Category Theory Course Notes

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Chapter 1

1.1 Definition of Category

A **category** (1-category) $\mathcal{C}$ consists of:

1 - A class $Ob(\mathcal{C})$ of objects of $\mathcal{C}$

2 - $\forall X, Y \in Ob(\mathcal{C})$.
a class $Hom_{\mathcal{C}}(X, Y)$ of **morphisms** from X to Y

3 - $\forall X \in Ob(\mathcal{C})$.
an **identity morphism** $id_X \in Hom_{\mathcal{C}}(X, X)$

4 - $\forall X, Y, Z \in Ob(\mathcal{C})$.
a **composition rule**:

$$Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$$
$$(g, f) \mapsto g \circ f$$

Such that it satisfies the following axioms:

1 - **Associativity of composition**:

$\forall X, Y, Z, W \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y), g \in Hom_{\mathcal{C}}(Y, Z), h \in Hom_{\mathcal{C}}(Z, W)$.
$h \circ (g \circ f) = (h \circ g) \circ f$

2 - **Neutrality**:

$\forall X, Y \in Ob(\mathcal{C})$.
$\forall f \in Hom_{\mathcal{C}}(X, Y)$.
$id_Y \circ f = f \wedge f \circ id_X = f$

1.2 Thin Categories

A category is **thin** if parallel morphisms are always the same, meaning that there is only one morphism between two objects.

In a thin category all morphisms are monic and epic.

1.3 Definition of Initial Object

An object I of a category $\mathcal{C}$ is **initial** (dual of terminal, special case of a colimit (of a functor from $\mathcal{C}$ to the empty category))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(I, X)$

1.4 Definition of Terminal Object

An object T of a category $\mathcal{C}$ is **terminal** (dual of initial, special case of limit (of a functor from the empty category to $\mathcal{C}$))
$\Updownarrow$
$\forall X \in Ob(\mathcal{C})$.
$\exists! f \in Hom_{\mathcal{C}}(X, T)$

1.5 Definition of Monomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is a **monomorphism** (or monic in $\mathcal{C}$) (dual of epimorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Z, X)$.
$f \circ p = f \circ q \implies p = q$

Example:
In **Set** monomorphisms are precisely the injective maps.

Monomorphisms "can be cancelled" from the left.

1.6 Definition of Split Monomorphism

A **split monomorphism** (dual of split epi) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$g \circ f = id_X$

Proposition: every split mono is a mono.
Proposition: in **Set**, every mono $f : X \to Y$ where X is inhabited is a split mono, assuming LEM holds.

1.7 Definition of Epimorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **epimorphism** (or epic in $\mathcal{C}$) (dual of monomorphism)
$\Updownarrow$
$\forall Z \in Ob(\mathcal{C}). \forall p, q \in Hom_{\mathcal{C}}(Y, Z).$
$p \circ f = q \circ f \implies p = q$

Example:
In **Set** epimorphisms are precisely the surjective maps.

Epimorphisms "can be cancelled" from the right.

1.8 Definition of Split Epimorphism

A **split epimorphism** (dual of split mono) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$f \circ g = id_Y$

Proposition: every split epi is an epi.
Proposition: in **Set**, every epi is a split epi $\iff$ assuming LEM holds.

1.9 Definition of Isomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in Hom_{\mathcal{C}}(X, Y)$) is an **isomorphism**
$\Updownarrow$
$\exists g \in Hom_{\mathcal{C}}(Y, X).$
$f \circ g = id_Y \land g \circ f = id_X$

$id_X \forall X \in Ob(\mathcal{C})$ is always an isomorphisms for every category $\mathcal{C}$.
Objects X and Y in a category $\mathcal{C}$ are **isomorphic**
$\Updownarrow$
there exists an isomorphism between X and Y ($X \cong Y$)

In **Set**, if there exists an isomorphism between X and Y, X and Y are called eqinumerous.

1.10 Definition of Opposite Category

“The mother of all dualities”

Let $\mathcal{C}$ be a category. Then its opposite category $\mathcal{C}^{op}$ is the following category:

- $Ob(\mathcal{C}^{op}) := Ob(\mathcal{C})$

- $Hom_{\mathcal{C}^{op}}(X, Y) := Hom_{\mathcal{C}}(Y, X)$

- identities and composition inherited from $\mathcal{C}$
  $id_X \in Hom_{\mathcal{C}}(X, X) = id_X^{op} \in Hom_{\mathcal{C}^{op}}(X, X)$
  $f \circ g := g^{op} \circ f^{op}$

Observations / Remarks:

- An object I of $\mathcal{C}$ is initial in $\mathcal{C}$
  $\Updownarrow$
  I is terminal when regarded as an object of $\mathcal{C}^{op}$

- A morphism in $\mathcal{C}$ is a monomorphism
  $\Updownarrow$
  it is an epimorphism in $\mathcal{C}^{op}$

1.11 Dualities?

1.12 Definition of Product

A **product** (special case of limit) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object P of $\mathcal{C}$

- a morphism $\pi_X : P \to X$ in $\mathcal{C}$

- a morphism $\pi_Y : P \to Y$ in $\mathcal{C}$

such that for every object Q of $\mathcal{C}$ together with morphisms $\varphi_X : Q \to X, \varphi_Y : Q \to Y$ there is exactly one morphism $Q \to P$ such that the following diagram commutes:

$\varphi_X = \pi_X \circ !$
$\varphi_Y = \pi_Y \circ !$

Remarks:
- Products are always associative and commutative up to isomorphism.
- There is also the notion of the (co) product of zero, one, three, four, ... objects.
- The zero case of a product is just a terminal object, an object with exactly one morphism from each object.

1.13 Definition of Coproducts

A **coproduct** (special case of colimits) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object C of $\mathcal{C}$

- a morphism $\iota_X : X \to C$ in $\mathcal{C}$

- a morphism $\iota_Y : Y \to C$ in $\mathcal{C}$

such that for every object D of $\mathcal{C}$ together with morphisms $\chi_X : X \to D, \chi_Y : Y \to D$ there is exactly one morphism $C \to D$ which renders the following diagram commutative:

$\chi_X = ! \circ \iota_X$
$\chi_Y = ! \circ \iota_Y$

Remarks:

- Products in $\mathcal{C}^{op}$ are precisely coproducts in $\mathcal{C}$

- The zero case of a coproduct is the same as an initial object.

1.14 Definition of Functor

A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

- an object $F(X) \in Ob(\mathcal{D})$ for each object $X \in Ob(\mathcal{C})$

- a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each morphism $f : X \to Y$ in $\mathcal{C}$

such that:

- $\forall X \in Ob(\mathcal{C}).\, F(id_X) = id_{F(X)}$

- $\forall X, Y, Z \in Ob(\mathcal{C}). \forall f : X \to Y \in \mathcal{C}, g : Y \to Z$ in $\mathcal{C}.\, F(g \circ f) = F(g) \circ F(f)$

Motto:
Functors $\mathcal{I} \to \mathcal{C}$ are $\mathcal{I}$-shaped **diagrams** in $\mathcal{C}$

1.15 Definition of Contravariant Functor

A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

1.16 Forgetful Functors?

1.17 Powerset Functor[s?]?

1.18 Definition of Discrete Category

The **discrete category** associated with a set X, written $\mathcal{D}(X)$, is a category containing all the objects of X as objects, and no morphisms between different objects, just the identity morphisms.

1.19 Definition of Induced Functors

Claim:
Any map between sets can be turned into a functor.

Let $f : X \to Y$ be a map between sets.

Consider the discrete categories $\mathcal{D}(X), \mathcal{D}(Y)$.

Then f induces the following functor $\mathcal{D}(X) \to D(Y)$:
$x \mapsto f(x)$
$id_x \mapsto id_{f(x)}$

1.20 Definition of the Walking Arrow?

1.21 Definition of the Walking Commutative Triagle?

1.22 Definition of Essentially Surjective Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** iff:

$$\forall Y \in Ob(\mathcal{D}).\, \exists X \in Ob(\mathcal{C}) | F(X) \cong Y$$

1.23 Definition of Faithful Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** iff:

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ \forall f, g : X \to Y \text{ in } \mathcal{C} \\ F(f) = F(g) \implies f = g \end{array}$$

Reformulation: iff

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)) \\ f \mapsto F(f) \end{array}$$

is injective.

1.24 Definition of Full Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** iff:

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ \forall g : F(X) \to F(Y) \text{ in } \mathcal{D} \\ \exists f : X \to Y \text{ in } \mathcal{C} | F(f) = g \end{array}$$

Reformulation: iff

$$\begin{array}{l} \forall X, Y \in Ob(\mathcal{C}). \\ Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)) \\ f \mapsto F(f) \end{array}$$

is surjective.

1.25 Definition of Fully Faithful Functor

A functor is **fully faithful** iff it is full and faithful.

Reformulation: iff

$$\forall X, Y \in Ob(\mathcal{C}).$$
$$Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$f \mapsto F(f)$$

is bijective.

1.26 Definition of Elementary Equivalence

An **elementary equivalence** is a fully faithful, essentially surjective functor.

1.27 Definition of Equivalence of Categories

Categories are called **equivalent** iff there is an elementary equivalence between them.

Remark: Equivalent categories have exactly the same categorical properties.

1.28 Definition of Natural Transformation

A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : C \to D$ consists of:

- for each object $X \in Ob(\mathcal{C})$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$

such that for all morphisms $f : X \to Y$ in $\mathcal{C}$, the **naturality square** commutes:

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

Motto:
Natural transformations are **uniform** families of morphisms.

1.29 Definition of Functor Category

Let $\mathcal{C}, \mathcal{D}$ be categories.
The **functor category** $[\mathcal{C}, \mathcal{D}]$ has:

- as objects: all functors $\mathcal{C} \to \mathcal{D}$

- as morphisms: $Hom_{[\mathcal{C},\mathcal{D}]}(F, G) \coloneqq \{h : F \Rightarrow G | h \text{ is a natural transformation}\}$

- as identity: for the object F, the identity $id_F : F \Rightarrow F$
  $(id_F)_X : F(X) \to F(X)$
  given by $id_{F(X)}$

- as composition rule:
  $(\omega \circ \eta)_X \coloneqq \omega_X \circ \eta_X$

  $\omega_X : G(X) \to H(X)$
  $\eta_X : F(X) \to G(X)$

and $\omega \circ \eta$ should be natural.

1.30 Definition of Small Category

A category $\mathcal{C}$ is small when $Ob(\mathcal{C})$ is just a set and not a proper class.

1.31 Definition of Category of Categories

The **1-category of 1-categories**, **Cat** has:

- as objects: all categories

- as morphisms: $Hom_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \coloneqq \{F : \mathcal{C} \to \mathcal{D} | F \text{ is a functor}\}$

- as identities Id_F (the identity functor?)

- as composition rule:
  $F : \mathcal{C} \to \mathcal{D}$
  $G : \mathcal{D} \to \mathbf{E}$

  $G \circ F : \mathcal{C} \to \mathbf{E}$
  $X \mapsto G(F(X))$
  $f \mapsto G(F(f))$

There are two issues with this definition:

- Size issue (in ZFC). (it's too big, the objects don't fit in a proper class?)
  Remedies:

  - just consider the category of small categories

  - switch foundations

- It ignores natural transformations
  Remedy:
  Consider the 2-category of 1-categories

  The 2-category of 1-categories has:

  - as objects: all 1-categories

  - as morphisms: functors

  - as -2-morphisms / 2-cells: natural transformations

1.32 Definition of Cone

A **cone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : A \to F(X)$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$\pi_Y = \pi_X \circ F(f)$

commutes.

1.33 Definition of Cocone

A **cocone** of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cocone)

- for each object $X \in Ob(\mathcal{C})$, a morphism $\pi_X : F(X) \to A$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

$$\pi_X = \pi_Y \circ F(f)$$

commutes.

1.34 Definition of Morphism Between Cones

A **morphism** between a cone $(A, (\pi_X)_X)$ and a further cone $(B, (\phi_X)_X)$ of a diagram $F : \mathcal{I} \to \mathcal{C}$ consists of a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\pi_X = \pi_Y \circ f$$

1.35 Definition of Limit

A **limit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is a **terminal cone** of F, that is, a terminal object in the category of of cones of cones of F.

1.36 Definition of Colimit

A **colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an **initial cocone** of F.

1.37 Definition of Equalizer of Two Set-Theoretic Maps

Let $f, g : X \to Y$. Then the **equalizer** of f and g is the following function:

$$Eq(f, g) = x \in X | f(x) = g(x)$$

1.38 Definition of Pullback

1.39 Definition of Pushout

1.40 Definition of Small Diagram

A **small diagram** in $\mathcal{C}$ is a diagram $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category.

1.41 Definition of Complete Cateogory

A category $\mathcal{C}$ is **complete** iff every small diagram in $\mathcal{C}$ has a limit (it has all small limits).

1.42 Definition of Cocomplete Category

A category $\mathcal{C}$ is **cocomplete** iff every small diagram in $\mathcal{C}$ has a colimit (it has all small colimits).

$\mathcal{C}$ complete $\iff$ $\mathcal{C}^{op}$ cocomplete.

1.43 Formula for Limits in Set

1.44 Formula for Colimits in Set

1.45 Definition of Presheaf

A **presheaf** (plural presheaves) on a category $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathbf{Set}$

Motto:
we picture a presheaf F on $\mathcal{C}$ as an "ideal, fictional, object of $\mathcal{C}$" in that we know its relation to actual objects of $\mathcal{C}$

1.46 Definition of $\hat{X}$

$\hat{X}$ (**X hat**) is a presheaf:

$\mathcal{C}^{op} \to \mathbf{Set}$
$T \mapsto Hom_{\mathcal{C}}(T, X)$

1.47 Definition of Yoneda Lemma

1.48 Yoneda Embedding

1.49 Yoneda Style Proofs

1.50 Definition of Representable Presheaf

A presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable iff:

$\exists X \in Ob(\mathcal{C}) : F \cong \hat{X}$

1.51 Definition of Adjoint Functors

1.52 Currying Adjunction

1.53 Adjunction of Logical Connectives

1.54 Monoids

1.55 Monoids Categorically

1.56 Definition of Monoidal Category

1.57 Monoidal Categories

1.58 Definition of Monad

1.59 Definition of Kleisli Category

1.60 Definition of Topological Quantum Field Theory

1.61 Definition of Cobordism Category