C_c^0 DENSITY IN L^p

Consider ($\mathbb{R}^n$, Lebesgue measure). We want to prove the following:

Thm. (Lusin):
Let $u : \Omega \to \mathbb{R}$ be Lebesgue-measurable with $\Omega \subseteq \mathbb{R}^n$ meas. and bounded. Then:

$$\forall \varepsilon > 0 \ \exists K \subset \Omega, K \text{ compact s.t.}$$
$$|\Omega \setminus K| < \varepsilon \wedge u|_K \text{ is continuous}$$

Proof:
Fix $s \in \mathbb{N}$, $s > 0$, and decompose $\mathbb{R} = \bigcup_{i=1}^{+\infty} I_{is}$, with I_{is} pairwise disjoint s.t. $|I_{is}| < \frac{1}{s}$ $\forall i$. Choose $y_{is} \in I_{is}$ and let $A_{is} = u^{-1}(I_{is})$, then $\Omega = \bigcup_{i=1}^{+\infty} A_{is}$, A_{is} pairwise disjoint, measurable. So $\exists K_{is}$ s.t. $K_{is} \subset A_{is}$, $|A_{is} \setminus K_{is}| < \frac{\varepsilon}{2^{i+s}}$, K_{is} compact $\forall i$. Then:

$$|\Omega \setminus \bigcup_{i=1}^{+\infty} K_{is}| < \frac{\varepsilon}{2^s}$$

In general $\bigcup_{i=1}^{+\infty} K_{is}$ is not compact, but $\exists N_s \in \mathbb{N}$ s.t.

$$|\Omega \setminus \bigcup_{i=1}^{N_s} K_{is}| < \frac{\varepsilon}{2^s}$$

and $\bigcup_{i=1}^{N_s} K_{is} =: K_s$ is compact. Define now $u_s : K_s \to \mathbb{R}$ by

$$u_s(x) = y_{is} \quad (x \in K_{is})$$

$\Rightarrow u_s$ is continuous (trivially: piecewise constant on sets with positive distance between each other). Moreover:

$$|u_s - u| < \frac{1}{s}$$

Consider now $K = \bigcap_{s=1}^{+\infty} K_s$, we have $|\Omega \setminus K| < \varepsilon$, K is compact and $u_s \to u$ uniformly on K, so $u|_K$ is continuous.

$\square$

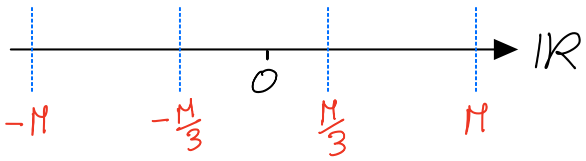
Thm. (Tietze):
Let $K \subseteq \mathbb{R}^n$ be compact, $u : K \to \mathbb{R}$ continuous. Then $\exists v : \mathbb{R}^n \to \mathbb{R}$ continuous s.t.:

$$v(x) = u(x) \ \forall x \in K \wedge \|v\|_{\infty, \mathbb{R}^n} = \|u\|_{\infty, K}$$

We can further require that $v \in C_c^0(\Omega)$, where Ω is any open set s.t. $K \subset \Omega$.

Proof:

Let $M = \|u\|_{\infty,K} = \max\{|u(x)| : x \in K\}$. Divide $[-M, M]$ in 3 equal parts $[-M, -\frac{M}{3}], [-\frac{M}{3}, \frac{M}{3}], [\frac{M}{3}, M]$ and consider $K_1 = u^{-1}([-M, -\frac{M}{3}])$, $K_2 = u^{-1}([\frac{M}{3}, M]) \Rightarrow K_1, K_2$ are compact and disjoint. Let $\delta = \mathrm{dist}(K_1, K_2) > 0$. Define $v_1(x) = -\frac{M}{3} + \frac{2M}{3\delta}\min\{\mathrm{dist}(x, K_1), \delta\}$



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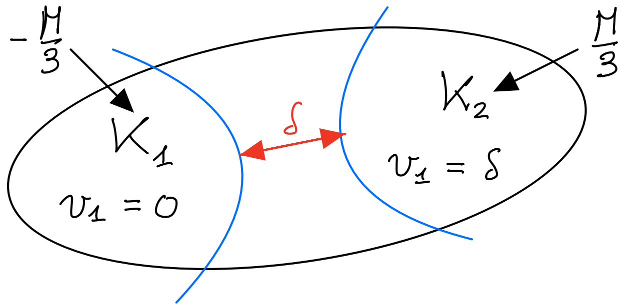

$\Rightarrow v_1 \in C^0(\mathbb{R}^n)$, $\|v_1\|_{\infty,\mathbb{R}^n} = \frac{M}{3}$, $\|u - v_1\|_{\infty,K} \leq \frac{2}{3}M$.

Apply the same construction with $u - v_1$ insteds of u, we get:

$$\exists v_2 \in C^0(\mathbb{R}^n) \text{ s.t. } \|v_2\|_{\infty,\mathbb{R}^n} = \frac{2}{3^2}M, \ \|(u - v_1) - v_2\|_{\infty,K} \leq \left(\frac{2}{3}\right)^2 M$$

Iterating the procedure we obtain:

$$\exists v_K \in C^0(\mathbb{R}^n) \text{ s.t. } \|v_K\|_{\infty,\mathbb{R}^n} = \frac{2^{K-1}}{3^K}M, \ \|(u - \sum_{i=1}^{K-1} v_i) - v_K\|_{\infty,K} \leq \left(\frac{2}{3}\right)^K M$$

$\Rightarrow$ take $v(x) := \sum_{K=1}^{+\infty} v_K(x)$, then $v \in C^0(\mathbb{R}^n) \wedge v(x) = u(x) \ \forall x \in K$

We now show that we can require $v \in C^0_c(\Omega)$. Take Ω' s.t. Ω' is open, $K \subset \Omega' \subset\subset \Omega$. Then:

$$\mathrm{dist}((\mathbb{R}^n \setminus \Omega'), K) > 0 \Rightarrow \exists f \in C^0(\mathbb{R}^n) \text{ s.t.}$$

$$f(x) = 0 \text{ on } \Omega \setminus \Omega' \wedge f(x) = 1 \text{ on } K \wedge 0 \leq f(x) \leq 1 \ \forall x$$

Now replace $v(x)$ with $v(x) \cdot f(x) \in C^0_c(\Omega)$

$\square$

Thm.:

Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < +\infty$, then $C^0_c(\Omega)$ is dense in $L^p(\Omega)$

Proof:

Let $u \in L^p(\Omega)$. Suppose, for the moment, that $\|u\|_{L^\infty(\Omega)} = M < +\infty$ and that Ω is bounded. By Lusin $\exists K \subset \Omega$ compact s.t. $|\Omega \setminus K| < \varepsilon \wedge u|_K$ is continuous. By Tietze $\exists v \in C^0_c(\Omega)$ s.t. $u(x) = v(x) \ \forall x \in K \wedge \|v\|_{L^\infty(\Omega)} = M$. Compute:

$$\|u-v\|^p_{L^p(\Omega)} = \int_\Omega |u(x)-v(x)|^p dx = \int_{\Omega \setminus K} \underbrace{|u(x)-v(x)|^p}_{\leq (2M)^p} dx \leq \varepsilon (2M)^p$$

Let now u be unbounded and consider the functions

$u_n(x) = \min\{n, \max\{-n, u(x)\}\} \Rightarrow \|u_n\|_{L^\infty} = n \wedge \lim_{n\to+\infty} u_n(x) = u(x)$

We have that $u_n \xrightarrow[L^p]{} u$ (Lebesgue).

Let now Ω be unbounded and consider:

$$u_n(x) = \begin{cases} u(x) & |x| < n \\ 0 & |x| \geq n \end{cases}$$

$\Rightarrow$ again we have that $u_n \xrightarrow[L^p]{} u$ (Lebesgue)

$\square$

Corollary:

Given Ω as above, $p \in [1, +\infty)$, $L^p(\Omega)$ is separable.

Corollary (Continuity of Translations in L^p):

Given $p \in [1,+\infty)$, $u \in L^p(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, we define the y-translation of u as:

$$\tau_y u(x) := u(x-y)$$

Then we have that $\lim_{\|y\|\to 0} \|\tau_y u - u\|_{L^p(\mathbb{R}^n)} = 0$

N.B.

This is trivially true for $u \in C^0_c$ because such u is uniformly continuous.

Consider now $\varphi : \mathbb{R}^n \to \mathbb{R}$ given by the following:

$$\varphi(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \quad \text{(Bump Function)}$$

$\Rightarrow \varphi \in C^\infty_c(\mathbb{R}^n) \wedge \operatorname{supp} \varphi = \overline{B_1(0)}$. We choose c s.t. $\int_{\mathbb{R}^n} \varphi(x)dx = 1$

We can re-scale φ as follows:

$$\varphi_k(x) = k^n \varphi(kx) : \mathbb{R}^n \to \mathbb{R}, \; k \text{ s.t. } \int_{\mathbb{R}^n} \varphi_k(x)\,dx = 1$$

$\Rightarrow \operatorname{supp} \varphi_k \subset \overline{B_{\frac{1}{k}}(0)}$

$\Rightarrow$ given $u \in L^p(\mathbb{R}^n)$ we can regularize it by convolution:

$$u_k(x) := \int_{\mathbb{R}^n} u(x-y)\,\phi_k(y)\,dy \text{ with } \begin{array}{l} u_k \in C^\infty(\mathbb{R}^n) \wedge u_k \xrightarrow[L^p(\Omega)]{} u, \\ \phi_k(y) = c_k\left(\frac{1+\cos x}{2}\right)^k, \int_{\mathbb{R}^n} \phi_k = 1 \end{array}$$

Indeed, change variables s.t.:

$$z = x-y \Rightarrow y = x-z,\ dy = dz \Rightarrow u_K(x) = \int_{\mathbb{R}^n} u(z)\phi_K(x-z)\,dz$$

<u>Proposition</u>:

Let $\phi \in C^1_c(\mathbb{R}^n)$, $u \in L^1_{loc}(\mathbb{R}^n)$ (the restriction of u on every compact set is L^1). Define $v(x) := \int_{\mathbb{R}^n} u(z)\phi(x-z)\,dz$. Then $v \in C^1(\mathbb{R}^n)$ and $\partial_{x_i} v(x) = \int_{\mathbb{R}^n} u(z)\partial_{x_i}\phi(x-z)\,dz$

<u>Proof</u>:

$$|v(x+y)-v(x)| = \left|\int_{\mathbb{R}^n} u(z)\big(\phi(x+y-z)-\phi(x-z)\big)dz\right|$$

$$\leq \int_{\mathbb{R}^n} |u(z)|\cdot|\phi(x+y-z)-\phi(x-z)|\,dz = \int_K \underbrace{|u(z)|\cdot \underbrace{|\phi(x+y-z)-\phi(x-z)|}_{\xrightarrow{y\to 0} 0}}_{\substack{\longrightarrow 0 \text{ pointwise} \\ (\text{dominated by } 2M\mathbb{1}_K)}}\,dz$$

$$\xrightarrow{y\to 0} 0$$

so v is continuous. Now we check the derivatives:

$$\partial_{x_i} v(x) = \lim_{h\to 0} \frac{v(x+he_i)-v(x)}{h}$$

$$= \lim_{h\to 0} \int_K u(z)\underbrace{\left(\frac{\phi(x+he_i-z)-\phi(x-z)}{h}\right)}_{\longrightarrow\ \partial_{x_i}\phi(x-z)}dz$$

□

The fact that $u_K \xrightarrow[L^p(\Omega)]{} u$ derives from the following:

$$\|u-u_K\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\big(u(x-y)-u(x)\big)\phi_K(y)\,dy\right|^p dx$$

$$\leq \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}|u(x-y)-u(x)|\underbrace{\phi_K(y)}_{=\phi_K(y)^{\frac{1}{p}}\cdot\phi_K(y)^{1-\frac{1}{p}}}dy\right|^p dx$$

$$\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|u(x-y)-u(x)|^p\phi_K(y)\,dy\right)dx = \underbrace{\int_{B_{\frac{1}{k}}(0)}\phi_K(y)\underbrace{\int_{\mathbb{R}^n}|u(x-y)-u(x)|^p dx}_{\xrightarrow{y\to 0} 0}dy}_{<\varepsilon \text{ for } K \text{ large enough}}$$

$< \varepsilon$ for K large enough.

Take now $\Omega \subseteq \mathbb{R}^n$ open, $u \in L^p(\Omega)$ (extend u to be 0 outside of Ω), then we know that $\forall \varepsilon > 0\ \exists v \in C^\infty_c(\Omega)$ s.t. $\|u-v\|_{L^p(\Omega)} < \varepsilon$. Let v_K be the regularization by convolution of v, then we have that $\|v_K - v\|_{L^p(\Omega)} \longrightarrow 0$. Moreover,

$v_K \in C_c^\infty(\Omega)$ for K large enough, so $\|v_K - v\|_{L^p(\Omega)} < \varepsilon$ $\wedge$ $\|v_K - u\|_{L^p(\Omega)} < 2\varepsilon$ for K large enough.

CONTINOUS FUNCTIONS' DENSITY IN $L^p(\mu)$

Suppose now to have (X, μ), X locally compact metric space, μ outer measure of X. Is it true that continuous functions are dense in $L^p(\mu)$??? NOT IN GENERAL !!! Yes, if μ is a RADON MEASURE.

<u>Def.</u> (Borel Measure, Borel regularity, Radon Measure):

μ is a BOREL MEASURE if Borel sets are μ-measurable.

μ is BOREL REGULAR if $\forall A \subset X$ $\exists B$ Borel s.t. $A \subset B$ $\wedge$ $\mu(A) = \mu(B)$ (B is a Borel Envelope of A).

μ is a RADON MEASURE if it is Borel regular and $\mu(K) < +\infty$ $\forall K \subset X$, K compact.