HILBERT SPACES

Proposition:

Let $(X, \|\cdot\|)$ be a normed space, $F: X \to \mathbb{R}$ convex and continuous, F is sequentially weakly lower semicontinuous:

$$\forall \bar{x} \in X,\ \forall \{x_k\} \subset X \text{ with } x_k \rightharpoonup \bar{x} \quad F(\bar{x}) \leq \liminf_{k \to +\infty} F(x_k)$$

Proof:

Let $\bar{x} \in X$, $x_k \rightharpoonup \bar{x}$, $\ell = \liminf\limits_{k \to +\infty} F(x_k)$. If $\ell = +\infty$ there's nothing to prove, so suppose $\ell < +\infty$. Let $s \in \mathbb{R}$ s.t. $s > \ell$ and consider $C_s := \{x \in X : F(x) \leq s\}$. C_s is closed (F is continuous) and convex (F is convex) so it is sequentially weakly closed. Up to subseq. $\lim\limits_{k \to +\infty} F(x_k) = \ell < s \Rightarrow x_k \in C_s$ for k large enough $\Rightarrow \bar{x} \in C_s$

$\Rightarrow F(\bar{x}) \leq s \Rightarrow F(\bar{x}) \leq \ell$

$\square$

Corollary:

Let $(X, \|\cdot\|)$ be a reflexive Banach space, $C \neq \emptyset$ closed, convex, $x_0 \in X \setminus C$. Then $\exists \bar{y} \in C$ s.t. $\|x_0 - \bar{y}\| = dist(x_0, C)$ ($\exists$ point of a convex set with minimum distance from the space).

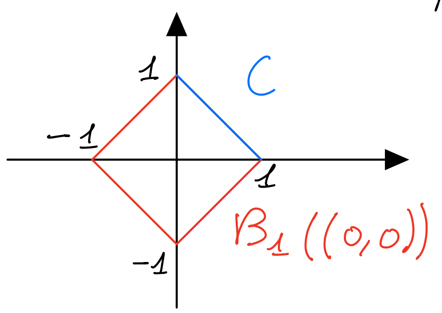
Proof

WLOG assume $x_0 = 0$. We look for a point in C with minimum norm. Let $\delta = \inf\{\|y\|, y \in C\}$. Choose $\{y_k\}_k \subset C$ s.t. $\|y_k\| \to \delta$ (minimizing sequence). Then y_k is bounded in norm, so by Banach-Alaoglu Thm, up to subseq, $\exists\ y_k$ subseq. s.t. $y_k \rightharpoonup \bar{y}$.
$\Rightarrow C$ is weakly closed $\wedge$ $\|\bar{y}\| \leq \liminf\limits_{k \to +\infty} \|y_k\|$ (via lower-semicontinuity.

$\square$

Counterexample of the (NON) uniqueness:

$(\mathbb{R}^2, |\cdot|_1)$, $|(x,y)|_1 := |x| + |y|$, C as in figure:



Def. (Hilbert Space):

A HILBERT SPACE is a vector space X over $\mathbb{R}$ equipped with a scalar product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ s.t. the induced norm is complete $\left(\|x\| := \langle x, x \rangle^{\frac{1}{2}} \right)$

Inequalities and Identities in Hilbert spaces

1) Cauchy-Schwarz inequality

$$\forall x, y \in X \quad |\langle x, y \rangle| \leqslant \|x\| \cdot \|y\|$$

2) Parallelogram Law:

$$\forall x, y \in X \quad \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Indeed: $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle = \langle x, x \rangle + \cancel{2\langle x, y \rangle} + \langle y, y \rangle + \langle x, x \rangle \cancel{- 2\langle x, y \rangle} + \langle y, y \rangle = 2\left(\|x\|^2 + \|y\|^2\right)$

3) Polarization Identity:

$$\forall x, y \in X \quad \langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right)$$

Proposition:

Let $(X, \|\cdot\|)$ be a normed space. Then $\|\cdot\|$ is induced by a scalar product $\Longleftrightarrow$ $\|\cdot\|$ satisfies the Parallelogram Law.

Proof:

$(\Rightarrow)$: we already know from above.

$(\Leftarrow)$: define $a(x,y) := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$, we show that it is a scalar product inducing $\|\cdot\|$ by proving bilinearity.

$$\forall x, y \in X \quad a(x,y) = a(y,x), \quad a(x,0) = 0,$$
$$a(-x,y) = -a(x,y).$$

We have:

$$a(x_1, y) + a(x_2, y) = \frac{1}{4}\left(\|x_1+y\|^2 - \|x_1-y\|^2 + \|x_2+y\|^2 - \|x_2-y\|^2 \right)$$
$$= \frac{1}{8}\left(\|x_1 + x_2 + 2y\|^2 + \cancel{\|x_1 - x_2\|^2} - \|x_1 + x_2 - 2y\|^2 - \cancel{\|x_1 - x_2\|^2} \right)$$
$$= \frac{1}{2} a(x_1 + x_2, 2y) \Rightarrow \text{take } x_2 = 0, \text{ then } a(x_1, y) + 0 = \frac{1}{2} a(x_1, 2y)$$
$$\Rightarrow \frac{1}{2} a(x_1 + x_2, 2y) = a(x_1 + x_2, y)$$

Moreover, $\forall n \in \mathbb{N} \quad a(nx, y) = \underbrace{a(x,y) + \dots + a(x,y)}_{n \text{ times}} = n\, a(x,y)$

$$\Rightarrow a\left(\frac{n}{2^k} x, y\right) = \frac{n}{2^k} a(x,y), \quad \frac{n}{2^k} \in \mathbb{Q} \text{ which is dense in } \mathbb{R}.$$

$\square$

Thm. (Projection on a closed convex set in a Hilbert space):
Let $(X, \langle\cdot,\cdot\rangle)$ be Hilbert, $C \neq \emptyset$ closed convex in X, $x_0 \in X$.
Then $\exists!\ \bar{y} \in C$ s.t. $\|x_0 - \bar{y}\| = \mathrm{dist}(x_0, C)$

Proof:

WLOG assume $x_0 = 0$. Let $\delta := \inf\{\|y\| : y \in C\}$, $\{y_k\} \subset C$ be s.t. $\|y_k\| \to \delta$. Then y_k is Cauchy:

$$\|y_n - y_m\|^2 \underset{\text{parallelogram law}}{=} 2\left(\|y_n\|^2 + \|y_m\|^2\right) - \|y_n + y_m\|^2$$

$$= 2\left(\|y_n\|^2 + \|y_m\|^2\right) - 4\Big\|\underbrace{\frac{y_n + y_m}{2}}_{\in C}\Big\|^2 \leq 2\left(\|y_n\|^2 + \|y_m\|^2\right) - 4\delta^2$$

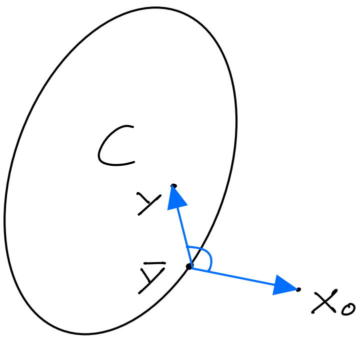
$$\xrightarrow[m,n\to+\infty]{} 0 \text{ for } m,n \text{ large enough}$$

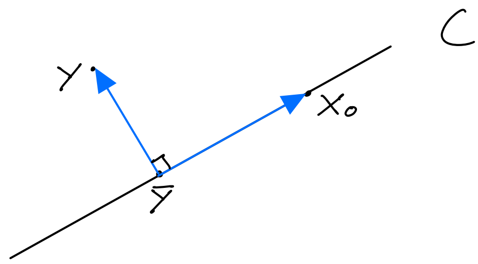
$\Rightarrow y_k \to \bar{y} \in C \Rightarrow \|\bar{y}\| = \delta$. We now prove uniqueness:

Let $\tilde{y} \in C$ with $\|\tilde{y}\| = \delta \Rightarrow \|\bar{y} - \tilde{y}\|^2 \leq 2(\underbrace{\|\bar{y}\|^2}_{=\delta^2} + \underbrace{\|\tilde{y}\|^2}_{=\delta^2}) - 4\delta^2 = 0$
$\Rightarrow \tilde{y} = \bar{y}$

□

Proposition:
Let $(X, \langle\cdot,\cdot\rangle)$, C, x_0 be as above. Then $\bar{y} \in C$ is the point of minimum distance from $x_0 \iff \langle x_0 - \bar{y}, y - \bar{y}\rangle \leq 0 \ \forall y \in C$ (1)
In particular, if C is a vector subspace of X we have that $\bar{y} \in C$ is the point of minimum distance from $x_0 \iff$
$\langle x_0 - \bar{y}, y - \bar{y}\rangle = 0 \quad \forall y \in C$ (2)
Geometrically this means the following:

1) 

2) 

Proof:

($\Leftarrow$): Let $y \in C$, then:

$$\|x_0 - y\|^2 = \|(x_0 - \bar{y}) + (\bar{y} - y)\|^2 = \langle (x_0 - \bar{y}) + (\bar{y} - y), (x_0 - \bar{y}) + (\bar{y}, y)\rangle$$

$$= \|x_0 - \bar{y}\|^2 + 2\underbrace{\langle x_0 - \bar{y}, \bar{y} - y\rangle}_{\geq 0} + \underbrace{\|\bar{y} - y\|^2}_{\geq 0} \geq \|x_0 - \bar{y}\|^2$$

$\Leftarrow$: Let $y \in C$, $t \in (0,1)$, then $\bar{y} + t(y-\bar{y}) \in C$ and:

$$\cancel{\|x_0 - \bar{y}\|^2} \leq \|x_0 - (\bar{y} + t(y-\bar{y}))\|^2 = \|(x_0-\bar{y}) - t(y-\bar{y})\|^2$$
$$= \cancel{\|x_0-\bar{y}\|^2} - 2t\langle x_0-\bar{y}, y-\bar{y}\rangle + t^2\|y-\bar{y}\|^2$$
$$\Rightarrow 2t\langle x_0-\bar{y}, y-\bar{y}\rangle \leq t^2\|y-\bar{y}\|^2 \Rightarrow \langle x_0-\bar{y}, y-\bar{y}\rangle \leq \frac{t}{2}\|y-\bar{y}\|^2 \xrightarrow[t\to 0^+]{} 0$$

$\square$

<u>Proposition</u> (Orthogonal decomposition of a closed subspace):
Let $(X, \langle\cdot,\cdot\rangle)$ be Hilbert, Y be a closed proper vector subspace of X. Given the orthogonal complement of Y, $Y^\perp = \{x \in X : \langle x,y\rangle = 0 \ \forall y \in Y\}$, we have that $X = Y \oplus Y^\perp$ with continuous projections $p: X \to Y$, $Id - p: X \to Y^\perp$ and:

$$\forall x \in X \quad \|x\|^2 = \|p(x)\|^2 + \|x - p(x)\|^2$$

(GENERALIZED PYTHAGORA'S THEOREM)

<u>Proof</u>:
$\forall x \in X$ let $p(x)$ be the point of Y having minimum distance from x (it is characterized by $\langle x - p(x), y\rangle = 0 \ \forall y \in Y$).
We now check the linearity of p:

$$p(x_1 + tx_2) \overset{?}{=} p(x_1) + t\,p(x_2)$$

$$\Rightarrow \langle x_1 + tx_2 - p(x_1) - tp(x_2), y\rangle = \underbrace{\langle x_1 - p(x_1), y\rangle}_{=0} + t\underbrace{\langle x_2 - p(x_2), y\rangle}_{=0}$$
$$\Rightarrow p(x_1 + tx_2) = p(x_1) + t\,p(x_2)$$

Now we write:

$$\langle x - p(x), \underbrace{p(x)}_{\in Y}\rangle = 0 \Rightarrow \|x\|^2 = \|(x - p(x)) + p(x)\|^2 = \|x - p(x)\|^2$$
$$+ 2\underbrace{\langle x - p(x), x\rangle}_{=0} + \|p(x)\|^2 \Rightarrow \|x\|^2 = \|x - p(x)\|^2 + \|p(x)\|^2$$

$\square$

N.B.
$\|p(x)\|^2 \leq \|x\|^2$ (p is continuous with norm ≤ 1)

<u>Remark</u>:
Y n-dim. subspace of X (Hilbert), $\{e_1, \dots, e_n\}$ orthonormal basis of Y (it $\exists$ by Gram-Schmidt). Then $p(x) = \sum_{k=1}^{n}\langle x, e_k\rangle e_k$
To prove it, write:

$$\langle x - p(x), y\rangle = 0 \ \forall y \in Y \Leftrightarrow \langle x - p(x), e_j\rangle = 0 \ \forall j = 1, \dots, n$$
$$\Leftrightarrow \langle x - \sum_{k=1}^{n}\langle x, e_k\rangle e_k, e_j\rangle = \langle x, e_j\rangle - \sum_{k=1}^{n}\langle x, e_k\rangle\langle e_k, e_j\rangle$$

$= \langle x, e_s\rangle - \langle x, e_s\rangle = 0$ ✓

$L^2(\mu)$ coupled with $\langle f, g\rangle = \int_X f(x)g(x)d\mu(x)$ is a Hilbert space.

ℓ^2 coupled with $\langle \{x_k\}, \{y_k\}\rangle = \sum_{k=1}^{+\infty} x_k y_k$ is a Hilbert space

Dual of a Hilbert Space

Thm. (Dual of a Hilbert space):
Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For $y \in X$ define the linear functional $T_y : X \to \mathbb{R}$ s.t. $T_y(x) = \langle x, y\rangle$. Then the map

$$\phi : X \to X' \qquad y \mapsto T_y$$

is an isometric isomorphism. So we can say that, if X is Hilbert, then $X' = X$

Proof:

1) We prove that ϕ is an isometry:
$|T_y(x)| = |\langle x, y\rangle| \leqslant \|y\|\cdot\|x\| \Rightarrow T_y \in X'$ with $\|T_y\|_{X'} \leqslant \|y\|$.
$|T_y(\frac{y}{\|y\|})| = \langle y, \frac{y}{\|y\|}\rangle = \|y\| \Rightarrow \|T_y\|_{X'} = \|y\| \Rightarrow \phi$ is an isometry

2) Surjectivity of ϕ:
$\Rightarrow$ Take $T \in X'$, let $Y = \operatorname{Ker} T$ closed vector subspace of X.
let $x_0 \in X \setminus Y$, $\bar{y}$ its projection on Y, and let $x \in X$. Then:

$$x - \frac{T(x)}{T(x_0-\bar{y})}(x_0-\bar{y}) \in Y \Rightarrow \langle x_0-\bar{y}, x - \frac{T(x)}{T(x_0-\bar{y})}(x_0-\bar{y})\rangle = 0$$

$$\Rightarrow \langle x_0-\bar{y}, x\rangle - \frac{T(x)}{T(x_0-\bar{y})}\|x_0-\bar{y}\|^2 = 0 \Rightarrow T(x) = \langle \underbrace{T(x_0-\bar{y})\frac{x_0-\bar{y}}{\|x_0-\bar{y}\|^2}}_{=:y}, x\rangle$$

$\Rightarrow$ take $T = T_y$

$\square$