# 7. EINHEIT - TRANSFORMATION

a) Ansatzfunktionen $N^d(\underline{\xi})$

$$N^1(\underline{\xi}) = (1-\xi_1)(1-\xi_2)$$

$$N^2(\underline{\xi}) = \xi_1(1-\xi_2)$$

$$N^3(\underline{\xi}) = \xi_1 \xi_2$$

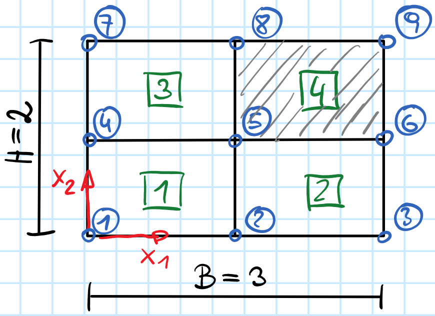
$$N^4(\underline{\xi}) = (1-\xi_1)\xi_2$$

b) Inverse Transformation $(T^4)^{-1}(\underline{\xi})$

Element $K=4$ mit Knotenpunkt = koordinaten:

$$\underline{x}^5 = \underline{\tilde{x}}^1 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \qquad \underline{x}^6 = \underline{\tilde{x}}^2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\underline{x}^9 = \tilde{x}^3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \underline{x}^8 = \underline{\tilde{x}}^4 = \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix}$$




c) Transformation $T^4(\underline{x})$

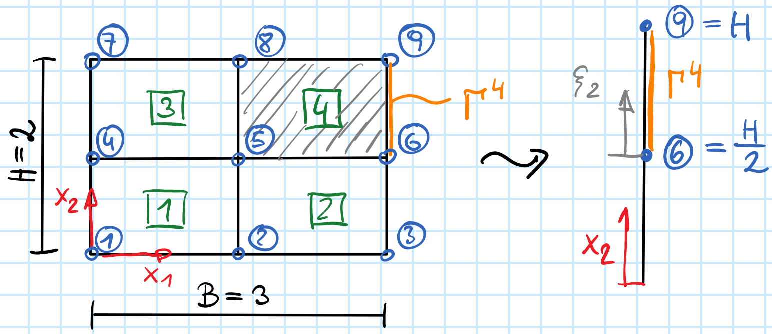
d) Ableitungen $\frac{\partial N^{4,d}(\underline{x})}{\partial x_i}$

1)
$$\frac{\partial N^d(\xi)}{\partial \xi_j} = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^1}{\partial \xi_2} \\ \frac{\partial N^2}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_2} \\ \frac{\partial N^3}{\partial \xi_1} & \frac{\partial N^3}{\partial \xi_2} \\ \frac{\partial N^4}{\partial \xi_1} & \frac{\partial N^4}{\partial \xi_2} \end{bmatrix} =$$

2)
$$\frac{\partial \xi_j}{\partial x_i} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} \end{bmatrix} =$$

$$\frac{\partial N^{K,d}(x)}{\partial x_i} = \begin{array}{c} d\downarrow \end{array} \begin{array}{c} \xrightarrow{j} \\ \begin{bmatrix} \xi_2 - 1 & \xi_1 - 1 \\ 1-\xi_2 & -\xi_1 \\ \xi_2 & \xi_1 \\ -\xi_2 & 1-\xi_1 \end{bmatrix} \end{array} \cdot \begin{array}{c} j\downarrow \end{array} \begin{array}{c} \xrightarrow{i} \\ \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} d\downarrow \end{array} \begin{array}{c} \xrightarrow{j} \\ \begin{bmatrix} \frac{2}{3}(\xi_2-1) & \xi_1 - 1 \\ \frac{2}{3}(1-\xi_2) & -\xi_1 \\ \frac{2}{3}\xi_2 & \xi_1 \\ -\frac{2}{3}\xi_2 & 1-\xi_1 \end{bmatrix} \end{array}$$

e) Transformation d. Elementrandes $\Gamma^4$




f) Lineares Gleichungssystem

aus Aufgabe 8c) $v(\underline{x}) \approx v^h(\underline{x}) = \sum_{I=1}^{N=9} u^I(\underline{x}) \xrightarrow{\text{konformer Ansatz}} u(\underline{x}) \approx u^h(\underline{x}) = \sum_{J=1}^{N=9} \hat{u}_J u^J(\underline{x})$

$$I = 1,\dots,9: \quad \sum_{J=1}^{N=9} \underbrace{\int_{\hat{\Omega}} u^J_{,i}(x)\, u^I_{,i}(x)\, d\hat{\Omega}}_{\text{globale Steifigkeitsmatrix } A^{IJ}}\ \hat{u}_J = \underbrace{\int_{\hat{\Omega}} f(x)\, u^I(x)\, d\hat{\Omega} + \int_{\partial\hat{\Omega}_N} \bar{g}(x)\, u^I(x)\, d\Gamma}_{\text{globaler Kraftvektor } F^I}$$

- Zusammenhang globaler / lokaler Ansatzfunktionen:

- Lineares Gleichungssystem:

g) Eintrag Elementsteifigkeitsmatrix

$$A^{k,\alpha;k,e} = \int_{\Omega^k} N_{,i}^{k,\alpha}(\underline{x})\, N_{,i}^{k,e}(\underline{x})\, d\Omega^k$$

h) Elementkraftvektor

$$F^{k,\alpha} = \int_{\Omega^k} f(\underline{x})\, N^{k,\alpha}(\underline{x})\, d\Omega^k + \int_{\Gamma^k} \bar{g}(\underline{x})\, N^{k,\alpha}(\underline{x})\, d\Gamma^k$$

## i) Steifigkeitsmatrix und Assemblierung

- Steifigkeitsmatrix $C^{IK,d}$

$$C^{IK,d} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

- Globale Freiheitsgrade $u^I(x) = C^{IK,d} N^{K,d}(x)$

$$\begin{bmatrix} u^1(x) \\ u^2(x) \\ u^3(x) \\ u^4(x) \\ u^5(x) \\ u^6(x) \\ u^7(x) \\ u^8(x) \\ u^9(x) \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix} N^{1,1}(x) \\ N^{1,2}(x) \\ N^{1,3}(x) \\ N^{1,4}(x) \\ N^{2,1}(x) \\ N^{2,2}(x) \\ N^{2,3}(x) \\ N^{2,4}(x) \\ N^{3,1}(x) \\ N^{3,2}(x) \\ N^{3,3}(x) \\ N^{3,4}(x) \\ N^{4,1}(x) \\ N^{4,2}(x) \\ N^{4,3}(x) \\ N^{4,4}(x) \end{bmatrix}$$

- Totale Steifigkeitsmatrix $A_{tot}^{K,d;L,e}$

$$A_{\text{tot}}^{K,d;L,e} = \begin{bmatrix}
\frac{13}{18} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{36} & \frac{13}{18} & -\frac{7}{18} & -\frac{13}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{13}{36} & -\frac{7}{18} & \frac{13}{18} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{7}{18} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{13}{18} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{36} & \frac{13}{18} & -\frac{7}{18} & -\frac{13}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{18} & \frac{13}{18} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{7}{18} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{18} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{13}{18} & -\frac{7}{18} & -\frac{13}{36} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{18} & \frac{13}{18} & \frac{1}{36} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{18} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{18} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{13}{18} & -\frac{7}{18} & -\frac{13}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{18} & \frac{13}{18} & \frac{1}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{18} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{18}
\end{bmatrix}$$

- Totalen Kraftvektor $F^{K,d}_{tot}$

$$F^{K,d}_{\text{tot}} = \begin{bmatrix} \frac{13}{64} \\ \frac{31}{64} \\ \frac{39}{64} \\ \frac{21}{64} \\ 1.27604166666667 \\ 2.953125 \\ 3.578125 \\ 1.06770833333333 \\ \frac{53}{64} \\ \frac{71}{64} \\ \frac{95}{64} \\ \frac{77}{64} \\ 0.901041666666667 \\ 5.078125 \\ 5.953125 \\ 0.942708333333333 \end{bmatrix}$$

- Assemblierte Steifigkeitsmatrix $A^{IJ} = C^{IK,d} A^{K,d;K,e} C^{JK,e} \Leftrightarrow \underline{\underline{A}} = \underline{\underline{C}} \cdot \underline{\underline{A}}_{tot} \cdot \underline{\underline{C}}^T$

$$A^{IJ} = \begin{bmatrix} \frac{13}{18} & \frac{1}{36} & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{13}{9} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{13}{18} & 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & 0 & 0 \\ -\frac{7}{18} & -\frac{13}{36} & 0 & \frac{13}{9} & \frac{1}{18} & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 \\ -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & \frac{1}{18} & \frac{26}{9} & \frac{1}{18} & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} \\ 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & \frac{1}{18} & \frac{13}{9} & 0 & -\frac{13}{36} & -\frac{7}{18} \\ 0 & 0 & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 & \frac{13}{18} & \frac{1}{36} & 0 \\ 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{9} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & \frac{1}{36} & \frac{13}{18} \end{bmatrix}$$

- Assemblierten Kraftvektor $F^I = C^{IK,d} F^{K,d} \Leftrightarrow \underline{F} = \underline{\underline{C}} \cdot \underline{F}_{tot}$

$$F^{I} = \begin{bmatrix} \frac{13}{64} \\ 1.76041666666667 \\ 2.953125 \\ \frac{37}{32} \\ 3.6875 \\ 8.65625 \\ \frac{77}{64} \\ 2.42708333333333 \\ 5.953125 \end{bmatrix}$$

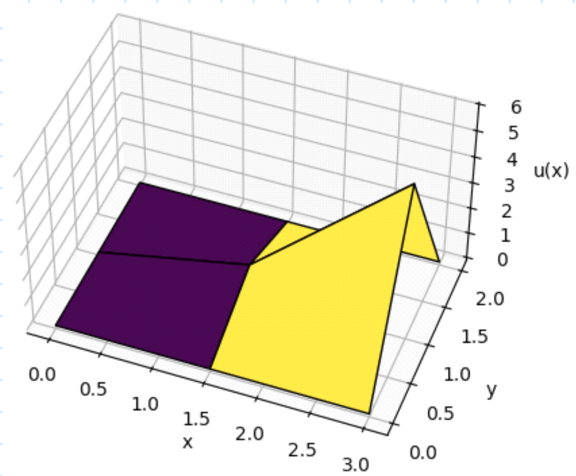
• Lineares Gleichungssystem $A^{IJ}\hat{u}_J = F^I \Leftrightarrow \underline{\underline{A}} \cdot \underline{\hat{u}} = \underline{F}$

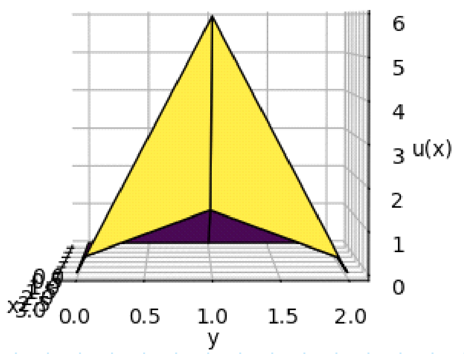
$$\begin{bmatrix} \frac{13}{18} & \frac{1}{36} & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{13}{9} & \frac{1}{36} & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{13}{18} & 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & 0 & 0 \\ -\frac{7}{18} & -\frac{13}{36} & 0 & \frac{13}{9} & \frac{1}{18} & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 \\ -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & \frac{1}{18} & \frac{26}{9} & \frac{1}{18} & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} \\ 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & \frac{1}{18} & \frac{13}{9} & 0 & -\frac{13}{36} & -\frac{7}{18} \\ 0 & 0 & 0 & -\frac{7}{18} & -\frac{13}{36} & 0 & \frac{13}{18} & \frac{1}{36} & 0 \\ 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & \frac{1}{36} & \frac{13}{9} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & \frac{1}{36} & \frac{13}{18} \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \end{bmatrix} = \begin{bmatrix} \frac{13}{64} \\ 1.7604166666667 \\ 2.953125 \\ \frac{37}{32} \\ 3.6875 \\ 8.65625 \\ \frac{77}{64} \\ 2.4270833333333 \\ 5.953125 \end{bmatrix}$$

j) Dirichlet-Randbedingungen + Lösung

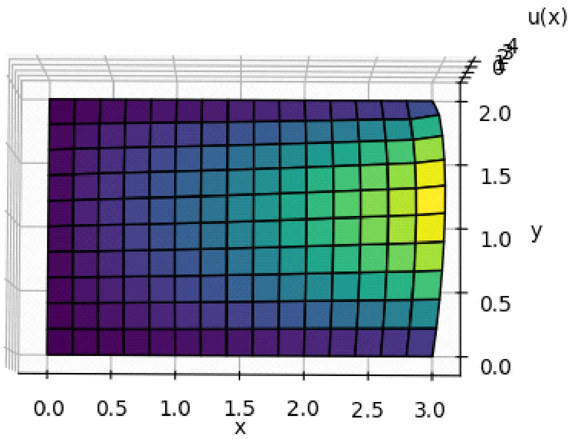
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} & \frac{1}{18} & \frac{26}{9} & \frac{1}{18} & -\frac{13}{36} & -\frac{7}{9} & -\frac{13}{36} \\ 0 & -\frac{13}{36} & -\frac{7}{18} & 0 & \frac{1}{18} & \frac{13}{9} & 0 & -\frac{13}{36} & -\frac{7}{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.6875 \\ 8.65625 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

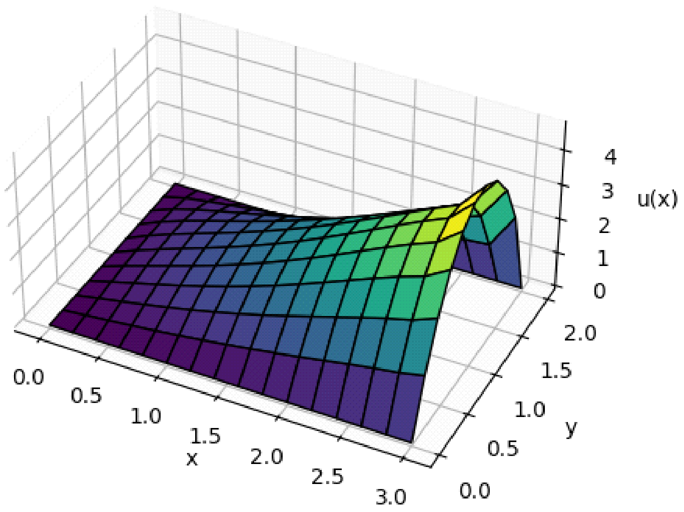
• Lösung: (2x2 Elemente wie in der Angabe)

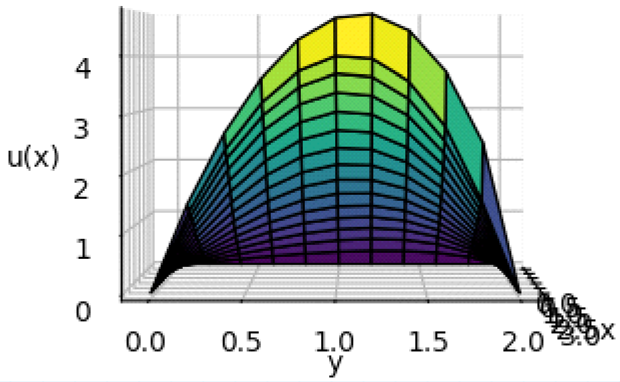


• Lösung (15x10 Elemente)



$$\begin{bmatrix} \frac{3}{2} - \frac{3}{2}\xi_2 - \frac{3}{2}\xi_1 + \cancel{\frac{3}{2}\xi_1\xi_2} + 3\xi_1 - \cancel{3\xi_1\xi_2} + \cancel{3\xi_1\xi_2} + \frac{3}{2}\xi_2 - \cancel{\frac{3}{2}\xi_1\xi_2} \\ 1 - \xi_2 - \cancel{\xi_1} + \cancel{\xi_1\xi_2} + \cancel{\xi_1} - \cancel{\xi_1\xi_2} + \cancel{2\xi_1\xi_2} + 2\xi_2 - \cancel{2\xi_1\xi_2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{3}{2}\xi_1 \\ 1 + \xi_2 \end{bmatrix}$$

$$= \int_{\xi_1=0}^{1} \int_{\xi_2=0}^{1} \left[ \frac{4}{9}\,\frac{3}{2}\left(\xi_2^2 - 2\xi_2 + 1\right) + \frac{3}{2}\left(\xi_1^2 - 2\xi_1 + 1\right) \right] d\xi_2\, d\xi_1$$

$$= \int_{\xi_1=0}^{1} \left[ \frac{2}{3}\left(\frac{\xi_2^3}{3} - \xi_2^2 + \xi_2\right) + \frac{3}{2}\left(\xi_1^2\xi_2 - 2\xi_1\xi_2 + \xi_2\right) \right]_{\xi_2=0}^{1} d\xi_1$$

$$= \int_{\xi_1=0}^{1} \left[ \frac{2}{9} + \frac{3}{2}\left(\xi_1^2 - 2\xi_1 + 1\right) \right] d\xi_1 = \left[ \frac{2}{9}\xi_1 + \frac{\xi_1^3}{2} - \frac{3}{2}\xi_1^2 + \frac{3}{2}\xi_1 \right]_{\xi_1=0}^{1} = \frac{13}{18}$$