Connection with the usual Fourier series

In the case $X = L^2(2\pi) = \{ f : \mathbb{R} \to \mathbb{R} : f \text{ is } 2\pi\text{-periodic, measurable, and } \int_{-\pi}^{\pi} |f|^2 dx < +\infty \}$, we have:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$\Rightarrow F = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos Kx}{\sqrt{\pi}}, \frac{\sin Kx}{\sqrt{\pi}} : K = 1, 2, 3, \ldots \right\}$ is a maximal orthonormal family (i.e. it is an Hilbert Basis) of $L^2(2\pi)$

In order to prove it, it is enough to show that the span of F (also called the set of trigonometric polynomials) is dense in $L^2(2\pi)$.

Remark:
Every polynomial in $\sin x$, $\cos x$ is a trigonometric polynomial

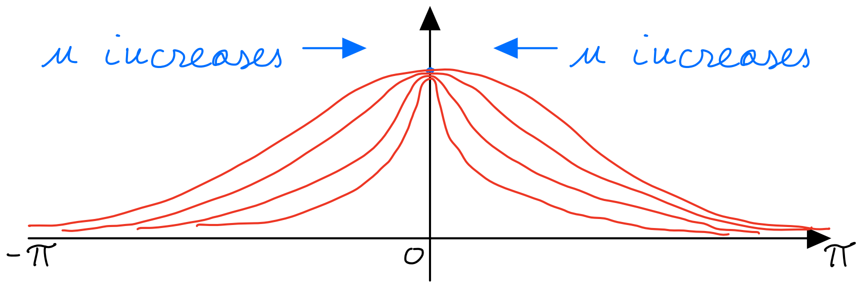
Thm. (Trigonometric Stone - Weierstraß):
Any function in $C^0(2\pi)$ can be uniformly approximated by a sequence of trigonometric polynomials.

Proof:
Consider the functions $\varphi_n \in C^0(2\pi)$ s.t.:

$$\varphi_n(x) := c_n \left(\frac{1 + \cos x}{2} \right)^n, \quad c_n \text{ s.t. } \int_{-\pi}^{\pi} \varphi_n(x)\, dx = 1$$

$\Rightarrow \varphi_n$ are even, non negative trigonometric polynomials.
We claim that $\exists K > 0$ s.t. $c_n \leq K\sqrt{n}$. Indeed:

$$\frac{1}{c_n} = \int_{-\pi}^{\pi} \left(\frac{1+\cos x}{2} \right)^n dx \geq \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \left(\frac{1+\cos x}{2} \right)^n dx \geq \frac{2}{\sqrt{n}} \underbrace{\left(\frac{1+\cos \frac{1}{\sqrt{n}}}{2} \right)^n}_{\to e^{-\frac{1}{4}}} \geq \frac{C}{\sqrt{n}}$$



Let $u \in C^0(2\pi)$ and define $u_n(x) := \int_{-\pi}^{\pi} u(x+y)\, \varphi_n(y)\, dy$. These u_n are actually trigonometric polynomials (to check it, set

$z = x+y, \quad y = z-x \Rightarrow u_n(x) = \int_{-\pi}^{\pi} u(z)\, \tau_n(x-z)\, dz$

$= \int_{-\pi}^{\pi} u(z) \cdot \left(A_0 + \sum_{k=1}^{n} A_k \cos k(x-z) + B_k \sin k(x-z)\right) dz\, \big)$. Consider now:

$$|u_n(x) - u(x)| = \left|\int_{-\pi}^{\pi} (u(x+y) - u(x))\, \tau_n(y)\, dy\right|$$

$$\leqslant \int_{-\pi}^{\pi} |u(x+y) - u(x)|\, \tau_n(y)\, dy = \int_{-\pi}^{-\delta} \cdots + \int_{-\delta}^{\delta} \cdots + \int_{\delta}^{\pi}$$

(u is uniformly continuous:
$\forall \varepsilon > 0\ \exists \delta$ s.t. $|y| < \delta \Rightarrow |u(x+y) - u(x)| < \varepsilon$)

$$= \int_{-\pi}^{-\delta} \cdots + \int_{-\delta}^{\delta} \underbrace{|u(x+y) - u(x)|}_{< \varepsilon}\, \tau_n(y)\, dy + \int_{\delta}^{\pi} \underbrace{|u(x+y) - u(x)|}_{< 2M,\ M = \|u\|_\infty}\, \tau_n(y)\, dy$$

$$\leqslant \int_{-\pi}^{-\delta} \cdots + \varepsilon + 2M \int_{\delta}^{\pi} c_n \left(\frac{1+\cos y}{2}\right)^n dy$$

$$\leqslant \int_{-\pi}^{-\delta} \cdots + \varepsilon + \underbrace{2M c_n \left(\frac{1+\cos\delta}{2}\right)^n}_{\leqslant C\sqrt{n}\left(\frac{1+\cos\delta}{2}\right)^n} \leqslant \underbrace{\int_{-\pi}^{-\delta} \cdots}_{\leqslant \varepsilon \text{ (analogue to } \int_{\delta}^{\pi} \cdots)} + \varepsilon + \varepsilon \leqslant 3\varepsilon$$

$\square$

Weak Convergence in Hilbert Spaces

Remind that $x_n \rightharpoonup x \Leftrightarrow \forall T \in X'\ T(x_n) \to T(x)$. In a Hilbert space this becomes the following:

$$x_n \rightharpoonup x \Leftrightarrow \forall y \in X \ \langle x_n, y\rangle \to \langle x, y\rangle$$

we also have:

$$x_n \rightharpoonup x \Rightarrow \|x\| \leqslant \liminf_{k\to+\infty} \|x_n\|$$

<u>Proposition</u>:

Given $(X, \langle\cdot,\cdot\rangle)$ Hilbert, $\{x_k\}_{k\in\mathbb{N}} \subset X$, $x \in X$, we have:

$$x_k \to x \Leftrightarrow x_k \rightharpoonup x \ \wedge \ \|x_k\| \to \|x\|$$

<u>Proof</u>:

($\Rightarrow$): trivial ✓

($\Leftarrow$): $\|x_k - x\|^2 = \langle x_k - x, x_k - x\rangle = \underbrace{\|x_k\|^2}_{\to \|x\|^2} - 2\underbrace{\langle x_k, x\rangle}_{\to \langle x, x\rangle} + \|x\|^2$

$\to 2\|x\|^2 - 2\|x\|^2 = 0$

$\square$

<u>Thm.</u> (Banach-Alaoglu in Hilbert Spaces):

If $\{x_k\} \subset X$ is bounded in norm then $\exists$ subsequence x_{k_h}, $\bar{x} \in X$ s.t. $x_{k_h} \rightharpoonup \bar{x}$

Proof:

Suppose (for the moment) that X is separable. Let $\{e_n\}_{n\in\mathbb{N}}$ be a Hilbert basis for X, consider the Fourier coefficients: $\langle x_k, e_1\rangle$ is a bounded sequence in $\mathbb{R} \Rightarrow \exists \{x_k^{(1)}\}_k$ subsequence s.t. $\langle x_k^{(1)}, e_1\rangle \to c_1$. Apply the same to $\langle x_k^{(1)}, e_2\rangle$ bounded in $\mathbb{R}$, then $\exists\{x_k^{(2)}\}_k$ subsequence s.t. $\langle x_k^{(2)}, e_2\rangle \to c_2$ etc...

At the n-th step we have:

$\langle x_k^{(n-1)}, e_n\rangle$ bounded in $\mathbb{R} \Rightarrow \exists\{x_k^{(n)}\}_k$ s.t. $\langle x_k^{(n)}, e_n\rangle \to c_n$

Moreover $\langle x_k^{(n)}, e_j\rangle \to c_j$, $j = 1,\dots,n$. Take the diagonal subsequence $\{\tilde{x}_k\}_k$ given by $\tilde{x}_k = x_k^{(k)}$, then:

$$\langle \tilde{x}_k, e_j\rangle \to c_j \quad \forall j\in\mathbb{N}$$

Take now $y \in \langle e_k\rangle_{k\in\mathbb{N}}$, then $y = \sum_{j=1}^{n}\lambda_j e_j \wedge \langle \tilde{x}_k, y\rangle \to T(y)$ with $T(y) = \sum_{j=1}^{n}\lambda_j c_j : Y \to \mathbb{R}$ linear, bounded. T can be extended to $\tilde{T}\in X'$ by H.-B. Thm. $\Rightarrow \exists \bar{x}\in X$ s.t. $T(y) = \langle \bar{x}, y\rangle$

So we have:

$$\langle \tilde{x}_k, y\rangle \to \langle \bar{x}, y\rangle \quad \forall y \in Y$$

and it is enough to check weak convergence in a dense subspace of X.

$\Rightarrow$ If X is not separable, consider $Z = \overline{\langle x_k\rangle_{k\in\mathbb{N}}}$. It is a closed subspace of X and it is separable.

$\Rightarrow \exists \{x_{k_h}\}$ subsequence, $\bar{x}\in Z$ s.t.

$$\langle x_{k_h}, y\rangle \to \langle \bar{x}, y\rangle \quad \forall y \in Z$$

$$\Rightarrow X = Z \oplus Z^\perp \Rightarrow y = \underset{\in Z}{y_1} + \underset{\in Z^\perp}{y_2}$$

$$\Rightarrow \langle x_{k_h}, y\rangle = \langle x_{k_h}, y_1\rangle + \underbrace{\langle x_{k_h}, y_2\rangle}_{=0\,=\,\langle 0, y_2\rangle} \to \langle \bar{x}, y\rangle$$

$\square$