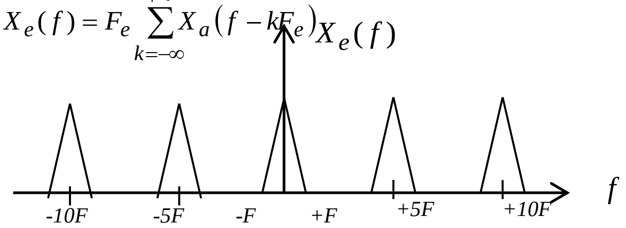
DIGITAL SIGNAL PROCESSING - CORRECTION TD N°1 -

EXERCISE N°1

$$X_e(f) = F_e \sum_{k=-\infty}^{+\infty} X_a(f - kF_e)$$



The sampled signal is written :

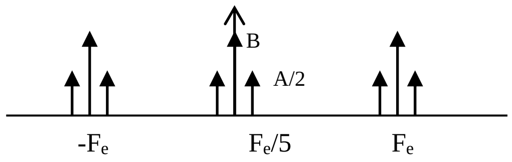
$$x(nT_e) = A\cos(2\pi f_0 n T_e) + B$$

The Fourier Transform is written :

$$X_a(f) = \frac{A}{2}[\delta(f - f_0) + \delta(f + f_0)] + B\delta(f)$$

The Discrete Fourier Transform is expressed :

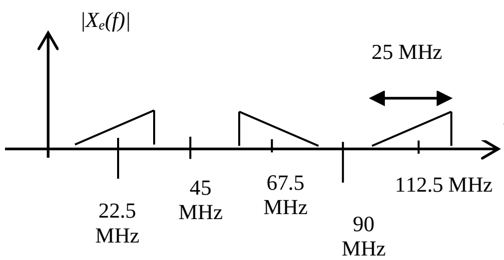
$$X_e(f) = \frac{A}{2}\left[W_{F_e}(f - f_0) + W_{F_e}(f + f_0)\right] + BW_{F_e}(f)$$



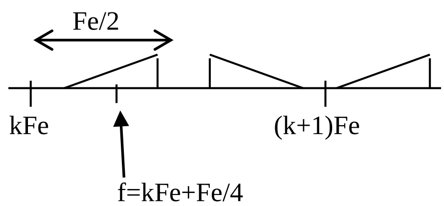
EXERCISE N°2

1. The Shannon theorem is respected because $F_e > 2B$ ($90\ MHz > 50\ MHz$)

2.



3.
The solution is not valid for 100 MHz and Fe=50 MHz.
A sufficient condition by down-sampling at $F_e > 2B$ and $f = kF_e + \frac{F_e}{4}$

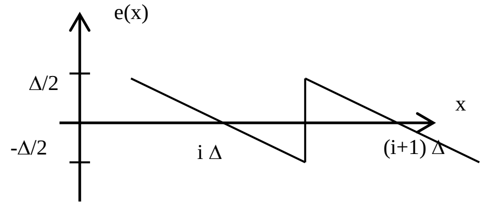


EXERCISE N°3

the maximal frequency value is 22 kHz. Thus, the minimal sampling frequency is 44 kHz. If we consider two microphones at 44 kHz and 16 bits, we obtain 1.408 Mbits/s. Consequently for 70 minutes, we obtain : 739.2 Mbytes.

Exercise n°4

1)



2) The mean :

$$m_e = \int_{-\infty}^{+\infty} p_e(u) u\, du = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \frac{1}{\Delta} u\, du = 0$$

The variance

$$\sigma_e^2 = E\left[e^2\right] = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} p_e(u) u^2 du = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \frac{1}{\Delta} u^2 du = \frac{\Delta^2}{12}$$

1) The signal-to-quantization noise ratio:

$$\Gamma = 10\log_{10}\left(\frac{12\sigma_x^2 2^{2b}}{A^2}\right),$$ d'où

$$\Gamma = 6.02b + 20\log_{10}\left(\frac{\sigma_x}{A}\right) + 10.8$$

6.02 dB per additional bit.

5) For a sinusoidal signal, the amplitude (Peak value) la is $\sqrt{2}\sigma_x$ and the condition of non-saturation can be written : $\sqrt{2}\sigma_x < \frac{A}{2}$. Thus, $\Gamma < 97.8\ dB$.

For a Gaussian signal with an estimation of its amplitude of $4\sigma_x$, we obtain: $4\sigma_x < \frac{A}{2}$, and consequently $\Gamma < 88.7\ dB$.

EXERCISE N°1

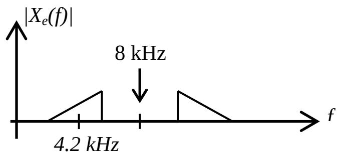
$\frac{F_e}{2} = 8kHz$, consequently $F_e = 16\ kHz$

Thus, $f_1 = f_{1n} \times F_e = 200\ Hz$

$y(nT_e) = x(nT_e)\cos(2\pi f_1 nT_e)$

$y(nT_e) = x(nT_e)\cos\left(\frac{\pi n}{40}\right)$

If we consider a narrow band, and we filter the signal around a frequency of 3.8 kHz, we obtain the following spectrum:



EXERCISE N°2

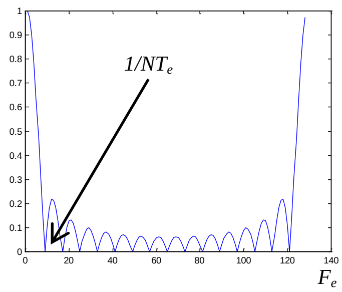
The analog signal has an unbounded spectrum (cardinal sine function), Thus, it is impossible to respect Shannon Theorem.

$$TF(\Pi) = \frac{1}{M}\sum_{i=0}^{N-1} e^{-j2\pi f iT_e}$$

$$= \frac{1}{M}\frac{1-e^{-j2\pi f NT_e}}{1-e^{-j2\pi f T_e}}$$

$$= \frac{1}{M}\frac{e^{-j\pi f NT_e}}{e^{-j\pi f T_e}}\frac{e^{j\pi f NT_e} - e^{-j\pi f NT_e}}{e^{j\pi f T_e} - e^{-j\pi f T_e}}$$

$$TF(\Pi) = \frac{1}{M} e^{-j\pi f (N-1)T_e}\frac{\sin \pi f NT_e}{\sin \pi f T_e}$$



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$$C(f) = \frac{1}{2}(X(f) + X(-f)) = \frac{1}{2}\left(X(f) + X(f)^*\right)$$

$$C(f) = Re\{X(f)\}$$

- In order to have C(f)=X(f), $X(f)$ must be real. Thus, $x(nT_e)$ is symmetric.

It is completely wrong to say that the Fourier transform is real because the signal x(nTe) is symmetric.

We have : $x_0 = x_4 = 0.2$, $x_1 = x_3 = 0.5$, $x_2 = 1$, $N = 5$

$$X\left(\frac{k}{5T_e}\right) = \sum_{n=0}^{4} x_n e^{-j2\pi\frac{nk}{5}T_e} = x_0\left(1 + e^{-j2\pi\frac{4k}{5}}\right)$$

$$+ x_1\left(e^{-j2\pi\frac{k}{5}} + e^{-j2\pi\frac{3k}{5}}\right) + x_2 e^{-j2\pi\frac{2k}{5}}$$

The computed expression is not real.

When we add a zero sample x_0=0, we obtain

$x_0 = 0$, $x_1 = x_5 = 0.2$, $x_2 = x_4 = 0.5$, $x_3 = 1$

$$X\left(\frac{k}{6T_e}\right) = \sum_{n=0}^{5} x_n e^{-j2\pi\frac{nk}{6}T_e} = x_0 + x_3$$

$$+ x_1\left(2\cos\left(\frac{2\pi k}{6}\right)\right) + x_2\left(2\cos\left(\frac{4\pi k}{6}\right)\right)$$

Th result is real.

EXERCISE N°3

The observation duration is = $10\ \mu s$ Thus, the frequency resolution is = $\frac{1}{10\mu s} = 100\ kHz$

An other method to calculate the frequency resolution :

By sampling a signal during $10\ \mu s$ at $10\ MHz$ we obtain 100 samples. Thus, the resolution is :

$\frac{F_e}{N} = \frac{10\ MHz}{100} = 100\ kHz$. Thus, the imprecision is $\frac{1}{2}\frac{F_e}{N}$.

The imprecision on frequency value is expressed as $\Delta f = f_0 \times \frac{\Delta v}{c}$. By using $f_0 = 5\ GHz$, the imprecision on the mobile speed can be calculated as $\Delta v = \pm 3\ km/s = \pm 10.800\ km/h$. This radar gives information about distance and never about the mobile speed.

For the mobile speed, we must observe the received signal much more time in order to decrease the imprecision on the frequency value.

CORRECTION OF TUTOPRIAL N°3

EXERCISE N°1

The transfer function of a given digital filter is given by :

$$H(f) = \sum_{k=0}^{N-1} h(kT_e) e^{-j2\pi f k T_e}$$

In general we consider the normalized frequency relative to the sampling frequency : $f_n = \dfrac{f}{F_e}$. When we consider the frequency 0 and 0.5, we talk about the normalized frequency.

$$H(f_n) = \sum_{k=0}^{N-1} h_k e^{-j2\pi f_n k}$$

$$H(0) = \sum_{k=0}^{N-1} h_k = 2$$

$$H(0.5) = \sum_{k=0}^{N-1} h_k e^{-jk\pi}$$

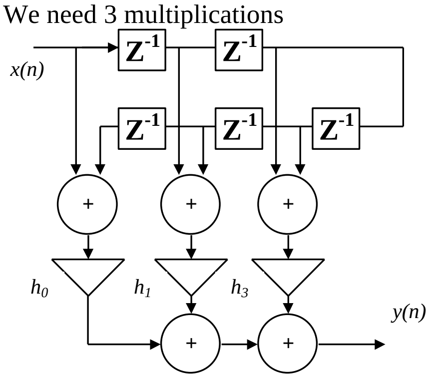
$$= h_0 - h_1 + h_2 - h_3 + h_4 - h_5 = 0$$

After changing the sign of odd index coefficients :

$$H'(0.5) = \sum_{k=0}^{N-1} h_k e^{-jk\pi}$$

$$= h_0 - (-h_1) + h_2 - (-h_3) + h_4 - (-h_5) = 2$$

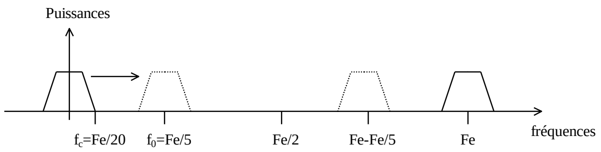
We need 3 multiplications



EXERCISE N°2

The multiplication in temporal domain is equivalent to a convolution operation in the frequency domain by $\delta(f + f_0)$ et $\delta(f - f_0)$

The low pass narrowband filter becomes a pass band filter around the frequency f_0



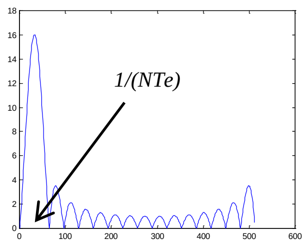
EXERCISE N°3

For each given real input $x(n)$, the computation of the filter output $y(n)$ requires 2 additions and two multiplications. By a confusion of the addition and the multiplication oeprations, the considered filter requires without any parallelisme process of theses operations, at minimum 40 Mops.

EXERCISE N°4

1) yes, it is the first output spectral component at frequency 0 of the signal FFT $\{x(n), x(n-1), \ldots x(n-63)\}$

2) cf Tutorial n°2 $\left| \dfrac{\sin \pi f N T_e}{N \sin \pi f T_e} \right|$



3) Yes, the same FFT but at the second output spectral components.

4) We compute an FFT 64 at each Te, i.e., with a sliding window of 64 values which shifts at each clock count.

CORRECTION OF TUTOPRIAL N°4

Infinite Impulse Response (IIR) Filters

EXERCISE N°1

1) $y(n) = K\left[x(n-1) - y(n-1)\right] + y(n-1)$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{KZ^{-1}}{1+(K-1)Z^{-1}}$$

2) the output of the filter to the unit level input is:

$$y(n) = K\frac{1-(1-K)^n}{1-(1-K)} = 1-(1-K)^n$$

3) The stability domain is given by :
$|K-1| < 1$; $0 < K < 2$

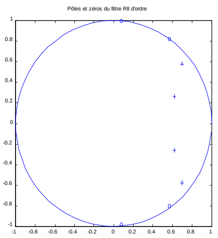
EXERCISE N°2

We have an IIR (Infinite Impulse response) filter because it contains both poles and zeros simultaneously.

The Z transform of the considered filter is :

$$H(Z) = \frac{\left(Z-Z_1\right)\left(Z-Z_1^*\right)\left(Z-Z_2\right)\left(Z-Z_2^*\right)}{\left(Z-P_1\right)\left(Z-P_1^*\right)\left(Z-P_2\right)\left(Z-P_2^*\right)}.$$

The numerator and the denominator are both of order 4.



The positions of the poles and zeros indicate that we have a low-pass filter.

EXERCISE N°3

1) We have an IIR filter of second order.
2) We have a pure phase shifter, $|H(\omega)| = 1 \ \forall\omega$.

3) $H(Z) = \dfrac{N(Z)}{D(Z)} = \dfrac{Z^{-2}\ D(Z^{-1})}{D(Z)}$

$\Rightarrow \varphi(\omega) = 2\,\varphi_D(\omega) - 2\,\omega$

EXERCISE N°4

$$x(n<0)=0, x(0)=1, x(1)=b_1, x(2)=b_2, x(n>2)=0$$

The input signal $x(n)$ is the Dirac function $\delta(n)$ filtered by the FIR filter with its Z transform

$G(Z) = 1 + b_1 Z^{-1} + b_2 Z^{-2}$. G(z) is the reverse filter of $H(Z)$, i.e. $G(Z).H(Z) = 1$

$$H(Z) = \frac{1}{1-b_1 Z^{-1} - b_2 Z^{-2}} = \frac{1}{\left(1-PZ^{-1}\right)\left(1-P^*Z^{-1}\right)}$$

$$H(Z) = \frac{A}{1-PZ^{-1}} + \frac{B}{1-P^*Z^{-1}}$$

By identification, $A = \dfrac{P}{P-P^*}$ et $B = \dfrac{P^*}{P^*-P}$

By performing the division :

$$H(Z) = \left(\frac{P}{P-P^*}\right)\sum_{n=0}^{\infty} P^n Z^{-n} + \left(\frac{P^*}{P^*-P}\right)\sum_{n=0}^{\infty} P^{*n} Z^{-n}$$

$$H(Z) = \sum_{n=0}^{\infty}\left(\left(\frac{PP^n}{P-P^*}\right) + \left(\frac{P^*P^{*n}}{P^*-P}\right)\right)Z^{-n}$$

Thus, $h_n = \left(\dfrac{PP^n}{P-P^*}\right) + \left(\dfrac{P^*P^{*n}}{P^*-P}\right)$

By using the polar coordinates $P = \rho.e^{j\theta}$, we obtain :

$$h_n = \rho^n \frac{\sin(n+1)\theta}{\sin\theta}$$

If the pole value is outside the unit circle, the function $\sum_{n=-\infty}^{n=+\infty} |h_n|$ diverges.

CORRECTION OF TUTOPRIAL N°5

Correlation of random variables

EXERCISE N°1

$r_{xx}(0) = 4.3\sigma^2$, $r_{xx}(1) = 0.88\sigma^2$

$r_{xx}(2) = -0.35\sigma^2$, $r_{xx}(3) = 0.2\sigma^2$

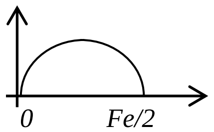
The signal $x(n)$ isn't white because it has been filtered. We note that its autorrelation function is not a dirac function.

The signal is a gaussian because it a linear combination of Gaussian signals.

$$R_3 = \sigma^2 \begin{pmatrix} 4.3 & 0.88 & -0.35 \\ 0.88 & 4.3 & 0.88 \\ -0.35 & 0.88 & 4.3 \end{pmatrix}$$

$H(Z) = 1 - Z^2 = (1 - Z^{-1})(1 + Z^{-1})$

$Z_0 = 1$, $Z_0 = -1$, the cut-off frequencies are 0 and $\frac{F_e}{2}$



The signal $y(n)$ is not white.

$r_{xy}(0) = 2.2\sigma^2$, $r_{xy}(1) = 0.4\sigma^2$, $r_{xy}(2) = -0.2\sigma^2$

EXERCISE N°2

Noticing that in this exercise the noise $b(n)$ is not white because it is not centred and it has a raie on zero frequency which is more important than other ones situated at other frequencies. The temporal autocorrelation function is not a dirac.

$$H(Z) = (1 - Z_0 Z^{-1})(1 - Z_0^* Z^{-1})$$
$$= 1 - \sqrt{2} Z^{-1} + Z^{-2}$$

$x(n) = b(n) - \sqrt{2} b(n-1) + b(n-2)$

$E(b(n)) = \frac{1}{2}$ et $E(b(n)^2) = \frac{1}{3}$

$$r_{xx}(0) = \frac{4}{3} + \frac{1}{2} - \sqrt{2} = 0.419$$

$$r_{xx}(1) = \frac{3}{2} - \frac{\sqrt{2}}{2} - \frac{2}{3}\sqrt{2} \approx -0.137$$

EXERCISE N°3

Express the Signa-to-Noise-Ratio on each sensor. We note $x_1(t)$ and $x_2(t)$ the corresponding signals for the two sensors.

$x_1(t) = s(t) + b_1(t)$

$x_2(t) = s(t) + b_2(t)$

$\frac{S}{B} = \frac{1}{\sigma^2}$ on each sensor.

$y(t) = 2s(t) + b_1(t) + b_2(t)$

$\frac{S}{B} = \frac{2}{\sigma^2}$ sur le signal $y(t)$

With the attenuation coefficient α :

$\frac{S}{B} = \frac{(1+\alpha)^2}{2\sigma^2}$ for the signal $y(t)$

$\alpha = 0$, waste of 3 dB

$\alpha \approx 0.414$, gain of 0 dB

$\alpha = 1$, gain of 3 dB