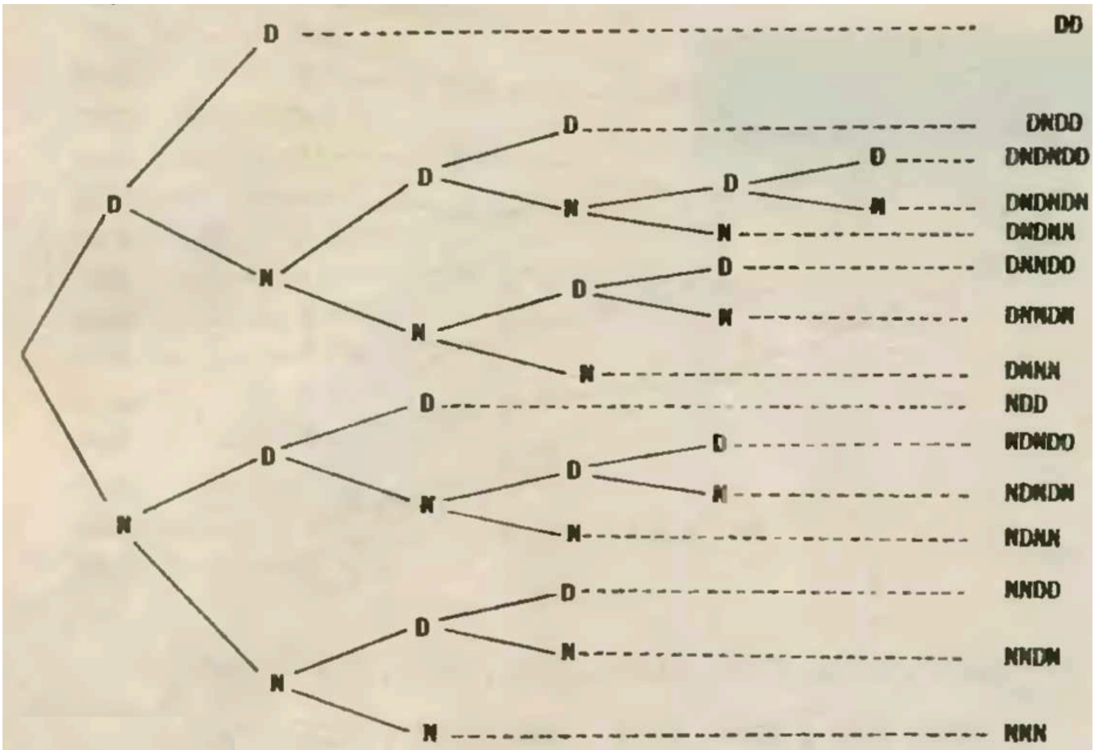
PROBABILITY THEORY – EXERCISES SESSION N° 1

SOLUTIONS

TD1 – 1 *:



TD1 – 2 :

a) Considering the total probability law:

$P(B) = P(B \cap A) + P(B \cap \bar{A})$,

then:

$$P(B \cap \bar{A}) = P(B) - P(B \cap A)$$

$P(B \cap \bar{A}) = P(B) - P(B)P(A)$ (because A and B are independent)

$$P(B \cap \bar{A}) = P(B)(1 - P(A) = P(B)P(\bar{A})$$

Then, $\bar{A}$ and B are independent.

b) Application

b.1) 0.045

b.2) 0.855

<u>TD1 – 3</u> :

a) For 4 tires 24 and for 5 tires 120
b) 336
c) 201376
d) 60

<u>TD1 – 4</u> :

a) 1/3

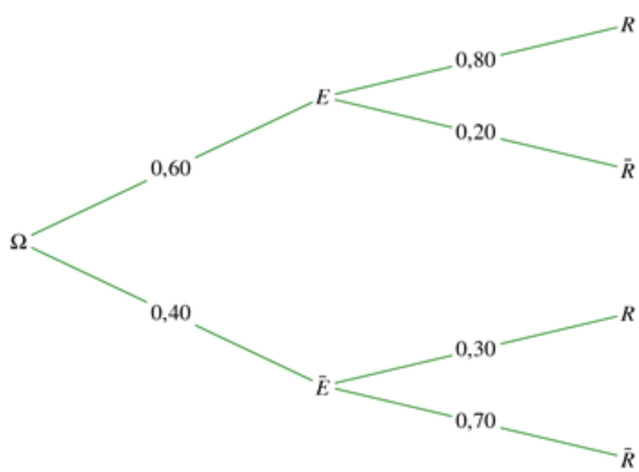
b) 3/11

<u>TD1 – 5</u> :

a) It's the probability that a call comes from outside and that Lisa refuses to transfer to her boss

b) 0,8.

c) The tree



d) 0.8

<u>TD1 – 6</u> :

a) 1/4

b) 5/8

<u>TD1 – 7</u> :

a) 0,435

b) 0, 517

c) 0,531

PROBABILITY THEORY – EXERCISES SESSION N° 2

TD2 – 1 :

a)

X is an integer between 0 and n

X/n is the relative frequency of A ;

b) $P(X=k)=C_n^k p^k(1-p)^{n-k}$ for k = 0, 1, …, n the binomial distribution

c) We have :

- $\forall k, P(X=k)\geq 0$ and
- the probability of the certain event is : $\sum_{k=1}^{n} P(X=k)=(p+1-p)^n=1$
- Then, : $\forall k, P(X=k)\leq 1$.

TD2 – 2 :

a) $P(X=k)=\frac{21-2k}{100}$ for k between 1 and 10

b) Let Y denote the expected gain E(Y). We have $E(Y)=\frac{3}{10}$

The game favorable to the player

TD2 – 3 :

If 0≤p<2/3 choose A. If p=2/3, choose either A or B, and if 2/3<p<1, choose B.

TD2 – 4 :

a) $P(X=Y)=\frac{1}{n}$

b) $P(X\geq Y)=\frac{1}{2}+\frac{1}{2n}$

c)

There are two possibilities :

a. If k ≤ n, $P(X+Y=k)=\frac{k-1}{n^2}$

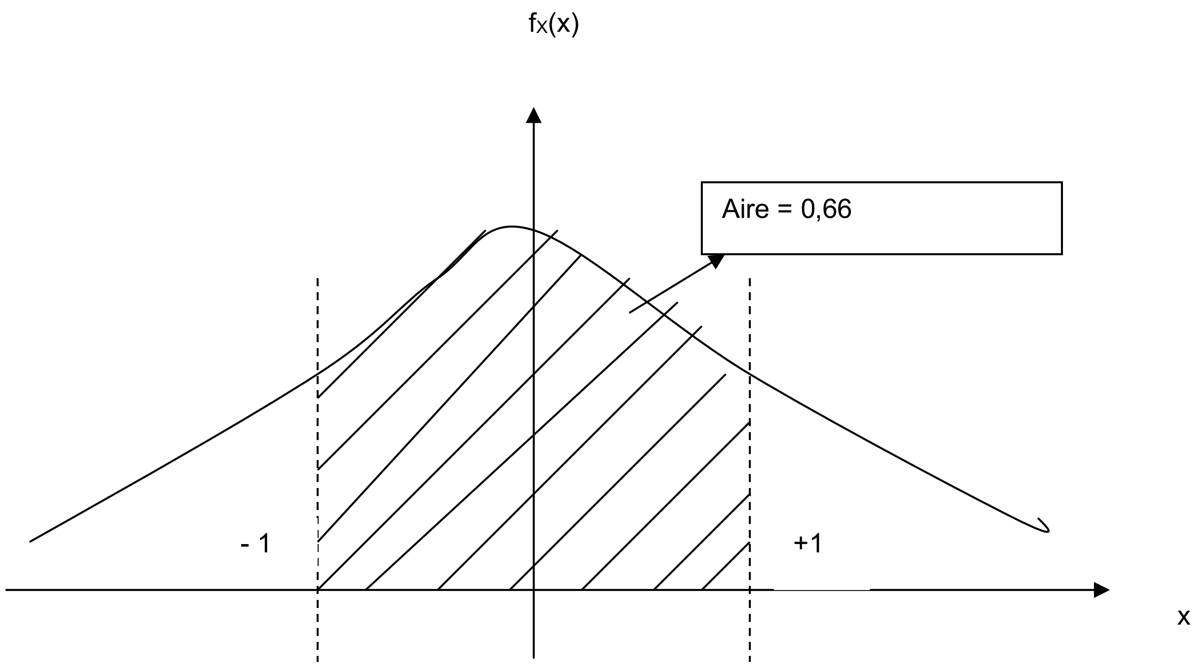
b. If k> n, $P(X+Y=k)=\frac{2n+1-k}{n^2}$

TD2 – 5 :

- greater than 20? 0.0099
- less than 5? 0,0038
- comprised between 12 and 14? 0,2586
- greater than 12? : 0,6293
- minimal threshold grade 16,84

TD2 – 6 :

a)



b)

Probability	Value
$P(0 \leq X \leq 1)$	0,33
$P(X \leq 1)$	0,83
$P(X > 1)$	0,17
$P(X = 1)$	0
$P(X \geq 1)$	0,17
$P(\lvert X \rvert \leq 1)$	0,66
$P(X^2 \leq 1)$	0,66
$P(X^3 \leq 1)$	0,83

PROBABILITY THEORY – EXERCISES SESSION N° 3

SOLUTIONS

TD3 – 1 :

a) Probability distribution of X

X	-2	1	2	3
P(x)	½	1/6	1/6	1/6

The cumulative distribution function (CDF) F_X :

$$F_X(x) = \begin{cases} 0 \; si \; x < -2, \\ \frac{1}{2} \; si - 2 \leq x < 1, \\ \frac{2}{3} \; si \; 1 \leq x < 2, \\ \frac{5}{6} \; si \; 2 \leq x < 3, \\ 1 \; si \; x \geq 3. \end{cases}$$

b) $E(X) = 0$

c) V(X)= $\frac{13}{3}$.

d) Sample set:

$$\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$

Possible outcomes :

(u, v)	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	3	3	3	-5	-5	-5
5	3	3	3	-5	-5	-5
6	3	3	3	-5	-5	-5

e)

Y	-5	1	2	3
P_Y(y)	9/36	6/36	6/36	15/36

$$E(Y) = \frac{1}{2}$$

f) Given that E(Y) > E(X), on average the second variant is more advantageous than the 1st one.

TD3 – 2 :

a) $p^2(3 - 2p)$.

b) $q^2(3q^2 - 8q + 6)$.

c) Si p = q the solution with redundant power supplies is more interesting.

TD3 – 3:

a) $e^{-4} \approx 0.0183$

b) $e^{-4}\frac{4^5}{5!} \approx 0.156$

c) $1 - e^{-4}\sum_{k=0}^{5}\frac{4^k}{k!} \approx 0.215$

TD3 – 4:

a) Conditions *f(x)* ≥ *0* et $\int_{-\infty}^{+\infty} f\,(x)dx\ =\ 1$

This implies $a \in \left[-\frac{3}{2}, 3\right]$ et $b = 1 - \frac{a}{3}$

b) *a=3* and *b=0*

c) $E(X) = \frac{3}{4}$ and $V(X) = \frac{3}{80}$

TD3 – 5:

a) $\frac{1}{4}$

b) 50 cl.

TD3 – 6:

The minimum average lifespan of a hard disk is 99,5 years !

TD3 – 7 *:

a) $f_X(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$ It is the Cauchy distribution.

a) Not, because de derivative of $\varphi_X(t)$ is undefined at the point X=0.

TD3 - 8 :

P(X<7 ou X>19) ≈ 0,0454

Using the Tchebychev inequality, we find the upper bound:

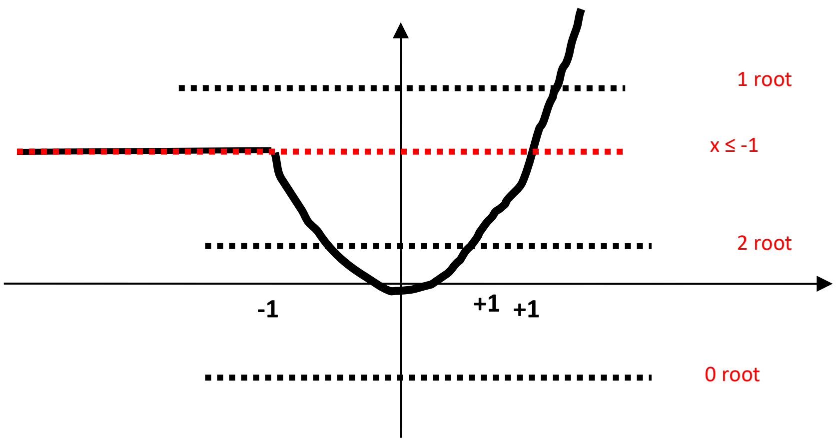
$$P(|X - 13| > 6) \leq \frac{3^2}{6^2} = 0{,}25$$

PROBABILITY THEORY – EXERCISES SESSION N° 4

SOLUTIONS

TD4 – 1 :

a)



For $y < 0$, $f_Y(y) = 0$

For $0 \leq y < 1$ $f_Y(y) = e^{-y}$

For $y = 1$ $P(Y = 1) = \frac{1}{2e}$

For $y > 1$ $f_Y(y) = \frac{1}{2}e^{-y}$

b)

For $y < 0$, $F_Y(y) = 0$

For $0 \leq y < 1$ $F_Y(y) = 1 - \frac{1}{e^y}$

For $y = 1$ $F_Y(y) = 1 - \frac{1}{2e}$

For $y > 1$ $F_Y(y) = 1 - \frac{1}{2e^y}$

Verify that this function lies between 0 and 1, it is monotonically non-decreasing and right continuous.

TD4 – 2 :

a) α = 0,05

b) Marginal of *X* ;

x	*x=-2*	*x=0*	*x=1*
P(X=x)	0.45	0.25	0.3

Marginal of Y:

y	*y=-1*	*y=1*	*y=2*
P(Y=y)	0.5	0.3	0.2

c) Counter-example, P(X=1,Y =1) =0 ≠ P(X=1)×P(Y =1)= 0,3×0,3.

d)

x	*2*	*0*	*1*
P(X=x| Y=1)	2/3	1/3	0

e)

x	*2*	*0*	*1*
P(X=x| Y≠2)	0.4/0.8=1/2	0.2/0.8=1/4	0.2/0.8=1/4

f)

P(Z =−3) = 0.2

P(Z =−1) = 0.3

P(Z =0) = 0.25

P(Z =1) = 0.1

P(Z =2) = 0.05

P(Z = 3) = 0.1

<u>TD4 – 3</u> :

a) A = 2, so

$$f_{X,Y}(x,y) = 2xy \text{ for } (x,y) \in D$$

b) Marginal density of X

$$f_X(x) = 4x^3 \text{ for } 0 \leq x \leq 1$$

Marginal density of Y

$$f_Y(y) = \frac{4y - y^3}{4} \text{ for } 0 \leq y \leq 2$$

For both functions you should verify that they are positive and their integral in their definition domain is equal to 1.