Introduction

1) Basic theory of Banach/Hilbert spaces (L^p spaces)
2) Measure theory
3) Sobolev spaces, basic PDE's applications
4) Operator theory
5) BV functions
6) Distributions

Brief recap on Measure Theory

An outer measure on a set X is a function

$$\mu : \mathcal{P}(X) \to [0,+\infty]$$

such that:

1) $\mu(\emptyset) = 0$
2) $\mu(A) \leq \sum_{k=1}^{+\infty} \mu(A_k)$ whenever $A, A_1, \dots, A_k \in \mathcal{P}(X)$ and $A \subset \bigcup_k A_k$

E.G.:

1) Lebesgue measure:

$X = \mathbb{R}^n$, if $A \subset \mathbb{R}^n \Rightarrow |A| = \inf\left\{ \sum_{k=1}^{+\infty} \underbrace{|I_k|}_{\substack{n\text{-th dim.}\\ \text{volume of } I_k}} : \begin{array}{l} I_k \text{ intervals,} \\ A \subset \bigcup_k I_k \end{array} \right\}$

2) δ mass:

X set, $X \neq \emptyset$, $x_0 \in X$, $A \subset X \Rightarrow \delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & \text{oth.} \end{cases}$

3) Counting measure:

X set, $A \subset X \Rightarrow \#(A) = \mathcal{H}^0(A) = \begin{cases} \#\text{ of elements of } A \text{ if } A \text{ is finite} \\ +\infty \quad \text{oth.} \end{cases}$

<u>Def.</u> (Measure set for an outer measure μ on the set X)

Charatheodory:

$A \subset X$ is measurable $\Leftrightarrow \mu(T) = \mu(T \cap A) + \mu(T \setminus A) \quad \forall T \subset X$

<u>Thm.</u>:

Given μ outer measure on a set X,

1) $\emptyset, X$ are μ-measurable, A is μ-measurable $\Rightarrow X \setminus A$

is μ-measurable

2) if $A_1, \dots, A_k, \dots$ are μ-measurable $\Rightarrow \bigcup_{k=1}^{+\infty} A_k$, $\bigcap_{k=1}^{+\infty} A_k$ is μ-measurable

3) if $A_1, \dots, A_k, \dots$ are μ-measurable and are pairwise disjointed $\Rightarrow \mu\left(\bigcup_{k=1}^{+\infty} A_k\right) = \sum_{k=1}^{+\infty} \mu(A_k)$ (COUNTABLE ADDITIVITY)

4) if $A_1 \subset \dots \subset A_k \subset \dots$ are μ-measurable $\Rightarrow \mu\left(\bigcup_{k=1}^{+\infty} A_k\right) = \lim_{k\to+\infty} \mu(A_k)$

5) if $A_1 \supset \dots \supset A_k \supset \dots$ are μ-measurable and $\mu(A_1) < +\infty$
$\Rightarrow \mu\left(\bigcap_{k=1}^{+\infty} A_k\right) = \lim_{k\to+\infty} \mu(A_k)$

<u>Def.</u>

Given X a set,

$\mathcal{C}$ σ-algebra of subsets of X

$\Leftrightarrow$

$\emptyset \in \mathcal{C}$, $A^c := X \setminus A \in \mathcal{C}$ $\forall A \in \mathcal{C}$, whenever $A_1, \dots, A_k, \dots \in \mathcal{C}$ we have $\bigcup_{k=1}^{+\infty} A_k \in \mathcal{C}$ $\left[\bigcap_{k=1}^{+\infty} A_k = \left(\bigcup_{k=1}^{+\infty} A^c\right)^c\right]$

<u>Def.</u> (Measure on a σ-algebra):

$\mathcal{C}$ a σ-algebra of subsets of X.

$\mu : \mathcal{C} \to [0, +\infty]$ is a measure $\Leftrightarrow \mu(\emptyset) = 0$, μ is countably additive

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MEASURABLE FUNCTIONS - INTEGRALS

(X, μ) outer measure on X (or a measure on a σ-algebra of subsets of X).

$f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is μ-meas. if $f^{-1}((a, +\infty]) = \{x \in X \mid f(x) > a\}$ is μ-meas. $\forall a \in \mathbb{R}$

$\Leftrightarrow$

$f^{-1}(U)$ is μ-meas. $\forall U \subset \overline{\mathbb{R}}$, U open.

N.B.

$\forall \varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ continuous, $f : X \to \overline{\mathbb{R}}$ μ-meas. $\Rightarrow \varphi \circ f$ μ-meas.

<u>Def.</u> (Simple function):

$s : X \to [0, +\infty)$ is simple $\Leftrightarrow s$ is μ-meas. and $s(X)$ is a finite set

$\Leftrightarrow s(x) = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j}(x)$ $c_1, \dots, c_n \in [0, +\infty)$, $A_1, \dots, A_n$ μ-meas., pairwise disjoint, $\bigcup_{j=1}^{n} A_j = X$

Def. (Integral of a simple function):

s a simple function $\Rightarrow \int_X s(x)\,d\mu(x) = \sum_{s=1}^{n} c_s\,\mu(A_s)$

N.B. (CONVENTION):

$0 \cdot +\infty = 0$!!!!

Given $f : X \to [0, +\infty]$ μ-meas. we define the integral of f:

$$\int_X f(x)\,d\mu(x) = \sup\left\{ \int_X s(x)\,d\mu(x) : s \text{ is simple}, s \leq f \right\}$$

Thm. (Approximation Theorem):

Let $f : X \to [0, +\infty]$ μ-meas. There are simple functions $s_1 \leq \ldots \leq s_k \leq \ldots$ s.t. $\lim_{k \to +\infty} s_k(x) = f(x)$ $\forall x \in X$

Thm. (Beppo Levi Theorem - Monotone Convergence):

If $f_1 \leq \ldots \leq f_k \leq \ldots$ are μ-meas., $f_k : X \to [0, +\infty]$, $f(x) = \lim_{k \to +\infty} f_k(x)$ then we have:

$$\int_X f(x)\,d\mu(x) = \lim_{k \to +\infty} \int_X f_k(x)\,d\mu(x)$$