Integrals (continued)

Thanks to the Beppo-Levi theorem we have:

$f, g : X \to [0, +\infty]$ μ meas. $\Rightarrow \int_X f + g \, d\mu(x) = \int_X f \, d\mu(x) + \int_X g \, d\mu(x)$

<u>Thm.</u> (Fatou's Lemma):

Let $f_k : X \to [0, +\infty]$ μ-meas. functions. Then:

$$\int_X \liminf_{k \to +\infty} f_k(x) \, d\mu(x) \leq \liminf_{k \to +\infty} \int_X f_k(x) \, d\mu(x)$$

Let now $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be μ-meas., then we define $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = -\min\{f(x), 0\}$. We have:

$$f(x) = f^+(x) - f^-(x) \quad \text{and} \quad |f(x)| = f^+(x) + f^-(x)$$

By definition we also have $\int_X f(x) \, d\mu(x) = \int_X f^+(x) \, d\mu(x) - \int_X f^-(x) \, d\mu(x)$

If $\int_X f(x) \, d\mu(x) < +\infty$, then f is called SUMMABLE.

If at least one of the 2 integrals $\int_X f^+(x) \, d\mu(x)$, $\int_X f^-(x) \, d\mu(x)$ is finite, then f is said to be INTEGRABLE

<u>Thm.</u> (Lebesgue's Dominated Convergence):

Let $f_k : X \to \overline{\mathbb{R}}$ μ-meas. be s.t.:

1) $\exists \, \varphi : X \to [0, +\infty]$ summable s.t.

$$|f_k(x)| \leq \varphi(x) \quad \text{for a.e. } x \in X, \ \forall k \in \mathbb{N}$$

2) for a.e. $x \in X$ $\exists \lim_{k \to +\infty} f_k(x) := f(x)$

Then $\lim_{k \to +\infty} \int_X |f_k(x) - f(x)| \, d\mu(x) = 0$ and $\int_X \lim_{k \to +\infty} f_k(x) \, d\mu(x) = \int_X f(x) \, d\mu(x)$

<u>Remark</u>

There are cases in which the Riemann int. $\exists$ and the Lebesgue int. $\nexists$!!!

e.g.:

$\int_0^{+\infty} \frac{\sin x}{x} \, dx$ $\exists$ finite as a generalized Riemann int.,
$\nexists$ as a Lebesgue int. because:

$$\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^- dx = \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^+ dx = +\infty$$

N.B.

It is obviously possible to integrate on a subset of X !!!

e.g.:

$f : A \to \overline{\mathbb{R}}$ μ-meas., $\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$

$\Rightarrow \int_A f(x)\,d\mu(x) := \int_X \tilde{f}(x)\,d\mu(x)$

Thm. (Fubini Theorem for multiple integrals):
Let $f : \mathbb{R}^{k+n} \to \overline{\mathbb{R}}$ Lebesgue meas.. If $f \geq 0$ or if f is summable then we have:

$$\int_{\mathbb{R}^{k+n}} f(x,y)\,\underbrace{dx\,dy}_{\substack{(k+n)\text{-dim.} \\ \text{Lebesgue meas.}}} = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^n} f(x,y)\,dx \right) dy$$

TOPOLOGICAL VECTOR SPACES

Let $L^p(\mu) = \{ f : X \to \overline{\mathbb{R}} : \int_X |f(x)|^p d\mu(x) < +\infty \}$ with $p \in [1, +\infty]$, μ outer meas. on X. $L^p(\mu)$ is called the Lebesgue Space

Def. (Topological Vector Space):
Let X be a vector spaces over $\mathbb{R}$ (or over $\mathbb{C}$). Let τ be a topology over X as a set (in the usual sense).
(X, τ) is a TOPOLOGICAL VECTOR SPACE if vector space operators are continuous for the topology τ

N.B.
The 2 operators on vector spaces are:

$$+ : \underset{\tau \times \tau}{X \times X} \to \underset{\tau}{X}, \quad \cdot : \underset{e \times \tau}{\mathbb{R} \times X} \to \underset{\tau}{X} \quad (e = \text{natural topology})$$

Thm (Characterization of Continuous Functions):
Given (X, ρ), (Y, τ) metric spaces, $f : X \to Y$, $x \in X$, f is continuous at x iff $\forall \{x_n\}_{n=1}^{\infty} \subset X$ s.t. $x_n \to x$ we have that $f(x_n) \to f(x)$. (Remind that $\{f(x_n)\}_{n=1}^{\infty} \subset Y$!!!!)
