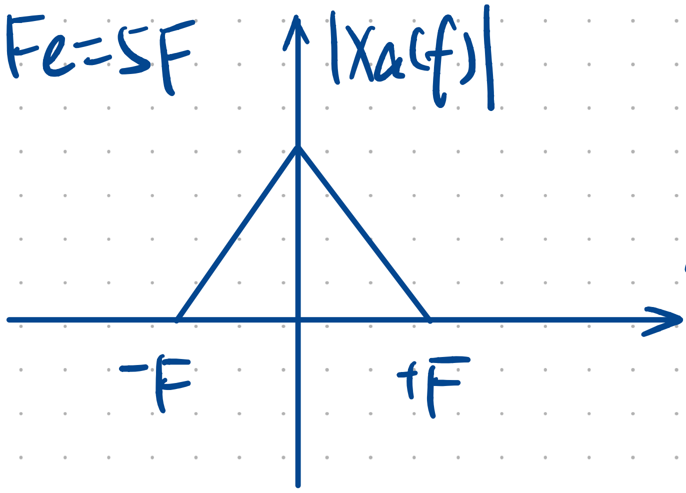
Exercise 1.1

$x_a(t) \xrightarrow{T_e} x_e(t) = x_a(t)$

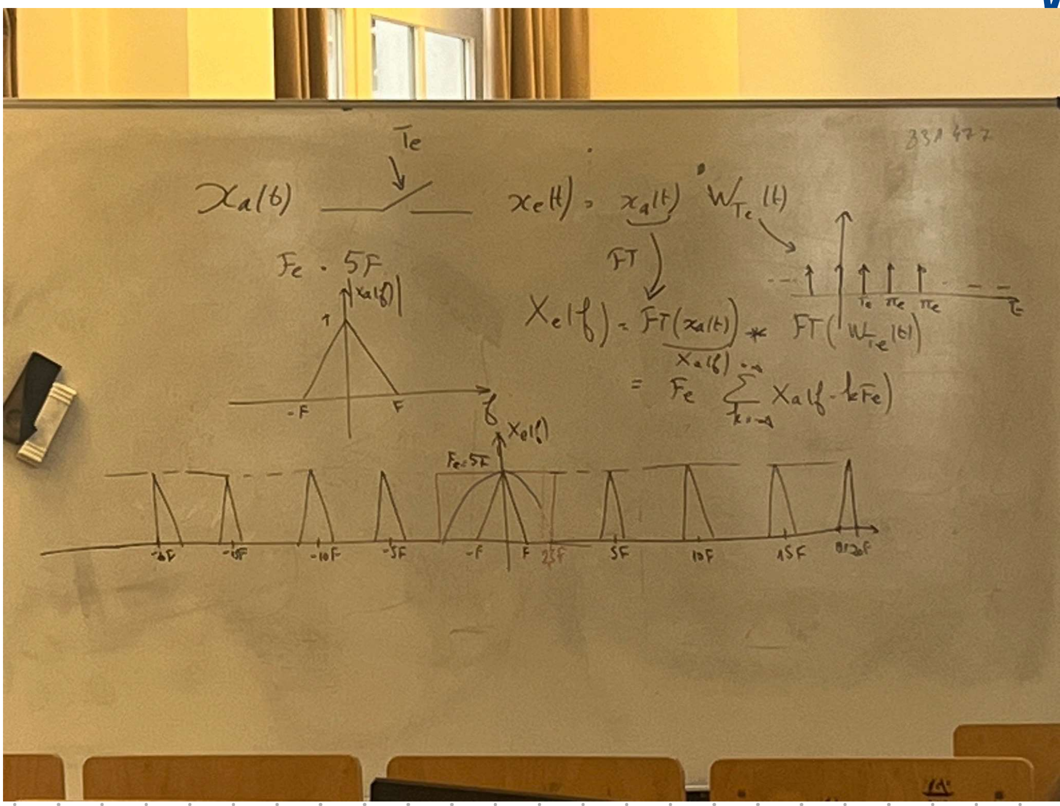
$F_e = 5F$



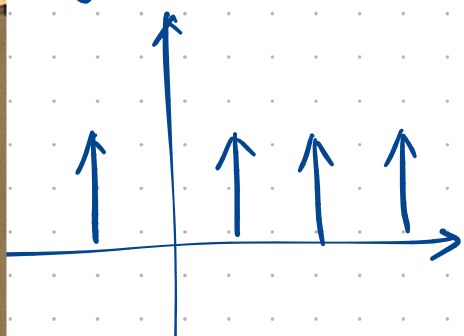
Ft

$X_e(f) = F_T(x_a(t))$

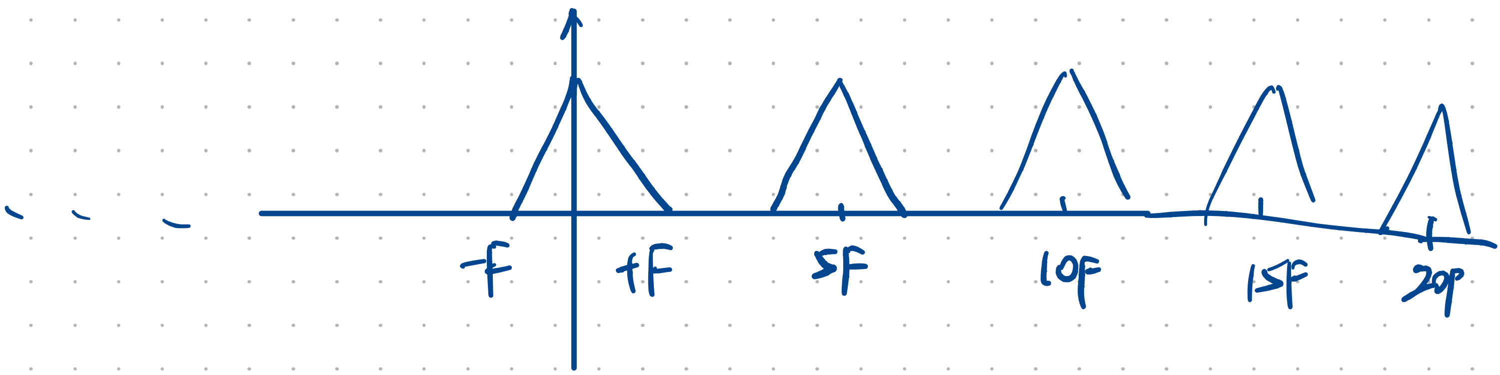
$= F_e \sum^{\infty} X_a(f-$



$W_{Te}(t)$



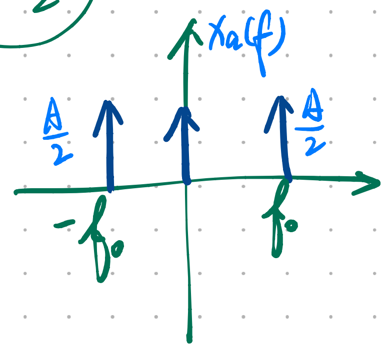
$Ft(W_{Te}(t))$



Exercice 1.

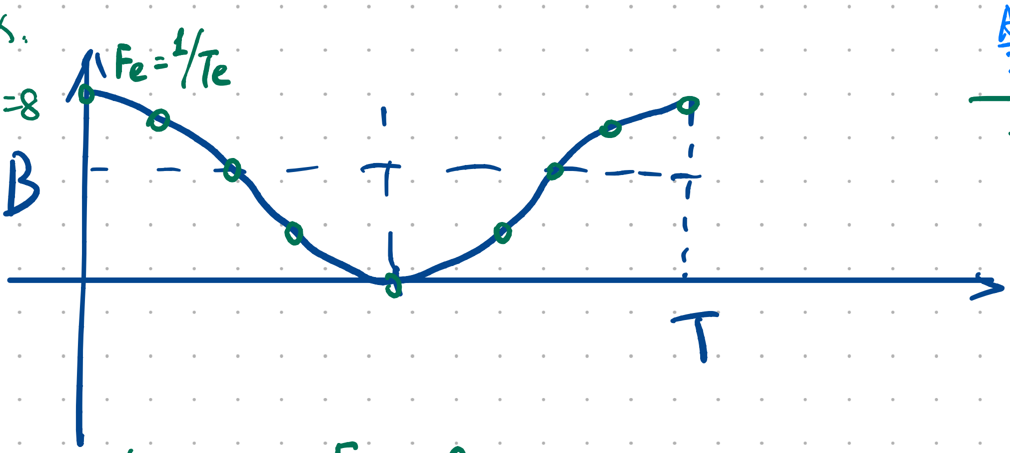
$$x_a(t) = A\cos(2\pi f_0 t) + B \xrightarrow{Ft} X_a(f) = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0) + B\,\delta(f)$$

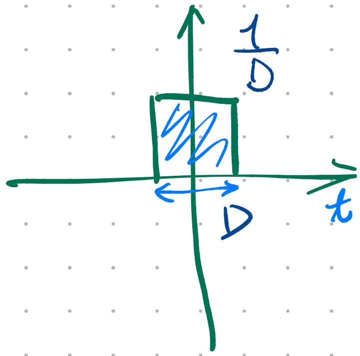
$$B = \frac{A}{2}$$



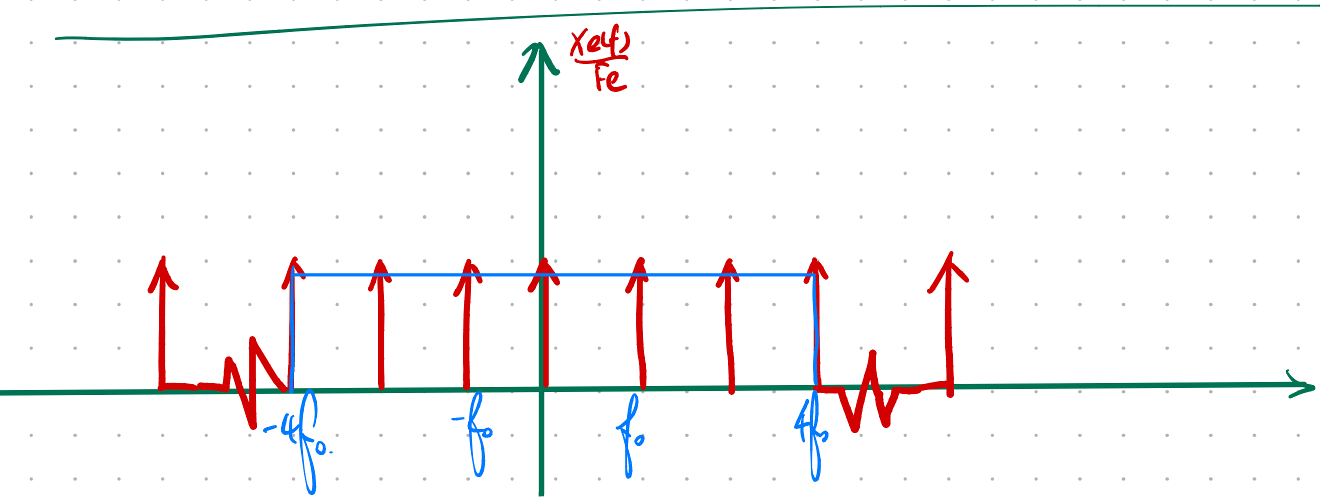
$$x(nT_e) = x_a(nT_e)$$

ex. $A = 8$



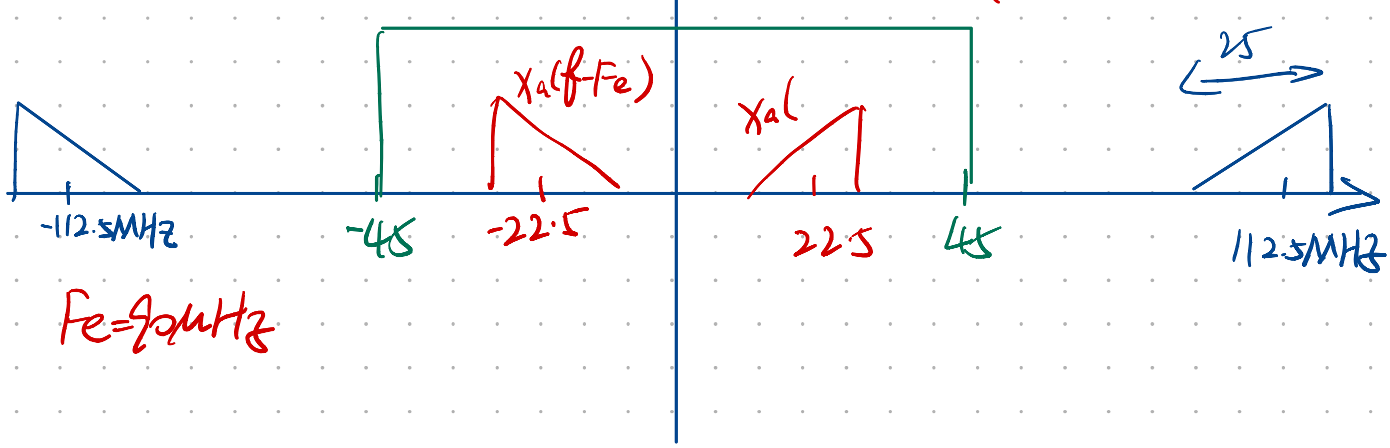


$(8T_e)^{-1} = (T)^{-1}$ $\quad \frac{F_e}{8} = f_0 \quad$ $F_e = 8f_0$

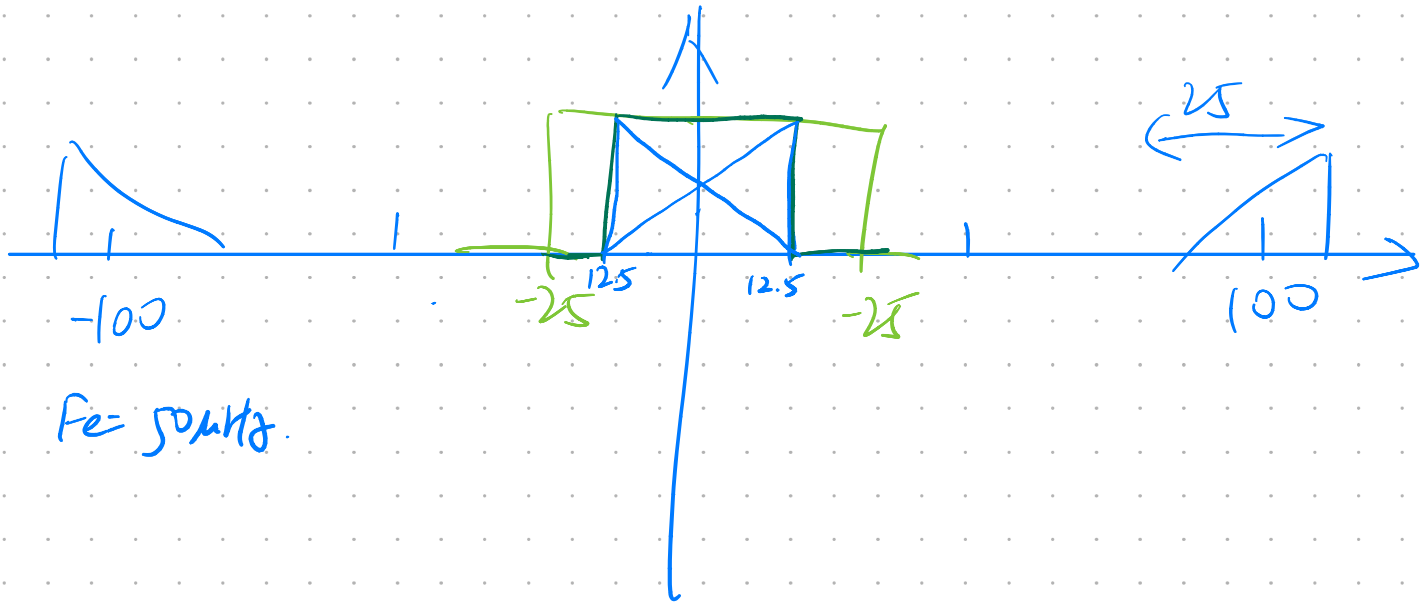


Exercise 2

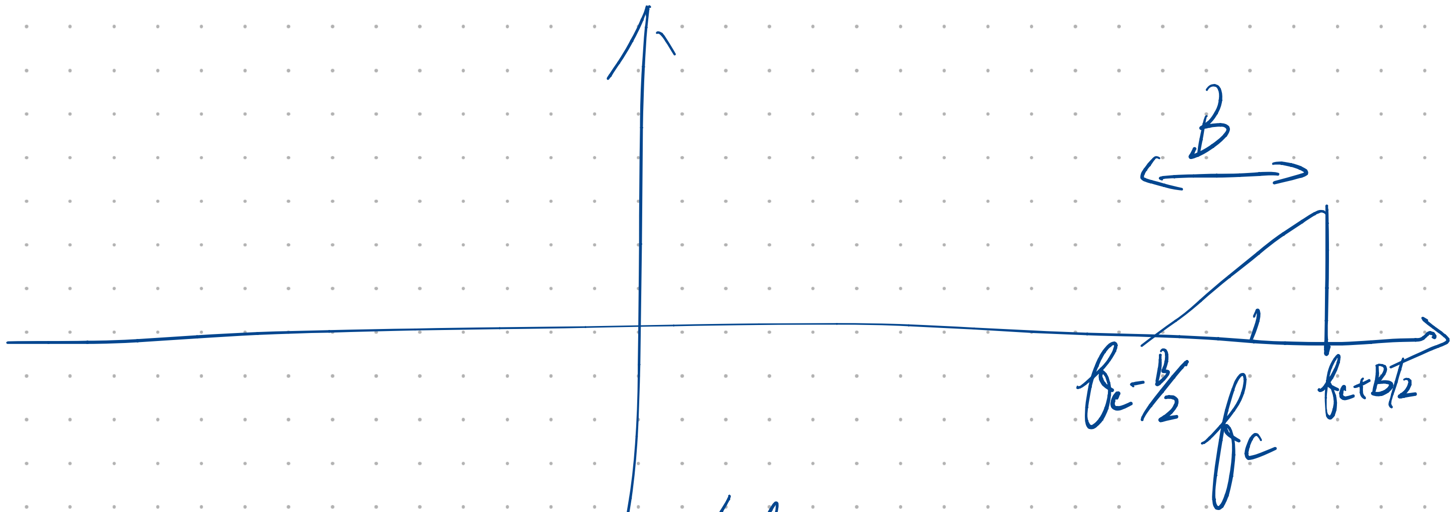
$X_e(f) = F_e \sum_{k=0}^{\infty} X_a(f - kF_e)$



Fe=90MHz



Fe= 50MHz.

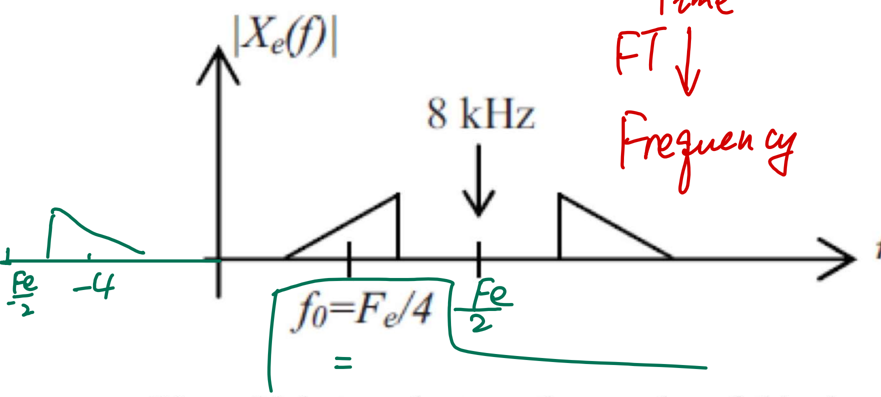


$$\begin{cases} f_c - kF_e - \frac{c}{2} > 0 \\ f_c - kF_e + \frac{B}{2} < \frac{F_e}{2} \\ B < \frac{F_e}{2} \end{cases}$$

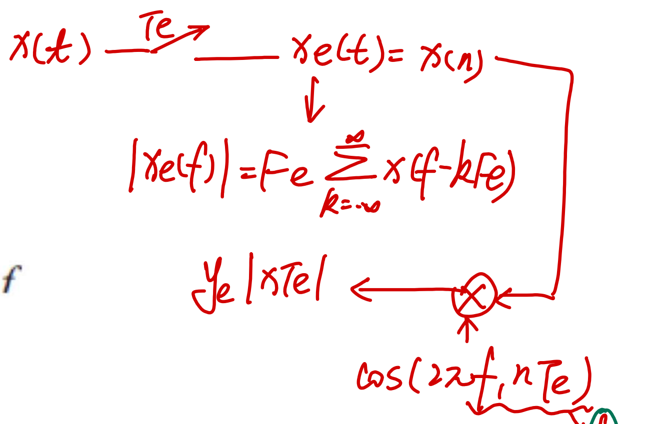
$f_c + kF_e + \frac{F_e}{4}$

Exercise 1

We consider a real digital signal x(nTe) with the following spectrum



Time
FT ↓
Frequency



We multiply term by term the samples of this signal by a cosine carrier with a normalized frequency f1n =0.0125 to obtain the signal y(nTe).

- Compute the value of the frequency f1,
- Express y(nTe) as function of x (nTe),
- Represent the spectrum of y(nTe) (we assume that the signal band is very weak compared to the frequency value f1).

$$f_{1n} = \frac{f_1}{Fe} \qquad 0.0125 = f_{1n}$$

① $f_{1n} = \frac{f_1}{Fe} = 0.0125$

$\Rightarrow f_1 = 0.0125\, Fe$

$= 0.0125 \times 16 \times 10^3\, Hz$

$= 200\ Hz$

② $y(nTe) = x(nTe) \cos(2\pi f_{1n} n)$

③ $B \ll 200\,Hz$

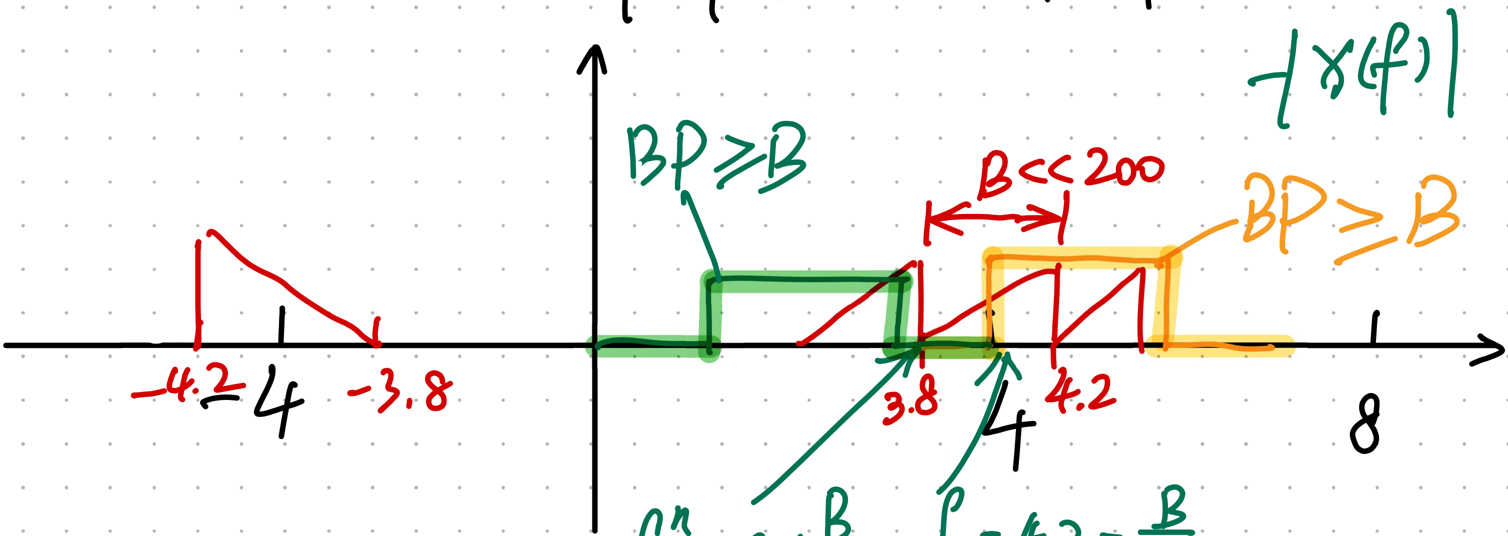
?

$$\boxed{\begin{array}{l} y_e(f) = F_T(y(nTe)) \\ \text{Plot} \end{array}}$$

$$FT \begin{cases} y(t) = x(t) \cos(2\pi f_1 t) \\ y(f) = x(f) * \left(\frac{1}{2}\delta(f-f_1) + \frac{1}{2}\delta(f+f_1)\right) \end{cases}$$

$$= \frac{1}{2} X(f) * \delta(f - f_1) + \frac{1}{2} X(f) * \delta(f + f_1)$$

$$= \frac{1}{2} X(f - f_1) + \frac{1}{2} X(f + f_1)$$

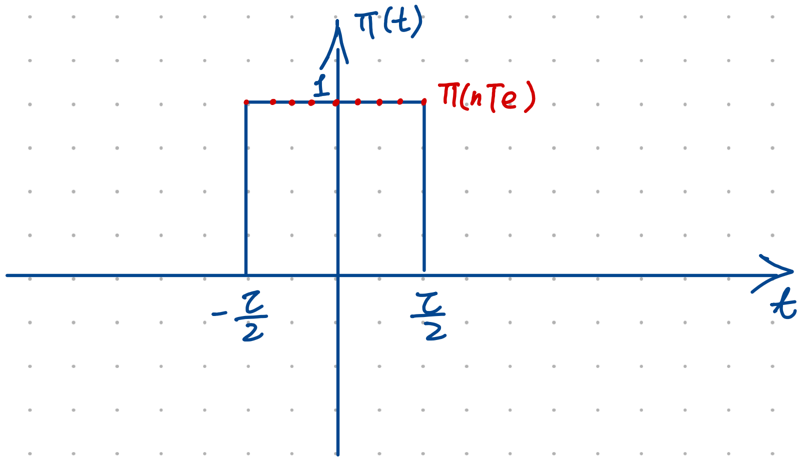


$f^n = 3.8 + \frac{B}{2}$ $f_1 = 4.2 - \frac{B}{2}$

$$f_1 - f^n = 0.4 - B$$
$$= 400\,Hz - B > 0$$

n°2/Ex 2.

1°



$$\Pi(f) = FT\cdot(\Pi(t)) = \tau\cdot \mathrm{sinc}(\pi f \tau)$$

$$= \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi f t} dt$$

$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \Pi(t) e^{-j2\pi f t} dt$$

$$= \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

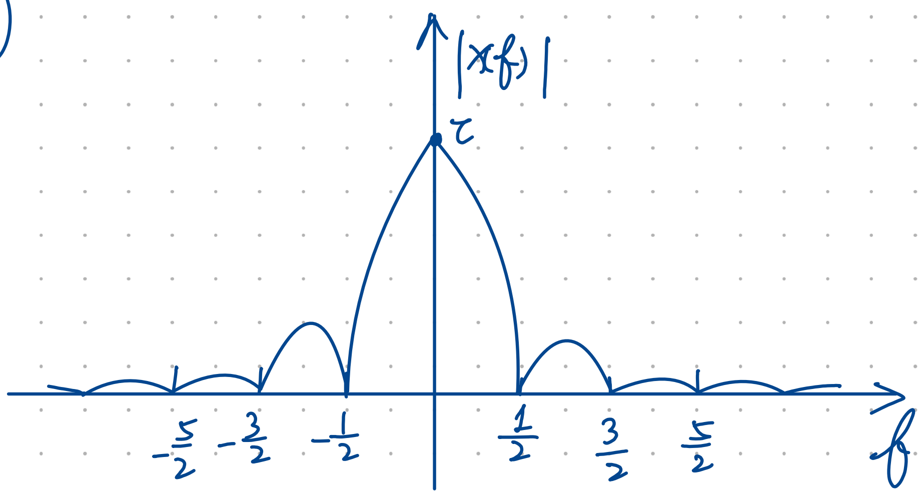
$$= \frac{1}{-j2\pi f}\left(e^{-j\pi f\tau} - e^{j\pi f \tau}\right)$$

$$= \frac{1}{-j2\pi f} * (-2j\sin(\pi f \tau))$$

$$= \tau \frac{\sin(\pi f \tau)}{\pi f \tau}$$

$$\pi f \tau = k\pi$$

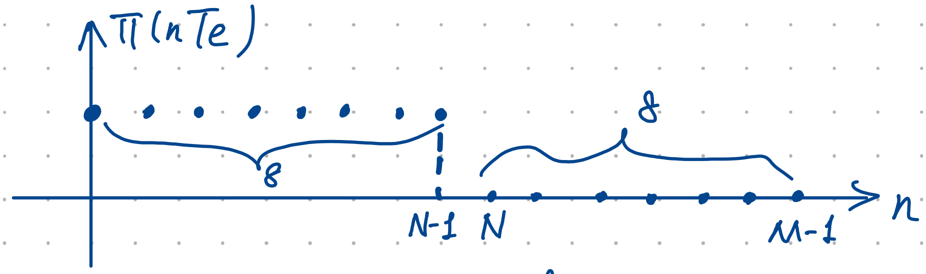
$$f = \frac{k}{2}$$



$$F_{max} = +\infty$$

$$F_e = +\infty$$

2°



$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\frac{\sin x}{x} = \operatorname{sinc} x$$

$$\Pi(f) = FT(\pi(t)) = \sum \pi_n e^{-j2\pi f n T_e}$$

$$= \sum_{n=0}^{N-1} 1 \cdot e^{-j2\pi f n T_e}$$

$$= \frac{1 - e^{-j2\pi f T_e \cdot N}}{1 - e^{-j2\pi f T_e}}$$

$$= e^{-j\pi f (n-1) T_e} * \frac{\cancel{2j} \sin(\pi f N T_e) / \pi f N T_e}{\cancel{2j} \sin(\pi f T_e) / \pi f N T_e}$$

$$= e^{-j\pi f (N-1) T_e} * N \cdot \frac{\operatorname{sinc}(\pi f N T_e)}{\operatorname{sinc}(\pi f T_e)}$$

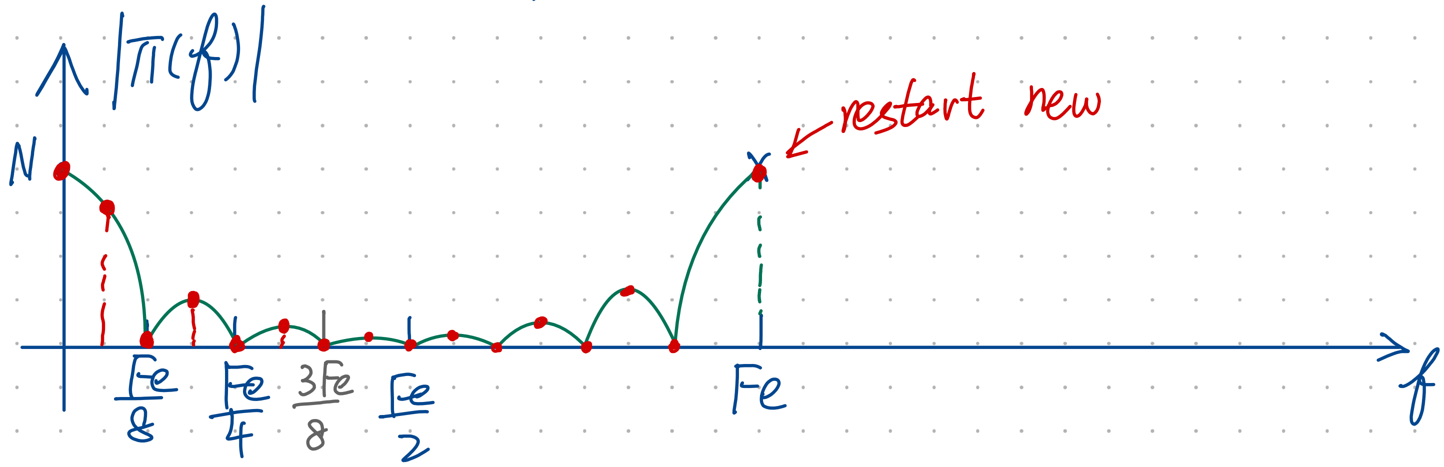
$\cancel{\pi} f N T_e = n \cancel{\pi}$

$f = \frac{n F_e}{N}$

$f = F_e$

$$\Pi(f) \xrightarrow[f = \frac{k F_e}{M}]{} \Pi(k) \quad , k = 0, \ldots, M-1$$

$$\Pi(k) = e^{-j\pi \frac{k(N-1)}{M}} * N \frac{\operatorname{sinc}\left(\frac{\pi k N}{M}\right)}{\operatorname{sinc}\left(\frac{\pi k}{M}\right)}$$



Exercise 1

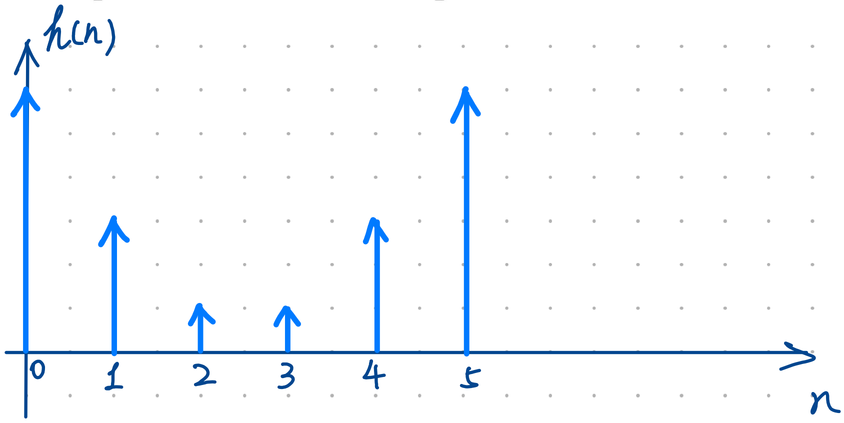
We consider a finite impulse response (FIR) filter with the following coefficients

$h_2 = h_3 = 0.1$

$h_1 = h_4 = 0.3$

$h_0 = h_5 = 0.6$

1. Compute the response of this filter at the null frequency f=0.
2. Compute the response of this filter at the frequency f=0.5.
3. We change the sign of odd index coefficients, what is the response of the new obtained filter at the frequency 0.5.
4. Give a scheme of realization of the filter given above. How many multiplication operations are required at each filter output?



$x(n) \longrightarrow \boxed{} \longrightarrow y(n) = \sum_{i=0}^{5} h_i \cdot x(n-i)$

$= 0.6\,x(n) + 0.3\,x(n-1) + 0.1\,x(n-2) + 0.1\,x(n-3)$

$+ 0.3\,x(n-4) + 0.6\,x(n-5)$

$h(n) = 0.6\,\delta(n) + \dots + 0.6\,\delta(n-5)$

$H(z) = 0.6 + 0.3\,z^{-1} + 0.1\,z^{-2} + 0.1\,z^{-3} + 0.3\,z^{-4} + 0.6\,z^{-5}$

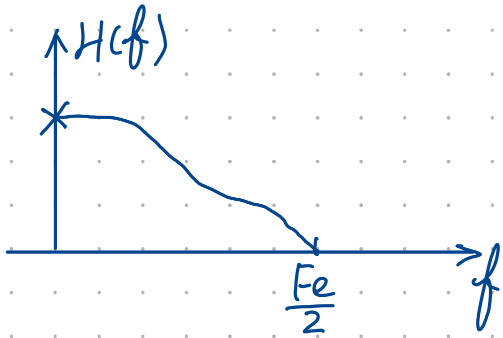
$H(f) = H(z)\Big|_{z = e^{j2\pi f T_e}}$

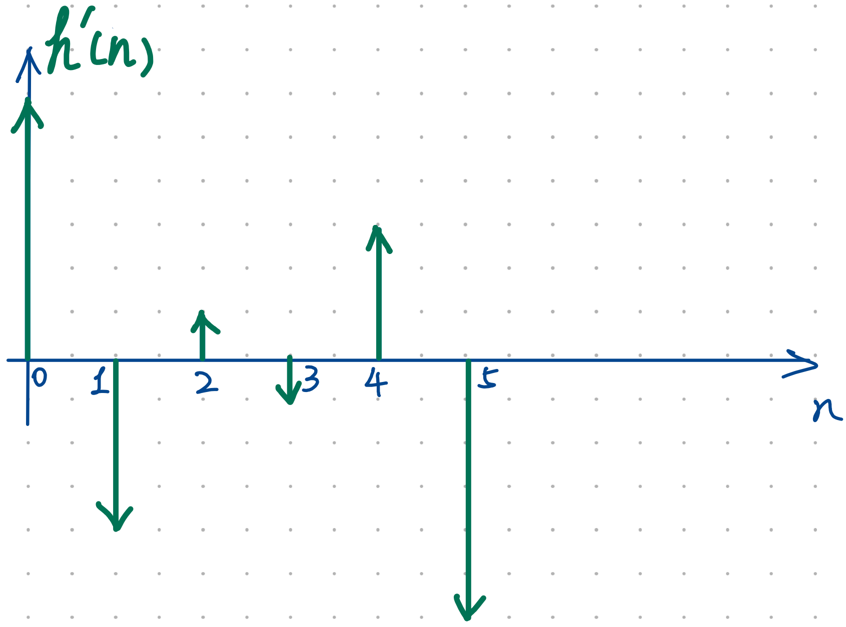
$= 0.6 + 0.3\,e^{-j2\pi f T_e} + 0.1\,e^{-j4\pi f T_e} + 0.1\,e^{-j6\pi f T_e}$

$+ 0.3\,e^{-j8\pi f T_e} + 0.6\,e^{-j10\pi f T_e}$

1) $H(f=0) = \sum_{i=0}^{5} h_i = 2$

2) $H(f = 0.5F_e) = 0.6 - 0.3 + 0.1 - 0.1 + 0.3 - 0.6 = 0$





3) $h'(n) = (-1)^n \cdot h(n)$

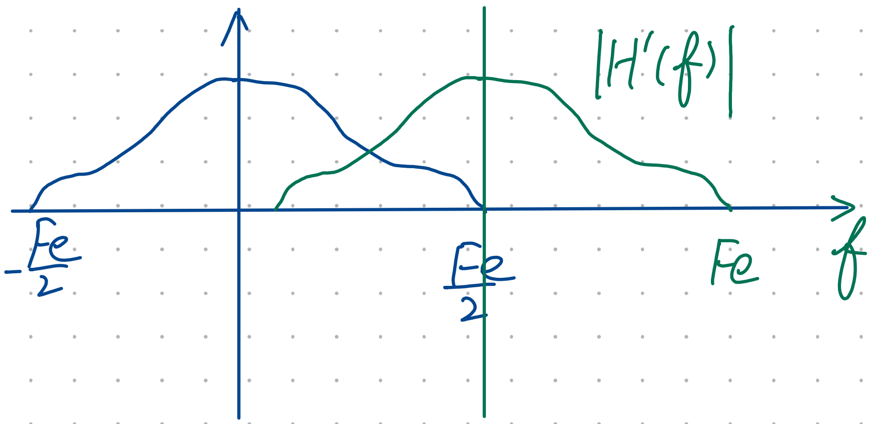
$$H'(f) = \sum_{n=0}^{5} h'(n)\, e^{-j2\pi f n T_e} = \sum_{n=0}^{5} h(n) \boxed{(-1)^n} e^{-j2\pi f n T_e}$$

$e^{j\pi n}$

$$= \sum_{n=0}^{5} h'(n) \boxed{e^{j\pi n}} e^{-j2\pi f n T_e} = \sum_{n=0}^{5} h'(n)\, e^{j2\pi n T_e (f - \frac{F_e}{2})}$$

$e^{2j\frac{\pi}{2} n T_e F_e}$ $\hookrightarrow T_e = \frac{1}{F_e}$

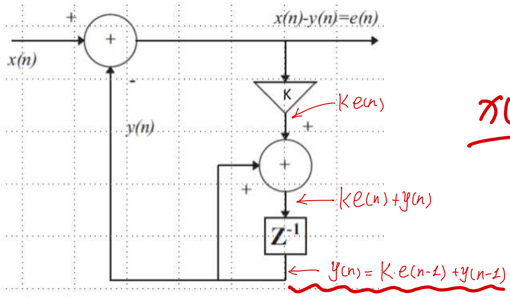
$$= H\left(f - \frac{F_e}{2}\right)$$



Tutorial session n°4:
Infinite Impulse Response (IIR) Filters

Exercise 1

A phase locked loop is modeled by the following circuit.



$x(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$

1- Compute the relationship between the input sequence x(n) and the output sequence y(n). Deduce the Z transform of the considered filer.

2- Compute the output of this filter to the unit level input. Check that it corresponds to a control of y(n) by using x(n), that is, it tends towards unity when n tends towards infinity.

3- The coefficient K represents the gain of the control loop. Compute the range of K values in order to obtain the system stability.

1° $K e(n-1) + y(n-1) = K[x(n-1) - y(n-1)] + y(n-1)$

$y(n) = K\,x(n-1) + (1-K)\,y(n-1)$

$\Big\downarrow ZT$

$Y(z) = K \cdot z^{-1} X(z) + (1-K) z^{-1} Y(z)$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{Kz^{-1}}{1-(1-K)z^{-1}}$

$ZT^{-1} \downarrow$

$h(n) = h_1 * h_2$

$= [(1-K)^n u(n)] * [K\,\delta(n-1)]$

$= K(1-K)^{n-1} u(n-1)$

$u(n) \xrightarrow{ZT} \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$

$\alpha^n u(n) \xrightarrow{ZT} \dfrac{z}{z-\alpha} = \dfrac{1}{1-\alpha z^{-1}} x(n)$

3° $H(z) = \dfrac{Kz^{-1}}{1-(1-K)z^{-1}} = \dfrac{K}{z-(1-K)}$

$P_1 \Rightarrow P_1 - (1-K) = 0$

$\Rightarrow P_1 = 1-K$

$|P_1| < 1 \Rightarrow |1-k| < 1 \Rightarrow 0 < k < 2$

2° $\lim_{n \to \infty} y(n)$

$Y(z) = H(z) X(z) = \frac{k}{z-(1-k)} * \frac{z}{z-1}$

$\lim_{z \to 1} (z-1) Y(z) = \lim_{z \to 1} \cdot \cancel{(z-1)} \frac{k}{z-(1-k)} * \frac{z}{\cancel{z-1}}$

$= \lim_{z \to 1} \frac{k \cdot z}{z-(1-k)} = \lim_{z \to 1} \frac{k \cdot z}{z-1+k} = 1$

Exercise 2

We consider the following digital filter defined by its zeros *Z1 = 0.09 ± j 0.99 ; Z2 = 0.58 ± j 0.81* and poles *P1 = 0.62 ± j 0.26* ; *P2 = 0.70 ± j 0.58*

1- Indicates the type of this filter (FIR or IIR).

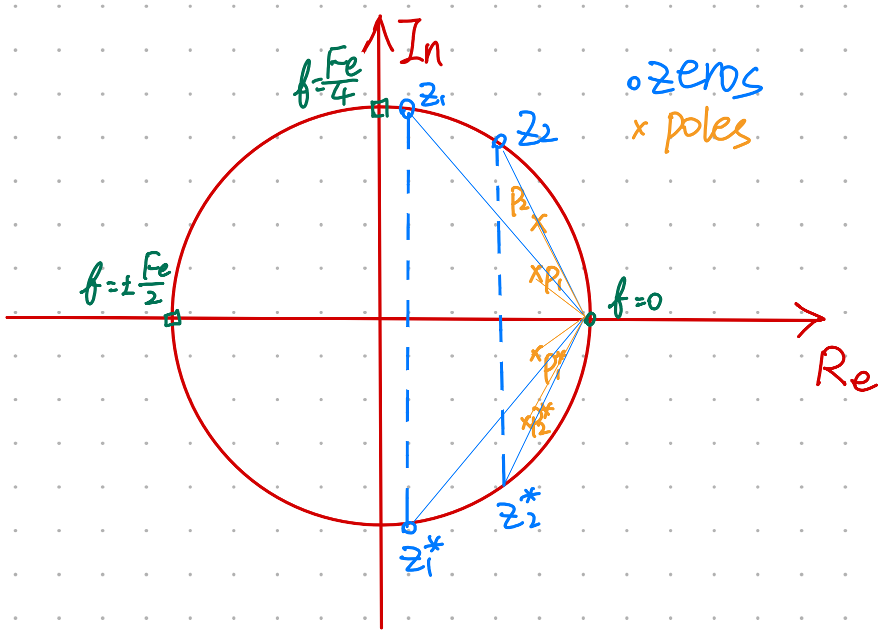
2- Determines its order.

3- Which is its transfer function (high pass filter, low pass filter, or pass-band filter)?

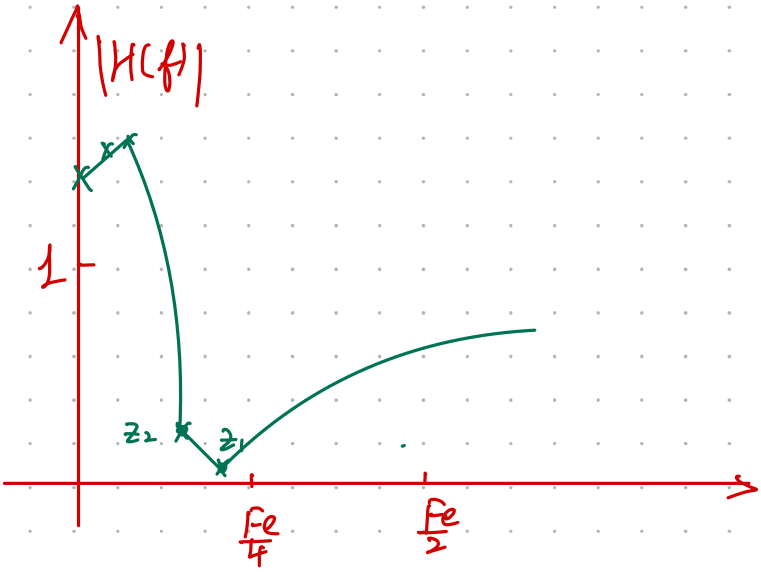
1° $H(z) = K_0 \frac{(z-z_1)(z-z_1^*)(z-z_2)(z-z_2^*)}{(z-P_1)(z-P_1^*)(z-P_2)(z-P_2^*)}$

$|H(f)| = \frac{MZ_1 MZ_2 MZ_3 M_4}{MP_1 MP_2 MP_3 MP_4}$

2°



order = 4



$$Z_1 = 0.09 \pm j\,0.99 \Rightarrow \alpha e^{j\varphi} \begin{cases} \alpha = \sqrt{0.09^2 + 0.99^2} \\ \varphi = \tan^{-1} \frac{0.99}{0.09} \end{cases}$$

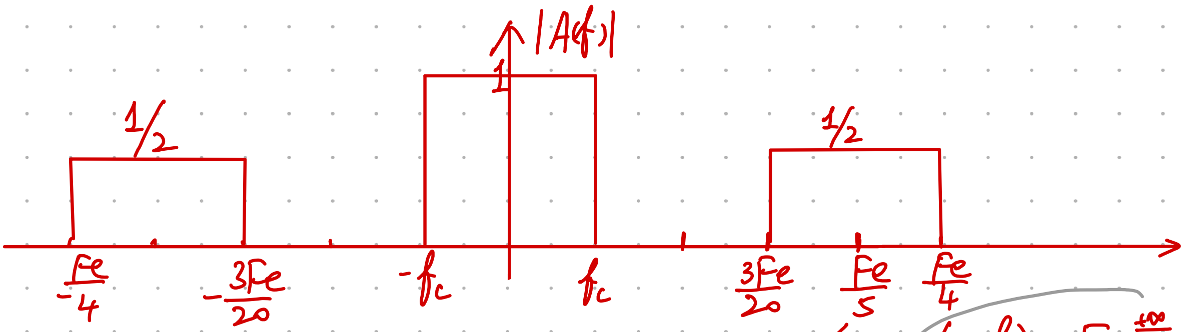
$$\varphi = \frac{2\pi f_{z_1}}{Fe}$$

Exercise 2

We consider a finite impulse response (FIR) of a digital filter with the following equation.

$$y(n) = \sum_{k=0}^{n-1} a_k x(n-k)$$

The filter coefficients a_k have been computed such that the considered filter is a low pass one with a cutoff frequency fc=Fe/20, (Fe is the sampling frequency).
What becomes this filter if we replace the coefficients a_k by $a_k\cos(2\pi f0kTe)$, f0=Fe/5.



f_0

$$X_e(f) = F_e \sum_{k=-\infty}^{+\infty} X(f - kF_e)$$

$$a(n) \xrightarrow{FT} A(f)$$

$$\cos(2\pi f_0 n T_e) \longrightarrow \otimes$$

$$a'(n) = a(n) * \cos(2\pi f_0 n T_e)$$

FT

$$A'(f) = A(f) *$$

$$\cos(2\pi f_0 t) \xrightarrow{FT} \frac{\delta(f-f_0)+\delta(f+f_0)}{2}$$

Te

$$\cos(2\pi f_0 n T_e) \longrightarrow$$

$$\frac{F_e}{2}\sum_{k=-\infty}^{\infty} \delta(f-kF_e-f_0)+\delta(f-kF_e+f_0)$$

Exercise 2

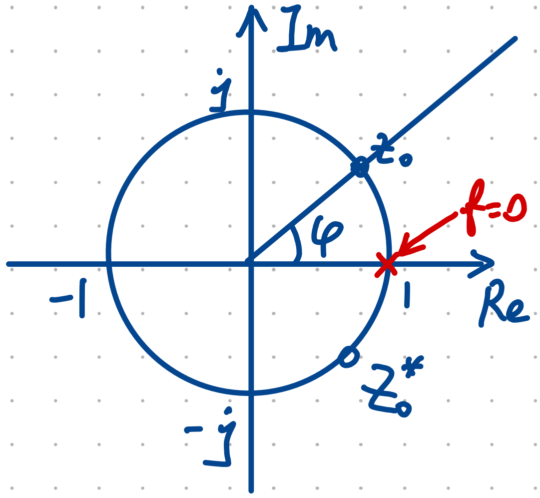
We consider a filter with only zeros: $Z_0 = e^{j\frac{\pi}{4}}$, $Z_0^* = e^{-j\frac{\pi}{4}}$,

A noise b(n) ~~uniformly distributed between [0,1]~~ is filtered by this filter. We note x(n) as the output of this filter.

b(n)= white zero mean signal with variance $= \sigma^2$

1- Compute x(n) in function of the input signal.

2- Calculate the autocorrelation coefficients $r_{xx}(0)$, $r_{xx}(1)$ of order 0 and 1 of x(n);



$z = e^{j2\pi f T_e}$

$$\left|\begin{array}{l} z = 1 \Rightarrow f = 0 \\ z = z_0 \Rightarrow e^{j2\pi f T_e} = e^{j\frac{\pi}{4}} \end{array}\right.$$

$\Rightarrow 2\pi T_e = \frac{\pi}{4}$

$\Rightarrow f = \frac{F_e}{8}$

$\varphi = 2\pi f T_e$

$H(z)$

$$b(n) \longrightarrow \boxed{H(z)} \longrightarrow x(n)$$

$H(z_0) = 0$

$H(z_0^*) = 0$

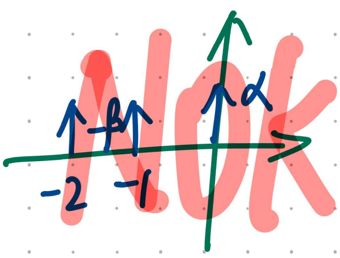
? $H(z) = (1 - z^{-1} z_0)(1 - z^{-1} z_0^*)$ Causal

$H(z) = (z - z_0)(z - z_0^*)$ not Causal

$= \left(z^2 - (z_0 + z_0^*) z + |z_0|^2\right) z^{-2}$

ZT^{-1}

$h(n) = \delta(n+2) - \underbrace{(z_0 + z_0^*)}_{\beta} \delta(n+1) + \underbrace{|z_0|^2}_{\alpha} \delta(n)$



$$H(z) = 1-(z_0+z_0^*)z^{-1} + |z_0|^2 z^{-2}$$

$$\downarrow ZT^{-1}$$

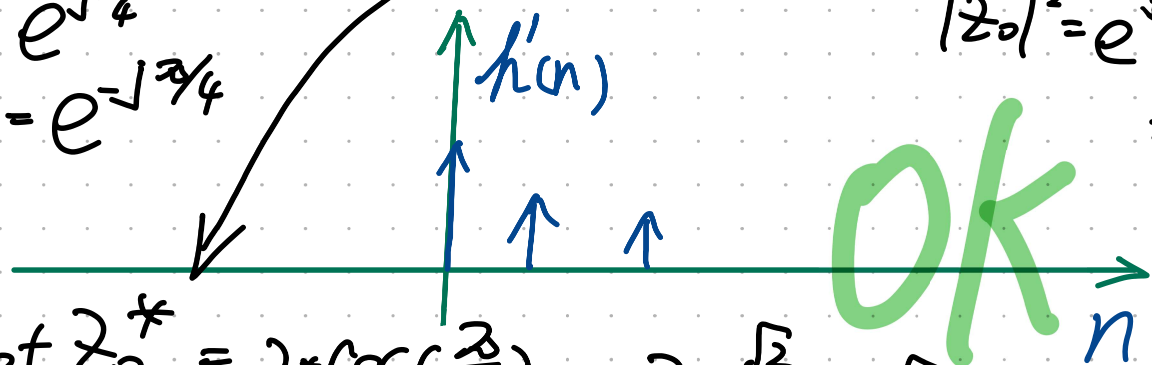
$$\boxed{h'(n) = 1-\underbrace{(z_0+z_0^*)}\delta(n-1) + \underbrace{|z_0|^2}\delta(n-2)}$$

$z_0 = e^{j\frac{\pi}{4}}$

$z_0^* = e^{-j\pi/4}$

$|z_0|^2 = e^{j\frac{\pi}{4}} * e^{-j\frac{\pi}{4}}$

$= 1$



$$z_0+z_0^* = 2*\cos\left(\frac{\pi}{4}\right) = 2*\frac{\sqrt{2}}{2} = \sqrt{2}$$

$$\to h'(n) = 1+\sqrt{2}*\delta(n-1) + 2\sqrt{2} + \sqrt{2}$$

$$x(n) = h'(n) * b(n)$$

$$= \left(1-\sqrt{2}\,\delta(n-1) + \delta(n-2)\right) * b(n)$$

$$x(n) = b(n) - \sqrt{2}\,b(n-1) + b(n-2)$$

2) $R_{xx}(p) = E[x(n)\,x(n-p)]$

$$= E[\underbrace{b(n) - \sqrt{2}\,b(n-1) + b(n-2))}_{x(n)}$$

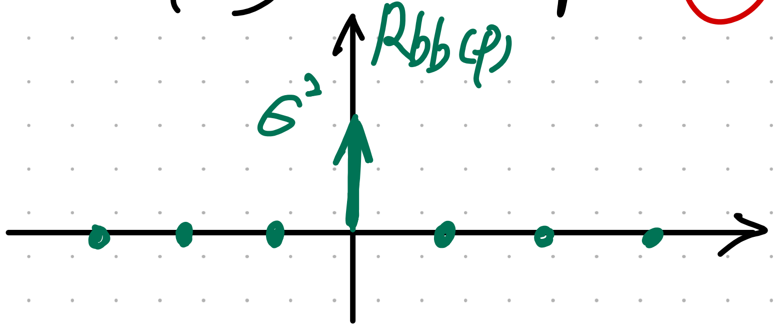
$$x(n-p) \quad *\left(b(n-p)+\sqrt{2}\,b(n-p-1)+b(n-p-2)\right)\Big]$$

$$R_{xx}(0) = E\Big[\left(b(n)-\sqrt{2}\,b(n-1)+b(n-2)\right)\left(b(n)-\sqrt{2}\,b(n-1)+b(n-2)\right)\Big]$$

$$\begin{aligned} R_{xx}(0) = E\Big\{ & b^2(n) - \sqrt{2}\,b(n)\,b(n-1) + b(n) \\ & * b(n-2) - \sqrt{2}\,b(n)\,b(n-1) \\ & + 2\,b^2(n-1) - \sqrt{2}\,b(n-1)\,b(n-2) \\ & + b(n)\,b(n-2) - \sqrt{2}\,b(n-1)\,b(n-2) \\ & + \sqrt{2}\,b^2(n-2) \Big\} \end{aligned} \quad ①$$

$$E\{a+b+c\} = E\{a\}+E\{b\}+E\{c\}$$

$$R_{bb}(p) = E\{b(n)\,b(n-p)\} = \sigma^2\delta(p) \quad ②$$



③ $E\{b^2(n)\} = \sigma^2 = E\{b^2(n-n_0)\}$

$$R_{xx}(0) = \sigma^2 + 2\sigma^2 + \sqrt{2}\sigma^2 = (3+\sqrt{2})\sigma^2$$

$$R_{xx}(1) = E\{x(n)\,x(n-1)\}$$

$$= E\{(b(n) - \sqrt{2}\,b(n-1) + b(n-2))(b(n-1) - \sqrt{2}\,b(n-2) + b(n-3)\}$$

$$= -\sqrt{2}\,E\{\underbrace{b^2(n-1)}_{\sigma^2}\} - \sqrt{2}\,E\{\underbrace{b^2(n-2)}_{\sigma^2}\}$$

$$= -2\sqrt{2}\,\sigma^2$$

$$R_{xx}(2) = E\{x(n)\,x(n-2)\}$$

$$= E\{(b(n) - \sqrt{2}\,b(n-1) + b(n-2)) * (b(n-2) - \sqrt{2}\,b(n-3) + b(n-4)\}$$

$$= E\{b^2(n-2)\} = \sigma^2$$

$$R_{xx}(3) = 0$$

$R_{xx}(p) = 0$ for $p \geq 3$

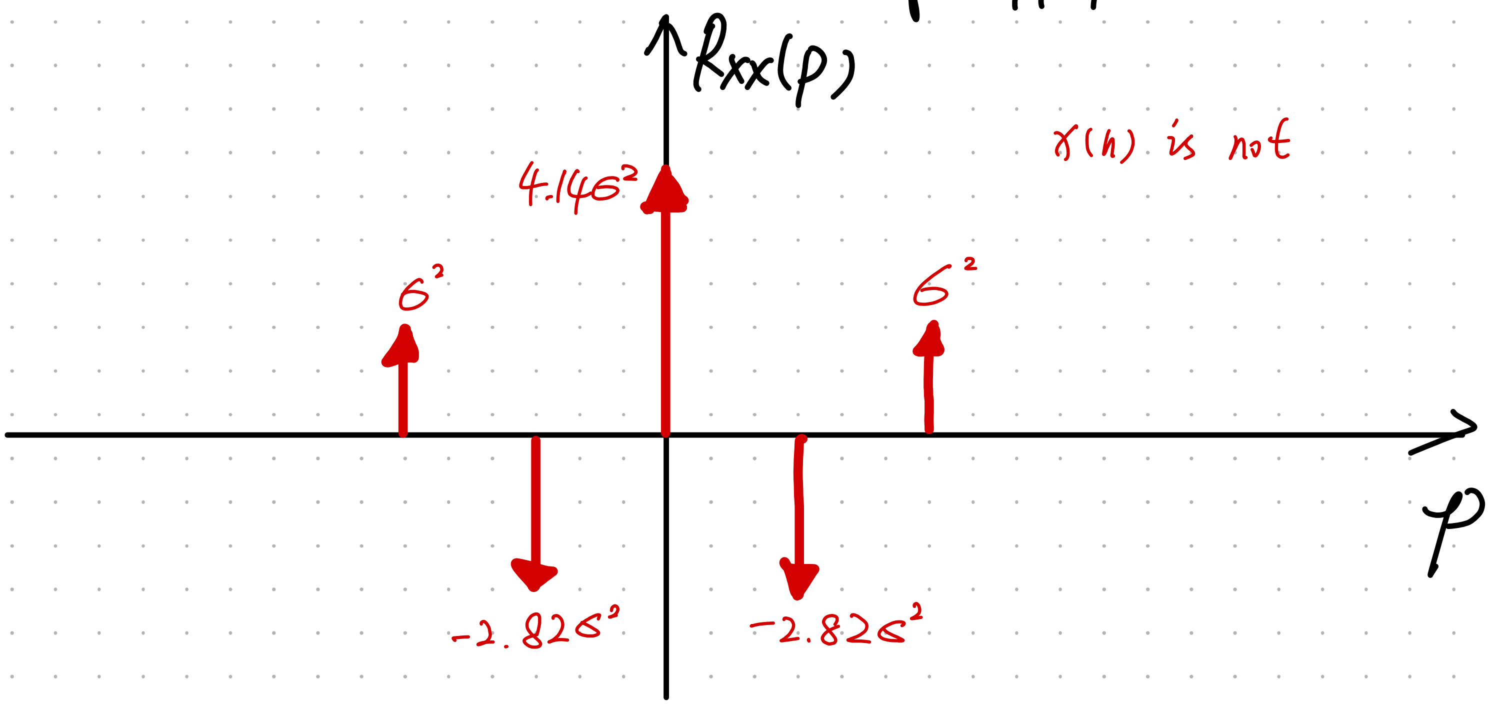
$$R_{xx}(-p) = E\{x(n)\underbrace{x(n+p)}_{n'}\}$$

$n = n' - p$

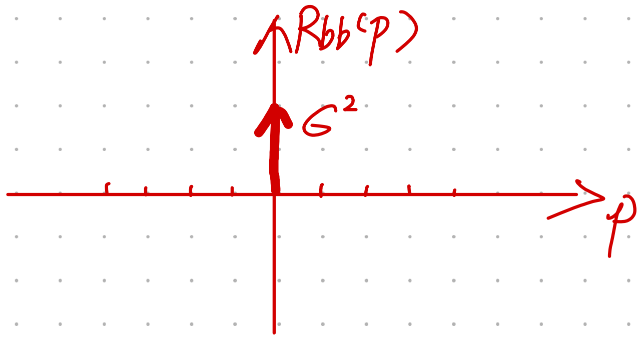
$$= E\{x(n'-p)\, x(n')\}$$

$$= E_{xx}(p)$$

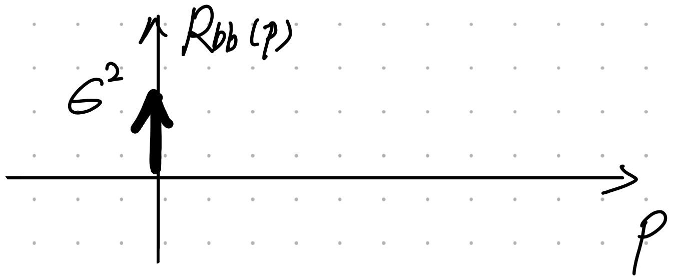
$$R_{xx}(p) = \begin{cases} (3+\sqrt{2})\sigma^2 & \text{if } p = 0 \\ -2\sqrt{2}\,\sigma^2 & \text{if } p = \pm 1 \\ \sigma^2 & \text{if } p = \pm 2 \\ 0 & \text{if } |p| > 2 \end{cases}$$



x(n) is not

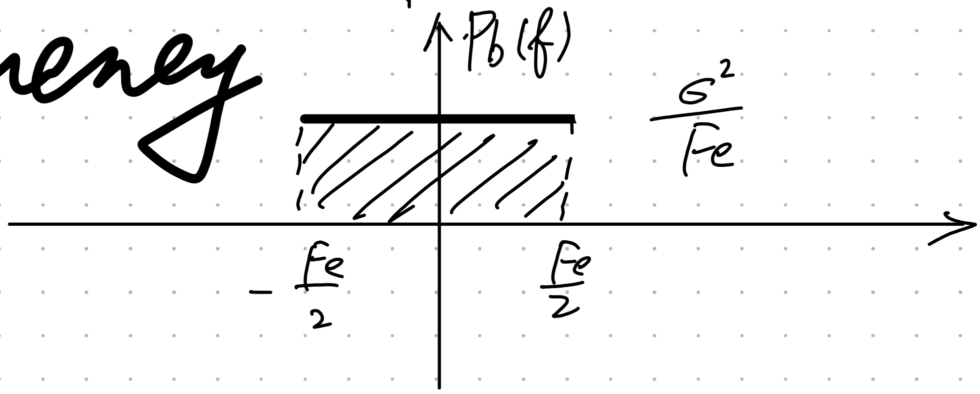


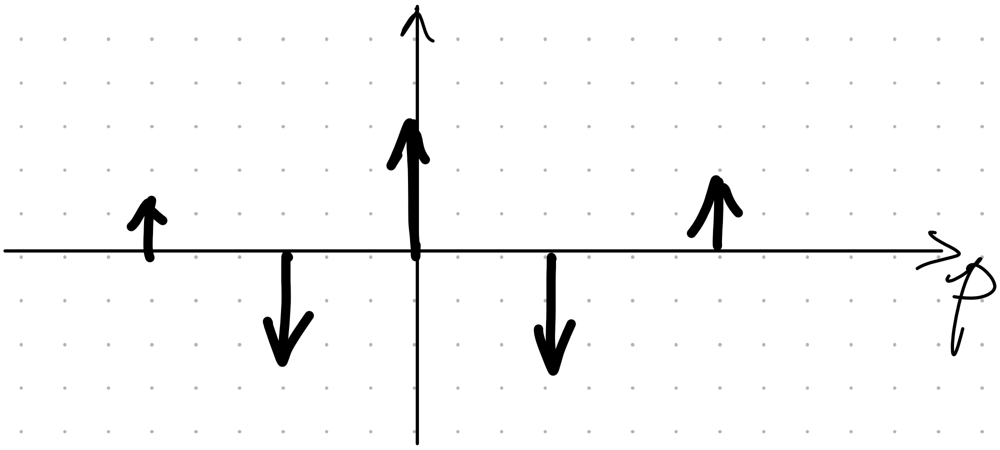
Time



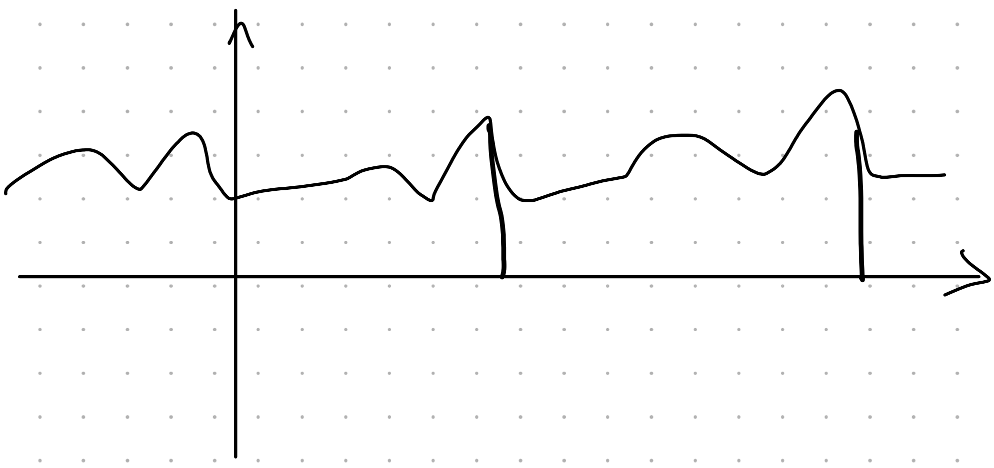
FT

frequency





FT



$$b(n) \longrightarrow \boxed{h'(n)} \longrightarrow x(n)$$

$$b(n) \downarrow$$

$$P_b(f) \qquad H'(f) \qquad P_x(f) = P_b(f)\,|H'(f)|^2$$

$$h'(n) = \delta(n) - \sqrt{2}\,\delta(n-1) + \delta(n-2)$$

$$\text{FT} \downarrow$$

$$H'(f) = \sum_{n=0}^{2} h'(n)\, e^{-j2\pi f n T_e}$$

$$= 1 - \sqrt{2}\, e^{-j2\pi f T_e} + e^{-j4\pi f T_e}$$

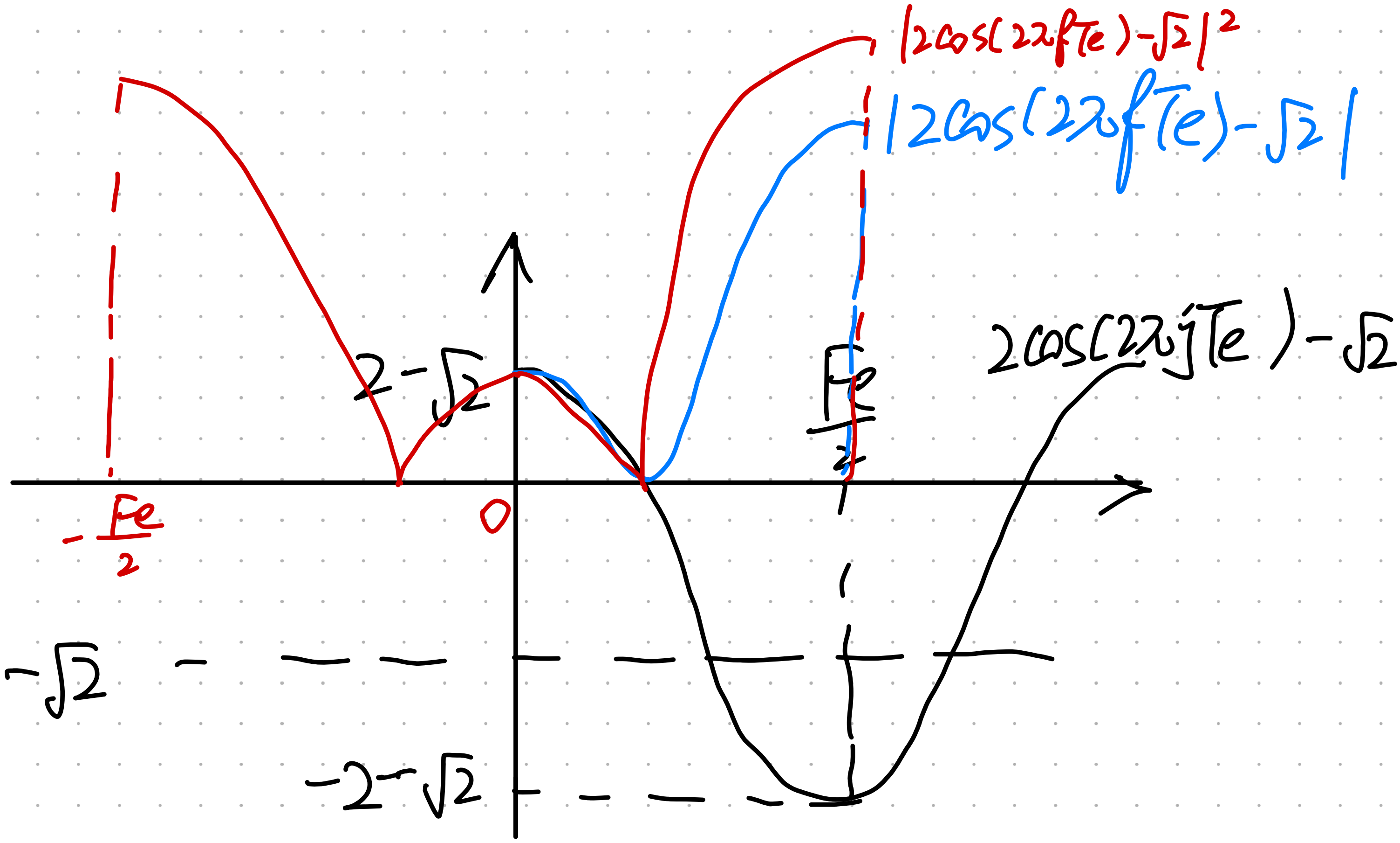
$$H'(f) = e^{-j2\pi f T_e}\left(e^{j2\pi f T_e} - \sqrt{2} + e^{-j2\pi f T_e}\right)$$

$$2\cos(2\pi f T_e)$$

$$H'(f) = e^{-j2\pi f T_e}\left(2\cos(2\pi f T_e) - \sqrt{2}\right)$$

$$|H'(f)|^2 = H'(f)\, H'^*(f)$$

$$= \left(2\cos(2\pi f T_e) - \sqrt{2}\right)^2$$



$|2\cos(2\pi f T_e) - \sqrt{2}|^2$
$|2\cos(2\pi f T_e) - \sqrt{2}|$
$2\cos(2\pi j T_e) - \sqrt{2}$
$2-\sqrt{2}$
$\frac{F_e}{2}$
$-\frac{F_e}{2}$
0
$-\sqrt{2}$
$-2-\sqrt{2}$