

# CATEGORY OF CATEGORIES

## Examples:

1)  $F: \text{Set} \rightarrow \text{Set}$

$$X \mapsto X \times X = \{(a, b) : a, b \in X\}$$

2)  $G: \text{Set} \rightarrow \text{Set}$

$$X \mapsto \{(a, b, c) : a, b, c \in X, c = a\}$$

3)  $H: \text{Set} \rightarrow \text{Set}$

$$X \mapsto \{\coprod_{\coprod} (a, b)_{\coprod} : a, b \in X\} \quad (\text{N.B. } (a, b) \neq \coprod_{\coprod} (a, b)_{\coprod})$$

$\Rightarrow$  Are  $F, G, H$  the same functor? No! (they have 3 different structures) BUT they are isomorphic. In which category? In the category of functors from  $\text{Set}$  to  $\text{Set}$ , which is denoted as  $[\text{Set}, \text{Set}]$ .

## Def. (Functor Category):

Let  $\mathcal{C}, \mathcal{D}$  be categories. The **FUNCTOR CATEGORY**  $[\mathcal{C}, \mathcal{D}]$  consists of:

1) Objects: all functors  $\mathcal{C} \rightarrow \mathcal{D}$

2) Morphism:

$$\text{Hom}_{[\mathcal{C}, \mathcal{D}]}(F, G) := \{\eta : \eta : F \Rightarrow G \text{ natural transformation}\}$$

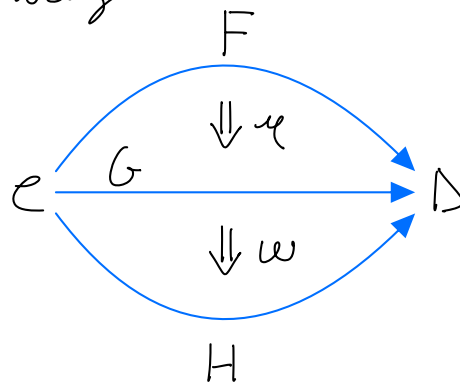
3) Identities:

$$\forall F \in \text{Ob}([\mathcal{C}, \mathcal{D}]) \quad \text{id}_F : F \Rightarrow F \text{ s.t.}$$

$$(\text{id}_F)_x : F(x) \rightarrow F(x) \text{ given by } \text{id}_{F(x)}$$

4) Composition rule:

we use the following:



$$\Rightarrow \omega \circ \eta : F \Rightarrow H, \quad (\omega \circ \eta)_x := \omega_x \circ \eta_x$$
$$\begin{array}{ccc} & \downarrow & \searrow \\ : F(x) \rightarrow H(x) & & : F(x) \rightarrow G(x) \\ & \downarrow & \\ & : G(x) \rightarrow H(x) & \end{array}$$

## Example

1) We check that the 3 functors  $F, G, H$  of the previous examples are isomorphic:

$$\eta: F \Rightarrow G$$

$$\eta_x: F(x) \rightarrow G(x)$$

$$(a, b) \mapsto (a, b, a)$$

$$\omega: G \Rightarrow F$$

$$\omega_x: G(x) \rightarrow F(x)$$

$$(a, b, c) \mapsto (a, b)$$

and similarly for  $F, H$  and  $G, H$

## Def. (Category of Categories):

The (1-) CATEGORY OF (1-) CATEGORIES,  $\text{Cat}$  and consists:

1) Objects:  $\text{Ob}(\text{Cat}) = \text{all } (1-) \text{ categories}$

2) Morphisms:  $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) = \{F: \mathcal{C} \rightarrow \mathcal{D} : F \text{ functor}\}$

3) Identities:  $\forall F \in \text{Hom}_{\text{Cat}}, \text{Id}_F$

4) Composition rule:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \Rightarrow G \circ F: \mathcal{C} \rightarrow \mathcal{E}$$

$$X \mapsto G(F(X))$$

$$f \mapsto G(F(f))$$

## Remark:

There are 2 issues with this definition:

1) It ignores the natural transformations. Remedy: consider the 2-Category of 1-categories, which consists of:

1) objects: all 1-categories

2) morphisms: all functors between 1-categories

3) 2-morphisms / 2-cells: all natural transformations

2) Size issues. Remedies:

1) switch foundations,

2) consider the category of small categories (its objects and morphisms only form a set, not a proper class)

## Limits

We want to generalize the limit concept in analysis to categorical structures.

We know that:

$$3, 3.1, 3.14, 3.141 \rightarrow \pi$$

Consider the following:

$$\mathbb{R}^0 \rightarrow \mathbb{R}^1 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^4 \dots \rightarrow \mathbb{R}^\infty$$

$$() \mapsto (x)$$

$$(x) \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \end{pmatrix} : x_i \in \mathbb{R}, \exists n \in \mathbb{N} : \forall l \geq n, x_l = 0 \right\}$$

We can also do the "opposite limit":

$$\mathbb{R}^\omega \rightarrow \dots \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^1 \rightarrow \mathbb{R}^0$$

Example

