Proper Classes and Sets

Every set is also a class, but not every class is a set. Classes which are not sets are called "proper classes"

Examples:

1) $V :=$ the class of all sets $= \{x : x = x\} = \{x : \top\}$ is proper
2) The class of all vector spaces is proper
3) The class of all ordinal numbers is proper:

$$0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \dots, \omega+\omega = 2\omega,$$
$$3\omega, \dots, \omega \cdot \omega = \omega^2, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots, \omega^{\omega^{\omega^{\omega^{\cdots}}}} = \varepsilon_0$$

etc. (ω is the first infinite ordinal)

4) A Category is small if its objects form a set, large if they form a proper class.
5) Set is large

Russell's Paradox:

Let $R := \{x : x \notin x\}$ be a class. We ask: $R \in R$?

$$R \in R \Leftrightarrow R \notin R \text{ ↯ hence } \bot \text{, indeed:}$$

1) Claim: $\neg(R \in R)$

Proof:

Assume $R \in R$. Show $\bot$

$\Rightarrow R \in R \Rightarrow R \notin R$ ↯ So $\bot$

2) Claim: $R \in R$

Proof:

We know by (1) $R \notin R$, $\Rightarrow R \in R$ by $\Leftarrow$

3) Claim: $\bot$

Proof: by (2) and (1)

Variant of the Russell's Paradox:

Define $\tilde{R} = \{x : x \in x \Rightarrow 2+2=5\}$. Using the same argument as above we obtain $2+2=5$

$\Rightarrow$ Russell's Paradox proved that Naive Set Theory, with its unrestricted principle of set comprehension, is INCONSISTENT

Resolution of the Russell's Paradox:

$\Rightarrow$ we solve Russell's Paradox by switching to formal proofs, where only a narrow list of logical inference rules and axioms may be used. The most well known system of said rules is ZFC (Zermelo-Fraenkel Set Theory with the axiom of choice). We have also extensions:

$$\begin{tikzcd}
ZFC+U & NBG & ZFC/S \\
& ZFC \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \\
& ZF \arrow[u, dash] & \\
& IZF \arrow[u, dash] & PA \arrow[ul, dash] \\
& PA \arrow[u, dash] & \\
& & HA \arrow[ul, dash] \arrow[uu, dash]
\end{tikzcd}$$

In ZFC we can't form the set $\{x : x \notin x\}$, R is therefore a proper class and not a set.

Dealing with Size Issues

0) Ignore them
1) Switch from ZFC to NBG, which natively supports both sets and proper classes. HOWEVER, in NBG only sets can be element of classes. In NBG, unlike in ZFC, we can formulate statements like "for every proper class...", "there is a proper class with ...". Moreover, for statements which are purely about sets NBG is conservative over ZFC (any NBG proof can be transformed into a ZFC proof).
2) Switch from ZFC to ZFC + "there $\exists$ a Grothendieck universe" also called ZFC + U.
   Note: ZFC + U is NOT conservative over ZFC
3) Try to fake classes in ZFC (standard approach for logicians). Regard classes as mere "syntectic sugar" (e.g. i++ for i=i+1 in C). For instance, "$\forall x \in V \ldots$"

is not a statement in ZFC (it uses the proper class V) But "$\forall x \ldots$" is. The same holds for "$\exists x \in On \ldots$" ($On :=$ proper class of ordinal numbers) and "$\exists x$ s.t. x is an ordinal number and ..."

4) Switch from ZFC to ZFC/S:

Recall: the cumulative hierachy!

$$V_0 = \phi, \quad V_1 = \mathcal{P}(V_0) = \{\phi\}, \quad V_2 = \mathcal{P}(V_1), \ldots,$$
$$V_\omega = \bigcup_{i=0} V_i, \ldots, V_{\omega+1} = \mathcal{P}(V_\omega), \ldots, V_{2\omega} = \bigcup_{i=0} V_i$$

<u>Thm.</u>:

$V = \bigcup_{\alpha \in On} V_\alpha$ (i.e. every set is contained in some V_α)

Reflection theory:

for any statement A in ZFC, ZFC proves that:

$$\exists \alpha \in On \text{ s.t. } A \Leftrightarrow A^{V_\alpha}$$

where A^{V_α} is the V_α relativization of A:

$$\text{"}\forall x \ldots\text{"}^{V_\alpha} = \text{"}\forall x \in V_\alpha \ldots\text{"},$$
$$\text{"}\exists x \ldots\text{"}^{V_\alpha} = \text{"}\exists x \in V_\alpha \ldots\text{"}$$

<u>N.B.</u>

$\forall \alpha \in On$ V_α is a set!!! Not a class!!!

ZFC/S is a variant of ZFC with reflection built into it:

ZFC/S = ZFC + \$, \$ is the set of all small sets

<u>Reflection axiom</u>:

For every statement A in ZFC, $A \Leftrightarrow A^{\$}$

ZFC/S is conservative over ZFC for statements which don't contain \$