R&D Networks under Heterogeneous Firm Productivities

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Abstract

We introduce heterogeneous R&D productivities into an endogenous R&D network formation model, generalizing the framework in Goyal and Moraga-Gonzalez (2001). Heterogeneous productivities endogenously create asymmetric gains for connecting firms: the less productive firm benefits disproportionately, while the more productive firm exerts greater R&D effort and incurs higher costs. For sufficiently large productivity gaps between two firms, the more productive firm experiences reduced profits from being connected to the less productive one. This overturns the benchmark results on pairwise stable networks: for sufficiently large productivity gaps, the complete network becomes unstable, whereas the Positive Assortative (PA) network—where firms cluster by productivity levels—emerges as stable. Simulations show that the PA structure delivers higher welfare than the complete network; nevertheless, welfare under PA formation follows an inverted U-shape in the fraction of high-productivity firms, reflecting crowding-out effects at high fractions. Altogether, a counterintuitive finding emerges: economies with higher average R&D productivity may exhibit lower welfare through (i) the formation of alternative stable R&D network structures or (ii) a crowding-out effect of high-productivity firms. Our findings highlight that productivity-enhancing policies should account for their impact on endogenous R&D alliances and effort, as such endogenous responses may offset or even reverse the intended welfare gains.

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1 Introduction

Innovation is widely acknowledged as a fundamental driver of economic growth in modern societies (Romer, 1990; Jones, 1995). A key engine of the innovation process is investment by private firms in R&D (Grossman and Helpman, 1991; Aghion and Howitt, 1992), a key component of which is collaborative R&D alliances (e.g., as empirically shown in Cassiman and Veugelers, 2002; Belderbos et al., 2004; Calero et al., 2007).

In this paper, we study collaborative R&D alliances and their implications for firms' profits and aggregate welfare in an R&D network formation model, where firms have heterogeneous R&D productivities. A growing theoretical literature, including Goyal and Moraga-Gonzalez (2001); Goyal and Joshi (2003); König et al. (2019), has studied R&D networks where firms improve their R&D efficiencies by forming collaborations while still competing with each other in the product market. However, these models typically assume that firms are homogeneous in their ability to transform R&D effort into cost-reducing innovations. While analytically convenient, this assumption is at odds with theoretical models and empirical evidence in the networks literature showing that capabilities vary substantially across firms (Magerman et al., 2016; Chen et al., 2023; Milan, 2024; Acemoglu and Tahbaz-Salehi, 2025).[1]

This paper contributes to three strands of the literature on R&D networks. First, it extends models of endogenous R&D network formation by incorporating firm-level heterogeneity in R&D productivities. Second, it shows how such heterogeneity leads to endogenous asymmetric benefits for connecting firms, which alter firms' incentives to collaborate and consequently generate a richer set of equilibrium network structures compared to the homogeneous benchmark. Third, it highlights a novel crowding-out effect: while increasing the fraction of high-productive firms in the economy initially raises welfare, beyond a certain threshold a larger fraction of high-productive firms reduces individual

[1] Further empirical studies show that the return on R&D is driven by unobserved heterogeneity in innovation efficiency (Lentz and Mortensen, 2008) and organizational competence (Henderson and Cockburn, 1994), rather than spending intensity alone. Indeed, the output elasticity of R&D is highly firm-specific (Knott, 2008) and depends on the interplay between a firm's size and innovation strategy (Akcigit and Kerr, 2018).

R&D efforts in stable network structures, dampening the overall welfare gains from R&D activities.

Our model generalizes the framework introduced in Goyal and Moraga-Gonzalez (2001) by allowing for heterogeneous firm capabilities to transform R&D efforts into cost reductions. In particular, each firm exerts costly R&D effort[2] that lowers the marginal cost of production, and firms form bilateral R&D collaborations[3] with each other to access the benefits of each other's R&D efforts, while still competing à la Cournot in a homogeneous product market. The introduced heterogeneity in our framework implies that, when firms form bilateral R&D collaborations, the marginal benefit a firm derives from its partner's effort depends on the partner's productivity. That is, when a firm with relatively higher productivity increases its effort, the resulting cost reduction for its collaborators is substantial. In contrast, a lower-productivity firm contributes less to its partner's cost reduction, even if it exerts the same level of R&D effort. Such heterogeneity in firm-level R&D productivities introduces asymmetries in equilibrium R&D efforts and the firm-level gains from R&D collaborations, thus unlocking new results on pairwise stable R&D networks and new transmission channels of these efforts into firm profits and social welfare.

Under heterogeneity, there are two distinct effects of linking with another firm: (i) the productivity effect, and (ii) the connectivity effect — firm-level outcomes depend not only on productivity differences but also on how these firms are linked to other firms in the R&D network. To isolate the effect of heterogeneous productivities from the R&D network effect, we first derive comparative statics on equilibrium efforts and profits for two firms with distinct productivities that occupy symmetric network positions, meaning they face the same network externalities. We show that when these firms are connected, the low-productivity firm experiences a relatively larger increase in profit than the high-productivity firm while exerting a lower R&D effort and therefore bearing

[2] Subject to decreasing marginal returns, in line with the evidence in Bloom et al. (2020).

[3] We assume that pairwise link formation is costless, as in Goyal and Moraga-Gonzalez (2001) and Zirulia (2012). Other work that considers linking costs includes Goyal and Joshi (2003); Galeotti et al. (2006); Westbrock (2010); Hsieh et al. (2024).

a lower R&D cost. Consequently, among two firms with symmetric network positions, whenever the high-productivity firm benefits from forming the link, the low-productivity firm also benefits, showing an ordering of asymmetric gains from linking in favour of the low-productivity firm. Furthermore, our simulation-based findings show that a low-productivity firm always benefits from connecting to a high-productivity firm—while the reverse does not necessarily hold—implying positive/negative benefits from such link formation for low-/high-productivity firms, showing that asymmetric returns go beyond level effects and create differences in link formation incentives. To study the equilibrium network configurations that arise from such asymmetries, we characterize these structures under the widely used equilibrium concept of pairwise stability introduced by Jackson and Wolinsky (1996), as also used in closely related R&D network models, e.g., Goyal and Moraga-Gonzalez (2001); Goyal and Joshi (2003); Zirulia (2012); Billand et al. (2019).

Our next result shows that the complete network—which is pairwise stable in the homogeneous framework—ceases to be stable once the productivity gap between firms becomes sufficiently large. Extending stability results to network structures beyond the complete network is known to be intractable due to network externalities, even under the homogeneous benchmark as discussed in Goyal and Moraga-Gonzalez (2001), and this intractability is further compounded under heterogeneity. We therefore adopt a simulation-based approach to extract insights on stable R&D networks. Our first key finding in this part shows that the benefit of linking with another firm is monotonically increasing in that other firm's R&D productivity, and that there exists a productivity threshold for a given firm such that it does not form bilateral links with another firm that has a productivity level below that threshold.

In the rest of the analysis, for simplicity of exposition, we focus on the simplest form of heterogeneity by restricting our setting to two firm types: high- and low-productivity firms, having productivity levels equal to 1 (normalized) and $\theta < 1$, respectively. By using simulations, we then characterize the full set of pairwise stable networks for a given number of firms and type distribution, showing that, under heterogeneity, the set of stable configurations extends beyond the complete network. In particular, while the complete network continues to be stable when the productivity gap between high- and low-type firms is sufficiently small, if this gap is sufficiently large, the class of Positive Assortative (PA) networks becomes stable. These networks are *clustered* by productivity type, such that firms form bilateral links with all firms of the same type only. Specifically, we establish the existence of a lower threshold productivity θ, denoted by $\underline{\theta}$, above which the complete network is stable, and an upper threshold, $\overline{\theta}$, below which the PA network is stable. This implies that in cases where $\underline{\theta} < \overline{\theta}$, which holds when the fraction of high-productivity firms is not extremely large, an intermediate range for θ exists for which both networks are pairwise stable. Furthermore, the PA/complete network is the unique stable structure for sufficiently large/small productivity gaps.

Having fully characterized the set of pairwise stable networks, the remainder of the paper shifts attention to a broader implication of heterogeneity beyond firm-level outcomes—its impact on aggregate welfare, measured by the sum of consumer and producer surplus. Simulation-based analysis uncovers a counterintuitive finding: a higher average R&D productivity in the economy —through a higher productivity of low-type firms (θ) or a higher fraction of high-type firms (ρ)— may exhibit lower welfare due to (i) the formation of alternative stable R&D network structures and/or (ii) a crowding-out effect of high-productivity firms. First, we find that the PA structure delivers higher welfare compared to the complete network. Consequently, comparing two economies that differ in their productivity gaps, the economy with a smaller gap (where the complete network is stable) exhibits lower welfare than the economy with a larger gap (where the PA network is stable). Second, we find that welfare under PA formation follows an inverted U-shape in the fraction of high-productivity firms: welfare increases with this fraction at low levels, peaks at an intermediate fraction, then declines at high fractions, highlighting a crowding-out effect. Thus, comparing two economies where the PA structure is stable (both having sufficiently large productivity gaps), the economy with the higher fraction of high-productivity firms exhibits lower welfare.

These counterintuitive findings contradict the conventional wisdom in economic models that increasing productivity in an economy is welfare-enhancing. When collaborations between firms and the resulting interactions in the overall economy are taken into account, higher average R&D productivity need not be welfare-enhancing if it alters stable network structures and their associated network effects. A relevant policy implication is that innovation and productivity-enhancing policies should account for their impact on endogenous R&D collaborations and effort, as endogenous responses in the network structure may offset or even reverse the intended welfare gains. Moreover, an avenue for future research is to empirically analyze these findings from a cross-country perspective or by examining changes in innovation productivities and collaborations between firms over time within a given region. Such analysis can offer an alternative explanation for the productivity puzzle observed in developed economies recently—while firms may become more productive in innovation at the individual level, this does not necessarily translate into higher aggregate welfare (or observed overall productivity) due to altered incentives in the innovation landscape.

Our approach differs from existing contributions by allowing for endogenous asymmetric benefits to link formation arising from productivity differentials. For example, Zirulia (2012) introduces pre-defined partner-specific spillovers, highlighting how firms with unique technological capabilities can become central hubs in their markets. In a similar direction, Billand et al. (2019) introduces connection-specific benefits from linking, where link-specific benefits are the same for a linked pair but can differ across the type of pairing. This existing work exhibits distinct mechanisms and results from ours. More specifically, in these models, the marginal cost reduction of pairwise linking is the same for two connecting firms, regardless of any underlying heterogeneity between them. In contrast to these models, our framework allows for such marginal cost reductions to be endogenously asymmetric, generating different network formation incentives and pairwise stable structures with distinct welfare implications. Additionally, while some other models explore the dynamic formation of the R&D network and evolution of technology diffusion (such as Bischi and Lamantia, 2012; Hsieh et al., 2024), our model focuses on a static environment that allows us to fully characterize the implications of heterogeneity for link formation, effort allocation, and aggregate welfare.

The remainder of the paper is structured as follows. Section 2 outlines the theoretical framework and the multi-stage game setup, providing our main theoretical findings. By simplifying to a two-type productivity setup, Section 3 provides a full characterization of pairwise stable networks for $n = 4$ firms, and extends insights on firm-level outcomes, pairwise stability and welfare through simulations. Section 4 provides our closing remarks and discusses how our model can have implications for industrial policy.

2 Model

There exists a set of firms $\mathcal{N} = \{1, \ldots, n\}$ competing in a single-product oligopolistic market and we employ a complete information multi-stage game setup similar to Goyal and Moraga-Gonzalez (2001). The timing of actions is as follows:

1. **Network Formation.** Firms engage in *bilateral* R&D alliances with one another, represented as an undirected link between two collaborators. Following Jackson and Wolinsky (1996), we use the standard notion of pairwise stability as the equilibrium concept. Letting $\mathcal{G}^n$ denote the set of all possible binary $n \times n$ adjacency matrices, an R&D network is represented by an zero-diagonal, symmetric adjacency matrix $\mathbf{G} \in \mathcal{G}^n$ in which G_{ij} equals 1 if there is a collaboration between firms i and j, and 0 otherwise.

2. **R&D Investment.** Given the network $\mathbf{G}$, firms choose their level of R&D effort $(\tilde{e}_i)$ with a quadratic cost $\mathrm{C}(\tilde{e}_i) = \phi_i \cdot \tilde{e}_i^2$, where $\phi_i > 0$ is the firm specific steepness of the cost function. Then, firms share their R&D efforts with their collaborators in the network. As a result, each firm's marginal cost of production (c_i) is:

$$c_i\left(\tilde{\mathbf{e}}, \mathbf{G}\right) = \bar{c} - \tilde{e}_i - \sum_{j \in \mathcal{N}_i(\mathbf{G})} \tilde{e}_j, \qquad (1)$$

where $\bar{c}$ is the baseline (pre-R&D) marginal cost common to all firms, and $\tilde{\mathbf{e}} := \{\tilde{e}_j\}_{j=1}^n$. We define $\mathcal{N}_i(\mathbf{G}) := \{j \mid G_{ij} = 1\}$ as the set of firm i's collaborators given the network $\mathbf{G}$.

3. **Product Market Competition.** Finally, firms compete in an oligopoly à la Cournot, where they produce a single homogeneous product, while facing different production costs under a linear demand curve characterized by $p(\mathbf{q}) = \alpha - \sum_{i=1}^n q_i$, where p is the price of the single product in the market, α represents the maximum price consumers are willing to pay, $q_i \in \mathbb{R}_+$ is firm i's level of output, and we collect $\mathbf{q} := \{q_j\}_{j=1}^n$.

In this three-stage game, firms maximize their profits π_i by sequentially choosing collaboration links, then R&D effort $(\tilde{e}_i)$, and then quantity (q_i), where:

$$\pi_i(\mathbf{q}, \tilde{\mathbf{e}}, \mathbf{G}) = q_i \cdot [p(\mathbf{q}) - c_i(\tilde{\mathbf{e}}, \mathbf{G})] - \phi_i \cdot \tilde{e}_i^2. \tag{2}$$

We depart from the literature by adding heterogeneity in R&D productivities into the baseline model described above as introduced by Goyal and Moraga-Gonzalez (2001). Specifically, while earlier studies assume that $\phi_i = \phi$ for all firms, in our framework ϕ_i captures the inverse of firm i's R&D productivity. A higher value of ϕ_i implies the firm must incur a greater R&D cost to achieve a given efficiency (or technology) level as measured by its marginal production cost c_i.

For ease of interpretation, we normalize R&D costs relative to the most R&D-efficient firm in the market. Let ϕ denote the cost coefficient of this most productive firm, defined as $\phi := \min\{\phi_1, \ldots, \phi_n\}$. We then define the *relative R&D productivity* of firm i with respect to the most productive firm as $\theta_i := \sqrt{\phi/\phi_i}$ and a rescaled *R&D effort* variable as $e_i := \tilde{e}_i/\theta_i$. Given this normalization, $\theta_i \in (0, 1]$ and the most productive firm(s) having $\max_{i\in\mathcal{N}} \theta_i = 1$.

These definitions enable us to rewrite the marginal cost function (1) and firm profits (2):

$$\begin{aligned} c_i(\mathbf{e}, \mathbf{G}) &= \bar{c} - \theta_i\, e_i - \sum_{j \in \mathcal{N}_i(\mathbf{G})} \theta_j\, e_j, \\ \pi_i(\mathbf{q}, \mathbf{e}, \mathbf{G}) &= q_i \cdot [p(\mathbf{q}) - c_i(\mathbf{e}, \mathbf{G})] - \phi \cdot e_i^2, \end{aligned} \tag{3}$$

where $\mathbf{e} := \{e_j\}_{j=1}^n$ is the vector of R&D efforts. Henceforth, we refer to e_i as firm i's R&D effort and to θ_i as its R&D productivity. A higher θ_i corresponds to greater cost reductions, equivalent to a lower ϕ_i coefficient.

2.1 Equilibrium Characterization

We solve for the equilibrium production level, R&D efforts, and collaboration links by performing backwards induction. First, we solve for the firm's maximization problem and obtain equilibrium production levels and R&D efforts, corresponding to the last two stages of the game. To this end, we provide Proposition 1 that guarantees the existence of a solution with strictly positive efforts and outputs.

These last two stages can be interpreted as the solution to a game with an exogenously given R&D network (a similar structure is analyzed in König et al., 2019, under the assumptions of homogeneous productivity and multiple markets). Finally, we solve for the equilibrium network structure invoking the concept of pairwise stability (Jackson and Wolinsky, 1996), characterizing the endogenous formation of R&D alliances for heterogeneous firms.

2.1.1 Market Competition

In the final (third) stage of the game, firms choose their production quantities conditional on their realized marginal costs c_i, taking the market demand function as given. The equilibrium production quantities and corresponding profits are given by[4]

$$
\begin{aligned}
q_i^*(\mathbf{e},\mathbf{G}) &= \frac{1}{n+1}\left[\alpha - n\cdot c_i(\mathbf{e},\mathbf{G}) + \sum_{k\neq i} c_k(\mathbf{e},\mathbf{G})\right],\\
\pi_i^*(\mathbf{e},\mathbf{G}) &= q_i^*(\mathbf{e},\mathbf{G})^2 - \phi\cdot e_i^2.
\end{aligned} \tag{4}
$$

Let $d_i(\mathbf{G}) \coloneqq |\mathcal{N}_i(\mathbf{G})| = \sum_{j\in\mathcal{N}} G_{ij}$ denote the degree of firm i in network $\mathbf{G}$. Substituting equation (3) into (4), the equilibrium output level of firm i can be rewritten as a function of the R&D collaboration network and effort profile:

$$
q_i^*(\mathbf{e},\mathbf{G}) = \underbrace{\frac{\alpha-\bar{c}}{n+1}}_{\text{baseline}} + \underbrace{\frac{n-d_i(\mathbf{G})}{n+1}\,\theta_i e_i}_{\text{own effort effect}} + \underbrace{\sum_{j\in\mathcal{N}_i(\mathbf{G})}\frac{n-d_j(\mathbf{G})}{n+1}\,\theta_j e_j}_{\text{neighbors' effort effect}} - \underbrace{\sum_{k\in\mathcal{N}_{-i}(\mathbf{G})}\frac{1+d_k(\mathbf{G})}{n+1}\,\theta_k e_k}_{\text{non-neighbors' effort effect}}, \tag{5}
$$

where $\mathcal{N}_{-i}(\mathbf{G}) \coloneqq \{j\in\mathcal{N} \mid G_{ij}=0\}$ denotes the set of firms that are not neighbors of firm i in the network $\mathbf{G}$. Equation (5) shows that a firm's optimal production level increases with its own R&D effort and decreases with the R&D efforts of its non-neighbors. The effect of R&D efforts by firm i's neighbors on its production is positive given that the degree of each firm $j\in\mathcal{N}$ always satisfies $0\leq d_j(\mathbf{G})\leq n-1<n$.

2.1.2 Equilibrium R&D Efforts

In the second stage, we solve for each firm's optimal R&D effort e_i, conditional on the collaboration network $\mathbf{G}$. Using (4), the first order condition (FOC) for profit maximization with respect to e_i is:

$$
\left.\frac{\partial\pi_i^*(\mathbf{e},\mathbf{G})}{\partial e_i}\right|_{e_i^*} = \frac{2q_i^*(\mathbf{e},\mathbf{G})}{n+1}\left(-n\left.\frac{\partial c_i(\mathbf{e},\mathbf{G})}{\partial e_i}\right|_{e_i{}^*} + \sum_{k\neq i}\left.\frac{\partial c_k(\mathbf{e},\mathbf{G})}{\partial e_i}\right|_{e_i{}^*}\right) - 2\phi\, e_i^* = 0,
$$

where $e_i^*(\mathbf{e}_{-i},\mathbf{G}) \coloneqq \arg\max_{e_i}\{\pi_i^*\}$ is firm i's best-response effort level, and $\mathbf{e}_{-i} \coloneqq \{e_j\}_{j\neq i}$ is the effort of all other firms. Let $\eta_i(\mathbf{G}) \coloneqq [n-d_i(\mathbf{G})]/(n+1)$ represent a strictly positive *sparsity* coefficient that decreases with the firm's degree $d_i(\mathbf{G})$. From equation (3), we know that $\partial c_i(\mathbf{e},\mathbf{G})/\partial e_i = -\theta_i$ and $\partial c_i(\mathbf{e},\mathbf{G})/\partial e_j = -\theta_j\, G_{ij}$, such that we can rewrite

[4] The derivations are provided in the Supplementary Appendix.

the FOC as:

$$e_i^*\left(\mathbf{G}, \mathbf{e}_{-i}\right)=\underbrace{\frac{\theta_i\;\eta_i(\mathbf{G})}{\phi-\theta_i^2\;\eta_i^2(\mathbf{G})}}_{\text{(i) sparsity}}\cdot\Bigg[\underbrace{\frac{\alpha-\bar{c}}{n+1}}_{\text{(ii) baseline}}+\underbrace{\sum_{j\in\mathcal{N}_i(\mathbf{G})}\eta_j(\mathbf{G})\;\theta_j\;e_j}_{\text{(iii) neighbor effect}}-\underbrace{\sum_{k\in\mathcal{N}_{-i}(\mathbf{G})}\left[1-\eta_k(\mathbf{G})\right]\;\theta_k\;e_k}_{\text{(iv) non-neighbor effect}}\Bigg], \tag{6}$$

The best-response effort of firms can be decomposed into four key factors. The first is the proportionality constant that is increasing in the firm's productivity θ_i, and its sparsity coefficient $\eta_i(\mathbf{G})$, and therefore, decreasing in the degree $d_i(\mathbf{G})$. The second is a baseline effort level that depends on market parameters α, $\bar{c}$, and n. The structures of the neighbor effect (iii) and non-neighbor effects (iv) mirror that in equation (5), indicating that a firm's optimal R&D effort is increasing in neighbors' efforts and productivities, and decreasing in neighbors' degrees as well as in non-neighbors' efforts, productivities and degrees.

Letting $\mathbf{1}_n$ denote an $n \times 1$ vector of ones, the FOCs in (6) yields a system of n linear equations in n unknowns:

$$\mathbf{A}(\mathbf{G})\;\mathbf{e}=(\alpha-\bar{c})\;\mathbf{1}_n, \quad \text{where} \quad A_{ij}(\mathbf{G})=\begin{cases}\frac{(n+1)^2\;\phi}{\theta_i\;[n-d_i(\mathbf{G})]}-\theta_i\;[n-d_i(\mathbf{G})] & \text{for } i=j,\\ [1+d_j(\mathbf{G})]\;\theta_j-(n+1)\;G_{ij}\;\theta_j & \text{for } j\neq i.\end{cases} \tag{7}$$

Proposition 1. *There exists a unique equilibrium effort profile* $\mathbf{e}^*(\mathbf{G})=[e_1^*(\mathbf{G}),\cdots,e_n^*(\mathbf{G})]^\top$ *with* $e_i^*(\mathbf{G})>0$ *for all* $i\in\mathcal{N}$ *that solves equation* (7) *if the cost parameter* ϕ *satisfies:*

$$\phi>\underline{\phi}:=\frac{n^3-n^2-n+2}{n^2+2n+1}. \tag{8}$$

Proposition 1 establishes that a unique and interior equilibrium exists for any network $\mathbf{G}\in\mathcal{G}^n$, provided the cost coefficient ϕ exceeds a threshold $\underline{\phi}$ that grows linearly with n. The proof of Proposition 1 relies on the Gershgorin Disk Theorem (Gershgorin, 1931). Specifically, we show that the matrix $\mathbf{A}$ is strictly diagonally dominant, which implies that it is non-singular and hence the FOC system admits a unique solution.

While existence and uniqueness of an equilibrium effort profile are guaranteed by Proposition 1, the equilibrium effort levels cannot be generally expressed in closed form, as they involve an intractable inverse of the non-symmetric matrix $\mathbf{A}(\mathbf{G})$. Nevertheless, we can derive equilibrium profits using equilibrium efforts as:

$$\pi_i^*(\mathbf{G}) = \left[\frac{\phi}{\theta_i^2\ \eta_i^2(\mathbf{G})} - 1\right] \phi\, e_i^{*2}(\mathbf{G})\,. \tag{9}$$

To uncover the black box of equilibrium efforts and profits under heterogeneous productivities, we characterize relative equilibrium efforts and profits of any two firms i and j that are symmetric with respect to their position in the network. This is formalized in the following definition:

Definition 1 (Symmetric position)**.** Firms $i, j \in \mathcal{N}$ have a *symmetric position* in a given network $\mathbf{G}$ if $\forall k \in \mathcal{N} \neq i, j$, $G_{ik} = G_{jk}$.

For two firms i, j that have a symmetric position, they hold identical links in the network $\mathbf{G}$. This directly implies their degrees are also the same, $d_i(\mathbf{G}) = d_j(\mathbf{G})$, and so are their sparsity coefficients $\eta_i(\mathbf{G}) = [n - d_i(\mathbf{G})]/(n+1) = [n - d_j(\mathbf{G})]/(n+1) = \eta_j(\mathbf{G})$.

Proposition 2. *Let firms $i, j \in \mathcal{N}$ have a symmetric position in a given network $\mathbf{G}$. Then, the ratio of their equilibrium efforts and profits are:*

$$\begin{aligned}
\frac{e_i^*(\mathbf{G})}{e_j^*(\mathbf{G})} &= \frac{\theta_i}{\theta_j}\frac{\phi - \theta_j^2\, \eta_j(\mathbf{G})\,(1 - G_{ij})}{\phi - \theta_i^2\, \eta_i(\mathbf{G})\,(1 - G_{ji})}, \\
\frac{\pi_i^*(\mathbf{G})}{\pi_j^*(\mathbf{G})} &= \frac{\phi - \theta_i^2\, \eta_i^2(\mathbf{G})}{\phi - \theta_j^2\, \eta_j^2(\mathbf{G})}\left[\frac{\phi - \theta_j^2\, \eta_j(\mathbf{G})\,(1 - G_{ij})}{\phi - \theta_i^2\, \eta_i(\mathbf{G})\,(1 - G_{ji})}\right]^2.
\end{aligned} \tag{10}$$

Proposition 2 provides an expression for the ratio of optimal efforts and profits for symmetrically positioned firms i and j. It shows that if $\theta_i > \theta_j$, then $e_i^*(\mathbf{G}) > e_j^*(\mathbf{G})$ always holds, regardless of the connection between i and j. That is, if firm i is more productive than firm j while having the same network position, then firm i commits higher R&D effort than j. In particular, let $\mathbf{G}_{+ij}$ denote the original network $\mathbf{G}$ where the link between firms i and j is present (i.e., $G_{ij} = G_{ji} = 1$), with all other links remaining unchanged; and let $\mathbf{G}_{-ij}$ denote the network where this link is not present (i.e., $G_{ij} = G_{ji} = 0$).

Then, the relative effort ratio given in Proposition 2 implies:

$$1 < \frac{\theta_i}{\theta_j} = \frac{e_i^*(\mathbf{G}_{+ij})}{e_j^*(\mathbf{G}_{+ij})} < \frac{e_i^*(\mathbf{G}_{-ij})}{e_j^*(\mathbf{G}_{-ij})},$$

meaning that when the link between i and j is removed, the relative effort of the higher productive firm i compared to the lower productive firm j rises.

The ranking of equilibrium profits based on their productivities, however, does depend on whether these two firms form a collaboration link or not. To better understand the incentives for forming a link between two firms with different productivity levels—under the assumption that they occupy symmetric positions in the network—we next compare their relative profits with and without such a link.

Corollary 1. *Let $i, j \in \mathcal{N}$ have a symmetric position in $\mathbf{G}$, and $\theta_i > \theta_j$. Then:*

(i) $\pi_i^*(\mathbf{G}_{-ij}) > \pi_j^*(\mathbf{G}_{-ij})$, that is, when i and j are not connected, the firm with higher productivity has higher profit.

(ii) $\pi_i^*(\mathbf{G}_{+ij}) < \pi_j^*(\mathbf{G}_{+ij})$, that is, when i and j are connected, the firm with higher productivity has lower profit.

Defining the link deviation operator Δ^{ij} on any function $f : \mathcal{G}^n \to \mathbb{R}$ as $\Delta^{ij} f(\mathbf{G}) := f(\mathbf{G}_{+ij}) - f(\mathbf{G}_{-ij})$, Corollary 1 also implies that if $\Delta^{ij}\pi_i^*(\mathbf{G}) > 0$ holds, then $\Delta^{ij}\pi_j^*(\mathbf{G}) > 0$ also holds. This means that if adding the link between i and j is beneficial (profit-increasing) for the firm with higher productivity, then it is always beneficial for the firm with lower productivity. However, whether each such firm benefits from forming a link is still ambiguous, and this problem remains intractable even under stronger assumptions on the network structure (e.g., complete or empty network) than the symmetric position assumption we use. Therefore, in Section 3 we take a simulation-based approach to uncover relevant theoretical insights into pairwise stability.

Finally, we provide a comparison of firm sizes as proxied by output levels. Rearranging the FOCs, we get the optimal quantity produced by any firm i as:

$$q_i^* (\mathbf{G}) = \frac{\phi}{\theta_i \; \eta_i(\mathbf{G})} \, e_i^* (\mathbf{G}) \, . \tag{11}$$

This can be compared with the result in Hsieh et al. (2024), showing $e_i^* = \theta q_i^*/(2\phi)$, with the relevant changes in notation. The differences in equilibrium effort and profit levels in these two setups arise from, first, differences in sources of heterogeneity (in baseline costs in their model vs. R&D productivities in our model) and also the solution methods implemented for equilibrium construction. Equation (11) implies that a firm's equilibrium size is positively related to its equilibrium R&D effort, consistent with empirical findings in Cohen and Klepper (1996). Additionally, we see from the expression that the optimal output level is negatively related to its productivity θ_i and its sparsity coefficient $\eta_i(\mathbf{G})$. Equations (10) and (11) imply that for $\theta_i > \theta_j$:

$$\frac{q_i^*(\mathbf{G}_{+ij})}{q_j^*(\mathbf{G}_{+ij})} = 1, \qquad \text{and} \qquad \frac{q_i^*(\mathbf{G}_{-ij})}{q_j^*(\mathbf{G}_{-ij})} = \frac{\phi - \theta_j^2 \; \eta_j(\mathbf{G}_{-ij})}{\phi - \theta_i^2 \; \eta_i(\mathbf{G}_{-ij})} > 1. \tag{12}$$

Equation (12) shows that although the less productive firm earns a higher profit than the more productive firm when they are connected, this ranking does not extend to output levels. Under the symmetric position assumption, the more productive firm always produces a weakly higher output, and strictly higher when the two firms are not connected. This result clarifies why the more productive firm may earn lower profits under connection: both firms face the same price and marginal cost of production leading to the same production level and therefore equal profits in the competitive market. However, the more productive firm incurs a higher R&D cost due to its greater effort level, as shown in Equation (10), leading to an overall lower profit.

2.1.3 R&D Network Formation

Pairwise stability requires that no firm has an incentive to sever an existing link, and that no two firms both benefit (with at least one strictly gaining) from forming a new one (Jackson and Wolinsky, 1996). In our case, the formal statement translates to the following: a network $\mathbf{G}$ is *pairwise stable* if and only if for all firms $i, j \in \mathcal{N}$,

- *Existing Alliances*: If $G_{ij} = 1$, then neither firm i nor firm j would benefit from dissolving the alliance, i.e., $\pi_i(\mathbf{G}) \geq \pi_i(\mathbf{G}_{-ij})$ and $\pi_j(\mathbf{G}) \geq \pi_j(\mathbf{G}_{-ij})$.

- *Potential Alliances*: If $G_{ij} = 0$, then at least one of the firms would incur a loss in profit by forming the alliance, i.e., if $\pi_i(\mathbf{G}_{+ij}) > \pi_i(\mathbf{G})$, then $\pi_j(\mathbf{G}_{+ij}) < \pi_j(\mathbf{G})$.

Let the complete network $\mathbf{G}^C$ be the one in which all firms are linked to every other firm ($\forall i \neq j \in \mathcal{N}, G^C_{ij} = 1$). Under homogeneity, Goyal and Moraga-Gonzalez (2001) shows that the complete network is pairwise stable. By generalizing to heterogeneous R&D productivities, we present our next result.

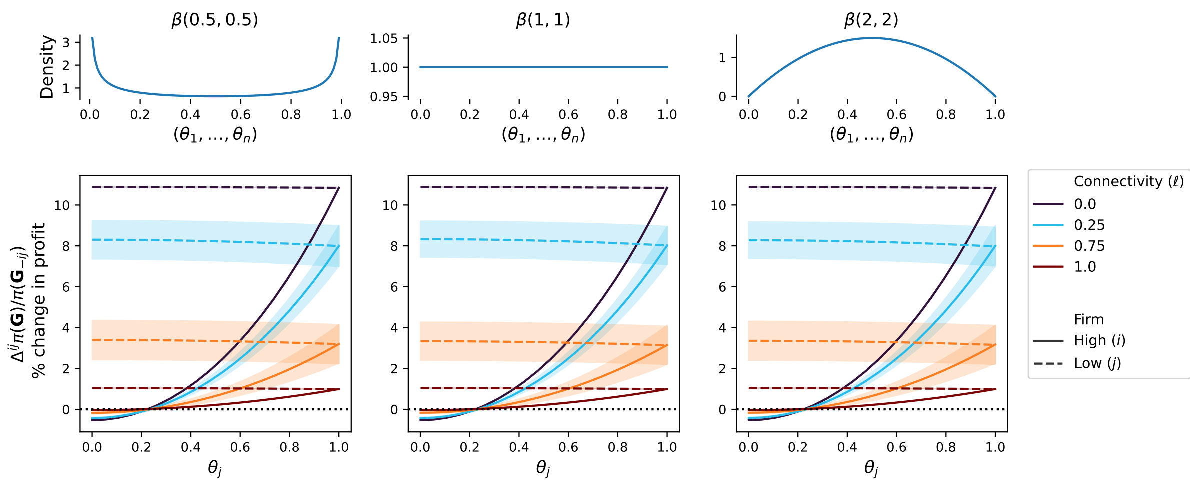
Proposition 3. *There exists $\theta^\star \in (0,1)$ such that if there is a firm j with $\theta_j = \theta^\star$, then the complete network $\mathbf{G}^C$ is not pairwise stable.*

Proposition 3 shows that the complete network, which is stable under homogeneous productivities, need not remain stable when productivities are heterogeneous. When a sufficiently large productivity gap exists between the most productive firm (say i) and any competitor j, a link between them cannot be sustained, leading to instability of the complete network. The proof of Proposition 3 proceeds by comparing the equilibrium profit of the most productive firm i (defined as having $\theta_i = 1$) in the complete network, $\pi^*_i(\mathbf{G}^C_{+ij})$, vs. in the network that severs the link between i and j, $\pi^*_i(\mathbf{G}^C_{-ij})$. We show that removing the link is profitable for the most productive firm as θ_j approaches 0. Conversely, when θ_j approaches 1, such a deviation is not profitable and the complete network is pairwise stable.

Finally, through simulations, we show that the relative profit change, $\Delta^{ij}\pi^*_i(\mathbf{G}^C)/\pi^*_i(\mathbf{G}^C_{-ij})$, is monotonically increasing in θ_j (R&D productivity of potential partner): a low-type firm j always benefits from creating the connection to the most productive firm i, whereas firm i has a threshold level θ^* such that it would deviate (increase its profits) by removing the link when $\theta_j < \theta^*$. This suggests the stronger result that, for any given network $\mathbf{G}$, a link G_{ij} between the most productive firm i and a competitor j cannot be sustained if there is a sufficiently large productivity gap $\theta_j < \theta^*$.

To obtain these insights, in Figure 1, we simulate random networks of $n = 20$ firms using an Erdos-Renyi generation (Erd6s and Rényi, 1960), with probability of connection $\ell \in [0, 1]$ (where $\ell = 0$ corresponds to the empty network and $\ell = 1$ to the complete network). We obtain productivity distributions $(\theta_1, \ldots, \theta_n)$ as random samples from Beta distributions with varying parameters. Finally, we fix the productivities of two firms i and j, having $\theta_i = 1$ and allow θ_j to vary between 0 and 1, and show how adding/severing the link between such i and j affects their profits, while changing the density of the network and the productivity distribution for (other) firms. Figure 1 plots the productivity distributions in the upper panel that are used to calculate profit changes $\Delta^{ij}\pi^*(\mathbf{G})/\pi^*(\mathbf{G}_{-ij})$ in the main (lower) panel, also varying the level of network connectivity ℓ within each case.

Figure 1: Percentage changes in profits of a high- and low-productivity firm after forming an R&D collaboration link between them in random networks with varying connection density and firm productivity distributions

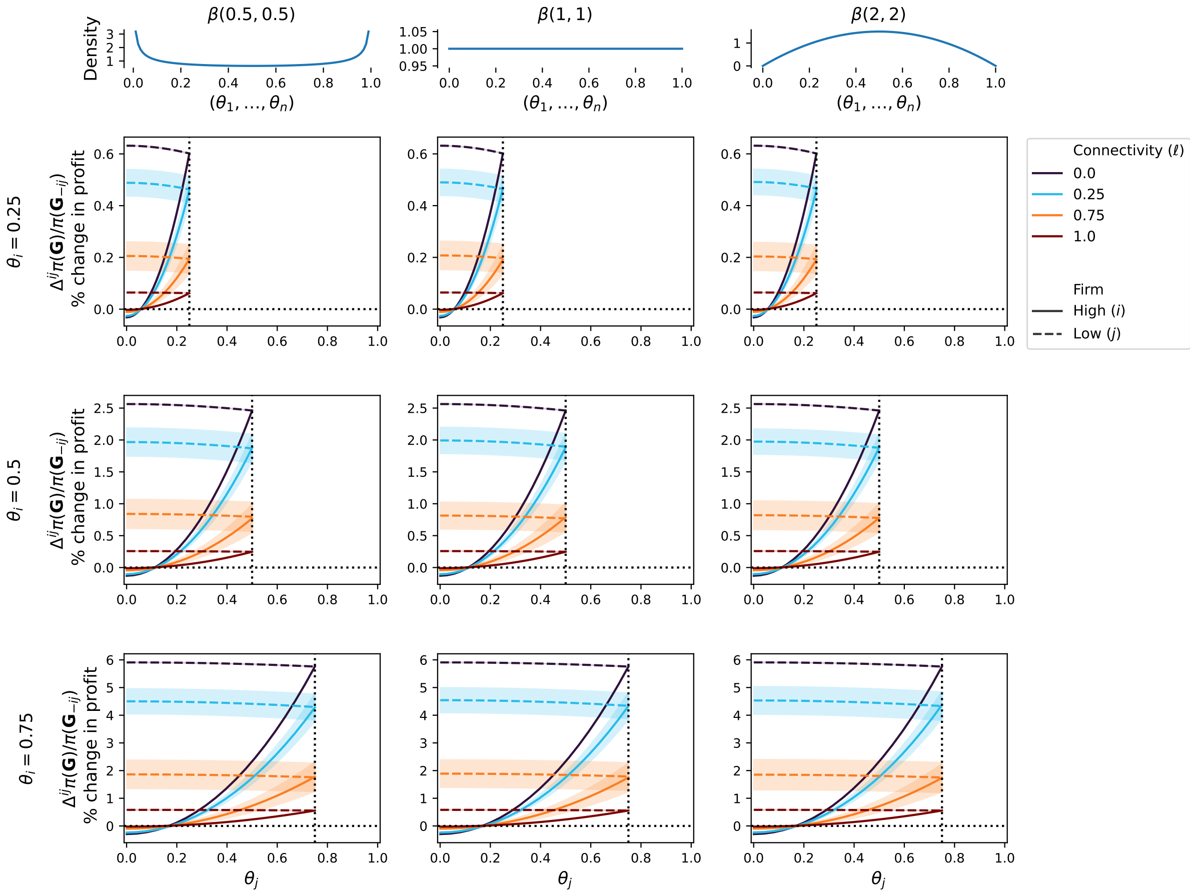


Note: Networks $\mathbf{G}$ with $n = 20$ firms generated as random binary adjacency matrices according to an Erdos-Renyi scheme with connectivity parameter $\ell \in \{0,\ 0.25,\ 0.75,\ 1\}$, represented by different colors. Productivity distributions $(\theta_1, \ldots, \theta_n)$ drawn as random samples from a Beta$(0.5, 0.5)$ distribution *(left)*; Beta$(1, 1)$ *(middle)*; and Beta$(2, 2)$ *(right)*, with corresponding density plotted in the panels above. The percentage change in profit of firm i having $\theta_i = 1$ (before and after adding a link to firm j) is drawn using a solid line, and for firm j using a dashed line. Percentage change in profits for firm i crosses 0 at a threshold $\theta^* \in (0, 1)$ for each productivity distribution and connectivity level cases shown.

Next, we provide further simulation-based findings in Figure 2 showing that for an arbitrary firm i with productivity θ_i, there exists a threshold $0 < \theta_i^* < \theta_i$, such that firm i

would not form a collaboration with a firm j having $\theta_j < \theta_i^*$; i.e., if firm j is sufficiently less productive than firm i. This generalizes the insights in Proposition 3 and Figure 1, shown only for the most productive firm, to any pair of randomly selected firms with arbitrary productivities.

Figure 2: Extension of Figure 1 to a general setting with two firms i and j having arbitrary productivities $\theta_i > \theta_j$



Note: Simulation setting is identical to that of Figure 1, except the productivity of the higher-productive firm i is chosen from $\theta_i \in \{0.25, 0.5, 0.75\}$. Figures are limited to the range $0 < \theta_j < \theta_i$ of each configuration to highlight the crossing at 0 for the higher-productive firm i.

3 Simulation Exercises for Stability and Welfare

The analysis so far shows that the complete network need not remain stable when productivities are heterogeneous, unlike Goyal and Moraga-Gonzalez (2001), where they also show that this structure is not welfare-maximizing in the sense that less connectivity is preferred in the social optimum. Building on these insights, in this section, we extend the analysis to investigate both the stability and welfare properties of alternative network structures in the heterogeneous-productivity setting.

Due to the intractable nature of the problem discussed earlier, we are unable to fully characterize the set of pairwise stable networks in general. Under homogeneity, Goyal and Moraga-Gonzalez (2001) encounter similar difficulties in determining whether the complete network is the unique stable (symmetric) configuration, and show this is the case by restricting the problem to $n = 4$ firms.[5] This challenge becomes much more pronounced under heterogeneity. Additionally, this intractability extends to making comparisons of equilibrium effort and profit levels across different network structures. Even after deriving closed-form solutions for efforts via Equation (7) in simple network configurations —found by inverting matrix $\mathbf{A}(\mathbf{G})$ that depends on the network— comparing levels of efforts across structures requires comparing rational functions of large degree in their denominators, which is again unfeasible even for small number of firms n.

To open this black box, we categorize firms into two types based on their R&D productivities: *high*- and *low*-productive firms— a simplified structure that makes the simulation exercise tractable for the remainder of the paper. The productivity of a high-type firm is therefore $\theta_H = 1$, and for a low-type firm it is denoted by $\theta := \theta_L \in (0, 1)$. We also define n_L and n_H as the number of low- and high-type firms in the economy, respectively, such that $\rho := n_H/n$ is the ratio of high-type firms. We set $\rho = 1/2$ for the initial simulations, and then we experiment with its value in extensions highlighting a novel crowding-out mechanism on welfare operating through high-type firms.

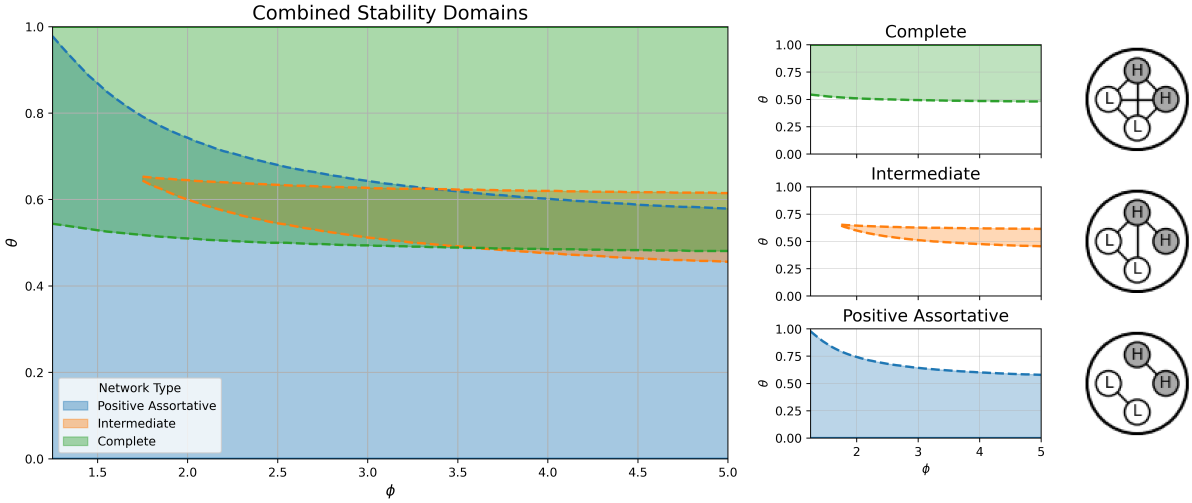
By using computer-assisted analysis, first, we fully characterize the set of stable networks for $n = 4$ in this setting.[6] Crucially, the analysis reveals that differences in productivity levels and the distribution of high- and low-type firms within the economy can sustain pairwise stable networks different from the complete network.

[5]Readers are referred to pp. 696 of Goyal and Moraga-Gonzalez (2001) for a detailed discussion on the intractability of the network formation problem in this class of models.

[6]For the full implementation of the computer-assisted setup, see the supplementary material.

Our first simulation documents a new fact: when introducing heterogeneity in firm R&D productivities, the *positive assortative* (PA) network $\mathbf{G}^{PA}$ —in which firms fully connect only to the others of their same type— becomes a pairwise stable configuration. In a two-types setting, a PA network is the one in which high- (low-) type firms are linked to every other high- (low-) type firms ($G_{ij}^{PA} = G_{ji}^{PA} = 1$ if i and j are of the same type, otherwise $G_{ij}^{PA} = G_{ji}^{PA} = 0$). This structure leads to two fully connected components of size $\rho \cdot n$ and $(1-\rho) \cdot n$ composed solely of firms of high- and low-type firms, respectively.

Figure 3: Pairwise stability domains for $n = 4$ firms with two high- and two low-productivity firms for different productivity gap levels



Note: The shaded region indicates the parameter combinations (θ, ϕ) under which the corresponding configuration is pairwise stable. This figure only plots those network structures that showed non-empty stability region in simulations. The figure for every possible structure is provided in the Supplementary Appendix.

Figure 3 fully characterizes the set of pairwise stable networks for $n = 4$ and $\rho = 1/2$. To construct this figure, we solve for the closed-form equilibrium profits of each starting possible network configuration (in a space of 2^6 starting structures) and numerically evaluate all possible link deviations from a given starting structure. We plot the contour of values for ϕ and θ that guarantee no profitable deviations exists from a starting network, which precisely corresponds to the full characterization of pairwise stable configurations.

Our results show that, for any fixed R&D effort cost coefficient ϕ, there exist threshold levels $\underline{\theta}$ (shown as the green dashed line) and $\overline{\theta}$ (shown as the blue dashed line) with $\underline{\theta} < \overline{\theta}$. The complete network is stable for $\underline{\theta} \leq \theta < 1$, and the PA network is stable for $0 < \theta \le \overline{\theta}$. These together imply both networks are stable in the interim region $\underline{\theta} \le \theta \le \overline{\theta}$. In addition, both thresholds are decreasing in ϕ, meaning that the stability area spanned by the complete/PA network is increasing/decreasing in R&D cost.

Furthermore, while the PA and complete networks span the entire parameter space for θ, there is one other configuration that emerges as stable for certain parameter values, which is shown in the middle right panel of Figure 3. This network is an intermediate between PA and complete, in the sense that it is denser/sparser than the PA/complete network through adding/severing links from a high-type firm to all low-type firms. Lastly, this full characterization shows that for sufficiently low/high θ, the PA/complete network is the unique stable structure.

3.1 Welfare Comparison

Following Goyal and Moraga-Gonzalez (2001), we define welfare as the total surplus generated in the economy by summing up producer and consumer surplus.[7] Producer surplus (PS) is the sum of profits earned by all firms in the network, given by $PS := \sum_{i=1}^{n} \pi_i^*$, and consumer surplus ($CS$) is given by $CS := (1/2)\left(\sum_{i=1}^{n} q_i^*\right)^2$, as derived from the linear demand specification $p(\mathbf{q})$. Adding up these components, total welfare of the given R&D network structure $\mathbf{G}$ is given by:

$$W(\mathbf{G}) = \frac{1}{2}\left[\sum_{i\in\mathcal{N}} q_i^*(\mathbf{G})\right]^2 + \sum_{i\in\mathcal{N}} \pi_i^*(\mathbf{G}). \tag{13}$$

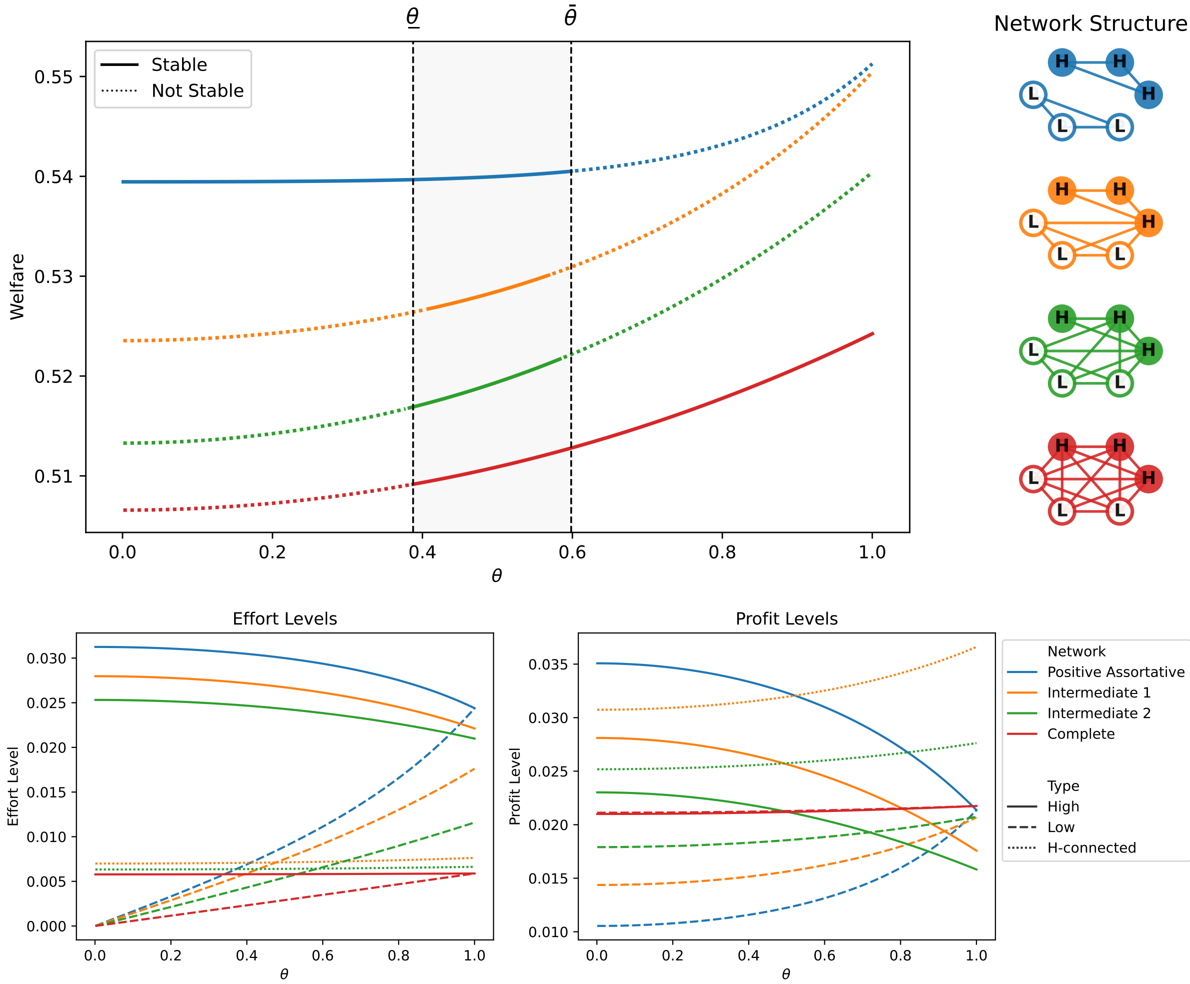
To compare the welfare of the stable network configurations that arise in our setting, as well as R&D efforts and profits of firms, in Figure 4 we next provide a simulation result increasing the number of firms to $n = 6$ while keeping the two-type firm setting fixed.[8]

The upper panel of Figure 4 shows four different stable network structures drawn in different colors. In these different networks, while low-type firms are always symmetric,

[7]For an alternative approach based on a utility function over consumption, see Hsieh et al. (2024).

[8]When n varies, we set $\phi = \underline{\phi}$, the lower bound of R&D cost that generates positive efforts as described in Proposition 1.

Figure 4: Welfare, profit, and effort comparison of pairwise stable structures in $n = 6$ setting, with $\rho = 1/2$



Note: This figure shows welfare (*top*), individual firm efforts (*bottom left*), and individual firm profits (*bottom right*) for all network structures that are stable under $n = 6$ and $\rho = 0.5$. Colors represent different stable network structures showing a high-type firm connectivity-based transition from the PA network (blue) to the complete (red) network. All outcomes are plotted against the productivity of low-productivity firms (θ). In the top panel, line style (dashed or solid) indicates stability, where only the structures with a non-zero stability region at $\phi = \underline{\phi}$ are included. In the lower panels, line style reflects firm type: solid for high-productivity firms, dashed for low-productivity firms, and dotted for the H-connected firms that are the high-productivity firms connected to all other firms in the intermediate structures.

high-type firms can have asymmetric connectivity with their own type. In particular, in the orange network, one high-type firm is connected to every other firm in the network; and in the green network, two high-type firms are connected to every other firm. In the figure, we label these high-type firms that become connected to the entire network as *H-connected*. These intermediate configurations are pairwise stable for some $0 < \theta < 1$ values as shown in the upper panel and are relevant as they create a transition from the PA to the complete structure by increasing the connectivity of high-productivity firms. On the other hand, there are no stable networks that feature *L-connected* firms —those where a low-type firm connects to all other firms in the network— and consequently, these are not drawn in the figure.

The welfare comparison reveals a clear ranking of network structures by their connectivity levels: the PA network delivers the highest welfare (stable for $\theta \leq \underline{\theta}$), while the complete network yields the lowest among the set of stable structures (stable for $\theta \geq \overline{\theta}$). However, this ranking holds only for intermediate values of θ. When θ is sufficiently low or high, the PA or complete network, respectively, becomes the unique stable structure.

This leads to a counterintuitive insight: although welfare increases monotonically with θ for any given stable network, the level of θ determines which structure is pairwise stable, resulting in a non-monotonic welfare effect. As an example, starting from a low level of θ (i.e., R&D productivity for low-type firms) where PA is the only pairwise stable configuration, welfare increases as θ rises up to a threshold $\bar{\theta}$. Further increases in θ past this threshold leads to a change in the stable structure from PA to complete, resulting in a discontinuous decrease in welfare that even falls below the starting point level. This observation has implications on how government and/or firm-level R&D policies can have negative welfare effects even if they increase average R&D productivity in the economy. Such non-monotonic effects of productivity are interesting, and further investigation on how R&D policies endogenously affect welfare is left for future research.

The lower panel of Figure 4 reports the efforts and profits of low-type, high-type, and H-connected type firms, labeled as dashed, solid, and dotted lines, respectively. The effort comparison shows that in intermediate configurations, high-type firms that only connect to other high-productivity firms have the highest effort level, whereas the H-connected firm has the lowest level of effort. On the other hand, in the PA and complete networks (where there is no H-connected type), it always holds that high-type firms have higher efforts than low-type firms. When profits are compared, in intermediate configurations, H-connected firms have the highest profits, but the ordering of low-type vs. high-type firms' profits depends on the θ level. In the PA network, it always holds that high-productivity firms have higher profits than low-productivity firms, with this ordering reversed for the complete network (as suggested by Corollary 1) but with much smaller differences in profits.

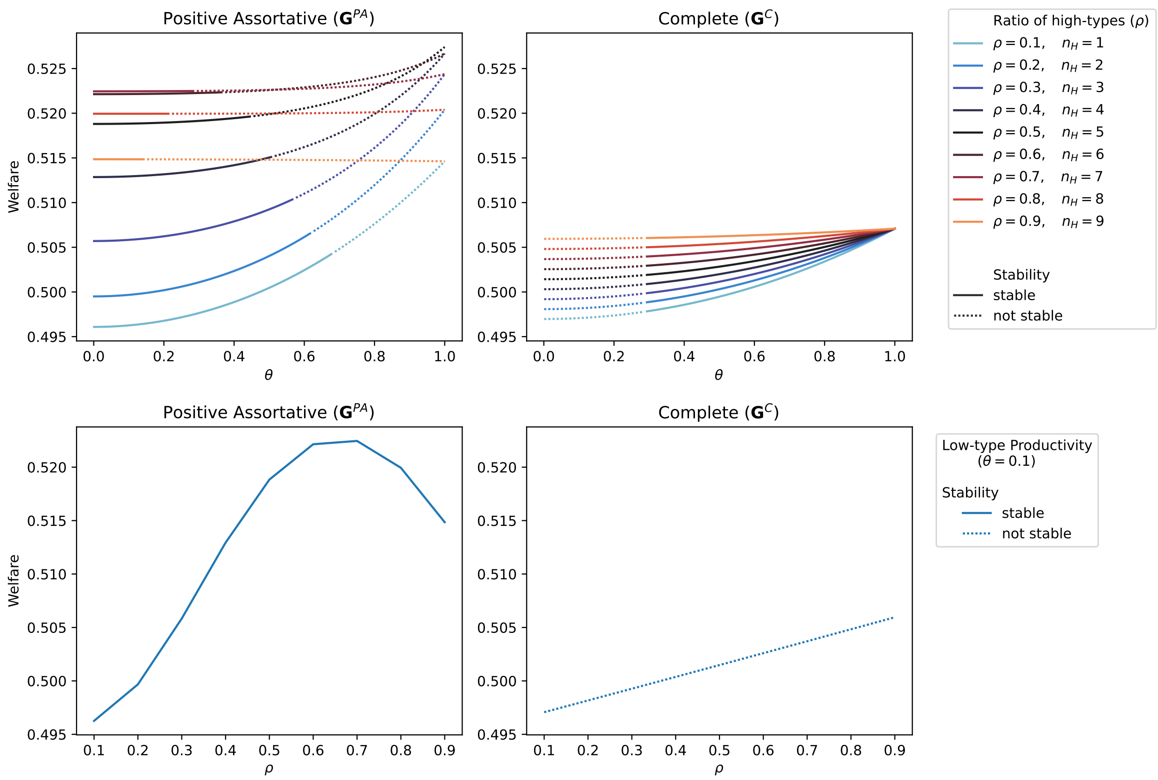
Finally, comparing the PA, complete, and other intermediate stable networks in Figure 4 reveals that group formation (under coalitional deviations) might serve as an alternative mechanism to study the network formation. While we employ the pairwise stability concept as commonly used in the R&D networks literature, group formation analysis and its welfare implications are left as another avenue for future research.

3.2 Crowding-out Effect

Our next exercise experiments with ρ, the ratio of the high-productivity firms in the economy, to showcase a novel channel: a crowding-out effect of high-productivity firms on welfare. Intuitively, one might expect that when comparing two economies while holding θ fixed, the one with the higher number of high-productivity firms (i.e., with a higher ρ) would also have a higher welfare, as it has a larger average productivity. However, our results indicate that this relationship is mediated by the R&D network. While welfare is monotonic in the fraction of high-type productivity firms for the complete network, it instead follows an inverted U-shape under the PA formation: both low and high fractions of high-productivity firms in the economy result in lower levels of welfare compared to an intermediate fraction of high-productivity firms.

Under the PA formation, firms are clustered by type, which means that the average productivity in the economy is higher for a larger ρ value, along with the size of the high-type connected component (with the low-type component shrinking in return) when we compare it with a setting with a lower ρ. This implies that an intermediate fraction of high-type firms in the economy maximizes welfare under the PA network formation process.

Figure 5: Crowding-out effect of high-productivity firms on welfare



Note: This figure reports welfare for the positive assortative (*left*) and complete (*right*) networks when $n = 10$. The top row plots welfare over low-type productivity θ, with colors indicating $\rho \in \{0.1, 0.2, \ldots, 0.9\}$ and (dashed) solid lines denoting (un)stability under $\phi = \underline{\phi}$. The top row shows that the PA network's stability region shrinks as ρ increases, whereas the stability region of the complete network remains unchanged. The bottom row plots welfare with ρ on the x-axis, for a single value of $\theta = 0.1$. For the PA network, welfare displays an inverted U-shape in ρ, capturing a crowding-out effect at higher values of ρ.

Figure 5 illustrates welfare levels for the PA and complete network structures as the fraction of high-productivity firms, ρ, increases. We modify the number of firms to $n = 10$ in our simulation results to provide a wider range of ρ values. Under the PA structure, for an arbitrary low-type productivity θ, welfare initially increases with the ratio of high-productivity firms but begins to decline once this ratio exceeds a certain threshold, as shown on the left side of the upper panel of Figure 5. As the PA structure is stable when an economy exhibits large productivity gaps, shown in the lower panel of Figure 5 by setting $\theta = 0.1$, the increasing ratio of high-productivity firms leads to an inverted U-shaped relationship with welfare in a network structure that remains stable. In other words, when firms exhibit substantial heterogeneity in their productivities, there is a crowding-out effect of high-productivity firms on welfare in the stable PA configuration.

Figure 5 provides an additional insight: as ρ rises, while the stability region of the complete network remains unchanged, this is not the case for the PA structure. Instead, the PA stability region shrinks as ρ increases. Formally, the upper threshold $\overline{\theta}$, which ensures the stability of the PA network for any $\theta \leq \overline{\theta}$, decreases in the fraction of high-productivity firms, while the lower threshold $\underline{\theta}$, guaranteeing the stability of the complete network for any $\theta \geq \underline{\theta}$, remains unaffected.[9]

As shown earlier in this Section (and illustrated in Figures 3 and 4), the PA and complete network configurations are the main pairwise-stable structures under heterogeneity: together, they span almost the entire parameter space, meaning that when one structure is not stable, the other is typically stable. Therefore, understanding the crowding-out patterns in the PA structure becomes especially relevant in alternative economies where the PA structure is stable.

Comparing the PA and complete networks across different values of ρ reveals an important distinction. Under the complete formation, the connectivity structure remains fixed—every firm is linked to every other—and only the productivity distribution varies. Under the PA formation, by contrast, the connectivity structure itself evolves with ρ. Specifically, both the total number of links (i.e., network density) and the firms that constitute the connected components also change with ρ.[10]

[9]For a comparison of how the stability regions of the PA and complete structures evolve under different settings, see Figure 9 in Appendix B.

[10]Appendix Figure 8 disentangles this "composition-only" channel by holding the network fixed at a two-clique structure (two fully connected components of size 5, as under PA when $\rho = 1/2$) and varying only the productivity distribution via ρ. Upgrading firms one-by-one from θ to 1 shows that the transitioning firm's profit gain is larger for lower θ and is typically increasing in ρ, with a kink around $\rho = 1/2$ when the fixed connectivity coincides with exact type clustering; the gain nevertheless remains positive. This exercise is illustrative rather than a stability result, since the imposed network is generally not pairwise stable away from $\rho = 1/2$ (and sufficiently low θ).

Specifically, the number of links in the PA structure is given by the sum of links within the two fully connected components—one with $n_H = \rho \cdot n$ high-productivity firms and the other with $n_L = (1 - \rho) \cdot n$ low-productivity firms:

$$\text{Total Number of Links in PA} = \frac{n_H\,(n_H - 1)}{2} + \frac{n_L\,(n_L - 1)}{2} = \frac{n\,(n-1)}{2} - n^2\,\rho\,(1-\rho).$$

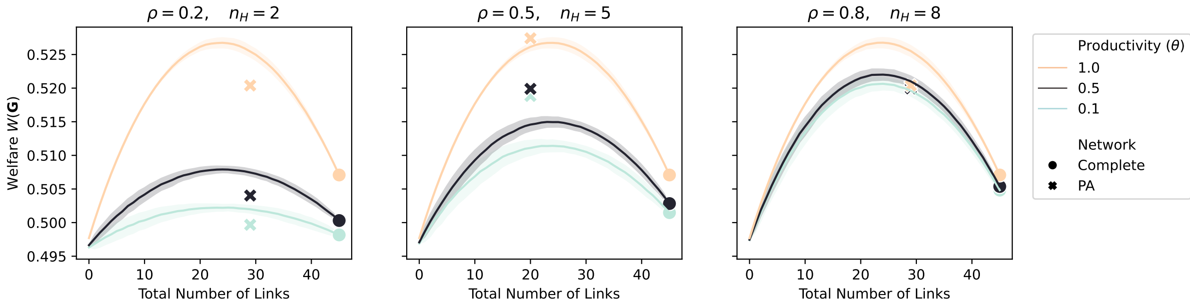
This implies that as ρ increases from 0 to 1, the number of links decreases until reaching a minimum at $\rho = 0.5$, symmetrically increasing thereafter. As Goyal and Moraga-Gonzalez (2001) show, welfare in symmetric networks under the homogeneous setting exhibits an inverted-U relationship with the number of links. This naturally suggests that the crowding-out effect we observe in the PA structure under heterogeneity may stem from a generalized version of their result, where welfare displays an inverted-U shape in the network's overall connectivity.

In the next section, we investigate this hypothesis by isolating the connectivity effect from the structural effect in the PA network. We find that although connectivity is correlated with the welfare pattern of the PA structure, it cannot fully account for the sharp increase at intermediate values or the decline at high values of ρ.

3.3 Network Structure vs. Density Effect

For symmetric networks with homogeneous firms, Goyal and Moraga-Gonzalez (2001) show that welfare is concave in the average degree of the network, implying that welfare is maximized at an intermediate level of connectivity—a network configuration between the empty and complete. In our heterogeneous setting, we find a similar and more general connectivity effect, based on simulations with randomly generated networks instead of the more restricted symmetric network case. In Figure 6, we depict the inverted U-shaped relationship between the network density and welfare for $n = 10$ firms, where, for a given total number of links (on the x-axis), we average welfare over 1000 replications of networks where the links are randomly assigned.

Figure 6: Welfare over Network Link Density for Random vs. PA/Complete Structures



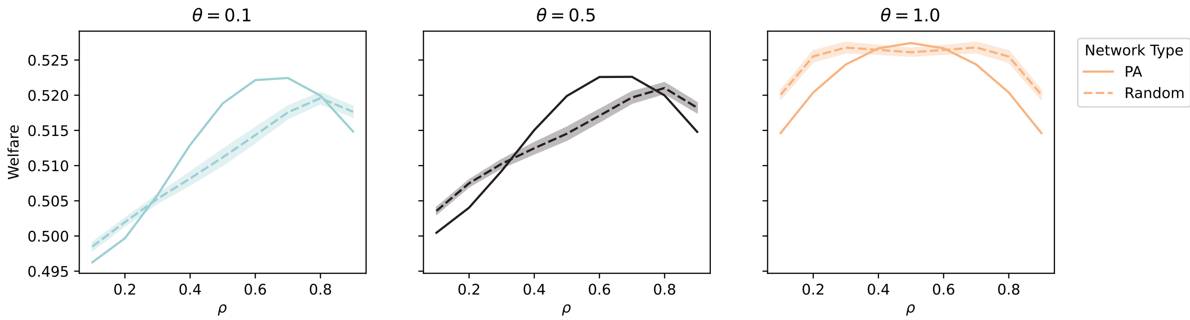
Note: Welfare is plotted against the total number of links in a randomly generated network for $n = 10$ under $\phi = \underline{\phi}$. Each panel corresponds to a different fraction of high-productivity firms in the economy, ρ: 0.2 (*left*), 0.5 (*middle*), and 0.8 (*right*). Colors represent different values of low-type productivity, $\theta \in \{0.1, 0.5, 1.0\}$. For each fixed total number of links (on the x-axis), networks are generated by randomly assigning links. Solid lines show average (across simulations) welfare, and shaded regions indicate plus and minus one standard deviation from the mean welfare. The welfare of the PA network and the complete network is marked with a cross and a circle, respectively. At the mid-level value of ρ, the PA network achieves a welfare level (for $\theta = 0.1$) that exceeds even the maximum average welfare obtained from the random networks when $\theta = 0.5$, highlighting the welfare advantage of the PA structure at intermediate values of ρ. This pattern does not occur for low or high values of ρ.

Furthermore, the simulations indicate that average welfare is positively correlated with the economy's average productivity, as the welfare curves shift upward as the productivity gap narrows (i.e., as θ increases) and as the proportion of high-productivity firms (ρ) rises.

However, when we compare the PA and complete structures to this average benchmark, a more complex picture emerges. While the complete network (circle markers in Figure 6) generally aligns with the shifts caused by θ and ρ described in the previous paragraph, the PA structure (cross markers in Figure 6) exhibits distinct relationships with these quantities. To isolate this effect, Figure 7 provides a comparison by plotting welfare against the fraction of high-productivity firms (ρ) for both the PA structure and a set of randomly generated networks constrained to have the exact same total number of links.

Together, Figures 6 and 7 indicate that the performance of the PA structure is highly non-monotonic relative to the average benchmark. At extreme values of ρ (either low or high), the welfare in PA network is lower relative to the average welfare in random networks with equal link density. Conversely, at intermediate levels of ρ, the PA structure generates a significant welfare premium. This premium, however, diminishes as the productivity gap closes (i.e., as θ increases). The network structure advantage at intermediate ρ is

Figure 7: Network Structure Effect — Welfare as a function of high-productivity firm ratio (ρ) in PA and Random Networks with Equal Total Number of Links



Note: Welfare is plotted for the positive assortative network alongside the average welfare from randomly generated networks with the same number of links for $n = 10$ and $\phi = \underline{\phi}$. Panels correspond to different values of low-type productivity θ: 0.1 (*left*), 0.5 (*middle*), and 1.0 (*right*). The solid lines show welfare under the PA structure. For each feasible ρ, the number of links in the corresponding PA network is computed, and random networks with the same ρ and the same number of links are generated. The dashed lines denote the average welfare of these random networks, and the shaded regions represent plus and minus one standard deviation from the mean welfare. Although welfare in the PA and random networks are positively correlated, the differences between them are substantial, highlighting the effect of network structure on welfare.

so pronounced in large productivity gap settings that it can override the effect of lower intrinsic productivity. For instance, at $\rho = 0.5$, the PA network with a large productivity gap ($\theta = 0.1$) achieves a welfare level exceeding even the maximum average welfare of random networks with a significantly smaller productivity gap ($\theta = 0.5$).

Our findings highlight that, under heterogeneity, the specific topology of the PA network creates an additional effect that goes beyond connection density, such that simple measures like the total number of links cannot capture the full effect.

4 Conclusion

In this paper, we study R&D network formation with heterogeneous productivities, generalizing the homogeneous framework in Goyal and Moraga-Gonzalez (2001). Firms differ in their cost of doing R&D, modeled as a reduction in the marginal cost of production in a Cournot competition setup.

Under such heterogeneity, first, we provide some comparative statics results on R&D

efforts and profits of firms. The findings show connectivity-based relative efforts and profits: when a low and a high-productivity firm (having a symmetric position in the R&D network) gets connected, the relative gain of the less productive firm is higher than the more productive firm. Moreover, in terms of efforts, when two firms (with symmetric network positions) are compared, the one with higher productivity has more R&D effort, and hence total R&D cost, than the one with lower productivity.

These benchmark comparisons show that low-productivity firms highly benefit from being connected to high-productivity firms, but the reverse does not necessarily hold, having explicit implications for link formation incentives. In particular, the complete network—that is pairwise stable in a homogeneous setting—ceases to be stable when the productivity gap is sufficiently large.

Our simulation results illustrates a stability feature: the positive assortative (PA) network structure, which is a productivity-level-based clustered network, and the complete network structure together almost covers the entire stability space, meaning that when one of these two structures is not stable, the other typically is. Therefore, understanding the welfare and stability implications of the PA structure is crucial. This alternative configuration has unique implications for welfare, defined as the aggregate consumer and producer surplus in the economy.

Our simulations show that the PA structure yields higher welfare in comparison to the complete structure. Furthermore, we uncover a counterintuitive effect: a higher average productivity in the economy could potentially lead to a lower welfare due to (i) emergence of alternative stable structures and (ii) crowding-out effect of high-productivity firms. Finally, we compare the welfare in the PA and complete networks with the average welfare in random networks in our simulations, isolating the welfare effects of the network structure from the welfare effects of network density measured by the number of links, providing further insights into how the connectivity structure together with productivity distribution and network density shape the welfare under heterogeneity.

Taken together, these findings suggest that economies with higher average firm-level R&D productivity—either through higher productivity of low-type firms or through a larger fraction of high-type firms—may exhibit lower welfare compared to economies with lower average R&D productivity. Therefore, policies aimed at enhancing firm level R&D productivities should account for their impact on endogenous R&D alliances and effort, as endogenous responses to productivity changes may offset or even reverse the intended welfare gains.

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A Proofs

Proof of Proposition 1. We use the Gershgorin's theorem: a matrix is nonsingular if every column is strictly diagonally dominant, i.e.

$$|A_{ii}| \; > \; \sum_{j \neq i} |A_{ji}| \qquad \text{for all } i.$$

For i we have

$$\begin{aligned}
\sum_{j \neq i} |A_{ji}| &= \sum_{j \neq i} \big| \theta_j \big\{ (n+1)\, G_{ij} - [1 + d_j(\mathbf{G})] \big\} \big| \\
&= \sum_{j \neq i} \theta_j \big| (n+1)\, G_{ij} - d_j(\mathbf{G}) - 1 \big| \\
&= \sum_{j \in \mathcal{N}_i(\mathbf{G})} \theta_j \big| n - d_j(\mathbf{G}) \big| \; + \sum_{j \in \mathcal{N}_{-i}(\mathbf{G})} \theta_j \big| - d_j(\mathbf{G}) - 1 \big|,
\end{aligned}$$

where $\mathcal{N}_i(\mathbf{G})$ denotes the neighbors of i and $\mathcal{N}_{-i}(\mathbf{G})$ its non-neighbors.

Bounding each term by its maximum gives

$$\sum_{j \neq i} |A_{ji}| < d_i(\mathbf{G}) \max_{j \in \mathcal{N}_i(\mathbf{G})} \big[n - d_j(\mathbf{G}) \big] + (n - 1 - d_i(\mathbf{G})) \max_{j \in \mathcal{N}_{-i}(\mathbf{G})} \big[d_j(\mathbf{G}) + 1 \big].$$

If $j \in \mathcal{N}_i(\mathbf{G})$, then $d_j(\mathbf{G}) \geq 1$ so

$$\max_{j \in \mathcal{N}_i(\mathbf{G})} \big[n - d_j(\mathbf{G}) \big] \; \leq \; n - 1.$$

If $j \in \mathcal{N}_{-i}(\mathbf{G})$, then $d_j(\mathbf{G}) \leq n - 2$ so

$$\max_{j \in \mathcal{N}_{-i}(\mathbf{G})} \big(d_j(\mathbf{G}) + 1 \big) \; \leq \; n - 1.$$

Therefore,

$$\sum_{j \neq i} |A_{ji}| < d_i(\mathbf{G})\, (n-1) + [n - 1 - d_i(\mathbf{G})]\, \big(n - 1 \big) = (n-1)^2. \tag{1}$$

Column i is strictly diagonally dominant if

$$\frac{(n+1)^2\phi}{\theta_i[n-d_i(\mathbf{G})]} - \theta_i[n-d_i(\mathbf{G})] > (n-1)^2.$$

Rearranging yields the sufficient bound

$$\phi > \theta_i\,[n-d_i(\mathbf{G})]\left(\frac{n-1}{n+2}\right)^2 + \theta_i^2\left(\frac{n-d_i(\mathbf{G})}{n+1}\right)^2. \tag{2}$$

To obtain a uniform sufficient condition, set $\theta_i \le \theta_{\max} = 1$ and $n - d_i(\mathbf{G}) \le n$. Therefore

$$\phi > \frac{n(n-1)^2 + n^2}{(n+1)^2} = \frac{n^3 - n^2 + n}{(n+1)^2}.$$

Thus if ϕ satisfies the above inequality, every column of A is strictly diagonally dominant, and by Gershgorin's theorem the matrix A is invertible.

□

Proof of Proposition 2. We can rewrite the FOC in equation (6) in equilibrium as

$$\left[\frac{\phi}{\theta_i\,\eta_i(\mathbf{G})} - \theta_i\,\eta_i(\mathbf{G})\right] e_i^*(\mathbf{G}) = \frac{\alpha-\bar{c}}{n+1} + \sum_{l\in\mathcal{N}_i(\mathbf{G})} \theta_l\,\eta_l(\mathbf{G})\,e_l^*(\mathbf{G}) - \sum_{k\in\mathcal{N}_{-i}(\mathbf{G})} [1-\eta_k(\mathbf{G})]\,\theta_k\,e_k^*(\mathbf{G}).$$

Lets define $\mathcal{N}_i^{(-j)}(\mathbf{G}) := \{k \mid G_{ik} = 1, k \neq i, j\}$, and $\mathcal{N}_{-i}^{(-j)}(\mathbf{G}) := \{k \mid G_{ik} = 0, k \neq i, j\}$ to be the set on neighbors and non-neighbors of firm i in network $\mathbf{G}$ excluding the firm j. Thus for every pair of firms i and j we can write:

$$\left[\frac{\phi}{\theta_i\,\eta_i(\mathbf{G})} - \theta_i\,\eta_i(\mathbf{G})\right] e_i^*(\mathbf{G}) - [\eta_j(\mathbf{G}) - 1 + G_{ij}]\ \theta_j e_j^*(\mathbf{G}) = \frac{\alpha-\bar{c}}{n+1}$$
$$+ \sum_{l\in\mathcal{N}_i^{(-j)}(\mathbf{G})} \theta_l\,\eta_l(\mathbf{G})\,e_l^*(\mathbf{G}) - \sum_{k\in\mathcal{N}_{-i}^{(-j)}(\mathbf{G})} [1-\eta_k(\mathbf{G})]\,\theta_k\,e_k^*(\mathbf{G}).$$

From the symmetric position definition in Definition 1 we know that if two firm i and j

have a symmetric position with respect to a network means:

$$\mathcal{N}_i^{(-j)}(\mathbf{G}) = \mathcal{N}_j^{(-i)}(\mathbf{G}), \quad \text{and} \quad \mathcal{N}_{-i}^{(-j)}(\mathbf{G}) = \mathcal{N}_{-j}^{(-i)}(\mathbf{G}).$$

Therefore the right hand side of the rewritten version of equilibrium FOCs for the symmetric firms i and j are equal. Therefore:

$$\left[\frac{\phi}{\theta_i\,\eta_i(\mathbf{G})} - \theta_i\,\eta_i(\mathbf{G})\right] e_i^*(\mathbf{G}) - [\eta_j(\mathbf{G}) - 1 + G_{ij}]\;\theta_j e_j^*(\mathbf{G})$$
$$=$$
$$\left[\frac{\phi}{\theta_j\,\eta_j(\mathbf{G})} - \theta_j\,\eta_j(\mathbf{G})\right] e_j^*(\mathbf{G}) - [\eta_i(\mathbf{G}) - 1 + G_{ji}]\;\theta_i e_i^*(\mathbf{G}). \quad (14)$$

Rearranging the equation, we get:

$$\left[\frac{\phi}{\theta_i\eta_i(\mathbf{G})} - \theta_i\,(1 - G_{ij})\right] e_i^*(\mathbf{G}) = \left[\frac{\phi}{\theta_j\eta_j(\mathbf{G})} - \theta_j\,(1 - G_{ji})\right] e_j^*(\mathbf{G}),$$

and since $\eta_i(\mathbf{G}) = \eta_j(\mathbf{G})$, we have:

$$\frac{e_i^*(\mathbf{G})}{e_j^*(\mathbf{G})} = \frac{\theta_i}{\theta_j} \cdot \frac{\phi - \theta_j^2\;\eta_j(\mathbf{G})\;(1 - G_{ij})}{\phi - \theta_i^2\;\eta_i(\mathbf{G})\;(1 - G_{ji})}.$$

Finally, substituting this into the equilibrium profit equation (9) we get:

$$\frac{\pi_i^*(\mathbf{G})}{\pi_j^*(\mathbf{G})} = \frac{\phi - \theta_i^2\;\eta_i^2(\mathbf{G})}{\phi - \theta_j^2\;\eta_j^2(\mathbf{G})} \cdot \left[\frac{\phi - \theta_j^2\;\eta_j(\mathbf{G})\;(1 - G_{ij})}{\phi - \theta_i^2\;\eta_i(\mathbf{G})\;(1 - G_{ji})}\right]^2.$$

□

Proof of Corollary 1. Showing that $\pi_i^*(\mathbf{G}_{+ij}) < \pi_j^*(\mathbf{G}_{+ij})$ is straightforward as we have:

$$\frac{\pi_i^*(\mathbf{G}_{+ij})}{\pi_j^*(\mathbf{G}_{+ij})} = \frac{\phi - \theta_i^2\;\eta_i^2(\mathbf{G}_{+ij})}{\phi - \theta_j^2\;\eta_j^2(\mathbf{G}_{+ij})}.$$

Since $\eta_i^2(\mathbf{G}_{+ij}) = \eta_j^2(\mathbf{G}_{+ij})$ due to the symmetric position assumption, the term $\phi - \theta_i^2\;\eta_i^2(\mathbf{G}_{+ij}) > \phi - \theta_j^2\;\eta_j^2(\mathbf{G}_{+ij})$, and $\pi_i^*(\mathbf{G}_{+ij})/\pi_j^*(\mathbf{G}_{+ij}) < 1$. Now for when $G_{ij} = 0$ we have:

$$\frac{\pi_i^*(\mathbf{G}_{-ij})}{\pi_j^*(\mathbf{G}_{-ij})} = \frac{\phi - \theta_i^2\,\eta_i^2(\mathbf{G}_{-ij})}{\phi - \theta_j^2\,\eta_j^2(\mathbf{G}_{-ij})} \left(\frac{\phi - \theta_j^2\,\eta_j(\mathbf{G}_{-ij})}{\phi - \theta_i^2\,\eta_i(\mathbf{G}_{-ij})}\right)^2$$
$$= \underbrace{\frac{\phi - \theta_i^2\,\eta_i^2(\mathbf{G}_{-ij})}{\phi - \theta_i^2\,\eta_i(\mathbf{G}_{-ij})} \cdot \frac{\phi - \theta_j^2\,\eta_j(\mathbf{G}_{-ij})}{\phi - \theta_j^2\,\eta_j^2(\mathbf{G}_{-ij})}}_{=f(\theta_i^2)/f(\theta_j^2)} \cdot \underbrace{\frac{\phi - \theta_j^2\,\eta_j(\mathbf{G}_{-ij})}{\phi - \theta_i^2\,\eta_i(\mathbf{G}_{-ij})}}_{>1},$$

where we define $f(x) := [\phi - x\,\eta_i^2(\mathbf{G}_{-ij})] \,/\, [\phi - x\,\eta_i(\mathbf{G}_{-ij})]$. The last fraction is greater that 1, as $\theta_i > \theta_j$. Now we only need to show that $f(x)$ is an strictly increasing function, which makes the first two fraction greater that 1 as well and results in $\pi_i^*(\mathbf{G}_{-ij}) > \pi_i^*(\mathbf{G}_{-ij})$.

Taking the derivative of f with respect to x, and by using the term $\zeta \coloneqq \eta_i(\mathbf{G}_{-ij})$ for convinience, we have:

$$\frac{\partial f(x)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\phi - \zeta^2\,x}{\phi - \zeta\,x}\right)$$
$$= \frac{-\zeta^2\,(\phi - \zeta\,x) + \zeta\,(\phi - \zeta^2\,x)}{(\phi - \zeta\,x)^2}$$
$$= \frac{\zeta\,(\phi - \zeta\,\phi)}{(\phi - \zeta\,x)^2} = \frac{\zeta\,(1 - \zeta)\,\phi}{(\phi - \zeta\,x)^2}.$$

Since $\zeta = \eta_i(\mathbf{G}_{-ij}) = \eta_j(\mathbf{G}_{-ij}) \in \left[\frac{2}{n+1}, \frac{n}{n+1}\right]$, we have $f'(x) > 0$, meaning that $f(x)$ is an strictly increasing function, and therefore:

$$\pi_i^*(\mathbf{G}_{-ij}) > \pi_i^*(\mathbf{G}_{-ij}).$$

□

Proof of Proposition 3. Start from a fully connected network $\mathbf{G}$ such that $\mathbf{G}_{ij} = 1$ for all $i, j \in \mathcal{N}$ with $i \neq j$. For this network to be pairwise stable, there must not be any profitable linking deviations. In the complete case, any deviation involves deleting a link. Without loss of generality, consider deleting $G_{1,2}$ by setting it equal to 0.

First, solve the system of equations in (7). Since $\forall i,j \in \mathcal{N}: G_{ij} = 1$, we have

$$\left[\frac{1}{\theta_i}\frac{(n+1)^2\phi}{n-d_i(\mathbf{G})} - (n+1)\theta_i\right] e_i = (\alpha - \bar{c}) + \sum_{j=1}^{n} \theta_j e_j \Big[(n+1) - (1+\delta d_j(\mathbf{G}))\Big].$$

Using the fact that in a complete network $d_i(\mathbf{G}^C) = n-1, \forall i \in \mathcal{N}$, we obtain

$$e_i^*(\mathbf{G}^C) = (\alpha - \bar{c}) \cdot \frac{\theta_i}{(n+1)^2\,\phi - \sum_{k\in\mathcal{N}}\theta_k^2}, \quad \forall i \in \mathcal{N}. \tag{15}$$

By deviating from the complete network and removing the link between agents i and j, we obtain

$$e_i^*(\mathbf{G}^C_{-ij}) = (\alpha - \bar{c}) \cdot \frac{\lambda}{\big[1-(1+\Delta)\,Q\big]\cdot\big[b_i\,\lambda - (n-1)\,\theta_j\big]}, \tag{16}$$

where

$$\lambda = \frac{\theta_i}{\theta_j}\cdot\frac{(n+1)^2\phi - 2\,\theta_j^2}{(n+1)^2\phi - 2\,\theta_i^2}, \qquad \Delta = \frac{2\,(\lambda\,\theta_i + \theta_j)}{b_i\,\lambda - (n-1)\theta_j},$$

$$Q = \sum_{k\in\mathcal{N}/\{i,j\}} \frac{\theta_k^2}{(n+1)^2\,\phi}, \qquad b_i = \frac{(n+1)^2\,\phi}{2\,\theta_i} - 2\,\theta_i.$$

Using the equilibrium profit in Equation (9), the profit for firm i under full connectivity is

$$\pi_i^*(\mathbf{G}^C) = \left[\frac{\phi}{\theta_i^2}\,(n+1)^2 - 1\right]\phi\left((\alpha-\bar{c})\cdot\frac{\theta_i}{(n+1)^2\,\phi - \sum_{k\in\mathcal{N}}\theta_k^2}\right)^2, \tag{17}$$

while after deleting the link (i,j) it becomes

$$\pi_i^*(\mathbf{G}^C_{-ij}) = \left[\frac{\phi}{\theta_i^2}\left(\frac{n+1}{2}\right)^2 - 1\right]\phi\left((\alpha-\bar{c})\cdot\frac{\lambda}{\big[1-(1+\Delta)\,Q\big]\cdot\big[b_i\,\lambda-(n-1)\,\theta_j\big]}\right)^2. \tag{18}$$

Assume that θ_i represents the productivity of the most productive firm, i.e. $\theta_i = 1$, and that θ_j is the least productive firm. In the limit $\theta_j \to 1$, all firms are homogeneous with $\theta_{\min} = \theta_{\max}$. This corresponds to the case in Goyal and Moraga-Gonzalez (2001), where it is shown that deviation from a complete network is not beneficial:

$$\lim_{\theta_j \to 1} \pi_i^*\big(\mathbf{G}^C\big) > \lim_{\theta_j \to 1} \pi_i^*\big(\mathbf{G}^C_{-ij}\big).$$

Now consider the opposite case $\theta_j \to 0$. We obtain

$$\lim_{\theta_j \to 0} \pi_i^*\big(\mathbf{G}^C\big) = \underbrace{(\alpha - \bar{c})^2}_{\text{constant}} \underbrace{\left[\phi(n+1)^2 - 1\right]\phi}_{\text{multiplier}} \underbrace{\left(\frac{1}{(n+1)^2\phi - \sum_{j=3}^{n} \theta_j^2}\right)^2}_{\text{target}},$$

$$\lim_{\theta_j \to 0} \lambda = \infty, \qquad \lim_{\theta_j \to 0} \Delta = \frac{4}{(n+1)^2\phi - 4}.$$

Therefore,

$$\lim_{\theta_j \to 0} \pi_i^*\big(\mathbf{G}^C_{-ij}\big) = \underbrace{(\alpha - \bar{c})^2}_{\text{constant}} \underbrace{\left[\phi\left(\tfrac{n+1}{2}\right)^2 - 1\right]\phi}_{\text{multiplier}} \underbrace{\left(\frac{2}{(n+1)^2\phi - \sum_{j=3}^{n} \theta_j^2}\right)^2}_{\text{target}}.$$

Comparing the two limits yields

$$\lim_{\theta_j \to 0} \pi_i^*\big(\mathbf{G}^C_{-ij}\big) > \lim_{\theta_j \to 0} \pi_i^*\big(\mathbf{G}^C\big).$$

Define

$$r(\theta_j) := \frac{\pi_i^*\big(\mathbf{G}^C\big)}{\pi_i^*\big(\mathbf{G}^C_{-ij}\big)}.$$

We have shown that

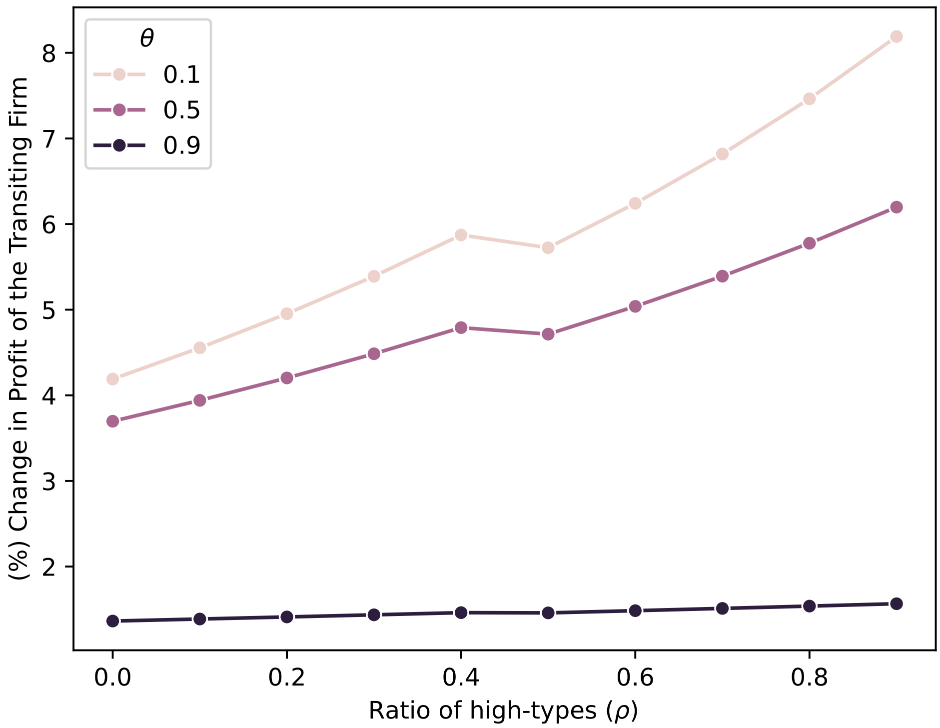
$$\lim_{\theta_j \to 0} r(\theta_j) < 1 \quad \text{and} \quad \lim_{\theta_j \to 1} r(\theta_j) > 1.$$

By continuity and monotonicity of $r(\theta_j)$, there exists a threshold $\bar{\theta}_2 \in (0,1)$ such that $r(\bar{\theta}_2) = 1$. Hence, for $\theta_j < \bar{\theta}_2$, deviation from the fully connected network is profitable, and the complete network is not pairwise stable.

□

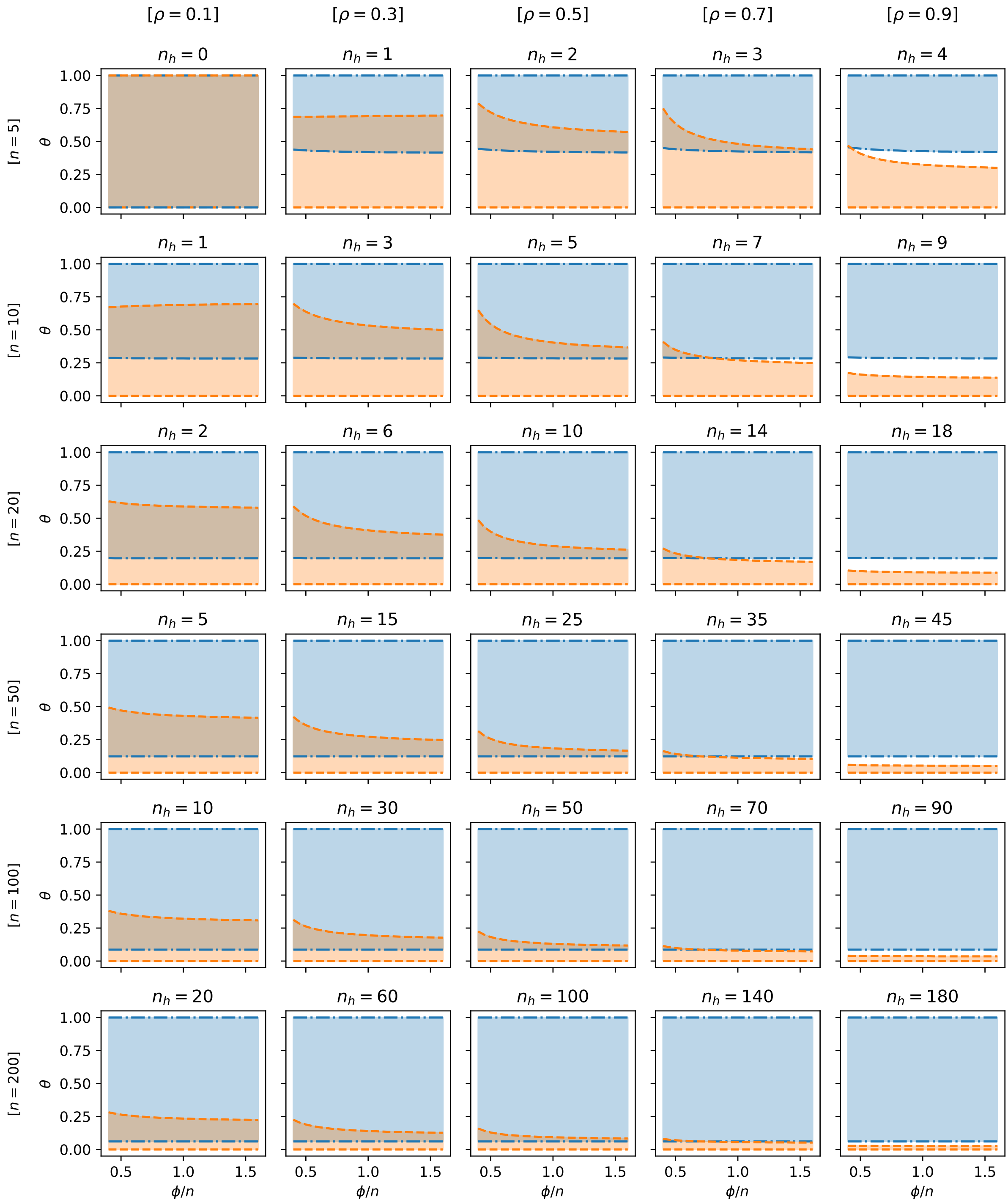
B Additional Simulation Results

Figure 8: Profit impact of a productivity upgrade under a fixed two-clique network



Note: This figure isolates the effect of changing the productivity composition while holding the network fixed. We consider $n = 10$ firms partitioned into two fully connected components (two cliques of size 5), mirroring the connectivity pattern of the Positive Assortative (PA) network at $\rho = \frac{1}{2}$. Starting from $\rho = 0$ (all firms have low productivity θ), we increase ρ in increments of 0.1 by upgrading one firm at a time from θ to 1 (high productivity). At each step, we record the change in profit for the *transitioning* firm (the firm whose productivity is upgraded in that step), holding all other firms' productivities and the network links fixed. The x-axis reports ρ (the share of high-productivity firms), and the y-axis reports the corresponding profit change of the transitioning firm. Curves are shown for three values of the low productivity parameter, $\theta \in \{0.1, 0.5, 0.9\}$. Two patterns emerge. First, the profit gain from upgrading is larger when θ is smaller, consistent with a larger productivity jump from θ to 1. Second, the profit gain is generally increasing in ρ, except around $\rho = \frac{1}{2}$, where the fixed network aligns with exact type clustering (all high types concentrated within one clique and all low types within the other); upgrading a low-type firm within the low-type clique yields a smaller marginal benefit than upgrading the remaining low-type firm in the high-type clique. In all cases the profit change remains positive. Finally, note that this fixed-network exercise is not, in general, a stability analysis: the imposed two-clique network need not be pairwise stable for the corresponding productivity composition, except at $\rho = \frac{1}{2}$ (and sufficiently low θ).

Figure 9: Stability Regions of PA and FC Networks in Large n Settings



Note: Each row corresponds to a different number of firms, ranging from $n = 5$ at the top to $n = 200$ at the bottom. Each column corresponds to a different proportion of high-productivity firms (ρ), increasing from 10% (left) to 90% (right). Orange regions indicate parameter combinations where the PA network is stable; blue regions indicate where the FC network is stable.