Convex Optimization: Homework #2

Due on October 23, 2017 at 11:59pm

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Problem 1

Textbook Exercises 3.2

Solution

According to these curves, we can get:

1. these sublevel sets are convex $\Longrightarrow f$ could be convex
2. these superlevel sets are not convex $\Longrightarrow f$ is not concave or quasiconcave

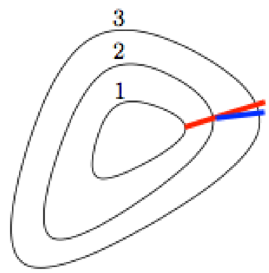

Figure 1:

As figure 1 shows, take two points at $f(x) = 1$ $f(x) = 2$ and draw a line. From $f(x) = 2$ to $f(x) = 3$, the relative below line is shorter. Thus, f is not convex.
For second level sets, we can see that the superlevel sets are convex but sublevel sets are not convex. Thus f could be concave and quasiconcave but could not convex.

Problem 2

Textbook exercise 3.3

Solution

From $g(f(x) = x$ we can get

$$g'(f(x)) = 1/f'(x), g''(f(x)) = -\frac{f''(x)}{f'(x)^3}$$

In addition, $\mathbf{dom}g$ is convex. Thus g is concave.

Problem 3

Textbook exercise 3.16

Solution

(a)

- e^x is convex. So $f(x)$ is convex
- and therefore it is quasiconvex.
- $-f(x) = 1 - e^x$ is not convex. Thus f is not concave.
- it is also qusiconcave because all of its super level sets are convex.

(b)

- The Hessian of f is
$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
Thus f is not convex or concave.
- its superlevel sets $\{(x_1, x_2) \in \mathbf{dom} f | x_1 x_2 \geq \alpha\}, \forall \alpha \in \mathbb{R}^2_{++}$ are convex. Thus f is quasiconcave.
- its sublevel sets are not convex. Thus f is not quasiconvex.

(c)

- The Hessian of f is
$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/x_1^2 & 1/x_1 x_2 \\ 1/x_1 x_2 & 2/x_2^2 \end{bmatrix}$$
where $x_1, x_2 \in \mathbb{R}_{++}$. Thus $\nabla^2 f(x) \succeq 0$ and f is convex and quasiconvex.
- its superlevel sets are not convex. Thus f is not quasiconcave.
- $-f$ is not convex. Thus f is not concave.

(d)

- The Hessian of f is
$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$
which is not positive semidefinite or negative semidefinite. Thus f is not convex or concave.
- its superlevel sets and sublevel sets are $\{(x_1, x_2) \in \mathbf{dom} f | x_1/x_2 \geq \alpha\}$ and $\{(x_1, x_2) \in \mathbf{dom} f | x_1/x_2 \leq \alpha\}$. Both are halfspaces and therefore convex. Thus f is quasiconvex and quasiconcave.

(e)

- The Hessian of f is
$$\nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} \succeq 0.$$
Thus f is convex and quasiconvex.
- obviously $-f$ is not convex and its superlevels sets are not convex. Thus f is not concave or quasiconcave.

Problem 3 continued on next page...

(f)

- The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} \alpha\,(\alpha-1)\,x_1^{\alpha-2}x_2^{1-\alpha} & \alpha\,(1-\alpha)\,x_1^{1-\alpha}x_2^{-\alpha} \\ \alpha\,(1-\alpha)\,x_1^{1-\alpha}x_2^{-\alpha} & -\alpha\,(1-\alpha)\,x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix} \preceq 0$$

  Thus f is concave and quasiconcave.

- f is not convex or quasiconvex.

Problem 4

Textbook exercise 3.24 a-f

Solution

(a)

$$\mathbf{E}(x) = \sum_{x=a_i}^{a_n} x p_i = \sum_{i=1}^{n} a_i p_i$$

is a affine function and therefore it is convex, quasiconvex, concave, and quasiconcave.

(b)

$$\mathbf{prob}(x \geq \alpha) = \sum_{i=j}^{n} a_i p_i,$$

where

$$a_j = min\{a_i \geq \alpha | i = 1, 2, \ldots, n\}$$

Similar to (a), it is also convex, quasiconvex, concave and quasiconcave.

(c)

$$\mathbf{prob}(\alpha \leq x \leq \beta) = \sum_{i=j}^{k} a_i p_i$$

where

$$a_j = min\{a_i \geq \alpha | i = 1, 2, \ldots, n\}, a_k = max\{a_i \leq \beta | i = 1, 2, \ldots, n\}$$

Similar to (a)(b), it is also convex, quasiconvex, concae, and quasiconcave.

(d) $\sum_{i=1}^{n} p_i \log p_i$ is convex and quasiconvex because negative entropy is convex and quasiconvex. And its superlevel sets are not convex. Thus it is not quasiconcave or concave.

(e)

$$var(x) = \sum_{i=1}^{n} a_i^2 p_i + \left(\sum_{i=1}^{n} a_i p_i \right)^2$$

is a quadratic function of p and therefore it's concave and quasiconcave.

(f)Its superlevel sets and sublevel sets are convex. Thus **quartile**(x) is quasiconvex and quasiconcave. And it is not continuous and therefore it is not convex or concave.