NON-EXACT FACTORIZATION METHODS

We have:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\Delta t} - \nu \Delta u^{n+1} + (u^* \cdot \nabla) u^{**} + \nabla p^{n+1} = f^{n+1} \\ \nabla u^{n+1} = 0 \end{cases}$$

$\Rightarrow$ algebrically counterpart:

$$\begin{pmatrix} C & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u}^{n+1} \\ \hat{p}^{n+1} \end{pmatrix} = \begin{pmatrix} \tilde{F}^{n+1} \\ \vec{0} \end{pmatrix}$$

$\Rightarrow$ semi-implicit:

$$C = \frac{M}{\Delta t} + \nu \mathbb{K} + N(\hat{u}^n), \quad \tilde{F}^{n+1} = F^{n+1} + \frac{M}{\Delta t} u^n$$

$\Rightarrow$ explicit:

$$C = \frac{M}{\Delta t} + \nu \mathbb{K}, \quad \tilde{F}^{n+1} = F^{n+1} + \frac{M}{\Delta t} \hat{u}^n - N(\hat{u}^n)\hat{u}^n$$

Exact LU factorization:

$$A = \begin{pmatrix} C & B^T \\ B & \vec{0} \end{pmatrix} = LU, \quad L = \begin{pmatrix} C & \vec{0} \\ B & -BC^{-1}B^T \end{pmatrix} \cdot \begin{pmatrix} Id & C^{-1}B^T \\ \vec{0} & Id \end{pmatrix}$$

<u>Remark:</u>

LU exact factorization is particularly expensive in computational terms.

$\Rightarrow$ we therefore introduce an non exact LU factorization:

$$A \approx \tilde{A} = \tilde{L}\tilde{U}, \quad \tilde{L} = \begin{pmatrix} C & \vec{0} \\ B & -BH_1B^{-1} \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} Id & H_2B^T \\ \vec{0} & Id \end{pmatrix}$$

with H_1, H_2 approximations of C^{-1}.

$$\Rightarrow C = \frac{M}{\Delta t} + \nu \mathbb{K} + N(\hat{u}^n) = \frac{M}{\Delta t}\Big(Id + \Delta t M^{-1} \underbrace{(\nu \mathbb{K} + N(\hat{u}^n))}_{W}\Big)$$

$$= \frac{M}{\Delta t}\left(Id + \Delta t M^{-1} W\right)$$

Let $\rho(\Delta t M^{-1} W) := \max\{|\lambda| : \lambda \text{ eigenvalue of } \Delta t M^{-1} W\}$ the spectral radius of $\Delta t M^{-1} W$

Recall that $\frac{1}{1+x} = \sum_{k \in \mathbb{N}} (-1)^k x^k$ if $|x| < 1$. If $\rho(\Delta t M^{-1} W) < 1$ we can write $(Id + \Delta t M^{-1} W)^{-1} = \sum_{k \in \mathbb{N}} (-1)^k (\Delta t M^{-1} W)^k$ (Neumann expansion).

Remark:

In general, for Δt sufficiently small, $\rho(\Delta t M^{-1} W) < 1$!!!
So we can write:

$$C^{-1} = \Delta t \left(Id + \Delta t M^{-1} W \right)^{-1} M^{-1} = \Delta t \sum_{k \in \mathbb{N}} (-1)^k (\Delta t M^{-1} W)^k M^{-1}$$
$$= \Delta t \, Id \, M^{-1} + \mathcal{O}(\Delta t) \approx \Delta t M^{-1}$$

Remark:

We can also use M_L^{-1}, with M_L being the "lumped" version of the matrix M !!!

$\Rightarrow$ we then take $H_1 = H_2 = \Delta t M_L^{-1}$ and plug them into the algebraic problem:

$$\begin{pmatrix} C & \vec{0} \\ B & \Delta t B M_L^{-1} B^T \end{pmatrix} \cdot \begin{pmatrix} Id & \Delta t M_L^{-1} B^T \\ \vec{0} & Id \end{pmatrix} \cdot \begin{pmatrix} \hat{u}^{n+1} \\ \hat{p}^{n+1} \end{pmatrix} = \begin{pmatrix} \widetilde{F}^{n+1} \\ \vec{0} \end{pmatrix}$$

and now we calculate the LU factorization:

1) $$\begin{pmatrix} C & \vec{0} \\ B & -B \Delta t M_L^{-1} B^T \end{pmatrix} \begin{pmatrix} \hat{\hat{u}} \\ \hat{\hat{p}} \end{pmatrix} = \begin{pmatrix} \widetilde{F}^{n+1} \\ \vec{0} \end{pmatrix}$$

2) $$\begin{pmatrix} Id & \Delta t M_L^{-1} B^T \\ \vec{0} & Id \end{pmatrix} \cdot \begin{pmatrix} \hat{u}^{n+1} \\ \hat{p}^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{\hat{u}} \\ \hat{\hat{p}} \end{pmatrix}$$

$\Rightarrow$ from step 1) we get:

$$\begin{cases} C \hat{\hat{u}} = \widetilde{F}^{n+1} \\ \Delta t B M_L^{-1} B^T \hat{\hat{p}} = B \hat{\hat{u}} \end{cases}$$

$\Rightarrow$ from step 2) we get:

$$\begin{cases} \hat{p}^{n+1} = \hat{\hat{p}} \quad (\text{tautological condition}) \\ \hat{u}^{n+1} + \Delta t M_L^{-1} B^T \hat{p}^{n+1} = \hat{\hat{u}} \end{cases}$$

$\Rightarrow$ so we have:

$$\begin{cases} C \hat{\hat{u}} = \widetilde{F}^{n+1} \\ \Delta t B M_L^{-1} B^T \hat{p}^{n+1} = B \hat{\hat{u}} \\ \hat{u}^{n+1} = \hat{\hat{u}} \cdot \Delta t M_L^{-1} B^T \hat{p}^{n+1} \end{cases}$$

Remark:

$B M_L^{-1} B^T$ is a discrete version of the Laplacian !!!

1) $C \hat{\hat{u}} = \widetilde{F}^{n+1} \iff \dfrac{\tilde{u} - \tilde{u}^n}{\Delta t} - \nu \Delta \tilde{u} + (\underset{\text{PREDICTION}}{\tilde{u}^{*}} \cdot \nabla) \tilde{u}^{**} = f^{n+1}$

2) $\Delta t\, B M_L^{-1} B^T \hat{p}^{n+1} = B\tilde{u} \iff \begin{cases} \Delta t\, \Delta p^{n+1} = \nabla \cdot \tilde{u} \\ \partial_{\hat{n}}\, p^{n+1}\big|_{\partial\Omega} = 0 \end{cases}$ PROJECTION

3) $\hat{u}^{n+1} = \hat{\tilde{u}} - \Delta t\, M_L^{-1} B^T \hat{p}^{n+1} \iff u^{n+1} = \tilde{u} - \Delta t\, \nabla p^{n+1}$ CORRECTION

$\Rightarrow$ it is a discretized version of the Chorin - Temam method !!!

Consider now $\hat{p}^{n+1} = \hat{p}^n + \delta p$:

$$\begin{pmatrix} C & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u}^{n+1} \\ \hat{p}^n + \delta p \end{pmatrix} = \begin{pmatrix} \tilde{F}^{n+1} \\ \vec{0} \end{pmatrix} \iff \begin{pmatrix} C & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u}^{n+1} \\ \delta p \end{pmatrix} = \begin{pmatrix} \tilde{F}^{n+1} - B^T \hat{p}^n \\ \vec{0} \end{pmatrix}$$

$\Rightarrow$ computing the non exact LU factorization we get :

$$\begin{cases} C\hat{\tilde{u}} = \tilde{F}^{n+1} - B^T \hat{p}^n & \text{PREDICTION} \\ \Delta t\, B M_L^{-1} B^T \delta\hat{p} = B\hat{\tilde{u}} & \text{PROJECTION} \\ \hat{u}^{n+1} = \hat{\tilde{u}} - \Delta t\, M_L^{-1} B^T \delta\hat{p} & \text{CORRECTION} \\ \hat{p}^{n+1} = \hat{p}^n + \delta p \end{cases}$$

which is the discretized version of the incremental Chorin - Temam method !!!

LAPLACIAN DISCRETIZATION:

$$\begin{cases} \Delta p = f \\ p = 0 & \Gamma_N \\ \partial_{\hat{n}} p = 0 & \Gamma_D \end{cases} \quad \text{with } p \in L^2 = Q$$

$$\Rightarrow \int_\Omega \Delta p \cdot q \, d\Omega = -\int_\Omega \nabla p \cdot \nabla q \, d\Omega + \cancel{\int_{\Gamma_D} q \underbrace{\nabla p \cdot \hat{n}}_{=0} d\Gamma}$$
