Convex Optimization: Homework #1

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Problem 1

Textbook Exercises 2.1

Proof. We use induction on k as the hint said. Suppose that the proposition is true when $k = m - 1 \in \mathbb{R}$ and $x_1, \cdots, x_m \in C$ and $_1 + \cdots + \theta_m = 1$ with $\theta_i \geq 0$. Then we have:

$$y = \theta_1 x_1 + \cdots + \theta_m x_m \tag{1.1}$$

we always can find a θ_r in $\theta_1, \cdots, \theta_m$ with $\theta_r \neq 1$. So we can get:

$$y = \theta_r x_r + (1-\theta_r)(\frac{\theta_1 x_1}{1-\theta_r} + \cdots + \frac{\theta_{r-1} x_{r-1}}{1-\theta_r} + \frac{\theta_{r+1} x_{r+1}}{1-\theta_r} + \cdots + \frac{\theta_m x_m}{1-\theta_r}) \tag{1.2}$$

And

$$\frac{\theta_1}{1-\theta_r} + \cdots + \frac{\theta_{r-1}}{1-\theta_r} + \frac{\theta_{r+1}}{1-\theta_r} + \cdots + \frac{\theta_m}{1-\theta_r} = \frac{1-\theta_r}{1-\theta_r} = 1$$

Also we have that the proposition is true when $k = m - 1$.

$$(\frac{\theta_1}{1-\theta_r} x_1 + \cdots + \frac{\theta_{r-1}}{1-\theta_r} x_{r-1} + \frac{\theta_{r+1}}{1-\theta_r} x_{r+1} + \cdots + \frac{\theta_m}{1-\theta_r} x_m) \in C$$

Applying the definition of convex set on equation 1.2, we can get $y \in C$. Thus, we can conclude that the proposition is true for arbitrary k. Proof is complete. $\square$

Problem 2

Textbook exercise 2.8

Solution

(a)Not solved yet.

(b) S is defined by linear equalities and inequalities. So it is a polyhedra and can be expressed as S where $A = -1$, $b = 0$, $F = \begin{pmatrix} \mathbf{1}^T \\ V_1^T \\ V_2^T \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ b_1 \\ b_2 \end{pmatrix}$, $V_1 = (a_1, \cdots, a_n)$, $V_2 = (a_1^2, \cdots, a_n^2)$

(c) $x^T y \leq 1 \Longrightarrow \|x\|_2 \cdot \|y\|_2 \leq 1$, applying $\|y\|_2 = 1$ we get $\|x\| \leq 1$. And $x \succeq 0$, S is the intersection of $\{x \mid \|x\|_2 \leq 1\}$ and $\mathbb{R}_+^n$. Thus it is not a polyhedra.

(d)We have $x \succeq 0$ and $\sum_{i=1}^n |y_i| = 1$. So we can proof that:

$$max\left(x^T y\right) = max\left(x_i\right), i = 1, \cdots, n$$

Thus, $x^T y \leq 1 \Longleftrightarrow max\left(x^T y\right) = max\left(x_i\right) \leq 1 \Longleftrightarrow |x_i| \leq 1$, means that S is a polyhedra.

$$x_i \geq 0$$

$$x_i \leq 1$$

$$i = 1, \cdots, n$$

Problem 3

Textbook exercise 2.12 a-e

Solution

(a)It is the solution set of $\alpha \le a^T x \le \beta$. Thus it is a polyhedra and convex.

(b)the same as (a), so it is convex.

(c)the same as (a)(b), so it is convex.

(d)

$$\begin{aligned}\|x - x_0\|_2 \le \|x - y\|_2 &\Longleftrightarrow (x - x_0)^T (x - x_0) \le (x - y)^T (x - y) \\ &\Longleftrightarrow x^T x - 2x_0^T x + x_0^T x_0 \le x^T x - 2x_i^T x + x_i^T x_i \\ &\Longleftrightarrow 2 (x_i - x_0) x \le x_i^T x_i - x_0^T x_0\end{aligned}$$

It is a halfspace given a fixed $y \in \mathbb{C}$. So the set is the intersection of multi-halfspace and thus it is convex.

(e)It is not convex.

Problem 4

Textbook exercise 2.15

Solution

(a)The set is solution set of two linear inequalities so it is convex.

(b)We set a_r is the minimum one that great than α.

$$\mathbf{prob}\,(x > \alpha) \leq \beta \Longleftrightarrow \sum_{i=r}^{n} p_i \leq \beta$$

Thus, it is convex.

(c)

$$\begin{aligned}\mathbf{E}|x^3| \leq \alpha\mathbf{E}|x| &\Longleftrightarrow \sum_{i=1}^{n} |a_i^3|p_i \leq \alpha \sum_{i=1}^{n} |a_i|p_i \\ &\Longleftrightarrow \sum_{i=1}^{n} \left(|a_i^3| - \alpha|a_i|\right) p_i \leq 0\end{aligned}$$

Thus, the set is convex.

(d)

$$\mathbf{E}x^2 \leq \alpha \Longleftrightarrow \sum_{i=1}^{n} x^2 p_i \leq \alpha$$

Also convex.

(e) similar with (d), this condition also define a convex set.

(f)

$$\mathbf{var}\,(x) \leq \alpha \Longleftrightarrow \sum_{i=1}^{n} a_i^2 p_i - \left(\sum_{i=1}^{n} a_i p_i\right)^2 \leq \alpha$$

We can take $n = 2, a_1 = 1, a_2 = 0$.The condition becomes:

$$p_1 - p_1^2 \leq \alpha$$

If we take $\alpha = \frac{1}{6}$, $p^1 = (1,0)$, $p^2 = (0,1)$ and $p^3 = \frac{1}{2}p^1 + \frac{1}{2}p^1 = \left(\frac{1}{2}, \frac{1}{2}\right)$, we can get:

$$p_1^1 - \left(p_1^1\right)^2 = 0 \leq \alpha = \frac{1}{6}$$

$$p_1^2 - \left(p_1^2\right)^2 = 0 \leq \alpha = \frac{1}{6}$$

$$p_1^3 - \left(p_1^3\right)^3 = \frac{1}{4} \geq \alpha = \frac{1}{6}, p^3 \notin setP$$

Thus,it is not convex.

(g)

Problem 4 continued on next page. . .

$$\mathbf{var}\,(x) \le \alpha \Longleftrightarrow \sum_{i=1}^{n} a_i^2 p_i - \left(\sum_{i=1}^{n} a_i p_i\right)^2 \ge \alpha \tag{4.1}$$

We can take p^1, p^2 that satisfy equation 4.1, i.e:

$$\sum_{i=1}^{n} a_i^2 p_i^1 - \left(\sum_{i=1}^{n} a_i p_i^1\right)^2 \ge \alpha$$

$$\sum_{i=1}^{n} a_i^2 p_i^2 - \left(\sum_{i=1}^{n} a_i p_i^2\right)^2 \ge \alpha$$

And we take $p^3 = \theta_1 p^1 + \theta_2 p^2$, where $\theta_1 + \theta_2 = 1$ and $\theta_1, \theta_2 \in [0, 1]$.

$$\begin{aligned}
\sum_{i=1}^{n} a_i^2 p_i^3 - \left(\sum_{i=1}^{n} a_i p_i^3\right)^2 &= \theta_1 \sum_{i=1}^{n} a_i^2 p_i^1 + \theta_2 \sum_{i=1}^{n} a_i^2 p_i^2 + \left(\theta_1 \sum_{i=1}^{n} a_i p_i^1 + \theta_2 \sum_{i=1}^{n} a_i p_i^2\right)^2 \\
&\ge \theta_1 \sum_{i=1}^{n} a_i^2 p_i^1 + \theta_2 \sum_{i=1}^{n} a_i^2 p_i^2 - \theta_1 \left(\sum_{i=1}^{n} a_i p_i^1\right)^2 - \theta_2 \left(\sum_{i=1}^{n} a_i p_i^2\right)^2 \\
&\ge \theta_1 \alpha + \theta_2 \alpha \\
&= \alpha
\end{aligned}$$

Thus, $p^3 \in \mathbf{P}$. The set is convex.

(h)$\mathbf{quartile}\,(x) \ge \alpha$ is equivalent to:

$$\mathbf{prob}\,(x \le \beta) \ge 0.25, \text{for all } \beta \ge \alpha \Longleftrightarrow \mathbf{prob}\,(x \le \beta) \le 0.25, \text{for all } \beta < \alpha$$

We can define $a_k = max\,\{a_i | a_i < \alpha\}$,then:

$$\begin{aligned}
\mathbf{prob}\,(x \le \beta) \le 0.25, \text{for all } \beta \ge \alpha &\Longleftrightarrow \mathbf{prob}\,(x \le \beta_{max}) \le 0.25 \\
&\Longleftrightarrow \mathbf{prob}\,(x \le a_k) \le 0.25 \\
&\Longleftrightarrow \sum_{i=1}^{k} p_i \le 0.25
\end{aligned}$$

It is a halfspace and convex.

(i)Similar with (h), this condition is equivalent to:

$$\mathbf{prob}\,(x \le \alpha) \ge 0.25 \Longleftrightarrow \sum_{i=1}^{k} p_i \ge 0.25$$