Convex Optimization: Homework #4

Due on November 20, 2017 at 11:59pm

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Problem 1

Textbook Exercises 5.1

Solution

(a) From the constrains, we can get the feasible set:

$$\{x|\, 2 \leq x \leq 4\}$$

And the optimal value:

$$p^* = 5 \text{ when } x^* = 2$$

(b)

$$\begin{aligned}
g(\lambda) &= \inf_x (x^2 + 1 + \lambda(x-2)(x-4)) \\
&= \inf_x ((\lambda+1)x^2 - 6\lambda x + 8\lambda + 1) \\
&= (\lambda+1)\left(\frac{3\lambda}{\lambda+1}\right)^2 - 6\lambda\left(\frac{3\lambda}{\lambda+1}\right) + 8\lambda + 1 \\
&= \begin{cases} \frac{9\lambda - \lambda^2 + 1}{\lambda+1} & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}
\end{aligned}$$

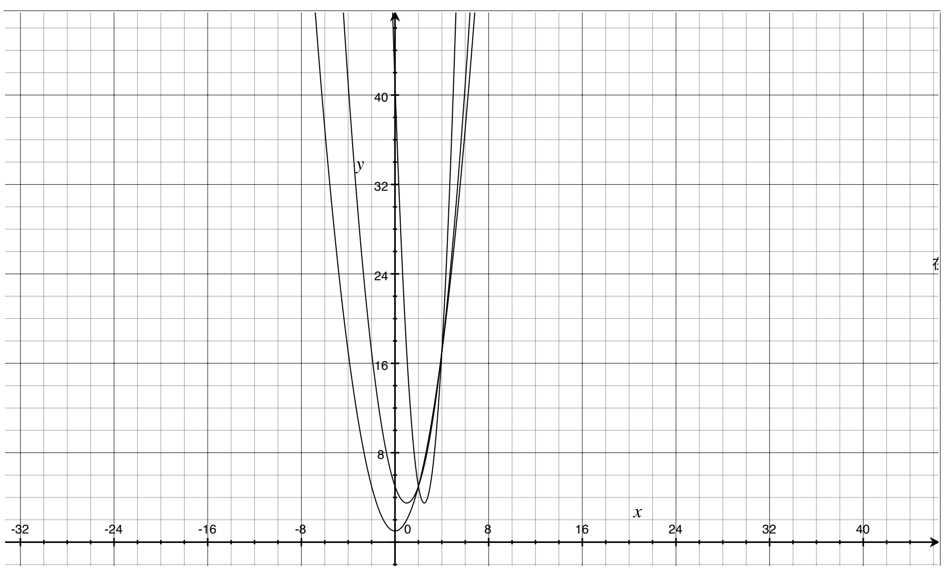

Figure 1: $L(x, \lambda)$ versus x with positive λ

(c) The Lagrange dual problem:

$$\begin{aligned}
\text{maximize} \quad & \frac{9\lambda - \lambda^2 + 1}{\lambda + 1} \\
\text{subject to} \quad & \lambda \geq 0
\end{aligned}$$

It is a concave maximization problem and strong duality holds for it. And the optimal value:

$$d^* = 5 \text{ when } \lambda = 2$$

Problem 1 continued on next page...

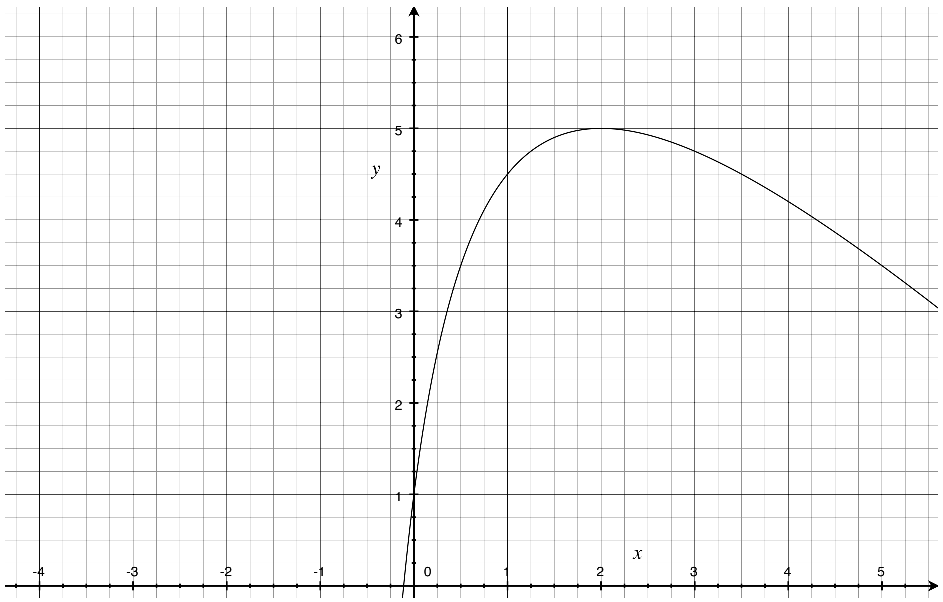

Figure 2: $g(\lambda)$ versus λ

(d) The feasible set is empty when $u < -1$. When $u \geq -1$, the feasible set is $[3 - \sqrt{1+u}, 3 + \sqrt{1+u}]$. And the $P^* = 11 + u - 6\sqrt{1+u}$ when $-1 \leq u \leq 8$, $P^* = 1$ when $u > 8$. Thus

$$p^* = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u > 8 \end{cases}$$

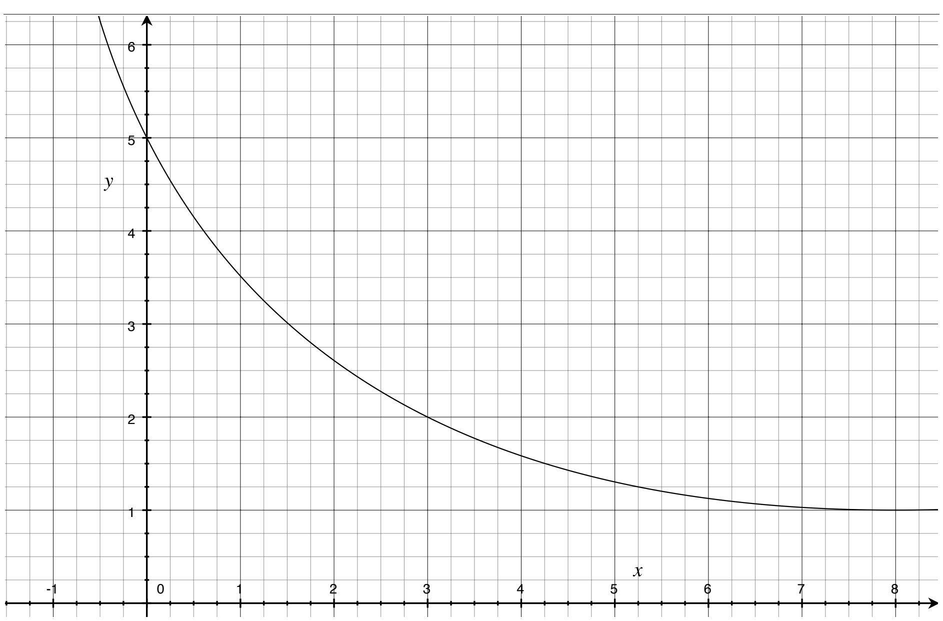

Figure 3: $p^*(u)$ versus u

Problem 2

Textbook exercise 5.13

Solution

(a) The Lagrange dual function is:

$$\begin{aligned} g(\lambda,\nu) &= \inf_x \left(c^T x + \lambda^T(Ax-b) + \sum_i \nu_i(x_i - x_i^2) \right) \\ &= \inf_x \left(c^T x + \lambda^T(Ax-b) + \nu^T x - x^T diag(\nu)x \right) \\ &= \inf_x \left(\sum_i^n ((c_i + a_i^T\lambda_i + \nu_i)x_i + \nu_i x_i^2) - \lambda^T b \right) \\ &= \begin{cases} \sum_i^n \frac{-(c_i + a_i^T\lambda_i - \nu_i)^2}{4\nu_i} - \lambda^T b & \nu_i \geq 0 \\ -\infty & o.w. \end{cases} \end{aligned}$$

And the dual problem is:

$$\begin{aligned} \text{maximize} \quad & \sum_i^n \frac{-(c_i + a_i^T\lambda_i - \nu_i)^2}{4\nu_i} - \lambda^T b \\ \text{subject to} \quad & \lambda \succeq 0,\ \nu \succeq 0 \end{aligned}$$

(b)The dual function of the LP relaxation is:

$$\begin{aligned} g(x,\lambda_1,\lambda_2,\lambda_3) &= \inf_x (c^T x + \lambda_1^T(Ax-b) - \lambda_2^T x + \lambda_3^T(x - \mathbf{1})) \\ &= \begin{cases} -\lambda_1^T b + \lambda_3^T \mathbf{1} & c + A^T\lambda_1 - \lambda_2 + \lambda_3 = 0 \\ -\infty & o.w. \end{cases} \end{aligned}$$

And the dual problem is:

$$\begin{aligned} \text{maximize} \quad & -\lambda_1^T b + \lambda_3^T \mathbf{1} \\ \text{subject to} \quad & c + A^T\lambda_1 - \lambda_2 + \lambda_3 = 0 \\ & \lambda_1 \succeq 0,\ \lambda_2 \succeq 0,\ \lambda_3 \succeq 0 \end{aligned}$$

Thus, they can give the same lower bound.

Problem 3

Textbook exercise 5.21

Solution

(a)It is a convex optimization problem. And the feasiable set is $\{(x, y) \mid x = 0,\, y > 0\}$. Thus the optimal value $p^* = 1$.
(b) The Lagrange dual function is:

$$\begin{aligned} g(\lambda) &= \inf_{x \in \mathbf{D}} (e^{-x} + \lambda x^2/y) \\ &= \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0 \end{cases} \end{aligned}$$

And the dual problem:

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & \lambda \geq 0 \end{array}$$

$d^* = 0$ and the duality gap $p^* - d^* = 1$
(c)it is not satisfied.
(d)

$$p^* = \begin{cases} -\infty & u < 0 \\ 1 & u = 0 \\ 0 & u > 0 \end{cases}$$

d^* is achieved when $\lambda \geq 0$. So the global sensitivity inequality does not hold apparently.

Problem 4

Textbook exercise 5.27

Solution

(a) The KKT conditions for this problem is:

$$G^T x^* - h = 0 \tag{1}$$

$$2A^T(Ax^* - b) + G^T\nu^* = 0 \tag{2}$$

The Lagrangian is

$$\begin{aligned} L(x,\nu) &= \|Ax - b\|_2^2 + \nu^T(Gx - h) \\ &= x^T A^T A x + (G^T\nu - 2A^T b)^T x - 2\nu^T h \end{aligned} \tag{3}$$

And the dual function is

$$g(\nu) = \inf_x (L(x,\nu)) \tag{4}$$

which is a object function of a unconstrained least-squares problem. Let $\nabla L(x,\nu) = 0$, we can get $x_1 = -\frac{1}{2}(A^T A)^{-1}(G^T\nu - 2A^T b)$. Thus, the dual function is equivalent to

$$g(\nu) = L(x_1, \nu) \tag{5}$$

From equation 1, we can get $x^* = (A^T A)^{-1}(A^T b - \frac{1}{2}G^T\nu^*)$, where ν^* satisfies $G^T(A^T A)^{-1}(A^T b - \frac{1}{2}G^T\nu^*) = h$

Problem 5

Textbook exercise 5.32

Solution

Let function $g(x, u, \nu) = f_0(x)$ and **dom** $g = \{x, u, \nu \mid f_i(x) \leq u_i,\ i = 1, \ldots, m,\ Ax - b = \nu\}$. Apparently the $g(x, u, \nu)$ is convex on its domain. Therefore, the minimization of convex function $p^*(u, \nu) = \inf_x g(x, u, \nu)$ is convex on its domain.

Problem 6

Textbook exercise 5.39

Solution

(a) If X is feasible, X can be expressed as xx^T because of **rank** $X = 1$. $X_{ii} = 1$ is equivalent to $x_i \in \{-1, 1\}$ because $(xx^T)_{ij} = x_i x_j$. $\mathbf{tr}\{WX\} = x^T W x$.
If $x_i^2 = 1, i = 1, \ldots, n$, let $X = xx^T$ and we can get $X_{ii} = x_i^2$, **rank** $X = 1$.
(b)Its optimal value gives a lower bound because we minimize the object function on a relaxed constrains. It's primal optimal when the **rank** $X^* = 1$.
(c)The SDP relaxation is equivalent to

$$\begin{array}{ll} \text{minimize} & 1^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \geq 0 \end{array}$$

Its dual function is

$$\begin{aligned} g(\nu, X) &= \inf_x (1^T - \mathbf{tr}(X(W + \mathbf{diag}(\nu))) \\ &= \inf_x (-\mathbf{tr}(XW) + \sum_{i=1}^n \nu_i (1 - X_{ii})) \\ &= \begin{cases} -\mathbf{tr}(XW) & X_{ii} = 1, i = 1, \ldots, n \\ -\infty & o.w. \end{cases} \end{aligned}$$

Thus, its dual problem is

$$\begin{array}{ll} \text{maximize} & -\mathbf{tr}(XW) \\ \text{subject to} & X \succeq 0 \\ & X_{ii} = 1, \ i = 1, \ldots, n \end{array}$$

which is equivalent to the (5.114). The lower bounds found by it is better than those found by (5.114).