DUAL OF $L^p(\mu)$

Let q be the conjugate exponent of p, $v \in L^q(\mu)$, consider the linear functional $T_v : L^p(\mu) \to \mathbb{R}$ defined by:

$$T_v(u) = \int_X u(x) v(x)\, d\mu(x) \in \mathbb{R}$$

$\Rightarrow$ (Holder's ineq.) $\int_X |u(x)| \cdot |v(x)|\, d\mu(x) \leq \|v\|_{L^q} \cdot \|u\|_{L^p} < +\infty$

$\Rightarrow |T_v(u)| \leq \|v\|_{L^q} \cdot \|u\|_{L^p} \Rightarrow T$ is continuous : $\|T_v\|_{(L^p)'} \leq \|v\|_{L^q}$

The map $\phi : L^q(\mu) \to (L^p(\mu))'$ is linear and continuous.

$$v \longmapsto T_v$$

In many cases, such ϕ is also an isometry:

CLAIM:

$1 \leq p < +\infty \Rightarrow \phi$ is a linear isometry:

$$\|v\|_{L^q(\mu)} = \|T_v\|_{(L^p(\mu))'}$$

Proof:

Consider indeed $u(x) = \text{sign}(v(x)) \dfrac{|v(x)|^{q-1}}{\|v\|^{q-1}_{L^q(\mu)}}$, $p = \dfrac{q}{q-1}$:

$$\|u\|_{L^p(\mu)} = \frac{1}{\|v\|^{q-1}_{L^q(\mu)}} \left(\int_X |v(x)|^q d\mu \right)^{\frac{q-1}{q}} = 1$$

$$\Rightarrow T_v(u) = \int_X \frac{v(x)\, \text{sign}\, v(x)\, |v(x)|^{q-1}}{\|v\|^{q-1}_{L^q}} d\mu(x) = \frac{\int_X |v(x)|^q d\mu}{\|v\|^{q-1}_{L^q}} = \|v\|_{L^q}$$

Case $p = 1 \Rightarrow q = +\infty$:

Assume μ satisfies the following hypothesis:

$\forall A$ μ-meas. set with $\mu(A) = +\infty$ $\exists B \subset A$ μ-meas. s.t. $0 < \mu(B) < +\infty$

Then $\phi : L^\infty(\mu) \to (L^1(\mu))'$ s.t. $\phi(v) = T_v$ is an isometry

Indeed, let $\varepsilon > 0 \Rightarrow \mu(\{x \in X : |v(x)| > \|v\|_{L^\infty} - \varepsilon\}) > 0$

$\Rightarrow \exists A$ μ-meas. with $0 < \mu(A) < +\infty$ s.t. $|v(x)| > \|v\|_{L^\infty} - \varepsilon$ $\forall x \in A$

Define

$$u(x) = \begin{cases} \dfrac{\text{sign}(v(x))}{\mu(A)} & x \in A \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow \|u\|_{L^1} = \int_A \frac{1}{\mu(A)} d\mu(x) = 1$, $T_v(u) = \frac{1}{\mu(A)} \int_A \text{sign}(v(x)) \cdot v(x)\, d\mu(x)$

$\geq \frac{1}{\mu(A)} \int_A (\|v\|_{L^\infty} - \varepsilon)\, d\mu(x) = \|v\|_{L^\infty} - \varepsilon$

The claim is false for $p = +\infty$: in general ϕ is an isometry, not necessarily surjective.

Example:

$\phi : \ell^1 \to \ell^\infty$ is NOT surjective: $\phi(\underset{\ell^1}{\underset{\cap}{\{y_k\}}}_{k\in\mathbb{N}}) = T_{\{y_k\}_k}$

with $T_{\{y_k\}}(\underset{\ell^\infty}{\underset{\cap}{\{x_k\}}}) = \sum_{k=1}^{+\infty} x_k y_k$

$\ell^\infty = \{\{x_k\}_k : \underbrace{\sup\{|x_k| : k\in\mathbb{N}\}}_{=:\,\|\{x_k\}_k\|_{\ell^\infty}} < +\infty\}$.

Consider $c = \{\{x_k\} \in \ell^\infty : \exists \lim_{k\to+\infty} x_k\}$ vector subspace and take $T : c \to \mathbb{R}$ with $T(\{x_k\}_k) = \lim_{k\to+\infty} x_k$. We have $\|T\|_{(c)'} = 1$ (indeed, $|T(\{x_k\}_k)| = |\lim_{k\to+\infty} x_k| \leq \|\{x_k\}_k\|_{\ell^\infty} \wedge$ if $\{x_k\}_k = \{1\}_k$ we get the equality).

$\Rightarrow$ We can apply Hahn-Banach Thm. and extend T to $T : \ell^\infty \to \mathbb{R}$ with $\|T\|_{(\ell^\infty)'} = 1$. Suppose by contradiction $\exists \{y_k\}_k \in \ell^1$ s.t. $T(\{x_k\}) = \sum_{k=1}^{+\infty} x_k y_k$. Let $e^n \in \ell^\infty$ be the seq. $\{e^n_k\}_k := \{\delta_{kn}\}_k \Rightarrow e^n \in c \wedge T(e^k) = 0 \wedge T(e^k) = \sum_{k=1}^{+\infty} \delta_{kn} y_k = y_n$

$\Rightarrow y_n = 0 \ \forall n \Rightarrow y_k = 0 \ \forall k$ ↯