SOBOLEV EMBEDDING THM.

Now we know that if $1 \leq p \leq n$, $\Omega \subseteq \mathbb{R}^n$, functions in $W^{1,p}(\Omega)$ are not necessarily continuous

Thm. (Meyers - Serrin):
$\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$

Proof:
We only prove the thm. in the case $\Omega = \mathbb{R}^n$.
Let $u \in W^{1,p}(\mathbb{R}^n)$. Define:

$$u_k(x) = \int_{\mathbb{R}^n} u(x-y) \underbrace{\phi_k(y)}_{\text{bump functions}} dy = \int_{\mathbb{R}^n} u(y)\, \phi_k(x-y)\, dy \xrightarrow[L^p(\mathbb{R}^n)]{} u$$

$\Rightarrow$ we know that $u_k \in C^\infty(\mathbb{R}^n)$. Derivate:

$$\partial_{x_i} u_k(x) = \int_{\mathbb{R}^n} u(y) \underbrace{\partial_{x_i} \phi_k(x-y)}_{-\partial_{y_i}\phi_k(x-y)} dy = -\int_{\mathbb{R}^n} u(y)\, \partial_{y_i} \phi_k(x-y)\, dy$$

$$= \int_{\mathbb{R}^n} \underbrace{D_i u(y)}_{\in L^p} \phi_k(x-y)\, dy \xrightarrow[L^p]{} D_i u$$

$\phi_k \in \mathcal{C}^\infty_c(\mathbb{R}^n)$

$\square$

Def. ($W_0^{1,p}(\Omega)$):
$W_0^{1,p}(\Omega)$ is the closure of $\mathcal{C}^1_c(\Omega)$ WRT the $W^{1,p}$ norm.

Thm. (Sobolev Embedding):
For $1 \leq p < n$ $\exists C$ (depending only on p) s.t. $\forall u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^{p^*}(\Omega)} \leq C \cdot \|\nabla u\|_{L^p(\Omega)}$$

where ∇u is the vector of the weak derivatives, $p^* = \frac{np}{n+p}$ (Sobolev Exponent).

N.B.-
$u \in \mathcal{C}^1_c(\mathbb{R}^n) \Rightarrow u \in W^{1,p}(\Omega)$ if $\operatorname{supp}(u) \subset \Omega$

Suppose for some universal constant C the following holds.

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$u \in \mathcal{C}^1_c(\mathbb{R}^n) \Rightarrow v(x) = u(\nu x) \in \mathcal{C}^1_c(\mathbb{R}^n)$ $(\nu > 0)$.

We know that $\|v\|_q \leq C \|\nabla u\|_{L^p} \Rightarrow \|v\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(rx)|^q dx\right)^{\frac{1}{q}}$
$= \left(r^{-n} \int_{\mathbb{R}^n} |u(y)|^q dy\right)^{\frac{1}{q}} = r^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$. At the same time:

$$|\nabla_x u(rx)| = r|\nabla_y u(y)|$$

$$\|\nabla v\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} r^p |\nabla u(y)|^p r^{-n} dy\right)^{\frac{1}{q}} = r^{1-\frac{n}{p}} \|\nabla u\|_{L^p}$$

$$\Rightarrow \|u\|_{L^q(\mathbb{R}^n)} \leq C r^{1+\frac{n}{q}-\frac{n}{p}} \|\nabla u\|_{L^p} \quad \forall r$$

$\Rightarrow$ it must be $1 + \frac{n}{q} - \frac{n}{p} = 0 \iff q = \frac{np}{n-p} = p^*$

Dirichlet Problem for Poisson Equation

Suppose $\Omega \subseteq_{op} \mathbb{R}^n$ bounded and regular. We want to solve the following problem:

$$(*) \begin{cases} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

A classical solution would be $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ s.t. it satisfies the above conditions. Take $\phi \in \mathcal{C}^1_c(\Omega)$ test function and consider:

$$\int_\Omega (\Delta u(x) - f(x))\phi(x)\, dx = 0$$

Integration by parts:

$$(**) \quad -\int_\Omega \big(\underbrace{\nabla u(x) \cdot \nabla \phi(x)}_{\substack{\text{we only have the} \\ 1^{st} \text{ derivative here !!!}}} + f(x)\,\phi(x)\big)\, dx = 0$$

<u>Def.</u> (Weak Solution of $(*)$):
Let $f \in L^2(\Omega)$. A function $u \in W^{1,2}_0(\Omega)$ is a Weak Solution of $(*)$ iff $(**)$ holds $\forall \phi \in \mathcal{C}^1_c(\Omega)$ $(\forall \phi \in W^{1,2}_0)$

<u>Corollary</u> (Poincaré Inequality):
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. $\exists C$ (depending on Ω) s.t. $\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$ $\forall u \in W^{1,p}_0$

<u>Proof</u>:
$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \cdot 1\, dx\right)^{\frac{1}{p}}$ (Hölder with $\frac{p^*}{p}$, $\frac{p^*}{p^*-p}$)

$$\leqslant \left(\int_\Omega |u|^{p^*} dx\right)^{\frac{1}{p^*}} \cdot |\Omega|^{\frac{p^*-p}{p \cdot p^*}} \leqslant C |\Omega|^{\frac{p^*-p}{pp^*}} \|\nabla u\|_{L^p(\Omega)}$$

□

Remember that in $W^{1,2}$ we have the scalar product:

$$\langle u, v \rangle_{W^{1,2}} = \int_\Omega (u \cdot v + \nabla u \cdot \nabla v)\, dx$$

In $W_0^{1,2}$ we have an equivalent scalar product:

$$((u,v)) := \int_\Omega (\nabla u(x) \cdot \nabla v(x))\, dx$$

It is non degenerate:

$$((u,u)) = \int_\Omega |\nabla u|^2 dx \geqslant C \|u\|_{L^2}$$

It is the most common scalar product in $W_0^{1,2}$!!!

$\Rightarrow (W_0^{1,2}, ((\cdot,\cdot)))$ is an Hilbert space.

Now, consider $(**)$ again. It can be written as:

$$((u,\phi)) = \int_\Omega \nabla u(x) \cdot \nabla \phi(x)\, dx = -\int_\Omega f(x) \phi(x)\, dx =: F(\phi)$$

$\Rightarrow F \in (W_0^{1,2})'$!!! Indeed:

$$|F(\phi)| = \left|\int_\Omega f(x)\phi(x)\, dx\right| \leqslant \|f\|_{L^2(\Omega)} \cdot \|\phi\|_{L^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)} \cdot \|\nabla \phi\|_{L^2(\Omega)}$$

$\Rightarrow$ By Riesz Thm. $\exists!\, u \in W_0^{1,2}$ s.t.:

$$((u,\phi)) = F(\phi) \quad \forall \phi \in W_0^{1,2}(\Omega)$$

$\Rightarrow$ what about regularity? In dim. 1 it's easy to get it:
suppose we have $u''(x) = f(x)$ in (a,b), $u(a) = u(b) = 0$.
A weak solution solves the following:

$$\int_a^b (u'(x)\phi'(x) + f(x)\phi(x))\, dx = 0 \quad \forall \phi \in W_0^{1,2}([a,b])$$

$\Rightarrow \int_a^b u'\phi'\, dx = -\int_a^b f\phi\, dx \Rightarrow f$ is the weak derivative of u' !!!

PROOF OF THE SOBOLEV EMBEDDING THM.

Generalization of Hölder's Inequality:

Notice that, given $f_i \geqslant 0$ measurable, we have:

$$\int_\Omega \prod_{i=1}^m f_i(x)\, dx \leqslant \prod_{i=1}^m \|f_i\|_{L^{p_i}(\Omega)}, \quad p_i \text{ s.t. } \frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$$

(a valid choiche would be, for example, $p_i = m$, $i = 1, \dots, m$)

Lemma (Gagliardo):
Given $f_i : \mathbb{R}^{n-1} \to [0,+\infty]$ measurable, $i=1,\dots,n$ in $\mathbb{R}^n$, we have the following:

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} f_i(\underbrace{\hat{x}_i}_{\text{"}x_i \text{ omitted"}})\,dx \leq \prod_{i=1}^{n}\left(\int_{\mathbb{R}^{n-1}} f_i^{\,n-1}(\hat{x}_i)\,d\hat{x}_i\right)^{\frac{1}{n-1}}$$

Proof (Sobolev Embedding Thm.):
Case $p=1$, $p^* = \frac{n}{n-1}$:

$$u(x) = \int_{-\infty}^{x_1} \partial_{x_1} u(t, x_2, \dots, x_n)\,dt = \int_{-\infty}^{x_1} \partial_{x_1} u(x)\,dx_1$$

suboptimal notation

$$\Rightarrow |u(x)| \leq \int_{\mathbb{R}} |\nabla u(x)|\,dx_1, \quad \text{in general: } |u(x)| \leq \int_{\mathbb{R}} |\nabla u(x)|\,dx_i \quad i=1,\dots,n$$

$$\Rightarrow |u(x)|^{\frac{1}{n-1}} \leq \left(\int_{\mathbb{R}} |\nabla u(x)|\,dx_i\right)^{\frac{1}{n-1}} \quad i=1,\dots,n$$

$$\Rightarrow \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} \underbrace{\left(\int_{\mathbb{R}} |\nabla u(x)|\,dx_i\right)^{\frac{1}{n-1}}}_{= g(\hat{x}_i)} dx$$

$$\underset{\text{(Gagliardo)}}{\leq} \prod_{i=1}^{n}\left(\int_{\mathbb{R}^n} |\nabla u(x)|\,dx\right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |\nabla u(x)|\,dx\right)^{\frac{n}{n-1}}$$

$$\Rightarrow \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n)}$$

Case $p>1$:
Define $v(x) = |u(x)|^{1+r} \in C_c^1(\mathbb{R}^n)$, $r>0$. Then:

$$\nabla v(x) = (1+r)\,|u(x)|^{r} \cdot |\nabla u(x)|$$

$$\Rightarrow \left(\int_{\mathbb{R}^n} |u|^{(1+r)\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq (1+r)\int_{\mathbb{R}^n} |u(x)|^{r} \cdot |\nabla u(x)|\,dx$$

$$\leq (1+r)\,\|\nabla u\|_{L^p} \left(\int_{\mathbb{R}^n} |u(x)|^{r\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}$$

We choose r as follows:

$$\Rightarrow (1+r)\frac{n}{n-1} = r\frac{p}{p-1} \Rightarrow \frac{n}{n-1} = r\left(\frac{p}{p-1} - \frac{n}{n-1}\right) \Rightarrow r = \dots = \frac{n(p-1)}{n-p}$$

$$\Rightarrow r\frac{p}{p-1} = \frac{np}{n-p} = p^*$$

so we have:

$$\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{(1-\frac{1}{n})-(1-\frac{1}{p})}=\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{\frac{1}{p^*}}\leq(1+\gamma)\|\nabla u\|_{L^p}$$

$\square$

Proof (Gagliardo):

We prove the result by induction over n:

$n=2$: trivial, it is Fubini's Thm.

$n \rightsquigarrow n+1$:

assume the result holds for n. We have:

$$\int_{\mathbb{R}^{n+1}}\prod_{i=1}^{n+1}f_i(\hat{x}_i)\,dx=\int_{\mathbb{R}^n}f_{n+1}(\hat{x}_{n+1})\,d\hat{x}_{n+1}\int_{\mathbb{R}}\prod_{i=1}^{n}f_i(\hat{x}_i)\,dx_{n+1}$$

$$\leq\int_{\mathbb{R}^n}f_{n+1}(\hat{x}_{n+1})\prod_{i=1}^{n}\left(\int_{\mathbb{R}}f_i^n(\hat{x}_i)\,dx_{n+1}\right)^{\frac{1}{n}}d\hat{x}_{n+1}$$

$$\leq\left(\int_{\mathbb{R}^n}f_{n+1}^n(\hat{x}_{n+1})\,d\hat{x}_{n+1}\right)^{\frac{1}{n}}\left(\int_{\mathbb{R}^n}\left(\prod_{i=1}^{n}\int_{\mathbb{R}}f_i^n(\hat{x}_i)\,dx_{n+1}\right)^{\frac{1}{n-1}}d\hat{x}_{n+1}\right)^{\frac{n-1}{n}}$$

$$\underset{\text{(inductive hypothesis)}}{\leq}\prod_{i=1}^{n+1}\left(\int_{\mathbb{R}^n}f_i^n(\hat{x}_i)\,d\hat{x}_i\right)^{\frac{1}{n}}$$

$\square$