# Category Theory Course Notes

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# Chapter 1

## 1.1 Definition of Category

A **category** (1-category) $\mathcal{C}$ consists of:

1 - A class $Ob(\mathcal{C})$ of objects of $\mathcal{C}$

2 - $\forall X, Y \in Ob(\mathcal{C})$.
a class $Hom_{\mathcal{C}}(X, Y)$ of **morphisms** from $X$ to $Y$

3 - $\forall X \in Ob(\mathcal{C})$.
an **identity morphism** $id_X \in Hom_{\mathcal{C}}(X, X)$

4 - $\forall X, Y, Z \in Ob(\mathcal{C})$.
a **composition rule**:

$$Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$$
$$(g, f) \mapsto g \circ f$$

Such that it satisfies the following axioms:

1 - **Associativity of composition**:

$\forall X, Y, Z, W \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y), g \in Hom_{\mathcal{C}}(Y, Z), h \in Hom_{\mathcal{C}}(Z, W)$.
$h \circ (g \circ f) = (h \circ g) \circ f$

2 - **Neutrality**:

$\forall X, Y \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y)$.
$id_Y \circ f = f \wedge f \circ id_X = f$

## 1.2 Thin Categories

A category is **thin** if parallel morphisms are always the same, meaning that there is only one morphism between two objects.

In a thin category all morphisms are monic and epic.

## 1.3 Definition of Initial Object

An object $I$ of a category $\mathcal{C}$ is **initial** (dual of terminal, special case of a colimit (of a functor from $\mathcal{C}$ to the empty category))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(I, X)$

## 1.4 Definition of Terminal Object

An object $T$ of a category $\mathcal{C}$ is **terminal** (dual of initial, special case of limit (of a functor from the empty category to $\mathcal{C}$))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(X, T)$

## 1.5 Definition of Monomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is a **monomorphism** (or monic in $\mathcal{C}$) (dual of epimorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Z, X)$.
$f \circ p = f \circ q \implies p = q$

Example:
In **Set** monomorphisms are precisely the injective maps.

Monomorphisms "can be cancelled" from the left.

## 1.6 Definition of Split Monomorphism

A **split monomorphism** (dual of split epi) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$g \circ f = id_X$

Proposition: every split mono is a mono.
Proposition: in **Set**, every mono $f : X \to Y$ where $X$ is inhabited is a split mono, assuming LEM holds.

## 1.7 Definition of Epimorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **epimorphism** (or epic in $\mathcal{C}$) (dual of monomorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Y, Z).$
$p \circ f = q \circ f \implies p = q$

Example:
In **Set** epimorphisms are precisely the surjective maps.

Epimorphisms "can be cancelled" from the right.

## 1.8 Definition of Split Epimorphism

A **split epimorphism** (dual of split mono) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$f \circ g = id_Y$

Proposition: every split epi is an epi.
Proposition: in **Set**, every epi is a split epi $\iff$ assuming LEM holds.

## 1.9 Definition of Isomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **isomorphism**
$\Updownarrow$
$\exists g \in Hom_{\mathcal{C}}(Y, X).$
$f \circ g = id_Y \wedge g \circ f = id_X$

$id_X \forall X \in Ob(\mathcal{C})$ is always an isomorphisms for every category $\mathcal{C}$.
Objects $X$ and $Y$ in a category $\mathcal{C}$ are **isomorphic**
$\Updownarrow$
there exists an isomorphism between $X$ and $Y$ ($X \cong Y$)

In **Set**, if there exists an isomorphism between $X$ and $Y$, $X$ and $Y$ are called eqinumerous.

## 1.10 Definition of Opposite Category

"The mother of all dualities"

Let $\mathcal{C}$ be a category. Then its opposite category $\mathcal{C}^{op}$ is the following category:

- $Ob(\mathcal{C}^{op}) := Ob(\mathcal{C})$

- $Hom_{\mathcal{C}^{op}}(X, Y) := Hom_{\mathcal{C}}(Y, X)$

- identities and composition inherited from $\mathcal{C}$
  $id_X \in Hom_{\mathcal{C}}(X, X) = id_X^{op} \in Hom_{\mathcal{C}^{op}}(X, X)$
  $f \circ g := g^{op} \circ f^{op}$

Observations / Remarks:

- An object $I$ of $\mathcal{C}$ is initial in $\mathcal{C}$
  $\Updownarrow$
  I is terminal when regarded as an object of $\mathcal{C}^{op}$

- A morphism in $\mathcal{C}$ is a monomorphism
  $\Updownarrow$
  it is an epimorphism in $\mathcal{C}^{op}$

## 1.11 Dualities?

## 1.12 Definition of Product

A **product** (special case of limit) of two objects $X$ and $Y$ in a category $\mathcal{C}$ consists of:

- an object $P$ of $\mathcal{C}$

- a morphism $\pi_X : P \to X$ in $\mathcal{C}$

- a morphism $\pi_Y : P \to Y$ in $\mathcal{C}$

such that for every object $Q$ of $\mathcal{C}$ together with morphisms $\varphi_X : Q \to X, \varphi_Y : Q \to Y$ there is exactly one morphism $Q \to P$ such that the following diagram commutes:

$\varphi_X = \pi_X \circ !$
$\varphi_Y = \pi_Y \circ !$

Remarks:
- Products are always associative and commutative up to isomorphism.
- There is also the notion of the (co) product of zero, one, three, four, ... objects.
- The zero case of a product is just a terminal object, an object with exactly one morphism from each object.

## 1.13 Definition of Coproducts

A **coproduct** (special case of colimits) of two objects $X$ and $Y$ in a category $\mathcal{C}$ consists of:

- an object $C$ of $\mathcal{C}$

- a morphism $\iota_X : X \to C$ in $\mathcal{C}$

- a morphism $\iota_Y : Y \to C$ in $\mathcal{C}$

such that for every object $D$ of $\mathcal{C}$ together with morphisms $\chi_X : X \to D, \chi_Y : Y \to D$ there is exactly one morphism $C \to D$ which renders the following diagram commutative:

$\chi_X =! \circ \iota_X$
$\chi_Y =! \circ \iota_Y$

Remarks:

- Products in $\mathcal{C}^{op}$ are precisely coproducts in $\mathcal{C}$

- The zero case of a coproduct is the same as an initial object.

## 1.14 Definition of Functor

A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

- an object $F(X) \in Ob(\mathcal{D})$ for each object $X \in Ob(\mathcal{C})$

- a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each morphism $f : X \to Y$ in $\mathcal{C}$

such that:

- $\forall X \in Ob(\mathcal{C}).\, F(id_X) = id_{F(X)}$

- $\forall X, Y, Z \in Ob(\mathcal{C}).\, \forall f : X \to Y \in \mathcal{C}, g : Y \to Z$ in $\mathcal{C}.\, F(g \circ f) = F(g) \circ F(f)$

Motto:
Functors $\mathcal{I} \to \mathcal{C}$ are $\mathcal{I}$-shaped **diagrams** in $\mathcal{C}$

## 1.15 Definition of Contravariant Functor

A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

## 1.16 Forgetful Functors?

## 1.17 Powerset Functor[s?]?

## 1.18 Definition of Discrete Category

The **discrete category** associated with a set $X$, written $\mathcal{D}(X)$, is a category containing all the objects of $X$ as objects, and no morphisms between different objects, just the identity morphisms.

## 1.19 Definition of Induced Functors

Claim:
Any map between sets can be turned into a functor.

Let $f : X \to Y$ be a map between sets.

Consider the discrete categories $\mathcal{D}(X), \mathcal{D}(Y)$.

Then $f$ induces the following functor $\mathcal{D}(X) \to D(Y)$:
$x \mapsto f(x)$
$id_x \mapsto id_{f(x)}$

## 1.20 Definition of the Walking Arrow?

## 1.21 Definition of the Walking Commutative Triagle?

## 1.22 Definition of Essentially Surjective Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** iff:

$$\forall Y \in Ob(\mathcal{D}). \exists X \in Ob(\mathcal{C}) | F(X) \cong Y$$

## 1.23 Definition of Faithful Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** iff:

$$\forall X, Y \in Ob(\mathcal{C}).$$
$$\forall f, g : X \to Y \text{ in } \mathcal{C}$$
$$F(f) = F(g) \implies f = g$$

Reformulation: iff

$$\forall X, Y \in Ob(\mathcal{C}).$$
$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$f \mapsto F(f)$$

is injective.

## 1.24 Definition of Full Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** iff:

$$\forall X, Y \in Ob(\mathcal{C}).$$
$$\forall g : F(X) \to F(Y) \text{ in } \mathcal{D}$$
$$\exists f : X \to Y \text{ in } \mathcal{C} | F(f) = g$$

Reformulation: iff

$$\forall X, Y \in Ob(\mathcal{C}).$$
$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$f \mapsto F(f)$$

is surjective.

## 1.25 Definition of Fully Faithful Functor

A functor is **fully faithful** iff it is full and faithful.

Reformulation: iff

$\forall X, Y \in Ob(\mathcal{C})$.
$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$
$f \mapsto F(f)$

is bijective.

## 1.26 Definition of Elementary Equivalence

An **elementary equivalence** is a fully faithful, essentially surjective functor.

## 1.27 Definition of Equivalence of Categories

Categories are called **equivalent** iff there is an elementary equivalence between them.

Remark: Equivalent categories have exactly the same categorical properties.

## 1.28 Definition of Natural Transformation

A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : C \to D$ consists of:

- for each object $X \in Ob(\mathcal{C})$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$

such that for all morphisms $f : X \to Y$ in $\mathcal{C}$, the **naturality square** commutes:

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

Motto:
Natural transformations are **uniform** families of morphisms.

## 1.29 Definition of Functor Category

Let $\mathcal{C}, \mathcal{D}$ be categories.
The **functor category** $[\mathcal{C}, \mathcal{D}]$ has:

- as objects: all functors $\mathcal{C} \to \mathcal{D}$

- as morphisms: $Hom_{[\mathcal{C},\mathcal{D}]}(F, G) \coloneqq \{h : F \Rightarrow G | h \text{ is a natural transformation}\}$

- as identity: for the object F, the identity $id_F : F \Rightarrow F$
  $(id_F)_X : F(X) \to F(X)$
  given by $id_{F(X)}$

- as composition rule:
  $(\omega \circ \eta)_X \coloneqq \omega_X \circ \eta_X$

  $\omega_X : G(X) \to H(X)$
  $\eta_X : F(X) \to G(X)$

and $\omega \circ \eta$ should be natural.

## 1.30 Definition of Small Category

A category $\mathcal{C}$ is small when $Ob(\mathcal{C})$ is just a set and not a proper class.

## 1.31 Definition of Category of Categories

The **1-category of 1-categories**, **Cat** has:

- as objects: all categories

- as morphisms: $Hom_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \coloneqq \{F : \mathcal{C} \to \mathcal{D} | F \text{ is a functor}\}$

- as identities $Id_F$ (the identity functor?)

- as composition rule:
  $F : \mathcal{C} \to \mathcal{D}$
  $G : \mathcal{D} \to \mathbf{E}$

  $G \circ F : \mathcal{C} \to \mathbf{E}$
  $X \mapsto G(F(X))$
  $f \mapsto G(F(f))$

There are two issues with this definition:

- Size issue (in ZFC). (it's too big, the objects don't fit in a proper class?)
  Remedies:

  - just consider the category of small categories

  - switch foundations

- It ignores natural transformations
  Remedy:
  Consider the 2-category of 1-categories

  The 2-category of 1-categories has:

  - as objects: all 1-categories

  - as morphisms: functors

  - as -2-morphisms / 2-cells: natural transformations

## 1.32 Definition of Cone

A **cone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : A \to F(X)$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$\pi_Y = \pi_X \circ F(f)$

commutes.

## 1.33 Definition of Cocone

A **cocone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cocone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : F(X) \to A$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$$\pi_X = \pi_Y \circ F(f)$$

commutes.

## 1.34 Definition of Morphism Between Cones

A **morphism** between a cone $(A, (\pi_X)_X)$ and a further cone $(B, (\phi_X)_X)$ of a diagram $F : \mathcal{I} \to \mathcal{C}$ consists of a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\pi_X = \pi_Y \circ f$$

## 1.35 Definition of Limit

A **limit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is a **terminal cone** of $F$, that is, a terminal object in the category of of cones of cones of $F$.

## 1.36 Definition of Colimit

A **colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an **initial cocone** of $F$.

## 1.37 Definition of Equalizer of Two Set-Theoretic Maps

Let $f, g : X \to Y$. Then the **equalizer** of $f$ and $g$ is the following function:

$$Eq(f, g) = x \in X | f(x) = g(x)$$

## 1.38 Definition of Pullback

## 1.39 Definition of Pushout

## 1.40 Definition of Small Diagram

A **small diagram** in $\mathcal{C}$ is a diagram $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category.

## 1.41 Definition of Complete Cateogory

A category $\mathcal{C}$ is **complete** iff every small diagram in $\mathcal{C}$ has a limit (it has all small limits).

## 1.42 Definition of Cocomplete Category

A category $\mathcal{C}$ is **cocomplete** iff every small diagram in $\mathcal{C}$ has a colimit (it has all small colimits).

$$\mathcal{C} \text{ complete } \iff \mathcal{C}^{op} \text{ cocomplete.}$$

## 1.43 Formula for Limits in Set

## 1.44 Formula for Colimits in Set

## 1.45 Definition of Yoneda Lemma

## 1.46 Definition of Presheaf

## 1.47 Definition of Representable Presheaf

## 1.48 Yoneda Embedding

## 1.49 Yoneda Style Proofs

## 1.50 Definition of Adjoint Functors

## 1.51 Currying Adjunction

## 1.52 Adjunction of Logical Connectives

## 1.53 Monoids

## 1.54 Monoids Categorically

## 1.55 Definition of Monoidal Category

## 1.56 Monoidal Categories

## 1.57 Definition of Monad

## 1.58 Definition of Kleisli Category

## 1.59 Definition of Topological Quantum Field Theory

## 1.60 Definition of Cobordism Category