HAUSDORFF MEASURES IN $\mathbb{R}^n$ (continued)

Recall the definition of the Hausdorff. It is clear that $H^\alpha(A) = \lim_{\delta \to 0^+} H^\alpha_\delta(A)$ actually exists.

Moreover, the Caratheodory criterion applies, so H^α is a Borel measure. We also define the 0-dim. Hausdorff measure to be the counting measure:

$$H^0(A) = \#(A)$$

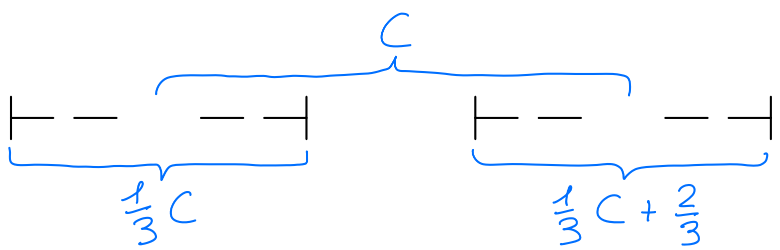
We now define the Hausdorff dim. of $A \subset \mathbb{R}^n$:

$$\dim_H(A) = \sup\{\alpha > 0 : H^\alpha(A) = +\infty\} = \inf\{\alpha > 0 : H^\alpha(A) = 0\}$$

N.B.
as α increases, $H^\alpha(A)$ decreases

Remark (Dimension of the Cantor Set):
The Cantor Set $C \subset [0, 1]$ has "fractal dimension" $0 < \frac{\log 2}{\log 3} < 1$. Indeed, assume $\exists \alpha$ s.t. $0 < H^\alpha(C) < +\infty$. Then:

$$\Rightarrow H^\alpha(\lambda A) = \lambda^\alpha H^\alpha(A)$$



$$\Rightarrow H^\alpha(C) = 2\left(\tfrac{1}{3}\right)^\alpha H^\alpha(C)$$
$$\Rightarrow 1 = 2\left(\tfrac{1}{3}\right)^\alpha$$
$$\Rightarrow 0 = \log 2 - \alpha \log 3$$
$$\Rightarrow \alpha = \frac{\log 2}{\log 3}$$

In the same way it is possible to show that the Hausdorff dim. of the Snowflake Curve is $\dim_H(\text{Snowflake curve}) = \frac{\log 3}{\log 4}$

RADON - NIKODYM THM.

Let X be a set, $\mathcal{S}$ a σ-algebra of subsets of X, $\mu : \mathcal{S} \to [0, +\infty]$ a measure. If $f \in L^1(\mu)$, $f \geq 0$, we can define a new measure $\nu : \mathcal{S} \to [0, +\infty)$ by:

$$\nu(A) := \int_A f(x)\, d\mu(x)$$

f is then called the DENSITY FUNCTION OF ν.
ν is countably additive by Beppo-Levi Thm. (Monotone convergence). But, when is it possible to write a given

measure as the $\int$ of a density function?

Remark:

If a measure $\nu : \mathcal{S} \to [0, +\infty)$ has a density f w.r.t. μ then $\mu(A) = 0 \Rightarrow \nu(A) = 0$

Def. (Absolute Continuity):

Let $\mu, \nu : \mathcal{S} \to [0, +\infty]$ be s.t. $\forall A \in \mathcal{S}$ $\mu(A) = 0 \Rightarrow \nu(A) = 0$.
We then say that ν is ABSOLUTELY CONTINUOUS WRT μ and we write $\nu << \mu$.

Thm. (Radon - Nikodym):

Let $\mu, \nu : \mathcal{S} \to [0, +\infty)$ be finite measures over the σ-algebra $\mathcal{S}$ of subsets of X. If $\nu << \mu$, then $\exists w \in L^1(\mu)$, $w \geq 0$ s.t.

$$\forall A \in \mathcal{S} \quad \nu(A) = \int_A w(x) \, d\mu(x)$$

Proof (by Von Neumann):

Define $\rho : \mathcal{S} \to [0, +\infty)$ as $\rho(A) = \mu(A) + \nu(A)$. Given $u \in L^1(\rho)$, define $T(u) = \int_X u(x) \, d\nu(x)$. Then $T : L^1(\rho) \to \mathbb{R}$ is linear and $T \in L^2(\rho)'$ (dual of $L^2(\rho)$) because:

$$u \in L^2(\rho) \Rightarrow T(u) = \int_X u \, d\nu \underset{\text{Hölder's ineq.}}{\leq} \|u\|_{L^2(\nu)} \cdot \nu(X)^{\frac{1}{2}} \leq \|u\|_{L^2(\rho)} \cdot \rho(X)^{\frac{1}{2}}$$

$\Rightarrow$ By Riesz $\exists v \in L^2(\rho)$ s.t. $T(u) = \langle u, v \rangle$ $\forall u \in L^2(\rho)$

$$\Rightarrow \int_X u \, d\nu = \int_X uv \, d\rho = \int_X uv \, d\mu + \int_X uv \, d\nu$$

$$\Rightarrow \int_X (1 - v) u \, d\nu = \int_X uv \, d\mu \quad \forall u \in L^2(\rho)$$

If we can choose $u(x) = \mathbb{1}_E(x) \cdot \frac{1}{1 - v(x)}$, $E \in \mathcal{S}$, we get:

$$\int_X \mathbb{1}_E(x) \, u(x) \, d\nu(x) = \int_X \frac{v(x)}{1 - v(x)} \mathbb{1}_E(x) \, d\mu(x)$$

$$\Rightarrow \nu(E) = \int_E \frac{v(x)}{1 - v(x)} \, d\mu(x)$$

Unfortunately we can't take such a $u(x)$, because it's not certain that it would be in $L^2(\rho)$.
Let $E \in \mathcal{S}$, $u = \mathbb{1}_E$. Then by the identity we found earlier we have:

$$0 \leq \nu(E) = \int_E v(x)\,d\mu(x) \leq \underset{\substack{\vee \\ 0}}{\rho(E)} \iff 0 \leq \underbrace{\frac{1}{\rho(E)}\int_E v(x)\,d\mu(x)}_{\text{Integral Average}} \leq 1$$

$\Rightarrow$ We claim that $0 \leq v(x) \leq 1$ for ρ-a.e. x. Indeed: fix $n = 1, 2, 3, \dots$ and consider $E_n = \{x \in X : v(x) \geq 1 + \frac{1}{n}\}$. If $\rho(E_n) > 0$ we get a contradiction because

$$\frac{1}{\rho(E_n)}\int_E v(x)\,d\rho \geq 1 + \frac{1}{n} \quad \lightning \quad \left(\frac{1}{\rho(E)}\int_E v(x)\,d\mu(x) \leq 1\right)$$

$\Rightarrow$ $\rho(E_n) = 0$ and we have $\{x \in X : v(x) > 1\} = \bigcup_{n=1}^{+\infty} E_n$

$\Rightarrow$ $v(x) \leq 1$ for ρ-a.e. x, and in the same way we prove that $v(x) \geq 0$

Consider now $F = \{x \in X : v(x) = 1\}$. Then take $u = \mathbb{1}_F$ and compute:

$$0 = \left(\int_X (1-v)u\,d\nu\right) = \int_X u v\,d\mu \underset{\int_X \mathbb{1}_F \cdot 1\,d\mu}{=} \mu(F) \Rightarrow \nu(F) = 0 \Rightarrow \rho(F) = 0$$

So actually we have $0 \leq v(x) < 1$ for ρ-a.e. x, we now use the geometric series of L^2 functions to approximate the following:

$$u = \mathbb{1}_E \cdot \frac{1}{1-v} \notin L^2(\rho) \text{ in general}$$

$$\Rightarrow u_n(x) := \mathbb{1}_E(x)\left(1 + v(x) + v^2(x) + \dots + v^n(x)\right) \in L^2(\rho)$$

$$\Rightarrow \int_E \underbrace{(1 - v(x)^{n+1})}_{\substack{\to 1 \text{ a.e.} \\ \text{increasing}}}\,d\nu(x) = \int_E \underbrace{(v + v^2 + \dots + v^{n+1})}_{\substack{\to \frac{v(x)}{1-v(x)} \text{ a.e.} \\ \text{increasing}}}\,d\mu(x)$$

$\Rightarrow$ apply Beppo-Levi Thm.:

$$\underbrace{\int_E (1 - v(x)^{n+1})\,d\nu(x)}_{\to \nu(E)} = \underbrace{\int_E (v + v^2 + \dots + v^{n+1})\,d\mu(x)}_{\to \int_E \frac{v(x)}{1-v(x)}\,d\mu(x)}$$

$$\Rightarrow \nu(E) = \int_E \frac{v(x)}{1-v(x)}\,d\mu(x) \Rightarrow \frac{v(x)}{1-v(x)} \text{ is the density function.}$$

$\square$

Radon-Nikodym can be extended to σ-finite measures:

Def. (σ-finite measure)

A measure $\mu : \mathcal{S} \to [0, +\infty]$ is σ-FINITE if it's possible to decompose $X = \bigcup_{k=1}^{+\infty} A_k$ s.t. $\mu(A_k) < +\infty \ \forall A_k \in \mathcal{S}$ (w.l.o.g. we can assume that A_k are pairwise disjoint.

e.g.:

The Lebesgue measure on $\mathbb{R}^n$ is σ-finite, the counting measure $\#$ on $\mathbb{R}$ is NOT σ-finite.

$\Rightarrow$ Radon-Nikodym Thm. also holds for σ-finite measures:
$\mu, \nu : \mathcal{S} \to [0, +\infty]$ σ-finite with $\nu \ll \mu \Rightarrow \exists\, w : X \to [0, +\infty]$ μ-measurable s.t. $\nu(E) = \int_E w(x)\, d\mu(x) \quad \forall E \in \mathcal{S}$