Signed Measures

Def. (Signed Measure):
Let X be a set, $\mathcal{S}$ a σ-algebra on X. A Signed Measure on X is a function $\mu : \mathcal{S} \to \mathbb{R}$ s.t. $\mu(\phi) = 0$ and it is countably additive:

$$E_1, \dots, E_n, \dots \in \mathcal{S} \text{ pairwise disjoint} \Rightarrow \mu\left(\bigcup_{k=1}^{+\infty}\right) = \sum_{k=1}^{+\infty} \mu(E_k)$$

N.B.
μ only takes finite values!!!

A signed measure is continuous under an increasing/decreasing sequence of measurable sets:

$$A_1 \subset \dots \subset A_n \subset \dots \in \mathcal{S} \Rightarrow \mu\left(\bigcup_{k=1}^{+\infty} A_k\right) = \lim_{k \to +\infty} \mu(A_k)$$

$$A_1 \supset \dots \supset A_n \supset \dots \in \mathcal{S} \Rightarrow \mu\left(\bigcup_{k=1}^{+\infty} A_k\right) = \lim_{k \to +\infty} \mu(A_k)$$

However, a signed measure fails on the monotonicity!!!

Def. (Positive/Negative/Null Set)
Let μ be a signed measure on the σ-algebra $\mathcal{S}$, $A \in \mathcal{S}$.

1) A is Positive if $\mu(E) \geq 0 \ \forall E \in \mathcal{S}$ s.t. $E \subseteq A$
2) A is Negative if $\mu(E) \leq 0 \ \forall E \in \mathcal{S}$ s.t. $E \subseteq A$
3) A is Null if $\mu(E) = 0 \ \forall E \in \mathcal{S}$ s.t. $E \subseteq A$

Thm (Hahn Decomposition):
Let μ be a signed measure on X. Then $X = P \cup N$ with $P, N \in \mathcal{S}$, $P \cap N \neq \phi$, P positive, N negative (Hahn Decomposition) and this decomposition is unique up to null sets

Corollary (Jordan Decomposition):
Define $\forall A \in \mathcal{S}$ $\mu^+(A) = \mu(A \cap P)$, $\mu^-(A) = \mu(A \cap N)$ (P, N as above). We have $\mu(A) = \mu^+(A) - \mu^-(A)$, $|\mu| := \mu^+(A) + \mu^-(A)$ (Total Variational Measure)

Proof (Hahn Decomposition):
Uniqueness up to null sets is trivial. We now prove existence
Claim:
$\forall M \in \mathcal{S}$ $\sup\{\mu(E) : E \subseteq M, E \in \mathcal{S}\} < +\infty$ (i.e. μ is bounded)

PROOF OF THE CLAIM:

By contradiction, suppose that

$$\forall M \in \mathcal{S} \quad \sup\{\mu(E) : E \subseteq M, E \in \mathcal{S}\} = +\infty$$

$\Rightarrow \exists A \subseteq M$ s.t. $\mu(E) \geq 1 + |\mu(M)|$. Let $B = M \setminus A$, then:

$$\mu(B) = \mu(M) - \mu(A) \leq \mu(M) - 1 - |\mu(M)| \leq -1$$

$\Rightarrow |\mu(A)| \geq 1$, $|\mu(B)| \geq 1$, moreover we must have that either $\sup\{\mu(E) : E \in \mathcal{S}, E \subseteq A\} = +\infty$ or $\sup\{\mu(E) : E \in \mathcal{S}, E \subseteq B\} = +\infty$. We suppose it is B, otherwise we swap A and B.

$\Rightarrow$ By iterating we compose $B = A_1 \cup B_1$ with A_1, A_2 meas., disjoint s.t. $A_1 \cap B_1 \neq \phi$, $|\mu(A_1)| \geq 1$, $\sup\{\mu(E) : E \subset B_1\} = +\infty$

$\Rightarrow$ We find a sequence of meas. sets, pairwise disjoint $A_1, A_2, \ldots \in \mathcal{S}$ s.t. $|\mu(A_K)| \geq 1$. But $\mu\left(\bigcup_{K=1}^{+\infty} A_K\right) = \sum_{K=1}^{+\infty} \mu(A_K)$ and $\sum_{K=1}^{+\infty} \mu(A_K)$ is NOT a convergence series!!! $\lightning$

CLAIM:

$\forall A \in \mathcal{S}, \forall \varepsilon > 0 \ \exists B \in \mathcal{S}, B \subseteq A$ s.t. $\mu(B) \geq \mu(A)$, $\forall E \in \mathcal{S}$, $E \in \mathcal{S}$, $\mu(E) > -\varepsilon$

PROOF OF THE CLAIM:

Let $c = \sup\{\mu(E) : E \subseteq A\}$. Choose $B \subseteq A$ s.t.

$$\mu(B) \geq \max\left\{\mu(A), c - \frac{\varepsilon}{2}\right\}$$

We claim that such B satisfies the claim: $\mu(E) > -\varepsilon \ \forall E \subseteq B$, otherwise $\exists E \subseteq B$ s.t. $\mu(E) \leq -\varepsilon$, $\mu(B \setminus E) = \mu(B) - \mu(E)$

$\Rightarrow \mu(B \setminus E) \geq c - \frac{\varepsilon}{2} + \varepsilon = c + \frac{3}{2}$ $\lightning$ (c is an upper bound)

CLAIM:

$\forall A \in \mathcal{S} \ \exists B \in \mathcal{S}$ s.t. $B \subseteq A$, $\mu(B) \geq \mu(A)$, B is positive

PROOF OF THE CLAIM:

Use the previous claim with $\varepsilon = \frac{1}{K} \Rightarrow A_1 \supseteq B_1 \supseteq B_2 \ \ldots \supseteq B_K$

$\Rightarrow \mu(A) \leq \ldots \leq \mu(B_K)$ and $\forall E \in \mathcal{S}$, $E \subset B_K$, $\mu(E) > \frac{1}{K}$.

Take $B = \bigcap_{K=1}^{+\infty} B_K$, then $\mu(B) \geq \mu(A)$ and B is a positive set.

Let now $S = \sup\{\mu(E) : E \in \mathcal{S}\} < +\infty$, by def. $\exists A_1, \ldots, A_n, \ldots \in \mathcal{S}$

s.t. $\mu(A_K) \to S$. By the previous claim $\exists B_1, \dots, B_n, \dots$ positive sets s.t. $\mu(B_K) \geq \mu(A_K) \Rightarrow \lim_{K \to +\infty} \mu(B_K) = S$. Then take $P = \bigcup_{K=1}^{+\infty} B_K$. Let $P_N = \bigcup_{K=1}^{N} B_K$, $\mu(P_N) \geq \mu(B_N) \to S$ $\Rightarrow \mu(P) = S \Rightarrow P$ is positive. Take now $N = X \setminus P$, then N is negative, otherwise $\exists E \subseteq N$ s.t. $\mu(E) > 0 \Rightarrow \mu(P \cup E) = \mu(P) + \mu(E) > S$ $\lightning$

$\square$

Remark:

We have the following:

Thm. (Radon-Nikodym for Signed Measures):

Let X be a set, $\mathcal{S}$ a σ-algebra on X, $\mu: X \to [0, +\infty)$ a positive and finite measure. Let $\nu: \mathcal{S} \to \mathbb{R}$ be a signed measure with $\nu \ll \mu$. Then $\exists w \in L^1(\mu)$ s.t.:

$$\nu(E) = \int_E w(x) d\mu(x) \quad \forall E \in \mathcal{S}$$

Proof:

Let $X = P \cup N$ be a Hahn decomposition for ν. Then let $\nu^+(A) := \mu(A \cap P)$, $\nu^- := \mu(A \cap N)$. We have $\nu^+ \ll \mu$, $\nu^- \ll \mu$

$\square$

We know that, given $p \in [1, +\infty]$, we can define a map $\phi: L^q(\mu) \to (L^p(\mu))'$ s.t. $\phi(v) = T_v$, $T_v(u) := \int_X u(x) v(x) d\mu(x)$ $\forall u \in L^q(\mu)$, and we also know that ϕ is an isometric injection (if $p = 1$, we make some additional assumption on μ). This map is actually an isomorphism, and now we can prove it.

Proposition:

If μ is a finite measure and $p \in [1, +\infty)$, then ϕ as above is surjective (therefore it is an isometric isomorphism).

Proof:

Take $T \in (L^p(\mu))'$. We need to show that $\exists v \in L^q(\mu)$ s.t. $T = T_v$. Let $E \in \mathcal{S}$, define $\nu(E) := T(\underset{L^p}{\underset{\Uparrow}{\mathbb{1}_E}})$. Then ν is a signed measure on $\mathcal{S}$:

$\nu(\emptyset) = 0$ (trivially),

$A, B \in \mathcal{S}$ s.t. $A \cap B = \emptyset \Rightarrow \nu(A \cup B) = T(\mathbb{1}_{A \cup B}) = T(\mathbb{1}_A + \mathbb{1}_B)$
$= T(\mathbb{1}_A) + T(\mathbb{1}_B) = \nu(A) + \nu(B)$,
$A_1, A_2, A_3, \ldots \in \mathcal{S}$ pairwise disjoint $\Rightarrow \mathbb{1}_{\bigcup_{k=1}^{+\infty} A_k} = \sum_{k=0}^{+\infty} \mathbb{1}_{A_k}$
(converges in $L^p(\mu)$)

$$\Rightarrow \nu\left(\bigcup_{k=0}^{+\infty} A_k\right) = T\left(\mathbb{1}_{\bigcup_{k=0}^{+\infty} A_k}\right) = \lim_{n \to +\infty} \mathbb{1}_{\bigcup_{k=1}^{n} A_k} = \lim_{n \to +\infty} T\left(\sum_{k=0}^{n} A_k\right)$$
$$= \lim_{n \to +\infty} \sum_{k=0}^{n} T(\mathbb{1}_{A_k}) = \lim_{n \to +\infty} \sum_{k=0}^{n} \nu(A_k) \Rightarrow \nu \ll \mu$$

$\Rightarrow \mu(E) = 0 \Rightarrow \mathbb{1}_E \sim 0 \Rightarrow T(\mathbb{1}_E) = T(0) = 0$
By Radon-Nikodym $\exists\, v \in L^1(\mu)$ s.t. $\nu(E) = T(\mathbb{1}_E)$
$$= \int_E v(x)\, d\mu(x) = \int_X v(x)\, \mathbb{1}_E(x)\, d\mu(x).$$
$\Rightarrow$ in general we have $T(s) = \int_X v(x) s(x)\, d\mu(x)$ $\forall s$ simple function (combination of characteristic functions)
$\Rightarrow$ simple functions are dense in $L^\infty(\mu)$, so the result still holds $\forall s \in L^\infty(\mu)$. We now prove that $v \in L^q(\mu)$ so that the result holds $\forall s \in L^p(\mu)$

CASE $1 < p < +\infty$:
Consider $E_n = \{x \in X : |v(x)| < n\}$, $s_n(x) = \mathbb{1}_{E_n}(x) \cdot |v(x)|^{q-1} \operatorname{sgn} v(x)$
$\Rightarrow s_n \in L^\infty(\mu)$, $p = \frac{q}{q-1}$, then we have:
$$T(s) = \int_{E_n} |v(x)|^q d\mu(x) \leqslant \|T\|_{(L^p)'} \cdot \|s_n\|_{L^p(\mu)}$$
$$= \|T\|_{(L^p)'} \left(\int_{E_n} |v(x)|^q d\mu(x)\right)^{\frac{q}{q-1}} \Rightarrow \left(\int_{E_n} |v(x)|^q d\mu(x)\right)^{\frac{1}{q}} \leqslant \|T\|_{(L^p)'}$$
$$\xrightarrow[n \to +\infty]{} \left(\int_X |v(x)|^q d\mu(x)\right)^{\frac{1}{q}} \Rightarrow \|v\|_{L^q(\mu)} \leqslant \|T\|_{(L^p)'}$$

CASE $p = 1$:
$$|T(\mathbb{1}_E)| = \left|\int_E v(x)\, dx\right| \leqslant \|T\|_{(L^1)'} \cdot \|\mathbb{1}_E\|_{L^1} = \|T\|_{(L^1)'} \cdot \mu(E)$$
$$\Rightarrow \left|\frac{1}{\mu(E)} \int_E v(x)\, dx\right| \leqslant \|T\|_{(L^1)'} \Rightarrow v(x) \leqslant \|T\|_{(L^1)'} \text{ for } \mu\text{-a.e. } x \in X$$
$$\Rightarrow \|v\|_{L^\infty(\mu)} \leqslant \|T\|_{(L^1)'}$$

$\square$