Ascoli-Arzelà Theorem

Corollary:
Let $(X, \|\cdot\|)$ be a normed space with $\dim_{\mathbb{R}} X = +\infty$. Then, if $K \subset X$ is compact, $\forall \varepsilon > 0 \ \exists Y$ vector subspace with $\dim_{\mathbb{R}} Y < +\infty$ s.t. $\text{dist}(x, Y) < \varepsilon \ \forall x \in K$.

Proof:
By total boundedness $\exists x_1, \dots, x_n \in X$ s.t. $\bigcup_{k=1}^{N} B_\varepsilon(x_k) \supset K$. It is enough to choose $Y = \langle x_1, \dots, x_n \rangle$

$\square$

Let $(A, d), (B, d)$ be compact metric spaces, and let $\mathcal{C}^0(A; B) = \{ f : A \to B : f \text{ is continuous} \}$. Then we can define $d_\infty(f, g) := \sup\{ d(f(x), g(x)) : x \in A \}$. This is a metric and it is also complete.

Thm. (Ascoli-Arzelà):
Given $(A, d_A), (B, d_B)$ compact metric spaces, $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{C}^0(A; B)$, assume that $\{f_k\}_k$ is equicontinuous:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x_1, x_2 \in A \text{ with } d_A(x_1, x_2) < \delta \text{ we have}$$
$$d_B(f_k(x_1), f_k(x_2)) < \varepsilon \quad \forall k \in \mathbb{N}$$

Then, $\exists$ subsequence $\{f_{k_h}\}_h$, $f \in \mathcal{C}^0(A; B)$ s.t. $f_{k_h} \xrightarrow{d_\infty} f$

N.B.
if $B = \mathbb{R}$, we require $\{f_k\}_k$ to be not only equicontinuous but also equibounded !!!

Proof:
It is enough to prove that $\mathcal{A} = \{ f_k : k \in \mathbb{N} \}$ is totally bounded in $(\mathcal{C}(A; B), d_\infty)$. Let $\varepsilon > 0$, then we can cover B with a finite number of balls $B_1, \dots, B_N$ of radius ε. Let $\delta > 0$ be the corresponding constant in the equicontinuity hypothesis. Then $\exists B_\delta(x_1), \dots, B_\delta(x_M)$ s.t. $A = \bigcup_{i=1}^{M} B_\delta(x_i)$. Consider a multiindex $(s_1, \dots, s_M) \in \{1, \dots, N\}^M$ (each index s_i is a number between 1 and N, there is a finite number of these) and the family of functions $W_{(s_1, \dots, s_M)} = \{ f \in \mathcal{A} : f(x_i) \in B_{s_i} \}$. We have that $\bigcup W_{(s_1, \dots, s_M)} = \mathcal{A}$.

Let $f, g \in W_{(\delta_1,\dots,\delta_M)}$: we show that $d_\infty(f,g) < 5\varepsilon$. Let $x \in A$, $\exists\, \bar{i} \in \{1,\dots,M\}$ s.t. $x \in B_\delta(x_{\bar{i}})$. Then:

$$d_B(f(x), g(x)) \leq \underbrace{d_B(f(x), f(x_{\bar{i}}))}_{\wedge\ \varepsilon} + \underbrace{d_B(f(x_{\bar{i}}), g(x_{\bar{i}}))}_{\wedge\ 2\varepsilon} + \underbrace{d_B(g(x_{\bar{i}}), g(x))}_{\wedge\ \varepsilon}$$

(by equicontinuity) $\quad$ (by equicontinuity)

$\downarrow$ $f(x_{\bar{i}}), g(x_{\bar{i}}) \in B_{\delta_{\bar{i}}}$

$$\Rightarrow d_B(f(x), g(x)) < 4\varepsilon \Rightarrow d_\infty(f,g) \leq 4\varepsilon$$

$\square$

<u>Application of Ascoli-Arzelà to $\mathbb{R}^n$ valued functions:</u>

Given $C^0(A, \mathbb{R}^n)$ with A usually a compact subset of $\mathbb{R}^n$, we have $\|f\|_\infty = \sup\{|f(x)| : x \in A\}$. If $\{f_k\}_{k\in\mathbb{N}} \in C^0(A;\mathbb{R}^n)$ is

equicontinuous:

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x_1, x_2 \in A\ d(x_1,x_2) < \delta \Rightarrow \|f(x_1) - f(x_2)\|_\infty < \varepsilon\ \forall k \in \mathbb{N}$

equibounded:

$\exists M > 0$ s.t. $|f_k(x)| \leq M\ \forall k \in \mathbb{N}\ \forall x \in A$

Then we can apply Ascoli-Arzelà Thm. to $\{f_k\}_{k\in\mathbb{N}}$.

WEAK CONVERGENCE - BANACH ALAOGLU THEOREM

<u>Def.</u> (Weak Convergence):

Given $(X, \|\cdot\|)$ a normed space, $\{x_k\}_k \subset X$, $\bar{x} \in X$, we say that $\{x_k\}_k$ CONVERGES WEAKLY to $\bar{x}$ iff $T(x_k) \to T(\bar{x})\ \forall T \in X'$.

In this case we write $x_k \rightharpoonup \bar{x}$

<u>Remark</u>:

1) $x_k \to \bar{x}$ $\left(\|x_k - \bar{x}\| \xrightarrow[k\to+\infty]{} 0 \right) \Rightarrow x_k \rightharpoonup \bar{x}$

2) If $\dim X < +\infty$ then $x_k \to \bar{x} \Leftrightarrow x_k \rightharpoonup \bar{x}$

3) Even if $\dim X = +\infty$ it could still happen that $x_k \to \bar{x} \Leftrightarrow x_k \rightharpoonup \bar{x}$ (e.g. $X = \ell^1$)

4) If X is reflexive, then $x_k \rightharpoonup \bar{x} \not\Rightarrow x_k \to \bar{x}$

<u>Thm.</u> (Banach-Alaoglu):

Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then $\overline{B_1(0)}$ is sequentially weakly compact:

$$\forall \{x_k\}_k \in \overline{B_1(0)}\ \exists \text{ subsequence } \{x_{k_h}\}_h,\ \bar{x} \in \overline{B_1(0)} \text{ s.t. } x_{k_h} \rightharpoonup \bar{x}$$

Remark:

1) If $x_k \rightharpoonup \bar{x}$, then $\{x_k\}_k$ is bounded in norm. Indeed:

$T(x_k) \to T(\bar{x}) \ \forall T \in X' \Rightarrow \{T(x_k)\}_k$ is bounded in $\mathbb{R}$ $\forall T \in X'$

$\Rightarrow$ by Banach-Steinhaus, $\{x_k\}$ is bounded

2) $x_k \rightharpoonup \bar{x} \Rightarrow \lim\limits_{k \to +\infty} \|x_k\| \geq \|\bar{x}\|$

Proposition:

Let $(X, \|\cdot\|)$ be a normed space, $C \neq \emptyset$ convex and closed. Then C is sequentially weakly closed: if $\{x_k\} \subset C$ is s.t. $x_k \rightharpoonup \bar{x}$, then $\bar{x} \in C$.

Proof:

By contradiction, suppose $\exists \bar{x} \in X \setminus C$ s.t. $\exists \{x_k\}_k \subset C$ with $x_k \rightharpoonup \bar{x}$

Then $C, \{\bar{x}\}$ are convex, closed and disjoint, $\{\bar{x}\}$ is compact so by Hahn-Banach $\exists T \in X'$ s.t. $T(x) < T(\bar{x}) - \varepsilon \ \forall x \in C$

$\Rightarrow T(x_k) < T(\bar{x}) - \varepsilon \wedge T(x_k) \to T(\bar{x})$ ↯

$\square$