Properties of Functors

Def. (Essential Surjectivity, Faithfulness, Fullness):
A functor $F: \mathcal{C} \to \mathcal{D}$ is:

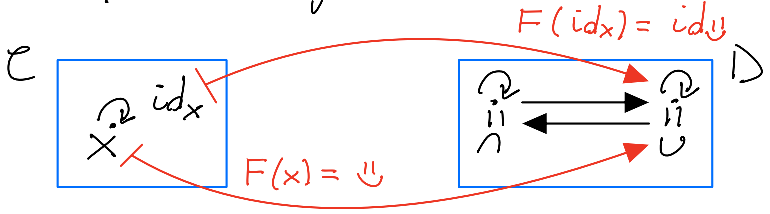
1) **Essentially Surjective** if:
$$\forall Y \in Ob(\mathcal{D}) \ \exists X \in Ob(\mathcal{C}) \text{ s.t. } F(X) \cong Y$$
2) **Faithful** if:
$$\forall f, g : X \to Y \text{ in } \mathcal{C} \text{ we have:}$$
$$F(f) = F(g) \Rightarrow f = g$$
3) **Full** if:
$$\forall X, Y \in Ob(\mathcal{C}) \ \forall g : F(X) \to F(Y) \text{ in } \mathcal{D}$$
$$\exists f : X \to Y \text{ in } \mathcal{C} \text{ s.t. } F(f) = g$$
4) **Fully Faithful** if it is both faithful and full

Proposition
Let $F: \mathcal{C} \to \mathcal{D}$ and consider the following map:
$$\begin{aligned} Hom_{\mathcal{C}}(X, Y) &\to Hom_{\mathcal{D}}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

Then F is:
1) faithful iff the map is injective $\forall X, Y \in Ob(\mathcal{C})$
2) full iff the map is surjective $\forall X, Y \in Ob(\mathcal{C})$
3) fully faithful if the map is bijective $\forall X, Y \in Ob(\mathcal{C})$

Def. (Elementary Equivalence, Equivalent Categories):
A functor is an **Elementary Equivalence** if it is fully faithful and essentially surjective. 2 categories are **Equivalent** if $\exists$ elementary equivalence between them.

example:
1) Given $\mathcal{C}, \mathcal{D}$ as follows:



the functor $F: \mathcal{C} \to \mathcal{D}$ s.t. $F(x) = ☺$, $F(id_x) = id_{☺}$
is an elementary equivalence. $\mathcal{C}, \mathcal{D}$ are equivalent
2) $Vect(\mathbb{R})$ is equivalent to the numerical category.

Remark:

Equivalent categories have exactly the same categorical properties (NO COMPARISON OF OBJECTS UP TO EQUALITY!!!)

Lemma:

Functors preserve commutative diagrams and isomorphisms

Proof:

1) if $g \circ f = h$, then $F(g) \circ F(f) = F(g \circ f) = F(h)$
2) if $f \circ g = id \wedge g \circ f = id$, then $F(f) \circ F(g) = F(id) = id \wedge$
$F(g) \circ F(f) = F(id) = id$

$\square$

NATURAL TRANSFORMATIONS

example:

1) $X \longrightarrow X$
$x \longmapsto x$

2) $List(X) \longrightarrow List(X)$
$[x_1, \dots, x_n] \longmapsto [x_n, \dots, x_1]$

Non example:

1) $X \longrightarrow X$
$x \longmapsto \begin{cases} 7 & x \in \mathbb{R} \\ x & \text{otherwise} \end{cases}$

2) $List(X) \longrightarrow List(X)$
$[x_1, \dots, x_n] \longmapsto \begin{cases} [5, \pi] & x \in \mathbb{R} \\ [x_1, \dots, x_n] & \text{otherwise} \end{cases}$

Natural Transformations are uniform families of morphisms

Def. (Natural Transformation):

A NATURAL TRANSFORMATION $\eta : F \Rightarrow G$ between the functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ ($\mathcal{C}, \mathcal{D}$ categories) consists of:

1) $\forall X \in Ob(\mathcal{C})$ a morphism $\eta_X : F(X) \longrightarrow G(X)$ in $\mathcal{D}$ s.t. $\forall f : X \to Y$ in $\mathcal{C}$ the naturality square commutes:

$$\begin{tikzcd} X \arrow[d, "\forall f"'] \\ Y \end{tikzcd} : \quad \begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "///"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd} \Rightarrow G(f) \circ \eta_X = \eta_Y \circ F(f)$$

example:

$X \longrightarrow X^3$
$x \longmapsto (x, x, x)$
is a natural transformation. More precisely

the map $\eta_{Set} : id_{Set} \Rightarrow G$ given by $\forall X \in Ob(Set)$ the maps

$$\eta_X : id_{Set}(X) \to G(X), \quad x \mapsto (x,x,x), \qquad id_{Set} : Set \to Set, \quad X \mapsto X, \quad f \mapsto f,$$

$$G : Set \to Set$$
$$X \mapsto X^3$$
$$f \mapsto ((x,y,z) \mapsto (f(x), f(y), f(z)))$$

is a natural transformation:
given $f : X \to Y$ an arbitrary map we have that the naturality square commutes:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X^3 \arrow[d] \\
Y \arrow[r, "\eta_Y"'] & Y^3
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\equiv"] & (x,x,x) \arrow[d, mapsto] \\
f(x) \arrow[r, mapsto] & (f(x), f(y), f(z))
\end{tikzcd}$$