Banach-Steinhaus Theorem

Corollary:
Let $(X, \|\cdot\|)$ be a normed space, $Y \subset X$ a proper vector subspace. Then:

$$Y \text{ is dense} \Leftrightarrow \forall T \in X' \text{ s.t. } T(y) = 0 \ \forall y \in Y, \ T = 0$$

Proof:

($\Rightarrow$): Y is dense $\Rightarrow \overline{Y} = X \Rightarrow$ let $x \in X$, $\exists \{y_k\}_k \subset Y$ s.t.
$y_k \to x \Rightarrow \underbrace{T(y_k)}_{=0} \to T(x) = 0$

($\Leftarrow$): Let Y be a vector subspace s.t. $\overline{Y} \subsetneq X$. Let $x_0 \in X \setminus \overline{Y}$ and consider the closed convex sets $\{x_0\}, \overline{Y}$: they are closed, disjoint and $\{x_0\}$ is compact $\Rightarrow \exists T \in X'$ s.t. $T(y) < T(x_0)$ $\forall y \in \overline{Y} \Rightarrow T(y) = 0 \ \forall y \in \overline{Y}$

$\square$

Thm. (Banach-Steinhaus Thm. - Uniform Boundedness):
Let $(X, \|\cdot\|)$ be a Banach-Space, $\{T_k\}_{k \in \mathbb{N}} \subset X'$ s.t.

$$\sup\{|T_k(x)| : k \in \mathbb{N}\} < +\infty \quad \forall x \in X$$

("T_k is bounded pointwise")

Then:

$$\sup\{\|T_k\|_{X'} : k \in \mathbb{N}\} < +\infty$$

Lemma (Baire):
Let (X, d) be a complete metric space, $\{F_k\}_{k \in \mathbb{N}}$ a sequence of closed sets with empty interiors. Then the union $\bigcup_{k=1}^{+\infty} F_k$ has empty interior.

$\Rightarrow$ **Application:**
$(X, \|\cdot\|)$ Banach, $\{F_k\}_{k \in \mathbb{N}}$ closed in X s.t. $\bigcup_{k=1}^{+\infty} F_k = X$
$\Rightarrow \exists \overline{k}$ s.t. $\mathring{F}_{\overline{k}} \neq \phi$

Proof (Baire's Lemma):
Take $\Omega \subset X$, Ω open, $\Omega \neq \phi$. We show that:
$\Omega \setminus \left(\bigcup_{k=1}^{+\infty} F_k\right) \neq \phi$:

$\Omega \setminus F_1 \neq \phi$, open $\Rightarrow \exists x_1, r_1 < 1$ s.t. $\overline{B_{r_1}(x_1)} \subset (\Omega \setminus F_1)$
$\Rightarrow B_{r_1}(x_1) \setminus F_2 \neq \phi$, open $\Rightarrow \exists x_2, r_2 < \frac{1}{2}$ s.t. $\overline{B_{r_2}(x_2)} \subset (B_{r_1}(x_1) \setminus F_2)$

$\Rightarrow B_{\mu_2}(x_2) \setminus F_3 \neq \phi$, open $\Rightarrow \exists\, x_3, \mu_3 < \frac{1}{3}$ s.t. . . . . .

$\Rightarrow B_{\mu_{k-1}}(x_{k-1}) \setminus F_k \neq \phi$, open $\Rightarrow \exists\, \mu_k, x_k < \frac{1}{k}$ s.t.

$$\overline{B_{\mu_k}(x_k)} \subset B_{\mu_{k-1}}(x_{k-1}) \setminus F_k$$

Then $\{x_k\}_k$ is a Cauchy Sequence $\Rightarrow x \xrightarrow{d} \bar{x} \in \Omega \setminus \left(\bigcup_{k=1}^{+\infty} F_k\right)$

□

Proof (Banach-Steinhaus Thm.):

Consider, for $n = 1, 2, 3, \dots$, the closed sets:

$$F_n = \{x \in X : |T_k(x)| < n \ \ \forall k \in \mathbb{N}\}$$

$\Rightarrow F_n$ is closed $\forall n \in \mathbb{N}$ and $\bigcup_{n=1}^{+\infty} F_n = X \Rightarrow$ (Baire's Lemma) $\exists\, \bar{n}$ s.t. $\mathring{F}_n \neq \phi \Rightarrow \exists\, \bar{x}, r > 0$ s.t. $B_r(\bar{x}) \subset F_{\bar{n}}$. Let $x \in X$ be s.t. $\|x\| \leq 1$, then:

$$|T_k(x)| = \left|\frac{2}{r} T_k\left(\frac{r}{2} x\right)\right| = \frac{2}{r}\left|T\big(\underbrace{\tfrac{r}{2}x + \bar{x}}_{\in B_n(\bar{x})}\big) - T(\bar{x})\right|$$

$$\leq \frac{2}{r}\Big(\underbrace{|T(\bar{x} + \tfrac{r}{2}x)|}_{\leq \bar{n}} + \underbrace{|T(\bar{x})|}_{\leq \bar{n}}\Big) \leq \frac{4}{r}\bar{n}$$

$\Rightarrow \|T_k\|_{X'} \leq \frac{4}{r}\bar{n}$

□

Bidual Space - Reflexive Spaces

Consider the Bidual space of X:

$(X, \|\cdot\|)$ normed space $\longrightarrow (X', \|\cdot\|_{X'})$ dual space $\longrightarrow (X'', \|\cdot\|_{X''})$ bidual

Claim:

$\exists\, J : X \to X''$ isometric injection with $J(x) = S_x$, $S_x : X' \to \mathbb{R}$, $S_x(\underbrace{T}_{\in X'}) = T(x)$

$$\|S_x\|_{X''} := \sup\{|S_x(T)| : T \in X', \|T\|_{X'} \leq 1\} = \sup\{|T(x)| : T \in X', \|T\|_{X'} \leq 1\} \overset{!}{=} \|x\|$$

Def. (Reflexive Space):

Given $(X, \|\cdot\|)$ Banach space, it is called a Reflexive Space if $J(X) = X''$

N.B. L^p is reflective $\forall p \in (1, +\infty)$, L^1, L^∞ are not reflexive

Proposition:

Given $(X, \|\cdot\|)$ a normed space, $\{x_k\}_{k\in\mathbb{N}} \subset X$, we have:

$\{x_k\}_k$ is bounded in norm $\Leftrightarrow \{T(x_k)\}_k$ is bounded in $\mathbb{R}$ $\forall T \in X'$

Corollary (trivial):

Given $(X, \|\cdot\|)$ a normed space, $A \subset X$, we have:

$$A \text{ is bounded in } X \Leftrightarrow T(A) \text{ is bounded in } \mathbb{R} \ \forall T \in X'$$

Proof (Proposition):

$(\Rightarrow)$: $\|x_k\| \leqslant C \ \forall k \in \mathbb{N}$, $|T(x_k)| \leqslant \|T\|_{X'} \cdot \|x_k\| \leqslant \|T\| \cdot C$

$(\Leftarrow)$: $T(x_k) = S_{x_k}(T)$, $\{S_{x_k}\}_k \subset X'' \Rightarrow S_{x_k}(T)$ is bounded

$\Rightarrow S_{x_k}$ is bounded pointwise $\Rightarrow \|S_{x_k}\|_{X''} = \|x_k\|$ is bounded

$\square$

Corollary (of Banach-Steinhaus)

Let $(X, \|\cdot\|)$ be a Banach space, $\{T_k\}_{k\in\mathbb{N}} \subset X'$ s.t. $\forall x \in X$ $\exists \lim_{k\to+\infty} T_k(x) =: T(x) \in \mathbb{R}$ (N.B. $T(x): X \to \mathbb{R}$ is clearly linear) then $\{T_k(x)\}_k$ is bounded in $\mathbb{R}$ $\forall x \in X \Rightarrow$ by Banach-Steinhaus $\|T_k\|_{X'} \leqslant C \wedge |T_k(x)| \leqslant \|T_k\| \cdot \|x\| \leqslant C\|x\| \wedge |T_k(x)| \to |T(x)|$

Ex.:

$$\|T\|_{X'} \leqslant \liminf_{k\to+\infty} \|T_k\|$$

Application of Banach-Steinhaus to Fourier Series

$f: \mathbb{R} \to \mathbb{R}$ 2π-periodic

$\Rightarrow$ Its Fourier series is $S(x) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos kx + b_k \sin kx$ with

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt \, dt, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt \, dt$$

If $f \in C^1$ then $S(x)$ converges uniformly to f!

Claim:

$\exists f \in C^0$ 2π-periodic s.t. $S(x)$ does not converge pointwise at 0

Proof:

Let $C^0(2\pi) = \{f: \mathbb{R} \to \mathbb{R} \text{ continuous, } 2\pi\text{-periodic}\}$ with $\|\cdot\|_\infty$ be a normed space. Define $S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\frac{1}{2}\right)(y-x)\right) \cdot \frac{1}{2\sin\left(\frac{y-x}{2}\right)} f(y)\,dy$ (Dirichlet's Formula).

We have:

$$S_N(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right)\cdot \frac{1}{2\sin\left(\frac{y}{2}\right)} f(y)\,dy =: T_N(f)$$

$\Rightarrow T_N : C^0(2\pi) \to \mathbb{R} \Rightarrow$ we call $g_N(y) := \sin\left(\left(N+\tfrac{1}{2}\right)y\right)\cdot \frac{1}{2\sin\left(\frac{y}{2}\right)}$ the DIRICHLET'S KERNEL and we have:

$$\|T_N\|_{(C^0(2\pi))'} = \|g_N\|_{L^1((-\pi,\pi))} \xrightarrow{N\to+\infty} +\infty$$

$\Rightarrow$ Then $S_N(0)$ doesn't converge to $f(0)$ $\forall f \in C^0(2\pi)$, otherwise T_N would be bounded pointwise and therefore bounded in norm by Banach-Steinhaus.

$\square$

OPEN MAPPING THEOREM

Thm. (Open Mapping):
Given $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ Banach spaces, $T : X \to Y$ linear, continuous and surjective. Then T is open (the image of any open set of X through T is open in Y). If T is also injective, then T^{-1} is continuous.

Proof:
It is enough to show that $\underbrace{T(\underset{\subseteq X}{B_1(0)})}_{\subseteq Y} \supset \underset{\subseteq Y}{B_r(0)}$ for some $r>0$

CLAIM:
$\overline{T(B_1(0))} \supset B_{2r}(0)$ for some $r>0$

Indeed,
$\Rightarrow \bigcup_{u=1}^{+\infty} \overline{T(B_u(0))} = Y \Rightarrow \exists\, \bar{u} \in \mathbb{N}$ s.t. $\mathring{\overline{T(B_{\bar{u}}(0))}} \neq \emptyset$ by Baire's Lemma. $\Rightarrow \mathring{\overline{T(B_1(0))}} \neq \emptyset$ and it is convex and symmetric WRT 0
$\Rightarrow 0 \in \mathring{\overline{T(B_1(0))}}$ ✓

We now work in order to remove the closure from the hypothesis

CLAIM:
$T(B_1(0)) \supset B_r(0)$ for some $r>0$

Indeed, we show that if $y \in B_r(0)$ then $\exists\, \bar{x} \in B_1(0)$ s.t. $T(\bar{x}) = y$ ($\Leftrightarrow \|y\| < r$). In particular:

$\overline{T(B_{\frac{1}{2}}(0))} \supset B_r(0)$ $\left(\Rightarrow \overline{T(B_{1/2^k}(0))} \supset B_{r/2^{k-1}}(0)\right)$

$\Rightarrow \exists\, z_1 \in B_{\frac{1}{2}}(0)$ s.t. $\|y - T(z_1)\| < r/2$

$\Rightarrow \exists\, z_2 \in B_{\frac{1}{4}}(0)$ s.t. $\|y - T(z_1) - T(z_2)\| < \frac{r}{4}$

$\Rightarrow \exists z_3 \in B_{\frac{1}{8}}(0)$ s.t. $\| y - T(z_1) - T(z_2) - T(z_3) \| < \frac{\nu}{8}$ etc.

$\Rightarrow \exists z_k \in B_{\frac{1}{2^k}}(0)$ s.t. $\| y - \sum_{i=1}^{k} T(z_i) \| < \frac{\nu}{2^k}$

$\Rightarrow \| y - \sum_{i=1}^{k} T(z_i) \| = \| y - T(\sum_{i=1}^{k} z_i) \| \wedge \sum_{k=1}^{+\infty} z_k$ converges because $\sum_{k=1}^{+\infty} \| z_k \| < +\infty$ (the sequence of the partial sums is Cauchy)

$\Rightarrow \bar{x} = \sum_{k=1}^{+\infty} z_k \Rightarrow \| y - T(\bar{x}) \| = 0 \Rightarrow y = T(\bar{x}) \wedge \| \bar{x} \| \leqslant \sum_{k=1}^{+\infty} \| z_k \| < \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1$

$\square$

<u>Corollary</u>

Given X a vector space, $\|\cdot\|$, $|||\cdot|||$ Banach norms on X s.t. $\exists C > 0$ s.t. $|||x||| \leqslant C \|x\| \; \forall x \in X$, then $|||\cdot|||$, $\|\cdot\|$ are equivalent

<u>Proof</u>

We apply the open mapping thm. to $Id: (X, \|\cdot\|) \to (X, |||\cdot|||)$

$\Rightarrow$ the inverse is continuous $\wedge \exists B > 0$ s.t. $\|x\| \leqslant B |||x||| \; \forall x \in X$

$\square$