1.6_More on linear Systems and Invertible Matrices

THEOREM 1.6.1 *A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

EXAMPLE 1 Solution of a Linear System Using A^{-1}

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$. ◀

THEOREM 1.6.2 *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

EXAMPLE 2 Solving Two Linear Systems at Once

Solve the systems

$$\text{(a)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

Solution The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced row echelon form yields (verify)

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

It follows from the last two columns that the solution of system (a) is $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ and the solution of system (b) is $x_1 = 2$, $x_2 = 1$, $x_3 = -1$. ◀

Properties of Invertible Matrices

Up to now, to show that an $n \times n$ matrix A is invertible, it has been necessary to find an $n \times n$ matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

The next theorem shows that if we produce an $n \times n$ matrix B satisfying *either* condition, then the other condition will hold automatically.

THEOREM 1.6.3 *Let A be a square matrix.*

(a) If B is a square matrix satisfying $BA = I$, *then* $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, *then* $B = A^{-1}$.

2. Solve the system by inverting the coefficient matrix and using the formula $x = A^{-1}b$

$\mathbf{4x_1 - 3x_2 = -3}$
$\mathbf{2x_1 - 5x_2 = 9}$

Solution: Given system of equation

$\mathbf{4x_1 - 3x_2 = -3}$
$\mathbf{2x_1 - 5x_2 = 9}$

The given system can be written in matrix form as $Ax = b$

Where, $A = \begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix}$; $x = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$ and $\quad b = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

At first, we invert the coefficient matrix A

Adjoining the unit matrix with coefficient matrix we get,

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right]$$

Interchanging R_1 and R_2

$$= \left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 4 & -3 & 1 & 0 \end{array}\right]$$

Applying $R_2' = R_2 - 2R_1$

$$= \left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{array}\right]$$

Applying $R_1' = \frac{1}{2}R_1$ and $R_2' = \frac{1}{7}R_2$ we get,

$$= \left[\begin{array}{cc|cc} 1 & -\frac{5}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right]$$

Applying $R_1' = R_1 + \frac{5}{2}R_2$ we get,

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{14} & -\frac{3}{14} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right]$$

$$\therefore \ A^{-1} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix}$$

Using the formula $x = A^{-1}b$ we get,

$$[x] = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$or, \ \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

The given system has exactly one solution which is $(x_1, x_2) = (-3, -3)$

4. Solve the system by inverting the coefficient matrix and using the formula $\boldsymbol{x = A^{-1}b}$

$$\boldsymbol{5x_1 + 3x_2 + 2x_3 = 4}$$
$$\boldsymbol{3x_1 + 3x_2 + 2x_3 = 2}$$

$$\boldsymbol{x_2 + \quad x_3 = 5}$$

Solution: Given system of equation

$$\boldsymbol{5x_1 + 3x_2 + 2x_3 = 4}$$
$$\boldsymbol{3x_1 + 3x_2 + 2x_3 = 2}$$
$$\boldsymbol{x_2 + \quad x_3 = 5}$$

The given system can be written in matrix form as $Ax = b$

Where, $A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$; $x = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix}$ $and \quad b = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

At first, we invert the coefficient matrix A

Adjoining the unit matrix with coefficient matrix we get,

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1' = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1' = \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2' = R_2 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 2 & -\frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2' = R_2 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_3' = R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}$$

Using the formula $x = A^{-1}b$ we get,

$$[x] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$or, \quad \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

The given system has exactly one solution which is $(x_1, x_2, x_3) = (1, -11, 16)$

6. Solve the system by inverting the coefficient matrix and using the formula $\boldsymbol{x = A^{-1}b}$

$$\boldsymbol{-x - 2y - 3z = 0}$$
$$\boldsymbol{w + x + 4y + 4z = 7}$$
$$\boldsymbol{w + 3x + 7y + 9z = 4}$$
$$\boldsymbol{-w - 2x - 4y - 6z = 6}$$

Solution: Given system of equation

$$-x - 2y - 3z = 0$$
$$w + x + 4y + 4z = 7$$
$$w + 3x + 7y + 9z = 4$$
$$-w - 2x - 4y - 6z = 6$$

The system can be arranged as follows

$$x + 2y + 3z = 0$$
$$w + x + 4y + 4z = 7$$
$$w + 3x + 7y + 9z = 4$$
$$w + 2x + 4y + 6z = -6$$

The given system can be written in matrix form as $Ax = b$

Where, $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 4 & 4 \\ 1 & 3 & 7 & 9 \\ 1 & 2 & 4 & 6 \end{bmatrix}$; $x = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ $and \quad b = \begin{bmatrix} 0 \\ 7 \\ 4 \\ -6 \end{bmatrix}$

At first, we invert the coefficient matrix A

Adjoining the unit matrix with coefficient matrix we get,

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Interchanging $R_1 \; and \; R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_3' = R_3 - R_4$ and $R_4' = R_4 - R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

Applying $R_3' = R_3 - 2R_4$ and $R_4' = R_4 - R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & -2 & 1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

Applying $R_3' = R_3 + R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1 \\ 0 & 0 & -2 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Applying $R_4' = R_4 + 2R_3$

$$\left[\begin{array}{cccc|cccc}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1\\ 0 & 0 & 0 & -1 & -6 & -1 & 2 & -1\end{array}\right]$$

Applying $R_1' = R_1 - R_2,\ R_2' = R_2 - 2R_3$

$$\left[\begin{array}{cccc|cccc}1 & 0 & 2 & 1 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 3 & 7 & 0 & -2 & 2\\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1\\ 0 & 0 & 0 & -1 & -6 & -1 & 2 & -1\end{array}\right]$$

Applying $R_4' = -R_4$

$$\left[\begin{array}{cccc|cccc}1 & 0 & 2 & 1 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 3 & 7 & 0 & -2 & 2\\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 6 & 1 & -2 & 1\end{array}\right]$$

Applying $R_1' = R_1 - 2R_3,\ R_2' = R_2 - 3R_4$

$$\left[\begin{array}{cccc|cccc}1 & 0 & 0 & 1 & 5 & 1 & -2 & 2\\ 0 & 1 & 0 & 0 & -11 & -3 & 4 & -1\\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 6 & 1 & -2 & 1\end{array}\right]$$

Applying $R_1' = R_1 - R_4$

$$\left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & -1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & -11 & -3 & 4 & -1\\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 6 & 1 & -2 & 1\end{array}\right]$$

$$\therefore\ \ A^{-1} = \begin{bmatrix}-1 & 0 & 0 & 1\\ -11 & -3 & 4 & -1\\ -3 & 0 & 1 & -1\\ 6 & 1 & -2 & 1\end{bmatrix}$$

Using the formula $x = A^{-1}b$ we get,

$$[x] = \begin{bmatrix} -1 & 0 & 0 & 1 \\ -11 & -3 & 4 & -1 \\ -3 & 0 & 1 & -1 \\ 6 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ -6 \end{bmatrix}$$

$$or, \quad \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix}$$

The given system has exactly one solution which is $(x_1, x_2, x_3, x_4) = (-6, 1, 10, -7)$