Categories

Observations:

1) $\min\{\underset{3}{a}, \underbrace{\min\{\underset{7}{b}, \underset{10}{c}\}}_{=7}\}$ with $\underbrace{\min\{a, \min\{b,c\}\}}_{=3}$, $\min\{\underbrace{\min\{\underset{3}{a}, \underset{7}{b}\}}_{=3}, \underset{10}{c}\}$ with $\underbrace{\min\{\min\{a,b\},c\}}_{=3}$ $\Rightarrow$ it's associative

2) $X\times(Y\times Z) = \{(x,(y,z)) : x\in X, y\in Y, z\in Z\}$,
$(X\times Y)\times Z = \{((x,y),z) : x\in X, y\in Y, z\in Z\}$
$\Rightarrow$ they are not identical, but $\exists$ bijection between them. Each element of a set can be reshuffled into an element of the other set.
$\Rightarrow X\times(Y\times Z)\cong(X\times Y)\times Z$

3) $X \amalg (Y\amalg Z) \cong (X\amalg Y)\amalg Z$ ($\amalg :=$ topological product)

$\shortparallel \qquad\qquad \shortparallel$

$X(YZ) \qquad (XY)Z$

$\Rightarrow$ for instances of a general categorical fact, there is associativity of the categorical product up to isomorphism

Remarks (Motivations for Category theory):

1) Conceptual clarity
2) Recognize analogies between analogies
3) Renders the trivial trivially trivial
4) Foundational language for algebraic topology, topos theory, combinatorics, quantum field theory, compositional systems, programming languages, algebraic geometry, logic, database theory ...

Def. (Category):

A CATEGORY $\mathcal{C}$ consists of:

1) A class $Ob(\mathcal{C})$ of objects of $\mathcal{C}$
2) $\forall X, Y \in Ob(\mathcal{C})$, a class $Hom_{\mathcal{C}}(X,Y)$ of morphisms for X, Y
3) $\forall X\in Ob(\mathcal{C})$, an identity morphism $id_X \in Hom_{\mathcal{C}}(X,X)$
4) $\forall X, Y, Z \in Ob(\mathcal{C})$, a composition rule:

$$\begin{aligned} Hom_{\mathcal{C}}(Y,Z)\times Hom_{\mathcal{C}}(X,Y) &\longrightarrow Hom_{\mathcal{C}}(X,Z) \\ (g,f) &\longmapsto g\circ f \end{aligned}$$

Such that it satisfies the following axioms:

1) Associativity of composition:

$\forall X, Y, Z, W \in Ob(\mathcal{C})$, $\forall f \in Hom_{\mathcal{C}}(X,Y)$, $\forall g \in Hom_{\mathcal{C}}(Y,Z)$, $\forall h \in Hom_{\mathcal{C}}(Z,W)$ we have:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2) Neutrality:

$\forall X, Y \in Ob(\mathcal{C})$, $\forall f \in Hom_{\mathcal{C}}(X,Y)$ we have:

$$id_Y \circ f = f \wedge f \circ id_X = f$$

Examples:

1) Category Set of sets and maps:
   1) objects: all sets
   2) morphisms: all maps between sets
   3) identity maps: usual identity maps $id_X : X \to X$, $id_X(x) = x$
   4) composition rule: usual composition of maps
2) Category Vect($\mathbb{R}$) of the $\mathbb{R}$-vector spaces:
   1) objects: all $\mathbb{R}$-vector spaces
   2) morphisms: all LINEAR maps
      $$Hom_{Vect(\mathbb{R})}(V,W) = \{f : V \to W : f \text{ linear}\}$$
   3) identity: usual identity maps
   4) composition rule: usual composition of maps
3) Numerical Category:
   1) objects: all natural numbers
   2) morphisms: $Hom(a,b) = \{A : A \in \mathbb{R}^{b \times a}\}$ set of $b \times a$ real matrices
   3) identity: $id_a := Id_{a\times a} = diag(1, \dots, 1)_{a \times a}$
   4) composition: multiplication between matrices

   $\Rightarrow$ the numerical category is equivalent to the full subcategory of Vect($\mathbb{R}$) of the finite-dimensional vector spaces.
4) Category of Pokémon:
   1) objects: all Pokémon
   2) morphisms: evolution maps
   3) identity: self-evolution maps
   4) composition: composition of evolution maps.

$\Rightarrow$ this category is thin i.e. between any objects there is at most 1 morphism

Initial and Terminal Objects

Def. (Initial Object):
An object I of a category $\mathcal{C}$ is INITIAL iff $\forall X \in Ob(\mathcal{C})$
$\exists!\, f \in Hom_{\mathcal{C}}(I, X)$

Def. (Terminal Object):
An object T of a category $\mathcal{C}$ is TERMINAL iff $\forall X \in Ob(\mathcal{C})$
$\exists!\, f \in Hom_{\mathcal{C}}(X, T)$

Examples:

1) In the Set Category, the singletons $\{x\}$ are terminal and the empty set ϕ is initial
2) In $Vect(\mathbb{R})$, $\{\vec{0}\}$ is initial and terminal
3) In the Pokémon category, $\nexists$ initial nor terminal objects

N.B.
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