

#### UNIT 1

Introduction Number Systems and Conversion



#### This chapter in the book includes:

- Objectives Study Guide
- 1.1 Digital Systems and Switching Circuits
- 1.2 Number Systems and Conversion
- 1.3 Binary Arithmetic
- 1.4 Representation of Negative Numbers
- 1.5 Binary Codes Problems



## Learning Objectives

#### Introduction:

- To be able to explain the difference between analog and digital systems, as well as why digital systems are capable of greater accuracy
- To be able to understand the difference between combinational and sequential circuits
- To be able to explain why two-valued signals and binary numbers are commonly used in digital systems

## Learning Objectives

#### **Number Systems and Conversion:**

To be able to solve the following types of problems:

- Given a positive integer, fraction, or mixed number in any base (2 through 16); convert to any other base. Justify the procedure using a power series expansion for the number.
- Add, subtract, multiply, and divide positive binary numbers. Explain the addition and subtraction process in terms of carries and borrows.
- Write negative binary numbers in sign and magnitude, 1's complement, and 2's complement forms. Add signed binary numbers using 1's complement and 2's complement arithmetic. Justify the methods used and identify overflows.
- Represent a decimal number in binary-coded-decimal (BCD), 6-3-1-1 code, excess-3 code, etc. Given a set of weights, construct a weighted code.

#### **Digital and Analog Systems:**

- Digital systems are used extensively in computation and data processing, control systems, communications, and measurement.
- In a digital system, physical quantities can only assume discrete values.
- In an analog system, physical quantities or signals can vary continuously over a specified range.
- In many cases digital systems can be designed so that for a given input, the output is exactly correct.
- The output of an analog system might have an error ranging from a fraction of one percent to a few percents because these systems do not work with discrete quantities.

#### Design of digital systems:

- System design: breaking the overall system into subsystems and specifying the characteristics of each subsystem.
- \* Logic design: determining how to interconnect basic logic building blocks to perform a specific function.
- Circuit design: specifying the interconnection of specific hardware components to form a gate, flip-flop or other logic building block.

#### **Switching circuits:**

- A switching circuit has one or more inputs and one or more outputs which take on discrete values.
- In combinational circuits, output values of circuit only depend on the present values of inputs, not past.
- In sequential circuits, output values of circuit depend on both past and present input values. For this reason, sequential circuits are said to have "memory", because they must remember past sequence of inputs.

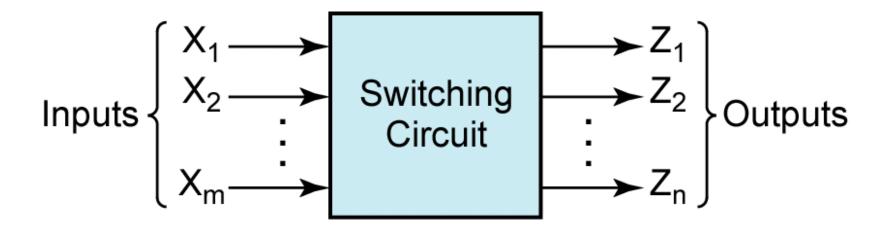


Figure 1-1: Switching circuit

#### **Building Blocks of Switching Circuits:**

- Basic building block of combinational circuits: logic gate
- Basic building block of sequential circuits: flip-flop
- Switching devices used in digital systems are usually two-state devices, meaning the output can only assume one of two discrete values, i.e. relays, diodes and transistors.
- Because the outputs of most switching devices assume only two different values, it is natural to use binary numbers internally in digital systems.

#### **Explanation of decimal and binary numbers:**

When we write decimal (base 10) numbers, we use a positional notation; each digit is multiplied by an appropriate power of 10 depending on its position in the number.

$$953.78_{10} = 9 \times 10^2 + 5 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} + 8 \times 10^{-2}$$

When we write binary (base 2) numbers, each digit is multiplied by an appropriate power of 2 on its position in the number.

$$1011.11_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2}$$
  
= 8 + 0 + 2 + 1 +  $\frac{1}{2}$  +  $\frac{1}{4}$  =  $11\frac{3}{4}$  =  $11.75_{10}$ 

## Power series expansion in any number system of base R:

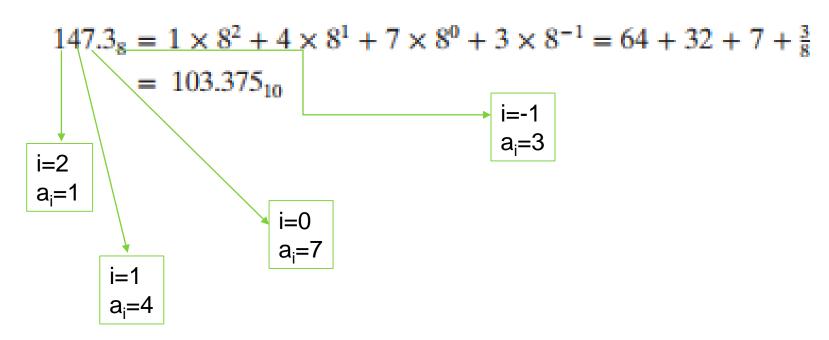
Any positive integer R (R > 1) can be chosen as the *radix* or *base* of a number system. If the base is R, then R digits  $(0,1,\ldots,R-1)$  are used. For example, if R = 8, then the required digits are 0,1,2,3,4,5,6, and 7. A number written in positional notation can be expanded in a power series in R. For example,

$$N = (a_4 a_3 a_2 a_1 a_0. a_{-1} a_{-2} a_{-3})_R$$
  
=  $a_4 \times R^4 + a_3 \times R^3 + a_2 \times R^2 + a_1 \times R^1 + a_0 \times R^0$   
+  $a_{-1} \times R^{-1} + a_{-2} \times R^{-2} + a_{-3} \times R^{-3}$ 

where  $a_i$  is the coefficient of  $R^i$  and  $0 \le a_i \le R - 1$ .

#### **Example:**

Using the previous slide, the following conversion can be made from base 8 to base 10.



#### Conversion to other bases:

The power series expansion can be used to convert to any base. For example, converting 147<sub>10</sub> to base 3 would be written as

$$147_{10} = 1 \times (101)^2 + (11) \times (101)^1 + (21) \times (101)^0$$

where all the numbers on the right-hand side are base 3 numbers. (*Note:* In base 3, 10 is 101, 7 is 21, etc.) To complete the conversion, base 3 arithmetic would be used.

#### Bases greater than 10:

For bases greater than 10, more than 10 symbols are needed to represent the digits.

In this case, letters are usually used to represent digits greater than 9. For example, in hexadecimal (base 16), A represents  $10_{10}$ , B represents  $11_{10}$ , C represents  $12_{10}$ , D represents  $13_{10}$ , E represents  $14_{10}$ , and E represents  $15_{10}$ . Thus,

$$A2F_{16} = 10 \times 16^2 + 2 \times 16^1 + 15 \times 16^0 = 2560 + 32 + 15 = 2607_{10}$$

## Conversion of a decimal integer to base R using the division method:

The base R equivalent of a decimal integer N is as follows (from previous slides):

$$N = (a_n a_{n-1} \cdot \cdot \cdot \cdot a_2 a_1 a_0)_R = a_n R^n + a_{n-1} R^{n-1} + \cdot \cdot \cdot + a_2 R^2 + a_1 R^1 + a_0$$

If we divide N by R, the remainder is  $a_0$ :

$$\frac{N}{R} = a_n R^{n-1} + a_{n-1} R^{n-2} + \dots + a_2 R^1 + a_1 = Q_1$$
, remainder  $a_0$ 

Then we divide the quotient  $Q_1$  by R:

$$\frac{Q_1}{R} = a_n R^{n-2} + a_{n-1} R^{n-3} + \dots + a_3 R^1 + a_2 = Q_2$$
, remainder  $a_1$ 

## Conversion of a decimal integer to base R using the division method (continued):

Next we divide  $Q_2$  by R:

$$\frac{Q_2}{R} = a_n R^{n-3} + a_{n-1} R^{n-4} + \dots + a_3 = Q_3$$
, remainder  $a_2$ 

This process is continued until we finally obtain  $a_n$ . Note that the remainder obtained at each division step is one of the desired digits and the least significant digit is obtained first.

#### **Example 1: Conversion of decimal integer to binary**

#### 

## Conversion of a decimal fraction F to base R using successive multiplications:

$$F = (a_{-1}a_{-2}a_{-3}\cdots a_{-m})_R = a_{-1}R^{-1} + a_{-2}R^{-2} + a_{-3}R^{-3} + \cdots + a_{-m}R^{-m}$$

Multiplying by R yields

$$FR = a_{-1} + a_{-2}R^{-1} + a_{-3}R^{-2} + \cdots + a_{-m}R^{-m+1} = a_{-1} + F_1$$

where  $F_1$  represents the fractional part of the result and  $a_{-1}$  is the integer part. Multiplying  $F_1$  by R yields

$$F_1R = a_{-2} + a_{-3}R^{-1} + \cdots + a_{-m}R^{-m+2} = a_{-2} + F_2$$

## Conversion of a decimal fraction F to base R using successive multiplications (continued):

Next, we multiply  $F_2$  by R:

$$F_2R = a_{-3} + \cdots + a_{-m}R^{-m+3} = a_{-3} + F_3$$

This process is continued until we have obtained a sufficient number of digits. Note that the integer part obtained at each step is one of the desired digits and the most significant digit is obtained first.

## Example 2: Conversion of a decimal fraction F to base R using successive multiplications

#### Example

Convert 0.625<sub>10</sub> to binary.

$$F = .625$$
  $F_1 = .250$   $F_2 = .500$   
 $\times 2$   $\times 2$   $\times 2$   
 $\frac{1.250}{(a_{-1} = 1)}$   $\frac{0.500}{(a_{-2} = 0)}$   $\frac{1.000}{(a_{-3} = 1)}$ 

 $.625_{10} = .101_2$ 

## Example 3: Conversion of a decimal fraction F to base R using successive multiplications

# Example Convert $0.7_{10}$ to binary. $\begin{array}{c} .7 \\ \underline{2} \\ (1).4 \\ \underline{2} \\ (0).8 \\ \underline{2} \\ (1).6 \\ \underline{2} \\ (1).2 \\ \underline{2} \\ (0).4 \\ \underline{2} \\ (0).4 \\ \underline{2} \\ (0).4 \\ \underline{2} \\ (0).4 \\ 0.7_{10} = 0.1 \\ 0110 \\ 0110 \\ 0110 \\ 0110 \\ 0.12 \end{array}$

## **Example 4: Conversion between two bases other than decimal**

Example	Convert 231.34 to base	e <b>7</b> .		
	2	$231.3_4 = 2$	× 16 + 3 ×	$4 + 1 + \frac{3}{4} = 45.75_{10}$
	7 <u>/45</u>		.75	
	7/6	rem. 3 rem. 6	$\frac{7}{(5).25}$	45.75 <sub>10</sub> = 63.5151 <sub>7</sub>
			7	,
			(1).75 7	
			(5).25	
			$\frac{7}{(1).75}$	

## Conversion from binary to hexadecimal and binary to octal:

Conversion from binary to hexadecimal (and conversely) can be done by inspection because each hexadecimal digit corresponds to exactly four binary digits (bits).

$$1001101.010111_2 = \frac{0100}{4} \quad \frac{1101}{D} \cdot \frac{0101}{5} \quad \frac{1100}{C} = 4D.5C_{16}$$

A similar conversion can be done from binary to octal, base 8 (and conversely), except each octal digit corresponds to three binary digits, instead of four.

$$100101101011010_2 = 100$$
  $101$   $101$   $011$   $010 = 45532_8$ 

#### Introduction and Binary Addition:

For simplicity, arithmetic in digital systems is performed in binary (base 2).

Addition table for binary:

```
0 + 0 = 0

0 + 1 = 1

1 + 0 = 1

1 + 1 = 0 and carry 1 to the next column
```

Carrying 1 to a column is equivalent to adding 1 to that column.

#### **Example 5: Binary addition**

#### Example

Add  $13_{10}$  and  $11_{10}$  in binary.

$$13_{10} = 1101$$

$$11_{10} = 1011$$

$$11_{10} = 1011$$

$$11000 = 24_{10}$$

#### **Binary Subtraction:**

Subtraction table for binary:

The subtraction table for binary numbers is

$$0 - 0 = 0$$
  
 $0 - 1 = 1$  and borrow 1 from the next column  
 $1 - 0 = 1$   
 $1 - 1 = 0$ 

Borrowing 1 from a column is equivalent to subtracting 1 from that column.

#### **Example 6: Subtractions in Binary**

Notice how in (b), the borrow propagates from column to column.

#### **Binary Multiplication:**

Multiplication table for binary:

$$0 \times 0 = 0$$

$$0 \times 1 = 0$$

$$1 \times 0 = 0$$

$$1 \times 1 = 1$$

#### **Example 7: Binary Multiplication**

13<sub>10</sub> by 11<sub>10</sub> in binary:

$$\begin{array}{r}
1101 \\
1011 \\
\hline
1101 \\
1101 \\
0000 \\
1101 \\
\hline
10001111 = 143_{10}
\end{array}$$

When doing binary multiplication, a common way to avoid carries greater than 1 is to add in the partial products one at a time as illustrated by the following example:

```
1111
                multiplicand
      1101
                multiplier
                1st partial product
      1111
                2nd partial product
      0000
     (01111)
                sum of first two partial products
     1111
                 3rd partial product
    (1001011)
                sum after adding 3rd partial
product
     1111
                4th partial product
    11000011
                final product (sum after adding 4th
                                 partial product)
```

#### **Binary Division:**

- Binary division is similar to decimal division, except it is much easier because the only two possible quotient digits are 0 and 1.
- We start division by comparing the divisor with the upper bits of the dividend.
- If we cannot subtract without getting a negative result, we move one place to the right and try again.
- If we can subtract, we place a 1 for the quotient above the number we subtracted from and append the next dividend bit to the end of the difference and repeat this process with this modified difference until we run out of bits in the dividend.

#### **Example 8: Binary Division**

145<sub>10</sub> by 11<sub>10</sub> in binary

The quotient is 1101 with a remainder of 10.

Common methods of representing both positive and negative numbers are:

- Sign and magnitude
- 2's complement
- 1's complement

In each of these methods, the leftmost bit of a number is 0 for positive numbers and 1 for negative numbers.

## 3 Systems for representing negative numbers in binary- Overview:

Sign & Magnitude: Most significant bit is the sign

$$Ex: -5_{10} = 1101_2$$

2's Complement:  $N^* = 2^n - N$ 

Ex: 
$$-5_{10} = 2^4 - 5 = 16 - 5 = 11_{10} = 1011_2$$

1's Complement:  $= (2^n - 1) - N$ 

Ex: 
$$-5_{10} = (2^4 - 1) - 5 = 16 - 1 - 5 = 10_{10} = 1010_2$$

#### Sign & Magnitude:

Most significant bit is the sign

 $Ex: -5_{10} = 1101_2$ 

TABLE 1-1
Signed Binary
Integers (word length: n = 4)

Cengage Learning 2014

	Positive		Negative Integers									
+N	Integers (all systems)	-N	Sign and Magnitude	2's Complement N*	1's Complement N							
+0	0000	-0	1000		1111							
+1	0001	-1	1001	1111	1110							
+2	0010	-2	1010	1110	1101							
+3	0011	-3	1011	1101	1100							
+4	0100	-4	1100	1100	1011							
+5	0101	<b>-</b> 5	1101	1011	1010							
+6	0110	-6	1110	1010	1001							
+7	0111	<b>-</b> 7	1111	1001	1000							
		<b>-</b> 8		1000								

#### 2's compliment:

- Positive number, N, is represented by a 0 followed by the magnitude of N as in the sign and magnitude system
- For negative numbers, -N, 2's compliment is represented by N\*, as follows:

$$N^* = 2^n - N$$

Where the word length is n bits.

#### **Example 9: 2's Compliment Addition**

**1.** Addition of two positive numbers, sum  $< 2^{n-1}$ 

$$+3 0011 +4 0100 +7 0111 (correct answer)$$

**2.** Addition of two positive numbers, sum  $\geq 2^{n-1}$ 

$$\begin{array}{ccc}
+5 & 0101 \\
+6 & 0110 \\
\hline
1011 & \longleftarrow \text{ wrong answer because of overflow (+11 requires 5 bits including sign)}
\end{array}$$

-5

## Representation of Negative Numbers

#### **Example 9: 2's Compliment Addition (continued)**

**3.** Addition of positive and negative numbers (negative number has greater magnitude)

$$\begin{array}{cccc}
 +5 & 0101 \\
 -6 & 1010 \\
 \hline
 -1 & 1111 & (correct answer)
 \end{array}$$

**4.** Same as case 3 except positive number has greater magnitude

$$+6$$
  $0110$   $+1$   $(1)0001$   $\leftarrow$  correct answer when the carry from the sign bit is ignored (this is *not* an overflow)

1011

#### Example 9: 2's compliment Addition (continued)

5. Addition of two negative numbers,  $|\text{sum}| \le 2^{n-1}$ 

**6.** Addition of two negative numbers,  $|\text{sum}| > 2^{n-1}$ 

#### 1's compliment:

$$\overline{N} = (2^n - 1) - N$$

Ex: 
$$-5_{10} = (2^4 - 1) - 5 = 16 - 1 - 5 = 10_{10} = 1010_2$$

**End around carry:** In one's compliment, the last carry is not discarded as it is in 2's compliment, but rather added to the n-bit sum in the position furthest to the right.

#### **Example 10: 1's compliment Addition**

**3.** Addition of positive and negative numbers (negative number with greater magnitude)

$$\begin{array}{ccc}
 +5 & 0101 \\
 -6 & 1001 \\
 \hline
 -1 & 1110 & (correct answer)
 \end{array}$$

**4.** Same as case 3 except positive number has greater magnitude

#### **Example 10: 1's compliment Addition (continued)**

**5.** Addition of two negative numbers,  $|\operatorname{sum}| < 2^{n-1}$ 

**6.** Addition of two negative numbers,  $|\text{sum}| \ge 2^{n-1}$ 

$$\begin{array}{c|c}
-5 & 1010 \\
\underline{-6} & 1001 \\
\hline
(1) & 0011 \\
\hline
& \underline{)} & 1 \\
\hline
0100 & (wrong answer because of overflow)
\end{array}$$

#### **Example 10: 1's compliment Addition (continued)**

1. Add -11 and -20 in 1's complement.

$$+11 = 00001011$$
  $+20 = 00010100$ 

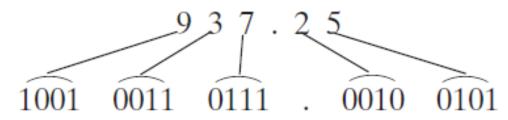
taking the bit-by-bit complement,

-11 is represented by 11110100 and -20 by 11101011

$$\begin{array}{ccc}
11110100 & (-11) \\
 & & \\
11101011 & +(-20) \\
\hline
(1) & 11011111 & \\
\hline
 & & \\
\hline
 & & \\
111000000 & = -31
\end{array}$$
 (end-around carry)

## **Binary Codes**

- Although most large computers work internally with binary numbers, the input-output equipment generally uses decimal numbers.
- Because most logic circuits only accept two-valued signals, the decimal numbers must be coded in terms of binary signals.
- In the simplest form of binary code, each decimal digit is replaced by its binary equivalent. For example, 937.25 is represented by:



## **Binary Codes**

8-4-2-1 Code (BCD)	6-3-1-1 Code	Excess-3 Code	2-out-of-5 Code	Gray Code
0000	0000	0011	00011	0000
0001	0001	0100	00101	0001
0010	0011	0101	00110	0011
0011	0100	0110	01001	0010
0100	0101	0111	01010	0110
0101	0111	1000	01100	1110
0110	1000	1001	10001	1010
0111	1001	1010	10010	1011
1000	1011	1011	10100	1001
1001	1100	1100	11000	1000
	Code (BCD) 0000 0001 0010 0011 0100 0111 0110 0111 1000	Code (BCD)  0000  0001  0001  0001  0010  0011  0011  0100  0101  0111  0110  1000  0111  1001  1001	Code (BCD)         6-3-1-1 Code         Excess-3 Code           0000         0000         0011           0001         0001         0100           0010         0011         0101           0011         0100         0110           0100         0101         0111           0101         0111         1000           0111         1001         1010           1000         1011         1010           1000         1011         1011	Code (BCD)         6-3-1-1 Code         Excess-3 Code         2-out-of-5 Code           0000         0000         0011         00011           0001         0001         0100         00101           0010         0011         0101         00110           0011         0100         0110         01001           0100         0101         0111         01010           0101         0111         1000         01100           0111         1001         1001         10010           0111         1001         10010         10010           1000         1011         1011         10100

Table 1–2. Binary Codes for Decimal Digits

## Binary Codes

	ASCII Code								ASCII Code								ASCII Code						
Character	$A_6$	A <sub>5</sub>	A <sub>4</sub>	A <sub>3</sub>	A <sub>2</sub>	A <sub>1</sub>	$A_0$	Character	$A_6$	$A_5$	$A_4$	A <sub>3</sub>	A <sub>2</sub>	$A_1$	$A_0$	Character	$A_6$	A <sub>5</sub>	$A_4$	A <sub>3</sub>	A <sub>2</sub>	A <sub>1</sub>	A <sub>0</sub>
space	0	1	0	0	0	0	0	@	1	0	0	0	0	0	0	,	1	1	0	0	0	0	0
1	0	1	0	0	0	0	1	Α	1	0	0	0	0	0	1	a	1	1	0	0	0	0	1
"	0	1	0	0	0	1	0	В	1	0	0	0	0	1	0	b	1	1	0	0	0	1	0
#	0	1	0	0	0	1	1	C	1	0	0	0	0	1	1	c	1	1	0	0	0	1	1
\$	0	1	0	0	1	0	0	D	1	0	0	0	1	0	0	d	1	1	0	0	1	0	0
%	0	1	0	0	1	0	1	E	1	0	0	0	1	0	1	e	1	1	0	0	1	0	1
&	0	1	0	0	1	1	0	F	1	0	0	0	1	1	0	f	1	1	0	0	1	1	0
,	0	1	0	0	1	1	1	G	1	0	0	0	1	1	1	g	1	1	0	0	1	1	1
(	0	1	0	1	0	0	0	Н	1	0	0	1	0	0	0	h	1	1	0	1	0	0	0
)	0	1	0	1	0	0	1	l l	1	0	0	1	0	0	1	i	1	1	0	1	0	0	1
*	0	1	0	1	0	1	0	J	1	0	0	1	0	1	0	j	1	1	0	1	0	1	0
+	0	1	0	1	0	1	1	K	1	0	0	1	0	1	1	k	1	1	0	1	0	1	1
,	0	1	0	1	1	0	0	L	1	0	0	1	1	0	0	- 1	1	1	0	1	1	0	0
-	0	1	0	1	1	0	1	M	1	0	0	1	1	0	1	m	1	1	0	1	1	0	1
:	0	1	0	1	1	1	0	N	1	0	0	1	1	1	0	n	1	1	0	1	1	1	0
/	0	1	0	1	1	1	1	0	1	0	0	1	1	1	1	0	1	1	0	1	1	1	1
0	0	1	1	0	0	0	0	P	1	0	1	0	0	0	0	р	1	1	1	0	0	0	0
1	0	1	1	0	0	0	1	Q	1	0	1	0	0	0	1	q	1	1	1	0	0	0	1
2	0	1	1	0	0	1	0	R	1	0	1	0	0	1	0	r	1	1	1	0	0	1	0
3	0	1	1	0	0	1	1	S	1	0	1	0	0	1	1	S	1	1	1	0	0	1	1
4	0	1	1	0	1	0	0	T U	1	0	1	0	1	0	0	t	1	1	1	0	1	0	0
5 6	0	1	1	0	1	0	1	V	1	0	1	0	1	0	1	u	1	1	1	0	1	0	1
7	0	1	1	0	1	1	0	_	1	0	1	0	1		0	V	1	1	1	0	1	1	0
_	0	1	1	0	0	1	1	w	1	0	1	0	1	1	1	w	1	1	1	0	1		1
8 9	0	1	1	1	0	0	0	X	1	0	1	1	0	0	0	X	1	1	1	1	0	0	0
	0	1	1	1	0	1	0	Z	i	0	1	1	0	1	0	y z	1	1	1	1	0	1	0
:	0	1	1	1	0	1	1	1	i	0	1	1	0	1	1	1	1	1	1	1	0	1	1
	0	1	1	1	1	0	0	L	1	0	1	1	1	0	0	1	1	1	1	1	1	0	0
<	0	1	1	1	1	0	1	ì	1	0	1	1	1	0	1	Į.	1	1	1	1	1	0	1
>	0	1	1	1	1	1	0	,	i	0	1	1	1	1	0	1	1	1	1	1	1	1	0
5	0	1	1	1	1	1	1		1	0	1	1	i	1	1	delete	1	1	1	1	1	1	1