ME570 Assignment 2

Zili Wang
zw2445@bu.edu

September 29, 2022

Contents

1 Question 1.1

Figure 1 gives the demonstration of rotation matrices, where black axes are the original system, and red dashed axes are the rotated system. The positive direction is defined by right hand rule, which is in an anticlockwise rotation around an axis from observer point of view.

(i) $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
As shown in Figure 1a, it represents rotating points anticlockwise by θ.
Specifically, the first column gives the coordinate in the original system by rotating the unit vector $e_1 = [1;\ 0]$ on x-axis anticlockwise by θ. And the second column gives the coordinate in the original system by rotating the unit vector $e_2 = [0;\ 1]$ on y-axis anticlockwise by θ.

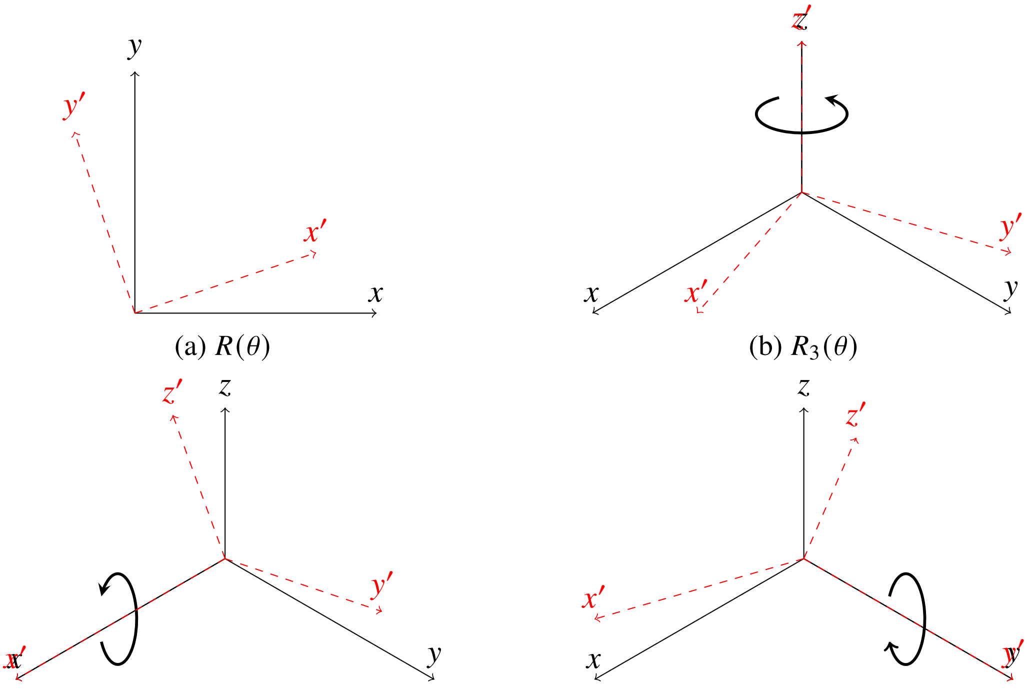
The 2D example gives a sense of rotations. In the 3D cases, each column is the result of rotating the unit vector on x-axis, y-axis, and z-axis respectively.

(ii) $R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$
As shown in Figure 1c, it represents rotating points around x-axis anticlockwise by θ.

(iii) $R_2(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$
As shown in Figure 1d, it represents rotating points around y-axis clockwise by θ.

(iv) $R_3(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
As shown in Figure 1b, it represents rotating points around z-axis anticlockwise by θ. In 2D, it is the same as Figure1a.

(v) $R_4(\theta) = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
It gives an opposite result to $R_3(\theta)$, which means it rotates the points around the z-axis anticlockwise by $(\pi + \theta)$.



(a) $R(\theta)$ (b) $R_3(\theta)$ (c) $R_1(\theta)$ (d) $R_2(\theta)$

Figure 1: Rotation matrices demo

2 Question 2.1

(i) ${}^{A}R_{B}$ means a rotation of reference frame B with respect to reference frame A.

$$
\begin{aligned}
{}^{W}p &= {}^{W}R_{B_1}\ {}^{B_1}p + {}^{W}T_{B_1} \\
&\downarrow \theta_1,\ {}^{W}T_{B_1} = 0,\ {}^{B_1}p = [{}^{B_1}p_x; {}^{B_1}p_y] \\
{}^{W}p &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} {}^{B_1}p \\
&= R(\theta_1)\ {}^{B_1}p \\
&= \begin{bmatrix} \cos(\theta_1)\ {}^{B_1}p_x - \sin(\theta_1)\ {}^{B_1}p_y \\ \sin(\theta_1)\ {}^{B_1}p_x + \cos(\theta_1)\ {}^{B_1}p_y \end{bmatrix}
\end{aligned}
$$

(ii)

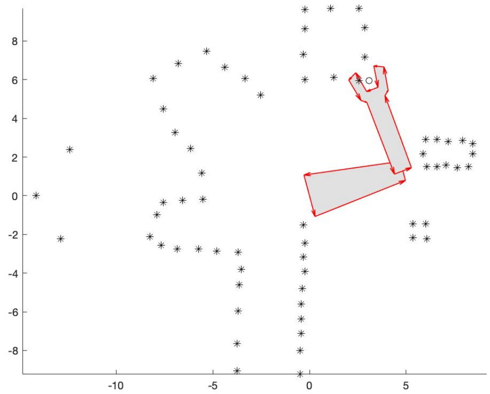
$$
\begin{aligned}
{}^{W}p &= {}^{W}R_{B_1}\ {}^{B_1}p + {}^{W}T_{B_1} \\
&= {}^{W}R_{B_1}\ ({}^{B_1}R_{B_2}\ {}^{B_2}p + {}^{B_1}T_{B_2}) + {}^{W}T_{B_1} \\
&= {}^{W}R_{B_1}\ {}^{B_1}R_{B_2}\ {}^{B_2}p + {}^{W}R_{B_1}\ {}^{B_1}T_{B_2} + {}^{W}T_{B_1} \\
&\downarrow \theta_1,\ \theta_2,\ {}^{W}T_{B_1} = 0,\ {}^{B_2}T_{B_1} = [5;0] \\
{}^{W}p &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}\ {}^{B_2}p \\
&\quad + \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
&\downarrow \text{trigonometric transformation formulas, } {}^{B_2}p = [{}^{B_2}p_x; {}^{B_2}p_y] \\
{}^{W}p &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}\ {}^{B_2}p + \begin{bmatrix} 5\cos(\theta_1) \\ 5\sin(\theta_1) \end{bmatrix} \\
&= R(\theta_1+\theta_2)\ {}^{B_2}p + R(\theta_1) \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2)\ {}^{B_2}p_x - \sin(\theta_1+\theta_2)\ {}^{B_2}p_y + 5\cos(\theta_1) \\ \sin(\theta_1+\theta_2)\ {}^{B_2}p_x + \cos(\theta_1+\theta_2)\ {}^{B_2}p_y + 5\sin(\theta_1) \end{bmatrix}
\end{aligned}
$$

3 Question 2.2

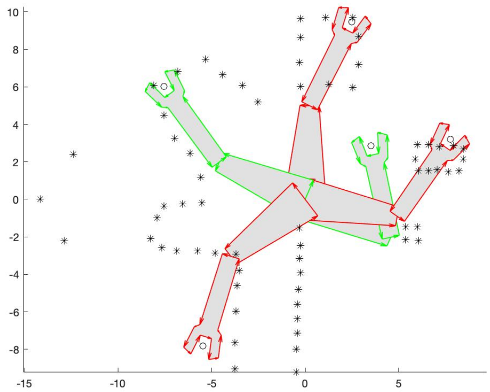
Code reference:

- rot2d.m
- twolink_polygon.m
- twolink_kinematicMap.m
- twolink_plotCollision.m
- twolink_plotCollision_test.m
- Homework 1

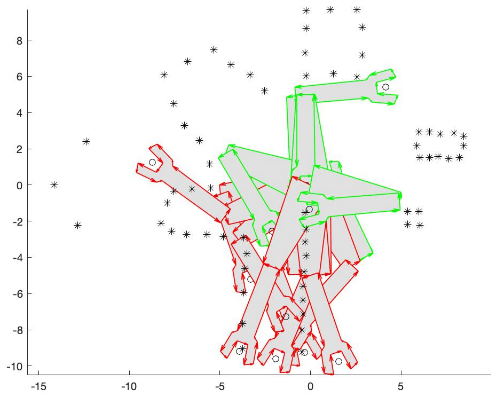
The result of twolink_plotCollision_test() is shown in Figure 2. Figure 2a, 2b, 2c, and 2d shows the result of link-obstacle collision test for 1, 5, 15, and 30 random configurations of two-link manipulator respectively. The black asterisks are the obstacle points, the red outline of links means a collision, and the green outline of links represent no collision.



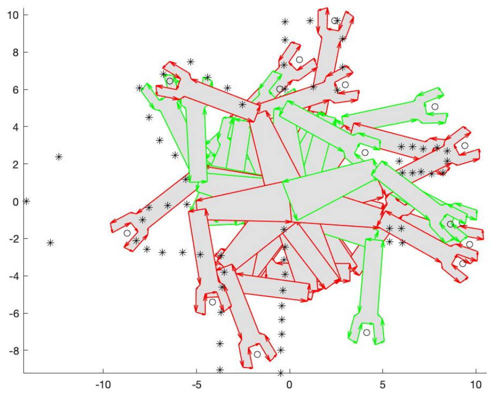
(a) 1 configuration



(b) 5 configurations



(c) 15 configurations



(d) 30 configurations

Figure 2: Two-link Manipulator Collision Test

4 Question (optional) 2.1

Code reference:

- twolink_freeSpace.m

Figure 3 shows the collision test result from the configurations $[\theta_1; \theta_2]$, where each angle linearly varies from 0 to 2π, with an interval of 2 degrees. The number of test configurations is (180×180). value one means there is a collision, and zero means no collision.

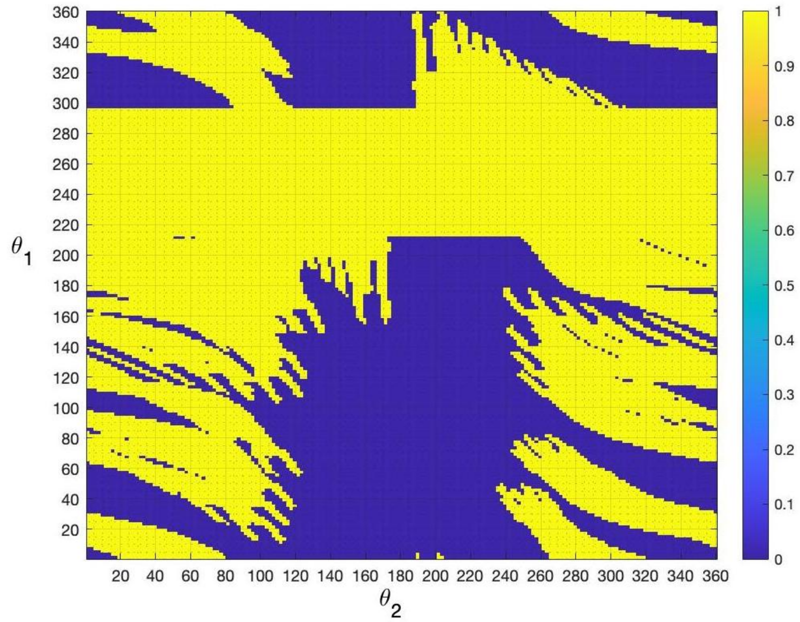


Figure 3: Configuration Space Collision Test

5 Question 4.1

As shown in Figure 1a, for $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, the first column represents rotating a unit vector on x to x', and the second column represents rotating a unit vector on y to y'. Therefore, it represents the rotations of a point from the origin. Mathematically,

(1) $R \in \mathbb{R}^{2\times 2}$

(2) $R(\theta)^T R(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3) $det(R(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$

So $R(\theta) \in SO(2)$.

$\phi_{circle}(\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, it represents a point on the unit circle which has the center at the origin ($\cos^2(\theta) + \sin^2(\theta) = 1$). So $\phi_{circle}(\theta) \in \mathbb{S}^1$ for all $\theta \in \mathbb{R}$.

6 Question (optional) 4.1

To cover the circle, two charts are needed as with a single chart, since it is open, there is a discontinuity at 2π and not all the space is covered. When using two open charts, the overlap can remove the discontinuity, and covers the entire space.

7 Question (optional) 4.2

Figure 4 shows ϕ_{circle} in $U_1 = (-\frac{3}{4}\pi; \frac{3}{4}\pi)$ with red circle, and in $U_2 = (\frac{1}{4}\pi; \frac{7}{4}\pi)$ with green cross. There is an overlap region for two intervals, when gives a closed space.

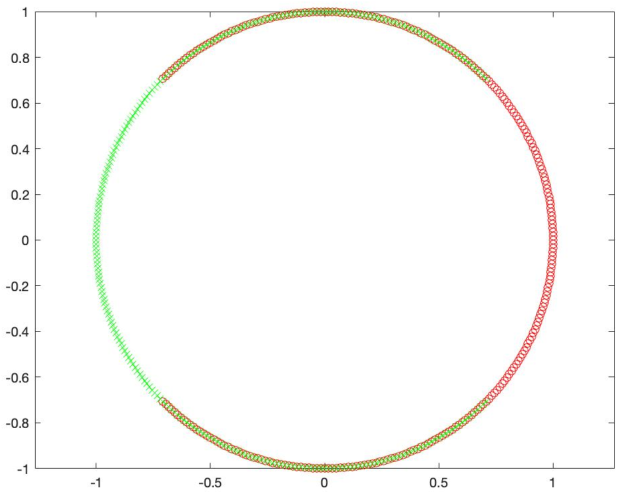


Figure 4: ϕ_{circle} in U_1 and U_2

8 Question 5.1

- chart_plot3.m

As shown in Figure 5, three regions in (a) (b) (c) can cover the entire flat representation of torus as in (d). Since for the square region, the opposite sides are glued together, the charts could be:

$$
\begin{aligned}
U_1 &= (0.2, 2\pi - 0.2) \times (0.2, 2\pi - 0.2)\\
U_2 &= \{[0, 0.3) \cup (0.5, 2\pi]\} \times \{[0, 0.5) \cup (2\pi - 0.4, 2\pi]\}\\
&= (0.5, 2\pi + 0.3) \times (-0.4, 0.5)\\
U_3 &= \{[0, 0.6) \cup (2\pi - 0.4, 2\pi]\} \times \{[0, 0.3) \cup (0.4, 2\pi]\}\\
&= (-0.4, 0.6) \times (0.4, 2\pi + 0.3)
\end{aligned}
$$

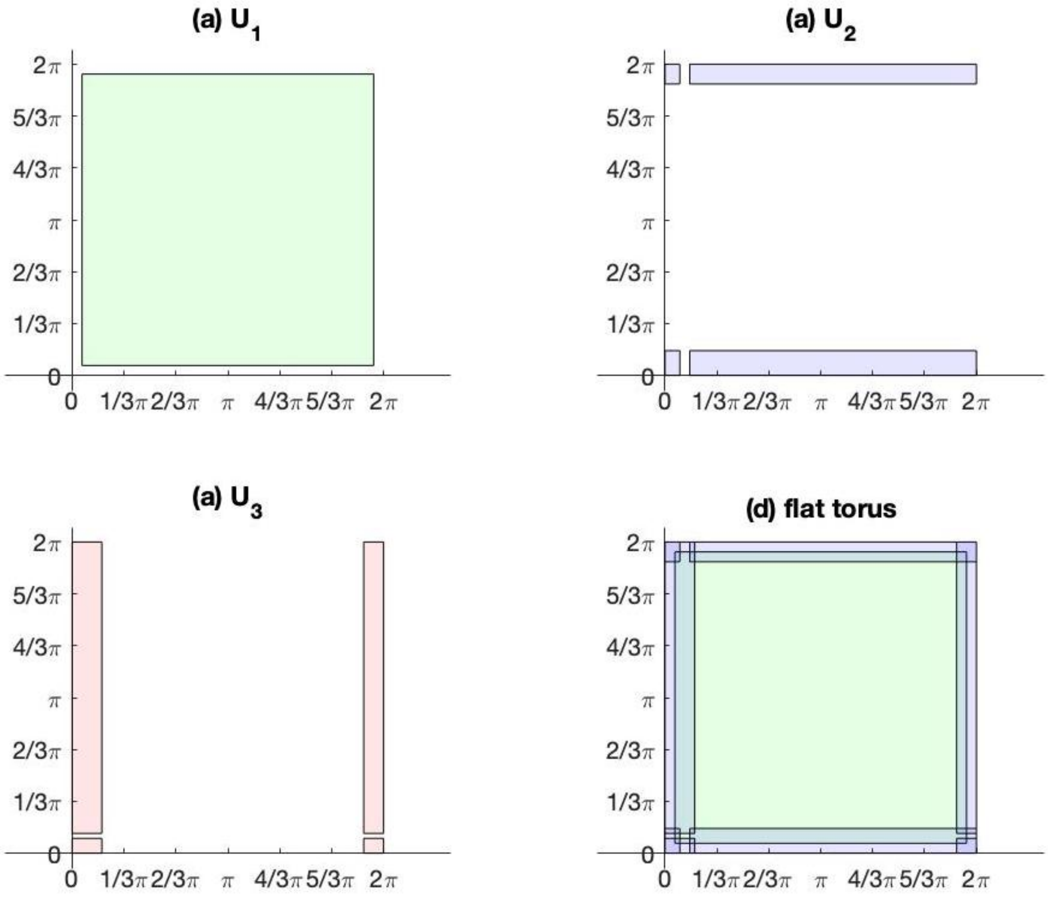


Figure 5: Torus Charts

9 Question 5.2

Figure 6 shows the results of transforming 2D angles to 3D euclidean space. U_1,U_2,and U_3 are incomplete region of the torus, while the union gives the complete torus.

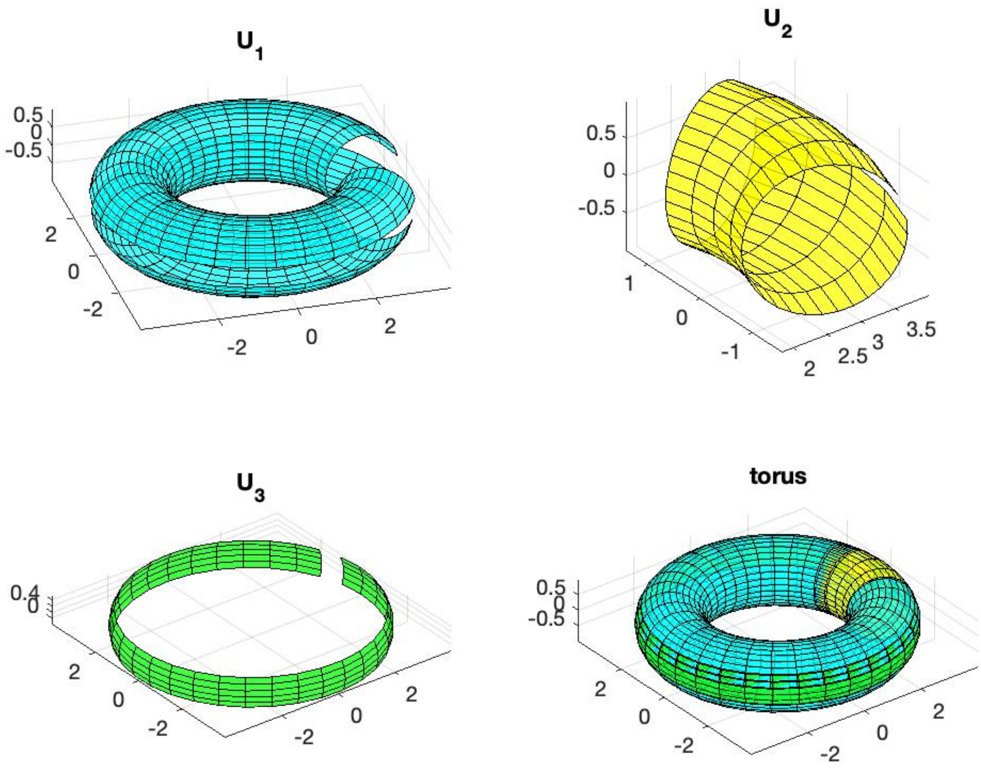


Figure 6: Torus in Chart Colors

10 Question 5.3

Since a chart is a homeomorphis from an open region in $\mathbb{R}^2$ to the topological space. There is no overlap in both x-axis and y-axis in a chart, so there is no same colored part overlap in the topological space. An atlas is a collection of charts of which the union covers the entire topological space, so the union of all the transform can give a complete torus.

11 Question 5.4

Given $\vec{\theta}(t) = \begin{bmatrix} a(1) \cdot t + b(1) \\ a(2) \cdot t + b(2) \end{bmatrix}$, $\dot{\theta}(t) = \begin{bmatrix} a(1) \\ a(2) \end{bmatrix}$

12 Question 5.5

- torus_phiPushCurve.m
- torus_plotChartsCurves.m

Figure 7 shows the output of torus_plotChartsCurves() . While the curve color for each tangent is:

- $a = [3/4\pi; 0]$: blue curve
- $a = [3/4\pi; 3/4\pi]$: black curve
- $a = [-3/4\pi; 3/4\pi]$: magenta curve
- $a = [0; 3/4\pi]$: red curve

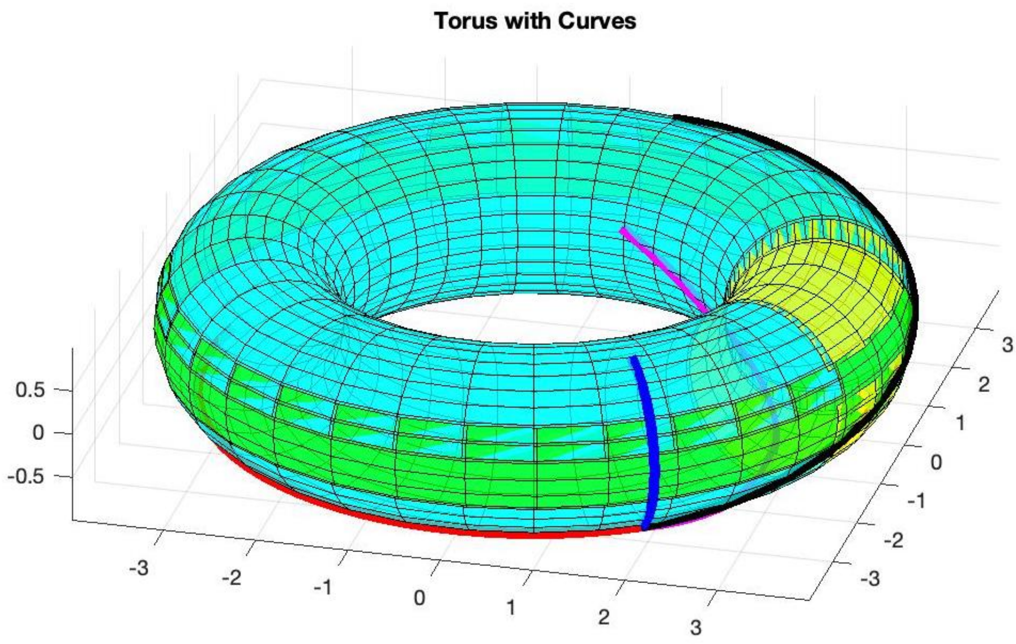


Figure 7: Configuration Curves on Torus

13 Question 6.1

As shown at the end of Question 2.1(ii),

$$
\begin{aligned}
{}^{W}p &= \begin{bmatrix} \cos(\theta_1+\theta_2)\ {}^{B_2}p_x - \sin(\theta_1+\theta_2)\ {}^{B_2}p_y + 5\cos(\theta_1) \\ \sin(\theta_1+\theta_2)\ {}^{B_2}p_x + \cos(\theta_1+\theta_2)\ {}^{B_2}p_y + 5\sin(\theta_1) \end{bmatrix} \\
&\downarrow\ {}^{B_2}p = [{}^{B_2}p_x; {}^{B_2}p_y] = {}^{B_2}p_{eff} = [5;0] \\
{}^{W}p_{eff} &= \begin{bmatrix} 5\cos(\theta_1+\theta_2) + 5\cos(\theta_1) \\ 5\sin(\theta_1+\theta_2) + 5\sin(\theta_1) \end{bmatrix} \\
\frac{d}{dt}({}^{W}p_{eff}) &= \begin{bmatrix} \frac{dp_x}{d\theta_1}\frac{d\theta_1}{dt} + \frac{dp_x}{d\theta_2}\frac{d\theta_2}{dt} \\ \frac{dp_y}{d\theta_1}\frac{d\theta_1}{dt} + \frac{dp_x}{d\theta_2}\frac{d\theta_2}{dt} \end{bmatrix} \\
&= \begin{bmatrix} -5\sin(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) - 5\sin(\theta_1)\dot{\theta_1} \\ 5\cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) + 5\cos(\theta_1)\dot{\theta_1} \end{bmatrix} \\
&= \begin{bmatrix} -5\sin(\theta_1+\theta_2) - 5\sin(\theta_1) & -5\sin(\theta_1+\theta_2) \\ 5\cos(\theta_1+\theta_2) + 5\cos(\theta_1) & 5\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta_1} \\ \dot{\theta_2} \end{bmatrix} \\
J &= \begin{bmatrix} -5\sin(\theta_1+\theta_2) - 5\sin(\theta_1) & -5\sin(\theta_1+\theta_2) \\ 5\cos(\theta_1+\theta_2) + 5\cos(\theta_1) & 5\cos(\theta_1+\theta_2) \end{bmatrix}
\end{aligned}
$$

Another way to think is:

$$
\begin{aligned}
{}^{W}p_{eff} &= R(\theta_1+\theta_2)\ {}^{B_2}p_{eff} + R(\theta_1)\begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
\frac{d}{dt}(R(\theta_1+\theta_2)) &= \begin{bmatrix} 0 & -(\dot{\theta_1}+\dot{\theta_2}) \\ (\dot{\theta_1}+\dot{\theta_2}) & 0 \end{bmatrix} R(\theta_1+\theta_2) \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R(\theta_1+\theta_2)\ (\dot{\theta_1}+\dot{\theta_2}) \\
\frac{d}{dt}(R(\theta_1)) &= = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R(\theta_1)\ \dot{\theta_1} \\
\frac{d}{dt}({}^{W}p_{eff}) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \{[R(\theta_1+\theta_2) + R(\theta_1)]\dot{\theta_1} + R(\theta_1+\theta_2)\dot{\theta_2}\} \begin{bmatrix} 5 \\ 0 \end{bmatrix}
\end{aligned}
$$

which gives the same answer.

14 Question 6.2

Code reference:

- twolink_jacobian.m
- twolink_plot.m
- twolink_jacobian_plot.m

(i) $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \dot{\theta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 0 & 0 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$
- $\theta = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} -5 & -5 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$
- $\theta = \begin{bmatrix} \pi \\ \frac{\pi}{2} \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 5 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$
- $\theta = \begin{bmatrix} \pi \\ \pi \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii) $a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \dot{\theta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 0 & 0 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$
- $\theta = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} -5 & -5 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$
- $\theta = \begin{bmatrix} \pi \\ \frac{\pi}{2} \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 5 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$
- $\theta = \begin{bmatrix} \pi \\ \pi \end{bmatrix}, \frac{d}{dt}({}^W p_{eff}) = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

The results are plotted as Figure 8. The first row (case 1-4) is with the configurations of $a = [1; 0]$, and the second row (case 5-8) is with the configurations of $a = [0; 1]$. As we could see from the equation, the first row is associated with the first column of the Jacobian. And the second row is associated with second column of the Jacobian, which is always perpendicular to the second link and with same magnitude.

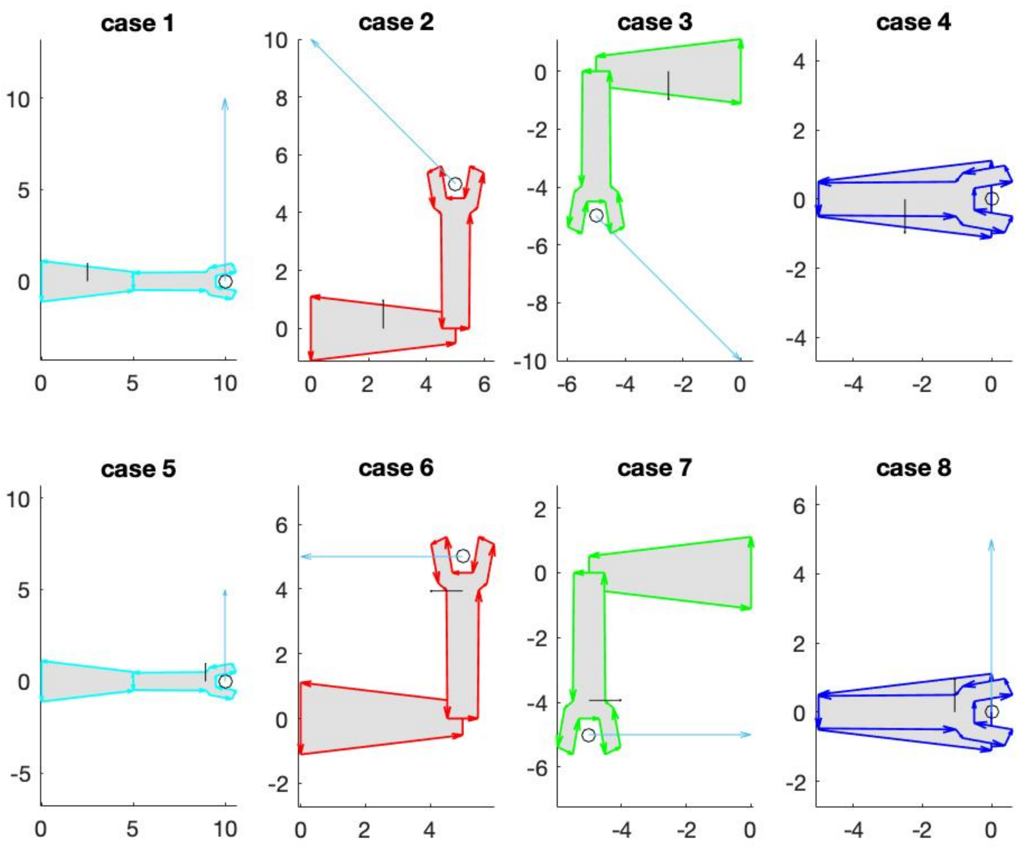


Figure 8: Tangents of Different Manipulator Configurations

15 Question 6.3

Code reference:

- torus_twokink_plotJacobian.m
- line_space.m
- twolink_plot.m

Figure 9 shows the result of torus_twokink_plotJacobian(), with the color consistent as in Figure 7.

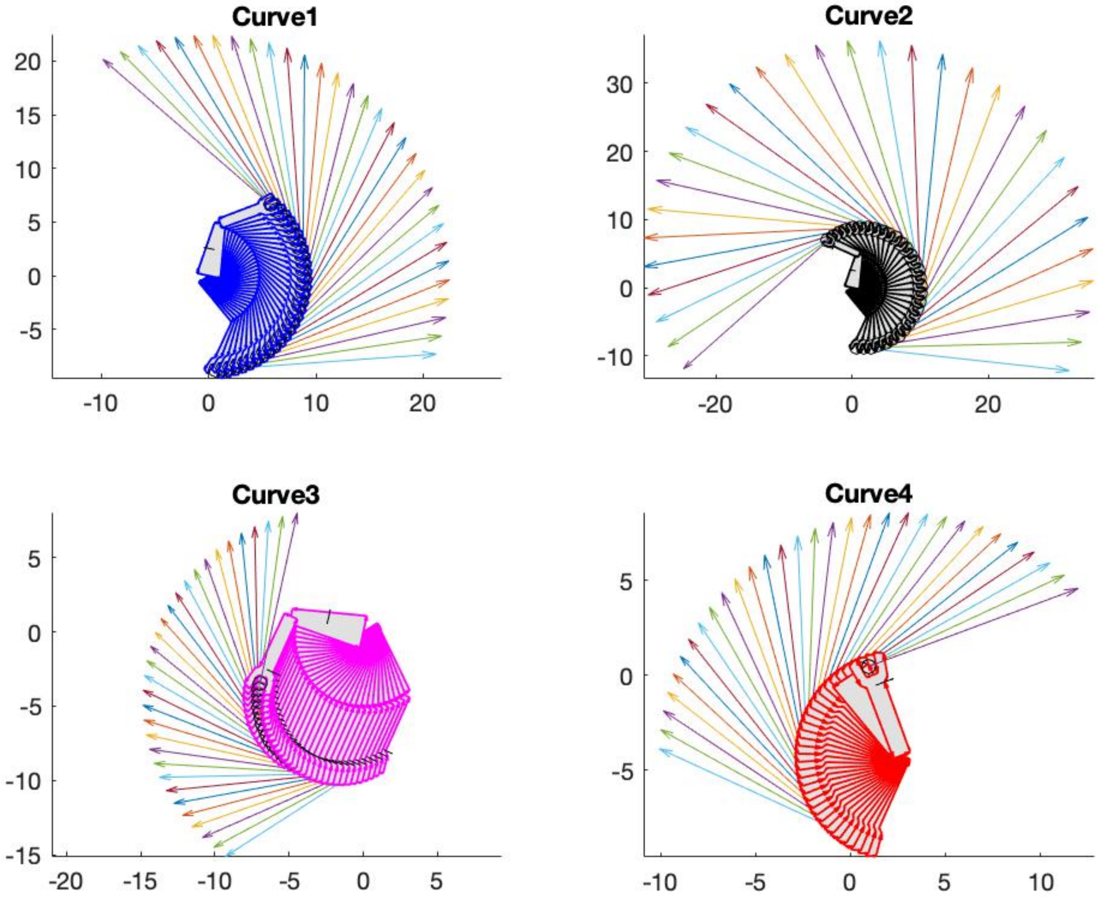


Figure 9: Tangents of Different Manipulator Configurations (Curves)

16 Question 6.4

For the four curves, Figure 7 shows them on the torus, while Figure 9 shows then on the 2D plane with the tangent of end effector.

- For curve 1 (in blue), only θ_1 is time-variant, since the derivative of θ_1 is positive, $(\dot{\theta}_1 + \dot{\theta}_2 = 3/4\pi > 0)$, it gives an anticlockwise rotation. This is visualized on the torus, where the curve starts at the bottom and grows around the cross plane circle anticlockwise.

- For curve 2 (in black), both θ_1 and θ_2 are time-variant, since the derivatives of both are positive, $(\dot{\theta}_1 + \dot{\theta}_2 = 3/2\pi > 0)$,it gives an anticlockwise and steeper rotation (compared to curve 1). This is visualized on the torus, where the curve starts at the bottom and grows around the cross plane circle as well as the flat plane circle anticlockwise.

- For curve 3 (in magenta), both θ_1 and θ_2 are time-variant, since the derivative of θ_1 is negative and of θ_2 is positive, ($\dot{\theta}_1 + \dot{\theta}_2 = 0$), it gives a clockwise rotation of link1 and constant move of link2. This is visualized on the torus, where the curve starts at the bottom and grows around the cross plane circle clockwise and the flat plane circle anticlockwise.

- For curve 4 (in red), only θ_2 is time-variant, since the derivative of θ_2 is negative, ($\dot{\theta}_1 + \dot{\theta}_2 = -3/4\pi < 0$), it gives a clockwise rotation of link while link1 remains static. This is visualized on the torus, where the curve starts at the bottom and grows around the flat plane circle clockwise.

17 Question 7.1

Estimate of hours taken to complete the homework:

- Programming and Debugs: 4-5 hours
- Report: 2 hours
- Total: 7-8hours

The hard part is with the understanding of tangents in 2D and 3D space.