Banach Spaces - Dual Spaces

We know that $C^0([a,b])$, coupled with the ∞-norm $\|f\|_\infty := \max\{|f(x)| : x \in [a,b]\}$, is a normed space.

<u>Def.</u> (Norm on a vector space):
Let X be a vector space over $\mathbb{R}$, a NORM ON X is a function $\|\cdot\| : X \to [0, +\infty)$ s.t.:

$$1)\ \|x\| = 0 \Leftrightarrow x = 0$$
$$2)\ \|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{R},\ \forall x \in X$$
$$3)\ \|x+y\| \leqslant \|x\| + \|y\| \quad \forall x,y \in X$$

$\Rightarrow$ Induced metric:

$$d(x,y) := \|x-y\|, \quad x,y \in X$$

$\Rightarrow$ since we have a metric, we also have a topology.

<u>Remark</u>:
$(X, \|\cdot\|)$ is a topological vector space $\Rightarrow$ its vector space operations are continuous.
Let $x,y \in X$, to check continuity of $+$ at the pair (x,y) it is enough to show that:

$$\forall x_n \to x, \quad (\Leftrightarrow \|x_n - x\| \to 0)$$
$$\forall y_n \to y \quad (\Leftrightarrow \|y_n - y\| \to 0)$$
$$\Rightarrow (x_n, y_n) \to (x+y)$$
$$(\|(x_n+y_n)-(x+y)\| \leqslant \|x_n - x\| + \|y_n + y\|)$$

<u>Def.</u> (Banach Space):
A vector space $(X, \|\cdot\|)$ is called a BANACH SPACE if it is complete with respect to its norm $\|\cdot\|$ (i.e. $\Leftrightarrow$ every Cauchy sequence converges)

<u>N.B.</u>
$\{x_n\}_{n\in\mathbb{N}} \subset X$ is Cauchy if $\forall \varepsilon > 0\ \exists \nu \in \mathbb{N}$ s.t.
$\forall m, n \geqslant \nu$ we have $\|x_n - x_m\| < \varepsilon$

<u>Examples:</u>
1) $(\mathbb{R}^n, |\cdot|)$ is a Banach space, $|(x_1, \dots, x_n)| = \sqrt{x_1^2 + \dots + x_n^2}$
2) $(\mathbb{R}^n, |\cdot|_1)$ is a Banach space, $|(x_1, \dots, x_n)|_1 = |x_1| + \dots + |x_n|$
   ($|\cdot|_1$ is called CITY BLOCK NORM)
3) $(\mathbb{R}^n, |\cdot|_\infty)$ is a Banach space, $|(x_1, \dots, x_n)|_\infty = \max_{i=1,\dots,n}\{|x_i|\}$

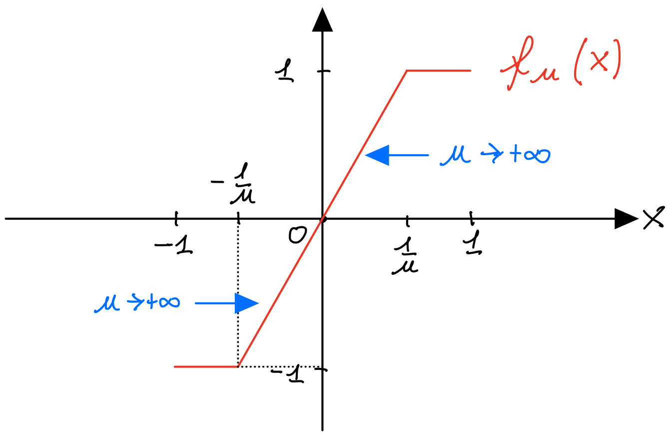
4) $(\mathbb{R}^n, |\cdot|_p)$, $p \in [1, +\infty)$ is a Banach space,
$|(x_1, \dots, x_p)| = \sqrt[p]{|x_1|^p + \dots + |x_n|^p}$

5) $(C^0([a,b]), \|\cdot\|_\infty)$ is a Banach space,
$\|f\|_\infty := \max\{|f(x)| : x \in [a,b]\}$

6) We define the L^1-norm of a function:

$$\|f\|_{L^1} = \int_a^b |f(x)|\, dx$$

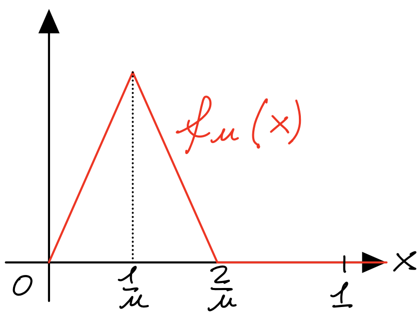
$\Rightarrow$ $(C^0([a,b]), \|\cdot\|_{L^1})$ is NOT a Banach space, $\|\cdot\|_{L^1}$ is not complete: take $[a,b] = [-1,1]$ and consider the following sequence, despite being Cauchy, it doesn't converge.



$$\Rightarrow \quad f_n(x) = \begin{cases} -1 & x \in [-1, -\frac{1}{n}] \\ nx & x \in (-\frac{1}{n}, \frac{1}{n}) \\ 1 & x \in [\frac{1}{n}, 1] \end{cases}$$

$$\Rightarrow f_n(x) \longrightarrow f(x) = \begin{cases} -1 & x \in [-1, 0) \\ 0 & x = 0 \\ 1 & x \in (0, 1] \end{cases}$$

Consider as well the following sequence on the interval $[a, b] = [0, 1]$:



$$\Rightarrow \|f_n(x)\|_{L^1} \longrightarrow 0 \quad \text{BUT} \quad \|f_n(x)\|_\infty \not\longrightarrow 0$$

N.B.

$Id : (C^0, \|\cdot\|_{L^1}) \longrightarrow (C^0, \|\cdot\|_\infty)$ is discontinuous at 0, this phenomena is totally general in infinite dimensional spaces !!!

<u>Proposition</u>:

Let $(X, \|\cdot\|)$ be a normed space s.t. $\dim_{\mathbb{R}} X = +\infty$. Then there $\exists\ T : X \longrightarrow \mathbb{R}$ linear and discontinuous.

N.B.
$\dim_{\mathbb{R}} X = +\infty$ means there is no finite maximal set of linear independent vectors.

<u>Proof.</u>:
Let $\mathcal{B} = \{x_\alpha\}_{\alpha \in I}$ be an (algebraic) basis for X ($|I| = +\infty$).
Let $\mathcal{B}' = \{\tilde{x}_k\}_{k \in \mathbb{N}} \subset \mathcal{B}$ be a countable subset of $\mathcal{B}$. WLOG we can assume that $\|x_\alpha\| = 1$ $\forall \alpha \in I$ (we can normalize $\mathcal{B}$).
We define $T(\tilde{x}_k) = k$ and $T(x_\alpha) = 0$ if $x_\alpha \in \mathcal{B} \setminus \mathcal{B}'$. Then T is discontinuous at 0:
we construct the sequence $y_k = \frac{\tilde{x}_k}{\sqrt{k}} \Rightarrow \|y_k\| = \frac{1}{\sqrt{k}} \longrightarrow 0$.
By linearity, however, we have:
$$T(y_k) = \frac{k}{\sqrt{k}} = \sqrt{k} \longrightarrow +\infty \neq T(0)$$
□

N.B.
a real-valued linear map is called a LINEAR FUNCTIONAL. The space of linear functionals is a vector space and is called the DUAL SPACE of X. Usually, the dual space of X only contains the continuous linear functionals.

<u>Def.</u> (Topological Dual):
Let $(X, \|\cdot\|)$ be a vector space over $\mathbb{R}$. Its TOPOLOGICAL DUAL is $X' = \{T : X \to \mathbb{R} : T \text{ is linear and continuous}\}$

<u>Proposition</u> (Linear Continuous Functionals on a normed v.s.):
Let $T : X \to \mathbb{R}$ be linear. The following are equivalent
1) T is continuous
2) T is continuous at 0
(*) 3) $\exists\, C > 0$ s.t. $|T(x)| \leq C \cdot \|x\|$ $\forall x \in X$ (BOUNDEDNESS)
4) $\operatorname{Ker} T$ is closed in X

<u>Proof.</u>:
$1 \Rightarrow 2$: obvious.
$2 \Rightarrow 1$: let $x \in X$. We must show that $\forall x_n \to x$ it holds that $T(x_n) \to T(x)$. We have:
$$T(x_n) - T(x) = T(\underbrace{x_n - x}_{\to 0}) = T(0) = 0$$

$3 \Rightarrow 2$: obvious.

$2 \Rightarrow 3$: continuity at 0 with $\varepsilon = 1$:

$\exists \delta > 0$ s.t. $\forall \|x\| \leqslant \delta \quad |T(x)| \leqslant 1$

Let $x \in X,\ x \neq 0 \Rightarrow \left\| \frac{\delta}{2} \cdot \frac{x}{\|x\|} \right\| = \frac{\delta}{2} < \delta$

$\Rightarrow \left| T\left(\frac{\delta}{2} \cdot \frac{x}{\|x\|} \right) \right| < 1 \Rightarrow \frac{\delta}{2\|x\|} |T(x)| < 1$

$\Rightarrow |T(x)| < \frac{2}{\delta} \|x\| \Rightarrow C = \frac{2}{\delta}$

$y \in X,\ y \neq 0$: $|T(y)| = \left| T\left(y \cdot \frac{\delta}{\|y\|} \cdot \frac{\|y\|}{\delta} \right) \right|$
$= \frac{\|y\|}{\delta} \left| T\left(y \cdot \frac{\delta}{\|y\|} \right) \right| \leqslant \frac{\|y\|}{\delta}$

$\left| T\left(y \frac{\delta}{\|y\|} \right) \right| \leqslant 1 \quad \left(\left\| \frac{y}{\|y\|} \delta \right\| \leqslant \delta \right)$

$1 \Rightarrow 4$: $\operatorname{Ker} T = T^{-1}(\underbrace{\{0\}}_{\text{closed in } \mathbb{R}})$ is closed because T is continuous.

$4 \Rightarrow 2$: contradiction:

suppose $\operatorname{Ker} T$ is closed but T is discontinuous at 0

$\Rightarrow \exists x_n \to 0$ s.t. $T(x_n) \not\to 0 \Rightarrow$ (up to subsequences)

$|T(x_n)| \geqslant C > 0$. Take $y \in X$ and consider $y_n := y - x_n \cdot \frac{T(y)}{T(x_n)}$

$\Rightarrow y_n \in \operatorname{Ker} T \wedge y_n \to y \Rightarrow y \in \overline{\operatorname{Ker} T} = \operatorname{Ker} T \Rightarrow T \equiv 0$ ↯

□

Def. (Dual Norm):

Let $T: X \to \mathbb{R}$ be linear, its dual norm is the smallest constant in the boundedness inequality $(*)$:

$$\|T\|_{X'} := \sup\left\{ \frac{|T(x)|}{\|x\|} : x \in X,\ x \neq 0 \right\}$$
$$= \sup\left\{ \left| T\left(\frac{x}{\|x\|} \right) \right| : x \in X,\ x \neq 0 \right\}$$
$$= \sup\left\{ |T(x)| : x \in X,\ \|x\| \leqslant 1 \right\}$$

N.B.

$\|T\|_{X'} < +\infty \Leftrightarrow T$ is continuous $\left(\|T\|_{X'} < +\infty \Rightarrow |T(x)| \leqslant \|T\|_{X'} \cdot \|x\|\ \forall x \in X \right)$

Proposition:

$(X', \|\cdot\|_{X'})$ is a Banach space

Proof.:

The dual norm is a norm! Let $\{T_n\}_{n \in \mathbb{N}}$ be a Cauchy seq. in X'. Then $\forall \varepsilon > 0\ \exists \nu \in \mathbb{N}$ s.t. $\forall n, m \geqslant \nu \quad \|T_n - T_m\|_{X'} \leqslant \varepsilon$

$\Rightarrow |T_n(x) - T_m(x)| \leqslant \varepsilon \|x\| \quad \forall x \in X$ (boundedness inequality with $C = \varepsilon$)

$\Rightarrow \{T_n(x)\}_n \subset \mathbb{R}$ is a Cauchy seq. in $\mathbb{R}$ and therefore it converges in $\mathbb{R}$

$\Rightarrow T_n(x) \longrightarrow T(x) \Rightarrow$ let $n \longrightarrow +\infty$ in $|T_n(x) - T_m(x)|$

$\Rightarrow |T_n(x) - T_m(x)| \leqslant \varepsilon \|x\| \ \forall n \geqslant \nu \Rightarrow \|T_n - T\|_{X'} \leqslant \varepsilon$

$\Rightarrow T_n \xrightarrow{\|\cdot\|_{X'}} T$

$\square$

Let now $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $L: X \longrightarrow Y$ linear, then:

$$L \text{ is continuous} \Leftrightarrow \exists C > 0 \text{ s.t. } \|L(x)\|_Y \leqslant C \|x\|_X$$

(BOUNDED)

we can define $\mathcal{L}(X;Y) := \{L : X \longrightarrow Y : L \text{ is linear and continuous}\}$ and $\|L\|_{\mathcal{L}(X;Y)} := \sup\{\|L(x)\|_Y : \|x\|_X \leqslant 1, x \in X\}$.

$\Rightarrow (\mathcal{L}(X;Y), \|\cdot\|_{\mathcal{L}(X;Y)})$ is Banach provided that $(Y, \|\cdot\|_Y)$ is Banach.