SOBOLEV SPACES IN DIM. N – COMPACTNESS FOR $W^{1,1}$

Def. (Weak Partial Derivative):
Given $\Omega \subseteq_{op} \mathbb{R}^n$, $p \in [1, +\infty]$, $u \in L^1(\Omega)$, we define the WEAK PARTIAL DERIVATIVE of u in the e_i direction as

$$\int_\Omega u(x)\, \partial_{x_i} \phi(x)\, dx = -\int_\Omega v(x)\, \phi(x)\, dx \quad \forall \phi \in C^1_c(\Omega)$$

N.B.
$$u \in C^1(\Omega) \Rightarrow \int_\Omega u(x)\, \partial_{x_i} \phi(x)\, dx = -\int_\Omega \partial_{x_i} u(x) \phi(x)\, dx$$

Remark:
The weak derivative is still unique in dim. n (the Fundamental Lemma of Calculus of Variation holds in dim n).

Def. (Sobolev Space (dim. n)):
We define the SOBOLEV SPACES IN DIM. n as:

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : \exists \text{ weak derivatives } \partial_{x_i} u \in L^p(\Omega) \}$$

They are Banach spaces with the following (equivalent) norms:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^n \|\partial_{x_i} u\|_{L^p(\Omega)},$$

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^n \|\partial_{x_i} u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and, for $p = 2$, we have:

$$\langle u, v \rangle = \int_\Omega u(x) v(x) + \sum_{i=1}^n \partial_{x_i} u(x)\, \partial_{x_i} v(x)\, dx, \quad u, v \in W^{1,2}(\Omega)$$

Thm. (Compactness in $W^{1,1}([a,b])$):
Let $p \in (1, +\infty)$, $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p}([a,b])$ s.t. $\exists C > 0$ s.t. $\|u_k'\|_{L^p([a,b])} \leq C$ $\forall k \in \mathbb{N}$ and either (1) $|u_k(a)| \leq C$ $\forall k \in \mathbb{N}$ or (2) $\|u_k\|_{L^p([a,b])} \leq C$ $\forall k \in \mathbb{N}$. Then $\exists \{u_{k_h}\}_{h \in \mathbb{N}}$, $u \in W^{1,p}([a,b])$ s.t. $u_{k_h} \to u$ uniformly on $[a,b]$ and $u_{k_h}' \rightharpoonup u'$ in $L^p([a,b])$.

Proof:
Suppose (1) holds, let $x, y \in [a,b]$:

$$|u_k(x) - u_k(y)| = \left| \int_x^y u_k'(t)\, dt \right| \leq \int_x^y |u_k'(t)|\, dt \overset{\text{Hölder}}{\leq} \|u_k'\|_{L^p([a,b])} \cdot |x-y|^{1-\frac{1}{p}}$$

$$\leq C \cdot |x-y|^{1-\frac{1}{p}} \text{ (Hölder Continuity of } u_k)$$

$\Rightarrow \{u_K\}$ are equicontinuous. They are also equibounded:

$$|u_K(x)| = \underbrace{|u_K(a)|}_{\leq C} + \underbrace{\left|\int_a^x u_K'(t)\,dt\right|}_{\leq C(b-a)^{1-\frac{1}{p}}} \leq C\left(1+(b-a)^{1-\frac{1}{p}}\right)$$

$\Rightarrow$ by Ascoli-Arzelà $\exists\, u_{K_h},\ u \in C^0([a,b])$ s.t. $u_{K_h} \longrightarrow u$ uniformly. By Banach-Alaoglu we also have $v \in L^p(a,b)$ s.t. $u_{K_h}' \rightharpoonup v$ in L^p.

We know that $\forall \phi \in C_c^1([a,b])$ $\int_a^b u_{K_h}(x)\phi'(x)\,dx = -\int_a^b u_{K_h}'(x)\phi(x)\,dx$

$$\longrightarrow \int_a^b u(x)\phi'(x)\,dx \qquad \longrightarrow -\int_a^b v(x)\phi(x)\,dx$$

$\Rightarrow \int_a^b u(x)\,\phi'(x)\,dx = -\int_a^b v(x)\,\phi(x)\,dx \Rightarrow u \in W^{1,p}$ and its weak derivative is v.

Suppose now that (2) holds. Then:

$$u_K(x) = u_K(a) + \int_a^x u_K'(t)\,dt \Rightarrow |u_K(a)| \leq |u_K(x)| + \int_a^b |u_K'(t)|\,dt$$

$\Rightarrow$ integrate over $[a,b]$:

$$(b-a)|u_K(a)| \leq \int_a^b |u_K(x)|\,dx + (b-a)\int_a^b |u_K'(x)|\,dx$$

$$\Rightarrow |u_K(a)| \leq \frac{1}{b-a}\cdot\int_a^b |u_K(x)|\,dx + \int_a^b |u_K'(x)|\,dx \leq C\|u\|_{W^{1,1}([a,b])}$$

and so (1) holds

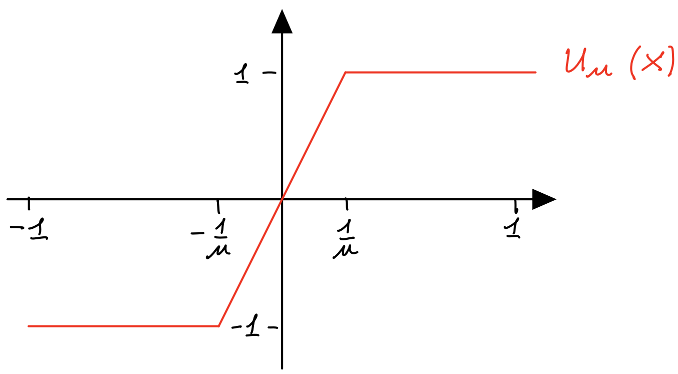
$\square$

<u>Remark</u>:

For $p=1$, the Thm. is false. A simple counterexample is the following:

$[a,b]=[-1,1]$, u_n as follows:

$$u_n(x) = \begin{cases} -1 & -1 \leq x \leq -\frac{1}{n} \\ nx & -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 1 \end{cases}$$



$$\Rightarrow \|u_n\|_{L^1} + \|u_n'\|_{L^1} \leq C$$

$$\Rightarrow \int_{-1}^1 |u_n'(t)|\,dt = \int_{-1}^1 u_n'(t)\,dt = u_n(1) - u_n(-1) = 2 \quad ↯$$

Case $p = +\infty$:

We have an even better statement in this case:

Def. (Weak * Convergence):

Given X separable and Banach, X' its dual, $\{v_k\}_k \subset X'$, $v \in X'$, we say that $v_k \xrightarrow{*} v$ in X' iff $v_k(x) \to v(x)$ $\forall x \in X$.

N.B. $X = L^1$, $X' = L^\infty \Rightarrow u_k \xrightarrow{*} u$ in $L^\infty \Leftrightarrow \int_a^b u_k v \to \int_a^b uv \quad \forall v \in L^1$

Thm. (Compactness in $W^{1,\infty}([a,b])$):

Given $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}([a,b])$ s.t. $\exists C > 0$ s.t. $\|u_k'\|_{L^p([a,b])} \leq C$ $\forall k \in \mathbb{N}$ and either (1) $|u_k(a)| \leq C$ $\forall k \in \mathbb{N}$ or (2) $\|u_k\|_{L^p([a,b])} \leq C$ $\forall k \in \mathbb{N}$. Then $\exists \{u_{k_h}\}$, $u \in W^{1,\infty}$ s.t. $u_{k_h} \to u$ uniformly and $u_{k_h}' \xrightarrow{*} u$ in $L^\infty([a,b])$.

Remark:

$W^{1,\infty}$ is the space of Lipschitz Continuous Functions:

$$u \in W^{1,\infty} \Rightarrow |u(x) - u(y)| = \left|\int_x^y u'(t)\,dt\right| \leq \|u'\|_{L^\infty} |x-y|$$

Conversely:

$$u \in Lip \Rightarrow |u'(x)| = \left|\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}\right| \leq L$$

We now give an example of a discontinuous Sobolev function in dim. > 1.

Example (Discontinuous Sobolev Function in $\mathbb{R}^2$):

$\Omega = B_1((0,0)) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$, $u(x,y) = \frac{1}{(x^2+y^2)^{\frac{1}{4}}}$

$\Rightarrow u(\rho,\theta) = \frac{1}{\sqrt{\rho}} \Rightarrow u$ is discontinuous at $(0,0)$. Still, $u \in W^{1,p}(\Omega)$ for $1 \leq p < \frac{4}{3}$. Indeed:

$$\|u\|^p_{L^p(\Omega)} = \int_0^{2\pi} d\theta \int_0^1 \rho^{-\frac{p}{2}} \rho\, d\rho = 2\pi \int_0^1 \rho^{1-\frac{p}{2}} d\rho \quad \Rightarrow 1 - \frac{p}{2} > -1 \Leftrightarrow p < 4$$

$$\partial_x u = -\frac{1}{2}\rho^{-\frac{3}{2}} \partial_x \rho = -\frac{1}{2}\rho^{-\frac{3}{2}}\cos\theta, \quad \partial_y u = -\frac{1}{2}\rho^{-\frac{3}{2}}\partial_y \rho = -\frac{1}{2}\rho^{-\frac{3}{2}}\sin\theta$$

$$\Rightarrow |\nabla u(x,y)| = \frac{1}{2}\rho^{-\frac{3}{2}} \Rightarrow \|\nabla u\|^p_{L^p(\Omega)} = \frac{1}{2^p}\int_0^{2\pi} d\theta \int_0^1 \rho^{-\frac{3}{2}p} \rho\, d\rho$$

$$\Rightarrow 1 - \frac{3}{2}p > -1 \Leftrightarrow p < \frac{4}{3}$$

Finally, we check that $\partial_x u$, $\partial_y u$ are actually not only the usual pointwise derivatives but also the weak derivatives:

$$u_n(x,y) = \frac{1}{(x^2+y^2+\frac{1}{n})^{\frac{1}{4}}} \in \mathcal{C}^1(\Omega)$$

for $1 \leqslant p < \frac{4}{3}$ we have:

$\Rightarrow u_n(x,y) \xrightarrow[L^p]{} u(x,y)$ (Lebesgue Dominated Convergence)

$\Rightarrow \nabla u_n(x,y) \xrightarrow[L^p]{} \nabla u(x,y)$ (Lebesgue Dominated Convergence)

$\Rightarrow \phi \in \mathcal{C}^1_c(\Omega)$:

$$\int_\Omega u_n \, \partial_x \phi \, dx = - \int_\Omega \partial_x u_n \, \phi \, dx$$