Gaussian Elimination

Reduced row echelon form:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***.

► In Exercises 1–2, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither. ◄

1. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

1. (a)-(f): Solution:

The given matrix satisfies all the properties of row reduced echelon form.
Therefore, it is both in row echelon form and reduced row echelon form.

1. (g) : Solution:

The given matrix satisfies all the properties of row echelon form.
Therefore, it is in row echelon form.

▶ In Exercises 3–4, suppose that the augmented matrix for a linear system has been reduced by row operations to the given row echelon form. Solve the system. ◀

3. (a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. a) Solution: Given augmented matrix is,

$$\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The linear system of equations are,

$x - 3y + 4z = 7$

$y + 2z = 2$

$z = 5$

Solving by back calculation we get,

$z = 5, \;\; y = -8, \qquad x = -37$

Solution of the given system is $x = -37, y = -8 \; and \; z = 5$

3. c) Solution: Given augmented matrix is,

$$\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system of equations are,

$$x_1 + 7x_2 - 2x_3 \qquad - 8x_5 = 3$$

$$x_3 + x_4 + 6x_5 = 5$$

$$x_4 + 3x_5 = 9$$

The system of equation has five variables with three equations. Hence, it has infinitely many solutions and two free variables. Let the free variables are x_5 and x_2.

Also let $x_5 = r$ and $x_2 = s$

$\therefore x_4 = 9 - 3r$

$x_3 = 5 - x_4 - 6x_5 = 5 - (9 - 3r) - 6r = -4 - 3r$

$x_1 = 3 - 7x_2 + 2x_3 + 8x_5 = 3 - 7s + 2(-4 - 3r) + 8r = 3 - 7s - 8 - 6r + 8r = -5 + 2r - 7s$

Solution of the given system is

$x_1 = 2r - 7s - 5$
$x_2 = s$
$x_3 = -3r - 4$
$x_4 = -3r + 9$
$x_5 = r$

3. d) Solution: Given augmented matrix is,

$$\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The linear system of equations are,
$x - 3y + 7z = 1$
$y + 4z = 0$
$0 = 1$
The system is inconsistent and has no solution.

▶ In Exercises 5–8, solve the linear system by Gaussian elimination. ◀

5.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

6.
$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

7.
$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

5. Solve the following linear system By Gaussian Elimination.

$$\boldsymbol{x_1 + x_2 + 2x_3 = 8}$$
$$\boldsymbol{-x_1 - 2x_2 + 3x_3 = 1}$$
$$\boldsymbol{3x_1 - 7x_2 + 4x_3 = 10}$$

Solution:

Given system of equations are

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

The augmented matrix for the system is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Applying $R_2' = R_2 + R_1$ *and* $R_3' = R_3 - 3R_1$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Applying $R_2' = -R_2$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Applying $R_3' = R_3 + 10R_2$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

Applying $R_3' = -\frac{1}{52}R_3$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ x_2 - 5x_3 &= -9 \\ x_3 &= 2 \end{aligned}$$

Solving we get,

$x_2 = 5x_3 - 9 = 5 \times 2 - 9 = 1$

$x_1 = 8 - x_2 - 2x_3 = 8 - 1 - 2 \times 2 = 3$

Hence the solution of the given system of equations are

$x_1 = 3, \quad x_2 = 1 \text{ and } x_3 = 2$

7. Solve the following linear system By Gaussian Elimination.

$$\begin{aligned} &\boldsymbol{x - y + 2z - w = -1} \\ &\boldsymbol{2x + y - 2z - 2w = -2} \\ &\boldsymbol{-x + 2y - 4z + w = 1} \\ &\boldsymbol{3x \qquad\qquad - 3w = -3} \end{aligned}$$

Solution:

Given system of equations are

$$\begin{aligned} &x - y + 2z - w = -1 \\ &2x + y - 2z - 2w = -2 \\ &-x + 2y - 4z + w = 1 \\ &3x \qquad\qquad - 3w = -3 \end{aligned}$$

The augmented matrix for the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

Applying $R_2' = R_2 - 2R_1$ and $R_3' = R_3 + R_1$, $R_4' = R_4 - 3R_1$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

Applying $R_2' = \frac{1}{3}R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

Applying $R_3' = R_3 - R_2$, $R_4' = R_4 - 3R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x - y + 2z - w &= -1 \\ y - 2z \quad &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Solving we get,

$x = -1 + y - 2z + w$

$or,\ \ x = -1 + w\ [\because y - 2z = 0]$

$y = 2z$

The system has two equations with four variables. So there are two free variables. Let the free variables are $z\ and\ w$. Also let $z = s\ and\ w = t$

Hence the solution of the given system of equations are

$x = -1 + t, \qquad y = 2s, \qquad z = s\ \ and\ w = t$

9. Solve the following linear system By Gauss-Jordan Elimination.

$\boldsymbol{x_1 + x_2 + 2x_3 = 8}$

$\boldsymbol{-x_1 - 2x_2 + 3x_3 = 1}$

$\boldsymbol{3x_1 - 7x_2 + 4x_3 = 10}$

Solution:

Given system of equations are

$x_1 + x_2 + 2x_3 = 8$

$-x_1 - 2x_2 + 3x_3 = 1$

$3x_1 - 7x_2 + 4x_3 = 10$

The augmented matrix for the system is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Applying $R_2' = R_2 + R_1\ and\ R_3' = R_3 - 3R_1$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Applying $R_2' = -R_2$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Applying $R_3' = R_3 + 10R_2$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

Applying $R_3' = -\frac{1}{52}R_3$ we get,

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the row echelon form of the augmented matrix.
Applying $R_1' = R_1 - R_2$ we get,

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Applying $R_1' = R_1 - 7R_3 \ and \ R_2' = R_2 + 5R_3$ we get,

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence the unique solution of the linear system is

$x_1 = 3, \qquad x_2 = 1 \ and \ x_3 = 2$

11. Solve the following linear system By Gauss-Jordan Elimination.

$\boldsymbol{x - y + 2z - w = -1}$
$\boldsymbol{2x + y - 2z - 2w = -2}$
$\boldsymbol{-x + 2y - 4z + w = 1}$
$\boldsymbol{3x \qquad\qquad - 3w = -3}$
Solution:

Given system of equations are
$x - y + 2z - w = -1$
$2x + y - 2z - 2w = -2$
$-x + 2y - 4z + w = 1$
$3x \qquad\qquad - 3w = -3$
The augmented matrix for the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

Applying $R_2' = R_2 - 2R_1 \ and \ R_3' = R_3 + R_1, \ \ R_4' = R_4 - 3R_1$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

Applying $R_2' = \frac{1}{3}R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

Applying $R_3' = R_3 - R_2,\ R_4' = R_4 - 3R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row echelon form of the augmented matrix.

Applying $R_1' = R_1 + R_2$ we get,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x \qquad\qquad - w &= -1 \\ y - 2z \qquad &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The system has two equations with four variables. So there are two free variables. Let the free variables are $z\ and\ w$. Also let $z = s\ and\ w = t$

Hence the solution of the given system of equations are

$x = -1 + t, \qquad y = 2s, \qquad z = s\ \ and\ w = t$

THEOREM 1.2.2 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = 0 \end{array}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

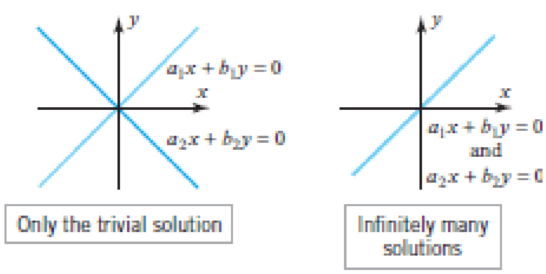
Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

In the special case of a homogeneous linear system of two equations in two unknowns, say

$$a_1x + b_1y = 0 \quad (a_1, b_1 \text{ not both zero})$$
$$a_2x + b_2y = 0 \quad (a_2, b_2 \text{ not both zero})$$

the graphs of the equations are lines through the origin, and the trivial solution corresponds to the point of intersection at the origin (Figure 1.2.1).



▶ Figure 1.2.1

13. Determine whether the following homogeneous system has nontrivial solutions by inspection (without pencil and paper)

$$\mathbf{2x_1 - 3x_2 + 4x_3 - x_4 = 0}$$
$$\mathbf{7x_1 + x_2 - 8x_3 + 9x_4 = 0}$$
$$\mathbf{2x_1 + 8x_2 + x_3 - x_4 = 0}$$

Solution:

The given homogeneous system is

$$\mathbf{2x_1 - 3x_2 + 4x_3 - x_4 = 0}$$
$$\mathbf{7x_1 + x_2 - 8x_3 + 9x_4 = 0}$$

$\mathbf{2x_1 + 8x_2 + x_3 - x_4 = 0}$

Since the number of variables is greater than the number of equations, the system has infinitely many solutions. Hence the system has both trivial and nontrivial solutions.

14. Determine whether the following homogeneous system has nontrivial solutions by inspection (without pencil and paper)

Solution:

The given homogeneous system is

$$\begin{aligned} \mathbf{x_1 + 3x_2 - x_3} &= \mathbf{0} \\ \mathbf{x_2 - 8x_3} &= \mathbf{0} \\ \mathbf{4x_3} &= \mathbf{0} \end{aligned}$$

Since the number of variables is equal to the number of equations, the system has trivial solution. It has no nontrivial solutions.

15. Solve the following linear system by any method

$$\begin{aligned} \mathbf{2x_1 + x_2 + 3x_3} &= \mathbf{0} \\ \mathbf{x_1 + 2x_2} &= \mathbf{0} \\ \mathbf{x_2 + x_3} &= \mathbf{0} \end{aligned}$$

Solution:

The augmented matrix of the given system of linear equations is

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_1' = R_1 - R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_2' = R_2 - R_1$ we get,

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_2' = \frac{1}{3} R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_3' = R_3 - R_2$ we get,

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

The solution is

From L_3:

$2x_3 = 0$

$or, \qquad x_3 = 0$

From L_2:

$x_2 - x_3 = 0$

$or, \qquad x_2 = 0$

From L_1:

$x_1 - x_2 + 3x_3 = 0$

$or, \qquad x_1 = 0$

17. Solve the following linear system by any method

$\boldsymbol{3x_1 + x_2 + x_3 + x_4 = 0}$

$\boldsymbol{5x_1 - x_2 + x_3 - x_4 = 0}$

The augmented matrix of the given system of linear equations is

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Applying $R_1' = 2R_1 - R_2$ we get,

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Applying $R_2' = R_2 - 5R_1$ we get,

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & -16 & -4 & -16 & 0 \end{bmatrix}$$

Applying $R_2' = -\frac{1}{16}R_2$ we get,

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$$

The system has two equations with four variables. So there are two free variables. Let the free variables are $\boldsymbol{x_3}$ $\boldsymbol{and}$ $\boldsymbol{x_4}$ Also let $\boldsymbol{x_3} = 4s\ and\ \boldsymbol{x_4} = t$

Hence the solutions are

From L_2:

$\boldsymbol{x_2 + \frac{1}{4}x_3 + x_4 = 0}$

$\boldsymbol{or}, \qquad \boldsymbol{x_2 = -\frac{1}{4}x_3 - x_4}$

$\boldsymbol{or}$, $\quad \boldsymbol{x_2 = -s - t}$

From L_1:

$\boldsymbol{x_1 + 3x_2 + x_3 + 3x_4 = 0}$

$\boldsymbol{or}$, $\quad \boldsymbol{x_1 = -3x_2 - x_3 - 3x_4}$

$\boldsymbol{or}$, $\quad \boldsymbol{x_1 = 3s + 3t - 4s - 3t}$

$\boldsymbol{or}$, $\quad \boldsymbol{x_1 = -s}$

19. Solve the following linear system by any method

$$\begin{aligned} \boldsymbol{2x + 2y + 4z} &= \boldsymbol{0} \\ \boldsymbol{w - \quad y - 3z} &= \boldsymbol{0} \\ \boldsymbol{2w + 3x + y + z} &= \boldsymbol{0} \\ \boldsymbol{-2w + x + 3y - 2z} &= \boldsymbol{0} \end{aligned}$$

Solution: Given system of equations are

$$\begin{aligned} x + y + 2z &= 0 \\ w - \quad y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

The augmented matrix of the given system of linear equations is

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$

Interchanging R_1 and R_2 we get,

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$

Applying $R_3' = R_3 - 2R_1$, $R_4' = R_4 + 2R_1$ we get,

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & -1 & 7 & 0 \\ 0 & 1 & 5 & -8 & 0 \end{bmatrix}$$

Applying $R_3' = R_3 - 3R_2$, $R_4' = R_4 - R_2$ we get,

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 4 & -10 & 0 \end{bmatrix}$$

Applying $R_4' = R_4 + R_3$ we get,

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -9 & 0 \end{bmatrix}$$

Applying $R_3' = -\frac{1}{4}R_3$, $R_4' = -\frac{1}{5}R_4$ we get,

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution is

From L_4:

$x_4 = 0$

From L_3:

$x_3 + x_4 = 0$

$or, \qquad x_3 = 0$

From L_2:

$x_2 + x_3 + x_4 = 0$

$or, \qquad x_2 = 0$

From L_1:

$x_1 + x_3 - 3x_4 = 0$

$or, \qquad x_1 = 0$

Hence the solution is $(x_{1,}x_2, x_3, x_4) = (0,0,0,0)$

In each part of Exercises 23–24, the augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

23. (a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$ (b) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$

24. (a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$ (d) $\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

23. Solution:

a) The system is consistent and has unique solution.

b) The system is consistent and has infinitely many solutions.

c) The system is inconsistent and has no solution.

d) The system is inconclusive.

24. Solution:

a) The system is consistent and has unique solution.

b) The system is consistent and has unique solution.

c) The system is inconclusive.

d) The system is consistent and has infinitely many solutions.

Exercise Set 1.2 (page 22)

1. **(a)** Both **(b)** Both **(c)** Both **(d)** Both **(e)** Both **(f)** Both **(g)** Row echelon form

3. **(a)** $x = -37,\ y = -8,\ z = 5$
   (b) $w = -10 + 13t,\ x = -5 + 13t,\ y = 2 - t,\ z = t$
   (c) $x_1 = -11 - 7s + 2t,\ x_2 = s,\ x_3 = -4 - 3t,\ x_4 = 9 - 3t,\ x_5 = t$
   (d) No solution

5. $x_1 = 3,\ x_2 = 1,\ x_3 = 2$ 7. $x = -1 + t,\ y = 2s,\ z = s,\ w = t$ 9. $x_1 = 3,\ x_2 = 1,\ x_3 = 2$

11. $x = -1 + t,\ y = 2s,\ z = s,\ w = t$ 13. Has nontrivial solutions 15. $x_1 = 0,\ x_2 = 0,\ x_3 = 0$

17. $x_1 = -\frac{1}{4}s,\ x_2 = -\frac{1}{4}s - t,\ x_3 = s,\ x_4 = t$ 19. $w = t,\ x = -t,\ y = t,\ z = 0$ 21. $I_1 = -1,\ I_2 = 0,\ I_3 = 1,\ I_4 = 2.$

23. **(a)** Consistent; unique solution
   (b) Consistent; infinitely many solutions
   (c) Inconsistent
   (d) Insufficient information provided

25. No solutions when $a = -4$;
   infinitely many solutions when $a = 4$;
   one solution for all values $a \neq -4$ and $a \neq 4$

27. $-a + b + c = 0$ 29. $x = \frac{2}{3}a - \frac{1}{9}b,\ y = -\frac{1}{3}a + \frac{2}{9}b$

31. E.g., $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (other answers are possible) 35. $x = \pm 1,\ y = \pm\sqrt{3},\ z = \pm\sqrt{2}$

37. $a = 1,\ b = -6,\ c = 2,\ d = 10$ 39. The nonhomogeneous system has only one solution.

True/False 1.2

(a) True **(b)** False **(c)** False **(d)** True **(e)** True **(f)** False **(g)** True **(h)** False **(i)** False