19/10/2022 Lezione 2

V.A continue

distrib.

$F_X(x) = P(X \leq x)$ dist. cumulat.

Attenzione: se X è una v.a continua $P(X = x) = 0$

$f_X(x) = \dfrac{d}{dx} F_X(x)$ funzione densità di probabilità

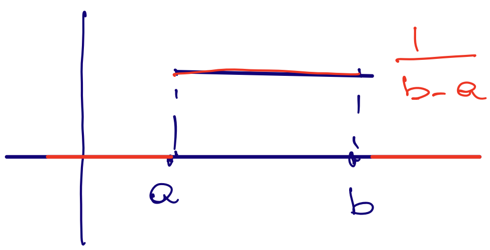
- Funzione generatrice
- Funzione caratteristica
- Momenti

Principali distribuz. continue

– Uniforme $\qquad U(a,b)$

$$f_U(u) = \begin{cases} \dfrac{1}{b-a} & u \in [a,b] \\ 0 & \text{altr.} \end{cases}$$

$$\underbrace{F_U(u)}_{P(U \leq u)} = \begin{cases} \displaystyle\int_a^u \frac{1}{b-a}\, du = \frac{u-a}{b-a} & u \in [a,b] \\ 0 & u < a \\ 1 & u > b \end{cases}$$

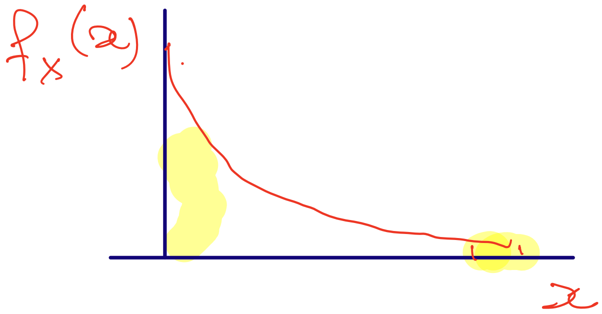


$a = 0 \quad b = 1$

$\hookrightarrow U(0,1)$

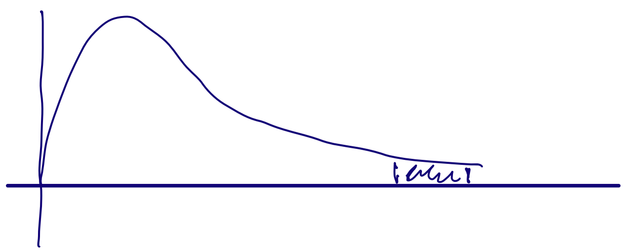
– V. a $\qquad$ esponenziale

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$
$$\lambda > 0$$



ha code leggere

$$F_X(x) = \begin{cases} \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$



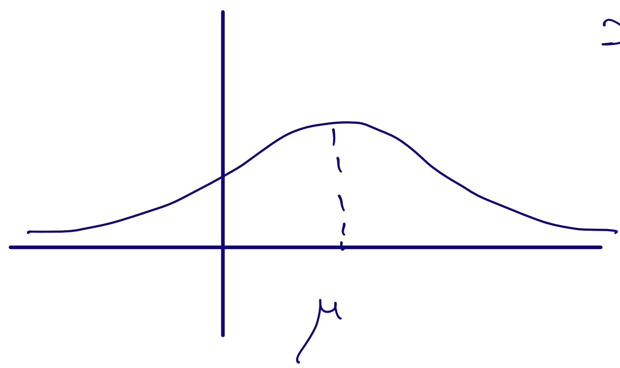
code pesante

- Normale (o Gaussiana)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(x-\mu)^2}{2\sigma^2} \right\}$$

$$\mathcal{N}(\mu, \sigma^2)$$

Gaussiana standard $\begin{cases} \mu = 0 \\ \sigma^2 = 1 \end{cases}$



$x \in \mathbb{R}$

- Gamma(k, θ)

$$f_X(x) = \begin{cases} k e^{-kx} \dfrac{(kx)^{\theta}}{\Gamma(\theta)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\Gamma(\theta) = \int_0^{\infty} e^{-x} x^{\theta-1} \, dx$$

$$\Gamma(n) = (n-1)!$$

Se $X_1, \dots, X_n$ sono v.a. indip.

esponenziali di parametro $\theta \Rightarrow$

$$Y = \sum_{i}^{n} X_i \sim \text{Gamma}(n, \theta)$$

Teorema del limite centrale

Se le v.a. $\{X_i\}_{i \geq 1}$ sono i.i.d.

allora

$$\frac{\sum_{i}^{n} X_i - E\left(\sum_{i}^{n} X_i\right)}{\sqrt{\text{Var} \sum_{i}^{n} X_i}} \xrightarrow{n \to \infty} Z$$

$$Z \sim \mathcal{N}(0,1)$$

MOMENTI di UNA V.A.

$$E[X^n] = \begin{cases} \sum x^n P(X = x) & \text{v.a. discrete} \\ \int x^n f_X(x)\,dx \end{cases}$$

V.a. $\quad U(a,b)$

$$\mathbb{E}X = \int_a^b \underbrace{\frac{1}{b-a}}_{f_X(x)} x \, dx =$$

$$= \frac{x^2}{2}\Big|_a^b \frac{1}{b-a} = \frac{1}{2}\,\frac{b^2-a^2}{b-a}$$

$$= \frac{b+a}{2}$$

V.a. $\quad$ esponenziale

$$\mathbb{E}X = \int_0^\infty x\,\lambda\, e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\mathbb{E}X^2 = \int_0^\infty \lambda\, x^2\, e^{-\lambda x}\, dx = \frac{2}{\lambda^2}$$

$\hookrightarrow \quad \mathrm{Var}\, X = \dfrac{1}{\lambda^2}$

ATTESA di funzioni di V.a.

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) P(X=x) \\ \int_{\mathbb{R}} g(x) f_X(x)\,dx \end{cases}$$

$Y = g(X)$ nuova v.a.

$$\mathbb{E}[g(X)] = \int y\, f_Y(y)\,dy$$

FUNZIONE GENERATRICE

$$\phi_X(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_x e^{tx} P(X=x) \\ \underbrace{\int e^{tx} f_X(x)\,dx}_{\text{se converge}} \end{cases}$$

TRASFORMATA di LAPLACE

$$\psi_X(t) = \mathbb{E}[e^{-tX}]$$

FUNZIONE CARATTERISTICA

$$\varphi_X(t) = \mathbb{E}[e^{itX}]$$

$$= \int_{\mathbb{R}} e^{itx} f_X(x)\,dx$$

OSSERVAZIONI

1

$$\frac{d}{dt} \varphi_X(t) = \int_{\mathbb{R}} i x e^{itx} f_X(x)\,dx$$

$$\frac{d}{dt} \varphi_X(t)\Big|_{t=0} = i \int_{\mathbb{R}} x \underbrace{1}_{e^{itx}|_{t=0}} f_X(x)\,dx$$

$$= i\,\mathbb{E}X$$

2

$$\frac{d}{dt} \Phi_X(t) = \frac{d}{dt} \int_{\mathbb{R}} e^{tx} f_X(x)\,dx$$

$$= \int x e^{tx} f_X(x)\,dx$$

$$\frac{d}{dt}\phi_X(t)\Big|_{t=0} = \int_{\mathbb{R}} x f_X(x)\,dx = EX$$

$$\frac{d^n}{dt^n}\phi_X(t)\Big|_{t=0} = \mathbb{E}X^n$$

DISTRIBUZIONI CONGIUNTE

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$
$$= P(\{X \leq x\} \cap \{Y \leq y\})$$

$$-\infty < x, y < +\infty$$

OSS.

$$F_{X,Y}(x,\infty) = F_X(x)$$

$$F_{X,Y}(\infty,y) = F_Y(y)$$

Se X e Y sono indip

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$$

$$F_{X,Y} = F_X\,F_Y$$

Caso discreto

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$

$$p_X(x) = \sum_y P(X=x, Y=y)$$

$$p_Y(y) = \sum_x P(X=x, Y=y)$$

Caso continuo

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{XY}(x,y)$$

∂

$$P(X\in A, Y\in B) = \int_B \int_A f_{X,Y}(x,y)\,dx\,dy$$

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y) \\ \int_{\mathbb{R}}\int_{\mathbb{R}} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy \end{cases}$$

$$\mathbb{E}[X+Y] =$$

$$\iint (x+y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \iint x\, f_{X,Y}(x,y)\, dx\, dy +$$

$$\iint y\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} dx\, x \overbrace{\int_{\mathbb{R}} f_{XY}(x,y)\, dy}^{f_X(x)} +$$

$$\int_{\mathbb{R}} dy\, y \underbrace{\int_{\mathbb{R}} f_{XY}(x,y)\, dx}$$

$$=$$

$$= \overbrace{\int_{\mathbb{R}} dx \; x \, f_X(x)}^{EX} + \underbrace{\int_{\mathbb{R}} dy \; y \, f_Y(y)}_{EY} =$$

$f_Y(y)$

$$\mathbb{E}X + \mathbb{E}Y$$

Generalizzando

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \, \mathbb{E}X_i$$

$$\int_{\mathbb{R}} f_{XY}(x,y)\,dy =$$

$$\int_{\mathbb{R}} \frac{\partial^2}{\partial x\,\partial y} P(X \le x, Y \le y)\,dy$$

$$= \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{\partial}{\partial y} P(X \le x, Y \le y)\,dy$$

$$\frac{\partial}{\partial x} P(X \le x, Y \le y)\Big|_{y \in \mathbb{R}}$$

$$\frac{\partial}{\partial x} P(X \le x) = f_X(x)$$

VARIABILI ALEATORIE INDIPENDENTI

$$P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y)$$

$$\forall (x,y) \in \mathbb{R}^2$$

<u>Teorema</u> Se X e Y sono v.a. indip. allora per ogni funzione h e g si ha

$$\mathbb{E}[h(X) g(Y)] = \mathbb{E}[h(X)] \mathbb{E}[g(Y)]$$

<u>Dim</u>

$$\mathbb{E}[h(X) g(Y)] =$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} g(x) h(y) f_{X,Y}(x,y) \, dx \, dy$$

$$\overset{\text{(ind.)}}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} g(x) h(y) f_X(x) f_Y(y) \, dx \, dy$$

$$g(x) f_X(x)\, dx$$

$$= \underbrace{\int_{\mathbb{R}} dx\, g(x) f_X(x)}_{E[g(X)]} \underbrace{\int_{\mathbb{R}} dy\, h(y) f_Y(y)}_{E[h(Y)]}$$

SIMULAZIONE di V.A continue

Metodo della trasformazione inversa

Teorema Se U è una v.a. Uniforme in $(0,1)$ e $F_X(x)$ è una funzione di distribuzione strettamente monotona e continua. Allora la v.a.

$$X = F_X^{-1}(U)$$

ha distribuzione F_X.

Dim

$$P(X \leq x) =$$

$$P(F_X^{-1}(U) \leq x) =$$

$$P(U \leq F_X(x)) = F_X(x)$$

Esempio

$$F(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$X \sim \exp(1)$$

$$U = 1 - e^{-x} \qquad x = -\ln(1-u)$$

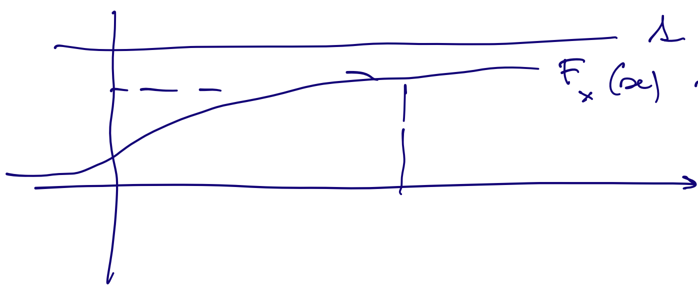
$$X = F^{-1}(U) = -\ln(1-U) \sim \exp(1)$$

Operativamente:

- Genero u_1 valore di una v.a. uniforme
- Calcolo $-\ln(1-u_1) = x_1$

  $-\ln(u_1)$

  x_1 è il valore assunto da una v.a. $\exp(1)$



$$F_x(x) = \int_{-\infty}^{x} \frac{e^{y^2/2}}{\sqrt{2\pi}}\,dy$$

- Metodo del rifiuto
- Per v.a. Gaussiane: Box e Muller

<u>Def</u>

$$\operatorname{Cov}(X,Y) = \mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]$$

$$\rho = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}X \cdot \operatorname{Var}Y}}$$

Se $\operatorname{cov}(X,Y) = 0$ le variabili si dicono non correlate

Se X e Y sono indip

$\longrightarrow$ $\rho = 0$ e $\operatorname{cov}(X,Y) = 0$

Non è vero il viceversa in generale

Se (X,Y) è gaussiana bivariata

$\operatorname{Cov}(X,Y) = 0 \iff X$ e Y indip.

Esempio

$X = \mathbb{1}_A \qquad Y = \mathbb{1}_B$

$$\mathbb{1}_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] =$$

$$= \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y) =$$

$$1 \cdot P(X=1, Y=1) - P(X=1)P(Y=1)$$

$$\mathrm{Cov}(X,Y) > 0 \iff P(X=1, Y=1) > P(X=1)P(Y=1)$$

$$\iff P(Y=1 \mid X=1) > P(Y=1)$$

VARIABILE INDICATORE

$$\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$$

$$\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

SOMME DI V.A. INDIPENDENTI

$X \sim f_X(x) \qquad Y \sim g_Y(y)$

$Z = X + Y$

$$F_{X+Y}(z) = P(X+Y \leq z)$$

$$= \int_{\mathbb{R}} \underbrace{P(X+Y \leq z \mid Y = y)}_{P(X \leq z-y \mid Y=y)} g_Y(y)\, dy$$

X, Y ind.

$$= \int_{\mathbb{R}} P(X \leq z - y)\, g_Y(y)\, dy$$

$$F_Z(z) = \int_{\mathbb{R}} F_X(z-y)\, g_Y(y)\, dy$$

deriviamo rispetto a z

$$f_Z(z) = \underbrace{\int_{\mathbb{R}} f_X(z-y)\, g_Y(y)\, dy}_{\text{integrale di convoluzione}}$$