Solving the Wave Equation with the Finite Difference Method

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Objective

The objective of this presentation is to simulate a physical system described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

using the Finite Difference Method (FDM), a technique for solving Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs).

Why?

For PDEs we do not have a standard technique that ensures to find an analytical solution. But using numerical techniques such as the FDM, we can find a numerical solution with a certain approximation by means of algorithmic computations.

Finite Difference Method

The Finite Difference Method follows two steps:

1. Replace the PDE with an algebraic equation
2. Solve the algebraic equation iteratively

Discretizing time and space

In order to apply the FDM, we have to evaluate the evolution of a function in a discrete manner. Sampling the time $[0, T]$ and space $[0, L]$ domain such that

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N_t - 1} < t_{N_t} = T$$

$$0 = x_0 < x_1 < x_2 < \ldots < x_{N_x - 1} < x_{N_x} = L$$

defines the **mesh** over the domain.

Discretizing time and space

Given an uniformly distributed mesh with the constant spacing Δx and Δt such that

$$x_i = i\Delta x, \quad i = 0, 1, ..., N_x$$

$$t_n = n\Delta t, \quad n = 0, 1, ..., N_t$$

we can evaluate a function $u(x, t)$ over the mesh points.
In our notation, n in the temporal index and i is the spacial index and for convenience $u(x_i, t_n) \rightarrow u_i^n$

Visualizing the mesh

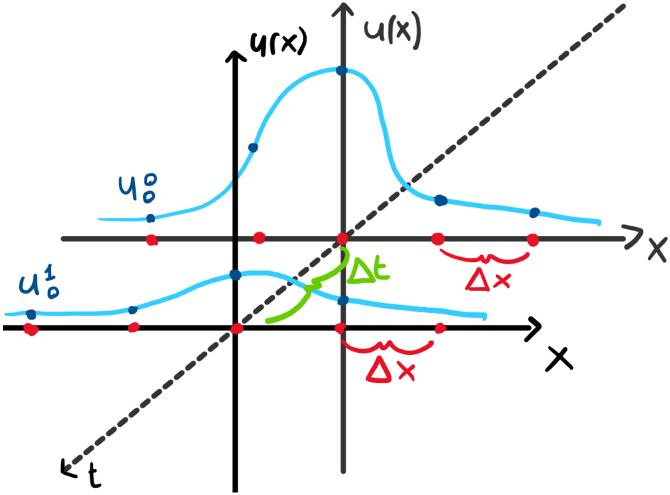

Figure: Sampling in space and time

Visualizing the mesh

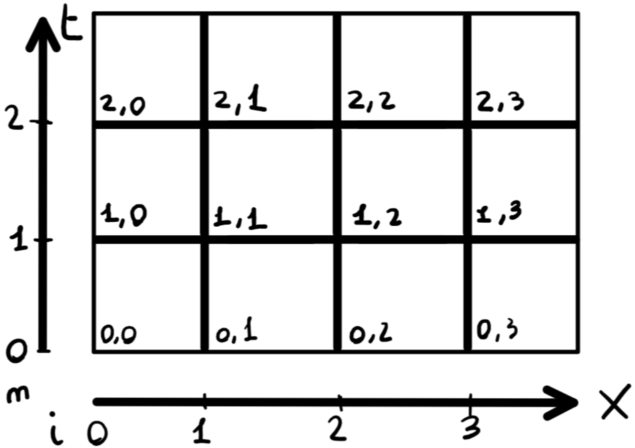

Figure: Mesh with $N_x = 3$ and $N_t = 2$

Approximating 1D PDEs (ODEs)

Aim: approximate
the tangent in $u(x_i)$

$$\frac{du(x_i)}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{u_i - u_{i-1}}{\Delta x}$$

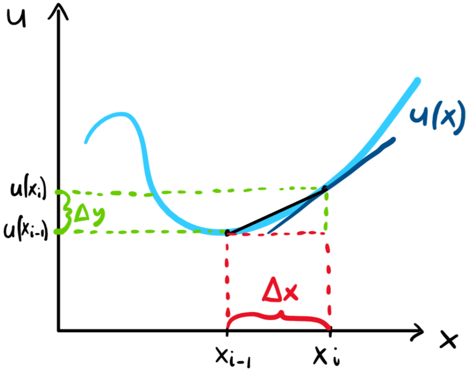

Figure: Derivative as the limit of the incremental ratio

Approximating 2D PDEs

There are three main techniques for approximating 2D PDEs:

1. Backward difference
2. Forward difference
3. Central difference

1.Backward difference

$$\frac{\partial u_i^n}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

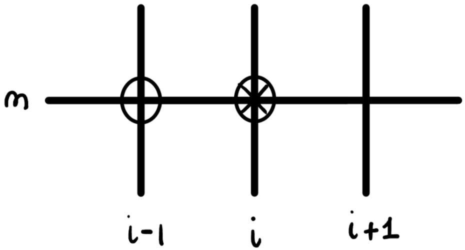

Figure: Stencil of the backward difference

2.Forward difference

$$\frac{\partial u_i^n}{\partial x} \approx \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

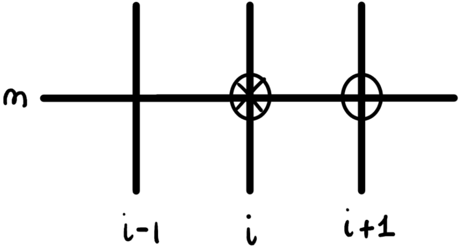

Figure: Stencil of the forward difference

3.Central difference

With a step of Δx

$$\frac{\partial u_i^n}{\partial x} \approx \frac{u_{i+1/2}^n - u_{i-1/2}^n}{\Delta x}$$

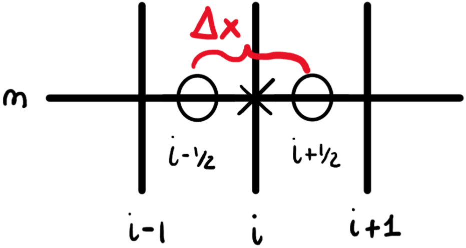

Figure: Stencil of the backward difference

Central difference for second order PDEs

$$\frac{\partial^2 u_i^n}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_i^n}{\partial x}\right) \approx \frac{\partial}{\partial x}\left(\frac{u_{i+1/2}^n - u_{i-1/2}^n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}\left(\frac{\partial u_{i+1/2}^n}{\partial x} - \frac{\partial u_{i-1/2}^n}{\partial x}\right)$$

$$\approx \frac{1}{\Delta x}\left(\frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{u_i^n - u_{i-1}^n}{\Delta x}\right)$$

$$= \frac{u_{i+1}^n - 2u_i^n - u_{i-1}^n}{\Delta x^2}$$

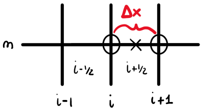

Figure: Stencil of the backward difference

Simulating waves on a string

Our system is a string of length L described by $u(x,t)$, the vertical displacement at time t, and the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Initial conditions:

$$u(x,0) = I(x)$$

$$\frac{\partial}{\partial x} u(x,0) = 0$$

Boundary conditions:

$$u(0,t) = 0$$

$$u(L,t) = 0$$

Finding the algebraic equation

From the discretized wave equation

$$\frac{\partial^2}{\partial t^2}u(x_i, t_n) = c^2\frac{\partial^2}{\partial x^2}u(x_i, t_n)$$

we rewrite the second order derivatives using the central differences

$$\frac{\partial^2}{\partial t^2}u(x_i, t_n) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}$$

$$\frac{\partial^2}{\partial x^2}u(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Finding the algebraic equation

Resulting in

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

where we can explicit the term u_i^{n+1}:

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

where $C = c\frac{\Delta t}{\Delta x}$ is the **Courant number**, a parameter that governs the quality and the stability of the approximation.

Algebraic equation

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

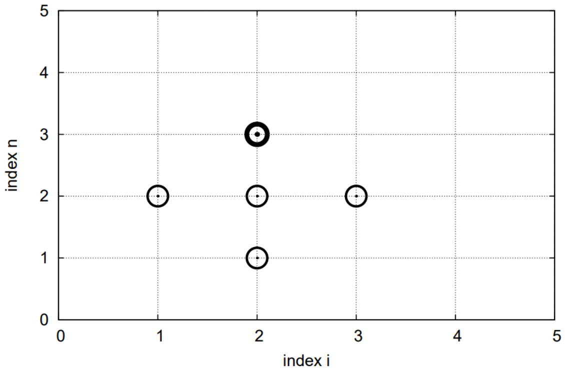

Figure: Mesh and stencil in space and time

Finding the algebraic equation

The initial condition for the velocity

$$\frac{\partial}{\partial t}u(x_i, 0) = 0$$

can be again approximated with central differences (by a step of $2\Delta t$)

$$\frac{\partial}{\partial t}u(x_i, 0) \approx \frac{u_i^1 - u_i^{-1}}{2\Delta t}$$

leading to

$$u_i^{-1} = u_i^1 \qquad i = 0, 1, ..., N_x$$

While the initial condition for the displacement is

$$u_i^0 = I(x_i) \qquad i = 0, 1, ..., N_x$$

First time step

If we apply the algebraic equation at the first time step $n = 0$

$$u_i^1 = -u_i^{-1} + 2u_i^0 + C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

where we notice that u_i^{-1} is not defined in our mesh, but by using the initial condition for the velocity

$$u_i^{-1} = u_i^1$$

we get

$$u_i^1 = u_i^0 + \frac{1}{2}C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

First time step

$$u_i^1 = u_i^0 + \frac{1}{2}C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

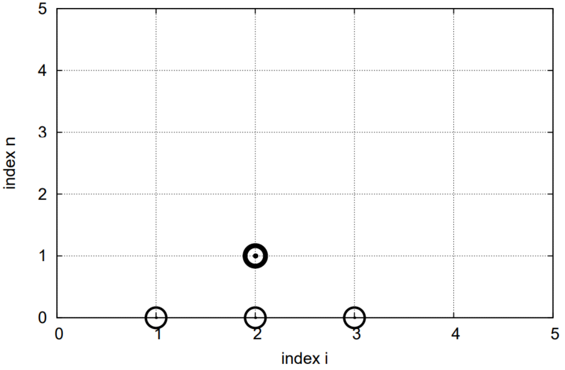

Figure: Modified stencil for the first time step

Other simulations

1. $u_{tt} = c^2 u_{xx} + f(x,t) \Rightarrow$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \Delta t^2 f_i^n$$

2. $u_{tt} = c^2(u_{xx} + u_{yy}) \Rightarrow$

$$\begin{aligned} u_{i,j}^{n+1} = -u_{i,j}^{n-1} + 2u_{i,j}^n &+ C_x^2(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) \\ &+ C_y^2(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) \end{aligned}$$