Sobolev Spaces

Proposition (Fundamental Lemma of Calculus of Variations):
Let $u \in L^1([a,b])$ s.t. $\int_a^b u(x)\varphi(x)dx = 0 \ \forall \varphi \in C_c^0((a,b))$,
then $u(x) = 0$ a.e.

Proof:
If we could take $\varphi(x) = \operatorname{sgn} u(x)$ we would get $\int_a^b |u(x)|dx = 0$
$\Rightarrow u(x) = 0$ a.e. We can't, but we can regularize such φ by convolution:

$$\exists \varphi_n \in C_c^0([a,b]) \text{ s.t. } \varphi_n \to \varphi(x) = \operatorname{sgn} u(x) \wedge -1 \leq \varphi_n \leq 1$$

$$\Rightarrow 0 = \int_a^b u(x)\varphi_n(x)dx \xrightarrow[n\to+\infty]{} \int_a^b |u(x)|dx$$

$\square$

Def. (Weak Derivative):
Let $u \in L^1(a,b)$. A WEAK DERIVATIVE of u is a function
$v \in L^1(a,b)$ s.t. $\int_a^b u(x)\varphi'(x)dx = -\int_a^b v(x)\varphi(x)dx \ \forall \varphi \in C_c^1((a,b))$

Remark:
$u \in C^1([a,b]) \Rightarrow u'$ is a weak derivative for u:

$$\int_a^b u(x)\varphi'(x)dx = \left[u(x)\varphi(x)\right]_a^b - \int_a^b u'(x)\varphi(x)dx$$

Remark:
If $u \in L^1(a,b)$ has a weak derivative $v \in L^1(a,b)$, then v is unique: indeed, let $\tilde{v} \in L^1(a,b)$ be another weak derivative for u. Then:

$$\int_a^b u\varphi' dx = -\int_a^b v\varphi\, dx = -\int_a^b \tilde{v}\varphi\, dx \Rightarrow \int_a^b (v-\tilde{v})\varphi\, dx = 0$$
$$\forall \varphi \in C_c^1([a,b])$$

$\Rightarrow$ by the Fundamental lemma $v(x) - \tilde{v}(x) = 0$ a.e.

Def (Sobolev Spaces (dim 1)):
We define the SOBOLEV SPACES as the following spaces:

$$W^{1,p}([a,b]) = \{u \in L^p([a,b]) : \exists u' \in L^p([a,b]) \text{ weak derivative}\}$$

where $p \in [1,+\infty]$

$\Rightarrow W^{1,p}$ is Banach with the norm

$$\|u\|_{W^{1,p}([a,b])} = \|u\|_{L^p([a,b])} + \|u'\|_{L^p([a,b])}$$

or, equivalently:

$$\|u\|_{W^{1,p}([a,b])} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{\frac{1}{p}}$$

We prove now that $W^{1,p}([a,b])$ is Banach:
Let $\{u_n\}_n \subset W^{1,p}([a,b])$ be a Cauchy sequence. Then $\{u_n\}_n, \{u_n'\}_n$ are Cauchy sequences in $L^p([a,b])$

$$\Rightarrow u_n \xrightarrow[L^p]{} u \wedge u_n' \xrightarrow[L^p]{} v \Rightarrow \int_a^b u_n(x)\varphi'(x)\,dx = -\int_a^b u_n'(x)\varphi(x)\,dx$$

$$\longrightarrow \int_a^b u(x)\varphi'(x)\,dx \qquad \longrightarrow -\int_a^b v(x)\varphi(x)\,dx$$

$\Rightarrow v = u' \Rightarrow$ by definition of $\|u\|_{W^{1,p}([a,b])}$ we conclude.

N.B.
If $p = 2$, $\underset{W^{1,2}}{\langle u,} \underset{W^{1,2}}{v\rangle} = \int_a^b (u(x)v(x) + u'(x)v'(x))\,dx$.

(It is quite common to denote $W^{1,2} = H^1$)

Def. (Absolutely Continuous Functions (Tonelli)):
A function $u: [a,b] \to \mathbb{R}$ is ABSOLUTELY CONTINUOUS if

$$\exists v \in L^1((a,b)) \text{ s.t. } u(x) = u(a) + \int_a^x v(t)\,dt$$

Thm.:
If $u: [a,b] \to \mathbb{R}$ is absolutely continuous then it is differentiable for a.e. $x \in [a,b]$ and $\forall x \in [a,b]$

$$u(x) = u(a) + \int_a^x u'(x)\,dx, \quad u' \in L^1$$

We write then that $u \in AC([a,b])$.

Proposition (Characterization of $AC([a,b])$ - Tonelli):
$u \in AC([a,b])$ iff the following holds:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall$ finite collection of intervals $[a_k, b_k]$, $k = 1, \dots, N$ pairwise disjoint s.t. $\sum_{k=1}^N (b_k - a_k) < \delta$ then $\sum_{k=1}^N |u(b_k) - u(a_k)| < \varepsilon$.

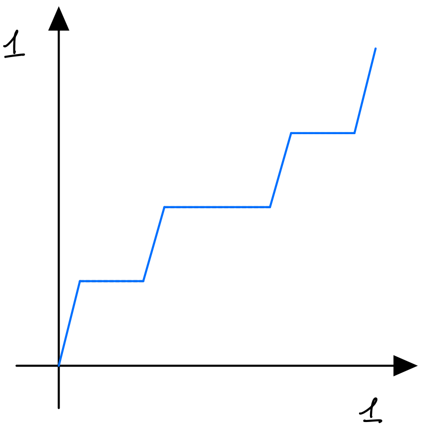
N.B.
u Lipschitz continuous $\Rightarrow u$ absolutely continuous.

Remark:

$u \in AC([a,b])$ is a much stronger condition than u being differentiable a.e..

Example (Cantor):

$\exists\, u:[0,1] \to \mathbb{R}$ increasing, continuous, s.t. $u(0)=1$, $u(1)=1$, differentiable a.e. with $u'(x)=0$ a.e. (Cantor Function or Devil's Staircase)



Thm.:

If $u \in AC([a,b])$, then $u \in W^{1,1}([a,b])$ and the weak derivative is the usual derivative. Converseley, if $u \in W^{1,1}([a,b])$ $\exists\, \tilde{u} \in AC([a,b])$ s.t. $\tilde{u}(x) = u(x)$ and

$u'(x) = \tilde{u}'(x)$ for a.e. $x \in [a,b]$

(WEAK) (POINTWISE)

N.B.

$u, v \in AC([a,b]) \Rightarrow u \cdot v \in AC([a,b])$,

$u \in C^1([a,b]) \Rightarrow$ (trivially) $u \cdot v \in AC([a,b])$

Proof:

1) $u \in AC([a,b])$, u' pointwise derivative. Consider $\phi \in C_0^1([a,b])$ and take $u\phi \in AC$:

$$\Rightarrow (u\phi)'(x) = u'(x)\phi(x) + u(x)\phi'(x)$$

$$0 = \int_a^b (u\phi)'(x)\,dx = \int_a^b (u'\phi + u\phi')\,dx$$

Fund. Thm. of Calculus

2) We use the following:

Lemma (Du Bois-Reymond):

Let $u \in W^{1,1}([a,b])$ s.t. the weak derivative $u'=0$ a.e. Then $\exists\, c \in \mathbb{R}$ s.t. $u = c$ a.e.

Proof:

$f \in C^0([a,b])$, $w(x) := f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds$. Define also $\phi(x) := \int_a^x w(t)\,dt \in C^1$. Then $0 = \phi(a) = \phi(b)$ so $\phi \in C_0^1([a,b])$

$\Rightarrow \int_a^b u(x)\phi'(x)\,dx = 0 \Rightarrow \int_a^b u(x)\left[f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dx = 0$

$\Rightarrow \int_a^b u(x)f(x)\,dx - \frac{1}{b-a}\int_a^b u(s)\,ds \int_a^b f(x)\,dx$

$= \int_a^b \left[u(x) - \frac{1}{b-a}\int_a^b u(s)\,ds\right] f(x)\,dx = 0$ and we conclude by the Fundamental Lemma of Calculus of Variations

$\square$

Define now $w(x) = \int_a^x u'(t)\,dt$. Then $w \in AC([a,b])$ and u' is its weak derivative $\Rightarrow u - w \in W^{1,1}([a,b])$ and $(u-w)' = 0$

$\Rightarrow$ by Du Bois-Raymond, $u(x) - w(x) = c \in \mathbb{R}$ for a.e. x

$\Rightarrow u(x) = \underbrace{w(x) + c}_{\tilde{u}(x)}$ for a.e. $x \in [a,b]$

$\square$