L^p Spaces

Def. (L^p Space, L^p norm):

Let μ be an outer measure on a set X, $p \in [1, +\infty)$. Then:

$$L^p(\mu) := \{ u : X \longrightarrow \overline{\mathbb{R}} \ \mu\text{-meas.} : \int_X |u(x)|^p d\mu(x) < +\infty \} / \sim$$

introducing $\sim$ s.t. $u \sim v \Leftrightarrow \mu(\{x \in X : u(x) \neq v(x)\}) \neq 0$,

we say that $u(x) = v(x)$ for μ-a.e. $x \in X$.

We also introduce the norm $\|u\|_{L^p(\mu)} := \left(\int_X |u(x)|^p d\mu(x) \right)^{\frac{1}{p}}$.

We extend the definition to the case $p = +\infty$:

$$\|u\|_{L^\infty(\mu)} := \text{ess sup}\{|u(x)| : x \in X\}$$

$$:= \inf\{t > 0 : \underbrace{\mu(\{x \in X : |u(x)| > t\}) = 0}_{|u(x)| \leq t \ \mu\text{-a.e.}}\}$$

Now we can also define $L^\infty(\mu) = \{ u : X \longrightarrow \overline{\mathbb{R}} \ \mu\text{-meas.} : \|u\|_{L^\infty(\mu)} < +\infty \}$

Remark:

$\mu(\{x \in X : |u(x)| > \|u\|_{L^\infty(\mu)}\}) = 0$ because, by definition:

$\mu(\{x \in X : |u(x)| > \|u\|_{L^\infty(\mu)} + \frac{1}{n}\}) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \bigcup_{n \in \mathbb{N}} \{x \in X : |u(x)| > \|u\|_{L^\infty(\mu)} + \frac{1}{n}\} = \{x \in X : |u(x)| > \|u\|_{L^\infty(\mu)}\}$

Particular case of L^p spaces: ℓ^p (space of the p-summable seq.)

$p \in [1, +\infty)$, $\ell^p = \{ \{x_k\}_{k \in \mathbb{N}} : \sum_{k=1}^{+\infty} |x_k|^p < +\infty \}$,

$\|\{x_k\}_k\|_{\ell^p} := \left(\sum_{k=0}^{+\infty} |x_k|^p \right)^{\frac{1}{p}}$

The extended case $p = +\infty$:

$$\ell^\infty = \{ \{x_k\}_{k \in \mathbb{N}} : \sup\{|x_k| : k \in \mathbb{N}\} < +\infty \}, \quad \|\{x_k\}\|_{\ell^\infty} = \sup\{|x_k| : k \in \mathbb{N}\}$$

These are $L^p(\mu)$ when $X = \mathbb{N}$ and μ is the counting measure

N.B.

Given the p-norm in $\mathbb{R}^N$:

$$|(x_1, \dots, x_N)|_p = \left(\sum_{k=1}^{N} |x_k|^p \right)^{\frac{1}{p}}$$

we can say that $|\cdot|_p : \{1, \dots, N\} \longrightarrow \mathbb{R}$

Claim:

Given the dual norm $\|T\|_{X'} = \sup\{|T(x)| : x \in X, \|x\| \leq 1\}$, the sup is NOT a maximum in general.

$\Rightarrow$ Counter example:

define $T : \ell^1 \to \mathbb{R}$ linear as follows:

$$\underset{\substack{\cap \\ \ell^1 \\ \left(\sum_{k=0}^{+\infty} |x_k| < +\infty\right)}}{T(\{x_k\}_{k\in\mathbb{N}})} = \sum_{k=1}^{+\infty} \underbrace{\left(1 - \frac{1}{k}\right)}_{<1} x_k \quad \text{converges absolutely}$$

we claim that $\|T\|_{(\ell^1)'} = 1$. Indeed:

$$|T(\{x_k\})| \leq \sum_{k=1}^{+\infty} \left|1 - \frac{1}{k}\right| |x_k| < \sum_{k=1}^{+\infty} |x_k| \leq \|\{x_k\}\|_{\ell^1}$$

$\Rightarrow \|T\|_{(\ell^1)'} \leq 1$

To show the opposite inequality, consider the canonical basis of ℓ^1:

$$\{e_k^m\}_k := \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases} = \delta_k^m$$

$\Rightarrow \|e_k^m\|_{\ell^1} = 1$, $(m \geq 1)$ $T(\{e_k^m\}) = 1 - \frac{1}{m} \xrightarrow{m \to +\infty} 1$

$\Rightarrow \|T\|_{(\ell^1)'} = 1$, BUT $\nexists \{x_k\}_k \in \ell^1$ s.t. $\|\{x_k\}_k\| \leq 1$ for which $T(\{x_k\}_k) = 1$

Example of Banach Space: $C^1([a,b])$

$C^1([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is differentiable}, f' \text{ is continuous}\}$

$\Rightarrow$ if $f \in C^1([a,b])$ we have:

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$$

Claim:

$(C^1, \|\cdot\|_{C^1})$ is a Banach space

Proof.

$\{f_k\}_{k\in\mathbb{N}} \subset C^1([a,b])$ Cauchy seq. $\Rightarrow \{f_k\}, \{f'_k\}$ are Cauchy in $(C^0, \|\cdot\|_\infty) \Rightarrow f_k \xrightarrow{\|\cdot\|_\infty} f \in C^0$, $f'_k \xrightarrow{\|\cdot\|_\infty} g \in C^0$

$\Rightarrow$ fund. thm. of Calculus:

$$f_k(x) = f_k(a) + \int_a^x f'_k(t)\,dt$$

$$\Rightarrow f(x) = f(a) + \int_a^x g(t)\,dt$$

$\Rightarrow f$ is differentiable and f' is continuous.

$\square$

<u>Remark</u>:

$(C^1([-1,1]), \|\cdot\|_\infty)$ is NOT Banach:
consider indeed the following sequence:

$$f_k(x) = \sqrt{x^2 + \tfrac{1}{k}} \xrightarrow[\|\cdot\|_\infty]{} |x| \notin C^1$$

Given X vector space, $\|\cdot\|$, $|||\cdot|||$ norms on X. The 2 norms induce the same topology iff $\exists\, c, C > 0$ s.t.:

$$c\|x\| \leq |||x||| \leq C\|x\| \quad \forall x \in X$$

$\Rightarrow$ if 2 norms are equivalent, they have the same Cauchy sequences.

<u>Proposition</u>:
All norms on $\mathbb{R}^N$ are equivalent

<u>Proof</u>:
It is enough to check that every norm $\|\cdot\|$ is equivalent to the euclidean norm $|\cdot|$. We know that $\|\cdot\|$ is continuous w. r. to the euclidean topology. In particular:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$$

$$\Rightarrow x = \sum_{k=1}^N x_k \cdot \vec{e}_k, \quad y = \sum_{k=1}^N y_k \cdot \vec{e}_k$$

$$\Rightarrow \big|\, \|x\| - \|y\| \,\big| \leq \|x-y\| = \Big\| \sum_{k=1}^N (x_k - y_k)\cdot \vec{e}_k \Big\| \leq \sum_{k=1}^N \underbrace{|x_k - y_k|}_{\leq |x-y|} \cdot \|e_k\|$$

$$\leq |x-y| \Big(\sum_{k=1}^N \|e_k\| \Big)$$

$\Rightarrow \|\cdot\|$ is Lipschitz-continuous w. r. to the euclidean topology.

$\Rightarrow S = \{x \in \mathbb{R}^N : |x| = 1\}$ is compact $\Rightarrow$ (Weierstraß thm.)

$$\exists M = \max\{\|x\| : x \in S\}, \; m = \min\{\|x\| : x \in S\} > 0$$

$\Rightarrow m \leq \|x\| \leq M \quad \forall x \in S \Rightarrow$ let $x \in \mathbb{R}^N$, $x \neq 0$, then $\frac{x}{|x|} \in S$

$\Rightarrow m \leq \left\| \frac{x}{|x|} \right\| < M \Rightarrow m|x| \leq \|x\| \leq M|x| \quad \forall x \in \mathbb{R}^N$

$\square$

<u>Corollary</u>:
Any 2 norms on a finite dim. vector space are equivalent