Closed Graph Theorem

Thm. (*Closed Graph Thm.*):
$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ Banach spaces, $T: X \to Y$ linear, then:
T is continuous $\Leftrightarrow$ $\Gamma_T = \{(x, T(x)) : x \in X\}$ is closed in $X \times Y$

Proof:
($\Rightarrow$): trivial, suppose T is continuous, consider $(\bar{x}, \bar{y}) \in \overline{\Gamma_T}$, then:

$$\exists \{x_k\} \subset X \text{ s.t. } (x_k, T(x_k)) \to (\bar{x}, \bar{y}) \wedge x_k \xrightarrow{X} \bar{x}$$

$$\Rightarrow T(x_k) \to T(\bar{x}) \Rightarrow \bar{y} = T(\bar{x}) \Rightarrow (\bar{x}, \bar{y}) \in \Gamma_T$$

($\Leftarrow$): suppose Γ_T is closed in $X \times Y$: it is a closed linear subspace of the Banach space $X \times Y \Rightarrow \Gamma_T$ is Banach. Consider the projection $p_1 : X \times Y \to X$, $p_1(x,y) = x$, it is linear, continuous of norm 1. Then $p_1 : \Gamma_T \to X$ is linear, continuous and an isomorphism. By the Open Mapping Thm., p_1 is open $\Rightarrow p_1^{-1}$ is continuous, but:

$$p_1^{-1} = \phi : X \to X \times Y, \quad \phi(x) = (x, T(x))$$

$$\Rightarrow T(x) = p_2 \circ \phi\,(x) \text{ is continuous}$$

$\square$

Compactness in ∞-dim. Spaces

"If $\dim X = +\infty$, then $(X, \|\cdot\|)$ does NOT look like $\mathbb{R}^n$"

Thm. (*Riesz*):
Let $(X, \|\cdot\|)$ be a normed space, $\overline{B_1(0)} = \{x \in X : \|x\| \leq 1\}$.
Then $\overline{B_1(0)}$ is compact $\Leftrightarrow \dim_{\mathbb{R}} X < +\infty$.

Proof:
($\Leftarrow$): trivial, $\dim_{\mathbb{R}} X = n \Rightarrow X \cong \mathbb{R}^n \wedge X, \mathbb{R}^n$ are isometric
($\Rightarrow$): we use the following:

Lemma (*Riesz's Lemma*):
Let $(X, \|\cdot\|)$ be a normed space, Y a closed proper vector subspace of X. Then $\exists \bar{x} \in X$ s.t. $\|\bar{x}\| = 1 \wedge \text{dist}(\bar{x}, Y) \geq \frac{1}{2}$

Proof:
Let $x_0 \in X \setminus Y$, $\delta = \text{dist}(x_0, Y) = \inf\{\|x_0 - y\|, y \in Y\} > 0$.
$\Rightarrow \exists \bar{y} \in Y$ s.t. $\|x_0 - \bar{y}\| = 2\delta$, define $\bar{x} = \dfrac{x_0 - \bar{y}}{\|x_0 - \bar{y}\|}$.

Let $y \in Y$, then $\|\bar{x} - y\| = \left\| \frac{x_0 - \bar{y}}{\|x_0 - \bar{y}\|} - y \right\| = \frac{1}{\|x_0 - \bar{y}\|} \|x_0 - \bar{y} - y\|x_0 - \bar{y}\|\|$

$= \frac{1}{\|x_0 - \bar{y}\|} \|x_0 - \underbrace{(\bar{y} + y\|x_0 - \bar{y}\|)}_{\in Y}\| \geq \frac{\delta}{2\delta} = \frac{1}{2}$

□

Choose a seq. of vector subspaces of X ($Y_1 \subset Y_2 \dots \subset Y_k$)
s.t. $\dim Y_k = k$. Let $x_1 \in Y_1$ s.t. $\|x_1\| = 1$, then $\exists x_2 \in Y_2$
s.t. $\|x_2\| = 1 \wedge \operatorname{dist}(x_2, Y_1) \geq \frac{1}{2}$.
We proceed in the same way:
$\exists x_k, Y_{k-1}$ s.t. $\|x_k\| = 1 \wedge \operatorname{dist}(x_k, Y_{k-1}) \geq \frac{1}{2}$
$\Rightarrow \{x_k\} \subset X \Rightarrow \exists n \neq m$ s.t. $\|x_n - x_m\| \geq \frac{1}{2}$

□

CHARACTERIZATION OF COMPACTNESS IN A METRIC SPACE:

Thm. (Compactness in a Metric Space):
Given (X, d) a metric space, $K \subset X$, the following are equivalent:
1) K is sequentially compact ($\forall \{x_k\}_k \subset K \ \exists \{x_{k_h}\}_h, \bar{x} \in K$ s.t. $x_{k_h} \to \bar{x}$)
2) K is topologically compact (each open covering of K has a finite subcovering)
3) K is complete and totally bounded ($\forall \varepsilon > 0$ there is a finite number of open balls with radius ε covering K)

Proof:
$1 \Leftrightarrow 2$ ✓.
$1, 2 \Rightarrow 3$:
Let K be compact, $\{x_k\}_k \subset K$ Cauchy seq. $\Rightarrow \exists \{x_{k_h}\}_h, \bar{x} \in K$ s.t. $x_{k_h} \to \bar{x} \Rightarrow x_k \to \bar{x} \Rightarrow K$ is complete.
Let $\varepsilon > 0$, then $\{B_\varepsilon(x) : x \in K\}$ is an open covering of K.
By topological compactness, $\exists$ finite number of balls covering $K \Rightarrow K$ is totally bounded.
$3 \Rightarrow 1, 2$:
We prove that K is sequentially compact. Let $\{x_k\}_{k \in \mathbb{N}} \subset K$.
By total boundedness we can cover K with a finite number of balls of radius 1
$\Rightarrow \exists B_1$ ball of radius 1 s.t. $x_k \in B_1$ for infinitely $k \in \mathbb{N}$.

Let $\{x_k^{(1)}\}_k$ be the subsequence of the elements in B_1.
We now cover K with a finite number of balls of radius $\frac{1}{2}$
$\Rightarrow \exists B_2$ ball of radius $\frac{1}{2}$ s.t. $x_k^{(1)} \in B_2$ for infinitely $k \in \mathbb{N}$.
Let $\{x_k^{(2)}\}_k$ be the subsequence of $\{x_k^{(1)}\}$ of the elements in B_2. etc.

$\Rightarrow$ cover K with a finite number of balls of radius $\frac{1}{n}$, choose B_n s.t. $\{x_k^{(n-1)}\} \subset B_n$ for infinitely $k \in \mathbb{N}$, and let $\{x_k^{(n)}\}$ be the corresponding subsequence of $\{x_k^{(n-1)}\}$.

$\Rightarrow$ Let $\{\tilde{x}_k\}_k$ be the diagonal subsequence $(\tilde{x} := x_k^{(k)})$: it is a Cauchy sequence in K.

$\square$