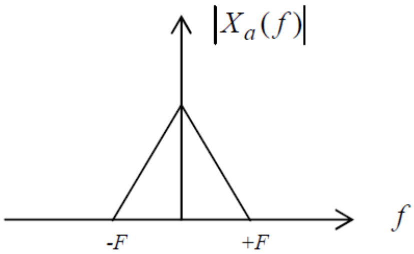
Tutorial n°1:
Signal Sampling and its spectrum

Exercise 1

We consider a continuous-time analog signal $x_a(t)$ with its spectrum, represented by the Fourier Transform module (TF) $X_a(f)$ which is bounded in the frequency range [- F, + F]. We sample $x_a(t)$ at the sampling frequency Fe=1/Te, where, Fe = 5F.

This sampling process, supposed ideal, is written using a Dirac function: x_e e(t)=x_a(t).W_{Te} (t)
We consider the following plot for the Fourier Transform of xa (t)?



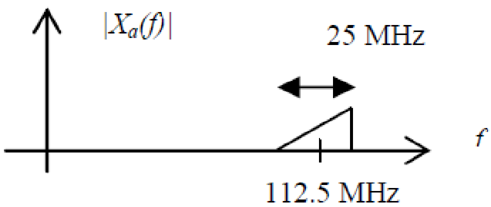
1. Plot the graph of |Xe(f)| for the frequency range [-20F, + 20F].

We sample the continuous time signal $x_a(t)$ given by, $x_a(t)$=Acos(2pf0t) + B. Thus, we obtain the sampled signal x(nTe) = x_a(nTe).

2. Determine the expression of x(nTe), express the Fourier Transform of this signal, and plot the graph of its spectrum.

Exercise 2

We consider a real-time continuous analog signal $x_a(t)$ with a bounded spectrum $X_a(f)$ of 25MHz and centered on 112.5MHz. The spectrum part corresponding to the "positive" frequencies is shown below.



An engineer has proposed to sample this signal at a frequency Fe = 90MHz. Most engineers interested by this proposal, claim that it does not respect the sampling theorem and do not see the interest of this solution.
1. Present an argument to defend the proposed solution by this engineer.
2. Represent the spectrum of the sampled signal,
3. The solution is it still valid, if the signal is centered on 100 MHz and sampled at Fe = 50MHz? could you express a sufficient condition for the proposed sampling such that this process stay always valid.

Exercise 3

A sound engineer records a concert with 2 microphones. It removes by filtering the signal components at frequencies above 22 kHz, then samples the signals from its microphones and quantifies the values on 16 bits. He wants to store digital signals on a CD-ROM. It is assumed that he does not perform any other processing on its data (no coding against possible errors for example).

What should be the capacity of his CD for 70 minutes of concert?

Exercise 4

In practice, the sampling process of a continuous time signal is followed by an encoding of each sample x into a digital value Q(x) (analog-to-digital conversion). This operation constitutes a digitization of the signal with a quantization step Δ that will be assumed constant. For $i\Delta \leq x \leq (i+1)\,\Delta$, Quantization method used in this exercise is a rounding of x value to $\pm\Delta/2$.

If a binary code on b bits, is used, the coding range is $A = 2b\ \Delta$. We define the quantization error by $e(x)=Q\,(x)-x$.

1. Determine the expression of the quantization error for x located in the i-th quantization interval and plot its corresponding graph.

2. We assumed that the quantization error e, also called quantization noise, is a continuous random variable, non-correlated with x, and whose probability density function is uniform. Prove that the signal is centered, and express its variance σ_e^2 versus Δ.

3. The signal-to-quantization noise ratio in dB is defined as:

$$\Gamma_{dB} = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right)$$

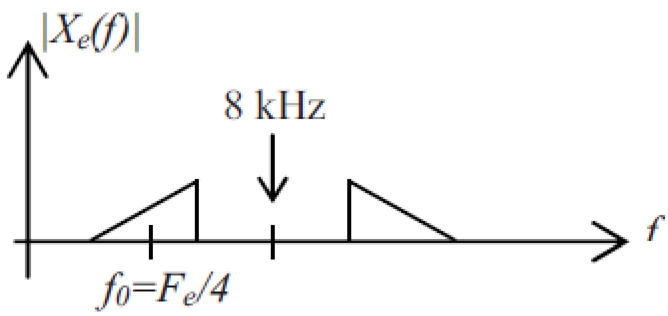
In this expression σ_x^2 represents the variance of x. The dynamic range of x is assumed to be in the coding range, $-A/2 \leq x \leq A/2$.

Determine the expression of Γ as a function of σ_x, the number of bits b, and the amplitude A of the coding range. What is the gain in decibel (dB) of an additional bit?

4. Calculate the maximum value of Γ before saturation for a 16-bit coding with the following supposed centered signals:
- a sinusoidal signal,
- a Gaussian signal with an estimated peak value to $4\sigma_x$.

Tutorial n°2:
FT, DFT and FFT

Exercise 1
We consider a real digital signal x(nTe) with the following spectrum



We multiply term by term the samples of this signal by a cosine carrier with a normalized frequency f1n =0.0125 to obtain the signal y(nTe).

- Compute the value of the frequency f1,
- Express y(nTe) as function of x (nTe),
- Represent the spectrum of y(nTe) (we assume that the signal band is very weak compared to the frequency value f1).

Exercise 2
We consider an analog signal $\Pi(t)$ defined as follows

$$\Pi(t) = 1,\ t \in \left[\frac{-\tau}{2}, \frac{\tau}{2}\right]$$
$$\Pi(t) = 0,\ t \notin \left[\frac{-\tau}{2}, \frac{\tau}{2}\right]$$

We digitize this signal with a sampling frequency Fe.

- In order to respect the Shannon's theorem, which is the condition that must be fulfilled by Fe?
- We consider now the digitized function composed of M samples given as follows:

$$\Pi(nT_e) = 1\ ,\ n \in [0, N-1]$$
$$\Pi(nT_e) = 0\ ,\ n \in [N, M-1]$$

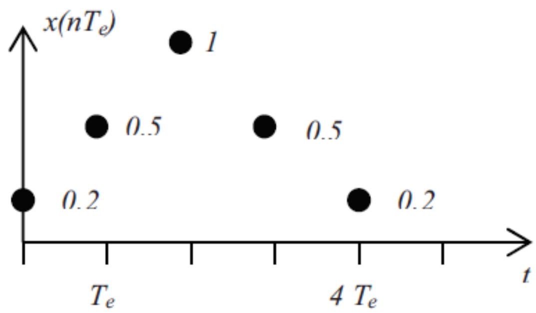
- Calculate the Discrete Fourier Transform of this digital function and plot the spectrum from 0 to Fe by specifying the null values of this spectrum.

We introduce the Discrete Cosine Transform as follows.

$$C(f) = \sum_{n=-\infty}^{+\infty} x(nT_e)\cos(2\pi f n T_e)$$

- Express C(f) in function of the Discrete Fourier Transform X(f). (We note that the temporal signal is real).
- Which condition must be fulfilled by the digital temporal signal x(nTe) in order to have C (f) = X (f).

Let consider N signal samples given as follows.



- The Discrete Fourier Transform of this signal, is it real?
- Compute the Discrete Fourier Transform values of this signal for the frequencies multiple of Fe/N.
- Which sample we could add to solve the problem?

Exercise 3

We observe for 10 µs a radio signal that we sample at 10 MHz rate. This signal is the echo of a radar pulse of 50 µs (i.e., a small window of sinusoid function) which had been transmitted a time ago to a moving vehicle.
The transmitted signal occupied, due to its special shape, a bandwidth of 100 kHz, and it was transposed into frequency at 5 GHz. It has been demodulated in baseband by frequency transposition.
- If we compute an FFT of the received signal, which precision we obtain on the frequency estimation?
- Compute the precision on the mobile speed by considering the Doppler effect.
- How could this accuracy be improved?

Tutorial n°3:
Finite Impulse Response (FIR) Filters

Exercise 1

We consider a finite impulse response (FIR) filter with the following coefficients
$h_2= h_3=0.1$
$h_1= h_4=0.3$
$h_0= h_5=0.6$

1. Compute the response of this filter at the null frequency f=0.
2. Compute the response of this filter at the frequency f=0.5.
3. We change the sign of odd index coefficients, what is the response of the new obtained filter at the frequency 0.5.
4. Give a scheme of realization of the filter given above. How many multiplication operations are required at each filter output?

Exercise 2

We consider a finite impulse response (FIR) of a digital filter with the following equation.

$$y(n)=\sum_{k=0}^{n-1} a_k x(n-k)$$

The filter coefficients a_k have been computed such that the considered filter is a low pass one with a cutoff frequency fc=Fe/20, (Fe is the sampling frequency).
What becomes this filter if we replace the coefficients a_k by $a_k\cos(2\pi f0kTe)$, f0=Fe/5.

Exercise 3

We consider a filter defined by the following equation.

$$y(n)=x(n)+0.7x(n-1)+0.9.x(n-2)$$

The sampling frequency is equal to 10MHz, the considered signals are real.
Which is the minimal computational power required for the realization of this filter?

Exercise 4

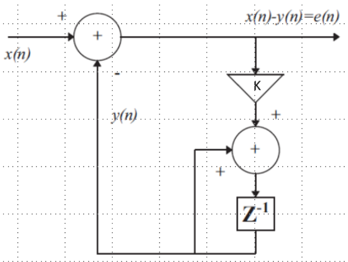
We consider a RIF digital filter A(z) with 64 coefficients equal to 1. We consider x(n) the filter input and y(n) the filter output at a given time n.
1. Is it possible to obtain the RIF coefficients among the FFT output of the input signal?
2. Plot the transfer function in frequency domain of the considered filter.

We multiply term by term the filter coefficients by $e^{j2\pi il/N}$.
3. Plot the transfer function in frequency domain of the obtained filter.
4. Is it possible to obtain the output filter at a given time n by using an FFT?
5. We want to decompose a signal by using a filter bank of 64 equidistant frequency filters. Give a solution by using an FFT.

Tutorial session n°4:
Infinite Impulse Response (IIR) Filters

Exercise 1

A phase locked loop is modeled by the following circuit.



1- Compute the relationship between the input sequence x(n) and the output sequence y(n). Deduce the Z transform of the considered filer.
2- Compute the output of this filter to the unit level input. Check that it corresponds to a control of y(n) by using x(n), that is, it tends towards unity when n tends towards infinity.
3- The coefficient K represents the gain of the control loop. Compute the range of K values in order to obtain the system stability.

Exercise 2

We consider the following digital filter defined by its zeros *Z1 = 0.09 ± j 0.99 ; Z2 = 0.58 ± j 0.81* and poles *P1 = 0.62 ± j 0.26* ; *P2 = 0.70 ± j 0.58*
1- Indicates the type of this filter (FIR or IIR).
2- Determines its order.
3- Which is its transfer function (high pass filter, low pass filter, or pass-band filter)?

Exercise 3

We consider the following digital filter defined by its Z transform given as follows.

$$H(Z) = \frac{b_2 + b_1 Z^{-1} + Z^{-2}}{1 + b_1 Z^{-1} + b_2 Z^{-2}}$$

1- Indicates the filter type (FIR or IIR), and its order?
2- Compute the Fourier Transform of this filter.
3- let $H(z) = \frac{N(z)}{D(z)}$ and we introduce $\varphi_D(\mathrm{w})$ as the phase response of the denominator $D(z)$. Compute the phase response of H(z) as a function of w and $\varphi_D(\mathrm{w})$.

Exercise 4

We consider the following digital filter defined as follows.

$$y(n) = x(n) - b_1 y(n-1) - b_2 y(n-2)$$

1- Which is the input signal x(n) that leads to an output $\delta(n)$. Discuss this result?

2- We note that a necessary and sufficient condition for digital filter stability is to have $\sum_{n=-\infty}^{n=+\infty} |h_n|$ as a bounded value, where h_n is the n^{th} value of the filter impulse response. We consider the case in which b1 and b2 are chosen such that $\frac{1}{1 + b_1 Z^{-1} + b_2 Z^{-2}}$ has two conjugates complex poles P and P^*.
3- Compute the impulse response of this filter,
4- Demonstrate that the stability condition consists in choosing the pole inside the unit circle. We can decompose the fraction into simple fraction elements and perform the division, (the use of the polar coordinates for the pole expression seems to be more advantageous for the calculation).

Tutorial session n°5:
Correlation of random signals

Exercise 1

We consider a random digital signal x(nTe). To simplify the notations, we suppose that Te=1, and we write the realization of the signal at the instant Te as x(n). The signal x(n) was obtained by filtering, using a FIR filter, a centered white Gaussian noise b(n) with variance σ^2. The equation of filtering is given as follows:

$$x(n) = 2.b(n) + 0.5.b(n-1) - 0.2.b(n-2) + 0.1.b(n-3)$$

1- Compute the autocorrelation coefficients of order 0, 1, 2, 3 in function of σ^2, denoted respectively, $r_{xx}(0)$, $r_{xx}(1)$, $r_{xx}(2)$, $r_{xx}(3)$.
2- The signal x(n), is it white?
3- The distribution of signal amplitude levels x(n), is it Gaussian? (Without justification).
4- Compute the autocorrelation matrix of order 3 that will be denoted R_3.

We assume that we have another signal denoted y(n), obtained also by noise filtering b (n), but it is written as follows:

$$y(n) = b(n) - b(n-2)$$

5- Compute the Z transform of the impulse response of the filter which has performed y(n) from b (n).
6- Place the zeros of this filter on a unit circle. Which are the cutoff frequencies of this filter? Plot the transfer function of this filter.
7- The signal y(n), is it white?

We consider now the inter-correlation coefficient between the signals x(n) and y(n), denoted by $r_{xy}(p)$ which is given by: $r_{xx}(p)=E[x(n)y(n-p)^*]$
8- Compute $r_{xy}(0)$, $r_{xy}(1)$, $r_{xy}(2)$.

Exercise 2

We consider a filter with only zeros: $Z_0 = e^{j\frac{\pi}{4}}$, $Z_0^* = e^{-j\frac{\pi}{4}}$,
A noise b(n) uniformly distributed between [0,1] is filtered by this filter. We note x(n) as the output of this filter.
1- Compute x(n) in function of the input signal.
2- Calculate the autocorrelation coefficients $r_{xx}(0)$, $r_{xx}(1)$ of order 0 and 1 of x(n);

Exercise 3

We consider a signal s(t) obtained by two sensors. On the first sensor, the signal is received with an additive noise $b_1(t)$. On the second one, the signal is received with an additive noise $b_2(t)$. These two noise variables are assumed to be independent, centered, Gaussian, and with the same variance σ^2. The signal s(t) is supposed to be centered and normalized, i.e., $E[s(t)]=0$ et $E[s^2(t)]=1$.
1- Express the Signal-to-Noise-Ratio (SNR) on each sensor. We denote by $x_1(t)$ and $x_2(t)$ the signals coming from the two sensors.

In order to improve this Signal-to-Noise Ratio, we decide to sum the signals coming from the two sensors. The obtained signal is denoted by $y(t) = x_1(t) + x_2(t)$.
2- Express the new Signal-to-Noise Ratio obtained by using y(t).

We suppose now that the second sensor works less well than the first one, and its received signal can be written as: $x_2(t)=\alpha\, s(t)+b_2(t)$, where α is an attenuation coefficient between 0 and 1.
3- Compute the Signal-to-Noise Ratio of the combination obtained by simple summation of the two signals by taking into account the attenuation coefficient α.
4- Plot the SNR versus the coefficient α between 0 and 1 and discuss the efficiency of this summation.