Products

Def. (Product):

A Product of 2 objects X, Y in a category $\mathcal{C}$ consists of:

1) an object P of $\mathcal{C}$
2) a morphism $\pi_X : P \to X$
3) a morphism $\pi_Y : P \to Y$

such that $\forall Q$ object of $\mathcal{C}$, $\varphi_X : Q \to X$, $\varphi_Y : Q \to Y$ morphisms, $\exists!\ Q \xrightarrow{!} P$ morphism s.t. the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[r, phantom, "\equiv"] & & Y \arrow[l, phantom, "\equiv"] \\ & Q \arrow[ul, "\varphi_X"] \arrow[ur, "\varphi_Y"'] \arrow[uu, dotted, "!"'] & \end{tikzcd} \Rightarrow$$

it commutes iff

$$\varphi_X = \pi_X \circ !$$
$$\wedge$$
$$\varphi_Y = \pi_Y \circ !$$

Examples:

1) In Set the product is given by the usual Cartesian product:
$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$
$$\pi_X(x,y) = x, \quad \pi_Y(x,y) = y$$
2) In Vect($\mathbb{R}$) the product is given by the outer direct sum:
$$V \oplus W = \{(v,w) : v \in V, w \in W\}$$
3) In Pokémon there is no product of Pikachu and Charmander, but there is the product of Pikachu and Pikachu, and it is Pikachu.
4) In the Divisibility category the product of x, y is given by $\gcd(x,y)$
5) In Vect$(\mathbb{R})^{op}$ the product is given again by the outer direct sum.
6) In Set^{op} the product is given by the disjoint union:
$$X \amalg Y = \{x : x \in X \vee y \in Y\}$$

Coproducts

Def. (Coproduct):
A COPRODUCT of objects X, Y in the category $\mathcal{C}$ consists of:

1) an object C of $\mathcal{C}$
2) a morphism $\iota_X : X \to C$ in $\mathcal{C}$
3) a morphism $\iota_Y : Y \to C$ in $\mathcal{C}$

such that $\forall D$ object of $\mathcal{C}$, $x_X : X \to D$, $x_Y : Y \to D$ morphisms in $\mathcal{C}$, $\exists!$ $C \xrightarrow{!} D$ morphism which renders the following diagram commutative:

$$
\begin{tikzcd}
 & C \arrow[dd, dashed, "!"] & \\
X \arrow[ur, "\iota_X"] \arrow[dr, "x_X"'] \arrow[r, phantom, "\equiv"] & & Y \arrow[ul, "\iota_Y"'] \arrow[dl, "x_Y"] \arrow[l, phantom, "\equiv"] \\
 & D &
\end{tikzcd}
\Rightarrow
$$

it commutes iff
$$x_X = ! \circ \iota_X$$
$$\wedge$$
$$x_Y = ! \circ \iota_Y$$

Proposition (Duality of Products/Coproducts):
Given a category $\mathcal{C}$, a product in $\mathcal{C}$ is a coproduct in $\mathcal{C}^{op}$

Example:
In the Divisibility category, the coproduct of x, y is given by $\operatorname{lcm}(x, y)$

Remark:
The notion of product/coproduct is extended to n-objects, but not to ∞-many objects.

Example:
1) The product of 0 many objects in $\mathcal{C}$ is a terminal object.
2) The coproduct of 0 many objects in $\mathcal{C}$ is an initial object.
3) The product of an arbitrary (or infinite) family of vector spaces $(V_i)_{i \in I}$ in $\mathrm{Vect}(\mathbb{R})$ is given by

$$\prod_{i \in I} V_i = \{(x_i)_{i \in I} : \forall i \in I \; x_i \in V_i\}$$

while the coproduct is given by:

$$\bigoplus_{i \in I} V_i = \{(x_i)_{i \in I} : \forall i \in I \; x_i \in V_i \wedge x_i \neq 0 \text{ only for finitely many } i \in I\}$$

FUNCTORS

Def. ((Covariant) Functor):
A (COVARIANT) FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

1) an object $F(X) \in Ob(\mathcal{D})$ $\forall X \in Ob(\mathcal{C})$
2) a morphism $F(f): F(X) \to F(Y)$ in $\mathcal{D}$ $\forall f: X \to Y$ morphism in $\mathcal{C}$

such that:

1) $\forall X \in Ob(\mathcal{C})$ $F(id_X) = id_{F(X)}$
2) $\forall X, Y, Z \in Ob(\mathcal{C})$, $\forall f \in Hom_{\mathcal{C}}(X, Y)$, $\forall g \in Hom_{\mathcal{C}}(Y, Z)$

$$F(g \circ f) = F(g) \circ F(f)$$

Def. (Contravariant Functor):
A CONTRAVARIANT FUNCTOR is a covariant functor $\mathcal{C}^{op} \to \mathcal{C}$

Examples:

1) The identity functor $Id_{\mathcal{C}}$ on a category $\mathcal{C}$:

$$\begin{aligned} Id_{\mathcal{C}}: \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \\ f &\mapsto f \end{aligned}$$

2) Let X_0 be an object of a category $\mathcal{C}$. The constant functor on X_0 is the functor:

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \\ f &\mapsto id_{X_0} \end{aligned}$$

3) The diagonal functor is the functor:

$$\begin{aligned} \Delta: Vect(\mathbb{R}) &\to Vect(\mathbb{R}) \\ V &\mapsto V \oplus V \\ (f: V \to W) &\mapsto \big(\Delta(f): V \oplus V \to V \oplus W \\ &\qquad\qquad (x, y) \mapsto (f(x), f(y))\big) \end{aligned}$$

4) Given the categories Field (Ob_{Field} = all the fields, $Hom_{Field}(X, Y)$ = homomorphisms from X to Y) and Grp (Ob_{Grp} = all the groups, $Hom_{Grp}(X, Y)$ = homomorphisms from X to Y), we have the following functor:

$$\begin{aligned} \mathrm{Field} &\longrightarrow \mathrm{Grp} \\ (K,0,1,+,\cdot) &\longmapsto (K^\times,1,\cdot) \\ \varphi: K\to L &\longmapsto (\varphi: K^\times \to L^\times) \end{aligned}$$

5) $$\begin{aligned} \text{Numerical Category} &\longrightarrow \mathrm{Vect}(\mathbb{R}) \\ n &\longmapsto \mathbb{R}^n \\ A\in\mathbb{R}^{n\times m} &\longmapsto (\mathbb{R}^m \to \mathbb{R}^n) \\ & \qquad x \mapsto Ax \end{aligned}$$

6) $$\begin{aligned} \mathrm{Ring} &\longrightarrow \mathrm{Ring} \\ R &\longmapsto R[X] \end{aligned}$$ the polynomial ring

7) $$\begin{aligned} GL_n: \mathrm{Field} &\longrightarrow \mathrm{Grp} \\ K &\longmapsto GL_n(K) \end{aligned}$$ the group of invertible $n\times n$ matrices

8) "forgetful" functors:

$$\begin{aligned} \mathrm{Vect}(\mathbb{R}) &\to \mathrm{Grp} \\ (V,0,+,\cdot) &\mapsto (V,0,+) \end{aligned}, \quad \begin{aligned} \mathrm{Vect}(\mathbb{R}) &\to \mathrm{Set} \\ (V,0,+,\cdot) &\mapsto V \end{aligned}, \quad \begin{aligned} \mathrm{AbGrp} &\to \mathrm{Grp} \\ G &\mapsto G \end{aligned}$$

9) Powerset functor:

$$\begin{aligned} \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \mathcal{P}(X) \\ (f: X\to Y) &\longmapsto (f[\ ]: \mathcal{P}(X) \to \mathcal{P}(Y)) \\ & \qquad\qquad U \mapsto f[U] = \{f(x) : x\in U\} \end{aligned}$$

10) $$\begin{aligned} \mathrm{Set}^{op} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \mathcal{P}(X) \\ (f: X\to Y) &\longmapsto (f^{-1}[\ ]: \mathcal{P}(Y) \to \mathcal{P}(X)) \\ & \qquad\qquad U \mapsto f^{-1}[U] = \{x\in Y : f(x)\in U\} \end{aligned}$$

11) Any ordinary map between sets is a functor. Let $f: X\to Y$ be a map between sets, consider the Discrete categories $D(X), D(Y)$ s.t. $Ob(D(X))$ = elements of X, $\mathrm{Hom}_{D(X)}$ = the identy morphisms (and the same for $D(Y)$). Then f induces the following functor from $D(X)$ to $D(Y)$:

$$\begin{aligned} D(X) &\longrightarrow D(Y) \\ x &\longmapsto f(x) \\ id_x &\longmapsto id_{f(x)} \end{aligned}$$

12) $\mathrm{Set} \longrightarrow \mathrm{Cat}$, $\mathrm{Cat} \longrightarrow \mathrm{Cat}$ where Cat is the category of the categories

$X \longmapsto D(X)$, $\mathcal{C} \longmapsto \mathcal{C}^{op}$

$f \longmapsto D(f)$

13) $\mathrm{Grp} \longrightarrow \mathrm{Field}$

$G \longmapsto \mathbb{R}$

$\varphi \longmapsto id_{\mathbb{R}}$

14) $\mathrm{Grp} \longrightarrow \mathrm{Grp}$ where G^{op} has the same elements of G and the same neutral element, but with a different multiplication:

$G \longmapsto G^{op}$

$\varphi \longmapsto \varphi^{op}$

$$\underset{\text{in } G^{op}}{g \bullet h} = \underset{\text{in } G}{h \circ g}$$

15) $\mathrm{Top} \longrightarrow \mathrm{Grp}$

$X \longmapsto H_n(X)$

Remarks (Split Mono/Epi-morphisms):

Def. (Split Monomorphism):
A morphism $f: X \to Y$ is a SPLIT MONOMORPHISM iff

$$\exists g: Y \to X \text{ s.t. } g \circ f = id_X$$

Def. (Split Epimorphism):
A morphism $f: X \to Y$ is a SPLIT EPIMORPHISM iff

$$\exists g: Y \to X \text{ s.t. } f \circ g = id_Y$$

Proposition:
Given f a morphism, we have that

$$f \begin{Bmatrix} \text{split monic} \\ \text{split epic} \end{Bmatrix} \Rightarrow f \begin{Bmatrix} \text{monic} \\ \text{epic} \end{Bmatrix}$$

AND the converse is NOT (generally) true !!!

Proposition:
Given $f: X \to Y$ a morphism in Set, we have:

1) f epic $\Rightarrow$ f split epic
(assuming the Axiom of choice holds)

2) f monic $\wedge$ X inhabited (i.e. X non empty) $\Rightarrow$ f split monic
(assuming LEM holds)

16) Let $\mathcal{I}$ be the following category:

$$\begin{tikzcd} & 3 \arrow[loop right, "id_3"] \arrow[d, "g"] \\ 1 \arrow[loop left, "id_1"] \arrow[r, "f"'] & 2 \arrow[loop right, "id_2"] \end{tikzcd}$$

and let $\mathcal{C}$ be an arbitrary category. How can we specify a functor $F: \mathcal{I} \to \mathcal{C}$?

$\Rightarrow$ it is enough to write:

1) 3 objects: $F(1)$, $F(2)$, $F(3)$
2) 2 morphisms:
$F(f): F(1) \to F(2)$,
$F(g): F(3) \to F(2)$

$$\begin{tikzcd} & F(3) \arrow[d, "F(g)"] \\ F(1) \arrow[r, "F(f)"] & F(2) \end{tikzcd}$$

$\Rightarrow$ Functors $\mathcal{I} \to \mathcal{C}$ are $\mathcal{I}$-shaped diagrams in $\mathcal{C}$

17) Walking Commutative Triangle:

Let $\mathcal{J}$ be the following category:

$$\begin{tikzcd} & 3 \arrow[loop right] \\ 1 \arrow[loop left] \arrow[ur, "h"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=4, "\equiv"] & 2 \arrow[loop right] \arrow[u, "g"'] \end{tikzcd}$$

$\Rightarrow g \circ f = h$

Given an arbitrary category, what does a functor $F: \mathcal{J} \to \mathcal{C}$ look like?

1) 3 objects: $F(1)$, $F(2)$, $F(3)$
2) 2 morphisms:
$F(f): F(1) \to F(2)$,
$F(g): F(2) \to F(3)$

$$\begin{tikzcd} & F(3) \\ F(1) \arrow[ur, "F(g)\circ F(f)"] \arrow[r, "F(f)"] & F(2) \arrow[u, "F(g)"'] \end{tikzcd}$$