1.1_Introduction to Systems of Linear Equations

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ (b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$ (d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$ (f) $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

Solution:

a) The equation is linear in $x_1, x_2, and\ x_3$ because power of the variables in each term is 1.
b) The equation is not linear in $x_1, x_2, and\ x_3$ because power of the variables in 3rd term is 2.
c) The equation is linear in $x_1, x_2, and\ x_3$ because power of the variables in each term is 1.
d) The equation is not linear in $x_1, x_2, and\ x_3$ because of the 1st term x_1^{-2}.
e) The equation is not linear in $x_1, x_2, and\ x_3$ because of the 1st term $x_1^{\frac{3}{5}}$.
f) The equation is linear in $x_1\ and\ x_2$ because power of the variables in each term is 1.

In each part of Exercises 5–6, find a linear system in the unknowns $x_1, x_2, x_3, \ldots$, that corresponds to the given augmented matrix.

5. (a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

6. (a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

Solution:

5. b) The linear system in the unknowns $x_1, x_2, x_3 - - - -$ that corresponds to te given augmented matrix is

$$\begin{aligned} 3x_1 \qquad - 2x_3 &= 5 \\ 7x_1 + x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 7 \end{aligned}$$

Solution:

6. b) The linear system in the unknowns $x_1, x_2, x_3 ----$ that corresponds to te given augmented matrix is

$$\begin{aligned} 3x_1 \quad\quad + x_3 - 4x_4 &= 3 \\ -4x_1 \quad\quad + 4x_3 + x_4 &= -3 \\ -x_1 + 3x_2 - 2x_4 \quad &= -9 \\ -x_4 \quad &= -2 \end{aligned}$$

7. a) Find the augmented matrix for the linear system

$$-2x_1 = 6$$
$$3x_1 = 8$$
$$9x_1 = -3$$

Solution:

Given linear system is

$$\left.\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}\right\} ------(i)$$

Augmented matrix for the linear system (i) is

$$\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$$

7. b) Find the augmented matrix for the linear system

$$6x_1 - x_2 + 3x_3 = 4$$
$$5x_2 - x_3 = 1$$

Solution:

Given linear system is

$$\left.\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}\right\} \text{---------------(i)}$$

Augmented matrix for the linear system (i) is

$$\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$$

7. c) Find the augmented matrix for the linear system

$$\begin{aligned} 2x_2 \quad\quad - 3x_4 + \ x_5 \ &= 0 \\ -3x_1 - x_2 + \ x_3 \quad\quad\quad\quad &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

Solution: Given linear system is

$$\left.\begin{array}{r} 2x_2 \qquad - 3x_4 + \ x_5 = 0 \\ -3x_1 - x_2 + \ x_3 \qquad\qquad = -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 \ = 6 \end{array}\right\} - - - - - - - - (i)$$

Augmented matrix for the linear system (i) is

$$\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$$

8. c) Find the augmented matrix for the linear system

$$\begin{array}{l} x_1 \qquad\quad = 1 \\ \quad x_2 \qquad = 1 \\ \qquad x_3 \qquad = 1 \end{array}$$

Solution: Given linear system is

$$\left.\begin{array}{r} x_1 \qquad\quad = 1 \\ x_2 \qquad = 1 \\ x_3 = 1 \end{array}\right\} - - - - - - - - (i)$$

Augmented matrix for the linear system (i) is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

10. a) Determine whether the 3-tuple $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ is a solution of the linear system

$\boldsymbol{x + 2y - 2z = 3}$
$\boldsymbol{3x - y + z = 1}$
$\boldsymbol{-x + 5y - 5z = 5}$

Solution:

Given linear system is

$$\left.\begin{array}{r} x + 2y - 2z = 3 \\ 3x - y + z = 1 \\ -x + 5y - 5z = 5 \end{array}\right\} - - - - - - (i)$$

Satisfying system (i) with the 3-tuple $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ we get,

$$\frac{5}{7} + 2 \times \frac{8}{7} - 2 \times 1 = 3$$
$$3 \times \frac{5}{7} - \frac{8}{7} + 1 = 1$$
$$-\frac{5}{7} + 5 \times \frac{8}{7} - 5 \times 1 = 5$$

Calculating we get,

$1 = 3$
$2 = 1$
$0 = 5$
Which is impossible.

Hence, the given 3-tuple $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ is not a solution of the linear system (i).

10. e) Determine whether the 3-tuple $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$is a solution of the linear system

$\boldsymbol{x + 2y - 2z = 3}$
$\boldsymbol{3x - y + z = 1}$
$\boldsymbol{-x + 5y - 5z = 5}$
Solution:

Solution:

Given linear system is

$$\left.\begin{array}{r} x + 2y - 2z = 3 \\ 3x - y + z = 1 \\ -x + 5y - 5z = 5 \end{array}\right\} ------(i)$$

Satisfying system (i) with the 3-tuple $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$ we get,

$$\frac{5}{7} + 2 \times \frac{22}{7} - 2 \times 2 = 3$$
$$3 \times \frac{5}{7} - \frac{22}{7} + 2 = 1$$
$$-\frac{5}{7} + 5 \times \frac{22}{7} - 5 \times 2 = 5$$
Calculating we get,

$3 = 3$
$1 = 1$
$5 = 5$
The given 3-tuple $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$ satisfies all the equations of the system.

Hence, the 3-tuple $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ is a solution of the linear system (i).

11. In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

(a) $3x - 2y = 4$
$6x - 4y = 9$

(b) $2x - 4y = 1$
$4x - 8y = 2$

(c) $x - 2y = 0$
$x - 4y = 8$

Solution:

11a) Given linear system is

$3x - 2y = 4 - - - -(i)$

$6x - 4y = 9 - - - -(ii)$

Applying $(ii) - 2 \times (i)$ in equation (ii) we get,

$3x - 2y = 4$

$0 = 1$, which is impossible

Hence the system of equations has zero solution.

11b) Given linear system is

$2x - 4y = 1 - - - -(i)$

$4x - 8y = 2 - - - -(ii)$

Applying $(ii) - 2 \times (i)$ in equation (ii) we get,

$2x - 4y = 1$

$0 = 0$

or, $2x - 4y = 1$

or, $x = \frac{4y+1}{2}$

or, $x = 2y + \frac{1}{2}$

Here, the system has 1 equation with two unknowns. Hence it has infinitely many solutions.

The parametric equations of this system of equations are

$$y = t, x = 2t + \frac{1}{2}$$

In each part of Exercises 13–14, use parametric equations to describe the solution set of the linear equation.

13. (a) $7x - 5y = 3$

(b) $3x_1 - 5x_2 + 4x_3 = 7$

(c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

(d) $3v - 8w + 2x - y + 4z = 0$

14. (a) $x + 10y = 2$

(b) $x_1 + 3x_2 - 12x_3 = 3$

(c) $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

(d) $v + w + x - 5y + 7z = 0$

Solution:

13. a) Given equation is

$7x - 5y = 3 \qquad or,\ x = \frac{3+5y}{7} \qquad$ or, $x = \frac{3}{7} + \frac{5}{7}y$

The parametric equations of the solution set is

$$y = t, \qquad x = \frac{3}{7} + \frac{5}{7}t$$

d) Given equation is

$3v - 8w + 2x - y + 4z = 0$

$or, 3v = 8w - 2x + y - 4z$

$$or, v = \frac{8}{3}w - \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3}z$$

The parametric equations of the solution set is

$or, v = \frac{8}{3}t_1 - \frac{2}{3}t_2 + \frac{1}{3}t_3 - \frac{4}{3}t_4\ ; \quad w = t_1,\ x = t_2,\ y = t_3,\ z = t_4$

16. The following linear system of equations has infinitely many solutions. Use parametric equations to describe its solution set.

b)

$\mathbf{2x - y + 2z = -4}$

$\mathbf{6x - 3y + 6z = -12}$

$\mathbf{-4x + 2y - 4z = 8}$

Solution:

Given system is

$$\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$$

Applying $(ii) - 3 \times (i)$ and $(iii) + 2 \times (i)$ the system becomes

$2x - y + 2z = -4$

$0 = 0$

$0 = 0$

$or, \qquad x = -2 + \frac{1}{2}y - z$

Here, the system has 1 equation with two unknowns. Hence it has infinitely many solutions.

The parametric equations of this system of equations are

$x = -2 + \frac{1}{2}r - s \; ; y = r, \;\; z = s$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

17. (a) $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

18. (a) $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

Solution:

17a) Given augmented matrix is

$$\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

Applying $R_1' = R_1 + 2R_2$ we get

$$\begin{bmatrix} 1 & -7 & 8 & 8 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

This is the required matrix.

17. b) Given augmented matrix is

$$\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$$

Interchanging the first and third row we get,

$$\begin{bmatrix} 1 & 4 & -3 & 3 \\ 2 & -9 & 3 & 2 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

This is the required matrix.

Solution:

18 a) Given augmented matrix is

$$\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$$

Applying $R_1' = \frac{1}{2}R_1$ we get

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$$

This is the required matrix.

Solution:

18 b) Given augmented matrix is

$$\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$$

Applying $R_1' = R_1 + R_3$ we get

$$\begin{bmatrix} 1 & -1 & -3 & 6 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$$

This is the required matrix.