PROBABILITY THEORY – EXERCISES SESSION N° 1

About chapters:

- Chapter 1: NOTION OF EVENT AND OF PROBABILITY
- Chapter 2: CONDITIONAL PROBABILITIES & INDEPENDENCE

TD1 – 1 * :

The Items obtained from a manufacturing chain must pass a pass/fail test. The inspector in charge of quality control examines each article, then classifies it. This process continues until either two consecutive defective items are observed or until three non-defective items are observed. Describe the set Ω of possible outcomes associated with this experiment.

TD1 – 2 :

1) Let A and B be two independent events associated to a random experiment. Demonstrate that the events $\overline{A}$ and B are also independent.

2) Application: Every morning of class, Peter, a student at ISEP, can be the victim of two independent events:

• R: "he does not hear his alarm clock ringing";

• S: “his scooter, poorly maintained, breaks down”.

For a given day of class, the probability of R is 0.1 and that of S is 0.05. If at least one of the two events occurs, Peter is late at ISEP, otherwise he is on time.

a) Calculate the probability that: "a given day Peter hears his alarm clock and his scooter breaks down”.
b) Calculate the probability that “Peter is on time at ISEP”.

TD1 – 3 :

Permutations – Combinations

a) In how many ways can 4 interchangeable tires be placed on 4 wheels of a car? What about 5 tires on 4 wheels?
b) What is the number of different “melodies” (series of different musical notes) of 3 notes that can be made from 8 different notes if no note is repeated?

c) 5 cards are taken randomly from a 32-card deck. How many different draws are possible?

d) We want to transmit coded information by arranging 6 colored electric lamps in a horizontal row. How many different messages are possible by changing the positions of 3 red, 2 green, one yellow lamps? (The lamps are all turned on at the same time).

<u>TD1 – 4</u> :

We roll two well-balanced dice and write S as the sum of the outcomes. Calculate the probability that S is at least equal to 10 if:

a) the first die's outcome is 5

b) at least one of the two dice's outcome is 5

<u>TD1 – 5</u> :

Lisa is a secretary, one of her tasks is telephone filtering. 60% of calls come from outside and 40% are internal to the company. Among the external calls, 80% are not transferred to her boss and among the internal ones, 30%.

Let P(E) be the probability that a call comes from outside the company and P(R) the probability that Lisa refuses to transfer the call to her boss.

a) Interpret the value of $P(E \cap R)$.

b) Calculate the probability $P(R / E)$

c) Sketch the probability-tree diagram

d) Given that Lisa has not transmitted the call, what is the probability that it came from outside?

<u>TD1 – 6</u> :

A bag contains two dice. One is perfectly balanced, but the other one gives a six once every two times (the other sides being assumed to be balanced). We draw a die at random from the bag and throw it.

a) The outcome is six. What is the probability that the thrown die is the balanced one?

b) The outcome is five. Same question as in a).

TD1 – 7 :

In the right pocket of your jacket, you have three €1 coins and four €2 coins. In your left pocket there are six €1 coins and three €2 coins. You randomly take 5 coins from the right pocket and put them into the left pocket. Then, you randomly draw a coin from the left pocket. Let us denote :

- B_i the event i coins of 1€ were taken from the right pocket to the left pocket.
- *M* the event "The coin drawn from your left pocket is worth 1€".

a) Calculate $P(B_i)$ for all possible values of i. What are the possible values of i ?

b) For all i calculate $P(M/B_i)$.

c) Calculate $P(M)$.

TD1 – 8 :

A bookcase has 3 shelves (high, medium, low). 30% of the books are on the high shelf, half on the middle shelf and one out of five on the low shelf.

There are two types of books, novels and math books. A quarter of the books on the upper shelf are novels, the middle shelf has 60% of math books, and on the lower shelf there are 4 novels for 1 math book.

We produced a catalog of the books in alphabetical order of titles, alongside with the genre (novel or mathematics book) as well as the storage shelf (high, medium, low).

a) By choosing a volume at random from the catalog, what is the probability of finding a novel?

b) By randomly choosing one of the three shelves, then choosing a volume at random from that shelf, what is the probability of finding a math book?

c) Choosing a volume at random from the catalog, you come across a math book: what is the probability that it comes from the middle shelf?

PROBABILITY THEORY – EXERCISES SESSION N° 2

About the chapters:

- Chapter 3: REAL-VALUED RANDOM VARIABLE

TD2 – 1 :

An event A has the probability P(A) = p of occurring. We perform n independent trials and denote X the random variable equal to the number of times the event A has occurred among the n trials.

a) What are the possible outcomes of X, what does the quantity X/n represent?
b) What is the probability distribution of X?
c) Verify that the distribution law satisfies the first 2 axioms of the definition of probability.

TD2 – 2 *:

An archer shoots arrows at a target made up of concentric rings, delimited by circles with radius 1,2, ..., 10 cm, and numbered respectively from 10 to 1. The probability of reaching the ring *k* is proportional to the area of the ring. We assume that the player hits the target. Let *X be* the random variable which associates the number of ring with each shot.

a) What is the probability law of X?
b) The player wins *k* euros if he reaches the ring numbered *k* for *k* between 6 and 10, while he loses 2 € if he reaches one of the peripheral rings numbered from 1 to 5. Is the game favorable to the player?
Hint: calculate the expected gain.

TD2 – 3 :

A and B are two airplanes having 4 and 2 engines respectively. The motors are assumed to be independent of each other, and they have a probability *p* of failing. Each plane arrives at its destination if less than half of its engines fail. Which plane would you choose? (Argue your choice for any value of p).

TD2 – 4 :

Let X, Y be two independent random variables following the uniform distribution on $\{1,\dots,n\}$.

a) Calculate $P(X = Y)$.
b) Calculate $P(X \geq Y)$.
c) * Calculate the probability law of $X + Y$.

TD2 – 5:

Let us suppose that the marks of an exam are distributed according to a Gaussian law of expected value 13 and standard deviation 3. What is the probability that a mark is:

- greater than 20?
- less than 5?
- comprised between 12 and 14?
- greater than 12?

We want to give an award to 10 % of the students in the class. What is the minimal threshold grade that a student should get to obtain the award?

TD2 – 6 :

We consider a continuous random variable X whose density is symmetric with respect to the origin, with $P(-1 \leq X \leq +1) = 0.66$

a) Sketch the characteristics of this distribution.
b) Calculate the probability of the following events by justifying your answer.

- $0 \leq X \leq 1$
- $X \leq 1$
- $X > 1$
- $X = 1$
- $X \geq 1$
- $|X| \leq 1$
- $X^2 \leq 1$
- $X^3 \leq 1$

PROBABILITY THEORY – EXERCISES SESSION N° 3

About the chapters :

- Chapter 3 : REAL-VALUED RANDOM VARIABLE
- Chapter 4 : TYPICAL VALUES OF A REAL-VALUED RANDOM VARIABLE.
- Chapter 5: CHARACTERISTIC FUNCTION

TD3 – 1 :

Consider the following game: the player first rolls a balanced die. If he rolls 1, 2 or 3, he wins the equivalent in euros (i.e. 1€ if he rolls 1, for example). Otherwise, he loses 2 €. We denote by X the random variable corresponding to the player's gain (negative in case of loss).

a) Calculate the probability distribution and the cumulative distribution function of X.
b) Calculate the expected value of X.
c) Calculate the variance of X.

The game is modified as follows: the gain remains the same for outcomes 1, 2 or 3, but if the player obtains something else, he throws the die once more. If the outcome is 3 or less, he wins 3€, otherwise he loses 5€.

d) * Formally describe the universe of the new game.
e) * Give the law of Y (which again designates the player's gain) and calculate his expectation.
f) * Which variant of the game is the most advantageous for the player?

TD3 – 2 :

To improve the dependability of a computer server, we are planning to introduce redundant components that can take over when a failure occurs. There are two possibilities:

- Use of three power supplies of 300 Watts each are used: the server can continue to operate even if one power supply fails because it consumes at most 500 Watts.
- The four hard disks are placed in a RAID 5 configuration: the server can continue to operate with one hard disk failure.

Assume that the probability of a power supply failure is p and that of a hard disk failure is q. We suppose that all components are independent.

a) Consider a server with redundant power supplies: calculate the probability of server failure assuming that no component other than the power supplies can fail.

b) Consider a server with RAID 5 hard disks: calculate the probability of server failure assuming that no component other than the hard disks can fail.
c) * If p = q, which redundancy solution is the most interesting?

TD3 – 3:

A supermarket receives on average 4 customers per day. The number of customers being distributed according to a Poisson distribution, calculate the probability that the store will be visited on Wednesday by:

a) no customer;
b) * 5 customers;
c) * at least 6 customers.

TD3 – 4:

Consider the function f defined by:

$$f(x) = \begin{cases} ax^2 + b & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

a) **Question 1** Under what conditions on a and b the function f is the probability density function of a continuous random variable?
b) **Question 2** Assume that a and b satisfy the conditions given in the answer to the previous question. Let X be a random variable of density f . Suppose that

   $P(X \geq 0.5) = 7/8$. Deduce the values of a and b.

c) **Question 3** Calculate E(X) and V(X) using the values of a and b obtained in the previous question.

TD3 – 5:

An experiment consists in filling a 20 cl glass with a random amount of water chosen uniformly between 0 and 20 cl:

a) what is the probability of getting less than 5 cl of water?
b) we empty 5 glasses that were filled that way in a huge basin. What is the average amount of water in the basin?

TD3 – 6:

Let us suppose that the lifetime of a hard disk follows an exponential law. The manufacturer wants to ensure that the probability of a disk to break down over 1 year is less than 0.01. What is the minimum average lifespan of a hard disk?

TD3 – 7 *:

Let X be a continuous random variable with characteristic function:

$$\varphi_X(t) \quad = \quad e^{-a|t|} \qquad\qquad (a > 0)$$

a) Calculate the probability density function of the random variable X,
b) Is it possible to use $\varphi_X(t)$ to calculate the moments of X ?

TD3 - 8 :

Consider the exercise TD2-5.

What is the probability of the event (X<7 or X>19) given that X follows a Gaussian distribution with mean 13 and standard deviation 3?

Use TChebyshev's theorem to determine an upper bound for the probability of this event. Interpret your result.

PROBABILITY THEORY – EXERCISES SESSION N° 4

About chapters :

- Chapter 5 (bis): TRANSFORMATION OF A REAL-VALUED RANDOM VARIABLE
- Chapter 6: TWO-DIMENSIONAL RANDOM VARIABLES

TD4 – 1 :

A random variable X has the following probability density function:

$$f_X(x) = |x| e^{-x^2} \quad with \quad x \in R$$

Consider the random variable Y, a transformation of X denoted g(X) and defined by:

$y = 1$ for $x \leq -1$

$y = x^2$ for $x \geq -1$

a) Represent this transformation in the Cartesian plane. Decompose the domain of variation of y as a function of the number of roots of the equation y =g(x) at y fixed. Calculate the probability density function of the random variable Y.
b) Calculate the cumulative distribution function of Y. Verify that this latter has the properties of a cumulative distribution function.

TD4 – 2:

This summer, John decided to play two games A and B. The random variables X and Y denote the number of points John can earn with the games A and B respectively. The joint probability distribution law of the pair of random variables (X, Y) is given in the following table:

$P(X=x, Y=y)$	$y=-1$	$y=1$	$y=2$
$x = -2$	0.2	0.2	α
$x = 0$	0.1	0.1	0.05
$x = 1$	0.2	0	0.1

For instance, the probability that John wins 1 point with game A and loses 1 point in game B is 0.2

From the table we notice that:

- With the game A John can earn -2, 0 or 1 point, or $X(\Omega) = \{-2, 0, 1\}$.
- With the game B John can earn -1, 1 or 2 points, or $Y(\Omega) = \{-1, 1, 2\}$.

a) What is the only possible value of α? Justify your answer.
b) Calculate the marginal distributions of *X* and *Y.*
c) Demonstrate that *X* and *Y* are not independent.
d) Calculate the conditional probability of *X* given $Y = 1$.
e) Calculate the conditional probability of *X* given $Y \neq 2$.
f) We define $Z = X + Y$ (total gain). Calculate the probability distribution of *Z*.

<u>TD4 – 3</u> :

We consider the probability density function given by :

$$f_{X,Y}(x,y) = A.xy \qquad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2x$$

$$f_{X,Y}(x,y) = 0 \qquad \text{elsewhere}$$

a) What is the value of A?

b) Calculate the marginal probability density of X and Y. Verify that these functions verify the properties of a probability density function.

PROBABILITY THEORY – EXERCISES SESSION N° 5

About chapters :

- Chapter 6 (continuation): GAUSSIAN RANDOM VARIABLES
- Chapter 7: EXPECTED VALUE, CHARACTERISTIC FUNCTION AND MOMENTS FOR TWO-DIMENSIONAL RANDOM VARIABLES

Reminder :

The term "Gaussian vector" represents a pair of Gaussian variables:

In general

- X and Y independent $\Rightarrow$ cov(X,Y)=0
- cov(X,Y)=0 $\nRightarrow$ X and Y independent

Definition: (X,Y) is a Gaussian vector $\Leftrightarrow \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$, $\boldsymbol{\alpha}X+\boldsymbol{\beta}Y$ is a Gaussian random variable.

- If (X,Y) is a Gaussian vector and X and Y are independent $\Leftrightarrow$ cov(X,Y)=0.
- (X,Y) Gaussian vector $\Rightarrow$ X Gaussian and Y Gaussian (particular case if $\boldsymbol{\beta}$=0 and $\boldsymbol{\alpha}$=0 respectively).
- (X Gaussian and Y Gaussian) $\nRightarrow$ (X,Y) Gaussian vector.
- (X Gaussian, Y Gaussian and (X,Y) independent) $\Rightarrow$ (X,Y) Gaussian vector. (the opposite is true iff cov(X,Y)=0)
- (X Gaussian, Y Gaussian and (X,Y) independent $\Rightarrow \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$, $\boldsymbol{\alpha}X+\boldsymbol{\beta}Y$ is a Gaussian random variable.

TD5 – 1 :

Let X be a normal random variable normal distributed with mean μ = 2 and standard deviation σ = 2.

a) Consider the random variable $T = \dfrac{X-\mu}{\sigma}$. What is the law of T?

b) Consider the event {X <1.5} What is the equivalent event for T? What is the corresponding probability?

c) Same question as in b) for the event {X> 2}

d) Consider the event {-1 $\leq$ T <1}. What is the equivalent event for X? What is its probability?

e) Calculate the values x of X such that $P(X<x) = 0.76$; $P(X \geq x) = 0.6$ and $P(0 \leq X<x) = 0.40$.

TD5 - 2 :

Let X and ε be two independent real random variables, such that $P(\varepsilon = +1) = P(\varepsilon = -1) = 1/2$ and X follows a standard normal distribution. We set $Y = \varepsilon X$.

a) Calculate the cumulative distribution function of Y. Deduce its probability distribution function.
b) Are the variables X and Y uncorrelated?
c) Calculate $P(X+Y = 0)$. Is the vector (X, Y) Gaussian? Are the variables X and Y independent?

TD5 – 3 :

Let (X,Y) *be* a centered Gaussian vector, with $E(X^2) = 4$ and $E(Y^2) = 1$, and such that the variables $(2X + Y)$ and $(X - 3Y)$ are independent.

a) Determine the law of (X, Y) and the correlation coefficient r(X, Y).
b) Demonstrate that the vector (X + Y, 2X - Y) is also Gaussian, then determine its covariance matrix and its law.

Hint : The covariance matrix of a Gaussian vector (X, Y), denoted V(X, Y) is a 2x2 matrix whose components are

$$V(X,Y) = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix}$$

TD5 – 4 :

Consider the exercise TD4-2:

This summer, John decided to play two games A and B. The random variables X and Y denote the number of points John can earn with the games A and B respectively. The joint probability distribution law of the pair of random variables (X, Y) is given in the following table:

P(X=x, Y=y)	*y=-1*	*y=1*	*y=2*
x = -2	0.2	0.2	0.05
x = 0	0.1	0.1	0.05
x =1	0.2	0	0.1

For instance, the probability that John wins 1 point with game A and loses 1 point in game B is 0.2

From the table we notice that:

- With the game A John can earn -2, 0 or 1 point, or $X(\Omega) = \{-2, 0, 1\}$.
- With the game B John can earn -1, 1 or 2 points, or $Y(\Omega) = \{-1, 1, 2\}$.

a) Calculate $E(X \mid Y = 1)$ (see question TD4-2.4 for the conditional distribution of X given $Y = 1$) .
b) Calculate $E(X \mid Y \neq 2)$ (see question TD4-2.5 for the conditional distribution of X given $Y \neq 2$). Compare the obtained result with that of the previous question.
c) Calculate $E(XY)$ as well as $Cov(X, Y)$ *from it.*
d) * We denote $Z = X+Y$. Calculate the probability distribution law of Z, $E(Z)$ and $V(Z)$.