IDEAL OBJECTS

Recall:

We picture a presheaf F on $\mathcal{C}$ (a functor $\mathcal{C}^{op} \to Set$) as an "ideal fictional object of $\mathcal{C}$". Just as we can probe actual objects A of $\mathcal{C}$, by morphisms $T \to A$ we can probe a presheaf "$Hom_{\mathcal{C}}(T, F) = F(T)$". Just as we can precompose a probe $T \xrightarrow{f} A$ with a morphism $T' \xrightarrow{g} T$ in $\mathcal{C}$, we can "precompose" a probe $s \in F(T)$ by such a morphism to get the probe $F(g)(s) \in F(T')$

Examples:

1) Given the functor

$$\begin{aligned} F: \mathcal{C}^{op} &\to Set \\ X &\to \{x\} \\ f &\to id_{\{x\}} \end{aligned}$$

for every actual object T of $\mathcal{C}$ we have:

$$\text{"}Hom_{\mathcal{C}}(T, F)\text{"} = F(T) = \{x\}$$

So F is a proxy/placeholder/next best substitute for a terminal object of $\mathcal{C}$

2) Let $A, B \in Ob(\mathcal{C})$, define the functor F as follows:

$$\begin{aligned} F: \mathcal{C}^{op} &\to Set \\ T &\to Hom_{\mathcal{C}}(T, A) \times Hom_{\mathcal{C}}(T, B) \end{aligned}$$

$\Rightarrow$ For every object $T \in Ob(\mathcal{C})$ we have:

$$\text{"}Hom_{\mathcal{C}}(T, F)\text{"} = Hom_{\mathcal{C}}(T, A) \times Hom_{\mathcal{C}}(T, B)$$

$\Rightarrow F$ is a proxy for a product of A, B

3) Consider the functor:

$$\begin{aligned} F: Vect(\mathbb{R})^{op} &\to Set \\ V &\to V \times V \times V = \{(x, y, z) : x, y, z \in V\} \end{aligned}$$

$\Rightarrow$ For every $V \in Vect(\mathbb{R})$ we have:

$$\text{"}Hom(V, F)\text{"} = V \times V \times V$$

$\Rightarrow F$ is a proxy for $\mathbb{R}^3$

Def. (Representable Presheaf):
A presheaf $F: \mathcal{C}^{op} \to Set$ is called REPRESENTABLE iff

$$\exists X \in Ob(\mathcal{C}) \text{ s.t. } F \cong \hat{X}: \begin{array}{l} \mathcal{C}^{op} \to Set \\ T \mapsto Hom_{\mathcal{C}}(T,X) \end{array}$$

Example:
The presheaf

$$F: (Ring^{op})^{op} \to Set$$
$$T \mapsto \{x \in T : x^5 = 0\}$$
$$\varphi \mapsto \{x \in T : x^5 = 0\}$$

is representable by $\underbrace{\mathbb{Z}/[X]/(X^5)}_{\text{this ring contains nilpotent}} \in Ob(Ring^{op})$

$$Hom_{Ring}(\mathbb{Z}/[X]/(X)^5, A) = \{x \in A : x^5 = 0\}$$
$$\|$$
$$Hom_{Ring^{op}}(A, \mathbb{Z}/[X]/X^5)$$

YONEDA EMBEDDING

Corollary (Yoneda Embedding):
The Yoneda Embedding $\mathcal{L}: \mathcal{C} \to Psh(\mathcal{C})$, $X \mapsto \hat{X}$ is fully faithful and cocontinuous

Proof:

$$Hom_{Psh(\mathcal{C})}(\underbrace{\mathcal{L}(X)}_{=\hat{X}}, \underbrace{\mathcal{L}(Y)}_{=\hat{Y}}) \underset{\text{Yoneda Lemma}}{\cong} \hat{Y}(X) \underset{\text{def. of } \hat{Y}}{=} Hom_{\mathcal{C}}(X,Y)$$

$\square$

Examples:
1) Let $\mathcal{C}$ be a category in which all products exist, $A, B \in Ob(\mathcal{C})$. Then $A \times B \cong B \times A$

Proof:
We just show $\mathcal{L}(A \times B) \cong \mathcal{L}(B \times A)$:

$$\mathcal{L}(A \times B) = \widehat{(A \times B)} = Hom_{\mathcal{C}}(\cdot, A \times B)$$
$$\cong Hom_{\mathcal{C}}(\cdot, A) \times Hom_{\mathcal{C}}(\cdot, B) \cong Hom_{\mathcal{C}}(\cdot, B) \times Hom_{\mathcal{C}}(\cdot, A)$$

$\cong \mathrm{Hom}(\cdot, B\times A) \cong \widehat{(B\times A)} \cong \text{よ}(B\times A)$

$\Rightarrow$ since よ is fully-faithful, we have $A\times B \cong B\times A$

$\square$

<u>Remark</u>:

The philosophy "relations already suffice to determine an object up to isomorphism" is implemented by the formal statement that よ is fully-faithful:

$$\underbrace{\text{よ}(X)}_{} \cong \text{よ}(Y) \Rightarrow X \cong Y$$

this presheaf encodes all relations with X:
$\text{よ}(X) : T \longmapsto \mathrm{Hom}_{\mathcal{C}}(T, X)$

<u>Example</u>:

Let X be a set. Which relations do we need to know to reconstruct X? We only need to know $\mathrm{Hom}_{Set}(\{x\}, X)$!!!
Indeed:

$$\begin{aligned} \mathrm{Hom}_{Set}(\{x\}, X) &\cong X \\ f &\longrightarrow f(x) \end{aligned}$$

So the relations of $\{x\}$ with X are enough (a small part of $\text{よ}(X)$, namely $\text{よ}(X), \text{よ}(\{x\})$)

<u>Remark</u>:

Table of analogies:

$\mathcal{C}$	$\mathbb{Q}$
$Psh(\mathcal{C})$	$\mathbb{R}$
よ	$i : \mathbb{Q} \to \mathbb{R},\ x \mapsto x$
よ fully faithful	i injective
よ functor	i monotone
よ NOT essentially surjective	i NOT surjective
よ dense	i dense
$\forall A, B \in Ob(\mathcal{C})$	$\forall a, b \in \mathbb{Q}\ (\forall c \in \mathbb{Q}$
$\text{よ}(A) \cong \text{よ}(B) \Rightarrow A \cong B$	$c \leq a \Leftrightarrow c \leq b) \Rightarrow a = b$
よ continuous	i continuous

Adjoints

Def. ((Left / Right) Adjoint):
Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be 2 functors. F is LEFT ADJOINT to G, $F \dashv G$ (or G is RIGHT ADJOINT to F, $F \vdash G$) iff $\forall X \in Ob(\mathcal{C}), \forall Y \in Ob(\mathcal{D})$ there $\exists$ isomorphism

$$Hom_{\mathcal{D}}(F(X), Y) \cong Hom_{\mathcal{C}}(X, G(Y))$$

Remark:
In Linear Algebra we have $\langle Mx, y\rangle = \langle x, M^T y\rangle$

Examples:
1) $\lceil\cdot\rceil : \mathbb{Q} \to \mathbb{Z}$, $\lfloor\cdot\rfloor : \mathbb{Q} \to \mathbb{Z}$, $i : \mathbb{Q} \to \mathbb{Z}$, $x \mapsto x$ induce functors:

$$B\lceil\cdot\rceil, B\lfloor\cdot\rfloor : B\mathbb{Q} \to B\mathbb{Z},$$
$$Bi : B\mathbb{Q} \to B\mathbb{Z}$$

$\Rightarrow Bi$ is not an elementary equivalence. Moreover:

$$B\lceil\cdot\rceil \dashv Bi \dashv B\lfloor\cdot\rfloor$$

N.B.
Still, $\cancel{B\lceil\cdot\rceil \dashv B\lfloor\cdot\rfloor}$!!! Adjointness is NOT a transitive property !!!

2) Let $F : Set \to Vect(\mathbb{R})$, $M \mapsto$ free vector space on M, $G : Vect(\mathbb{R}) \to Set$, $V \mapsto V$

Then $F \dashv G$

Proposition:
Adjoints are unique up to isomorphism