Essentials of Functional Analysis

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We will work in ($\Omega \subseteq \mathbb{R}^d$):

1) $L^2(\Omega) = \{v : \Omega \to \mathbb{R}^2 : \int_\Omega v^2 \, d\Omega < +\infty\}$
2) $H^1(\Omega) = \{v : \Omega \to \mathbb{R}^2 : \int_\Omega (v^2 + |\nabla v|^2) \, d\Omega < +\infty\}$
3) $H^1_\Gamma(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$, $\Gamma \subseteq \Omega$

Remark:

$\Gamma = \partial\Omega \Rightarrow H^1_\Gamma(\Omega) = H^1_0(\Omega)$

$\langle f, g \rangle_{L^2} = \int_\Omega fg \, d\Omega$, $\quad \|f\|^2_{L^2} = \int_\Omega f^2 \, d\Omega$

$\langle f, g \rangle_{H^1} = \int_\Omega (fg + \nabla f \cdot \nabla g) \, d\Omega$,

$\|v\|^2_{H^1} = \int_\Omega (v^2 + |\nabla v|^2) d\Omega = \|v\|^2_{L^2} + \|\nabla v\|^2_{L^2}$

We will use the following inequalities:

1) Poincaré (P):

$v \in H^1_\Gamma(\Omega) \Rightarrow \|v\|_{L^2} \leq C_p \|\nabla v\|_{L^2}$

Corollary:

$\|\nabla v\|^2_{L^2} \leq \|v\|^2_{H^1} = \|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} \leq (1 + C_p^2) \|\nabla v\|^2_{L^2}$

2) Cauchy-Schwarz (CS):

V Hilbert Space $\Rightarrow \langle x, y \rangle_V \leq \|x\|_V \cdot \|y\|_V \quad \forall x, y \in V$

3) Young (Y):

$a, b \in \mathbb{R}^{>0}$, $p, q \in [1, +\infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

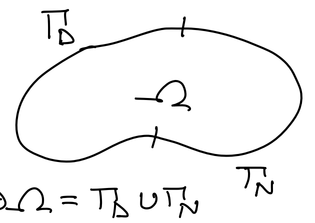
Corollary:

$$\forall \varepsilon > 0 \qquad ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$$

Stokes Equation

We want to find 2 functions $u : \Omega \to \mathbb{R}^d$ (velocity) and $p : \Omega \to \mathbb{R}$ (pressure) s.t. they satisfy the following integro-differential problem:

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = g & \text{in } \Gamma_D \\ \nu\nabla u\cdot\hat{n} + p\cdot\hat{n} = d & \text{in } \Gamma_N \end{cases}$$



$\partial\Omega = \Gamma_D \cup \Gamma_N$

with $V = [H^1(\Omega)]^d$ (velocity space), $Q = L^2(\Omega)$ (pressure space), $V_0 = [H^1_{\Gamma}]^d$

<u>Remark</u> (Weakening a Strong formulation):

Given the Poisson eq.:

$$\begin{cases} -\nu\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_D \\ \nabla u\cdot\hat{n} = d & \text{on } \Gamma_N \end{cases}, \quad V = H^1(\Omega),\ V_0 = H^1_{\Gamma_D}(\Omega)$$

$\Rightarrow$ taking $v \in V_0$ and integrating we have:

$$\int_\Omega -\nu\Delta u\cdot v\, d\Omega = \int_\Omega \nu\nabla u\cdot\nabla v\, d\Omega - \int_{\Gamma_N}\nu\nabla u\cdot\hat{n}\, v\, d\Gamma$$
$$= \int_\Omega \nu\nabla u\cdot\nabla v\, d\Omega - \int_{\Gamma_N} d v\, d\Gamma$$

We apply the same procedure to the Stokes Equation, so the problem is to find $u \in V$, $u|_{\Gamma_D} = g$ s.t.:

$$\int_\Omega \nu\nabla u\cdot\nabla v\, d\Omega = \int_\Omega f v\, d\Omega + \int_{\Gamma_N} d v\, d\Gamma \quad \forall v \in V_0$$

$\Rightarrow u \in V$, $v \in V_0$ s.t. $\quad\left(A:B = \sum_{i,j}^{m} A_{ij}B_{ij}\right)$

$$\int_\Omega -\nu\Delta u\cdot v\, d\Omega = \int_\Omega \nu\nabla u : \nabla v\, d\Omega - \int_{\Gamma_N}\nu\nabla u\cdot\hat{n}\cdot v\, d\Gamma$$
$$\int_\Omega \nabla p\cdot v\, d\Omega = -\int_\Omega p\,\nabla\cdot v\, d\Omega + \int_{\Gamma_N} p\hat{n}\cdot v\, d\Gamma$$
$$\Rightarrow \int_{\Gamma_N} p\hat{n}\cdot v\, d\Gamma - \int_{\Gamma_N}\nu\nabla u\cdot\hat{n}\cdot v\, d\Gamma = \int_{\Gamma_N} d\cdot v\, d\Gamma$$

Weak Formulation of the Stokes Equation:

Find $(v,p) \in V\times Q$, $v|_{\Gamma_D} = g$ s.t.:

$$\begin{cases} \int_\Omega \nu\nabla u : \nabla v\, d\Omega - \int_\Omega p\,\nabla\cdot v\, d\Omega = \int_\Omega f\cdot v\, d\Omega + \int_{\Gamma_N} d\cdot v\, d\Gamma & \forall v \in V_0, \\ \int_\Omega \nabla\cdot u\, q\, d\Omega = 0 & \forall q \in Q \end{cases}$$

We introduce the bilinear forms:

$$a : V \times V \to \mathbb{R} \text{ s.t. } a(u,v) = \int_\Omega \nu \nabla u : \nabla v \, d\Omega$$

$$b : V \times Q \to \mathbb{R} \text{ s.t. } b(v,q) = -\int_\Omega q \nabla \cdot v \, d\Omega$$

Then the problem becomes:

$$\begin{cases} a(u,v) + b(v,p) = F(v) \\ b(v,q) = 0 \end{cases} \qquad (\text{SADDLE POINT PROBLEM} - *)$$

where $F : V_0 \to \mathbb{R}$ s.t. $F(v) = \int_\Omega f \cdot v \, d\Omega + \int_{\Gamma_N} d \cdot v \, d\Gamma$.

Proposition:

If the following conditions hold:

1) a is continuous and coercive on V_0:

$$a(u,v) \leqslant C \|\nabla u\|_{L^2} \cdot \|\nabla v\|_{L^2} \quad (\text{continuity})$$

$$a(u,v) \geqslant \tilde{C} \|\nabla u\|^2_{L^2} \quad (\text{coercivity})$$

2) b is continuous on $V_0 \times Q$:

$$b(v,q) \leqslant C \|\nabla v\|_{L^2} \cdot \|q\|_{L^2}$$

3) The inf-sup condition holds:

$$\exists \beta > 0 \text{ s.t. } \inf_{q \in Q} \sup_{v \in V_0} \frac{b(v,q)}{\|\nabla v\|_{L^2} \cdot \|q\|_{L^2}} \geqslant \beta$$

then $\exists!$ sol. of the weak Stokes problem $(*)$, and the following estimate holds:

$$\|\nabla u\|_{L^2} + \|p\|_{L^2} \leqslant C \left(\|f\|_{L^2} + \|g\|_{H^{\frac{1}{2}}(\Gamma_D)} + \|d\|_{H^{\frac{1}{2}}(\Gamma_N)} \right)$$

Remark:

1) fully Dirichlet problems ($\Gamma_N = \phi$): if p is sol. of $(*)$ then $p + c$ is also sol. of $(*)$ $\forall c \in \mathbb{R}$, so we take:

$$Q = L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\Omega = 0 \right\}$$

and we add a compatibility condition on the data:

$$0 = \int_\Omega \nabla \cdot u \, d\Omega = \int_{\partial\Omega} u \cdot \hat{n} \, d\Gamma = \int_{\partial\Omega} g \cdot \hat{n} \, d\Gamma$$

2) fully Neumann problems ($\Gamma_D = \phi$): if u is sol. of $(*)$ then $u + c$ is also sol. of $(*)$ $\forall c \in \mathbb{R}$, so we take:

$$V = \left\{ v \in [H^1(\Omega)]^d : \int_\Omega v \, d\Omega = 0 \right\}$$

and we add a compatibility condition on the data:

$$\int_\Omega f\, d\Omega = -\int_{\partial\Omega} d\, d\Gamma$$

Galerkin Approximation

Let $V_h \subset V$, $Q_h \subset Q$ be discrete, finite spaces. Then:

$$N_u = \dim V_h \,,\quad N_p = \dim Q_h$$

$\Rightarrow$ the problem formulation becomes:

$$(u_h, p_h) \in V_h \times Q_h \,,\quad u_h|_{\Gamma_D} = g_h \quad \text{s.t.}$$

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = F(v_h) & \forall v_h \in V_{h0} \\ b(u_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

We have that conditions 1), 2) are trivially met.

Condition 3):

$$\exists \beta_h > 0 \quad \text{s.t.} \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|\nabla v_h\|_{L^2} \cdot \|q_h\|_{L^2}} \geq \beta_h$$

Remark:

If $\beta_h \to 0$ as $h \to 0$ there are stability problems !!!

So we require $\beta_h \not\propto h \Rightarrow \beta_h \to \beta$ as $h \to 0$.

Let V_h, Q_h be inf-sup incompatible. Then $\exists\, q_h^* \in Q_h$ s.t. $b(u_h, q_h^*) = 0$. If p_h is sol., then $p_h + q_h^*$ is sol:

$$a(u_h, v_h) + b(v_h, p_h + q_h^*) = a(u_h, v_h) + b(v_h, p_h) + b(\cancel{v_h}, q_h^*)$$
$$= F(v_h)$$

$\Rightarrow$ uniqueness is lost !!!

Remark:

q_h^* is called spourious pressure mode.

Let now $V_h = \text{span}\{\phi_i\}_{i=1}^{N_u}$, $Q_h = \text{span}\{\psi_k\}_{k=1}^{N_p}$. Then:

$$u_h = \sum_{i=1}^{N_u} \hat{u}_i \phi_i \,,\quad p_h = \sum_{k=1}^{N_p} \hat{p}_k \psi_k$$

$\Rightarrow$ we plug these expressions in $(*)$ and get:

$$\begin{cases} \displaystyle\sum_{i=1}^{N_u} \hat{u}_i\, a(\phi_i, \phi_j) + \sum_{k=1}^{N_p} \hat{p}_k\, b(\phi_j, \psi_k) = F(\phi_j) & j = 1, \dots, N_u \\ \displaystyle\sum_{i=1}^{N_u} \hat{u}_i\, b(\phi_i, \psi_\ell) = 0 & \ell = 1, \dots, N_p \end{cases}$$

We introduce

$$A \in \mathbb{R}^{N_U \times N_U} \quad \text{s.t.} \quad A_{ij} = a(\phi_i, \phi_j),$$
$$B \in \mathbb{R}^{N_U \times N_P} \quad \text{s.t.} \quad B_{jk} = b(\phi_j, t_k),$$
$$F \in \mathbb{R}^{N_U} \quad \text{s.t.} \quad F_j = F(\phi_j)$$

and the problem becomes:

$$\begin{cases} A\hat{u} + B^T\hat{p} = F \\ B\hat{u} = \vec{0} \end{cases} \iff \begin{pmatrix} A & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} F \\ \vec{0} \end{pmatrix}$$

$\Rightarrow$ suppose that we have:

$$\Omega = \bigcup_{\Omega_i \in \mathcal{T}_h} \Omega_i, \quad V_h \subseteq [H^1(\Omega)]^d,$$

$\Rightarrow$ we introduce the space of polynomials of deg $= r$:

$$X_h^r = \{ v \in C^0(\Omega) : v|_{\Omega_i} \in \mathbb{P}^r(\Omega_i) \ \forall \Omega_i \in \mathcal{T}_h \}$$

$$\Rightarrow V_h = [X_h^r]^d \Rightarrow X_h^r = \text{span}\{\phi_i\}_{i=1}^N, \quad N_U = d \cdot N$$

$$\Rightarrow V_h = \text{span}\Bigg\{ \underbrace{\begin{pmatrix} \phi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \phi_N \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{1^{st} \text{ component}}, \dots, \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_N \end{pmatrix}}_{d^{th} \text{ component}} \Bigg\}$$

$\Rightarrow$ if we are in $d = 3$ (3D):

$$u_h = \sum_{i=1}^N \hat{u}_i \begin{pmatrix} \phi_i \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^N \hat{u}_{N+i} \begin{pmatrix} 0 \\ \phi_i \\ 0 \end{pmatrix} + \sum_{i=1}^N \hat{u}_{2N+i} \begin{pmatrix} 0 \\ 0 \\ \phi_i \end{pmatrix}$$

$\Rightarrow A_{ij} = \int_\Omega \nu \nabla\phi_i \cdot \nabla\phi_j \, d\Omega$, define

$$K \in \mathbb{R}^{N \times N} \quad \text{s.t.} \quad K_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \quad \text{(stiffness matrix)}$$

$\Rightarrow A = \nu \begin{pmatrix} K & & 0 \\ & \ddots & \\ 0 & & K \end{pmatrix}$ and it's symmetric !!!

<u>Remark</u> (inf-sup at discrete level):

$c \in \mathbb{R}^n$ s.t. $\|c\|^2 = c \cdot c$, $Mc \in \mathbb{R}^n$ s.t. $\|Mc\|^2 = c^T M^T M c$

$$b(u_h, q_h) = \sum_{i=1}^{N_U} \sum_{k=1}^{N_P} \hat{u}_i \, b(\phi_i, t_k) \hat{p}_k = \hat{u} B^T \hat{p}$$

$$\Rightarrow \nu \|\nabla u_h\|_{L^2}^2 = a(u_h, u_h) = \sum_{i,j=1}^N \hat{u}_i \, a(\phi_i, \phi_j) \hat{u}_j = \hat{u}^T A \hat{u} = \|A^{\frac{1}{2}} \hat{u}\|$$

$$\Rightarrow \|p_h\|_{L^2}^2 = \int_\Omega p_h \cdot p_h \, d\Omega = \sum_{k,l=1}^{N_P} \hat{p}_k \int_\Omega t_k t_l \, d\Omega \, \hat{p}_l = \hat{p}^T M_p \hat{p} = \|M_p^{\frac{1}{2}} \hat{p}\|$$

where $M_p \in \mathbb{R}^{N_P \times N_U}$ s.t. $M_{p_{kl}} = \int_\Omega t_k t_l \, d\Omega$ (pressure mass matrix). We have:

$$\forall p_h \in Q_h \quad \sup_{u_h \in V_0} \frac{b(u_h, p_h)}{\|\nabla u_h\|_{L^2}} = \sup_{\hat{u} \in \mathbb{R}^N} \frac{\hat{u}^T B^T \hat{p}}{\|A^{\frac{1}{2}} \hat{u}\| / \sqrt{\nu}}$$

$$= \sup_{\hat{u} \in \mathbb{R}^N} \frac{\hat{u}^T A^{-\frac{1}{2}} A^{\frac{1}{2}} B^T \hat{p}}{\|A^{\frac{1}{2}} \hat{u}\| / \sqrt{\nu}} \underset{CS}{\leqslant} \cancel{\sup_{\hat{u}}} \frac{\|\cancel{\hat{u}^T A^{\frac{1}{2}}}\| \cdot \|A^{-\frac{1}{2}} B^T \hat{p}\|}{\|\cancel{A^{\frac{1}{2}} \hat{u}}\| / \sqrt{\nu}}$$

$$= \sqrt{\nu} \sqrt{\hat{p}^T B A^{-1} B^T \hat{p}}$$

$\Rightarrow$ take $\hat{u} = A^{-1} B^T \hat{p}$. We have:

$$\sup_{u_h \in V_{h0}} \frac{b(u_h, p_h)}{\|\nabla u_h\|_{L^2}} = \sup_{\hat{u} \in \mathbb{R}^N} \frac{\hat{u}^T B^T \hat{p}^T}{\|A^{-\frac{1}{2}} \hat{u}\| / \sqrt{\nu}} \leqslant \frac{(\hat{p}^T B A^{-1}) B^T p}{\sqrt{\hat{p}^T B A^{-1} A A^{-1} B^T \hat{p}} / \sqrt{\nu}}$$

$$= \sqrt{\nu} \frac{\hat{p}^T B^T A^{-1} B^T \hat{p}}{\sqrt{\hat{p}^T B A^{-1} B^T \hat{p}}} = \sqrt{\nu} \sqrt{\hat{p}^T B A^{-1} B^T \hat{p}}$$

$$\Rightarrow \beta_h^2 \leqslant \left(\inf_{p_h \in Q_h} \sup_{u_h \in V_h} \frac{b(u_h, p_h)}{\|\nabla u_h\|_{L^2} \|p_h\|_{L^2}} \right)^2 = \inf_{\hat{p} \in \mathbb{R}^N} \frac{\nu \hat{p}^T B A^{-1} B^T \hat{p}}{\hat{p}^T M_p \hat{p}}$$

This inequality is equivalent to the eigen problem:

$$B A^{-1} B^T \hat{p} = \lambda \frac{M_p}{\nu} \hat{p}, \quad \lambda \text{ eigenvalue of } B A^{-1} B^T \text{ w.r.t. } \frac{M_p}{\nu}$$

$\Rightarrow 0 < \beta_h^2 \leqslant \lambda_{min} \Rightarrow B A^{-1} B^T$ is symmetric pos. def. wrt M_p/ν

then:

$$b(u_h, p_h) = -\int_\Omega p_h \nabla \cdot u_h \, d\Omega \underset{CS}{\leqslant} \|p_h\|_{L^2} \cdot \|\nabla u_h\|_{L^2}$$

$$\leqslant \|p_h\|_{L^2} \cdot \sqrt{d} \, \|\nabla u_h\|_{L^2} = \sqrt{d} \, \|A^{-\frac{1}{2}} \hat{u}\| \cdot \|M_p^{\frac{1}{2}} \hat{p}\|$$

$$\Rightarrow \forall p_h \in Q_h \quad \sup_{u_h \in V_h} \frac{b(u_h, p_h)}{\|\nabla u_h\|_{L^2}} = \sqrt{\nu} \sqrt{\hat{p}^T B A^{-1} B^T \hat{p}}$$

$$\leqslant \frac{\sqrt{d} \, \|\cancel{A^{\frac{1}{2}} \hat{u}}\| \cdot \|M_p^{\frac{1}{2}} \hat{p}\|}{\|\cancel{A^{\frac{1}{2}} \hat{u}}\| / \sqrt{\nu}}$$

$$\Rightarrow \frac{\hat{p}^T B A^{-1} B^T \hat{p}}{\hat{p}^T \frac{M_p}{\nu} \hat{p}} \leqslant d \Rightarrow \beta_h^2 \leqslant \lambda \leqslant d$$