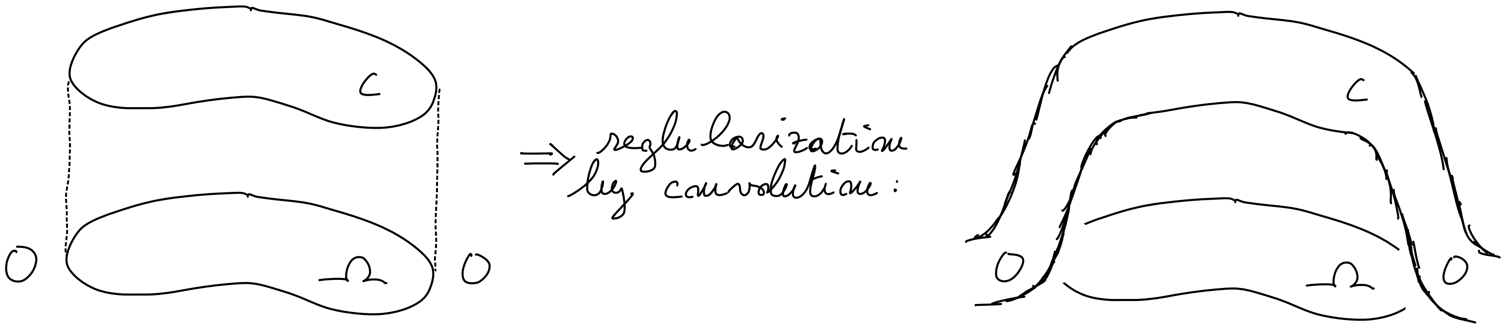
EXTENSION OF $W^{1,p}(\Omega)$

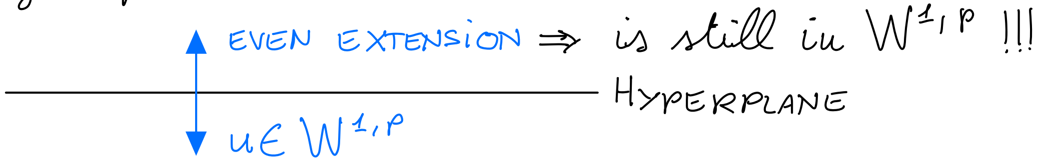
If $\Omega \subseteq \mathbb{R}^n$ open, bounded, constants are in $W^{1,p}(\Omega)$ BUT they don't satisfy the Sobolev / Morrey's embedding Thm.
Let $\Omega \subseteq_{op} \mathbb{R}^n$ sufficiently "smooth" (i.e. differentiable manifold) and let Ω' s.t. $\Omega \subset\subset \Omega'$. Then we can extend $u \in W^{1,p}(\Omega)$ to $\tilde{u} \in W^{1,p}(\Omega')$.

<u>Thm.</u>:
Let $\Omega \subset \mathbb{R}^n$ be regular, bounded and open (i.e. each point of $\partial\Omega$ can be described as the graph of a $\mathcal{C}^\infty$ function). Let also Ω' open s.t. $\Omega \subset\subset \Omega'$. Given $1 \leq p \leq +\infty$, $\forall u \in W^{1,p}(\Omega)$ $\exists \tilde{u} \in W_0^{1,p}(\Omega')$ extension of u and $\exists C > 0$ (independent on u, dependent on p, Ω, Ω') s.t. $\|\tilde{u}\|_{W^{1,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$

N.B.



Extension by reflection:



$\Rightarrow u \in W^{1,p}(\Omega) \rightsquigarrow \tilde{u} \in W_0^{1,p}(\Omega')$. We have:

$1 \leq p < n$:

$$\|u\|_{L^{p^*}(\Omega)} \leq \|\tilde{u}\|_{L^{p^*}(\Omega')} \leq C \|\nabla \tilde{u}\|_{L^p(\Omega')} \leq C \|u\|_{W^{1,p}(\Omega)}$$

$n < p \leq +\infty$:

$$[u]_\alpha \leq C \|u\|_{W^{1,p}(\Omega)}$$

N.B.
$p = n$ is called the CRITICAL CASE FOR SOBOLEV FUNCTIONS and it is the worst case possible. In general, functions in $W^{1,n}$ are not continuous nor bounded !!!

COMPACTNESS in dim. n

Thm. (Rellich):
Let $\Omega \subseteq_{op} \mathbb{R}^n$ bounded and regular, $1 \leq p < n$. Then bounded sets in $W^{1,p}(\Omega)$ are precompact in $L^s(\Omega)$, $1 \leq s < p^*$. In particular $\exists$ subsequence converging in norm in $L^1, L^2, \dots, L^s$

Thm. (Egorov):
Let μ be a finite measure on X, $\{u_k\}_{k \in \mathbb{N}}$, $u_k : X \to \mathbb{R}$ a sequence of μ-meas. functions s.t. $\exists u : X \to \mathbb{R}$ with $u_k(x) \to u(x)$ for μ-a.e. $x \in X$. Then $\forall \varepsilon > 0 \ \exists C \subset X$, C μ-meas., s.t. $\mu(C) < \varepsilon$ and $u_k \to u$ uniformly on $X \setminus C$.

Proof:
Fix $s = \in \mathbb{N}^{>0}$ and define $E_{n,s} := \left\{ x \in X : |u_k(x) - u(x)| \geq \frac{1}{s} \text{ for some } k \geq n \right\}$
Then $E_{n,s}$ is decreasing in n and $\mu\left(\bigcap_{n=1}^{+\infty} E_{n,s}\right) = 0$
$\Rightarrow \exists n_s \in \mathbb{N}$ s.t. $\mu(E_{n_s,s}) < \frac{\varepsilon}{2^s}$ for $\varepsilon > 0$. Take $\bigcup_{s=1}^{+\infty} E_{n_s,s}$, then $\mu(C) < \varepsilon$ and $u_k \to u$ uniformly on $X \setminus C$

$\square$

Proposition (Compactness criterion in $L^p(\Omega)$):
Given $\Omega \subseteq_{op} \mathbb{R}^n$ bounded, $1 \leq p < +\infty$, $\mathcal{F} \subset L^p(\Omega)$ family of functions bounded in norm, let ϕ_k be the usual sequence of bump functions. Then $\forall u \in \mathcal{F}$ $u_k := u * \phi_k \xrightarrow{L^p(\Omega)} u$ and, if this convergence is uniform for $u \in \mathcal{F}$, then $\mathcal{F}$ is (strongly) precompact in $L^p(\Omega)$

N.B. uniform convergence:
$$\forall \varepsilon > 0 \ \exists \bar{k} \in \mathbb{N} \text{ s.t. } \forall u \in \mathcal{F} \ \forall k \geq \bar{k} \ \|u_k - u\|_{L^p(\Omega)} \leq \varepsilon$$

Proof:
Fix $\varepsilon > 0$, we know that $\|u_{\bar{k}} - u\|_{L^p(\Omega)} < \varepsilon \ \forall u \in \mathcal{F}$. Define:
$$\mathcal{F}_{\bar{k}} := \left\{ u * \phi_{\bar{k}} : u \in \mathcal{F} \right\} \subset \mathcal{C}^\infty(\Omega)$$
If we prove that $\mathcal{F}_{\bar{k}}$ is totally bounded in $L^p(\Omega)$, then $\mathcal{F}$ is also totally bounded (and so precompact). So, it is enough to show that $\mathcal{F}_{\bar{k}}$ is precompact in $\mathcal{C}^0(\overline{\Omega})$. By Ascoli-Arzelà, it is enough to show that $\mathcal{F}_{\bar{k}}$ is equibounded in L^∞ and equi-

-Lipschitz. We have:

$$u_K(x) = \int_\Omega \phi_{\bar K}(x-y)\, u(y)\, dy \;\wedge\; \nabla u_{\bar K}(x) = \int_\Omega \nabla\phi_{\bar K}(x-y)\, u(y)\, dy$$

$\Rightarrow |u_{\bar K}(x)| \leq M \cdot \|u\|_{L^1(\Omega)} \leq M \cdot |\Omega|^{1-\frac{1}{p}} \cdot \|u\|_{L^p(\Omega)} \leq C$ independent on everything !!!

and the same holds for $\nabla u_{\bar K}$:

$\Rightarrow |\nabla u_{\bar K}(x)| \leq L \cdot \|u\|_{L^1(\Omega)} \leq C$ independent on everything !!!

$\square$

<u>Proof</u> (Rellich's Thm.):

Ω bounded, regular, $1 \leq p < n$, $F \subset W^{1,p}(\Omega)$ bounded in norm. We first prove the Thm. for $s=1$. For $u \in F$ let $u_K := u * \phi_K$, we show that $u_K \xrightarrow[L^1(\Omega)]{} u$ uniformly for $u \in F$. We have:

$$\|u_K - u\|_{L^1(\Omega)} = \int_\Omega \left| \int_{B_{\frac{1}{K}}(0)} (u(x-y) - u(x))\phi_K(y)\, dy \right| dx$$

$$\leq \int_{B_{\frac{1}{K}}(0)} \phi_K(y) \int_\Omega |u(x-y) - u(x)|\, dx\, dy$$

$\Rightarrow \int_\Omega |u(x-y)-u(x)|dx = \int_U |u(x-y)-u(x)|dx + \int_{\Omega\setminus U} |u(x-y)-u(x)|dx$

with U "large", open, $U \subset\subset \Omega$

$\Rightarrow \int_{\Omega\setminus U} |u(x-y)-u(x)|dx \leq |\Omega\setminus U|^{1-\frac{1}{p^*}} \cdot 2\|u\|_{L^{p^*}(\Omega)}$ (Hölder)

$\leq C \cdot \|u\|_{W^{1,p}(\Omega)} \cdot |\Omega\setminus U|^{1-\frac{1}{p^*}} \leq C \cdot |\Omega\setminus U|^{1-\frac{1}{p^*}} < \varepsilon$ for U "large"

$\Rightarrow \int_U |u(x-y) - u(x)|\, dx = \int_U \left| \int_0^1 \frac{d}{dt}(u(x-ty))dt \right| dx = \int_U \left| \int_0^1 \nabla u(x-ty)\cdot y\, dt \right| dx$

$\leq \frac{1}{K} \int_0^1 dt \int_U |\nabla u(x-ty)|\, dx \leq \frac{1}{K} |\Omega|^{1-\frac{1}{p}} \cdot \|\nabla u\|_{L^p(\Omega)} \leq \frac{C}{K} < \varepsilon$ for K "large"

$\Rightarrow \int_\Omega |u(x-y)-u(x)|dx < 2\varepsilon$ for K large enough

$\Rightarrow \|u_K - u\|_{L^1(\Omega)} < 2\varepsilon \Rightarrow$ we have precompactness in L^1. Now, we prove it for $s \in [1, p^*)$: any sequence $\{u_s\} \subset F$ has a subseq. s.t. $u_s \xrightarrow[L^1(\Omega)]{} u$ (and a.e.). We claim that $\{u_s\}$ is Cauchy in L^s

$\Rightarrow \{u_s\}$ is bounded in $L^{p^*}(\Omega) \Rightarrow \forall A$ measurable, ????

$$\|u_s\|_{L^s(A)} \leq |A|^{\frac{1}{s}-\frac{1}{p^*}} \cdot \underbrace{\|u_s\|_{L^{p^*}(\Omega)}}_{\leq C} < \varepsilon \text{ for } |A| < \delta$$

$\Rightarrow$ By Egorov, $\exists C$ measurable, $|C| < \delta$ s.t.:

$$u_s \longrightarrow u \text{ uniformly in } \Omega \setminus C$$

$\Rightarrow u_s \xrightarrow[L^s(\Omega\setminus C)]{} u \Rightarrow \exists \nu \in \mathbb{N}$ s.t. $\forall s, k \geq \nu \quad \|u_s - u_k\|_{L^s(\Omega \setminus C)} < \varepsilon$

$\Rightarrow \|u_s - u_k\|_{L^s(\Omega)} \leq \|u_s - u_k\|_{L^s(\Omega\setminus C)} + \|u_s\|_{L^s(C)} + \|u_k\|_{L^s(C)} < 3\varepsilon$

$\square$

<u>Corollary</u> (Weak Compactness in $W^{1,p}(\Omega)$):

Let Ω be bounded, open in $\mathbb{R}^n$, $1 < p < +\infty$, $\{u_k\} \subset W^{1,p}(\Omega)$ bounded in norm. Then $\exists$ subsequence, $u \in W^{1,p}(\Omega)$, s.t.:

1) If $1 < p < n \Rightarrow \nabla u_k \xrightharpoonup[L^p(\Omega)]{} \nabla u \wedge u_k \xrightarrow[L^s(\Omega)]{} u$, $1 \leq s < p^*$
2) If $n < p \Rightarrow \nabla u_k \xrightharpoonup[L^p(\Omega)]{} \nabla u \wedge u_k \longrightarrow u$ uniformly