CARATHEODORY CRITERION

Thm.
(X,μ) with X locally compact, separable metric space, μ a Radon measure. Then $\forall A \subset X$ s.t. A is Borel $\forall \varepsilon > 0\ \exists U, K$ with U open, K compact, $K \subset A \subset U \wedge \mu(U \setminus K) < \varepsilon$

$\Rightarrow$ Lusin Thm. applies for any Borel function in this setting!

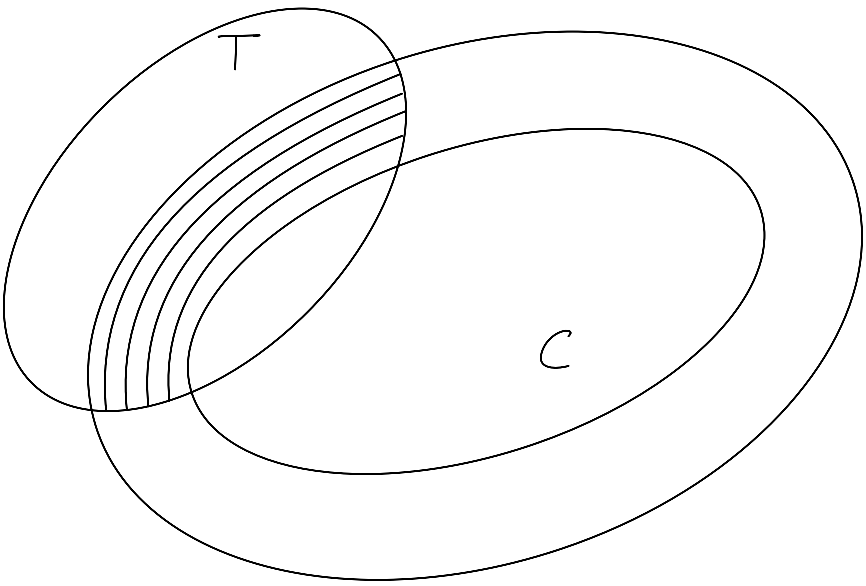
Thm. (Caratheodory Criterion):
Given (X,d) locally compact, separable metric space, μ an outer measure on X. If $\forall A, B \subset X$ s.t. $\text{dist}(A,B) > 0$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$ then μ is Borel

Proof:
It is enough to check that the closed sets are μ-measurable
$\Rightarrow \forall T \subset X \quad \mu(T) \overset{?}{\geq} \mu(T \cap C) + \mu(T \setminus C)$. Assume $\mu(T) < +\infty$.
For $s \in \mathbb{N}^{>0}$, define $C_s = \{x \in X : \text{dist}(x, C) \leq \frac{1}{s}\}$. Then:

$$\mu(T) \geq \mu((T \cap \underbrace{C) \cup (T \setminus C_s}_{C,\, C_s \text{ have positive dist}})) = \mu(T \cap C) + \mu(T \setminus C_s)$$

We now simply show that $\mu(T \setminus C_s) \longrightarrow \mu(T \setminus C)$:
define $R_k = \{x \in T : \frac{1}{k+1} < \text{dist}(x, C) \leq \frac{1}{k}\}$, then:

$$T \setminus C = (T \setminus C_s) \cup \bigcup_{k=s}^{+\infty} R_k$$



$$\Rightarrow \mu(T \setminus C) \leq \mu(T \setminus C_s) + \underbrace{\sum_{k=s}^{+\infty} \mu(R_k)}_{\xrightarrow[s \to +\infty]{} 0 \text{ since:}}$$

$$\sum_{k=1}^{+\infty} \mu(R_{2k}) = \lim_{n \to +\infty} \sum_{i=1}^{n} \mu(R_{2k}) = \lim_{n\to+\infty} \mu\left(\bigcup_{k=1}^{n} R_{2k}\right) \leq \mu(T) \wedge$$
$$\sum_{k=0}^{+\infty} \mu(R_{2k+1}) \leq \mu(T)$$

$\square$

Hausdorff Measures in $\mathbb{R}^n$

The α-dimensional Hausdorff measure is H^α and it is defined as follows ($\alpha > 0$):

(Premeasure) $H^\alpha_\delta(A) = c(\alpha) \inf \left\{ \sum_{s=1}^{+\infty} \operatorname{diam}(C_s)^\alpha : \right.$
$\left. C_s \text{ closed}, \bigcup_{s=1}^{+\infty} C_s \supseteq A, \operatorname{diam}(C_s) \leq \delta \ \forall s \right\}$ $(\delta > 0)$

$\Rightarrow H^\alpha(A) = \lim_{\delta \to 0^+} H^\alpha_\delta(A)$ (Hausdorff Measure)

If $\alpha \in \mathbb{N}$, $1 \leq \alpha \leq n$, then $c(\alpha)$ is the α-dim. Lebesgue measure of a α-dim. ball of radius 1. In particular:

$$c(\alpha) = \Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right) 2^\alpha}$$