Minkowski Functional

Def. (Minkowski Functional of a Convex set):
$(X, \|\cdot\|)$ normed vector space, $C \subseteq X$ convex, open s.t. $0 \in C$.
(N.B. C is convex $\Leftrightarrow$ $tx + (1-t)y \in C$ $\forall x,y \in C$ $\forall t \in [0,1]$), then the Minkowski Functional of C is defined as follows:

$$p_C(x) = \inf\{t>0 : \tfrac{x}{t} \in C\}$$

Proposition:
Given C, p_C as above, we have:
1) $p_C : X \to [0,+\infty)$ is well defined
2) $\exists K>0$ s.t. $p_C(x) \leqslant K\|x\|$ $\forall x \in X$
3) $C = \{x \in X : p_C(x) < 1\}$
4) p_C is a quasi-norm

Proof:
Positive homogeneity is obvious.
$\exists r>0$ s.t. $B_r(0) \subset C$. Let $x \in X$, $x \neq 0$, then $\frac{x}{\|x\|}\cdot\frac{r}{2} \in B_r(0)$
$\Rightarrow \frac{x}{\|x\|}\frac{r}{2} \in C \Rightarrow p_C(x) \leqslant \frac{2}{r}\|x\|$ so we take $K = \frac{2}{r}$ and this also defines p_C well. Let $x \in C$, then $\exists \delta>0$ s.t. $(1+\delta)x \in C$ (C is open in X) $\Rightarrow p_C(x) \leqslant \frac{1}{1+\delta} < 1$. Consider now x s.t. $p_C(x) < 1$, then $\exists t \in (0,1)$ s.t. $\frac{x}{t} \in C \Rightarrow x = t\underbrace{\left(\frac{x}{t}\right)}_{\in C} + (1-t)\cdot\underbrace{0}_{\in C} \in C$
We are now left to prove the triangle ineq:
Let $x,y \in X$, $\varepsilon>0$, we have:

$$\frac{x}{p_C(x)+\varepsilon} \in C, \quad \frac{y}{p_C(y)+\varepsilon} \in C \Rightarrow t = \frac{p_C(x)+\varepsilon}{p_C(x)+p_C(y)+2\varepsilon} \in (0,1)$$

$$\Rightarrow 1-t = \frac{p_C(y)+\varepsilon}{p_C(x)+p_C(y)+2\varepsilon} \Rightarrow C \ni t\frac{x}{p_C(x)+\varepsilon} + (1-t)\frac{y}{p_C(y)+\varepsilon} = \frac{x+y}{p_C(x)+p_C(y)+2\varepsilon}$$

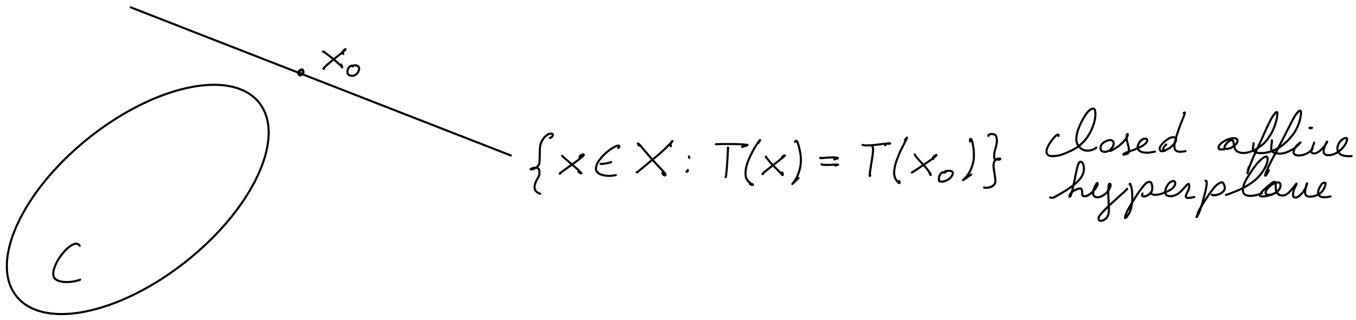
$\Rightarrow p_C(x+y) \leqslant p_C(x) + p_C(y) + 2\varepsilon$ (by def. of $p_C(x+y)$)

$\square$

Separation Theorems

Lemma:
Let $(X, \|\cdot\|)$ be a normed space, C convex, open, $\neq \emptyset$, $x_0 \in X \setminus C$, then $\exists T \in X'$ s.t. $T(x) < T(x_0)$ $\forall x \in C$

Geometrical meaning:



Proof:

WLOG $0 \in C$, consider p_C the Minkowski Functional of C.
$Y = \{ t x_0 : t \in \mathbb{R} \}$ is a 1-dim. subspace of X. Define
$T : Y \to \mathbb{R}$ s.t. $T(tx_0) = t\, p_C(x_0)$, then $p_C(x_0) \geq 1$ because $x_0 \notin C$.
Moreover $T(tx_0) \leq p_C(tx_0)$ $\forall t \in \mathbb{R}$. Extend T to $T : X \to \mathbb{R}$ s.t.
$T(x) \leq p_C(x)$ $\forall x \in X$ by Hahn Banach Thm. Then:

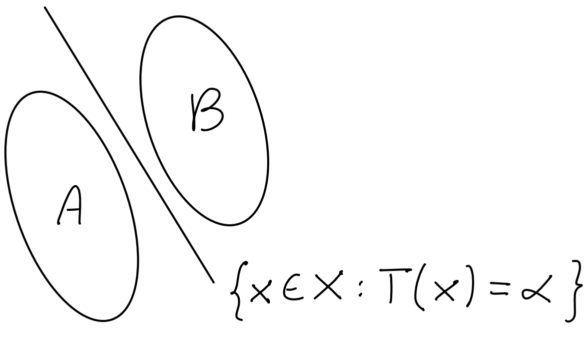
$$T(x) \leq p_C(x) \leq K\|x\| \quad \forall x \in X \Rightarrow T \in X'$$

Moreover, if $x \in C$ then $T(x) \leq p_C(x) < 1$

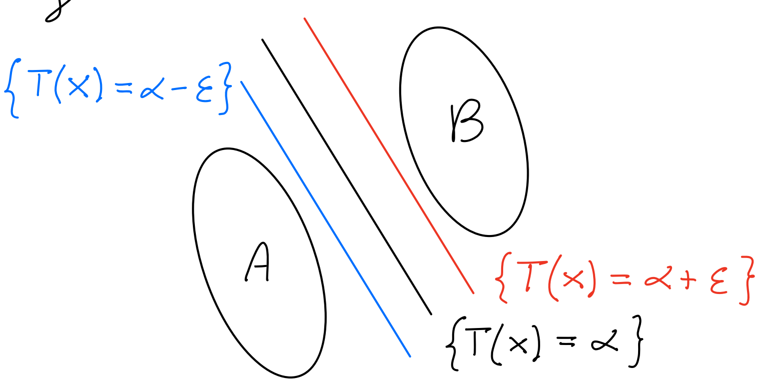
$\square$

Thm. (Geometric version of Hahn-Banach Thm.):
$(X, \|\cdot\|)$ normed space, A, B convex, $\neq \emptyset$, disjoint. Then:

1) A is open $\Rightarrow \exists T \in X'$, $T \neq 0$, $\alpha \in \mathbb{R}$ s.t. $T(x) \leq \alpha \leq T(y)$ $\forall x \in A$ $\forall y \in B$
$\Rightarrow$ geometrically:



2) A, B are closed, A compact
$\Rightarrow \exists T \in X'$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$ s.t. $T(x) \leq \alpha - \varepsilon$ $\forall x \in A$, $T(y) \geq \alpha + \varepsilon$ $\forall y \in B$
$\Rightarrow$ geometrically:



Proof:

1) Define $C = A - B = \{x-y : x \in A, y \in B\}$, C is convex, open and $0 \notin C$, indeed:

1) $x_1 - y_1, x_2 - y_2 \in C$
$\Rightarrow t(x_1-y_1) + (1-t)(x_2-y_2) = (tx_1 + (1-t)x_2) - (ty_1 + (1-t)y_2) \in C$

2) $C = \bigcup_{y \in B} (A - \{y\})$, A is open, $A - \{y\}$ is open (it is a translation of A), so C is union of open sets.

3) A, B are disjoint.

$\Rightarrow$ we can apply the lemma:

$\exists T \in X'$ s.t. $T(z) < T(0) = 0 \quad \forall z \in C$

$\Rightarrow z = x - y,\ x \in A, y \in B \Rightarrow T(x) - T(y) < 0 \quad \forall x \in A \ \forall y \in B$

$\Rightarrow T(x) < T(y) \quad \forall x \in A \ \forall y \in B$

$\Rightarrow$ we take $\alpha = \sup\{T(x) : x \in A\}$ or $\alpha = \inf\{T(y) : y \in B\}$

2) For $\varepsilon > 0$ define $A_\varepsilon = A + B_\varepsilon(0) = \{x + v : x \in A, \|v\| < \varepsilon\}$, $B_\varepsilon = B + B_\varepsilon(0) = \{y + w : y \in A, \|w\| < \varepsilon\}$. Then $A_\varepsilon, B_\varepsilon$ are convex, non empty and, if ε is small enough, they are disjoint: otherwise $\forall n \in \mathbb{N}$ $A_{\frac{1}{n}} \cap B_{\frac{1}{n}} \neq \emptyset \Rightarrow \exists z_n \in A_{\frac{1}{n}} \cap B_{\frac{1}{n}}$

$\Rightarrow z_n = x_n + v_n = y_n + w_n,\ x_n \in A, y_n \in B, \|v_n\|, \|w_n\| < \frac{1}{n}$

$\Rightarrow$ since A is compact, up to subsequences $x_n \to \bar{x} \in A$ but $y_n = x_n + v_n - w_n \to \bar{x} \in B$ $\lightning$ (A, B are disjoint).

$A_\varepsilon, B_\varepsilon$ are also open (e.g. $A_\varepsilon = \bigcup_{x \in A} B_\varepsilon(x)$), so we can apply (1):

$\Rightarrow \exists T \in X', T \neq 0, \alpha \in \mathbb{R}$ s.t. $T(x+v) \leq \alpha \leq T(y+w) \quad \forall x \in A, \forall y \in B, \forall v, w \in B_\varepsilon(0)$.

$\Rightarrow \alpha \geq \sup\{T(x+v) : v \in B_\varepsilon(0)\} = T(x) + \varepsilon \|T\|_{X'}$

$\Rightarrow T(x) \leq \alpha - \varepsilon \|T\|_{X'}$

$\Rightarrow \alpha \leq \inf\{T(y+w) : w \in B_\varepsilon(0)\} = T(y) - \varepsilon \|T\|_{X'}$

$\Rightarrow T(y) \geq \alpha + \varepsilon \|T\|_{X'}$

$\square$