ISEP
IIF.1105 - IF.2301 Data Science Fundamentals
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Tutorial course 0 : Linear Algebra for Data Science

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1 Exercises

Exercise 1. Given the following sets of vectors, determine if they are linearly independent or not. for each case, justify your answer.

1. $\left\{ \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \right\}$.
2. $\left\{ \mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \right\}$.
3. $\left\{ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}$.
4. $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$.
5. $\left\{ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix} \right\}$

Exercise 2. Consider the matrix $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$

1. Verify that $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ and $\mathbf{P}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.
2. Calculate $\mathbf{D}^2$, $\mathbf{D}^3$ and $\mathbf{D}^k \quad \forall k, \in \mathbb{R}$.
3. Calculate $\mathbf{A}^k \quad \forall k, \in \mathbb{R}$.
4. Application : The calculator with two digits
A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = (1 - p)$ that it will not.
The process of transmission constitutes a *Markov chain*, that is a stochastic process where at any stage the outcome or state depends only on the outcome or state of the previous stage.

   (a) The transition probability matrix of the Markov chain is defined by a square matrix of general term denoted $p_{ij} = P(j/i)$ is the probability of moving from state i to state j in one step. How many states does the process of transmission have ? Which ones ? What is the matrix of transition probabilities ?

(b) Now, assuming that the process begins in state 0 and moves through two stages of transmission. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit) ?

(c) After n stages of transmission, what is the probability that the machine produces the correct digit ?

Exercise 3. Is it possible to diagonalize the following matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$?

If so, diagonalize it.

Exercise 4. An Introduction to Principal Component Analysis (PCA)
The following table contains the final marks obtained by 5 ISEP students in three subjects : mathematics (X_1), English language (X_2) and Computer Science (X_3). For all the subjects, the marks were centered to have mean 0.

Student	X_1	X_2	X_3
1	$\frac{1}{\sqrt{6}} + \frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
2	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$
3	$-\frac{1}{\sqrt{6}} + \frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{6}}$
4	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	0
5	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$

In this exercise you are going to perform *Principal Component Analysis (PCA)*. *PCA* is a dimensionality reduction method widely used for visualization. In this exercise you will represent all the points in 2D (instead of 3D) without losing any information. Follow the steps :

1. Calculate the variance matrix of these variables, that is $\mathbf{V} = \frac{1}{n}\mathbf{X}^T\mathbf{X}$, where n is the number of observations.
2. Calculate the eigenvectors of the variance matrix. Are all eigenvalues positive ? If not, interpret this result.
3. From the previous step, consider all the eigenvectors associated to non-null eigenvalues. Orthogonally project the five points onto these new axis. These will be the new coordinates of the points.
4. Plot the observations according to their new coordinates and interpret the results.

TD1-1

1° $\left\{ v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \right\}$

$2v_1 = v_2$

$\Rightarrow$ not linearly independent.

2. $\left\{ v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \right\}$

$$\begin{vmatrix} 3 & 6 \\ 2 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \to \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$r = 0 < \min(2, 2)$

$\Rightarrow$ linerly independent

3° $\left\{ v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}$.

$$\begin{pmatrix} 2a + 4b - 2c \\ a - b + 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \end{vmatrix}$$

$$\downarrow$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \leftarrow \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} \leftarrow \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$\hookrightarrow r = 2 = \min(3, 2) \Rightarrow$ independent

4. $\left\{ v_1=\begin{pmatrix}1\\3\\2\end{pmatrix},\ v_2=\begin{pmatrix}2\\6\\4\end{pmatrix},\ v_3=\begin{pmatrix}1\\0\\4\end{pmatrix} \right\}$

$$\begin{vmatrix}1&2&1\\3&6&0\\2&4&4\end{vmatrix}=\begin{vmatrix}1&2&1\\0&0&-3\\0&0&2\end{vmatrix}=\begin{vmatrix}1&2&1\\0&0&-3\\0&0&0\end{vmatrix}$$

$\Rightarrow r=2 \Rightarrow$ not independent

5. $\left\{ v_1=\begin{pmatrix}2\\3\\5\end{pmatrix},\ v_2=\begin{pmatrix}0\\0\\0\end{pmatrix},\ v_3=\begin{pmatrix}1\\1\\8\end{pmatrix} \right.$

$$\begin{pmatrix}2&0&1\\3&0&1\\5&0&8\end{pmatrix}\Rightarrow\begin{pmatrix}1&0&\frac{1}{2}\\3&0&1\\5&0&8\end{pmatrix}\Rightarrow\begin{pmatrix}1&0&\frac{1}{2}\\0&0&-\frac{1}{2}\\0&0&\frac{11}{2}\end{pmatrix}=\begin{pmatrix}1&0&\frac{1}{2}\\0&0&-\frac{1}{2}\\0&0&0\end{pmatrix}$$

$\Rightarrow r=2<\min(3,3)$ not independent

TD 1-2

P、P^{-1}用于将对角矩阵D、A 互相转化

1

$$A=\begin{pmatrix}6&-1\\2&3\end{pmatrix}$$

$$PDP^{-1}=\begin{vmatrix}1&1\\1&2\end{vmatrix}\begin{vmatrix}5&0\\0&4\end{vmatrix}\begin{vmatrix}2&-1\\-1&1\end{vmatrix}$$

$$=\begin{vmatrix}5+0&0+4\\5+0&0+8\end{vmatrix}\begin{vmatrix}2&-1\\-1&1\end{vmatrix}$$

$$=\begin{vmatrix}5&4\\5&8\end{vmatrix}\begin{vmatrix}2&-1\\-1&1\end{vmatrix}$$

$$=\begin{vmatrix}10-4&-5+4\\10-8&-5+8\end{vmatrix}=\begin{vmatrix}6&-1\\2&3\end{vmatrix}=A$$

② $D^2 = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 25 & 0 \\ 0 & 16 \end{vmatrix}$

$D^3 = \begin{vmatrix} 25 & 0 \\ 0 & 16 \end{vmatrix} \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 125 & 0 \\ 0 & 64 \end{vmatrix}$

$D^k = \begin{vmatrix} 5^k & 0 \\ 0 & 4^k \end{vmatrix}$

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③ $A = \begin{vmatrix} 6 & -1 \\ 2 & 3 \end{vmatrix}$

$A^2 = \begin{vmatrix} 6 & -1 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} 6 & -1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 36-2 & -6-3 \\ 12+6 & -2+9 \end{vmatrix}$

$= \begin{vmatrix} 34 & -9 \\ 18 & 7 \end{vmatrix}$

$A = PDP^{-1}$

$A^k = PD^kP^{-1} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 5^k & 0 \\ 0 & 4^k \end{vmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 5^k & 4^k \\ 5^k & 2\cdot 4^k \end{vmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 2\cdot 5^k - 4^k & -5^k + 4^k \\ 2\cdot 5^k - 2\cdot 4^k & -5^k + 2\cdot 4^k \end{vmatrix}$

4 a) $P = \begin{bmatrix} q & p \\ p & q \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$

TD1-3.

Exercise 3. Is it possible to diagonalize the following matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$?

If so, diagonalize it.

$$A = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$P(\lambda) = \Delta(A - \lambda I) = \Delta \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda) \times \left[(2-\lambda)(1-\lambda)\right]$$

$$= (2-\lambda)^2 (1-\lambda) = 0$$

$\therefore \lambda_1 = 1$ or $\lambda_2 = 2$

① $\lambda = 1$

$$\vec{x}(A - I) = 0$$

$$\vec{x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = 0$$

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2. Calculate the eigenvectors of the variance matrix. Are all eigenvalues positive ? If not, interpret this result. $\det(V-\lambda I)=0$ $\quad$ $\det(M)=0$
3. From the previous step, consider all the eigenvectors associated to non-null eigenvalues. Orthogonally project the five points onto these new axis. These will be the new coordinates of the points.
4. Plot the observations according to their new coordinates and interpret the results.

1)

$$X = \begin{bmatrix} \frac{1}{\sqrt{6}}+\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}}-\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}}+\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}}-\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$X^T = \begin{bmatrix} \frac{1}{\sqrt6}+\frac{1}{2\sqrt3} & \frac{1}{\sqrt6} & \frac{1}{-\sqrt6}+\frac{1}{2\sqrt3} & \frac{1}{-\sqrt3} & \frac{1}{-\sqrt6} \\ \frac{1}{\sqrt6}-\frac{1}{2\sqrt3} & \frac{1}{\sqrt6} & \frac{1}{-\sqrt6}-\frac{1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{-\sqrt6} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & 0 & -\frac{1}{\sqrt6} \end{bmatrix}$$

$$V = \frac{1}{5} \times X^T \cdot X$$

$$C_{ij} = \sum_{k=1}^{5} X^T_{ik} * X_{kj}$$

$$C_{11} = \left(\frac{1}{\sqrt6}+\frac{1}{2\sqrt3}\right)^2 + \left(\frac{1}{\sqrt6}-\frac{1}{2\sqrt3}\right)^2 + \left(-\frac{1}{\sqrt6}+\frac{1}{2\sqrt3}\right)^2 + \left(\frac{1}{-\sqrt3}\right)^2 + \left(\frac{1}{-\sqrt6}\right)^2$$

$C_{12} =$

C_{13}

$C_{21} =$

C_{22}

C_{23}

C_{31}

C_{32}

C_{33}
