Asymptotics

① Pointwise convergence

$f_n \to f$ ptwise in D if $\forall x \in D$

$\lim_{n\to\infty} |f_n - f| = 0$

$f_n \to f$ ptwise in D if $\forall x \in D$ & $\forall \varepsilon > 0$

$\exists N$ s.t. if $n \geq N$ then $|f_n - f| < \varepsilon$

② Asymptotic convergence

Fix $N \geq 1$. $f(x) \sim \sum_{n=0}^{N} a_n \phi_n(x)$ where $\{\phi_n\}$ are well-ordered

i.e. ($\phi_{n+1} = o(\phi_n)$) as $x \to x_0$ if for $m \leq N$

$$\left| \frac{f(x) - \sum_{n=0}^{M} a_n \phi_n(x)}{\phi_M(x)} \right| \to 0 \text{ as } x \to x_0$$

in terms of functions : $f \sim g$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f}{g} = 1$

③ $\exists$ a function f & a set of functions $\{f_n(x)\}$

$f_n(x) \to f(x)$ ptwise

if $\forall x \in D$ $\lim_{n\to\infty} f_n(x) = f(x)$

$f_n(x) \to f(x)$ asymptotically as $x \to x^*$

if $\forall n$, $\lim_{x \to x^*} f_n(x) = f(x)$

④ uniform : $\forall \varepsilon > 0$ $\exists N$. s.t. $\forall n \geq N, x \in D \Rightarrow (\forall x \in D)$

$|f_n(x) - f(x)| < \varepsilon$

ptwise : $\forall x \in D$ $\forall \varepsilon > 0$ $\exists N$. s.t. $\forall n \geq N$

$|f_n(x) - f(x)| < \varepsilon$

Scanned with CamScanner

$$a_N = \lim_{x \to x_0} \frac{y(x) - \sum_{n=0}^{N-1} a_n (x-x_0)^{\alpha n}}{(x-x_0)^{\alpha N}}$$

proof: $a_0 = \lim_{x \to x_0} y(x)$ $\quad a_1 = \lim_{x \to x_0} \frac{y(x) - a_0}{(x-x_0)^{\alpha}}$

⑤ Uniqueness: $f \sim \sum a_n f_n(x)$, $f \sim \sum b_n f_n(x)$ $\Rightarrow a_n = b_n$.

⑥ Nonuniqueness: $\frac{1}{1-x} \sim \sum x^n$ $\qquad \frac{1}{1+x} \sim \boxed{\sum_{n=1} (-1)^{n+1} x^{-n}}$

$\frac{1}{1-x} \sim \sum (x+1) x^{2n}$. $\qquad \frac{1}{1+x} \sim \boxed{\frac{1+e^{-x}}{1+x}}$.

⑦ Arithmatic: ① $\alpha f + \beta g \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n$.

prove by definition.

② $fg \sim f_1 g_1$ if $f_1 \sim f$ $g_1 \sim g$.

use $\lim_{x \to x_0} \frac{f}{g} = 1$ if $f \sim g$

to prove.

③ $f_1 \sim f$ $\cancel{g_1} \sim g$

$f_1 + g_1 \sim f+g$.

Goal: $\lim_{x \to x_0} \frac{f_1 + g_1}{f+g} = 1$.

④ $f \sim g$ $g \sim h$ $\Rightarrow f \sim h$.

⑧ AC NOT PC:

$$f(x) = \frac{e^{-x}}{x} - \int_x^{\infty} \frac{e^{-t}}{t^2} dt$$

$$f(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right) + \int_x^{\infty} \frac{(-1)^n n! e^{-t}}{t^{n+1}} dt$$

Is $R_n(x) = o(\phi_n(x))$ where $\phi_n(x) = \frac{e^{-x}}{x^n}$ as $x \to \infty$

$$\lim_{x \to \infty} \left| \frac{R_n(x)}{\phi_n(x)} \right| = \lim_{x \to \infty} \left| \frac{\int_x^{\infty} \frac{(-1)^n n! e^{-t}}{t^{n+1}} dt}{\frac{e^{-x}}{x^n}} \right| \le \lim_{x \to \infty} \frac{n! x^n}{e^{-x}} \int_x^{\infty} \left| \frac{e^{-t}}{t^{n+1}} \right| dt.$$

$$\le \lim_{x \to \infty} \frac{n! x^n}{e^{-x} x^{n+1}} \int_x^{\infty} e^{-t} dt = \lim_{x \to \infty} \left| \frac{n!}{x} \right| = 0 \Rightarrow AC$$

Scanned with CamScanner

⑧ $f(x) \sim \sum_{n=0}^{N-1} \frac{(-1)^n e^{-x} n!}{x^{n+1}}$ as $x \to \infty$.

Do ratio test on series $\Rightarrow \infty \nless 1 \Rightarrow$ not PC.

⑨ uniform AC : $f(x,t) \sim g(x,t)$ uniformly as $x \to x_0$
for $t \in [a,b]$ if $\lim_{x \to x_0} \frac{f}{g} = 1 \quad \forall t \in [a,b]$

⑩. Invertible ? No ! $\Rightarrow$ not unique. = not 1 to 1.

⑪ Bijective ? No ! 1-1 $\neq$ no. not unique.
onto. ✓ Yes. they are asym to themselves

injective : 1-1 : no. Because one function can have many asymptotic series ✓

surjective : onto : $\exists$ at least one asym for every function ✓
(themselves).

⑫ Differentiation : NO :

Proof : Let $g = f + e^{-\frac{1}{(x-x_0)^2}} \sin(e^{\frac{1}{(x-x_0)^2}})$

$f \sim g \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n$.

BUT $f' \nsim g' \Rightarrow g' = f' - 2(x-x_0)^{-3} \cos(e^{\frac{1}{(x-x_0)^2}})$
$+ 2(x-x_0)^{-3} e^{\frac{-1}{(x-x_0)^2}} \sin(e^{\frac{1}{(x-x_0)^2}})$

In complex However : if f is analytic

$\theta_1 \le \arg(z-z_0) \le \theta_2 \quad 0 < |z-z_0| < R$

$f \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$f' \sim \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$

Scanned with CamScanner

Integration

13. ① $f \sim g$ as $x \to \infty$ $\int_a^\infty g(t)\,dt = \infty$.

$\Rightarrow \int_a^x f \sim \int_a^x g$

② Is $\int_a^b g(x,t)\,dt \sim \int_a^b g_0(t)$

$\begin{cases} ① \int_a^b g_0(t) \text{ has to be nonzero \& finite} \\ ② g(x,t) \xrightarrow{UA} g_0(t) \text{ as } x \to x_0 \end{cases}$

Then True

③ $\int_a^b g(x,t)\,dt \sim \int_a^b g_0(x,t)\,dt$

~~False~~ : $x\sin t + 1 \overset{UA}{\sim} x\sin t$

$x \to \infty$.

Proof : WTS : $\int_{x_0}^x f(t)\,dt \sim \sum_{n=0}^\infty \frac{a_n}{n+1}(x-x_0)^{n+1}$

Def : $|f - \sum_{n=0}^N a_n(x-x_0)^n| << (x-x_0)^N$

So : $\forall \varepsilon > 0 \ \exists R$ s.t. $|x-x_0| \le R$

$\Rightarrow |f - \sum_{n=0}^N a_n(x-x_0)^n| \le \varepsilon |x-x_0|^N$

Therefore : $\left|\int_{x_0}^x [f(t) - \sum_{n=0}^N a_n(t-x_0)^n]\,dt\right| \le \int_{x_0}^x |f(t) - \sum_{n=0}^N a_n(t-x_0)^n|\,dt$

$\le \varepsilon \int_{x_0}^x |t-x_0|^N\,dt = \frac{\varepsilon}{N+1}|x-x_0|^{N+1}$, $|x-x_0| \le R$

Hence : $\left| \frac{\int_{x_0}^x f(t)\,dt - \sum_{n=0}^N [a_n/(n+1)](x-x_0)^{n+1}}{(x-x_0)^{N+1}} \right| \le \frac{\varepsilon}{N+1}$

Since ε is arbitrary :

$\left|\int_{x_0}^x f(t)\,dt - \sum_{n=0}^N \frac{a_n}{n+1}(x-x_0)^{n+1}\right| << (x-x_0)^{N+1}$

as $x \to x_0$

Scanned with CamScanner

(14) Function composition.

① If it $\forall A$, then yes.

② If $f \sim g \Rightarrow e^f \sim e^g \Rightarrow \log f \sim \log g$.

But, $f = x + x^2$, $g = x^2$ $\quad f \sim g$ as $x \to \infty$.

but $\lim_{x \to \infty} \frac{e^{x+x^2}}{e^{x^2}} = e^x \to \infty$.

③ But: $f(F(x)) \sim g(F(x))$ if $\lim_{x \to a} F(x) = x_0$.

$\lim_{x \to x_0} \frac{f}{g} = 1$. then $\lim_{x \to a} \frac{f(F(x))}{g(F(x))} = 1$

④ A weakly pseudo-regularly varying function is a function that preserves function asymptotic composition

Def of WPRV: $\lim_{a \to 1} \sup f^*(c) = 1$

$\Downarrow$

imply uniform convergence.

$\left\{ \begin{array}{l} \text{Karamata's representation theorem.} \\ \text{Slowly varying functions} \end{array} \right.$

Scanned with CamScanner

(15) ① If we have a Laurent series expression for an $\boxed{\text{essential singularity}}$ then we do not have a well-ordered family of functions. $\Rightarrow$ no AE

② If we have a pole point, we can still have a well-ordered functions. therefore the Laurent series will be an A.E

① $f = \sum_{n=-\infty}^{\infty} a_n z^n$ $\{z^n\}$ not well ordered $z^n = 0$

② $f = \sum_{n=-N}^{\infty} a_n z^n$ $\{z^n\}$ well ordered

$f \sim \sum_{n=-N}^{\infty} a_n z^n$
$z \to z_0$

(16) Distributive : $f(g+h) \sim f_1(g_1+h_1)$ (if : $f \sim f_1$, $g \sim g_1$, $h \sim h_1$)

Transitivity : $f \sim g$ $g \sim h$ $\Rightarrow f \sim h$

not provable → (???) Associativity : $(f+g)+h = f+(g+h)$

(17) $\frac{1}{x} + \frac{1}{x^3} \sim \frac{1}{x} - \frac{1}{x^3}$.

It is only unique at the $\boxed{\text{same order}}$

$\sin(\frac{1}{x}) \sim \frac{1}{x}$ but $\sin(\frac{1}{x}) \nsim \frac{2}{x}$.

$\sin(\frac{1}{x}) \sim \frac{1}{x} - \frac{1}{3!}\frac{1}{x^3}$.

(18) $f = \sum_{j=0}^{N} \frac{1}{x^j}$ as $N \to \infty$ do not converge

but as $x \to \infty$ $f(x) \sim 1 + \frac{1}{x}$.

So AC but not PC.

(19) $f = \cos x$ $g = 1 - \frac{x^2}{2} + \dots$

$f \sim g$ as $x \to 0$

$f \nsim g$ as $x \to a$

$f \nsim g$ as $x \to b \pm 0$. $\forall N$

Scanned with CamScanner