FUNDAMENTAL THM. OF ABSTRACT FOURIER SERIES

Thm. (Fundamental Thm. of Abstract Fourier Series):
Given $(X, \langle \cdot, \cdot \rangle)$ Hilbert, $\{e_\alpha\}_{\alpha \in I}$ orthonormal family, the following are equivalent:

1) $\{e_\alpha\}_{\alpha \in I}$ is maximal
2) $\langle e_\alpha \rangle_{\alpha \in I}$ is dense in X
3) $\forall x \in X \quad \|x\|^2 = \sum_{\alpha \in I} \langle x, e_\alpha \rangle^2$
(Parseval's equality)

Proof:

1) $\Rightarrow$ 2):

Suppose that $\langle e_\alpha \rangle_{\alpha \in I}$ is not dense in X, then

$$Y = \overline{\langle e_\alpha \rangle_{\alpha \in I}} \subsetneq X$$

$\Rightarrow$ let $x_0 \in X \setminus Y$, $p(x_0)$ its orthogonal projection on Y.

$$\Rightarrow \langle x_0 - p(x_0), y \rangle = 0 \quad \forall y \in Y$$
$$\Rightarrow \langle \underbrace{x_0 - p(x_0)}_{\neq 0}, e_\alpha \rangle = 0 \quad \forall \alpha \in I$$

$\Rightarrow \dfrac{x_0 - p(x_0)}{\|x_0 - p(x_0)\|}$ can be added to $\{e_\alpha\}_{\alpha \in I}$ and it gives a larger orthonormal family $\lightning$

2) $\Rightarrow$ 3):

$\langle e_\alpha \rangle_{\alpha \in I}$ is dense in $X \Rightarrow x \in X$, $\varepsilon > 0$ $\exists \alpha_1, \dots, \alpha_N \in I$, $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ s.t. $\|x - \lambda_1 e_{\alpha_1} - \dots - \lambda_N e_{\alpha_N}\|^2 \leq \varepsilon$

$\Rightarrow Y = \langle e_{\alpha_1}, \dots, e_{\alpha_N} \rangle$, consider $p(x)$ the orthogonal projection on Y. Then $\|x - p(x)\|^2 \leq \varepsilon$

$$\|x - p(x)\|^2 = \|x\|^2 - \|p(x)\|^2 = \|x\|^2 - \sum_{k=1}^{N} \langle x, e_{\alpha_k} \rangle^2$$

$$\Rightarrow \|x\|^2 - \varepsilon \leq \sum_{k=1}^{N} \langle x, e_\alpha \rangle^2 \leq \sum_{\alpha \in I} \langle x, e_\alpha \rangle^2 \leq \|x\|^2$$

3) $\Rightarrow$ 1):

Let $x \in X$ be s.t. $\langle x, e_\alpha \rangle = 0 \ \forall \alpha \in I$. Then:

$$\|x\|^2 = \sum_{\alpha \in I} \langle x, e_\alpha \rangle^2 = 0 \Rightarrow x = 0$$

$\square$

Remarks:

1) If $\{e_\alpha\}_{\alpha\in I}$ is a maximal orthonormal family, then:

$$\forall x \in X \qquad x = \sum_{\alpha\in I} \langle x, e_\alpha \rangle e_\alpha$$

2) $(X, \langle\cdot,\cdot\rangle)$ is separable (i.e. it has a countable dense subset) $\Leftrightarrow$ X is an at most countable maximal orthonormal family.

Proof of ②:

$\Leftarrow$: $\exists \{e_k\}_{k\in\mathbb{N}} \subset X$ maximal orthonormal family $\Rightarrow \langle e_k \rangle_{k\in\mathbb{N}}$ is dense in X $\Rightarrow$ rational linear combination of the e_k are dense in X.

$\Rightarrow$: Let $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a dense subset, and apply the Gram-Schmidt algoritm.

$\square$

Def. (Hilbert Basis):

A maximal orthonormal family is called a HILBERT BASIS of the Hilbert space X

Remark:

If the orthonormal family $\{e_\alpha\}_{\alpha\in I}$ is NOT maximal then the Fourier Series $\sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha$ converges to $p(x)$, where p is the orthonormal projection on $Y = \overline{\langle e_\alpha \rangle_{\alpha\in I}}$

Indeed, $\{e_\alpha\}_{\alpha\in I}$ is (by definition) maximal in Y (which is Hilbert). Moreover:

$$X = Y \oplus Y^\perp \Rightarrow x = \underbrace{p(x)}_{\in Y} + \underbrace{(x - p(x))}_{\in Y^\perp}$$