Spazi normati

- $\|\ldots\|_X$ "norma" in X sp. vettoriale,
  è una funzione $\|\ldots\|: X \to [0,+\infty)$ con certe prop....
- la coppia $(X, \|\ldots\|_X)$ si dice "spazio normato".
- $\forall x_1, x_2 \in X$: "distanza" tra x_1 e x_2 $= \mathrm{dist}(x_1, x_2) \overset{def.}{=} \|x_1 - x_2\|_X$
- $\underbrace{\lim_{n\to+\infty} x_n = x}_{\left(\substack{\text{indicato anche:} \\ x_n \to x}\right)}$ risp. $\|\ldots\|_X \overset{def.}{\iff} \lim_{n\to+\infty} \|x_n - x\|_X = 0$

Sistemi (operatori) continui tra spazi normati

- $A: (X, \|\ldots\|_X) \to (Y, \|\ldots\|_Y)$ "cont." risp. a $\|\ldots\|_X$ e $\|\ldots\|_Y$

$$\Updownarrow \text{def.}$$

$$\underset{\text{risp. } \|\ldots\|_X}{x_n \to x \text{ in } X} \implies \underset{\text{risp. } \|\ldots\|_Y}{Ax_n \to Ax \text{ in } Y}$$

- Nota $A: (X, \|\ldots\|_X) \to (Y, \|\ldots\|_Y)$ cont.
  significa che:
  $$A\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Ax_n$$
- Prop Sia $A: (X, \|\ldots\|_X) \to (Y, \|\ldots\|_Y)$ lineare, allora

$$A \text{ cont.}$$
$$\Updownarrow$$
$$\exists C \geq 0, \forall x \in X : \|Ax\|_Y \leq C\|x\|_X$$

$\left(\right.$ $\boxed{\Uparrow}$ Infatti, sia $\|x_n - x\|_X \to 0$ allora
$\|Ax_n - Ax\|_Y = \|A(x_n - x)\|_Y \leq C\|x_n - x\|_X \to 0$
$\boxed{\Downarrow}$ non la dim. $\left.\right)$

Serie in spazi normati

- $x = \sum_{n=1}^{+\infty} x_n$ risp. $\|\dots\|_X$ $\overset{\text{def.}}{\Longleftrightarrow}$ $x = \lim_{n\to+\infty} \underbrace{\sum_{j=1}^{n} x_j}_{\text{"somme parziali"}}$ risp. $\|\dots\|_X$

- Nota: analogo se $\sum_{n=0}^{+\infty} x_n$
(o in generale $\sum_{n=n_0}^{+\infty} x_n$)

- Sia $A : (X, \|\dots\|_X) \to (Y, \|\dots\|_Y)$ sist. lin. cont.
allora, se $x_n \in X$ e $a_n \in \mathbb{C}$ o $\mathbb{R}$, abbiamo

$$\boxed{A\left(\sum_{n=1}^{\infty} a_n x_n\right) = \sum_{n=1}^{\infty} a_n A x_n}$$

(Infatti:
$$A\left(\sum_{n=1}^{\infty} a_n x_n\right) = A\left(\lim_{n} \sum_{j=1}^{n} a_j x_j\right) = \lim_{n\to\infty} A\left(\sum_{j=1}^{n} a_j x_j\right)$$
$$= \lim_{n\to\infty} \sum_{j=1}^{n} a_j A x_j = \sum_{n=1}^{\infty} a_j A x_j$$
)

- Nota $\sum_{n=0}^{-\infty} x_n \overset{\text{def.}}{=} \sum_{n=0}^{+\infty} x_{-n}$ e

$$\sum_{n=-\infty}^{+\infty} x_n \overset{\text{def.}}{=} \sum_{n=1}^{+\infty} x_n + \sum_{n=0}^{-\infty} x_n$$