PROPER CLASSES AND SETS

Every set is also a class, but not every class is a set. Classes which are not sets are called "proper classes"

Examples:

1)  $\forall := the class of all sets = \{x : x = x\} = \{x : T\}$  is proper

2) The class of all vector spaces is proper

3) The class of all ordinal numbers is proper:

 $O,\ 1,2,3,\ldots,\ \omega\ ,\omega+1,\omega+2,\ldots,\ \omega+\omega=2\omega,$ 

etc. (w is the first infinite ndival)

4) A Category is small if its objects from a set, large if they from a proper class.

5) Set is large

RUSSELL'S PARADOX.

Let  $R := \{x : x \notin x\}$  be a class. We ask:  $R \in R$ ?  $R \in R \Leftrightarrow R \notin R \not\downarrow$  hence  $\bot$ , indeed:

1) Claim: 7 (RER)

Proof:

Assume RER. Show 1

⇒ RER ⇒ R&R & Sor L

2) Claim: RER

Brank:

3) Claim: 1

Proof: lry (2) and (1)

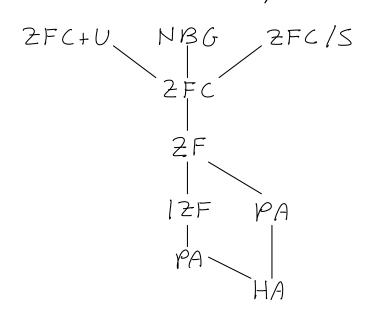
Variout of the Russell's Paradox:

Define  $R = \{x : x \in x \Rightarrow 2 + 2 = 5\}$ . Using the same argument as above we obtain 2 + 2 = 5

⇒ Russell's Paradox proved that Naive Set Theny, with its werestricted principle of set comprehension, is inconsistent

## RESOLUTION OF THE RUSSELL'S PARADOX:

⇒ we salve Russell's Paradox by switching to formal proofs, where only a norrow list of logical inference rules and axious may be used. The most well known system of said rules is ZFC (Zermelor - Fraenkel Set Theory with the axiom of choice). We have also extensions:



In ZFC we cou't four the set {x: x ∉x}, R is therefore a proper class and not a set.

## DEALING WITH SIZE ISSUES

- O) Igune them
- 1) Switch from ZFC to NBG, which natively supports both sets and proper classes. However, in NBG only sets can be element of classes. In NBG, unlike in ZFC, we can formulate statements like "for every proper class...", "there is a proper class with ...". Moreover, for statements which are purely about sets NBG is conservative over ZFC (any NBG proof can be transformed into a ZFC proof).
- 2) Switch from ZFC to ZFC + "there I a Glotheuclieck universe" also called ZFC + U.
  - Note: ZFC+U is not conservative over ZFC
- 3) Try to fake classes in ZFC (standard approach for logicious). Regard classes as mere "syntectic sugar" (e.g. i++ for i=i+1 in C). For instance, "Y×EV..."

is not a statement in ZFC (it uses the proper class V) Bur "Yx ... " is. The same holds for " Ix E On ... " (On := proper class of ndival numbers) and " ]x s.t. × is an ordinal number and ... " 4) Switch from ZFC to ZFC/S: Recall: the cumulative hierachy!  $V_o = \phi$ ,  $V_1 = \mathcal{P}(V_o) = \{\phi\}$ ,  $V_2 = \mathcal{P}(V_1)$ ,...,  $V_{\omega} = \bigcup_{i=0}^{\infty} V_{i}, \dots, V_{\omega+1} = \mathcal{O}(V_{\omega}), \dots, V_{2\omega} = \bigcup_{i=0}^{\infty} V_{i}$ <u>Thue:</u>: V = U V2 (i.e. every set is contained in some V2) Reflection theory: In any statement A in ZFC, ZFC proves that: ∃2 ∈ On 1. t. A ⇔ A V2 where  $A^{\vee_{\alpha}}$  is the  $V_{\alpha}$  relativization of A:  $"\forall \times ... "V_{\alpha} = "\forall \times \in V_{\alpha} ... ",$   $"\exists \times ... "V_{\alpha} = "\exists \times \in V_{\alpha} ... "$ ∀2 ∈ On V2 is a set!!! Not a class!!!

ZFC/S is a variout of ZFC with seflection built into it: ZFC/S = ZFC + S, S is the set of all small sets

Reflection axiom: For every statement A in ZFC,  $A \Leftrightarrow A^{\$}$ ZFC/S is conservative over ZFC for statements which Lou't contain S