MORREY'S EMBEDDING THM.

N.B. for $W^{1,p}(\Omega)$, we can prove the Sobolev Embedding Thm. if Ω is regular enough

Now we investigate the case $p>n$

Thm. (Morrey's Embedding):
Let $p>n$. Then $\exists\, C$ (depending only on p) s.t. $\forall u \in W_0^{1,p}(\Omega)$

$$[u]_\alpha \leqslant C \|\nabla u\|_{L^p(\Omega)}$$

where $[u]_\alpha := \sup\left\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} : x,y\in\Omega,\ x\neq y \right\}$, $\alpha = 1-\frac{n}{p}$

Proof:
It is enough to prove the thm. for $u\in C_c^1(\mathbb{R}^n)$:

Fix $x,y\in\mathbb{R}^n$, let $|x-y| =: \delta>0$

$$|u(x)-u(y)| \leqslant |u(z)-u(x)| + |u(z)-u(y)|$$

$\Rightarrow$ let $S = B_\delta(x)\cap B_\delta(y)$ and integrate both sides:

$$\underbrace{|S|}_{=k\cdot\delta^n}\cdot|u(x)-u(y)| \leqslant \int_S |u(z)-u(x)|dz + \int_S |u(z)-u(y)|dz$$

$\Rightarrow$ we have:

$$|u(z)-u(x)| = \left|\int_0^1 \frac{d}{dt}u(x+t(z-x))dt\right| = \left|\int_0^1 \nabla u(x+t(z-x))\cdot(z-x)\,dz\right|$$

$$\leqslant \delta\int_0^1 |\nabla u(x+t(z-x))|\,dt$$

$$\Rightarrow \delta\int_S\int_0^1 |\nabla u(x+t(z-x))|\,dt\,dz = \delta\int_0^1\int_S |\nabla u(x+t(z-x))|\,dz\,dt$$

$$\leqslant \delta\int_0^1 t^{-n}\int_{B_{\delta t}(x)} |\nabla u(w)|\,dw\,dt \leqslant \delta\int_0^1 t^{-n}\|\nabla u\|_{L^p(\mathbb{R}^n)}\cdot(\underbrace{\omega_n}_{\text{measure of the 1-radius ball in } \mathbb{R}^n}\delta^n t^n)^{1-\frac{1}{p}}dt$$

$w = x+t(z-x)$
$dw = t^n\,dz$

$$= \omega_n^{1-\frac{1}{p}}\cdot\delta^{1+n-\frac{n}{p}}\cdot\|\nabla u\|_{L^p(\mathbb{R}^n)}\cdot\underbrace{\int_0^1 t^{-\frac{n}{p}}\,dt}_{<+\infty}$$

and the same holds for the 2^{nd} integral. We have:

$$|u(x)-u(y)| \leqslant C\,\delta^{1-\frac{n}{p}}\|\nabla u\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}}\cdot\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$\square$

Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be equibounded, $p>n$, then we can get a compactness thm. for $W^{1,p}(\Omega)$ which is almost identical to the result we got in dim. 1.

N.B.

In the space $W^{1,p}(\Omega)$ the Sobolev Embedding Thm. and the Morrey's Thm. become:

$$\begin{cases} \|u\|_{L^{p^*}(\Omega)} \leq C_\Omega \cdot \|u\|_{W^{1,p}(\Omega)} & (1 \leq p < n) \\ [u]_{1-\frac{n}{p}} \leq C_\Omega \cdot \|u\|_{W^{1,p}(\Omega)} & (p > n) \end{cases}$$