Category Theory Course Notes

Ettore Forigo

Chapter 1

1.1 Definition of Category

A **category** (1-category) $\mathcal{C}$ consists of:

1 - A class $Ob(\mathcal{C})$ of objects of $\mathcal{C}$

2 - $\forall X, Y \in Ob(\mathcal{C})$.
a class $Hom_{\mathcal{C}}(X, Y)$ of **morphisms** from X to Y

3 - $\forall X \in Ob(\mathcal{C})$.
an **identity morphism** $id_X \in Hom_{\mathcal{C}}(X, X)$

4 - $\forall X, Y, Z \in Ob(\mathcal{C})$.
a **composition rule**:

$$Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$$
$$(g, f) \mapsto g \circ f$$

Such that it satisfies the following axioms:

1 - **Associativity of composition**:

$\forall X, Y, Z, W \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y), g \in Hom_{\mathcal{C}}(Y, Z), h \in Hom_{\mathcal{C}}(Z, W)$.
$h \circ (g \circ f) = (h \circ g) \circ f$

2 - **Neutrality**:

$\forall X, Y \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y)$.
$id_Y \circ f = f \wedge f \circ id_X = f$

1.2 Thin Categories

A category is **thin** if parallel morphisms are always the same, meaning that there is only one morphism between two objects.

In a thin category all morphisms are monic and epic.

1.3 Definition of Initial Object

An object I of a category $\mathcal{C}$ is **initial** (dual of terminal, special case of a colimit (of a functor from $\mathcal{C}$ to the empty category))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(I, X)$

1.4 Definition of Terminal Object

An object T of a category $\mathcal{C}$ is **terminal** (dual of initial, special case of limit (of a functor from the empty category to $\mathcal{C}$))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(X, T)$

1.5 Definition of Monomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is a **monomorphism** (or monic in $\mathcal{C}$) (dual of epimorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Z, X)$.
$f \circ p = f \circ q \implies p = q$

Example:
In **Set** monomorphisms are precisely the injective maps.

Monomorphisms "can be cancelled" from the left.

1.6 Definition of Split Monomorphism

A **split monomorphism** (dual of split epi) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$g \circ f = id_X$

Proposition: every split mono is a mono.
Proposition: in **Set**, every mono $f : X \to Y$ where X is inhabited is a split mono, assuming LEM holds.

1.7 Definition of Epimorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **epimorphism** (or epic in $\mathcal{C}$) (dual of monomorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Y, Z).$
$p \circ f = q \circ f \implies p = q$

Example:
In **Set** epimorphisms are precisely the surjective maps.

Epimorphisms "can be cancelled" from the right.

1.8 Definition of Split Epimorphism

A **split epimorphism** (dual of split mono) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$f \circ g = id_Y$

Proposition: every split epi is an epi.
Proposition: in **Set**, every epi is a split epi $\iff$ assuming LEM holds.

1.9 Definition of Isomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **isomorphism**
$\Updownarrow$
$\exists g \in Hom_{\mathcal{C}}(Y, X).$
$f \circ g = id_Y \wedge g \circ f = id_X$

$id_X \forall X \in Ob(\mathcal{C})$ is always an isomorphisms for every category $\mathcal{C}$.
Objects X and Y in a category $\mathcal{C}$ are **isomorphic**
$\Updownarrow$
there exists an isomorphism between X and Y ($X \cong Y$)

In **Set**, if there exists an isomorphism between X and Y, X and Y are called eqinumerous.

1.10 Definition of Opposite Category

"The mother of all dualities"

Let $\mathcal{C}$ be a category. Then its opposite category $\mathcal{C}^{op}$ is the following category:

- $Ob(\mathcal{C}^{op}) := Ob(\mathcal{C})$

- $Hom_{\mathcal{C}^{op}}(X, Y) := Hom_{\mathcal{C}}(Y, X)$

- identities and composition inherited from $\mathcal{C}$
  $id_X \in Hom_{\mathcal{C}}(X, X) = id_X^{op} \in Hom_{\mathcal{C}^{op}}(X, X)$
  $f \circ g := g^{op} \circ f^{op}$

Observations / Remarks:

- An object I of $\mathcal{C}$ is initial in $\mathcal{C}$
  $\Updownarrow$
  I is terminal when regarded as an object of $\mathcal{C}^{op}$

- A morphism in $\mathcal{C}$ is a monomorphism
  $\Updownarrow$
  it is an epimorphism in $\mathcal{C}^{op}$

1.11 Dualities?

1.12 Definition of Product

A **product** (special case of limit) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object P of $\mathcal{C}$

- a morphism $\pi_X : P \to X$ in $\mathcal{C}$

- a morphism $\pi_Y : P \to Y$ in $\mathcal{C}$

such that for every object Q of $\mathcal{C}$ together with morphisms $\varphi_X : Q \to X, \varphi_Y : Q \to Y$ there is exactly one morphism $Q \to P$ such that the following diagram commutes:

$\varphi_X = \pi_X \circ !$
$\varphi_Y = \pi_Y \circ !$

Remarks:
- Products are always associative and commutative up to isomorphism.
- There is also the notion of the (co) product of zero, one, three, four, ... objects.
- The zero case of a product is just a terminal object, an object with exactly one morphism from each object.

1.13 Definition of Coproducts

A **coproduct** (special case of colimits) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object C of $\mathcal{C}$

- a morphism $\iota_X : X \to C$ in $\mathcal{C}$

- a morphism $\iota_Y : Y \to C$ in $\mathcal{C}$

such that for every object D of $\mathcal{C}$ together with morphisms $\chi_X : X \to D, \chi_Y : Y \to D$ there is exactly one morphism $C \to D$ which renders the following diagram commutative:

$\chi_X = ! \circ \iota_X$
$\chi_Y = ! \circ \iota_Y$

Remarks:

- Products in $\mathcal{C}^{op}$ are precisely coproducts in $\mathcal{C}$

- The zero case of a coproduct is the same as an initial object.

1.14 Definition of Functor

A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

- an object $F(X) \in Ob(\mathcal{D})$ for each object $X \in Ob(\mathcal{C})$

- a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each morphism $f : X \to Y$ in $\mathcal{C}$

such that:

- $\forall X \in Ob(\mathcal{C}).\, F(id_X) = id_{F(X)}$

- $\forall X, Y, Z \in Ob(\mathcal{C}).\forall f : X \to Y \in \mathcal{C}, g : Y \to Z$ in $\mathcal{C}.\, F(g \circ f) = F(g) \circ F(f)$

Motto:
Functors $\mathcal{I} \to \mathcal{C}$ are $\mathcal{I}$-shaped **diagrams** in $\mathcal{C}$

1.15 Definition of Contravariant Functor

A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

1.16 Forgetful Functors?

1.17 Powerset Functor[s?]?

1.18 Definition of Discrete Category

The **discrete category** associated with a set X, written $\mathcal{D}(X)$, is a category containing all the objects of X as objects, and no morphisms between different objects, just the identity morphisms.

1.19 Definition of Induced Functors

Claim:
Any map between sets can be turned into a functor.

Let $f : X \to Y$ be a map between sets.

Consider the discrete categories $\mathcal{D}(X), \mathcal{D}(Y)$.

Then f induces the following functor $\mathcal{D}(X) \to D(Y)$:
$x \mapsto f(x)$
$id_x \mapsto id_{f(x)}$

1.20 Definition of the Walking Arrow?

1.21 Definition of the Walking Commutative Triagle?

1.22 Definition of Essentially Surjective Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** iff:

$$\forall Y \in Ob(\mathcal{D}).\, \exists X \in Ob(\mathcal{C}) | F(X) \cong Y$$

1.23 Definition of Faithful Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** iff:

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ \forall f, g : X \to Y \text{ in } \mathcal{C} \\ F(f) = F(g) \implies f = g \end{array}$$

Reformulation: iff

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)) \\ f \mapsto F(f) \end{array}$$

is injective.

1.24 Definition of Full Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** iff:

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ \forall g : F(X) \to F(Y) \text{ in } \mathcal{D} \\ \exists f : X \to Y \text{ in } \mathcal{C} | F(f) = g \end{array}$$

Reformulation: iff

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)) \\ f \mapsto F(f) \end{array}$$

is surjective.

1.25 Definition of Fully Faithful Functor

A functor is **fully faithful** iff it is full and faithful.

Reformulation: iff

$\forall X, Y \in Ob(\mathcal{C})$.
$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$
$f \mapsto F(f)$

is bijective.

1.26 Definition of Elementary Equivalence

An **elementary equivalence** is a fully faithful, essentially surjective functor.

1.27 Definition of Equivalence of Categories

Categories are called **equivalent** iff there is an elementary equivalence between them.

Remark: Equivalent categories have exactly the same categorical properties.

1.28 Definition of Natural Transformation

A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : C \to D$ consists of:

- for each object $X \in Ob(\mathcal{C})$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$

such that for all morphisms $f : X \to Y$ in $\mathcal{C}$, the **naturality square** commutes:

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

Motto:
Natural transformations are **uniform** families of morphisms.

1.29 Definition of Functor Category

Let $\mathcal{C}, \mathcal{D}$ be categories.
The **functor category** $[\mathcal{C}, \mathcal{D}]$ has:

- as objects: all functors $\mathcal{C} \to \mathcal{D}$

- as morphisms: $Hom_{[\mathcal{C},\mathcal{D}]}(F, G) := \{h : F \Rightarrow G | h \text{ is a natural transformation}\}$

- as identity: for the object F, the identity $id_F : F \Rightarrow F$
  $(id_F)_X : F(X) \to F(X)$
  given by $id_{F(X)}$

- as composition rule:
  $(\omega \circ \eta)_X := \omega_X \circ \eta_X$

  $\omega_X : G(X) \to H(X)$
  $\eta_X : F(X) \to G(X)$

and $\omega \circ \eta$ should be natural.

1.30 Definition of Small Category

A category $\mathcal{C}$ is small when $Ob(\mathcal{C})$ is just a set and not a proper class.

1.31 Definition of Category of Categories

The **1-category of 1-categories**, **Cat** has:

- as objects: all categories

- as morphisms: $Hom_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) := \{F : \mathcal{C} \to \mathcal{D} | F \text{ is a functor}\}$

- as identities Id_F (the identity functor?)

- as composition rule:
  $F : \mathcal{C} \to \mathcal{D}$
  $G : \mathcal{D} \to \mathbf{E}$

  $G \circ F : \mathcal{C} \to \mathbf{E}$
  $X \mapsto G(F(X))$
  $f \mapsto G(F(f))$

There are two issues with this definition:

- Size issue (in ZFC). (it's too big, the objects don't fit in a proper class?)
  Remedies:

  - just consider the category of small categories

  - switch foundations

- It ignores natural transformations
  Remedy:
  Consider the 2-category of 1-categories

  The 2-category of 1-categories has:

  - as objects: all 1-categories

  - as morphisms: functors

  - as -2-morphisms / 2-cells: natural transformations

1.32 Definition of Cone

A **cone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : A \to F(X)$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$\pi_Y = \pi_X \circ F(f)$

commutes.

1.33 Definition of Cocone

A **cocone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cocone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : F(X) \to A$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$$\pi_X = \pi_Y \circ F(f)$$

commutes.

1.34 Definition of Morphism Between Cones

A **morphism** between a cone $(A, (\pi_X)_X)$ and a further cone $(B, (\phi_X)_X)$ of a diagram $F : \mathcal{I} \to \mathcal{C}$ consists of a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\pi_X = \pi_Y \circ f$$

1.35 Definition of Limit

A **limit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is a **terminal cone** of F, that is, a terminal object in the category of of cones of cones of F.

1.36 Definition of Colimit

A **colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an **initial cocone** of F.

1.37 Definition of Equalizer of Two Set-Theoretic Maps

Let $f, g : X \to Y$. Then the **equalizer** of f and g is the following function:

$$Eq(f, g) = x \in X | f(x) = g(x)$$

1.38 Definition of Pullback

1.39 Definition of Pushout

1.40 Definition of Small Diagram

A **small diagram** in $\mathcal{C}$ is a diagram $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category.

1.41 Definition of Complete Cateogory

A category $\mathcal{C}$ is **complete** iff every small diagram in $\mathcal{C}$ has a limit (it has all small limits).

1.42 Definition of Cocomplete Category

A category $\mathcal{C}$ is **cocomplete** iff every small diagram in $\mathcal{C}$ has a colimit (it has all small colimits).

$\mathcal{C}$ complete $\iff$ $\mathcal{C}^{op}$ cocomplete.

1.43 Formula for Limits in Set

1.44 Formula for Colimits in Set

1.45 Definition of Presheaf

A **presheaf** (plural presheaves) on a category $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathbf{Set}$

Motto:
we picture a presheaf F on $\mathcal{C}$ as an "ideal, fictional, object of $\mathcal{C}$" in that we know its relation to actual objects of $\mathcal{C}$

1.46 Definition of $\hat{X}$

$\hat{X}$ (**X hat**) is a presheaf:

$\mathcal{C}^{op} \to \mathbf{Set}$
$T \mapsto Hom_{\mathcal{C}}(T, X)$

1.47 Definition of Yoneda Lemma

1.48 Yoneda Embedding

1.49 Yoneda Style Proofs

1.50 Definition of Representable Presheaf

A presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable iff:

$\exists X \in Ob(\mathcal{C}) : F \cong \hat{X}$

1.51 Definition of Adjoint Functors

Let $F : C \to D$, $G : D \to C$

Then, $F \dashv G$ "F is **left adjoint** to G"
(or $G \vdash F$ ("G is **right adjoint** to F"))

iff for every object $X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$ there is an isomorphism:

$$Hom_{\mathcal{D}}(F(X), Y) \cong Hom_{\mathcal{C}}(X, G(Y))$$

naturally in X and Y.

1.52 Currying Adjunction

1.53 Adjunction of Logical Connectives

1.54 Monoids

A **monoid** consists of:
- a set M

- an element $e \in M$

- an operation $\circ : M \times M \to M$

such that:

- $\forall x \in M.\, x \circ e = x = e \circ x$

- $\forall x, y, z \in M.\, (x \circ y) \circ z = x \circ (y \circ z)$

1.55 Monoids Categorically

Equivalently, a **monoid** consists of:

- an object M

- a morphism 1 from a terminal object to every other object.

- a map $M \times M \to M$

such that

1.56 Definition of Monoidal Category

A **monoidal category** consists of:

- a category $\mathcal{C}$

- a functor $* : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

- an object $1 \in Ob(\mathcal{C})$

- natural isomorphisms:

  - $1 * X \cong X$

  - $X * 1 \cong X$

  - $X * (Y * Z) \cong (X * Y) * Z$

such that certain coherence conditions are satisfied.

Remark:
In any monoidal category one can speak of **monoid objects**.

Monoidal categories are sometimes called tensor categories.

1.57 Definition of Monad

A **monad** over a category $\mathcal{C}$ consists of:

- a functor $M : \mathcal{C} \to \mathcal{C}$

- a natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow M$

- a natural transformation $\mu : M \circ M \Rightarrow M$

such that

1.58 Definition of Kleisli Category

The **Kleisli category** $\mathcal{C}_M$ of a monad M in a category $\mathcal{C}$ is the following category:

- objects: objects of $\mathcal{C}$

- morphisms: $Hom_{\mathcal{C}_M}(X, Y) \coloneqq Hom_{\mathcal{C}}(X, M(Y))$

1.59 Definition of Cobordism Category

The category $nCob$ ("the **cobordism category**") has:

- as objects $(n-1)$-dimensional oriented manifolds

- as morphisms: n-dimesional cobordisms between those

1.60 Definition of Category of Hilbert Spaces

$Hilb$ is the **category of Hilbert spaces** (vector spaces with additional structure).

1.61 Definition of Topological Quantum Field Theory

A **topological quantum field theory** (in spacetime dimension n) is a monoidal functor between the monoidal categories $nCob$ and $Hilb$:

$Z : nCob \to Hilb$