Abstract Fourier Series in Hilbert Spaces

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\{e_\alpha\}_{\alpha \in I} \subset X$ an orthonormal family. Is it true that $\forall x \in X \quad x = \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha$? Yes, provided that $\{e_\alpha\}_{\alpha \in I}$ is maximal.

In every Hilbert space there are several maximal orthonormal families !!!

Infinite Sums:

Let $\{a_\alpha\}_{\alpha \in I}$ be a family of real numbers. We define:

$$\sum_{\alpha \in I} |a_\alpha| := \int_I a_\alpha \, d\underbrace{\#}_{\text{counting measure}}(\alpha) = \sup\left\{\sum_{\alpha \in I'} |a_\alpha| : I' \subset I,\ I' \text{ finite}\right\}$$

Remark:

If $\sum_{\alpha \in I} |a_\alpha| < +\infty$ then $\{\alpha \in I : a_\alpha \neq 0\}$ is at most countable.

Indeed $\forall n \in \mathbb{N}, n \geq 1 \quad \{\alpha \in I : |a_n| \geq \frac{1}{n}\}$ is finite

Def. ($\ell^2(I)$):

Let I be a (possibly uncountable) set. We define the space:

$$\ell^2(I) := \left\{\{c_\alpha\}_{\alpha \in I} : \sum_{\alpha \in I} c_\alpha^2 < +\infty\right\}$$

with norm: then to

$$\|\{c_\alpha\}\|_{\ell^2(I)} := \sqrt{\sum_{\alpha \in I} c_\alpha^2}$$

and scalar product:

$$\langle \{a_\alpha\}_{\alpha \in I}, \{b_\alpha\}_{\alpha \in I} \rangle := \sum_{\alpha \in I} a_\alpha b_\alpha$$

$\ell^2(I)$ is a Hilbert space and every Hilbert space is isomorphic and isometric to it. If an Hilbert space is separable (i.e. it has a countable dense subset) it is isometrically isomorphic to ℓ^2.

Def. (Fourier Coefficients):

Let $(X, \langle \cdot, \cdot \rangle)$ be an Hilbert space, $\{e_\alpha\}_{\alpha \in I}$ an orthonormal family in X. We call the FOURIER COEFFICIENTS of $x \in X$ the numbers $\langle x, e_\alpha \rangle$

Proposition (Bessel's Inequality):

$\{\langle x, e_\alpha \rangle\}_{\alpha \in I} \in \ell^2(I)$ and $\sum_{\alpha \in I} \langle x, e_\alpha \rangle^2 \leq \|x\|^2$

Proof:

Let $J \subset I$, J finite and consider $Y = \langle e_\alpha \rangle_{\alpha \in J}$. The orthogonal projection $p: X \longrightarrow Y$ is given by $p(x) = \sum_{\alpha \in J} \langle x, e_\alpha \rangle e_\alpha$. We have:

$$\sum_{\alpha \in J} \langle x, e_\alpha \rangle^2 = \|p(x)\|^2 \underset{\uparrow}{\leqslant} \|x\|^2$$

orthogonal decomposition of Y

$\square$

Thm. (Riesz - Fischer):

Given $X, \{e_\alpha\}_{\alpha \in I}$ as above, we have:

$$\forall \{c_\alpha\}_{\alpha \in I} \in \ell^2(I) \ \exists x \in X \text{ s.t. } \langle x, e_\alpha \rangle = c_\alpha \ \forall \alpha \in I$$

In other words, the map

$$\begin{aligned} \phi: X &\longrightarrow \ell^2(I) \\ x &\longmapsto \{\langle x, e_\alpha \rangle\}_{\alpha \in I} \end{aligned}$$

is surjective.

Proof:

Let $\{c_\alpha\}_{\alpha \in I} \in \ell^2(I)$, then the set $I' = \{\alpha \in I : c_\alpha \neq 0\}$ is at most countable, so we can enumerate it: $I' = \{\alpha_k : k = 1, \dots\}$. Idea:

$$x \overset{?}{=} \left(\sum_{\alpha \in I'} c_\alpha e_\alpha \right) \overset{?}{=} \sum_{k=1}^{+\infty} c_{\alpha_k} e_{\alpha_k}$$

$\Rightarrow x_N = \sum_{k=1}^{N} c_{\alpha_k} e_{\alpha_k}$ is Cauchy:

$$\|x_{N+H} - x_N\|^2 = \left\|\sum_{k=N+1}^{N+H} c_{\alpha_k} e_{\alpha_k}\right\|^2 = \sum_{k=N+1}^{N+H} c_{\alpha_k}^2 \leqslant \underbrace{\sum_{k=N+1}^{+\infty} c_{\alpha_k}^2}_{\substack{= N\text{-th remainder} \\ \text{of a convergent} \\ \text{series}}} \xrightarrow{N \to +\infty} 0$$

Compute:

$$\langle x, e_{\alpha_j} \rangle = \lim_{N \to +\infty} \underbrace{\langle x_N, e_{\alpha_j} \rangle}_{= c_{\alpha_j} \text{ if } N \geqslant j} = c_{\alpha_j},$$

$$\underset{\alpha \in I \setminus I'}{\langle x, e_\alpha \rangle} = \lim_{N \to +\infty} \langle x_N, e_\alpha \rangle = 0 = c_\alpha$$

$\square$