Limits (continued) – Colimits

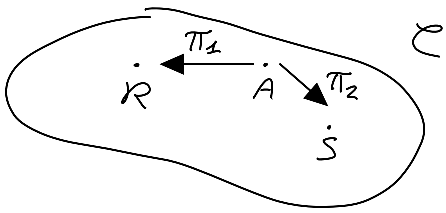
Def. (Cone)
A **Cone** of a diagram $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

1) an object $A \in Ob(\mathcal{C})$ ("tip" of the cone)
2) $\forall X \in Ob(\mathcal{I})$ a morphism $\pi_X : A \to F(X)$ s.t. $\forall f : X \to Y$ in $\mathcal{I}$ the following triangle commutes

$$\begin{tikzcd} & A \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[d, phantom, "\equiv"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$

A **Co-cone** of F is a cone of the induced functor $\mathcal{I}^{op} \xrightarrow{F^{op}} \mathcal{C}^{op}$

Example:
$\mathcal{I} = \quad \underset{1}{\cdot} \quad \underset{2}{\cdot}$, $F : \mathcal{I} \to \mathcal{C}$, A as follows:
$1 \to R$
$2 \to S$



$\Rightarrow (A, \pi_1, \pi_2)$ is a cone

Def. (Morphism between cones):
A **Morphism between cones** $(A, (\pi_X)_X)$, $(B, (\varphi_X)_X)$ of a diagram $F : \mathcal{I} \to \mathcal{C}$ consists of a morphism $A \to B$ in $\mathcal{C}$ s.t. the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\pi_X"'] & \arrow[d, phantom, "\equiv"] & B \arrow[dl, "\varphi_X"] \\ & F(X) & \end{tikzcd}$$

Def. (Limit):
A **Limit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is a terminal cone of F i.e. a terminal object in the category of cones of F.

Remark:
A limit of F is a product of R and S

Def. (Colimit):
A **Colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an initial co-cone of F

Examples:

1) I empty category, $\mathcal{C}$ an arbitrary category, $F : I \to \mathcal{C}$ then a limit of F is any terminal object in $\mathcal{C}$ and a colimit of F is any initial object of $\mathcal{C}$

2) Let I be as follows:

$$\begin{tikzcd} 2 \arrow[r, "g"] \arrow[d, "f"'] & 3 \\ 1 & \end{tikzcd}$$

and $F : I \to \mathrm{Top}$ ($\mathrm{Top} :=$ category of topological spaces, morphisms are continuous maps)

$1 \to$ 

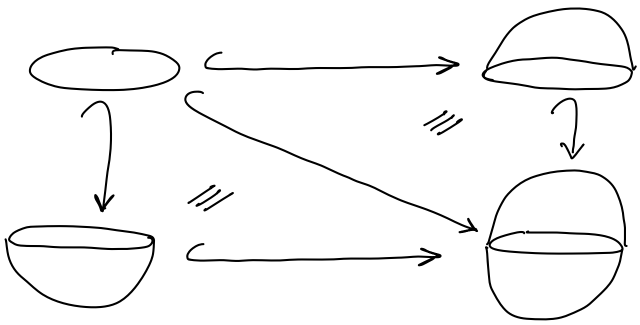
$2 \to$ 

$3 \to$ 

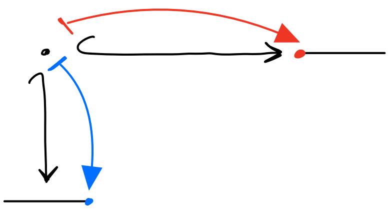
$f \to$ inclusion of  in 

$g \to$ inclusion of  in 

then the colimit of F is . Indeed:

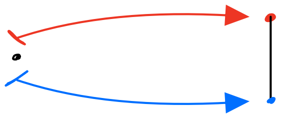


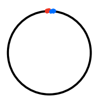
3) Given:



$\Rightarrow$ the colimit is

4) Given:



$\Rightarrow$ the colimit is 

5) The limit of the following diagram in Set

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is $\{x \in X : f(x) = g(x)\} = \mathrm{Eq}(f,g)$ the equalizer of f,g

6) Let $f: V \to W$ be a linear transformation of vector spaces. Then the limit of the following diagram

$$\begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & W \end{tikzcd}$$

is $\operatorname{Ker} f$

7) The limit of the following diagram in Ring

$$\begin{array}{ccccccc} \cdots\cdots \longrightarrow & \mathbb{Z}/(10^3) & \longrightarrow & \mathbb{Z}/(10^2) & \longrightarrow & \mathbb{Z}/(10) \\ & xyz & \longrightarrow & yz & \quad xy \longrightarrow & y \end{array}$$

is called $\mathbb{Z}_{10}$ (the 10-adic integers) and it is given by $\mathbb{Z}_{10} = \{ \dots x_2 x_1 x_0 : x_i \in \{0, \dots, 9\}\}$

8)
$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$
where $A \times_B C = \{(x,y) : x \in A, y \in C, f(x) = g(y)\}$ is the Fiber product of A and C over B

9) Given $U \subseteq Y$ in Set,

$$\begin{tikzcd} f^{-1}U \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \\ U \arrow[r, hook, "i"] & Y \end{tikzcd}$$
where $f^{-1}U = \{x \in X : f(x) \in U\}$, g is the pullback of f along i

Def. (Complete / Cocomplete Category):
A category $\mathcal{C}$ is COMPLETE iff every small diagram in $\mathcal{C}$ (i.e. a functor $I \to \mathcal{C}$ where I is a small category) has a limit. $\mathcal{C}$ is COCOMPLETE iff it has all small colimits.

Remark:
$\mathcal{C}$ complete $\iff$ $\mathcal{C}^{op}$ cocomplete

Remark:
Assuming LEM and AC, the only categories which have all limits or all colimits are (some) thin categories (i.e. parallel morphisms are equal)

Examples
The following categories are complete and cocomplete:
Set, Vect($\mathbb{R}$), Grp, AbGrp, Top

$\mathrm{Vect}(\mathbb{R})_{\text{Finite dim.}}$, Pokémon, Numerical Category are NOT complete

Proposition:

Let $F: I \to \mathrm{Set}$ be an arbitrary diagram, I small. Then a limit of F is given as follows:

tip: $L := \{(s_X)_{X \in Ob(I)} : s_X \in F(X), \forall f: X \to Y \text{ in } I \ F(f)(s_X) = s_Y\}$

projection morphisms:

$$\begin{aligned} \pi_X : L &\to F(X) \\ (s) &\to s_X \end{aligned}$$

Remark:

The formula shows that in Set limits are subsets of products

Proposition:

Let $F: I \to \mathrm{Set}$ be an arbitrary diagram, I small. Then a colimit of F is given as follows:

tip: $K := \left(\coprod_{X \in Ob(I)} F(X)\right) / \sim$,

where $\sim$ is the finest equivalence relation generated by the following:

$$\forall f: X \to Y \text{ in } I \quad \forall s \in F(X)$$
$$s \sim \underbrace{F(f)(s)}_{\in F(Y)}$$

Yoneda's Lemma

Lemma (Yoneda):

Let $\mathcal{C}$ be a category, $X \in Ob(\mathcal{C})$, F a presheaf of $\mathcal{C}$. Then $\exists$ bijection $\mathrm{Hom}_{\mathrm{Psh}}(\hat{X}, F) \cong F(X)$, and this bijection is natural in X and F.

Corollary:

The Yoneda Embedding $\quad \begin{aligned} \text{よ}: \mathcal{C} &\to \mathrm{Psh}(\mathcal{C}) \\ X &\to \hat{X} \end{aligned} \quad$ is fully faithful and cocontinuous

Remark:

Compare this with the embedding $\mathbb{Q} \xhookrightarrow{i} \mathbb{R}$:

it is monotonic and continuous. $\mathbb{Q}$ is not complete ($\exists$ Cauchy sequences without a limit in $\mathbb{Q}$. $\mathbb{R}$ is complete.

Def. (Presheaf):
A PRESHEAF on a category $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathrm{Set}$

Interpretation:
We picture a presheaf F on $\mathcal{C}$ as an ideal fictional object of $\mathcal{C}$, in that we know its relation to actual objects of $\mathcal{C}$:

$$\text{"}\mathrm{Hom}_{\mathcal{C}}(X, F) := F(X)\text{"}$$

Indeed let "$s : X \to F$" (i.e. $s \in F(X)$) be a "morphism" and let $f : Y \to X$ be an (actual) morphism. Then there should $\exists$ a "morphism" "$Y \xrightarrow[s \circ f]{} F$". Indeed there $\exists$:

$$\underbrace{F(f)}_{F(X) \to F(Y)}(s) \in F(Y)$$