A is singular whenever $v^T u = -1$. We now look for the Null space. That is $Ay = 0$ for $y \neq 0$.

$$Ay = (I + uv^T)y = y + uv^T y = 0 \implies -y = uv^T y$$

Since A is singular for $v^T u = -1$, let $y = ku$ for $k \in \mathbb{R}$, then

$$\begin{aligned} -ku &= uv^T ku \\ -ku &= kuv^T u \\ -ku &= -ku \end{aligned}$$

So the choice $y = ku$ is correct. k is arbitrary so we conclude that $\mathrm{Null}(A) = \mathrm{Span}(A)$.

Question 3

a)

The system $Ax = b$ that determines the LSP is

$$\begin{pmatrix} f_1(y_1) & f_2(y_1) \\ f_1(y_2) & f_2(y_2) \\ \vdots & \vdots \\ f_1(y_5) & f_2(y_5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_5 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & y_1 \\ 1 & y_2 \\ \vdots & \vdots \\ 1 & y_5 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_5 \end{pmatrix}}_{b}$$

b)

The system $Ax = b$ that determines the LSP is

$$\begin{pmatrix} f_1(y_1) & f_2(y_1) & f_3(y_1) \\ f_1(y_2) & f_2(y_2) & f_3(y_2) \\ \vdots & \vdots & \\ f_1(y_5) & f_2(y_5) & f_3(y_5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_5 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & y_1 & {y_1}^2 \\ 1 & y_2 & {y_2}^2 \\ \vdots & \vdots & \vdots \\ 1 & y_5 & {y_5}^2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_5 \end{pmatrix}}_{b}$$

Question 4

a)

Let us consider the following system, and deduce the algorithmic formula

$$\begin{pmatrix} r_{11} & 0 & \dots & 0 \\ r_{21} & r_{22} & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Given the system above, the corresponding equations are

$$\begin{aligned} x_1 &= \frac{b_1}{r_{11}} \\ x_2 &= \frac{(b_2 - r_{21}x_1)}{r_{22}} \\ x_3 &= \frac{(b_3 - r_{31}x_1 - r_{32}x_2)}{r_{33}} \\ &\vdots \\ x_n &= \frac{(b_n - r_{n1}x_1 - r_{n2}x_2 - \dots - r_{n,n-1}x_{n-1})}{r_{nn}} \end{aligned} \qquad \Longrightarrow \qquad \boxed{x_i = \frac{1}{r_{ii}}\left(b_i - \sum_{j=1}^{i-1} r_{ij}x_j\right) \text{,for} \quad i = 1, \dots, n}$$

To calculate the total number of steps, we consider the algorithm for the upper triangular case. Given the range $i = n, n-1, \dots, 2, 1$, we have for each value an fixed number of operations ;

$$\begin{aligned} i = n: & \qquad \frac{1}{r_{nn}}(b_n) & ;\ 1 \text{ operation} \\ i = n-1: & \qquad \frac{1}{r_{n-1,n-1}}(b_{n-1} - r_{11}x_1) & ;\ 3 \text{ operations} \end{aligned}$$

$$
\begin{array}{lll}
i = n-2: & \dfrac{1}{r_{n-2,n-2}}(b_{n-2} - (r_{11}x_1 + r_{22}x_2)) & ;\ \text{5 operations} \\
\vdots & \vdots & \\
i = n-(n-1) = 1: & \dfrac{1}{r_{11}}(b_1 - (r_{11}x_1 + \cdots + r_{n-1,n-1}x_{n-1})) & ;\ 2n-1 \text{ operations}
\end{array}
$$

Thus,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2,$$

is the number of operations.

b)

The algorithm was stated in class ;

$$r_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} {r_{ji}}^2} \ \text{ for } \ i = 1, \ldots, n \qquad \text{where } \ r_{ij} = \frac{1}{r_{ii}}\left(a_{ij} - \sum_{k=1}^{i-1} r_{ki}r_{kj}\right)$$
$$\text{for } j = i+1, \ldots, n.$$

We look for the number of operations.

(Outer k loop:)

$$\sum_{i=1}^{n}\sum_{k=1}^{i-1} 2 = 2\sum_{i=1}^{n}(i-1) = n(n-1) = n^2 - n.$$

(Inner k loop)

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{i-1} 2 &= 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}(i-1) \\
&= 2\sum_{i=1}^{n}(n-i)(i-1) \\
&= 2n\sum_{i=1}^{n}(i-1) - 2\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i \\
&= n^2(n-1) - \frac{n(n+1)(2n+1)}{3} + n(n+1) = \frac{n^3 - 3n^2 + 2n}{3}.
\end{aligned}
$$

(Divisions:)

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n}(n-i) = \frac{n(n-1)}{2}.$$

(Square root:)

$$\sum_{i=1}^{n} 1 = n.$$

Thus, the total amounts to

$$n^2 - n + \frac{n^3 - 3n^2 + 2n}{3} + \frac{n^2 - n}{2} + n = \frac{n}{6}(2n^2 + 3n + 1).$$

c)

By Property, given that A is non-singular then $A^T A$ is positive definite. Then, following the Cholesky decomposition theorem ,there exists a unique decomposition $A^T A = R^T R$ where R is upper triangular. We use the Cholesky decomposition algorithm to find R and thereby R^T. Then we solve $R^T y = A^T b$ with backwards substitution. And then, we finally solve $Rx = y$ for x ,through forward substitution. The operations required for each steep are

$$\left.\begin{array}{l} 1.\ \ \frac{n}{6}(2n^2+3n+1) \\ 2.\ \ n^2 \\ 3.\ \ n^2 \end{array}\right\} \longrightarrow \text{most expensive : 1. } \frac{n}{6}(2n^2+3n+1),$$

ordered respectively.

d)

As before, we consider

$$\begin{pmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{pmatrix} = \begin{pmatrix} r_{11} & 0 & 0 & 0 \\ r_{12} & r_{22} & 0 & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ r_{14} & r_{24} & r_{34} & r_{44} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{pmatrix},$$

which gives the equations

$$\begin{aligned} 16 &= {r_{11}}^2 && \Longrightarrow r_{11} = 4 && , 8 = r_{12}r_{13} + r_{22}r_{23} && \Longrightarrow r_{23} = 2 \\ 4 &= r_{11}r_{12} && \Longrightarrow r_{12} = 1 && , 4 = r_{12}r_{14} + r_{22}r_{24} && \Longrightarrow r_{24} = 1 \\ 8 &= r_{11}r_{13} && \Longrightarrow r_{13} = 2 && , 12 = {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 && \Longrightarrow r_{33} = 2, \\ 4 &= r_{11}r_{14} && \Longrightarrow r_{14} = 1 && , 10 = r_{13}r_{14} + r_{23}r_{24} + r_{33}r_{34} && \Longrightarrow r_{34} = 3 \\ 10 &= {r_{12}}^2 + {r_{22}}^2 && \Longrightarrow r_{22} = 3 && , 12 = {r_{14}}^2 + {r_{24}}^2 + {r_{34}}^2 + {r_{44}}^2 && \Longrightarrow r_{44} = 1 \end{aligned}$$

which results in

$$R^T = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}.$$

Here we solve $R^T y = A^T b$ through the method outlined in Question 4.

$$\left.\begin{aligned} y_1 &= \tfrac{1}{4}(32) = 8 \\ y_2 &= \tfrac{1}{3}(26-(1)y_1) = 6 \\ y_3 &= \tfrac{1}{2}(38-(2)y_1-(2)y_2) = 17 \\ y_4 &= \tfrac{1}{1}(30-(1)y_1-(1)y_2-(3)y_3) = -35 \end{aligned}\right\} \qquad y = \begin{pmatrix} 8 \\ 6 \\ 17 \\ -35 \end{pmatrix}$$

Now through forward substitution we solve $Rx = y$;

$$\left.\begin{aligned} x_4 &= -\tfrac{35}{1} = -35 \\ x_3 &= \tfrac{1}{2}(17-3(x_4)) = 61 \\ x_2 &= \tfrac{1}{3}(6-2(x_3)-1(x_4)) = -27 \\ x_1 &= \tfrac{1}{4}(8-1(x_2)-2(x_3)-1(x_4)) = -13 \end{aligned}\right\} \qquad x = \begin{pmatrix} -35 \\ 61 \\ -27 \\ -13 \end{pmatrix}$$

Question 5

a)

$$\begin{aligned} \|x_1 + x_2\|_2{}^2 &= \sum_{i=1}^{m} |(x_1+x_2)_i|^2 \\ &= \sum_{i=1}^{m} (x_1+x_2)_i{}^2 \\ &= \sum_{i=1}^{m} x_{1,i}{}^2 + x_{2,i}{}^2 - 2\sum_{i=1}^{m} x_{1,i}x_{2,i} \\ &= \sum_{i=1}^{m} x_{1,i}{}^2 + x_{2,i}{}^2 - 2x_1{}^T x_2 \end{aligned}$$

Given that all vectors are orthogonal we have that $x_i{}^T x_j = 0$, so

$$\begin{aligned} &= \sum_{i=1}^{m} x_{1,i}{}^2 + x_{2,i}{}^2 \\ &= \sum_{i=1}^{m} |x_{1,i}|^2 + |x_{2,i}|^2 = \|x_1\|_2{}^2 + \|x_2\|_2{}^2 \end{aligned}$$

b)

<u>$n = 1:$</u>

This case is trivial since in both equalities the sum over $n = 1$ vanishes to 1, therefore the

equality is automatically satisfied.
$\underline{(n-1) \implies (n):}$ Define $y := x_1 + \cdots + x_{n-1} \in \mathbb{R}^m$, which is also orthogonal to x_n by property of orthogonality. Then,

$$\begin{aligned}\left\| \sum_{i=1}^{n} x_i \right\|_2^2 &= \|x_1 + \cdots + x_{n-1} + x_n\| \\ &= \|y + x_n\|_2{}^2\end{aligned}$$

The theorem holds for two vector addition as it was shown in part a), thence

$$\begin{aligned}&= \|y\|_2{}^2 + \|x_n\|_2{}^2 \\ &= \|x_1 + \cdots + x_{n-1}\|_2{}^2 + \|x_n\|_2{}^2\end{aligned}$$

Let $y = x_1 + \cdots + x_{n-2}$, we repeat the exact same process. After $n-1$ such processes we are left off with

$$\begin{aligned}&= \|x_1\|_2{}^2 + \cdots + \|x_n\|_2{}^2 \\ &= \sum_{i=1}^{n} \|x_i\|_2{}^2\end{aligned}$$

Question 6

a)

Let $R_1 = (a)_{ij}$ and $R_2 = (b)_{ij}$ along with $C = R_1 R_2 = (c)_{ij} = (a)_{ik}(b)_{kj}$. Then, by definition of upper triangular, $(a)_{ij} = 0$ for $i > j$ which implies that $a_{ik} = 0$ for $i > k$. Similarly, $(b)_{ij} = 0$ for $k > j$ which then implies $(b)_{kj} = 0$ for $k > j$.

So then, $(c)_{ij} = 0$ if $i > k$ or $k > j$ which in return imply that $i > k > j \implies i > j$. So $(c)_{ij} = R_1 R_2$ is upper triangular as well.

b)

Let the operations associated with the 0s be ignored , then

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\text{Multiplications}} \quad \underbrace{\begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}}_{\text{Additions}}$$

So we have the sequence

$$\left(\sum_{k=1}^{n} k + \sum_{k=1}^{n-1} k + \cdots + \sum_{k=1}^{1} k\right) + \left(\sum_{k=1}^{n-1} k + \cdots + \sum_{k=1}^{1} k\right) = \sum_{j=1}^{n}\sum_{k=1}^{j} k + \sum_{j=1}^{n-1}\sum_{k=1}^{j} k$$
$$= \frac{n(n+1)(n+2)}{6} + \frac{n(n-1)(n+1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6}$$

c)

$$R_2 x = \begin{pmatrix} \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & & \\ \vdots & & \ddots & \\ 0 & & & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix} \longrightarrow \underbrace{\begin{cases} n & (n-1) \\ (n-1) & (n-2) \\ (n-2) & (n-3) \\ \vdots & \vdots \\ 1 & 0 \end{cases}}_{\text{Multiplications, additions}}$$

$$\implies \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2.$$

d)

Here we consider two possibilities ; $(R_1R_2)x$ or $R_1(R_2x)$, both of which hold under matrix associativity but have different operations cost.

$$(R_1R_2)x \ : \quad \frac{n(n+1)(2n+1)}{6} + n^2, \qquad R_1(R_2x) \ : \quad n^2 + n^2.$$

The second option is much more efficient.