CATEGORY OF CATEGORIES

Examples:

1) $F : \mathrm{Set} \longrightarrow \mathrm{Set}$

$\qquad X \longmapsto X \times X = \{(a,b) : a,b \in X\}$

2) $G : \mathrm{Set} \longrightarrow \mathrm{Set}$

$\qquad X \longmapsto \{(a,b,c) : a,b,c \in X,\ c = a\}$

3) $H : \mathrm{Set} \longrightarrow \mathrm{Set}$

$\qquad X \longmapsto \{$☺(a,b)☺$: a,b \in X\}$ (N.B. $(a,b) \neq$ ☺(a,b)☺)

$\Rightarrow$ Are F, G, H the same functor? No! (they have 3 different structures) BUT they are isomorphic. In which category? In the category of functors from Set to Set, which is denoted as $[\mathrm{Set}, \mathrm{Set}]$.

Def. (Functor Category):
Let $\mathcal{C}, \mathcal{D}$ be categories. The FUNCTOR CATEGORY $[\mathcal{C}, \mathcal{D}]$ consists of:

1) Objects: all functors $\mathcal{C} \longrightarrow \mathcal{D}$

2) Morphisms:

$$\mathrm{Hom}_{[\mathcal{C},\mathcal{D}]}(F,G) := \{\eta : \eta : F \Rightarrow G \text{ natural transformation}\}$$

3) Identities:

$\forall F \in \mathrm{Ob}([\mathcal{C},\mathcal{D}])\quad \mathrm{id}_F : F \Rightarrow F$ s.t.

$(\mathrm{id}_F)_X : F(X) \longrightarrow F(X)$ given by $\mathrm{id}_{F(X)}$

4) Composition rule:

we use the following:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D}
\arrow[Rightarrow, from=F, to=G, "\eta"] \arrow[Rightarrow, from=G, to=H, "\omega"]
\end{tikzcd}$$

$$\Rightarrow\ \omega \circ \eta : F \Rightarrow H,\quad \underbrace{(\omega \circ \eta)_X}_{: F(X) \to H(X)} := \underset{\substack{\downarrow \\ : G(X) \to H(X)}}{\omega_X} \circ \underset{\substack{\searrow \\ : F(X) \to G(X)}}{\eta_X}$$

Example

1) We check that the 3 functors F, G, H of the previous examples are isomorphic:

$$\eta : F \Rightarrow G \qquad\qquad \omega : G \Rightarrow F$$

$$\eta_X : F(X) \longrightarrow G(X) \qquad\qquad \omega_X : G(X) \longrightarrow F(X)$$

$$(a,b) \longmapsto (a,b,a) \qquad\qquad (a,b,c) \longmapsto (a,b)$$

and similarly for F, H and G, H

Def. (Category of Categories):

The (1-) CATEGORY OF (1-) CATEGORIES, Cat and consists:

1) Objects: $Ob(Cat)$ = all (1-) categories
2) Morphisms: $Hom_{Cat}(\mathcal{C}, \mathcal{D}) = \{F : \mathcal{C} \to \mathcal{D} : F \text{ functor}\}$
3) Identities: $\forall F \in Hom_{Cat}$, Id_F
4) Composition rule:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \Rightarrow \quad G \circ F : \mathcal{C} \longrightarrow \mathcal{E}$$

$$X \longmapsto G(F(X))$$

$$f \longmapsto G(F(f))$$

Remark:

There are 2 issues with this definition:

1) It ignores the natural transformations. Remedy: consider the 2-Category of 1-categories, which consists of:
   1) objects: all 1-categories
   2) morphisms: all functors between 1-categories
   3) 2-morphisms / 2-cells: all natural transformations

2) Size issues. Remedies:
   1) switch foundations,
   2) consider the category of small category (its objects and morphisms only form a set, not a proper class)

LIMITS

We want to generalize the limit concept in analysis to categorical structures.

We know that:

$$3,\ 3.1,\ 3.14,\ 3.141 \longrightarrow \pi$$

Consider the following:

$$\mathbb{R}^0 \to \mathbb{R}^1 \to \mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}^4 \quad \ldots\ldots \quad \to \mathbb{R}^\infty$$

$$() \mapsto (x)$$

$$(x) \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ \vdots \end{pmatrix} : \begin{array}{l} x_i \in \mathbb{R}, \exists n \in \mathbb{N} : \\ \forall l \geq n \quad x_l = 0 \end{array} \right\}$$

We can also do the "opposite limit":

$$\mathbb{R}^\omega \to \quad \ldots\ldots \to \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0$$

<u>Example</u>

$$\begin{tikzcd}
 & \{0\} \arrow[r, hook] & \{0,1\} \arrow[r, hook] & \{0,1,10\} \arrow[dr, hook] & & \\
\phi \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \{1\} \arrow[ur, hook] & & & \{0,1,2,3,\ldots,12,\ldots\} \arrow[r, hook] & \mathbb{N} \\
 & \{2\} \arrow[r, hook] & \{2,5\} \arrow[urr, hook] & & &
\end{tikzcd}$$