MATH 475 Weekly Work 4

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Question 1

By Duhamel method, let v solve

$$\begin{cases} v_t - kv_{xx} = 0 & \text{in } \ 1_T, \\ v(0,t) = 0 \quad , v(1,t) = 0 & \text{in } \ (0,T], \\ v(x,0;s) = s\sin(2\pi x) & \text{in } \ (0,1). \end{cases}$$

We use separation of variables

$$\begin{aligned} &v(x,t) = e^{-k\lambda^2 t}(A\cos\lambda x + B\sin\lambda x) \\ &v(0,t) = 0 \implies A = 0 \\ &v(1,t) = 0 \implies \lambda = n\pi. \end{aligned}$$

We write the general solution

$$\begin{aligned} v(x,t;s) &= \sum_{n=0}^{\infty} B_n e^{-kn^2\pi^2 t}\sin(n\pi x) \\ v(x,0;s) &= s\sin(2\pi x) \implies B_1 = 0 \ \text{ and } \ B_2 = s \end{aligned}$$

Finally,

$$v(x,t-s;s) = se^{-k4\pi^2(t-s)}\sin(2\pi x).$$

Let $\alpha = 4k\pi^2$ then we proceed with the method

$$u(x,t) = \int_0^t se^{-\alpha(t-s)} sin(2\pi x)\ ds$$

Let $u = -\alpha t + \alpha s \implies du = \alpha ds$ and $s = (u + \alpha t/\alpha)$, thus

$$\begin{aligned} &= \frac{\sin(2\pi x)}{\alpha^2}\left(\int_{-\alpha t}^{0} ue^u\ du + \int_{-\alpha t}^{0} e^u \alpha t\ du\right) \\ &= \frac{(2\pi x)}{\alpha^2}\left[e^u u\Big|_{-\alpha t}^{0} - \int_{-\alpha t}^{0} e^u\ du + \int_{-\alpha t}^{0} e^u\alpha t\ du\right] \\ u(x,t) &= \frac{\sin(2\pi x)(-1 + e^{-\alpha t} + \alpha t)}{\alpha^2}, \end{aligned}$$

for $\alpha = 4k\pi^2$.

Question 2

Let $u(x,t) = v(x,t) + w(x,t)$ and let $w(x,t) = A(t)x + B(t)$ be a solution. The function that satisfies the given boundary conditions for $w(x,t)$ is $(t^2-1)x+1$. The $v(x,t) = u(x,t) - w(x,t)$ solves

$$\begin{cases} v_t - kv_{xx} = u_x - ku_{xx} - ((t^2-1)x+1)_t - ((t^2-1)x+1)_{xx} = t\sin(2\pi x) & \text{in } \ 1_T, \\ v(0,t) = v(1,t) = 1 - 1 = 0 & \text{in } \ (0,T], \\ v(x,0) = u(x,0) - w(x,0) = 1 - x - (-x+1) = 0 & \text{in } \ (0,1). \end{cases}$$

Then we apply Duhamel's method for $v'(x,t;s)$, the result of this particular situation is already computed in Question 1 :

$$\begin{aligned} v'(x,t-s;s) &= se^{-k4\pi^2(t-s)}\sin(2\pi x) \\ \implies v(x,t) &= \int_0^t se^{-k4\pi^2(t-s)}\sin(2\pi x)\ ds \\ &= \frac{\sin(2\pi x)(-1+e^{-\alpha t}+\alpha t)}{\alpha^2}, \end{aligned}$$

And so finally,

$$u(x,t) = v(x,t) + w(x,t) = \frac{\sin(2\pi x)(-1e^{-\alpha t}+\alpha t)}{\alpha^2} + (t^2-1)x + 1.$$

Question 3

Let $u(x,t) = A(t)x + B(t)$, then from the BVP's boundary condition, $A(t) = t^2$. We then can write $B(t) = u(x,t) - t^2(x)$. Then $B(t)$ solves

$$\begin{cases} B_t - B_{xx} = u_t - u_{xx} - (t^2x)_t - (t^2x)_{xx} = -2tx & \text{in } \ (0,\infty)\times(0,T], \\ B(x,0) = 0 & \text{in } \ (0,\infty), \\ B_x(0,t) = t^2 - t^2 = 0 & \text{in } \ (0,T]. \end{cases}$$

By Duhamel's method, we let v solve

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \ (0,\infty)\times(0,T], \\ v(x,0;s) = -2sx & \text{in } \ (0,\infty), \\ v_x(0,t) = 0 & \text{in } \ (0,T]. \end{cases}$$

We can solve this for the half-line using the reflection method. Neumann boundaries so we take g_{even}. Let

$$g_{\text{even}}(x) = \begin{cases} -2sx & , \quad x \geq 0 \\ 2sx & , \quad x \leq 0 \end{cases}$$

As it was shown in Weekly worksheet 3, for even function

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}} \right) g(y)\ dy,$$

therefore in our case ,

$$v(x,t-s;s) = \frac{-2s}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left(e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}} \right) y\ dy,$$

Since $u(x,t) = B(t) + t^2(x)$, it follows that

$$u(x,t) = \int_0^t \frac{-2s}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left(e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}} \right) y\ dy\ ds + t^2 x.$$