Monomorphisms - Epimorphisms - Isomorphisms

Def. (Monomorphism):
A morphism $f: X \to Y$ in a category $\mathcal{C}$ is a Monomorphism iff $\forall Z \in Ob(\mathcal{C})$, $\forall p, q \in Hom_{\mathcal{C}}(Z, X)$ we have:

$$f \circ p = f \circ q \Rightarrow p = q$$

Def. (Epimorphism):
A morphism $f: X \to Y$ in a category $\mathcal{C}$ is an Epimorphism iff $\forall Z \in Ob(\mathcal{C})$, $\forall p, q \in Hom_{\mathcal{C}}(X, Z)$ we have:

$$p \circ f = p \circ f \Rightarrow p = q$$

Def. (Isomorphism):
A morphism $f: X \to Y$ in a category $\mathcal{C}$ is an Isomorphism iff $\exists Y \in Hom_{\mathcal{C}}(Y, X)$ s.t.:

$$f \circ g = id_Y \wedge g \circ f = id_X$$

Examples:

1) In Set, the monomorphisms are exactly the injective maps, while the epimorphisms are exactly the surjective maps.

Proof:
Let f be injective $\Rightarrow (\forall x, x' \in X \; f(x) = f(x') \Rightarrow x = x')$
$\Rightarrow$ let $p, q: Z \to X$ s.t. $f \circ p = f \circ q \Rightarrow$ we need to show that $\forall z \in Z$ $p(z) = q(z)$:
$\Rightarrow f \circ p = f \circ q \Rightarrow (f \circ p)(z) = (f \circ q)(z) \Rightarrow f(p(z)) = f(q(z))$
$\Rightarrow p(z) = q(z)$ (f is injective)

$\square$

2) The same as above holds for Vect($\mathbb{R}$), Group category, Monoid category, but not (for instance) in the Pokémon category

3) In the Pokémon category all morphisms are mono- and epimorphisms because it is thin (i.e. parallel morphisms are equal)

4) In Set, isomorphisms are exactly the bijective maps

and the same holds for Vect($\mathbb{R}$), Group category, Monoid category

5) In the Pokémon category, only the identities are isomorphisms

6) Consider the Divisibility Category which has the integers $\mathbb{Z}$ as objects and the following morphisms:

$$a \to b \iff b \text{ is integral multiple of } a$$

The isomorphisms are precisely the identities and all the morphisms of the form $x \to -x$

<u>Def.</u> (Isomorphic Objects):
Objects X, Y are ISOMORPHIC iff $\exists$ isomorphism between them. We write then $X \cong Y$

<u>Examples</u>:
1) In the Divisibility Category 2 objects are isomorphic iff they agree in absolute value:

$$x \cong y \iff |x| = |y|$$

2) In Set, $X \cong Y \iff X, Y$ contain the same number of elements
3) In Vect($\mathbb{R}$), $V \cong W \iff \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} W$

OPPOSITE CATEGORY - DUALITY

<u>Def.</u> (Opposite Category):
Let $\mathcal{C}$ be a category. Its OPPOSITE CATEGORY $\mathcal{C}^{op}$ is the following category:

1) $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$
2) $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X) \quad \forall X, Y \in Ob(\mathcal{C}^{op})$
3) Identity: inherited from $\mathcal{C}$
4) Composition rule: inherited from $\mathcal{C}$ AND with the arguments swapped !!! ($f \circ g$ in $\mathcal{C} \Rightarrow g \circ f$ in $\mathcal{C}^{op}$)

<u>Example</u>:
The map $\mathbb{R}^{>0} \xrightarrow{f} \mathbb{R}$ s.t. $f(x) = \sqrt{x}$ is a morphism $\mathbb{R} \to \mathbb{R}^{>0}$ in Set^{op}

<u>Proposition</u> (Duality of Initial - Terminal Objects):
Let $\mathcal{C}$ be a category, $I \in Ob(\mathcal{C})$. Then we have:

$$I \text{ is initial in } \mathcal{C} \iff I \text{ is terminal in } \mathcal{C}^{op}$$

Proposition (Duality of Epimorphisms - Monomorphisms):
Let $\mathcal{C}$ be a category. Then a morphism in $\mathcal{C}$ is monic (i.e. monomorphism) in $\mathcal{C}$ iff it is epic (i.e. epimorphism) in $\mathcal{C}^{op}$

Remark:
The Duality concept is incredibly deep and a fundamental one in Category Theory: for instance, the fact that an entity has a dual means that if we prove something which involves the entity then we also automatically prove something which involves its dual! Therefore, any property can be extended by duality!

Example:
The set theoretic map P: set of persons $\to$ set of strings
$p \mapsto$ name of p
is a morphism $P \to S$ in Set
and a morphism $S \to P$ in Set^{op}

N.B.
$g \circ f$ in $\mathcal{C}^{op} := f \circ g$ in $\mathcal{C}$, but the 2 $\circ$ ARE NOT IDENTICALLY the same!!!

Examples (Duality):

injectivity	$\xleftrightarrow{\text{dual}}$	surjectivity
$\leqslant$	$\xleftrightarrow{\text{dual}}$	$\geqslant$
gcd	$\xleftrightarrow{\text{dual}}$	lcm
$\cap$	$\xleftrightarrow{\text{dual}}$	$\cup$
$\{x\}$	$\xleftrightarrow{\text{dual}}$	$\emptyset$
subset	$\xleftrightarrow{\text{dual}}$	quotient set
$X \times Y$	$\xleftrightarrow{\text{dual}}$	$X \amalg Y$
$f \circ g$	$\xleftrightarrow{\text{dual}}$	$g \circ f$
$\{\vec{0}\}$	$\xleftrightarrow{\text{dual}}$	$\{\vec{0}\}$