Natural Transformations (continued)

Remark:

Is this category $\mathcal{C}$ $\bullet \to \bullet \to \bullet \to \bullet$ equivalent to $\mathcal{C}^{op}$ $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$? Yes! But: given $\mathcal{D}$ $\bullet \to \bullet \to \bullet \to \bullet \dots$ and $\mathcal{D}^{op}$ $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \dots$ (∞ objects) they are NOT equivalent, since $\mathcal{D}$ has an initial object and $\mathcal{D}^{op}$ has no initial objects.

The numerical category is equivalent to its opposite (map n to n and $A \in \mathbb{R}^{m \times n}$ to $A^T \in \mathbb{R}^{n \times m}$), but Set is NOT equivalent to Set^{op}, and $\text{Vect}(\mathbb{R})$ is NOT equivalent to $\text{Vect}(\mathbb{R})^{op}$. For example, in Set every morphism into the initial object $\emptyset$ is an isomorphism (the only map into $\emptyset$ is its identity morphism $id_\emptyset : \emptyset \to \emptyset$) but in Set^{op} that is not the case: in Set^{op}, every morphism out of the terminal object $\{x\}$ is not necessarily an isomorphism. Set^{op} is equivalent to the completely atomized algebras

examples:

1) length:

$$\left.\begin{array}{l} \text{List}(X) \longrightarrow \mathbb{N} \\ [x_1, \dots, x_n] \longmapsto n \end{array}\right\} \text{ is a natural transformation } \eta : \text{List} \Rightarrow K_{\mathbb{N}}$$

where $K_{\mathbb{N}}$ is the constant functor:

$$K_{\mathbb{N}} : \text{Set} \to \text{Set}$$
$$X \mapsto \mathbb{N}$$
$$f \mapsto id_{\mathbb{N}}$$

2) $\det(\cdot)$ is a natural transformation:

$$\eta_{\mathbb{F}} : \begin{array}{l} \mathbb{F}^{n \times n} \longrightarrow \mathbb{F} \\ A \longmapsto \det(A) \end{array} \Rightarrow \eta : H \Rightarrow L \text{ where } H, L \text{ are as follows:}$$

$$H : \text{Field} \to \text{Mon}, \quad \mathbb{F} \mapsto \mathbb{F}^{n \times n}, \qquad L : \text{Field} \to \text{Mon}, \quad \mathbb{F} \mapsto (\mathbb{F}, 1, \cdot)$$

Indeed, given $\varphi : \mathbb{F} \to \mathbb{G}$ field homomorphism, we have:

$$\det \circ (H(\varphi)) = \det \circ \varphi = \varphi \circ \det$$

3) $\left.\begin{array}{rcl} X & \longrightarrow & X \times X \\ x & \longmapsto & (x,x) \end{array}\right\}$ is a natural transformation $\eta : \mathrm{Id}_{Set} \Rightarrow \mathrm{Pair}$

where:

$$\begin{array}{rrcl} \mathrm{Pair}: & Set & \longrightarrow & Set \\ & X & \longmapsto & X \times X \\ & f & \longmapsto & f \times f \end{array} \quad , \quad f \times f : (x,y) \longmapsto (f(x), f(y))$$

4) $\eta : \mathrm{Id}_{Set} \Rightarrow \mathcal{P}$ is a natural transformation:

$$\begin{array}{rrcl} \eta_X : & X & \longrightarrow & \mathcal{P}(X) \\ & x & \longmapsto & \{x\}, \emptyset, X \end{array}$$

even if η_X is as follows:

$$\begin{array}{rrcl} \eta_X : & X & \longrightarrow & \mathcal{P}(X) \\ & x & \longmapsto & \begin{cases} \{x\} & CH \\ X & \neg CH \end{cases} \end{array} \qquad CH := \begin{array}{l} \text{Continuum} \\ \text{Hypothesis} \end{array}$$

η is still a natural transformation.
Instead, if η_X is as follows:

$$\begin{array}{rrcl} \eta_X : & X & \longrightarrow & \mathcal{P}(X) \\ & x & \longmapsto & \begin{cases} \{x\} & X = \mathbb{N} \\ X & \text{otherwise} \end{cases} \end{array}$$

then η is not a natural transformation anymore