Inf - Sup Compatible Spaces

Let V_h, Q_h not be inf-sup compatible, then $\exists p_h^* \in Q_h$ s.t. $b(u_h, p_h^*) = 0 \quad \forall u_h \in V_{h,0} \iff \exists \hat{p}^* \in \mathbb{R}^{N_p}$ s.t. $\hat{u}^T B^T \hat{p}^* = 0$ $\forall \hat{u} \in \mathbb{R}^{N_u} \iff \hat{p}^* \in \text{Ker } B^T$

<u>Remark</u>:

If inf-sup isn't valid, the spurious pressure mode are in $\text{Ker } B^T$

If V_h, Q_h are inf-sup compatible then:

$$\text{Ker } B^T = \{\vec{0}\} \iff \text{Im } B = \mathbb{R}^{N_p} \iff \forall \hat{u} \in \mathbb{R}^{N_u} \; \exists \hat{p} \in \mathbb{R}^{N_p} \text{ s.t. } B\hat{u} = \hat{p}$$
$$\iff \text{rank } B = \text{rank } B^T = N_p, \quad B \in \mathbb{R}^{N_p \times N_u}, \quad B: \mathbb{R}^{N_u} \longrightarrow \mathbb{R}^{N_p}$$
$$\Rightarrow \dim \text{Ker } B = N_u - N_p > 0$$

<u>Remark</u>:

1) If inf-sup holds, there $\exists N_u - N_p$ solenoidal velocities
2) When V_h, Q_h are inf-sup compatible, $\dim V_h > \dim Q_h$!!!

Convergence Analysis

If V_h, Q_h are inf-sup comp., then the following holds:

$$\|\nabla(u - u_h)\|_{L^2} \leq 2\left(1 + \frac{\sqrt{d}}{\beta_h}\right) \inf_{w_h \in V_h} \|\nabla(u - w_h)\|_{L^2} + \frac{\sqrt{d}}{\nu} \inf_{\pi_h \in Q_h} \|p - \pi_h\|_{L^2}$$

$$\|p - p_h\|_{L^2} \leq \frac{2\nu}{\beta_h}\left(1 + \frac{\sqrt{d}}{\beta_h}\right) \inf_{w_h \in V_h} \|\nabla(u - w_h)\|_{L^2} + \left(1 + \frac{2\sqrt{d}}{\beta_h}\right) \inf_{\pi_h \in Q_h} \|p - \pi_h\|_{L^2}$$

<u>Remark</u>:

If $\beta_h \xrightarrow[h \to 0]{} 0$, convergence is not granted !!!

Inf - Sup Compatible FE:

1) $V_h = [X_h^r]^d, \; Q_h = X_h^r \Rightarrow V_h, Q_h$ are NOT inf-sup compatible
2) $V_h = \mathbb{P}_1, \; Q_h = \mathbb{P}_0 \Rightarrow V_h, Q_h$ are NOT inf-sup compatible
3) $V_h = \mathbb{P}_{k+1}, \; Q_h = \mathbb{P}_k, \; k \geq 1 \Rightarrow V_h, Q_h$ are inf-sup compatible with order of convergence $k+1$
4) $V_h = \mathbb{P}_1$ iso $\mathbb{P}_2, \; Q_h = \mathbb{P}_1 \Rightarrow V_h, Q_h$ are inf-sup compatible with order of convergence $\underline{1}$
5) $V_h = \mathbb{P}_1$ bubble, $Q_h = \mathbb{P}_1 \Rightarrow V_h, Q_h$ are inf-sup compatible with order of convergence $\underline{1}$

where

$\mathbb{P}_K :=$ polynomials of deg $= K$,

$\mathbb{P}_1$ bubble $:= \{ 0$ at $\partial\Omega_i$, polynomials of deg $= 3$ in $\Omega_i \}$

Stabilization Methods

Let $s: Q_h \times Q_h \to \mathbb{R}$ pos. def. We find $(u_h, p_h) \in V_h \times Q_h$ s.t. $u_h = g_h$ on Γ_D and:

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = F(v_h) & \forall v_h \in V_h \\ b(u_h, q_h) - s(p_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

$\Rightarrow$ Introduce $S \in \mathbb{R}^{N_p \times N_p}$, $S_{ke} = s(\psi_k, \psi_e)$, then we have:

$$\begin{pmatrix} A & B^T \\ B & -S \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} F \\ \vec{0} \end{pmatrix}$$

Remarks:

1) Well posedness holds for general inf-sup condition:

$$\sup_{v_h \in V_{h,0}} \frac{b(v_h, q_h)}{\|\nabla v_h\|_{L^2}} + s(q_h, q_h) \geq \beta_h > 0 \quad \forall p_h \in Q_h$$

2) Compatibility:

$$s(q_h, q_h) \xrightarrow[h \to 0]{} 0$$

Optimal convergence holds if $s \xrightarrow[h \to 0]{} 0$ at least at the same order as the method. Generally this is true if strong continuum compatibility holds:

$$s(p, p) = 0$$

Stabilizers:

1) Brezzi - Pitkeranta:

$$s(p_h, q_h) = \delta \sum_{\Omega_i \in \mathcal{T}_h} h_i^3 \Big|_{\Omega_i} \nabla p_h \cdot \nabla q_h \, d\Omega$$

$\Rightarrow$ uniformly stable if $\dim Q_h = \dim V_h$ with convergence rate 1 independently from the convergence order of the method.

2) Interior - Penalty (IP):

$$s(p_h, q_h) = \delta \sum_{\Omega_i \in \mathcal{T}_h} h_i^3 \Big|_{\partial\Omega_i \setminus \partial\Omega} [\nabla p_h] \cdot [\nabla q_h] \, d\Gamma$$

where $[f] = f^+ - f^-$

$\Rightarrow s(p,p) = 0 \quad \forall p \in H^2(\Omega)$

$\Rightarrow$ Optimal convergence $\forall K$, $\mathbb{P}_K / \mathbb{P}_K$

Resolution of the Galerkin problem

1) Pressure Method:

$$\begin{cases} A\hat{u} + B^T\hat{p} = F \\ B\hat{u} - S\hat{p} = \vec{0} \end{cases} \Rightarrow \hat{u} = A^{-1}(F - B^T\hat{p})$$

$$\Rightarrow BA^{-1}(F - B^T\hat{p}) - S\hat{p} = \vec{0} \Rightarrow \underbrace{(BA^{-1}B^T + S)}_{\Sigma = \text{Schurr Component}}\hat{p} = BA^{-1}F$$

$$\Rightarrow \Sigma\hat{p} = BA^{-1}F$$

1) $S = \vec{0}$:

$\Rightarrow \Sigma = BA^{-1}B^T \Rightarrow$ we have:

$$\frac{C_1}{\nu} h_{min}^d \|\hat{p}\| \leqslant \hat{p} \frac{M_p}{\nu} \hat{p} \leqslant \frac{C_2}{\nu} h_{max}^d \|\hat{p}\|$$

$$\frac{C_1}{\nu} h_{min}^d \beta_h^2 \leqslant \lambda(BA^{-1}B^T) \leqslant \frac{C_2}{\nu} h_{max}^d d$$

$$K(BA^{-1}B^T) \leqslant \frac{C_2}{C_1} \frac{\beta_h^2}{d} \left(\frac{h_{max}}{h_{min}}\right)^d$$

$\Rightarrow$ take $P = \frac{M_p}{\nu}$, then:

$$P^{-1} BA^{-1}B^T v = \lambda v \Rightarrow (\nu M_p^{-1}) BA^{-1}B^T v = \lambda v$$

$$\Rightarrow BA^{-1}B^T = \lambda \frac{M_p}{\nu} v \quad \wedge \quad \beta_h^2 \leqslant \lambda \leqslant d$$

and the same holds for $P = \frac{\mathrm{diag}\, M_p}{\nu}$

2) Monolithic Approach:

$$\begin{cases} A\hat{u} + B^T\hat{p} = F \\ \mp B\hat{u} \pm S\hat{p} = \vec{0} \end{cases} \Rightarrow \text{we have:}$$

$$\underbrace{\begin{pmatrix} A & B^T \\ B & -S \end{pmatrix}}_{=\Sigma_1} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} F \\ \vec{0} \end{pmatrix} \quad \vee \quad \underbrace{\begin{pmatrix} A & B^T \\ -B & S \end{pmatrix}}_{=\Sigma_2} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} F \\ \vec{0} \end{pmatrix}$$

1) Σ_1 is symmetric but it is indefinite:

$$(\hat{u} \;\; \hat{p})^T \Sigma_1 \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \underbrace{\hat{u}^T A \hat{u}}_{\geqslant 0} + \underbrace{2\hat{p}^T B \hat{u}}_{?} - \underbrace{\hat{p}^T S \hat{p}}_{\leqslant 0}$$

2) Σ_2 is not symmetric but it is semi-pos. def.:

$$(\hat{u} \;\; \hat{p})^T \Sigma_2 \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \hat{u}^T A \hat{u} + \hat{p}^T S \hat{p} \geqslant 0$$

$\Rightarrow$ take $P = \begin{pmatrix} A & \vec{0} \\ \vec{0} & \frac{M_p}{\nu} \end{pmatrix}$, then we have:

$$P^{-1}\Sigma v = \mu_i v \,, \quad \Sigma = \begin{pmatrix} A & B^T \\ B & \vec{0} \end{pmatrix}, \quad i = 1, \dots, N_u + N_p$$

$$\Rightarrow \begin{pmatrix} A & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \mu_i \begin{pmatrix} A & \vec{0} \\ \vec{0} & \frac{M_p}{\nu} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix}$$

1) $\hat{u} \in \operatorname{Ker} B$, $\hat{p} = \vec{0}$ $\Rightarrow$ $\dim \operatorname{Ker} B = N_u - N_p$

$$\begin{cases} A\hat{u} = \mu_i A u \\ \vec{0} = \vec{0} \end{cases} \Rightarrow \exists\, N_u \cdot N_p \text{ eigenvalues } \mu_i, \ 2N_p \text{ remain !!!}$$

2) $\hat{u} \notin \operatorname{Ker} B$, $\hat{p} \neq \vec{0}$:

$$\begin{cases} A\hat{u} + B^T\hat{p} = \mu_i A\hat{u} \\ B\hat{u} = \mu_i \frac{M_p}{\nu} \hat{p} \end{cases} \Rightarrow A(1-\mu_i)\hat{u} = -B^T\hat{p}$$

$$\Rightarrow A\hat{u} = \frac{1}{\mu_i - 1} B^T \hat{p} \Rightarrow \frac{1}{\mu_i - 1} B A^{-1} B^T \hat{p} = \mu_i \frac{M_p}{\nu} \hat{p}$$

$$\Rightarrow B A^{-1} B^T \hat{p} = \mu_i(\mu_i - 1)\frac{M_p}{\nu}\hat{p}$$

$$\Rightarrow \mu_i(\mu_i - 1) = \lambda_i \,, \quad 0 < \beta_h^2 \leqslant \lambda_i \leqslant d$$

$$\Rightarrow \mu_i = \frac{1 \pm \sqrt{1 + 4\lambda_i}}{2}$$

1) N_p are negative, $\mu_i \in \left[\frac{1 - \sqrt{1+4d}}{2}, \frac{1 + \sqrt{1 + 4\beta_h^2}}{2} \right]$

2) N_p are positive, $\mu_i \in \left[\frac{1 + \sqrt{1+4\beta_h^2}}{2}, \frac{1 + \sqrt{1+4d}}{2} \right]$

<u>Remark</u>:

$\begin{pmatrix} A & \vec{0} \\ \pm B & \frac{M_p}{\nu} \end{pmatrix}$ is also a good preconditioner $(+ : \Sigma_1, \ - : \Sigma_2)$

$\Rightarrow$ In this approach, we have the following system:

$$\begin{pmatrix} \underbrace{\nu K \quad 0}_{K_\nu} & B_1^T \\ 0 \quad \nu K & B_2^T \\ B_1 \quad B_2 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{p} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} K_\nu \hat{u}_1 + B_1^T \hat{p} = F_1 \\ K_\nu \hat{u}_2 + B_2^T \hat{p} = F_2 \\ B_1 \hat{u}_1 + B_2 \hat{u}_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{u}_1 = K_\nu^{-1}(\cancel{F_1} - B_1^T \hat{p}) \\ \hat{u}_2 = K_\nu^{-1}(\cancel{F_2} - B_2^T \hat{p}) \end{cases} \Rightarrow (B_1 K_\nu^{-1} B_1^T + B_2 K_\nu^{-1} B_2^T)p = 0$$

$$\Rightarrow b = B_1^T \hat{p} \,, \quad K_\nu z = b$$