Chapter 3

Governing Equations

3.1 Introduction

The governing equations are basically a set of equations which model a physical phenomena. In our context, they will be a set of partial differential equations (PDEs), also called a system of PDEs. These equations define how different properties of interest vary within space and time. The properties of interest are called the dependent variables; and the space and time are called independent variables. The space is spanned by three mutually perpendicular axis denoted by the variables x, y, z and the variable for time is t. Choosing an origin point in space time and arbitrary values for variables x, y, z and t, the complete three dimensional space and time can be spanned. When we talk about solving the governing equations we need to restrict ourselves within a well defined bounded region in space and time. This is called the solution space. At the boundary of solution space we have to define additional conditions to obtain unique solutions.

The dependent variables are properties like pressure, density, temperature, velocity etc. The distribution of these properties is the solution that we are interested in for, say, designing a product or understanding the physics. The system of PDEs is said to be closed when the number of dependent variables are equal to the number of equations. In many cases, the number of equations are lesser than the number of dependent variables. We have to then use models relating different properties, thus adding more equations, to close the system of governing equations. These models are mostly based on empirical relations or simplified assumptions.

3.2 Generic form of system of PDEs

The governing PDEs used for simulation of various physical processes can be cast into a generic form,

$$\frac{\partial W}{\partial t} + \frac{\partial F_C}{\partial x} + \frac{\partial G_C}{\partial y} + \frac{\partial H_C}{\partial z} = \frac{\partial F_D}{\partial x} + \frac{\partial G_D}{\partial y} + \frac{\partial H_D}{\partial z} + S \ . \tag{3.1}$$

Here, W is called the "conservative variable vector" or simply "conservative vector". It is named so because the variables in this vector are an outcome of application of conservation laws. The vectors F_C, G_C and H_C are called the "convective flux vectors". They are named so because they contain the product of velocity with terms from conservative vector. Therefore capturing the phenomena of translation of conserved quantities due to convection currents. Most importantly, the convective flux vector does not contain any terms with space derivatives and is purely a function of the conservative vector. The vectors F_D, G_D and H_D are called the "diffusion flux vectors". All the terms with space derivatives are placed in these vectors. The diffusion flux vector is called so due to the fact that the spatial derivatives cause spreading (diffusion) or concentration of the quantities, even when the bulk fluid velocity is zero. The vector S is called the "source vector". All the remaining terms in the governing equation, which can be integrated over finite control spatial volume, are clubbed together in this vector. Take a look at the example PDEs in section 3.5 to have a better understanding of these terms.

In many practical applications the solution at the steady state is sought, which implies that the time derivative term tends to zero. In such applications, solving time accurate PDEs is unnecessary and probably inefficient. We may therefore, include an additional artificial time derivative term which can be tweaked for efficient computations. The governing equations therefore becomes,

$$\frac{\partial U}{\partial \tau} + \frac{\partial W}{\partial t} + \frac{\partial F_C}{\partial x} + \frac{\partial G_C}{\partial y} + \frac{\partial H_C}{\partial z} = \frac{\partial F_D}{\partial x} + \frac{\partial G_D}{\partial y} + \frac{\partial H_D}{\partial z} + S \ . \tag{3.2}$$

Here, τ is called the artificial or pseudo time variable and t is the real time variable. The vector U is called the artificial or pseudo conservative vector and to distinguish from this vector, W is now onwards called the real conservative vector. The units of U and W are the same, however the actual terms may be completely different. It is a common practice to write $U = P\,W$, where P is a pre conditioner introduced to make the numerical calculations efficient and accurate. The flux and source vectors are functions of the artificial conservative vector. For steady state simulations the term $\partial W/\partial t = 0$ and for unsteady simulations $\partial W/\partial t \neq 0$. In both the cases (steady and unsteady simulations), the term $\partial U/\partial \tau$ must tend to zero for accepting the computational result.

There is another minor detail worth mentioning for the reason of completeness. The form in which the equations are written above is called the "conservative form", not to be confused with the conservative vector. In this form, the space and time derivatives do not have any coefficients. The conservative form naturally arises when deriving the PDEs using conservation principles and applying the Gauss's theorem. This form is important for using the finite volume method, which will be explained in a later chapter.

The advantage of writing the governing equations in a generic form is that, now, it is possible to discuss the various CFD techniques in a well defined setting. Also, this approach makes it possible to define computational models and write efficient computer programs which cater to a wide variety of applications which fit within this generic form.

3.3 Different types of variables

When we say that we are solving the governing PDEs, it basically means that we are solving for the conservative vector in space and time with specified boundary and/or initial conditions. The variables in the conservative vector (called "conservative variables") are generally not very intuitive to us humans. Instead, we like to deal with quantities that we can measure and have a feel for. For example, the quantity – 'density × velocity' – which is the conservative variable in the momentum equation is difficult to interpret as a whole, but we understand 'density' and 'velocity' separately. The so called "primitive variables" are simpler variables, introduced for this purpose. When we provide the inputs to the program or get outputs from the program, it is easier to deal with primitive variables.

The method of lines is a technique which is extensively used in the solution of time dependent PDEs. In this method it is useful to work with the so called "characteristics variables". Even though in computer programs we do not very often use these variables, they form the basis of many CFD techniques. Also in some boundary conditions, such as non reflective outlet boundary condition, the characteristics variables play an important role.

3.4 Representation of governing equations in code

To solve the system of PDEs, we need to first come up with a way to represent the equations in code. The chosen way has to be generic and at the same time efficient and easy to integrate into the computational methods. As discussed above, different types of variables may be required at various places in the code and hence we need to convert between the different types. This is incorporated in the code through functions,

$$U \mapsto W, \quad U \mapsto V, \quad V \mapsto U \quad \text{with} \quad U, W, V \in \mathbb{R}^N, \tag{3.3}$$

where, V is the vector of primitive variables. U and W are the artificial and real conservative variables respectively, as described earlier, and N is the number of PDEs in the governing equations. In addition, the variables names will be needed for user interaction for input and output. The convective flux and the source vector can be described easily in terms of the artificial conservative vector U and therefore we can define functions for them as,

$$U \mapsto (F_C, G_C, H_C)\,, \quad U \mapsto S \quad \text{with} \quad U, F_C, G_C, H_C, S \in \mathbb{R}^N. \tag{3.4}$$

The definition of functions for the diffusion flux vector requires not only the vector U, but also spatial derivatives. Therefore, it is necessary to calculate the spatial variation of variables before calculation of the diffusive flux. The functions for calculation of the diffusive flux therefore are defined as,

$$(U, \nabla U) \mapsto (F_D, G_D, H_D) \quad \text{with} \quad U, \nabla U, F_D, G_D, H_D \in \mathbb{R}^N. \tag{3.5}$$

Also, the characteristics of the convective flux and the material properties are necessary for the complete definition of the governing equations. There will also be some special cases when additional constitutive relations will be needed. All the above will be incorporated into the code by using an interface (abstract class). Hence making it extremely simple to adapt to any physical PDEs expressed in the generic form of the governing equations (3.2).

3.5 Few common governing equations

Let us look at some of the governing equations that fit into the generic form (3.2). Many of these equations form the building blocks of research and are widely used in industry for design purposes. The derivation or detailed description of the governing physics is beyond the scope of this book. For that it is recommended to refer to complete text on the specific subject. In the equations below, ρ is the mass density, $V = (u, v, w)$ is the velocity vector is a 3D Cartesian space, p is the absolute pressure, E is the total energy per unit mass (specific total energy), T is the temperature measured in Kelvin, $\vec{g} = (g_x, g_y, g_z)$ is the body force vector per unit mass, $\dot{q}$ is the rate of heat generation per unit mass, μ is the coefficient of dynamic viscosity, k is the coefficient of heat transfer due to conduction. We will start with an elaborate discussion on Navier Stokes as it forms a base for derivation of many other system of PDEs to follow.

3.5.1 Navier-Stokes equations

These equations are derived by applying the conservation principles of mass, momentum and energy to a fluid flow of single phase (gas or liquid). Furthermore, since the fluids are compressible, in general, the density is considered as a variable. The complete Navier Stokes equations in three spatial dimensions comprises of 5 equations. They can be written in the generic form (3.2) by substituting the following vectors,

$$
\begin{gathered}
W = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad F_C = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(\rho E + p) \end{bmatrix}, \quad G_C = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(\rho E + p) \end{bmatrix}, \quad H_C = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(\rho E + p) \end{bmatrix}, \\
F_D = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ \Theta_x \end{bmatrix}, \quad G_D = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ \Theta_y \end{bmatrix}, \quad H_D = \begin{bmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ \Theta_z \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \rho g_x \\ \rho g_y \\ \rho g_z \\ \rho(g \cdot V) + \rho\dot{q} \end{bmatrix}.
\end{gathered} \tag{3.6}
$$

Some of the variables in these equations are already described before, the remaining are described here. The specific total energy, E, consists of internal energy and kinetic energy, which can be written as,

$$
E = e + \frac{|V|^2}{2}, \tag{3.7}
$$

where, the internal energy, e, is a function of flow properties like pressure and temperature, $e = e(p, T)$, and $|V| = \sqrt{u^2 + v^2 + w^2}$. The internal energy for gases can be written as $e = c_v T$, based on the thermodynamic relation $de = c_v dT$, assuming the specific heat capacity at constant volume, c_v, as a constant and the constant of integration as zero.

The viscous stress tensor can be written as τ_{ij} with $i = x, y, z$ and $j = x, y, z$. The first subscript denotes the direction of the viscous force and the second subscript denotes the face on which the stress is acting. The nine components of the tensor can be combined in a matrix as,

$$
\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{3.8}
$$

In case of Newtonian fluids, these viscous stresses are assumed to be linearly proportional to gradients of velocity at the point. Thus providing with simplified relations for shear stresses as,

$$
\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right) + \delta_{ij} \lambda \nabla \cdot V. \tag{3.9}
$$

Here, u_i denotes the velocity component, with $u_x = u, u_y = v, u_z = w$. The symbol δ_{ij} is the Kronecker delta function which is defined as,

$$
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{3.10}
$$

the symbol λ is called the second coefficient of viscosity. Based on Stokes hypothesis, its value is generally taken as,

$$\lambda = -\frac{2}{3}\mu \tag{3.11}$$

and $\nabla \cdot V$ is the divergence of the velocity field.

The energy diffusion terms come about due to the work done by the viscous stresses and the diffusion of energy due to conduction. They are related to the conservative vector with the following relations,

$$\begin{aligned}
\Theta_x &= u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + k\frac{\partial T}{\partial x} \\
\Theta_y &= u\tau_{yx} + v\tau_{yy} + w\tau_{yz} + k\frac{\partial T}{\partial y} \quad . \\
\Theta_z &= u\tau_{zx} + v\tau_{zy} + w\tau_{zz} + k\frac{\partial T}{\partial z}
\end{aligned} \tag{3.12}$$

The diffusion coefficients like μ and k are not constants over a wide range of flow quantities. The coefficient of viscosity, for example, for gases is highly influenced by the temperature. These coefficients can be calculated using lookup tables for flow variables, such as temperature, pressure etc. These lookups can be done very efficiently using search algorithms and linear interpolation. It is also common to use empirical relations, such as the Sutherland formula for coefficient of viscosity. For gases the Sutherland formula can be written as,

$$\mu = \mu_0 \frac{T_0 + C}{T + C}\left(\frac{T}{T_0}\right)^{3/2} .$$

This can be simplified to,

$$\mu = C_0 \frac{T^{3/2}}{T + C} \quad \text{with} \quad C_0 = \mu_0 \frac{T_0 + C}{T_0^{3/2}} .$$

The dynamic viscosity for air as function of temperature (in Kelvin) turns out to be,

$$\mu = 1.512 \frac{T^{3/2}}{T + 120} \times 10^{-6} \text{ Pa s.}$$

In case of gases, the relation of Prandtl number is used for definition of k as,

$$k = \frac{\mu\, c_p}{Pr} ,$$

and the Prandtl number, Pr, is assumed to be constant for gases. In case of air, the Prandtl number is taken to be 0.72 in the complete domain. Similarly, empirical relations are available for other gases and liquids for calculation of coefficients as a function of local flow variables.

The source term in the momentum equations is a result of any body forces, such as gravity force, with force vector per unit mass given by $\vec{g} = (g_x, g_y, g_z)$. The source term in the energy equation is a result of the work done by the body forces and heat generation within the fluid due to sources, such as chemical reactions or radiation heat transfer.

Now, if we count the number of unknown variables, we can see that we have 6 unknowns – ρ, u, v, w, p, T – and only 5 equations. Therefore, we need another equation to close the system of equations. This is done by using the equation of state relating the pressure, density and temperature, $p = p\,(\rho, T)$. For an ideal gas the equation of state is given as,

$$p = \rho \frac{R_u}{\dot{m}} T . \tag{3.13}$$

where, $R_u = 8.314 \times 10^3$ J/kmol K, is the universal gas constant and $\dot{m}$ is the molar mass of the gas measured in kg/kmol (same as g/mol – unit generally used in reference tables). In practice, the ratio $R_u/\dot{m}$ is replaced by a single gas constant R. Thus, we get a simpler equation of state,

$$p = \rho R T . \tag{3.14}$$

In case of real gases at high temperatures or chemically reacting species, the equation of state will be more com plicated. It is common to eliminate the specific heat capacity c_v by introducing another constant γ, which is the ratio of specific heat capacity,

$$\gamma = \frac{c_p}{c_v} . \tag{3.15}$$

Here, c_p is the specific heat capacity at constant pressure. Using the thermodynamic relation, $R = c_p - c_v$, the constant c_v can be written in terms of γ and R as,

$$c_v = \frac{R}{\gamma - 1} \; . \tag{3.16}$$

The vector $U = PW$, with P as a pre conditioner matrix. The matrix P is chosen as an identity for high Mach flows, where the density changes are prominent. For low Mach flows (generally the case with liquid flows) there can be different ways of choosing a pre conditioner. One of them is called the artificial compressibility method (given in another example later). More sophisticated methods, called the all Mach number flow solvers, use a pre conditioner as a function of the Mach number and therefore maintain the system well conditioned in the complete domain and a wide range of Mach numbers. Putting it all together,

$$\boxed{\begin{aligned}
&W = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad F_C = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u\left(\rho E + p\right) \end{bmatrix}, \quad G_C = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v\left(\rho E + p\right) \end{bmatrix}, \quad H_C = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w\left(\rho E + p\right) \end{bmatrix}, \\
&F_D = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ \Theta_x \end{bmatrix}, \quad G_D = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ \Theta_y \end{bmatrix}, \quad H_D = \begin{bmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ \Theta_z \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \rho g_x \\ \rho g_y \\ \rho g_z \\ \rho\left(g \cdot V\right) + \rho \dot{q} \end{bmatrix}. \\
&\text{with, } E = \tfrac{R}{\gamma-1}T + \tfrac{u^2+v^2+w^2}{2}, \quad p = \rho RT, \quad \tau_{ij} = \mu\left(\tfrac{\partial u_i}{\partial j} + \tfrac{\partial u_j}{\partial i}\right) - \delta_{ij}\left(\tfrac{2}{3}\mu \nabla \cdot V\right), \\
&\Theta_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + k\tfrac{\partial T}{\partial x}, \quad \Theta_y = u\tau_{yx} + v\tau_{yy} + w\tau_{yz} + k\tfrac{\partial T}{\partial y}, \quad \Theta_z = u\tau_{zx} + v\tau_{zy} + w\tau_{zz} + k\tfrac{\partial T}{\partial z} \; .
\end{aligned}}$$

3.5.1.1 Non-dimensional form

It is very common to use the non dimensional form of the Navier Stokes equations. This has a dual advantage of reducing the numerical errors and making the solutions applicable to a wider range of scaled dimensional problems. The common method of non dimensionalization is to scale the variables by an appropriate free stream quantity. One approach could be to scale variables using the free stream velocity: V_∞, free stream dynamic pressure: $\rho_\infty V_\infty^2$, free stream density: ρ_∞, the characteristic length of the domain: L_∞ and time scale: L_∞ / V_∞. This results in the following definition of non dimensional quantities.

$$\begin{gathered}
\hat{x} = \frac{x}{L_\infty}, \quad \hat{y} = \frac{y}{L_\infty}, \quad \hat{z} = \frac{z}{L_\infty}, \quad \hat{t} = \frac{V_\infty}{L_\infty}t, \quad \hat{\tau} = \frac{V_\infty}{L_\infty}\tau \; , \\
\hat{\rho} = \frac{\rho}{\rho_\infty}, \quad \hat{u} = \frac{u}{V_\infty}, \quad \hat{v} = \frac{v}{V_\infty}, \quad \hat{w} = \frac{w}{V_\infty}, \quad \hat{p} = \frac{p}{\rho_\infty V_\infty^2} \; , \\
\hat{R} = \frac{R}{R_\infty} = 1, \quad \hat{\mu} = \frac{\mu}{\mu_\infty}, \quad \hat{k} = \frac{k}{k_\infty}, \quad \hat{\vec{g}} = \frac{L_\infty}{V_\infty^2}\vec{g} = \frac{1}{Fr^2}\frac{\vec{g}}{|\vec{g}|}, \\
Re_\infty = \frac{\rho_\infty V_\infty L_\infty}{\mu_\infty}, \quad Pr_\infty = \frac{\mu_\infty c_{p\infty}}{k_\infty} = \frac{\mu_\infty R_\infty \gamma}{k_\infty\left(\gamma - 1\right)}, \quad \hat{\dot{q}} = \frac{L_\infty}{V_\infty^3}\dot{q} \; .
\end{gathered} \tag{3.17}$$

Using the ideal gas equation of state we can write,

$$T = \frac{p}{\rho R} = \frac{\hat{p}\rho_\infty V_\infty^2}{\hat{\rho}\rho_\infty \hat{R} R_\infty} = \frac{\hat{p}}{\hat{\rho}\hat{R}}\frac{V_\infty^2}{R_\infty} = \hat{T}\frac{V_\infty^2}{R_\infty}$$

$$\implies \hat{T} = \frac{R_\infty}{V_\infty^2}T \; . \tag{3.18}$$

Now, let us substitute these in the original (dimensional) governing equation. Thus, the mass equation becomes,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{3.19}$$

$$\implies \frac{\partial\left(\hat{\rho}\rho_\infty\right)}{\partial\left(\hat{t}L_\infty/V_\infty\right)} + \frac{\partial\left(\hat{\rho}\rho_\infty\hat{u}V_\infty\right)}{\partial\left(\hat{x}L_\infty\right)} + \frac{\partial\left(\hat{\rho}\rho_\infty\hat{v}V_\infty\right)}{\partial\left(\hat{y}L_\infty\right)} + \frac{\partial\left(\hat{\rho}\rho_\infty\hat{w}V_\infty\right)}{\partial\left(\hat{z}L_\infty\right)} = 0$$

$$\implies \frac{\partial\left(\hat{\rho}\rho_\infty V_\infty\right)}{\partial\left(\hat{t}L_\infty\right)} + \frac{\partial\left(\hat{\rho}\hat{u}\rho_\infty V_\infty\right)}{\partial\left(\hat{x}L_\infty\right)} + \frac{\partial\left(\hat{\rho}\hat{v}\rho_\infty V_\infty\right)}{\partial\left(\hat{y}L_\infty\right)} + \frac{\partial\left(\hat{\rho}\hat{w}\rho_\infty V_\infty\right)}{\partial\left(\hat{z}L_\infty\right)} = 0$$

$$\frac{\partial\hat{\rho}}{\partial\hat{t}} + \frac{\partial\hat{\rho}\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{\rho}\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{\rho}\hat{w}}{\partial\hat{z}} = 0 \tag{3.20}$$

We can see that, the mass equation remains identical to the dimensional form. Substituting the definitions of non dimensional variables (3.17) in x momentum equation we can write,

$$\frac{\partial\rho u}{\partial t} + \frac{\partial\left(\rho u^2 + p\right)}{\partial x} + \frac{\partial\rho uv}{\partial y} + \frac{\partial\rho uw}{\partial z} = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + \rho g_x \tag{3.21}$$

$$\frac{\partial\rho u}{\partial t} + \frac{\partial\left(\rho u^2 + p\right)}{\partial x} + \frac{\partial\rho uv}{\partial y} + \frac{\partial\rho uw}{\partial z} = \frac{\partial\left(2\mu\left(\frac{\partial u}{\partial x}\right)\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)}{\partial y} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right)}{\partial z} + \frac{\partial\left(\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)}{\partial x} + \rho g_x$$

$$\begin{aligned}
\implies & \frac{\partial\left(\hat{\rho}\rho_\infty\hat{u}V_\infty\right)}{\partial\left(\hat{t}L_\infty/V_\infty\right)} + \frac{\partial\left(\hat{\rho}\rho_\infty\left(\hat{u}V_\infty\right)^2 + \hat{p}\rho_\infty V_\infty^2\right)}{\partial\left(\hat{x}L_\infty\right)} + \frac{\partial\hat{\rho}\rho_\infty\left(\hat{u}V_\infty\right)\left(\hat{v}V_\infty\right)}{\partial\left(\hat{y}L_\infty\right)} + \frac{\partial\hat{\rho}\rho_\infty\left(\hat{u}V_\infty\right)\left(\hat{w}V_\infty\right)}{\partial\left(\hat{z}L_\infty\right)} \\
&= \frac{\partial\left(2\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial(\hat{x}L_\infty)}\right)\right)}{\partial\left(\hat{x}L_\infty\right)} + \frac{\partial\left(\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial(\hat{y}L_\infty)} + \frac{\partial\hat{v}V_\infty}{\partial(\hat{x}L_\infty)}\right)\right)}{\partial\left(\hat{y}L_\infty\right)} + \frac{\partial\left(\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial(\hat{z}L_\infty)} + \frac{\partial\hat{w}V_\infty}{\partial(\hat{x}L_\infty)}\right)\right)}{\partial\left(\hat{z}L_\infty\right)} \\
&\quad + \frac{\partial\left(\lambda\left(\frac{\partial\hat{u}V_\infty}{\partial(\hat{x}L_\infty)} + \frac{\partial\hat{v}V_\infty}{\partial(\hat{y}L_\infty)} + \frac{\partial\hat{w}V_\infty}{\partial(\hat{z}L_\infty)}\right)\right)}{\partial\left(\hat{x}L_\infty\right)} + \hat{\rho}\rho_\infty\hat{g}_x\frac{V_\infty^2}{L_\infty}
\end{aligned}$$

$$\begin{aligned}
\implies & \frac{\rho_\infty V_\infty^2}{L_\infty}\frac{\partial\hat{\rho}\hat{u}}{\partial\hat{t}} + \frac{\rho_\infty V_\infty^2}{L_\infty}\frac{\partial\left(\hat{\rho}\hat{u}^2 + \hat{p}\right)}{\partial\hat{x}} + \frac{\rho_\infty V_\infty^2}{L_\infty}\frac{\partial\hat{\rho}\hat{u}\hat{v}}{\partial\hat{y}} + \frac{\rho_\infty V_\infty^2}{L_\infty}\frac{\partial\hat{\rho}\hat{u}\hat{w}}{\partial\hat{z}} \\
&= \frac{\mu_\infty V_\infty}{L_\infty^2}\frac{\partial\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)}{\partial\hat{x}} + \frac{\mu_\infty V_\infty}{L_\infty^2}\frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}} + \frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)}{\partial\hat{y}} + \frac{\mu_\infty V_\infty}{L_\infty^2}\frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}} + \frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)}{\partial\hat{z}} \\
&\quad + \frac{\mu_\infty V_\infty}{L_\infty^2}\frac{\partial}{\partial\hat{x}}\left[\left(-\frac{2}{3}\hat{\mu}\right)\left(\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{w}}{\partial\hat{z}}\right)\right] + \frac{\rho_\infty V_\infty^2}{L_\infty}\hat{\rho}\hat{g}_x
\end{aligned}$$

writing, $\hat{\lambda} = -\frac{2}{3}\hat{\mu}$ and dividing through by $\rho_\infty V_\infty^2/L_\infty$,

$$\begin{aligned}
\implies & \frac{\partial\hat{\rho}\hat{u}}{\partial\hat{t}} + \frac{\partial\left(\hat{\rho}\hat{u}^2 + \hat{p}\right)}{\partial\hat{x}} + \frac{\partial\hat{\rho}\hat{u}\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{\rho}\hat{u}\hat{w}}{\partial\hat{z}} \\
&= \frac{\mu_\infty}{\rho_\infty V_\infty L_\infty}\left(\frac{\partial\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)}{\partial\hat{x}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}} + \frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)}{\partial\hat{y}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}} + \frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)}{\partial\hat{z}} + \frac{\partial}{\partial\hat{x}}\left[\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]\right) \\
&\quad + \frac{L_\infty}{\rho_\infty V_\infty^2}\frac{\rho_\infty V_\infty^2}{L_\infty}\hat{\rho}\hat{g}_x
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial\hat{\rho}\hat{u}}{\partial\hat{t}} + \frac{\partial\left(\hat{\rho}\hat{u}^2 + \hat{p}\right)}{\partial\hat{x}} + \frac{\partial\hat{\rho}\hat{u}\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{\rho}\hat{u}\hat{w}}{\partial\hat{z}} \\
&= \frac{1}{Re_\infty}\left(\frac{\partial\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)}{\partial\hat{x}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}} + \frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)}{\partial\hat{y}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}} + \frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)}{\partial\hat{z}} + \frac{\partial}{\partial\hat{x}}\left[\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{v}}{\partial\hat{y}} + \frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]\right) + \hat{\rho}\hat{g}_x
\end{aligned} \tag{3.22}$$

and similarly the non dimensional y momentum equation can be written as,

$$\frac{\partial\hat{\rho}\hat{v}}{\partial\hat{t}} + \frac{\partial\hat{\rho}\hat{u}\hat{v}}{\partial\hat{x}} + \frac{\partial\left(\hat{\rho}\hat{v}^2+\hat{p}\right)}{\partial\hat{y}} + \frac{\partial\hat{\rho}\hat{v}\hat{w}}{\partial\hat{z}}$$
$$= \frac{1}{Re_\infty}\left(\frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{v}}{\partial\hat{x}}+\frac{\partial\hat{u}}{\partial\hat{y}}\right)\right)}{\partial\hat{x}} + \frac{\partial\left(2\hat{\mu}\left(\frac{\partial\hat{v}}{\partial\hat{y}}\right)\right)}{\partial\hat{y}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{v}}{\partial\hat{z}}+\frac{\partial\hat{w}}{\partial\hat{y}}\right)\right)}{\partial\hat{z}} + \frac{\partial}{\partial\hat{y}}\left[\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]\right) + \hat{\rho}\hat{g}_y \quad (3.23)$$

and the non dimensional z momentum equation can be written as,

$$\frac{\partial\hat{\rho}\hat{w}}{\partial\hat{t}} + \frac{\partial\hat{\rho}\hat{u}\hat{w}}{\partial\hat{x}} + \frac{\partial\hat{\rho}\hat{v}\hat{w}}{\partial\hat{y}} + \frac{\partial\left(\hat{\rho}\hat{w}^2+\hat{p}\right)}{\partial\hat{z}}$$
$$= \frac{1}{Re_\infty}\left(\frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{w}}{\partial\hat{x}}+\frac{\partial\hat{u}}{\partial\hat{z}}\right)\right)}{\partial\hat{x}} + \frac{\partial\left(\hat{\mu}\left(\frac{\partial\hat{w}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{z}}\right)\right)}{\partial\hat{y}} + \frac{\partial\left(2\hat{\mu}\left(\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right)}{\partial\hat{z}} + \frac{\partial}{\partial\hat{z}}\left[\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]\right) + \hat{\rho}\hat{g}_z \quad (3.24)$$

The non dimensional momentum equations above look similar to the dimensional form, except the scaling of the viscous flux by the Reynolds number Re_∞. The energy equation can be non dimensionalized similarly as shown below,

$$\frac{\partial\rho E}{\partial t} + \frac{\partial\left(u\left(\rho E+p\right)\right)}{\partial x} + \frac{\partial\left(v\left(\rho E+p\right)\right)}{\partial y} + \frac{\partial\left(w\left(\rho E+p\right)\right)}{\partial z} = \frac{\partial\Theta_x}{\partial x} + \frac{\partial\Theta_y}{\partial y} + \frac{\partial\Theta_z}{\partial z} + \rho\left(g\cdot V\right) + \rho\dot{q} \quad (3.25)$$

Let us look at one term at a time. The specific total energy can be written as,

$$E = c_vT + \frac{u^2+v^2+w^2}{2} = \frac{RT}{\gamma-1} + \frac{u^2+v^2+w^2}{2}$$
$$\implies E = \frac{R\hat{T}\frac{V_\infty^2}{R_\infty}}{\gamma-1} + \frac{\left(\hat{u}V_\infty\right)^2+\left(\hat{v}V_\infty\right)^2+\left(\hat{w}V_\infty\right)^2}{2} = V_\infty^2\left(\frac{\hat{T}}{\gamma-1}+\frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2}\right)$$
$$\implies E = V_\infty^2\left(\frac{\hat{T}}{\gamma-1}+\frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2}\right) \quad (3.26)$$

The diffusion terms can be written as,

$$\Theta_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + k\frac{\partial T}{\partial x}$$

$$\implies \Theta_x = \hat{u}V_\infty\left(2\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial\left(\hat{x}L_\infty\right)}\right)\right) + \hat{v}V_\infty\left(\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial\left(\hat{y}L_\infty\right)}+\frac{\partial\hat{v}V_\infty}{\partial\left(\hat{x}L_\infty\right)}\right)\right)$$
$$+ \hat{w}V_\infty\left(\hat{\mu}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial\left(\hat{z}L_\infty\right)}+\frac{\partial\hat{w}V_\infty}{\partial\left(\hat{x}L_\infty\right)}\right)\right) + \hat{u}V_\infty\hat{\lambda}\mu_\infty\left(\frac{\partial\hat{u}V_\infty}{\partial\left(\hat{x}L_\infty\right)}+\frac{\partial\hat{v}V_\infty}{\partial\left(\hat{y}L_\infty\right)}+\frac{\partial\hat{w}V_\infty}{\partial\left(\hat{z}L_\infty\right)}\right) + \hat{k}k_\infty\frac{\partial\left(\hat{T}\frac{V_\infty^2}{R_\infty}\right)}{\partial\left(\hat{x}L_\infty\right)}$$
$$\implies \Theta_x = \frac{V_\infty^2}{L_\infty}\left(\mu_\infty\left[\hat{u}\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)+\hat{v}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)+\hat{w}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}}+\frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)+\hat{u}\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]+\hat{k}\frac{k_\infty}{R_\infty}\frac{\partial\hat{T}}{\partial\hat{x}}\right)$$
$$\implies \Theta_x = \frac{V_\infty^2}{L_\infty}\mu_\infty\left(\left[\hat{u}\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)+\hat{v}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)+\hat{w}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}}+\frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)+\hat{u}\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]+\hat{k}\frac{k_\infty}{\mu_\infty c_{p\infty}}\frac{\gamma}{(\gamma-1)}\right.$$
$$\implies \Theta_x = \frac{V_\infty^2}{L_\infty}\mu_\infty\left\{\left[\hat{u}\left(2\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)\right)+\hat{v}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{x}}\right)\right)+\hat{w}\left(\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}}+\frac{\partial\hat{w}}{\partial\hat{x}}\right)\right)+\hat{u}\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]+\left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\right.$$
$$\implies \frac{\partial\Theta_x}{\partial x} = \frac{V_\infty^2\mu_\infty}{L_\infty^2}\frac{\partial}{\partial\hat{x}}\left\{\left[2\hat{u}\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{x}}\right)+\hat{v}\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{x}}\right)+\hat{w}\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{z}}+\frac{\partial\hat{w}}{\partial\hat{x}}\right)+\hat{u}\hat{\lambda}\left(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{\partial\hat{w}}{\partial\hat{z}}\right)\right]+\left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial\hat{T}}{\partial\hat{x}}\right\} \quad (3.27)$$

similarly,

$$\frac{\partial \Theta_y}{\partial y} = \frac{V_\infty^2 \mu_\infty}{L_\infty^2} \frac{\partial}{\partial \hat{y}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) + 2\hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{y}} \right) + \hat{w}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + \hat{v}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{y}} \right\}, \tag{3.28}$$

$$\frac{\partial \Theta_z}{\partial z} = \frac{V_\infty^2 \mu_\infty}{L_\infty^2} \frac{\partial}{\partial \hat{z}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}} \right) + \hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + 2\hat{w}\hat{\mu} \left(\frac{\partial \hat{w}}{\partial \hat{z}} \right) + \hat{w}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{z}} \right\}. \tag{3.29}$$

The term $\rho\,(g \cdot V)$ becomes,

$$\rho\,(\vec{g} \cdot V) = \hat{\rho}\rho_\infty \left(\left(\hat{\vec{g}} \frac{V_\infty^2}{L_\infty} \right) \cdot V \right) = \frac{\rho_\infty V_\infty^3}{L_\infty} \hat{\rho}\,(\hat{u}\hat{g}_x + \hat{v}\hat{g}_y + \hat{w}\hat{g}_z)\,. \tag{3.30}$$

Substituting all the above in the energy equation results in,

$$\begin{aligned}
&\frac{\rho_\infty V_\infty^3}{L_\infty} \frac{\partial \left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right]}{\partial \hat{t}} + \frac{\rho_\infty V_\infty^3}{L_\infty} \frac{\partial \left(\hat{u} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{x}} \\
&\quad + \frac{\rho_\infty V_\infty^3}{L_\infty} \frac{\partial \left(\hat{v} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{y}} + \frac{\rho_\infty V_\infty^3}{L_\infty} \frac{\partial \left(\hat{w} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{z}} \\
&= \frac{V_\infty^2 \mu_\infty}{L_\infty^2} \frac{\partial}{\partial \hat{x}} \left\{ \left[2\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{x}} \right) + \hat{v}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) + \hat{w}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}} \right) + \hat{u}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{x}} \right\} \\
&+ \frac{V_\infty^2 \mu_\infty}{L_\infty^2} \frac{\partial}{\partial \hat{y}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) + 2\hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{y}} \right) + \hat{w}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + \hat{v}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{y}} \right\} \\
&+ \frac{V_\infty^2 \mu_\infty}{L_\infty^2} \frac{\partial}{\partial \hat{z}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}} \right) + \hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + 2\hat{w}\hat{\mu} \left(\frac{\partial \hat{w}}{\partial \hat{z}} \right) + \hat{w}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{z}} \right\} \\
&\qquad + \frac{\rho_\infty V_\infty^3}{L_\infty} \hat{\rho}\,(\hat{u}\hat{g}_x + \hat{v}\hat{g}_y + \hat{w}\hat{g}_z) + \rho_\infty \hat{\rho}\dot{q}\;.
\end{aligned}$$

Dividing the equation by $\rho_\infty V_\infty^3 / L_\infty$ results in,

$$\begin{aligned}
&\frac{\partial \left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right]}{\partial \hat{t}} + \frac{\partial \left(\hat{u} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{x}} \\
&\quad + \frac{\partial \left(\hat{v} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{y}} + \frac{\partial \left(\hat{w} \left(\left[\hat{\rho} \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2} \right) \right] + \hat{p} \right) \right)}{\partial \hat{z}} \\
&= \frac{\mu_\infty}{\rho_\infty V_\infty L_\infty} \frac{\partial}{\partial \hat{x}} \left\{ \left[2\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{x}} \right) + \hat{v}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) + \hat{w}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}} \right) + \hat{u}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{x}} \right\} \\
&+ \frac{\mu_\infty}{\rho_\infty V_\infty L_\infty} \frac{\partial}{\partial \hat{y}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) + 2\hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{y}} \right) + \hat{w}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + \hat{v}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{y}} \right\} \\
&+ \frac{\mu_\infty}{\rho_\infty V_\infty L_\infty} \frac{\partial}{\partial \hat{z}} \left\{ \left[\hat{u}\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}} \right) + \hat{v}\hat{\mu} \left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}} \right) + 2\hat{w}\hat{\mu} \left(\frac{\partial \hat{w}}{\partial \hat{z}} \right) + \hat{w}\hat{\lambda} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] + \left[\frac{1}{Pr_\infty} \frac{\gamma}{(\gamma - 1)} \right] \hat{k} \frac{\partial \hat{T}}{\partial \hat{z}} \right\} \\
&\qquad + \hat{\rho}\,(\hat{u}\hat{g}_x + \hat{v}\hat{g}_y + \hat{w}\hat{g}_z) + \frac{L_\infty}{V_\infty^3} \hat{\rho}\dot{q}\;.
\end{aligned}$$

Introducing the Reynolds number produces the final non dimensional form,

$$
\begin{aligned}
&\frac{\partial \hat{\rho}\hat{E}}{\partial \hat{t}} + \frac{\partial\left(\hat{u}\left(\hat{\rho}\hat{E}+\hat{p}\right)\right)}{\partial \hat{x}} + \frac{\partial\left(\hat{v}\left(\hat{\rho}\hat{E}+\hat{p}\right)\right)}{\partial \hat{y}} + \frac{\partial\left(\hat{w}\left(\hat{\rho}\hat{E}+\hat{p}\right)\right)}{\partial \hat{z}} \\
&= \frac{1}{Re_\infty}\frac{\partial}{\partial \hat{x}}\left\{\left[2\hat{u}\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right) + \hat{v}\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right) + \hat{w}\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}}\right) + \hat{u}\hat{\lambda}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right)\right] + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{x}}\right\} \\
&+ \frac{1}{Re_\infty}\frac{\partial}{\partial \hat{y}}\left\{\left[\hat{u}\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right) + 2\hat{v}\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{y}}\right) + \hat{w}\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}}\right) + \hat{v}\hat{\lambda}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right)\right] + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{y}}\right\} \\
&+ \frac{1}{Re_\infty}\frac{\partial}{\partial \hat{z}}\left\{\left[\hat{u}\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}}\right) + \hat{v}\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}}\right) + 2\hat{w}\hat{\mu}\left(\frac{\partial \hat{w}}{\partial \hat{z}}\right) + \hat{w}\hat{\lambda}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right)\right] + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{z}}\right\} \\
&+ \hat{\rho}\left(\hat{u}\hat{g}_x + \hat{v}\hat{g}_y + \hat{w}\hat{g}_z\right) + \hat{\rho}\hat{\dot{q}}\ , \qquad (3.31)
\end{aligned}
$$

with,

$$\hat{E} = \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2}\right).$$

This non dimensionalized equation for energy maintains a similar form as the original equation but introduces one more non dimensional number called the Prandtl number Pr_∞. Putting it all together,

$$\frac{\partial W}{\partial \hat{t}} + \frac{\partial F_C}{\partial \hat{x}} + \frac{\partial G_C}{\partial \hat{y}} + \frac{\partial H_C}{\partial \hat{z}} = \frac{\partial F_D}{\partial \hat{x}} + \frac{\partial G_D}{\partial \hat{y}} + \frac{\partial H_D}{\partial \hat{z}} + S\ .$$

$$W = \begin{bmatrix} \hat{\rho} \\ \hat{\rho}\hat{u} \\ \hat{\rho}\hat{v} \\ \hat{\rho}\hat{w} \\ \hat{\rho}\hat{E} \end{bmatrix}, \quad F_C = \begin{bmatrix} \hat{\rho}\hat{u} \\ \hat{\rho}\hat{u}^2+\hat{p} \\ \hat{\rho}\hat{u}\hat{v} \\ \hat{\rho}\hat{u}\hat{w} \\ \hat{u}\left(\hat{\rho}\hat{E}+\hat{p}\right) \end{bmatrix}, \quad G_C = \begin{bmatrix} \hat{\rho}\hat{v} \\ \hat{\rho}\hat{u}\hat{v} \\ \hat{\rho}\hat{v}^2+\hat{p} \\ \hat{\rho}\hat{v}\hat{w} \\ \hat{v}\left(\hat{\rho}\hat{E}+\hat{p}\right) \end{bmatrix}, \quad H_C = \begin{bmatrix} \hat{\rho}\hat{w} \\ \hat{\rho}\hat{u}\hat{w} \\ \hat{\rho}\hat{v}\hat{w} \\ \hat{\rho}\hat{w}^2+\hat{p} \\ \hat{w}\left(\hat{\rho}\hat{E}+\hat{p}\right) \end{bmatrix},$$

$$F_D = \tfrac{1}{Re_\infty}\begin{bmatrix} 0 \\ \hat{\tau}_{xx} \\ \hat{\tau}_{yx} \\ \hat{\tau}_{zx} \\ \hat{\Theta}_x \end{bmatrix}, \quad G_D = \tfrac{1}{Re_\infty}\begin{bmatrix} 0 \\ \hat{\tau}_{xy} \\ \hat{\tau}_{yy} \\ \hat{\tau}_{zy} \\ \hat{\Theta}_y \end{bmatrix}, \quad H_D = \tfrac{1}{Re_\infty}\begin{bmatrix} 0 \\ \hat{\tau}_{xz} \\ \hat{\tau}_{yz} \\ \hat{\tau}_{zz} \\ \hat{\Theta}_z \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \hat{\rho}\hat{g}_x \\ \hat{\rho}\hat{g}_y \\ \hat{\rho}\hat{g}_z \\ \hat{\rho}\left(\hat{g}\cdot\hat{V}\right) + \hat{\rho}\hat{\dot{q}} \end{bmatrix}.$$

with, $\hat{E} = \left(\frac{\hat{T}}{\gamma-1} + \frac{\hat{u}^2+\hat{v}^2+\hat{w}^2}{2}\right)$, $\quad \hat{p} = \hat{\rho}\hat{R}\hat{T} = \hat{\rho}\hat{T}$, $\quad \hat{\tau}_{ij} = \hat{\mu}\left(\frac{\partial \hat{u}_i}{\partial \hat{j}} + \frac{\partial \hat{u}_j}{\partial \hat{i}}\right) - \delta_{ij}\left(\frac{2}{3}\hat{\mu}\hat{\nabla}\cdot\hat{V}\right)$,
$\hat{\nabla} = \left(\partial/\partial\hat{x}, \partial/\partial\hat{y}, \partial/\partial\hat{z}\right)$, $\quad \hat{V} = (\hat{u}, \hat{v}, \hat{w})$, $\quad \hat{\Theta}_x = \hat{u}\hat{\tau}_{xx} + \hat{v}\hat{\tau}_{xy} + \hat{w}\hat{\tau}_{xz} + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{x}}$,
$\hat{\Theta}_y = \hat{u}\hat{\tau}_{yx} + \hat{v}\hat{\tau}_{yy} + \hat{w}\tau_{yz} + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{y}}$, $\quad \hat{\Theta}_z = \hat{u}\hat{\tau}_{zx} + \hat{v}\hat{\tau}_{zy} + \hat{w}\hat{\tau}_{zz} + \left[\frac{1}{Pr_\infty}\frac{\gamma}{(\gamma-1)}\right]\hat{k}\frac{\partial \hat{T}}{\partial \hat{z}}$.

The system of equations derived above forms the complete set of non dimensionalized Navier Stokes equations for compressible flows with $\hat{\rho}, \hat{u}, \hat{v}, \hat{w}, \hat{p}$ and $\hat{T}$ being the unknowns. It must be noted that the initial and boundary conditions have to be specified in terms of the non dimensional variables using the equations (3.17) given above. Also, the mesh has to scaled according to the chosen length scale L_∞.

In the preceding discussion, only one way of choosing reference quantities for non dimensionalization is presented. However, in practice it may be beneficial to use other reference quantities, for normalizing the flow equations, depending upon the of flow physics.

3.5.2 Euler equations

The Euler equations are ideal fluid flow equations, where the viscous effects are negligible compared to convective effects, thus the diffusion terms tend to zero. These equations can be obtained by setting μ equal to zero in the Navier Stokes equations. Equivalently, we can set the Reynolds number tending to positive infinity in the non dimensional form of Navier Stokes equations. In the generic form (3.2),

$$W = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad F_C = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(\rho E + p) \end{bmatrix}, \quad G_C = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(\rho E + p) \end{bmatrix}, \quad H_C = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(\rho E + p) \end{bmatrix},$$
$$F_D = G_D = H_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \rho g_x \\ \rho g_y \\ \rho g_z \\ \rho(g \cdot V) + \rho\dot{q} \end{bmatrix}. \tag{3.32}$$

Furthermore, if the effects due to gravity, heat sources and chemical reactions are neglected, then the source vector can also be set to zero. In the non dimensional form the Euler equations are same as the dimensional equations given above, with all dimensional variables replaced by the corresponding non dimensional variables.

3.5.3 Stokes flow equations

Stokes flow equations are approximations for very slow moving highly viscous flows. Here, the viscous effects are predominant compared to the convective effects. Hence, the convective terms can be set to zero. The changes in density are insignificant, due to low Mach number, unless it is a multiphase flow. Also, the change in energy is negligible if there is no heat source, which results in the generic form (3.2) with following vectors,

$$W = \begin{bmatrix} 0 \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad F_C = \begin{bmatrix} u \\ p \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G_C = \begin{bmatrix} v \\ 0 \\ p \\ 0 \\ 0 \end{bmatrix}, \quad H_C = \begin{bmatrix} w \\ 0 \\ 0 \\ p \\ 0 \end{bmatrix},$$
$$F_D = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ 0 \end{bmatrix}, \quad G_D = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ 0 \end{bmatrix}, \quad H_D = \begin{bmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \rho g_x \\ \rho g_y \\ \rho g_z \\ \rho\dot{q} \end{bmatrix}.$$

Here, it is necessary to use a preconditioner, P, different from identity, for developing a time evolution algorithm.

TODO

shallow water
Axi symmetric flow
1D area varying nozzle
artificial compressibility
Incompressible with and without heat transfer
multiphase flow
Turbulence
Chemical reaction
Road traffic
Electromagnetism

3.6 Important properties of physical PDEs

TODO

Rotational invariance of system of PDEs
Homogeneity of flux function
Physical bounds on solutions