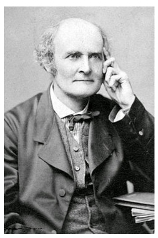
6. Linear Transformation



"As for everything else, so for a mathematical theory: beauty can be perceived but not explained"

— Arthur Cayley

6.1 Linear Mapping

In previous course of mathematics we have studied about the function, like $f(x) = x^2 + 3x$ e.t.c., where the function's domain and codomain are either $\mathbb{R}$ or subset of $\mathbb{R}$.
In this section we are going to study some special type of function whose domain and codomain are vector spaces or a subspaces of a vector space with some specific properties.

Let see it with an example

Suppose there are 3 shops in a locality, the original shop A and two other shops B and C. The price of rice, dal and oil in each of these shops are as given in the table below:

	Rice (per kg)	Dal (per kg)	Oil (per kg)
Shop A	45	125	150
Shop B	40	120	170
Shop C	50	130	160

Based on these prices, how will we decide from which shop to buy your groceries?

To get this idea, lets write the expression for the total cost of buying x_1 kg of rice, x_2 kg of dal and x_3 kg of oil for each of the three shops and try compare them. So total cost of buying x_1 kg of rice, x_2 kg of dal and x_3 kg of oil from shop A is $45x_1 + 125x_2 + 150x_3$.

This can be thought as a function C_A and viewed as a matrix multiplication.

$$C_A(x_1, x_2, x_3) = 45x_1 + 125x_2 + 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now, observe that the function C_A holds special properties:

$$\begin{aligned} C_A(\alpha(x_1, x_2, x_3)) &= (\alpha x_1, \alpha x_2, \alpha x_3) \\ &= 45\alpha x_1 + 125\alpha x_2 + 150\alpha x_3 \\ &= \alpha(45x_1 + 125x_2 + 150x_3) \\ &= \alpha C_A(x_1, x_2, x_3) \end{aligned}$$

$$\begin{aligned} C_A((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= 45(x_1 + y_1) + 125(x_2 + y_2) + 150(x_3 + y_3) \\ &= (45x_1 + 125x_2 + 150x_3) + (45y_1 + 125y_2 + 150y_3) \\ &= C_A(x_1, x_2, x_3) + C_A(y_1, y_2, y_3) \end{aligned}$$

These properties called linearity property of the function C_A. This property can be verified using above matrix multiplication expression of C_A also.

Similarly, for shops B and C , we get function C_B and C_C (holds the same properties as function C_A) whose expression and matrix form are:

$$C_B(x_1, x_2, x_3) = 40x_1 + 120x_2 + 170x_3 = \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_C(x_1, x_2, x_3) = 50x_1 + 130x_2 + 160x_3 = \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Comparing these expression, it is clear that for any quantities x_1, x_2, x_3 that one would buy (i.e., when x_1, x_2, x_3 are positive), the third expression yield larger values than fist one.

However, the comparison between the second expression and the others depends on the quantities of the item bought, i.e., on x_1, x_2, x_3.

A natural way to make this comparison would be the to create a vector of costs i.e., $(C_A(x_1, x_2, x_3), C_B(x_1, x_2, x_3), C_C(x_1, x_2, x_3))$.

We can think of the cost vector a function c from $\mathbb{R}^3$ to $\mathbb{R}^3$ by setting these expression as the coordinates in $\mathbb{R}^3$ i.e.,

$$c(x_1, x_2, x_3) = (C_A(x_1, x_2, x_3), C_B(x_1, x_2, x_3), C_C(x_1, x_2, x_3))$$

$$c(x_1, x_2, x_3) = (45x_1 + 125x_2 + 150x_3, 40x_1 + 120x_2 + 170x_3, 50x_1 + 130x_2 + 160x_3)$$

We can use matrix multiplication to express the cost function c in a compact form and extract its properties:

$$c(x_1, x_2, x_3) = \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As we have seen the linearity of the cost function C_A, C_B, C_C, the cost function $c : \mathbb{R}^3 \to \mathbb{R}^3$ also follows linearity property:

$$\begin{aligned}
c(\alpha(x_1, x_2, x_3) + (y_1, y_2, y_3)) &= c((\alpha x_1 + y_1, \alpha x_2 + y_2, \alpha x_3 + y_3)) \\
&= c(C_A(\alpha x_1 + y_1, \alpha x_2 + y_2, \alpha x_3 + y_3), C_B(\alpha x_1 + y_1, \alpha x_2 + y_2, \alpha x_3 + y_3), \\
&\quad C_C(\alpha x_1 + y_1, \alpha x_2 + y_2, \alpha x_3 + y_3)) \\
&= c(\alpha C_A(x_1, x_2, x_3) + C_A(y_1, y_2, y_3), \alpha C_B(x_1, x_2, x_3) + C_B(y_1, y_2, y_3), \\
&\quad \alpha C_C(x_1, x_2, x_3) + C_C(y_1, y_2, y_3)) \\
&= \alpha c(C_A(x_1, x_2, x_3), C_B(x_1, x_2, x_3), C_C(x_1, x_2, x_3)) \\
&\quad + c(C_A(y_1, y_2, y_3), C_B(y_1, y_2, y_3), C_C(y_1, y_2, y_3)) \\
&= \alpha c(x_1, x_2, x_3) + c(y_1, y_2, y_3)
\end{aligned}$$

We can verify it using above matrix multiplication expression of c.

So we can say that the cast function c is linear mapping.

6.1.1 Linear Mapping: The formal definition

Definition 6.1.1. A linear mapping f from $\mathbb{R}^3 \to \mathbb{R}^3$ can be defined as follows:

$$f(x_1, x_2, \ldots x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j \right)$$

Where the coefficient a_{ij} are real numbers (scalars). A linear mapping can be thought of as a collection of linear combinations.

We can write the expression on the RHS in matrix form as Ax, where

$$A = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

So using the matrix multiplication expression or directly from the expression $f(x_1, x_2, \dots x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \dots, \sum_{j=1}^{n} a_{nj}x_j\right)$, we can easily check that the function f holds linearity:

$$f(\alpha(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)) = A(\alpha x + y)$$

$$= \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \alpha f((x_1, x_2, \dots, x_n) + f(y_1, y_2, \dots, y_n))$$

6.1.2 Exercises

A book shop is organizing an year end sale. Price of any Bengali, Hindi, Tamil, and Urdu book is fixed as ₹200, ₹180, ₹230, and ₹250, respectively. Let $T(x, y, z, w)$ denote the total price of x number of Bengali books, y number of Hindi books, z number of Tamil books, and w number of Urdu books. Table 6.1 shows the numbers of books of different languages purchased by some customers.

	Bengali	Hindi	Tamil	Urdu
Samprita	3	0	0	2
Srinivas	0	1	2	1
Anna	0	1	0	3
Tiyasha	2	2	0	1
Hasan	2	2	1	1

Table 6.1:

Answer the below questions from the given data.

(1) What will be the correct expression for $T(x, y, z, w)$?

- Option 1: $T(x, y, z, w) = (200 + 180 + 230 + 250)(x + y + z + w)$
- Option 2: $T(x, y, z, w) = x + y + z + w$
- Option 3: $T(x, y, z, w) = 200x + 230y + 180z + 250w$
- Option 4: $T(x, y, z, w) = 200x + 180y + 230z + 250w$

(2) Which of the following expressions represents the total price of the books purchased by Samprita?

- Option 1: $T(3, 0, 0, 2)$
- Option 2: $T(3, 0, 0, 2, 2)$
- Option 3: $T(3, 2)$
- Option 4: $T(5)$

(3) What will be total price (in ₹) of the books purchased by Tiyasha?

(4) What will be total price (in ₹) of the books purchased by Hasan?

(5) Which of the following expressions represent the total price of the books purchased by Srinivas?

- Option 1: $2T(0, 1, 1, 0) + T(0, 0, 0, 1)$
- Option 2: $T(0, 1, 0, 0) + T(0, 0, 1, 1)$
- Option 3: $T(0, 1, 0, 0) + 2T(0, 0, 1, 0) + T(0, 0, 0, 1)$
- Option 4: $T(0, 1, 1, 0) + T(0, 0, 1, 1)$

(6) Which of the following expressions represents the total price of the books purchased by Anna?

- Option 1: $T(0, 1, 0, 0) + T(0, 0, 0, 1)$

- Option 2: $T(0,1,0,0) + T(0,0,1,0)$
- Option 3: $T(0,1,0,0) + 3T(0,0,0,1)$
- Option 4: $T(0,1,1,0) + 2T(0,0,1,0)$

(7) Which of the following expressions represent the total price of the books purchased by Srinivas and Anna together?

- Option 1: $T(0,2,2,4)$
- Option 2: $T(2,2,4)$
- Option 3: $T(0,2,2,1)$
- Option 4: $T(0,1,2,1) + T(0,1,0,3)$

(8) Which of the following expressions represent the difference between the total price of the books purchased by Samprita and Tiyasha?

- Option 1: $|T(5,2,0,3)|$
- Option 2: $|T(1,-2,0,1)|$
- Option 3: $|T(3,0,0,2) - T(2,2,0,1)|$
- Option 4: $|T(3,0,0,2) + T(-2,-2,0,-1)|$

6.2 Linear Transformation

Definition 6.2.1. A function $T : V \to W$ between two vector spaces V and W is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold:

- $T(v_1 + v_2) = T(v_1) + T(v_2)$
- $T(cv_1) = cT(v_1)$

Form the definition of linear transformation it is clear that linear transformation is linear mapping.

Example 6.2.1. Consider a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$T(x, y) = (2x, y)$$

Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, then

$$T(v_1 + v_2) = T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2)$$

$$\implies T(v_1 + v_2) = (2(x_1 + x_2), (y_1 + y_2)) = ((2x_1 + 2x_2), (y_1 + y_1))$$

$$\implies T(v_1 + v_2) = (2x_1, y_1) + (2x_2, y_2) = T(v_1) + T(v_2)$$

Let $c \in \mathbb{R}$, then

$$T(cv_1) = T(c(x_1, y_1)) = T(cx_1, cy_1) = (2cx_1, cy_1) = c(2x_1, y_1) = cT(v_1)$$

Hence, T is a linear transformation.

We now state the following proposition which is quite evident from the definition of linear transformations.

Proposition 6.2.1. *A linear transformation $T : V \to W$, where V, W are vector spaces, $T(0) = 0$ i.e., Image of the zero vector of V is the zero vector of W.*

Proof. Since T is a linear transformation so we can write

$$T(0 + 0) = T(0) + T(0)$$

$$\implies T(0) = T(0) + T(0)$$

$$\implies T(0) = 0$$

□

6.2.1 Exercises:

Check which of the following mapping is linear transformation

1. Consider a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that
$$T(x, y) = (2x, 0)$$
2. Consider a mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$T(x, y, z) = (\frac{x}{2}, 3y, 5z)$$
3. Consider a mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that
$$T(x, y, z) = (4y - z, 3y + \frac{11}{19}z, 5x - 2z, 23y)$$
4. Consider a mapping $T : \mathbb{R} \to \mathbb{R}^3$ such that
$$T(t) = (t, 3t, \frac{23}{89}t)$$
5. Consider a mapping $T : \mathbb{R}^2 \to \mathbb{R}$ such that
$$T(x, y) = x$$

6.2.2 Images of the vectors in the basis of a vector space

Example 6.2.2. Let us choose the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. Define the linear transformation as follows:

$$T:\mathbb{R}^2\to\mathbb{R}^2$$
$$T((1,0))=(2,0)$$
$$T((0,1))=(0,1)$$

Using the information we have the following:

$$\begin{aligned}T(x,y) &= T(x(1,0)+y(0,1))\\ &= T(x(1,0))+f(y(0,1))\\ &= xT(1,0)+yT(0,1)\\ &= x(2,0)+y(0,1)\\ &= (2x,y)\end{aligned}$$

Hence the explicit definition of T is given by

$$T:\mathbb{R}^2\to\mathbb{R}^2$$

$$T(x,y)=(2x,y)$$

Example 6.2.3. If we choose a different basis for $\mathbb{R}^2$, then we will get different linear transformation. Let us choose $\{(1,0),(1,1)\}$ to be a basis for $\mathbb{R}^2$. Define the linear transformation as we have defined earlier:

$$f:\mathbb{R}^2\to\mathbb{R}^2$$
$$f((1,0))=(2,0)$$
$$f((1,1))=(1,1)$$

Now we have,

$$\begin{aligned}f(x,y) &= f((x-y)(1,0)+y(1,1))\\ &= f((x-y)(1,0))+f(y(1,1))\\ &= (x-y)f(1,0)+yf(1,1)\\ &= (x-y)(2,0)+y(1,1)\\ &= (2x-2y,0)+(y,y)\\ &= (2x-y,y)\end{aligned}$$

Hence the explicit definition of T is given by

$$T:\mathbb{R}^2\to\mathbb{R}^2$$

$$T(x,y)=(2x-y,y)$$

We end this section with the following remark which is visualised by the above examples.

Remark 6.2.1. Let $T : V \to W$ be a linear transformation. It is enough to know the image of the basis elements of V to get the explicit definition of T.

6.2.3 Exercises

1 Choose the set of correct options.

- Option 1: Let $u = (3, 1, 0)$, $v = (0, 1, 7)$, and $w = (3, 0, -7)$. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(u) = T(v) = (0, 0, 0)$ and that $T(w) = (5, 1, 0)$.
- **Option 2:** Let $u = (2, 1, 0)$, $v = (1, 0, 1)$, and $w = (3, 5, 6)$. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(u) = (1, 0)$, $T(v) = (0, 1)$ and that $T(w) = (5, 6)$.
- Option 3: Let $u = (1, 0, 0)$, $v = (1, 0, 1)$, and $w = (0, 1, 0)$. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(u) = (0, 0, 0)$, $T(v) = (0, 0, 0)$, $T(w) = (0, 0, 0)$, and $T(1, 4, 2) = (2, 4, 1)$.
- **Option 4:** Let $u = (1, 0)$, and $v = (0, 1)$,. There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u) = (\pi, 1)$, and $T(v) = (1, e)$.

6.3 Injective and surjective linear transformations

In our earlier courses we have seen that a function is called injective if there are no two elements from the domain which map to a same image and a function is called surjective if every element of the codomain of the function has a pre-image. Similarly,

1. a linear transformation $T : V_1 \to W$, where V, W are vector spaces, is called a **monomorphism**, if T is an injective map from V to W, i.e., $T(v_1) = T(v_2) \implies v_1 = v_2$.
2. a linear transformation $T : V_1 \to W$, where V, W are vector spaces, is called an **epimorphism**, if T is a surjective map from V to W, i.e., for every $w \in W$ there exists $v \in V$ such that $T(v) = w$.
3. a linear transformation $T : V \to W$ is **isomorphism** if it is both injective (monomorphism) and surjective (epimorphism).

Example 6.3.1. Consider the following linear transformation T defined as follows:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x, y) = (2x, y)$$

Checking of injectivity: If $T(x_1, y_1) = T(x_2, y_2)$, then $(2x_1, y_1) = (2x_2, y_2)$. Hence $x_1 = x_2$ and $y_1 = y_2$, i.e., $(x_1, y_1) = (x_2, y_2)$. So, T is an injective linear transformation.
Checking of surjectivity: If $(u, v) \in \mathbb{R}^2$, then $(\frac{u}{2}, v) \in \mathbb{R}^2$ such that $T(\frac{u}{2}, v) = (u, v)$. So T is a surjective linear transformation.
Hence T is a bijective linear transformation, hence an isomorphism.

Example 6.3.2. Consider the following linear transformation T defined as follows:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x, y) = (2x, 0)$$

Checking of injectivity: If $T(x_1, y_1) = T(x_2, y_2)$ implies $(2x_1, 0) = (2x_2, 0)$, hence $x_1 = x_2$. But it does not guarantee that $y_1 = y_2$. As for example, $(1, 2)$ and $(1, 3)$ have the same image $(2, 0)$. Hence T is not injective.
Checking of surjectivity: There is no pre-image for the vector (u, v), where v is non-zero. So T is not surjective.

6.3.1 Null space and Range space of a linear transformation

Definition 6.3.1. Kernel or nullspace of a linear transformation: Let $T : V \to W$ be a linear transformation. We define kernel of T (denoted by $ker(T)$) to be the set of all vectors v in V such that $T(v) = 0$.

$$Ker(T) = \{v \in V \mid T(v) = 0\}$$

Definition 6.3.2. Image or range space of a linear transformation: Let $T : V \to W$ be a linear transformation. We define image of T (denoted by $Im(T)$) as follows:

$$Im(T) = \{w \in W \mid \exists v \in V \text{ for which } T(v) = w\}$$

Example 6.3.3. Let us consider the same example we considered before.

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x, y) = (2x, y)$$

Let $v = (x, y)$ such that $T(v) = 0$

$$\implies T(x, y) = (0, 0)$$

$$\implies (2x, y) = (0, 0)$$

$$\implies x = 0 \text{ and } y = 0$$

$$\implies v = (x, y) = (0, 0)$$

$Ker(T) = \{(0, 0)\}$. Hence $ker(T)$ is a singleton set which contains only the zero vector the space $\mathbb{R}^2$.
$Im(T) = \mathbb{R}^2$. Hence $Im(T)$ is the whole space $\mathbb{R}^2$, as we have observed earlier it is surjective.

Theorem 6.3.1. *Kernel or nullspace of a linear transformation $T : V \to W$ is a vector subspace of V.*

Proof. Let $v, v' \in Ker(T)$. We have $T(v + v') = T(v) + T(v') = 0 + 0 = 0$. So, $v + v' \in Ker(T)$.
Moreover, $T(cv) = cT(v) = c0 = 0$, so $cv \in Ker(T)$ for all $c \in \mathbb{R}$.
Hence, $Ker(T)$ is a vector subspace of V.

□

Theorem 6.3.2. *Image or range space of a linear transformation $T : V \to W$ is a vector subspace of W.*

Proof. Let $w, w' \in Im(T)$. Then there exists $v, v' \in V$ such that, $T(v) = w$ and $T(v') = w'$. So we have $T(v+v') = T(v)+T(v') = w+w'$. Hence $w+w' \in Im(T)$. Similarly, $T(cv) = cT(v) = cw$, for all $c \in \mathbb{R}$. Hence $cw \in Im(T)$ for all $c \in \mathbb{R}$. So, $Im(T)$ is a vector subspace of W. □

Definition 6.3.3. Dimension of kernel or nullspace of linear transformation T is defined as $Nullity(T)$ and dimension of image or range space of linear transformation T is defined as $Rank(T)$.

Theorem 6.3.3. *T is an injective linear transformation if and only if $Ket(T) = \{0\}$.*

Proof. First assume that T is injective. Let $v \in Ker(T)$. Hence $T(v) = 0 = T(0)$, as T is injective then $v = 0$. Hence we can conclude $Ker(T) = \{0\}$.
Conversely, Let $Ker(T) = \{0\}$. Let for the vectors v_1 and v_2 in V, we have $T(v_1) = T(v_2)$. Then $T(v_1 - v_2) = T(v_1) - T(v_2) = 0$. Hence $v_1 - v_2 \in ker(T)$. Which implies $v_1 - v_2 = 0$ i.e., $v_1 = v_2$. Hence T is injective.

□

Remark 6.3.1. A linear transformation $T : V \to W$ is surjective if and only if $Im(T) = W$

Corollary 6.3.4. *$T : V \to W$ is an injective (resp. surjective) linear transformation if and only if $Nullity(T) = 0$ (resp. $Rank(T) = dim(W)$).*

Definition 6.3.4. We define two vector spaces V and W are isomorphic to each other iff there exists an isomorphism $T : V \to W$.

At this point we mention an important theorem with the sketch of the proof of it. We want the readers to complete the proof.

Theorem 6.3.5. *Any n dimensional vector space is isomorphic to $\mathbb{R}^n$.*

Proof. Let V be a vector space of dimension n. Consider a basis $\{v_1, v_2, \ldots, v_n\}$ of V. We can define a linear transformation $T : V \to \mathbb{R}^n$ as follows:

$$T : V \to \mathbb{R}^n$$
$$T(v_i) = e_i$$

Where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. 1 at the i-th coordinate and 0 elsewhere. Let $v \in V$ such that $T(v) = 0$. As $v \in V$, then v can be written as $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$, for some a_i's in $\mathbb{R}$. Hence $T(v) = 0$ implies

$$T(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = 0$$

$$a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = 0$$

$$a_1e_1 + a_2e_2 + \ldots + a_ne_n = 0$$

$$(a_1, a_2, \ldots, a_n) = 0$$

i.e, $a_i = 0$ for all $i = 1, 2, \ldots, n$. Hence, $v = 0$.
So $Ker(T) = \{0\}$, which implies that T is injective.
We leave the checking of surjectivity to the readers. □

6.4 Matrix representation of linear transformation

We begin this section with some examples.

Example 6.4.1. Let us consider the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x, y) = (2x, y)$$

Now if we consider a basis $\{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$ both for the domain and the codomain. Then we can represent it as a matrix as follows:

$$T(1, 0) = (2, 0) = 2(1, 0) + 0(0, 1)$$
$$T(0, 1) = (0, 1) = 0(1, 0) + 1(0, 1)$$

The matrix representation of the linear transformation is given by,

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 6.4.2. Now for the same transformation T, suppose we consider a different basis $\{(1, 0), (1, 1)\}$ of $\mathbb{R}^2$, both for the domain and the codomain. Then we can represent it as a matrix as follows:

$$T(1,0) = (2,0) = 2(1,0) + 0(1,1)$$
$$T(1,1) = (2,1) = 1(1,0) + 1(1,1)$$

Hence the matrix representation of the linear transformation is given by,

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Note: It is important to note that changing basis gives us different matrices corresponding to same linear transformation.

Definition 6.4.1. Let $T : V \to W$ be a linear transformation. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis of V and $\gamma = \{w_1, w_2, \ldots, w_m\}$ be a basis of W. So each $T(v_i)$ can be uniquely written as linear combination of w_j's, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \\ &\vdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m \end{aligned}$$

The matrix corresponding to the linear transformation f with respect to the bases β and γ is given by,

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \vdots & \vdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Example 6.4.3. Consider the following linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x, y, z) = (x - z, 2x + 3y + z, 3y + 3z)$$

We consider the standard ordered basis of $\mathbb{R}^3$ for both the domain and codomain.

$$\begin{aligned} T((1,0,0)) &= (1,2,0) = 1(1,0,0) + 2(0,1,0) + 0(0,0,1) \\ T((0,1,0)) &= (0,3,3) = 0(1,0,0) + 3(0,1,0) + 3(0,0,1) \\ T((0,0,1)) &= (-1,1,3) = -1(1,0,0) + 1(0,1,0) + 3(0,0,1) \end{aligned}$$

The matrix representation of T with respect to the standard ordered basis of $\mathbb{R}^3$ for both the domain and codomain.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

Example 6.4.4. Consider the following linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x, y, z) = (x - z, 2x + 3y + z, 3y + 3z)$$

We consider the ordered basis $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ of $\mathbb{R}^3$ for both the domain and codomain.

$$\begin{aligned} T((1,1,0)) &= (1,5,3) = \left(\frac{3}{2}\right)(1,1,0) + \left(\frac{7}{2}\right)(0,1,1) + \left(-\frac{1}{2}\right)(1,0,1) \\ T((0,1,1)) &= (-1,4,6) = \left(-\frac{3}{2}\right)(1,1,0) + \left(\frac{11}{2}\right)(0,1,1) + \left(\frac{1}{2}\right)(1,0,1) \\ T((1,0,1)) &= (0,3,3) = 0(1,1,0) + 3(0,1,1) + 0(1,0,1) \end{aligned}$$

The matrix representation of T with respect to the standard ordered basis of $\mathbb{R}^3$ for both the domain and codomain.

$$\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ \frac{7}{2} & \frac{11}{2} & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Example 6.4.5. Consider the following linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x, y, z) = (x - z, 2x + 3y + z, 3y + 3z)$$

We consider the ordered basis $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ of $\mathbb{R}^3$ for the domain and the standard ordered basis for the codomain.

$$\begin{aligned} T((1,1,0)) &= (1,5,3) = 1(1,0,0) + 5(0,1,0) + 3(0,0,1) \\ T((0,1,1)) &= (-1,4,6) = -1(1,0,0) + 4(0,1,0) + 6(0,0,1) \\ T((1,0,1)) &= (0,3,3) = 0(1,0,0) + 3(0,1,0) + 3(0,0,1) \end{aligned}$$

The matrix representation of T with respect to the standard ordered basis of $\mathbb{R}^3$ for both the domain and codomain.

$$\begin{bmatrix} 1 & -1 & 0 \\ 5 & 4 & 3 \\ 3 & 6 & 3 \end{bmatrix}$$

6.4.1 Exercises

1) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, such that $T(x, y) = (x, 0)$. Which of the following options are correct?

- Option 1: The matrices corresponds to T with respect to the standard ordered basis of $\mathbb{R}^2$, i.e., $\{(1,0),(0,1)\}$, for both the domain and co-domain is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- **Option 2:** The matrices corresponds to T with respect to the standard ordered basis of $\mathbb{R}^2$, i.e., $\{(1,0),(0,1)\}$, for both the domain and co-domain is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
- **Option 3:** T is neither one-one nor onto.
- Option 4: T is one-one but not onto.

Let $W = \{(x, y, z) \mid x = 2y+z\}$ be a subspace of $\mathbb{R}^3$. Let $\beta = \{(2,1,0),(1,0,1)\}$ be a basis of W. Let $T : W \to \mathbb{R}^2$ be a linear transformation such that $T(2,1,0) = (1,0)$ and $T(1,0,1) = (0,1)$. Answer the questions 2, 3 and 4 using the given information.

2) Which of the following is the appropriate definition of T?

   - Option 1: $T(x, y, z) = (x, y)$
   - Option 2: $T(x, y, z) = (x, z)$
   - **Option 3:** $T(x, y, z) = (y, z)$
   - **Option 4:** $T(x, y, z) = (x - y - z, x - 2y)$

3) Choose the correct options.

   - Option 1: T is one to one but not onto.
   - Option 2: T is onto but not one to one.
   - Option 3: T is neither one to one nor onto.
   - **Option 4:** T is an isomorphism.

4) What will be the matrix representation of T with respect to the basis β for W and $\gamma = \{(1,1),(1,-1)\}$ for $\mathbb{R}^2$?

   - **Option 1:** $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   - Option 2: $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   - Option 3: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   - Option 4: $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

5) Which option represents the kernel and image of the following linear transformation?

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x, y) = (x, 0)$$

- Option 1: $ker(T) = Span\{(1, 0)\}, Im(T) = Span\{(1, 0)\}$.
- Option 2: $ker(T) = Span\{(1, 0)\}, Im(T) = Span\{(0, 1)\}$.
- **Option 3:** $ker(T) = Span\{(0, 1)\}, Im(T) = Span\{(1, 0)\}$.
- Option 4: $ker(T) = Span\{(0, 1)\}, Im(T) = Span\{(0, 1)\}$.

6.5 Finding basis for null space and range space by Row reduced echelon form

Let $T : V \to W$ be a linear. Follow the steps below to find bases for the null space and the range space of T.

Step 1: Find the matrix A corresponding to T with respect to some standard ordered bases $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$ for V and W respectively.

Step 2: Use row reduction on A to obtain the matrix R which is in reduced row echelon form.

Step 3: The basis of the solution space of $Rx = 0$ is the basis of null space of matrix A and can be obtained by finding the pivot and non-pivot columns (dependent and independent variables) as seen earlier.

Step 4: The vectors $\begin{bmatrix} c_{11} \\ c_{12} \\ . \\ . \\ . \\ c_{1n} \end{bmatrix}, \begin{bmatrix} c_{21} \\ c_{22} \\ . \\ . \\ . \\ c_{2n} \end{bmatrix}, \dots, \begin{bmatrix} c_{k1} \\ c_{k2} \\ . \\ . \\ . \\ c_{kn} \end{bmatrix}$ form a basis of the null space of A precisely when the vectors $v_1', v_2', \dots, v_k' \in Ker(T)$ where $v_i' = \sum_{j=1}^{n} c_{ij} v_j$, form a basis for $\ker(T)$. Use the basis obtained in step 3 to thus get a basis for $Ker(T)$.

Step 5: Recall that if $i_1, i_2, \dots, i_r$ are the columns of R containing the pivot elements, then the same columns of A form a basis for the column space of A.

Step 6: The vectors $\begin{bmatrix} d_{11} \\ d_{12} \\ . \\ . \\ . \\ d_{1m} \end{bmatrix}, \begin{bmatrix} d_{21} \\ d_{22} \\ . \\ . \\ . \\ d_{2m} \end{bmatrix}, \dots, \begin{bmatrix} d_{r1} \\ d_{r2} \\ . \\ . \\ . \\ d_{rm} \end{bmatrix}$ form a basis of the column space of A precisely when the vectors $w'_1, w'_2, \dots, w'_r \in im(T)$ where $w'_i = \sum_{j=1}^{m} d_{ij} w_j$, form a basis for $Im(T)$. Use the basis obtained in step 5 to thus get a basis for $Im(T)$.

Using these steps we will find out the basis of nullspace and range space of a linear transformation in the following example:

Example 6.5.1. Consider the following linear transformation:

$$T : \mathbb{R}^4 \to \mathbb{R}^3$$

$$T(x_1, x_2, x_3, x_4) = (2x_1 + 4x_2 + 6x_3 + 8x_4, x_1 + 3x_2 + 5x_4, x_1 + x_2 + 6x_3 + 3x_4)$$

The matrix corresponding to the standard basis is as follows:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$$

The row reduced echelon form of the above matrix is,

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The required null space will the set of all vectors such that the following holds:

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Which gives us $x_1 = -9x_3 - 2x_4$ and $x_2 = 3x_3 - x_4$

Hence the null space is spanned by the vectors $\{(-9, 3, 1, 0), (-2, -1, 0, 1)\}$. Moreover, the first and second coloumn of the row reduced echelon form contains the pivot elements. Hence the range space is spanned by the vectors $\{(2, 1, 1), (4, 3, 1)\}$.

6.6 Rank-nullity theorem

We state the theorem without any proof.

Theorem 6.6.1. *Let $T : V \to W$ be a linear transformation. Then*

$$Rank(T) + Nullity(T) = dim(V).$$

We end this chapter with stating some immediate corollaries of Rank nullity theorem.

Corollary 6.6.2. *Let $T : V \to W$ be a linear transformation.*

- *If T is injective, then $Rank(T) = dim(V)$.*
- *If T is an isomorphism, then $dim(W) = dim(V)$.*

Proof. If T is injective, then $Nullity(T) = 0$. Hence from rank nullity theorem, $Rank(T) + 0 = dim(V)$, i.e., $Rank(T) = dim(V)$.
Moreover if T is surjective then, $dim(W) = Rank(T)$. Hence if T is an isomorphism, i.e., T is both injective and surjective, then $dim(W) = Rank(T) = dim(V)$. □

Example 6.6.1. Find the rank and nullity of the the linear transformation

$$\begin{array}{cccc} T: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x, y) & \mapsto & (0, x). \end{array}$$

and verify the rank nullity theorem.

Solution: To find the rank and nullity, we need to find the range and kernel, respectively. By definition, range space of T is

$$\begin{aligned} \text{Range}(T) :&= \{T(x, y) \mid x, y \in \mathbb{R}\} \\ &= \{(0, x) \mid x \in \mathbb{R}\} \\ &= \{x(0, 1) \mid x \in \mathbb{R}\} \\ &= \text{Span}\{(0, 1)\}. \end{aligned}$$

Since $\{(0, 1)\}$ is a basis for Range(T), the Rank$(T) = \dim(\text{Range}(T))$ is 1.
Similarly, by definition, kernel of T is

$$\begin{aligned} \text{Ker}(T) :&= \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (0, 0)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid (0, x) = (0, 0)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid x = 0\} \\ &= \{(0, y) \mid y \in \mathbb{R}\} \\ &= \text{Span}\{(0, 1)\}. \end{aligned}$$

Since $\{(0,1)\}$ is also a basis for Ker(T), the Nullity(T) = dim(Ker(T)) is 1. Hence we have the following equality:

$$\text{Rank}(T) + \text{Nullity}(T) = 1 + 1 = \dim(V) = \dim(\mathbb{R}^2) = 2.$$

Example 6.6.2. Consider a linear transformation $T : V \to W$ such that $\dim(V) = 5$, $\dim(W) = 7$ and T is one-to-one. Find Rank(T) and Nullity(T).

Solution: Since T is one-to-one, one concludes that $\text{Ker}(T) = \{0\}$. Therefore the Nullity(T) = dim(Ker(T)) is 0. From the rank-nullity theorem, we obtain;

$$\begin{aligned} \dim(V) &= \text{Nullity}(T) + \text{Rank}(T) = 0 + \text{Rank}(T) \\ \Rightarrow \text{Rank}(T) &= \dim(V) = 5. \end{aligned}$$

Example 6.6.3. Consider a linear transformation $T : V \to W$ such that $\dim(V) = 5$, $\dim(W) = 3$ and dim(Ker(T)) is 2. Then show that T is onto.

Solution: By rank nullity theorem, we have;

$$\begin{aligned} \dim(V) &= \text{Nullity}(T) + \text{Rank}(T) = \dim(\text{Ker}(T)) + \text{Rank}(T) = 2 + \text{Rank}(T) \\ \Rightarrow \text{Rank}(T) &= \dim(V) - 2 = 3. \end{aligned}$$

Since Rank(T) is same as the dimension of W, the range of T is same as W. Therefore T is on-to.

Example 6.6.4. Validate the statement: There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with kernel $\{0\}$.

Solution: Since image of T is a subspace of $\mathbb{R}^3$, Rank(T)= dim(Range (T)) ≤ 3. Now, by rank nullity:

$$\begin{aligned} \text{Nullity}(T) + \text{Rank}(T) &= \dim(\mathbb{R}^4) = 4 \\ \Rightarrow \text{Nullity}(T) &= 4 - \text{Rank}(T) \geq 1 \quad (\because \text{Rank}(T) \leq 3). \end{aligned}$$

Therefore T can not be one-one, and hence the given statement is false.

6.6.1 Exercises

Consider the following linear transformation:

$$\begin{gathered} T : \mathbb{R}^3 \to \mathbb{R}^2 \\ T(x,y,z) = (2x+3z, 4y+z) \end{gathered}$$

Answer questions 1,2,3 and 4, using the information given above.

1) Which of the following matrices corresponds to the given linear transformation T with respect to the standard ordered basis for $\mathbb{R}^3$ and the standard ordered basis for $\mathbb{R}^2$?

   – Option 1: $\begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 0 \end{bmatrix}$

   – Option 2: $\begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$

   – **Option 3:** $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

   – Option 4: $\begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$

2) Which of the following represents a basis of the kernel of T?

   – Option 1: $\{(-\frac{3}{2}, 0, 1), (0, -\frac{1}{4}, 1)\}$.
   – **Option 2:** $\{(-\frac{3}{2}, -\frac{1}{4}, 1)\}$.
   – Option 3: $\{(-\frac{3}{2}, -\frac{1}{4}, 2)\}$.
   – Option 4: $\{(2, 0, 3), (0, 4, 1)\}$.

3) What will be the dimension of the subspace $Im(T)$? [Answer: 2]

4) Choose the correct option.

   – Option 1: T is an isomorphism.
   – Option 2: T is one to one but not onto.
   – **Option 3:** T is onto but not one to one.
   – Option 4: T is neither one to one, nor onto.

5) Choose the set of correct options.

   – **Option 1:** Nullity and rank of the identity transformation on a vector space of dimension n are 0 and n respectively.
   – Option 2: Nullity and rank of the identity transformation on a vector space of dimension n are 1 and $n-1$ respectively.
   – Option 3: Nullity and rank of the identity transformation on a vector space of dimension n are n and 0 respectively.
   – Option 4: Nullity and rank of an isomorphism between two vector spaces V and W (both of dimension n) are n and 0 respectively.

- **Option 5:** Nullity and rank of an isomorphism between two vector spaces V and W (both of dimension n) are 0 and n respectively.
- **Option 6:** There cannot exist an isomorphism between two vector spaces whose dimensions are not the same.

6) Choose the set of correct options.

- **Option 1:** Any injective linear transformation between any two vector spaces which have the same dimensions, must be an isomorphism.
- **Option 2:** Any surjective linear transformation between any two vector spaces which have the same dimensions, must be an isomorphism.
- **Option 3:** There does not exist any surjective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.
- Option 4: There does not exist any injective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

7) Choose the set of correct options.

- **Option 1:** There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that Image(T)= Kernel(T).
- Option 2: There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that Image(T)= Kernel(T).
- Option 3: If $T : V \to V$ is a linear transformation and $v_1,v_2 \in V$ are linearly independent then $T(v_1)$, $T(v_2)$ are also linearly independent.
- **Option 4:** If $T : V \to V$ is a linear transformation and $T(v_1)$, $T(v_2)$ are linearly independent then v_1, $v_2 \in V$ are also linearly independent.