5. Rank and Nullity of a matrix

"If only I had the Theorems! Then I should find the proofs easily enough"

— Bernhard Riemann

5.1 Introduction

In this chapter, we will define what the rank of a matrix is and outline the procedure to calculate the rank of a matrix. Then, we move on to define the concept of null space of a matrix and outline a procedure to calculate its basis and dimension, namely, the nullity of the matrix. We end this chapter by stating the very important rank-nullity theorem.

5.2 Rank of a matrix

Suppose a person X starts moving from position A. After 10 minutes, he arrives at a position B that is 3km to the west of A and 4km to the south of A. This information helps us to identify the position of X after 10 minutes. In addition to this, if we also say that X is 5km to the south west of the starting point A, we still arrive at the same location B. Hence this information is redundant. This is because in a two dimensional plane, with two directions we can identify the position of an object completely (recall that the dimension of $\mathbb{R}^2$ is 2). The role of linearly independent vectors is highlighted here. In this sense, if we look to find the number of "directions" needed to completely understand a matrix, we arrive at the concept of the rank of a matrix.

Definition 5.2.1. Let A be an $m \times n$ matrix. The rank of the matrix A is the number of linearly independent columns of A.

The subspace spanned by the columns of A is called the **column space** of A and the dimension of the column space is called the **column rank** of A. Similarly, the subspace spanned by the rows of A is called the **row space** of A and the dimension of the row space is called the **row rank** of A.

It is a very important fact that the row rank and the column rank of a matrix are always equal and is called the rank of the matrix.

Example 5.2.1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \\ 4 & -2 & 2 \end{pmatrix}$. The column space of A is the subspace spanned by $\{(1, -1, 4), (2, 0, -2), (3, -1, 2)\}$. Note that third vector is the sum of the first two vectors and hence the column rank (=dimension of the column space) of A is 2.

The row space is the subspace spanned by $\{(1, 2, 3), (-1, 0, -1), (4, -2, 2)\}$. Clearly, $(4, -2, 2) = -(1, 2, 3) - 5(-1, 0, -1)$ and hence the row rank (=dimension of the row space) of A is also 2.

A simple procedure to calculate the rank of a matrix is by converting the matrix into row echelon form or the reduced row echelon form and counting the number of non-zero rows of the matrix. Performing elementary row operations does not affect the number of linearly independent rows and hence the rank of the matrix.

The row echelon form of the matrix in the above example is $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus the rank, which is equal to the number of non-zero rows of the row echelon form of A is 2.

Example 5.2.2. Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 5 & 3 & 0 & 0 \\ -1 & 0 & 1 & 2 \end{pmatrix}$. The row echelon form of A is $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{5}{7} & 0 \\ 0 & 0 & 1 & \frac{7}{5} \end{pmatrix}$. Hence the rank (the number of non-zero rows in the row echelon form) of A is 3. Note that the column space is a subset of $\mathbb{R}^3$ and the row space is a subset of $\mathbb{R}^4$.

Example 5.2.3. If rank of the matrix $\begin{bmatrix} 2 & -3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix}$ is 2 then what is the value of a? Note that if the rank of the matrix is 3, then the dimension of the column space is 3 and hence the column vectors are linearly independent. This happens when the determinant of the matrix is non-zero. But, now since it is given that the rank is 2, we have only two linearly independent vectors and one the vectors is dependent on the other two. This forces the determinant of the matrix to be 0. By equating the determinant of the matrix to 0, we are making the rank to be atmost 2. $det(A) = 2a - 10$ and for rank to be less than 3, $det(A) = 0$ and thus we get $a = 5$. By putting $a = 5$ and reducing A to REF, verify that the rank$(A) = 2$.

From the above explanation of the rank of a matrix, it is clear that the rank cannot exceed the number of columns or the number of rows of a matrix. Thus

- $\text{rank}(A) \leq \min\{m, n\}$, where A is an $m \times n$ matrix.
- $\text{rank}(AB) \leq \min\{\text{ rank}(A), \text{rank}(B)\}$.

5.2.1 Exercise

Question 104. Choose the set of correct options from the following.

- **Option 1:** Row rank and column rank of a matrix is always the same.
- **Option 2:** The rank of a zero matrix is always 0.
- **Option 3:** The rank of a matrix, all of whose entries are the same non-zero real number, must be 1.
- Option 4: Rank of the $n \times n$ identity matrix is 1 for any $n \in \mathbb{N}$.

Question 105. Find the rank of $A = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 2 & 0 & 3 \end{bmatrix}$. [Answer: 3]

Question 106. Consider the following upper triangular matrix to choose the correct options.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$.

- **Option 1:** If $f = 0$, then the rank of the matrix must be less than or equal to 2.
- Option 2: If $f = 0$, then the rank of the matrix must be exactly 2.
- **Option 3:** If a, b, c, d, e, f are all non-zero then the rank of the matrix must be 3.
- **Option 4:** If a, d, f are non-zero then the rank of the matrix must be 3.

Question 107. What is the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}$, where x, y, z are distinct non-zero integers? [Answer: 3]

5.3 Nullity of a matrix

Let A be an $m \times n$ matrix. The subspace $\{x \in \mathbb{R}^n : Ax = 0\}$ is called the null space of the matrix A. This is nothing but the solution space of the homogeneous system of linear equations $Ax = 0$. (One can verify quickly that the null space is actually a subspace.) In other words, the null space contains those vectors of $\mathbb{R}^n$ that are "killed" by the matrix A, i.e., taken to the zero vector by the matrix A. The dimension of the null space of A is the nullity of the matrix A.

Example 5.3.1. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 4 & -1 \\ 3 & 3 & -1 \end{pmatrix}$. The null space of A is nothing but the solutions of the system $Ax = 0$. Recall that we can solve a system of linear equations using Gaussian elimination. Since we have a homogeneous system, the last column of the augmented matrix is not going to be affected by any row operations and will remain zero throughout and hence can be ignored. Since this is our first example, we shall anyway include the last column too. The augmented matrix is $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 3 & -1 & 0 \end{pmatrix}$. The reduced row echelon form is $\begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. x_3 is an independent variable as there is no pivot in the third column of the reduced row echelon form. Thus the solution set (null space of A) is $\{(-\frac{k}{6}, -\frac{k}{6}, k) : k \in \mathbb{R}\}$. The dimension of the null space is 1, that is the nullity of A is 1.

Note that we had one independent variable in the reduced row echelon form in the previous example and the nullity was equal to 1. It is in fact true in general that the nullity of the matrix is equal to the number of independent variables in the reduced row echelon form of the matrix.

We got the complete solution set of the null space by doing the following:

- Assign an arbitrary value k_i to the independent variables.
- Compute the value of the dependent variables in terms of these k_i's from the unique row they occur in.
- The set of all solutions is obtained by letting k_i's vary over $\mathbb{R}$.

Once we have the subspace, it is easy to get a basis for the null space. By substituting $k_i = 1$ and $k_j = 0$ for all $j \neq i$, as i varies, we get a basis for the null space of A.

Example 5.3.2. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$. To find the null space of A, we find the solutions of $Ax = 0$. We do row operations on the augmented matrix (but recall the last column of zeros are not going to be affected by any of the row operations and hence can be ignored) and obtain the reduced form as $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. There is only one column with a pivot. x_1 is dependent and x_2 and x_3 are independent. So the nullity is 2. Recall the procedure given above. We just assign arbitrary values to the independent variables x_2 and x_3, say $x_2 = t$ and $x_3 = s$. Thus we have, from the reduced system, $x_1 = -2t - 3s$. Hence the null space of A is $\{(-2t-3s, t, s) : t, s \in \mathbb{R}\}$. We can easily get a basis, by putting $t = 1$ and $s = 0$ and a second vector by putting $t = 0$ and $s = 1$, i.e., $\{(-2, 1, 0), (-3, 0, 1)\}$ forms a basis for the null space of A.

Example 5.3.3. Let $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & 1 & 0 \\ 3 & 3 & 1 & 4 \end{pmatrix}$. First we find the reduced form of the matrix. $\begin{pmatrix} 1 & 0 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$. x_1 and x_2 are dependent, whereas x_3 and x_4 are independent. We assign $x_3 = t$ and $x_4 = s$. Thus the null space is $\{(-\frac{2}{3}t + \frac{4}{3}s, \frac{1}{3}t - \frac{8}{3}s, t, s : t, s \in \mathbb{R}\}$. Clearly nullity is 2 since there are two independent variables. A basis can be got by putting $t = 1$, $s = 0$ and another vector by putting $t = 0$, $s = 1$. $\{(-\frac{2}{3}, \frac{1}{3}, 1, 0), (\frac{4}{3}, -\frac{8}{3}, 0, 1)\}$ is a basis for the null space obtained by the above procedure.

5.3.1 Exercise

Question 108. The null space of a matrix $A_{4\times 3}$ is

- Option 1: the subspace $W = \{x \in \mathbb{R}^4 \mid Ax = 0\ \}$ of $\mathbb{R}^4$.
- **Option 2**: the subspace $W = \{x \in \mathbb{R}^3 \mid Ax = 0\ \}$ of $\mathbb{R}^3$.
- Option 3: the subspace $W = \{x \in \mathbb{R}^2 \mid Ax = 0\ \}$ of $\mathbb{R}^2$.
- Option 4: the first column of the matrix $A_{4\times 3}$.

Question 109. Nullity of a matrix $A_{3\times 4}$ is

- Option 1: 3
- Option 2: 4

- **Option 3:** the number of independent variables in the system of linear equations $Ax = 0$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.
- Option 4: the number of dependent variables in the system of linear equations $Ax = 0$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Question 110. Choose the correct set of options from the following.

- Option 1: The nullity of a non-zero scalar matrix of order 3 must be 3.
- **Option 2:** The nullity of a non-zero scalar matrix of order 3 must be 0.
- Option 3: The nullity of a non-zero diagonal matrix of order 3 must be 3.
- **Option 4:** The nullity of a non-zero diagonal matrix of order 3 can be at most 2.

Question 111. Consider the coefficient matrix A of the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 - x_3 + x_4 &= 0 \end{aligned}$$

Which one of the following vector spaces represents the null space of A?

- Option 1: $\{(t_1 + t_2, t_1, t_1, t_2) \mid t_1, t_2 \in \mathbb{R}\}$
- **Option 2:** $\{(t_1 - t_2, -t_1, t_1, t_2) \mid t_1, t_2 \in \mathbb{R}\}$
- Option 1: $\{(t_1 + t_2, -t_1, t_1, t_2) \mid t_1, t_2 \in \mathbb{R}\}$
- Option 1: $\{(t_1 - t_2, -t_1, t_1, -t_2) \mid t_1, t_2 \in \mathbb{R}\}$

Also, find a basis for the null space.

5.4 The rank-nullity theorem

In this section, we state one of the very important theorems of linear algebra, namely, the rank-nullity theorem. Here, we state the theorem for matrices. A similar version for linear transformations will be stated in Chapter 6.

Theorem 5.4.1. *Let A be an $m \times n$ matrix. Then rank(A)+nullity$(A) = n$.*

We shall not prove the theorem, but it can be easily see from the following observation. The rank of the matrix is nothing but the number of non-zero rows in the (reduced) row echelon form of the matrix. This also denotes the number of dependent variables in the reduced form of the homogeneous system $Ax = 0$, because the number of non-zero rows will be equal to the number of rows with pivot elements. We also noticed earlier that the nullity of the matrix is equal to the number of independent variables of the reduced form of the homogeneous system $Ax = 0$. Thus the sum of dependent variables (rank) and independent variables (nullity) gives the total number of variables (which equals the number of columns of the matrix).

This theorem simplifies our job of finding the rank and nullity of a matrix. Given a matrix, once we know the rank, we get the nullity using this theorem and vice versa.

Example 5.4.1. Let nullity of the matrix $A_{3\times 5}$ be 2. Then the rank of A can be got using the rank-nullity theorem. rank(A) + nullity$(A) = 5$. Thus rank$(A) = 2$.

Checking linear dependence of n vectors in the vector space $\mathbb{R}^n$

Recall that the dimension of $\mathbb{R}^n$ is n and we have calculated different bases for $\mathbb{R}^n$. The most commonly used one is the standard basis consisting of n vectors with the i^{th} vector having 1 at the i^{th} position and zero at all other positions.

In general, if a set of n vectors in $\mathbb{R}^n$ is linearly independent, then it must be a basis for $\mathbb{R}^n$. Recall the procedure to check whether or not a set of vectors is linearly independent. We try to equate the linear combination of those vectors to 0 and solve for the coefficients. Basically, we try to solve a homogeneous system whose coefficient matrix A is the matrix whose columns are the given vectors. If the system has a unique solution (that is the zero solution), this means that the coefficients are all zero and hence set of vectors in independent. In the case of infinitely many solutions, the set of vectors become linearly dependent. Note that since we are dealing with a set of n vectors in $\mathbb{R}^n$, the coefficient matrix of the system will be a square matrix. Thus the homogeneous system $Ax = 0$ has a unique solution if $det(A) \neq 0$. In this case, the set of vectors are linearly independent and hence form a basis for $\mathbb{R}^n$. In the case when $det(A) = 0$, the set of vectors are linearly dependent and do not form a basis for $\mathbb{R}^n$.

Example 5.4.2. Consider the set of vectors $\{(1, 2, 3), (1, 0, 1), (4, 5, 7)\}$. Since we have a set of 3 vectors, it forms a basis for $\mathbb{R}^3$ only if it is linearly independent (in this case, it will be maximal linearly independent!). We check if this forms a basis for $\mathbb{R}^3$ by finding the determinant of the matrix A whose columns are the given vectors. That is $A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 0 & 5 \\ 3 & 1 & 7 \end{pmatrix}$. $det(A) = 4 \neq 0$ and hence the set is linearly independent. But this is enough to check that it is a basis for $\mathbb{R}^3$ (Why?).

5.4.1 Exercise

Question 112. Which of the following options are correct for a square matrix A of order $n \times n$, where n is any natural number?

- **Option 1:** If the determinant is non-zero, then the nullity of A must be 0.
- Option 2: If the determinant is non-zero, then the nullity of A may be non-zero.
- **Option 3:** If the nullity of A is non-zero, then the determinant of A must be 0.
- Option 4: If the nullity of A is non-zero, then the determinant of A may be non-zero.

Question 113. Choose the set of correct statements.

- **Option 1:** If nullity of a 3×3 matrix is c for some natural number c, $0 \leq c \leq 3$, then the nullity of $-A$ will also be c.
- Option 2: $nullity(A + B) = nullity(A) + nullity(B)$.
- **Option 3:** Nullity of the zero matrix of order $n \times n$, is n.
- Option 4: Nullity of the zero matrix of order $n \times n$, is 0.
- **Option 5:** There exist square matrices A and B of order $n \times n$, such that nullity of both A and B is 0, but the nullity of $A + B$ is n.

Question 114. If A is a 3×4 matrix, then which of the following options are true?

- **Option 1:** $rank(A)$ must be less than or equal to 3.
- **Option 2:** $nullity(A)$ must be greater than or equal to 1.
- **Option 3:** If A has 2 columns which are non-zero and not multiples of each other, while the remaining columns are linear combinations of these 2 columns, then $nullity(A) = 2$.

- Option 4: If A has 2 columns which are non-zero and not multiples of each other, while the remaining columns are linear combinations of these 2 columns, then $nullity(A) = 1$.

Question 115. Let $Ax = 0$ be a homogeneous system of linear equations which has infinitely many solutions, where A is an $m \times n$ matrix ($m > 1, n > 1$). Which of the following statements are possible?

- **Option 1:** $rank(A) = m$ and $m < n$.
- Option 2: $rank(A) = m$ and $m > n$.
- Option 3: $rank(A) = m$ and $m = n$.
- **Option 4:** $nullity(A) = n$.
- **Option 5:** $nullity(A) \neq 0$.

Question 116. Let $Ax = 0$ be a homogeneous system of linear equations which has a unique solution, where $A \in \mathbb{R}^{n \times n}$. What is the nullity of A? [Answer : 0]

Question 117. Suppose $x_1 \neq 0$ solves $Ax = 0$, where $A \in \mathbb{R}^{2\times4}$. What is the minimum number of elements in a linearly independent subset of null space of A that also spans the set of solutions of $Ax = 0$? [Answer : 2]