Regularization Techniques

Def. (Difference Quotients):
Given $u \in L^p(\Omega)$ we define the DIFFERENCE QUOTIENTS of u in direction $i \in \{1, \dots, n\}$ as $\tau_{h,i} u(x) := \dfrac{u(x+he_i)-u(x)}{h}$

Proposition:
Given $u \in L^p(\Omega)$, $1 \leq p \leq +\infty$, if $u \in W^{1,p}(\Omega)$ and $\Omega' \subset\subset \Omega$ then $\exists\, C>0, h_0>0$ s.t. if $h \in \mathbb{R}, |h|<h_0$ then $\|\tau_{h,i} u\|_{L^p(\Omega')} \leq C$
Conversely, if $1<p\leq +\infty$ and $u \in L^p_{loc}(\Omega)$ s.t. $\forall \Omega' \subset\subset \Omega$ $\exists\, h_0>0, C>0$ s.t. $\|\tau_{h,i} u\|_{L^p(\Omega')} \leq C$ $\forall |h|<h_0$, $i=1,\dots,n$, then $u \in W^{1,p}_{loc}(\Omega)$ and $\tau_{h,i} u \xrightarrow[L^p(\Omega')]{} \partial_{x_i} u$

Proof:

$$\frac{u(x+he_i)-u(x)}{h} = \frac{1}{h}\int_0^h \partial_{x_i} u(x-te_i)\,dt \quad \text{so we have:}$$

$$\|\tau_{h,i} u\|^p_{L^p(\Omega')} = \int_{\Omega'} \left|\frac{1}{h}\int_0^h \partial_{x_i} u(x-te_i)\,dt\right|^p dx$$

$$\leq \int_{\Omega'} \frac{1}{h}\int_0^h |\partial_{x_i} u(x-te_i)|^p\,dt\,dx = \frac{1}{h}\int_0^h \int_{\Omega} |\partial_{x_i} u(x-te_i)|^p\,dt\,dx$$

$$\leq \int_{\Omega} |\nabla u(x)|^p dx$$

$\Rightarrow \exists$ subseq. $h_j \to 0$ s.t. $\tau_{h_j,i} u \xrightarrow[L^p(\Omega')]{} g_i$ (Banach-Alaoglu)

Consider now, given $\phi \in C^1_c(\Omega')$:

$$\begin{array}{ccc}
\displaystyle\int_{\Omega'} \tau_{h_j,i} u(x)\,\phi(x)\,dx & \longrightarrow & \displaystyle\int_{\Omega'} g_i(x)\,\phi(x)\,dx \\
\| & & \| \\
\displaystyle -\int_{\Omega'} u(x)\,\underbrace{\tau_{-h_j,i}\,\phi(x)}_{\to\ \partial_{x_i}\phi(x)}\,dx & \longrightarrow & \displaystyle -\int_{\Omega'} u(x)\,\partial_{x_i}\phi(x)\,dx
\end{array}$$

We also have $\tau_{h,i} u(x) = \frac{1}{h}\int_0^h \partial_{x_i} u(x_i+te_i)\,dt$, compute now:

$$\|\tau_{h,i} u - \partial_{x_i} u\|^p_{L^p(\Omega')} = \int_{\Omega'} \left|\frac{1}{h}\int_0^h \left(\partial_{x_i} u(x+te_i) - \partial_{x_i} u(x)\right) dt\right|^p dx$$

$$\underset{\text{Hölder, Fubini}}{\leq} \frac{1}{h}\int_0^h \underbrace{\int_{\Omega'} |\partial_{x_i} u(x+te_i) - \partial_{x_i} u(x)|^p\,dx}_{\to\ 0 \text{ (continuity of translations in } L^p)}\,dt \longrightarrow 0$$

$\square$

Remark:

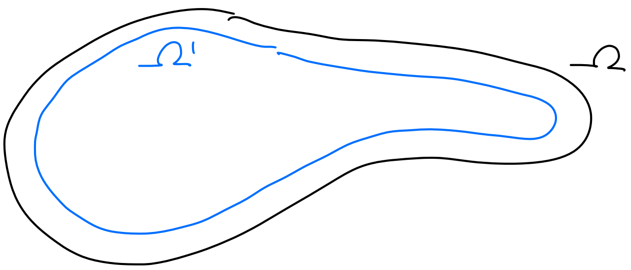
$u \in W_0^{1,2}(\Omega) \Rightarrow \int_\Omega (\nabla u \cdot \nabla \phi + f\phi)\,dx = 0 \quad \forall \phi \in W_0^{1,2}(\Omega)$

$\Rightarrow$ plug $\phi = u$ into the Poisson equation:

$$\int_\Omega (|\nabla u|^2 + fu)\,dx = 0 \Rightarrow \int_\Omega |\nabla u|^2 dx \leqslant -\int_\Omega f u\,dx \leqslant \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)}$$

$$\leqslant \frac{1}{2}\int_\Omega (|f|^2 + |u|^2)\,dx$$ (trivial case of the Caccioppoli Estimate)

We need a similar estimate for $u \in W^{1,2}(\Omega)$ satisfying $\int_\Omega (\nabla u \cdot \nabla \phi + f\phi)\,dx = 0 \ \forall \phi \in \mathcal{C}_c^1(\Omega)$ ($\phi \in W_0^{1,2}(\Omega)$). The idea is the following:



$\Omega' \subset\subset \Omega$

let $\eta \in \mathcal{C}_c^1(\Omega)$ s.t. $0 \leqslant \eta(x) \leqslant 1 \wedge \eta(x) = 1 \ \forall x \in \Omega'$ (cutoff function) and plug $\phi = \eta^2(x)u(x)$ into the equation:
the gradient of ϕ is: $\nabla\phi = \eta^2(x)\nabla u(x) + 2\eta(x)\cdot u(x)\cdot\nabla\eta(x)$

$$\Rightarrow \int_\Omega \eta^2 |\nabla u|^2 dx = -\int_\Omega \Big(2\eta(x)u(x)\underbrace{\nabla\eta(x)}_{\text{bounded}}\cdot\nabla u(x) + f\eta^2 u\Big)dx$$

$$\leqslant C\|\eta\nabla u\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)}$$

$$\leqslant C\varepsilon^2 \int_\Omega \eta^2 |\nabla u|^2 dx + \frac{C}{\varepsilon^2}\int_\Omega |u|^2 dx + C\int_\Omega |f|^2 dx$$

$$\Rightarrow (1 - C\varepsilon^2)\int_\Omega \eta^2|\nabla u|^2 dx \leqslant C\int_\Omega (|f|^2 + |u|^2)\,dx$$ ("HOLE - FILLING TECHNIQUE")

We have the CACCIOPPOLI ESTIMATE:

$$\int_{\Omega'} |\nabla u|^2 dx \leqslant \int_\Omega \eta^2 |\nabla u|^2 \leqslant C\int_\Omega (|u|^2 + |f|^2)\,dx$$

Let now $u \in W^{1,2}(\Omega)$ be a weak solution:

$$\int_\Omega (\nabla u(x)\cdot\nabla\phi(x) + f(x)\phi(x))\,dx = 0 \quad \forall \phi \in \mathcal{C}_c^1(\Omega')$$

$$\Rightarrow \int_\Omega (\nabla u(x + he_i))\cdot\nabla\phi(x) + f(x+he_i)\phi(x))\,dx = 0$$

$$\Rightarrow \int_\Omega (\nabla \tau_{h,i} u(x)\cdot\nabla\phi(x) + \tau_{h,i} f(x)\phi(x))\,dx = 0$$

Suppose $f \in W^{1,2}(\Omega)$, $\Omega' \subset\subset \Omega$, then by Caccioppoli we have:

$$\int_{\Omega''} |\tau_{h,i}\, u(x)|^2 dx \leq C \int_{\Omega'} (|\tau_{h,i} u|^2 + |\tau_{h,i} f|^2)\, dx$$

$$\int_{\Omega''} |\tau_{h,i}(\nabla u)|^2 dx \leq C \int_{\Omega} (|\partial_{x_i} u|^2 + \partial_{x_i} f|^2)\, dx \leq C$$

$\Rightarrow \nabla u \in W^{1,2}(\Omega'')$, $\Omega'' \subset\subset \Omega' \Rightarrow u \in W^{2,2}_{loc}(\Omega)$. We also have:

$$\int_\Omega (\partial_{x_i}(\nabla u(x)) \cdot \nabla \phi + \partial_{x_i} f \phi)\, dx = 0 \quad \forall \phi \in \mathcal{C}^1_c(\Omega')$$

$\Rightarrow$ if $f \in W^{2,2}(\Omega)$, then $\partial_{x_i} u \in W^{2,2}_{loc}(\Omega)$ and $u \in W^{3,2}_{loc}(\Omega)$ and in general we have:

$$f \in W^{K,2}(\Omega) \Rightarrow u \in W^{K+1,2}_{loc}(\Omega)$$

Suppose $u \in L^2(\Omega')$, $\nabla u \in L^2(\Omega')$, $D^2 u \in L^2(\Omega')$. Then, by the Sobolev Embedding we have:

$$\Rightarrow \nabla u \in W^{1,2}(\Omega') \Rightarrow u \in L^{2^*},\ \nabla u \in L^{2^*}(\Omega'),\ 2^* = \frac{2n}{n-2}$$

$\Rightarrow$ if all the derivatives of u are in $L^2(\Omega)$, then $u \in \mathcal{C}^\infty(\Omega)$!

Elliptic Equations in Divergence Form

They are equations of the form:

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $A(x) \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, strictly uniformly elliptic:

A symmetric, $\exists \nu > 0$ s.t. $\xi^T A \xi \geq \nu |\xi|^2 \ \forall \xi \in \mathbb{R}^n \ \forall x \in \Omega$

$\Rightarrow$ the weak solution is $u \in W^{1,2}_0(\Omega)$ s.t.:

$$\int_\Omega (A(x)\nabla u \cdot \nabla \phi + f\phi)\, dx = 0 \quad \forall \phi \in W^{1,2}_0(\Omega)$$

$\Rightarrow$ we can define the scalar product on $W^{1,2}_0(\Omega)$

$$((u,\phi)) := \int_\Omega (A(x)\nabla u(x)) \cdot \nabla \phi(x)\, dx$$

which is equivalent to the standard one!

<u>Proposition</u>:

Let $\Omega \subseteq_{op} \mathbb{R}^n$ be bounded, regular, $1 \leq p < +\infty$. Then $\exists C > 0$ (dependent on Ω, p, not on u) s.t. $\forall u \in W^{1,p}(\Omega)$ we have:

$$\|u-\bar{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

where $\bar{u} = \frac{u}{|\Omega|} \int_\Omega u(x)\,dx$ is the integral average of u.

<u>Proof:</u>

We prove this by contradiction:

suppose $\exists \{u_k\}_k \subset W^{1,p}(\Omega)$, $\bar{u}_k = 0$ $\forall k$ s.t. $\|\nabla u_k\|_{L^p} \to 0$ BUT $\|u_k\|_{L^p} = 1$. Then, by Rellich's Thm. we have that, up to subsequences, $u_k \xrightarrow[L^p]{} u$ pointwise. But $\nabla u = 0$, so u is constant with $\bar{u} = 0 \Rightarrow u = 0$ $\lightning$

$\square$