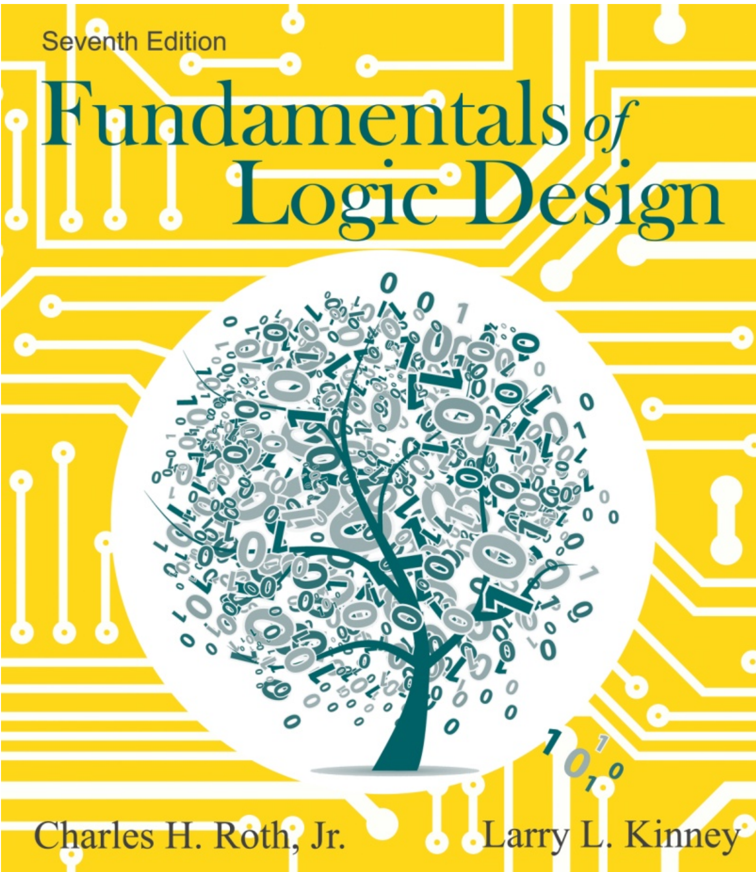


UNIT 2

Boolean Algebra

This chapter includes:

2.1 Introduction
2.2 Basic Operations
2.3 Boolean Expressions and Truth Tables
2.4 Basic Theorems
2.5 Commutative, Associative, Distributive and DeMorgan's Laws
2.6 Simplification Theorems
2.7 Multiplying Out and Factoring
2.8 Complementing Boolean Expressions

Learning Objectives

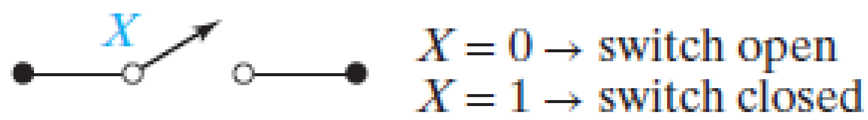
- Understand the basic operations and laws of Boolean algebra.
- Relate these operations and laws to circuits composed of AND gates, OR gates, INVERTERS and switches.
- Prove any of these laws in switching algebra using a truth table.
- Apply these laws to the manipulation of algebraic expressions including: obtaining a sum of products or product of sums, simplifying an expression and/or finding the complement of an expression

Introduction

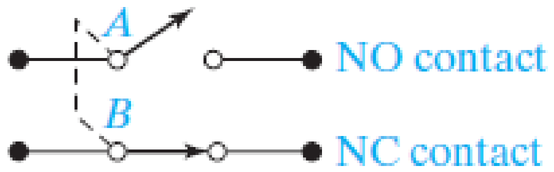
- All switching devices we will use are two-state devices, so we will emphasize the case in which all variables assume only one of two values.
- Boolean variable X or Y will be used to represent input or output of switching circuit.
- Symbols “0” and “1” represent the two values any variable can take on. These represent states in a logic circuit, and do not have numeric value.
- Logic gate: 0 usually represents range of low voltages and 1 represents range of high voltages
- Switch circuit: 0 represents open switch and 1 represents closed
- 0 and 1 can be used to represent the two states in any binary valued system.

Basic Operations

- The basic operations of Boolean (switching) algebra are called AND, OR, and complement (or inverse).
- To apply switching algebra to a switch circuit, each switch contact is labeled with a variable. See diagram below:



- NC (normally closed) and NO (normally open) contacts are always in opposite states.

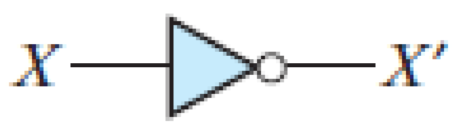


- If variable X is assigned to NO contact, then X’ will be assigned for NC.

Basic Operations

Complementation/ Inversion:

- Prime (’) denotes complementation.
- 0’=1 and 1’=0
- For a switching variable, X:
  - X’=1 if X=0 and X’=0 if X=1’
- Complementation is also called inversion. An inverter is represented as shown below, where circle at the output denotes inversion:



Basic Operations

Series Switching Circuits/ AND Operation:

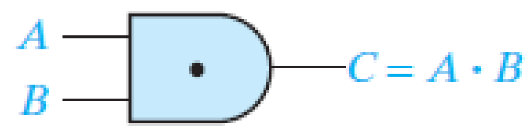
Series:
A) Truth table B) Logic gate diagram
C) Switch circuit diagram

The operation defined by the table is called AND. It is written algebraically as *C=A·B. We will usually write AB instead of A·B. The AND operation is also referred to as logical (or Boolean) multiplication.*

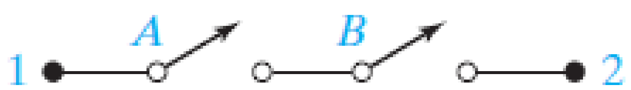
A)

A	B	$C = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

B)



C)



$C = 0 \rightarrow$ open circuit between terminals 1 and 2
$C = 1 \rightarrow$ closed circuit between terminals 1 and 2

Basic Operations

Parallel Switching Circuits/ OR Operation:

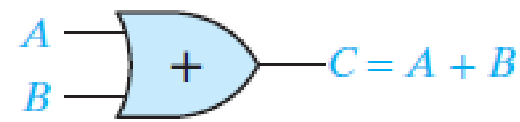
Series:
A) Truth table B) Logic gate diagram
C) Switch circuit diagram

The operation defined by the table is called OR. It is written algebraically as *C=A+B. The OR operation is also referred to as logical (or Boolean) addition.*

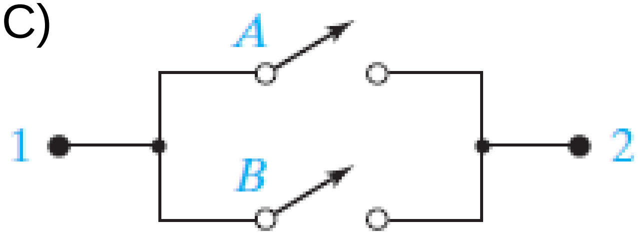
A)

A B	C = A + B
0 0	0
0 1	1
1 0	1
1 1	1

B)



C)



If switches A and B are connected in parallel, there is a closed circuit if either A or B, or both, are closed and an open circuit only if A and B are both open.

Boolean Operations and Truth Tables

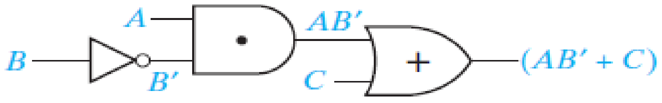
Examples of Boolean Expressions and Corresponding Diagrams:

Expressions

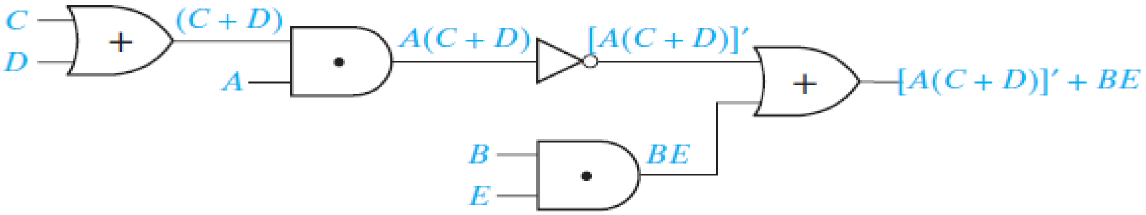
$$AB' + C \tag{2-1}$$
$$[A(C + D)]' + BE \tag{2-2}$$

Order of operations- Parentheses, Inversion, AND, OR

Logic Diagrams



(a)



(b)

Boolean Operations and Truth Tables

Truth Tables:

A truth table specifies the values of a Boolean expression for every possible combination of values of the variables in the expression.

FIGURE 2-2
Two-Input Circuit and Truth Table

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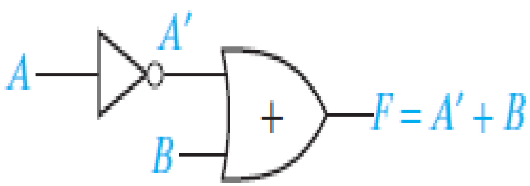

(a)

A	B	A′	F = A′ + B
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

(b)

Boolean Operations and Truth Tables

Equal Boolean Expressions:

Two boolean expressions are said to be **equal** if they have the same value for every possible combination of the variables.

TABLE 2-1

A B C	B'	AB'	$AB' + C$	$A + C$	$B' + C$	$(A + C)(B' + C)$
0 0 0	1	0	0	0	1	0
0 0 1	1	0	1	1	1	1
0 1 0	0	0	0	0	0	0
0 1 1	0	0	1	1	1	1
1 0 0	1	1	1	1	1	1
1 0 1	1	1	1	1	1	1
1 1 0	0	0	0	1	0	0
1 1 1	0	0	1	1	1	1

$$AB' + C = (A + C)(B' + C) \tag{2-3}$$

An n-variable expression will have 2^n rows in its truth table.

Basic Theorems

Single Variable Basic Theorems:

Operations with 0 and 1:

$X + 0 = X$	(2-4)	$X \cdot 1 = X$	(2-4D)
$X + 1 = 1$	(2-5)	$X \cdot 0 = 0$	(2-5D)

Idempotent laws:

$X + X = X$	(2-6)	$X \cdot X = X$	(2-6D)

Involution law:

$(X')' = X$	(2-7)

Laws of complementarity:

$X + X' = 1$	(2-8)	$X \cdot X' = 0$	(2-8D)

See book for switch circuit diagrams that illustrate these basic theorems (Roth-Kinney 7th edition, pg 42-43).

Commutative, Associative, Distributive and DeMorgan's Laws

Commutative and Associative Laws:

Commutative: Order in which variables are written does not affect result of applying AND and OR operations.

$$XY = YX \text{ and } X+Y = Y+X$$

Associative: Result of AND and OR operations is independent of which variables we associate together first.

$$(XY)Z = X(YZ) = XYZ$$

$$(X+Y)+Z = X+(Y+Z) = X+Y+Z$$

Commutative, Associative, Distributive and DeMorgan's Laws

Distributive Law:

The distributive law of boolean algebra is as follows:

$$X(Y+Z)= XY+XZ$$

Furthermore, a second distributive law is valid for Boolean algebra but not ordinary algebra:

$$X+YZ= (X+Y)(X+Z)$$

Proof of this second distributive law can be found on page 45.

Commutative, Associative, Distributive and DeMorgan's Laws

DeMorgan's Laws:

DeMorgan's Law is stated as follows:

$$(X+Y)'=X'Y'$$

$$(XY)'=X'+Y'$$

Truth table proof of DeMorgan's Laws is shown below:

X	Y	X'	Y'	X + Y	(X + Y)'	X'Y'	XY	(XY)'	X' + Y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Commutative, Associative, Distributive and DeMorgan's Laws

Laws of Boolean Algebra-(Table 2-3):

Operations with 0 and 1:

1. $X + 0 = X$ 1D. $X \cdot 1 = X$
2. $X + 1 = 1$ 2D. $X \cdot 0 = 0$

Idempotent laws:

3. $X + X = X$ 3D. $X \cdot X = X$

Involution law:

4. $(X')' = X$

Laws of complementarity:

5. $X + X' = 1$ 5D. $X \cdot X' = 0$

Commutative, Associative, Distributive and DeMorgan’s Laws

Laws of Boolean Algebra- ~~Reference~~ (continued):

Commutative laws:

6. $X + Y = Y + X$ 6D. $XY = YX$

Associative laws:

7. $(X + Y) + Z = X + (Y + Z) = X + Y + Z$ 7D. $(XY)Z = X(YZ) = XYZ$

Distributive laws:

8. $X(Y + Z) = XY + XZ$ 8D. $X + YZ = (X + Y)(X + Z)$

DeMorgan’s laws:

9. $(X + Y)' = X'Y'$ 9D. $(XY)' = X' + Y'$

Simplification Theorems

Simplification theorems:

Theorems used to replace an expression with a simpler expression are called **simplification theorems**.

Uniting theorems:

1. $XY + XY' = X$ 1D. $(X + Y)(X + Y') = X$

Absorption theorems:

2. $X + XY = X$ 2D. $X(X + Y) = X$

Elimination theorems:

3. $X + X'Y = X + Y$ 3D. $X(X' + Y) = XY$

Duality:

4. $(X + Y + Z + \cdots)^D = XYZ\ldots$ 4D. $(XYZ\ldots)^D = X + Y + Z + \cdots$

Theorems for multiplying out and factoring:

5. $(X + Y)(X' + Z) = XZ + X'Y$ 5D. $XY + X'Z = (X + Z)(X' + Y)$

Consensus theorems:

6. $XY + YZ + X'Z = XY + X'Z$ 6D. $(X + Y)(Y + Z)(X' + Z) = (X + Y)(X' + Z)$

Simplification Theorems

Proof of Simplification Theorems:

❖Using switching algebra, the theorems on the previous slide can be proven using truth tables.

❖In general Boolean algebra, these theorems must be proven algebraically starting with basic theorems.

Proof of (2-15): $XY + XY' = X(Y + Y') = X(1) = X$

Proof of (2-16): $X + XY = X \cdot 1 + XY = X(1 + Y) = X \cdot 1 = X$

Proof of (2-17): $X + X'Y = (X + X')(X + Y) = 1(X + Y) = X + Y$

Proof of (2-18): $XY + X'Z + YZ = XY + X'Z + (1)YZ =$

$XY + X'Z + (X + X')YZ = XY + XYZ + X'Z + X'YZ =$

$XY + X'Z$ (using absorption twice)

❖Using duality property, (2-15D)-(2-18D) can be proven

Multiplying Out and Factoring

Sum of Products:

An expression is said to be in *sum-of-products* (SOP) form when all products are the products of single variables. This form is the end result when an expression is fully multiplied out.

For example:

$$AB' + CD'E + AC'E'$$

$$ABC' + DEFG + H$$

Multiplying Out and Factoring

Product of Sums:

An expression is in *product-of-sums* (POS) form when all sums are the sums of single variables. It is usually easy to recognize a product-of-sums expression since it consists of a product of sum terms.

For example:

$$(A + B')(C + D' + E)(A + C' + E')$$

$$(A + B)(C + D + E)F$$

Multiplying Out and Factoring

Examples:

Example 1

Factor $A + B'CD$. This is of the form $X + YZ$ where $X = A$, $Y = B'$, and $Z = CD$, so

$$A + B'CD = (X + Y)(X + Z) = (A + B')(A + CD)$$

$A + CD$ can be factored again using the second distributive law, so

$$A + B'CD = (A + B')(A + C)(A + D)$$

Example 2

Factor $AB' + C'D$.

$AB' + C'D = (AB' + C')(AB' + D)$ ← note how $X + YZ = (X + Y)(X + Z)$ was applied here

$= (A + C')(B' + C')(A + D)(B' + D)$ ← the second distributive law was applied again to each term

Complementing Boolean Expressions

Using DeMorgan's Laws to find Inverse Expressions:

- The complement or inverse of any Boolean expression can be found using DeMorgan's Laws.
- DeMorgan's Laws for n-variable expressions:

$$(X_1 + X_2 + X_3 + \cdots + X_n)' = X_1'X_2'X_3' \ldots X_n' \qquad (2\text{-}25)$$

$$(X_1X_2X_3 \ldots X_n)' = X_1' + X_2' + X_3' + \cdots + X_n' \qquad (2\text{-}26)$$

For example, for $n = 3$,

$$(X_1 + X_2 + X_3)' = (X_1 + X_2)'X'_3 = X'_1X'_2X'_3$$

- The complement of the product is the sum of the complements.
- The complement of the sum is the product of the complements.

Complementing Boolean Expressions

Examples:

Example 1

To find the complement of $(A' + B)C'$, first apply (2-13) and then (2-12).

$$[(A' + B)C']' = (A' + B)' + (C')' = AB' + C$$

Example 2

$$\begin{aligned}
[(AB' + C)D' + E]' &= [(AB' + C)D']'E' && \text{(by (2-12))} \\
&= [(AB' + C)' + D]E' && \text{(by (2-13))} \\
&= [(AB')'C' + D]E' && \text{(by (2-12))} \\
&= [(A' + B)C' + D]E' && \text{(by (2-13))} \quad \text{(2-27)}
\end{aligned}$$

Note that in the final expressions, the complement operation is applied only to single variables.