Lezione 4

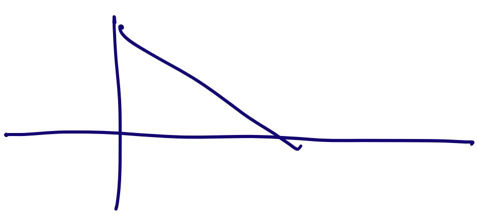
$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dx\,dy =$$

$$\int_{\mathbb{R}} dx \underbrace{\int f(x,y)\,dy}_{g(x)} =$$

$$\int dy \underbrace{\int f(x,y)\,dx}_{h(y)}$$

$$\int_{\mathbb{R}} \int_{v^2}^{\infty} f(x,v)\,dx\,dv =$$

$$= \int_{\mathbb{R}} dv \underbrace{\int_{v^2}^{\infty} f(x,v)\,dx}_{m(v)}$$



$$x+y \le z$$

$$x \le z - y$$

$E[X]$

$$\underbrace{E[X \mid Y = y]}_{\varphi(y)} \begin{cases} \text{discreto} & \sum x P(X=x \mid Y=y) \\ \text{continuo} & \int x f_{X|Y}(x|y)\,dx \end{cases}$$

$E[X \mid Y] = \varphi(Y)$ è v.a.

$$E[X] = E[E(X \mid Y)]$$

Esempio

X_i ident. dis

N v.a. indip. delle X_i

$$Y = \sum_{i=1}^{N} X_i$$

Os.

$$\sum_{i=1}^{3} X_i = T$$

$$E(T) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \underbrace{E X_i}_{\mu} = n\mu$$

$\varphi(N)$

$$\mathbb{E}Y = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}\left[\mathbb{E}\left(\sum_1^N X_i \mid N\right)\right]$$

$$= \sum_m \underbrace{\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N=m\right]}_{\varphi(m)} P(N=m)$$

$$= \sum_m \underbrace{\mathbb{E}\left[\sum_{i=1}^{m} X_i \mid N=m\right]}_{X_i \text{ e } N \text{ indip}} P(N=m)$$

$$= \sum_m \mathbb{E}\left[\sum_{i=1}^{m} X_i\right] P(N=m) =$$

$$\sum_m \underbrace{\sum_i \mathbb{E}X_i}_{m \underbrace{\mathbb{E}X_i}_{\mu}} P(N=m)$$

$$= \mathbb{E}X \underbrace{\sum m\, P(N=m)}_{\mathbb{E}N} = \mathbb{E}X\, \mathbb{E}N$$

$$= \mathbb{E}\left(\sum_i^N X_i\right)$$

Oss. $\sum_1^m X_i \neq m X_i$

Aggiungo un'ipotesi: $\{X_i\}$ siano indipendenti

$$\mathrm{Var}\left(\sum_1^N X_i\right) = \mathbb{E}N\,\mathrm{Var}(X) + [\mathbb{E}(X)]^2\,\mathrm{Var}(N)$$

$$\mathrm{Var}\left(\sum^N X_i\right) = \mathbb{E}\left[\left(\sum^N X_i\right)^2\right] - \left[\mathbb{E}\left(\sum X_i\right)\right]^2$$

$$\mathbb{E}\left[\left(\sum^N X_i\right)^2\right] = \mathbb{E}\left[\mathbb{E}\left(\sum^N X_i\right)^2 \Big| N\right]$$

Esempio

Prove indipendenti che proseguono fino a k successo consecutivo

Sia
$$N_k = \text{numero di tali prove}$$

$$M_k := \mathbb{E}N_k$$

Sol.

$$M_k = \mathbb{E}N_k = \mathbb{E}\left[\mathbb{E}(N_k \mid N_{k-1})\right]$$

$$\mathbb{E}[N_k \mid N_{k-1}] = N_{k-1} + \underbrace{1}_{p\,1+(1-p)\,1=1} + (1-p)\underbrace{\mathbb{E}N_k}_{\mu_k}$$

$$E N_k = \mathbb{E}\Big(\underbrace{\mathbb{E}[N_k \mid N_{k-1}]}_{N_{k-1}+1+(1-p)EN_k}\Big) =$$

$$= \mathbb{E}(N_{k-1}) + 1 + (1-p)\mathbb{E}N_k$$

$$\mu_k = \mu_{k-1} + 1 + (1-p)\mu_k$$

$$\mu_k = \frac{1}{p} + \frac{1}{p}\mu_{k-1}$$

$k=1$ $\qquad \mu_1 = \frac{1}{p}$ $\qquad$ (è il caso geometrico)

$$\mu_2 = \frac{1}{p} + \frac{1}{p}\mu_1 = \frac{1}{p} + \frac{1}{p^2}$$

$$\mu_3 = \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p^2}\right) =$$

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}$$

$$M_k = \sum_{i=1}^{k} \frac{1}{p^i}$$

Teorema

$$\operatorname{Var} X = \mathbb{E}[\operatorname{Var}(X|Y)] + \operatorname{Var}[\mathbb{E}(X|Y)]$$

Esercizio

Calcolare la varianza della distribuzione geometrica

Sol.

$$N \sim Geo(p)$$

$$Y = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$$\operatorname{Var} N = \mathbb{E}(N^2) - \underbrace{[\mathbb{E} N]^2}_{1/p^2}$$

$$\mathbb{E}[N^2] = \mathbb{E}[\mathbb{E}(N^2|Y)]$$

$$\underbrace{\mathbb{E}[N^2|Y=1]}_{1}\underbrace{P(Y=1)}_{p} +$$

$$\underbrace{\mathbb{E}[N^2|Y=0]}_{\mathbb{E}[(N+1)^2]}\underbrace{P(Y=0)}_{1-p} =$$

$$p + (1-p)\,\mathbb{E}[(N+1)^2] =$$

$$p + (1-p)\,\mathbb{E}(N^2 + 1 + 2N) =$$

$$\cancel{p} + (1-p)\,\mathbb{E}(N^2) + (1-\cancel{p}) + 2(1-p)\underbrace{\frac{1}{p}}_{\mathbb{E}N}$$

$$= 1 + \frac{2}{p}(1-p) + (1-p)\,\mathbb{E}(N^2)$$

$$= \mathbb{E}N^2$$

$$\hookrightarrow p\,\mathbb{E}N^2 = \frac{p+2-2p}{p}$$

$$\mathbb{E}N^2 = \frac{2-p}{p^2}$$

$$\mathrm{Var}\,N = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Se $I_A = \begin{cases} 1 & \text{se } A \text{ si verifica} \\ 0 & \text{altiv.} \end{cases}$

$$\mathbb{E}[I_A] = 1 \cdot P(A) + 0\, P(\bar{A})$$
$$= P(A)$$

$$A = \{X \leq x\}$$

$$\mathbb{E}[I_{\{X \leq x\}}] = P(X \leq x)$$

$$\mathbb{E}[I_A \mid Y = y] = P(A \mid Y = y)$$

$$\mathbb{E}[I_A \mid Y] = \underbrace{P(A \mid Y)}_{\text{v.a}}$$

$$\mathbb{E}[\,\mathbb{E}[I_A \mid Y]\,] =$$

$$\begin{cases} \sum_y P(A \mid Y=y)\, P(Y=y) \\ \int_{\mathbb{R}} P(A \mid Y=y)\, f_Y(y)\, dy \end{cases}$$

Teorema doppia attesa implica il teorema delle probabilità totali

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Esempio

X, Y v.a. continue indip.

Calcolare $P(X<Y)$

<u>Sol.</u>

$$P(X<Y) = \iint_{x,y\,:\,x<y} f(u,v)\,du\,dv$$

non ci piace!

$$P(X<Y) = \int_{\mathbb{R}} P(X<Y \mid Y=y)\, f_Y(y)\,dy$$

$Y \in (y, y+h)$

$P(Y \in (y, y+h))$

$$= \int_{\mathbb{R}} \underbrace{P(X<y \mid Y=y)}_{\text{indip.}} f_Y(y)\,dy =$$

$$P(X<y)$$

$$\int_{\mathbb{R}} \underbrace{P(X<y)}_{F_X(y)} f_Y(y)\,dy$$

$$= \int_{\mathbb{R}} F_X(y)\, f_Y(y)\,dy$$

Esempio $\qquad Z = X + Y \qquad X, Y$ ind.

$$P(Z \le z) = \int_{\mathbb{R}} P(Z \le z \mid Y = y) f_Y(y)\,dy$$

$$= \int_{\mathbb{R}} P(X + Y \le z \mid Y = y) f_Y(y)\,dy$$

$$= \int_{\mathbb{R}} P(X \le z - y \mid Y = y) f_Y(y)\,dy$$

$$F_Z(z) \overset{(\text{ind.})}{=} \int_{\mathbb{R}} \underbrace{P(X \le \overset{t}{z - y})}_{F_X(z-y)} f_Y(y)\,dy$$

$$f_Z(z) = \int_{\mathbb{R}} f_X(z - y) f_Y(y)\,dy$$

Int. di convoluz

$$P(X \le z - y) = \int_{-\infty}^{z-y} f_X(x)\,dx$$

Esercizio

Ogni persona che entra in un'agenzia di assicurazioni sottoscrive una polizza con probabilità p. Se il n° di clienti che entra in un giorno è una v. a. di Poisson (λ), indip. da ciò che decide di fare sulla sottoscrizione, si determino

1. la probab. di non avere sottoscrizioni in un giorno
2. la probab. che vengano sottoscritte k polizze in un giorno.

<u>Sol.</u> X : {n° di polizze vendute in un giorno}

$$P(x=0) = \sum_{n=0}^{\infty} \underbrace{P(x=0 \mid N=n)}_{(1-p)^n} P(N=n)$$

$$= \sum_{n=0}^{\infty} (1-p)^n \frac{\lambda^n e^{-\lambda}}{n!} =$$

$$= e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \frac{[\lambda(1-p)]^n}{n!}}$$

PROMEMORIA $\underbrace{\sum_{0}^{\infty} \frac{x^n}{n!}}_{1 + x + \frac{x^2}{2} + \dots} = e^x$

$$= e^{-\lambda} e^{\lambda(1-p)} = e^{-\lambda p}$$

2. $$P(x=k) = \sum_{\substack{\cancel{n=0} \\ n=k}}^{\infty} \underbrace{P(x=k \mid N=n)}_{\binom{n}{k} p^k (1-p)^{n-k}} P(N=n)$$

$$= \sum_{\cancel{m=0}\ m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \frac{\lambda^m}{m!} e^{-\lambda}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{\cancel{m=0}\ m=k}^{\infty} \frac{\cancel{m!}}{(m-k)!} \frac{\lambda^{m-k}}{\cancel{m!}} (1-p)^{m-k}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{\cancel{m=0}\ m=k}^{\infty} \frac{[\lambda(1-p)]^{m-k}}{(m-k)!} =$$

$n = m-k$

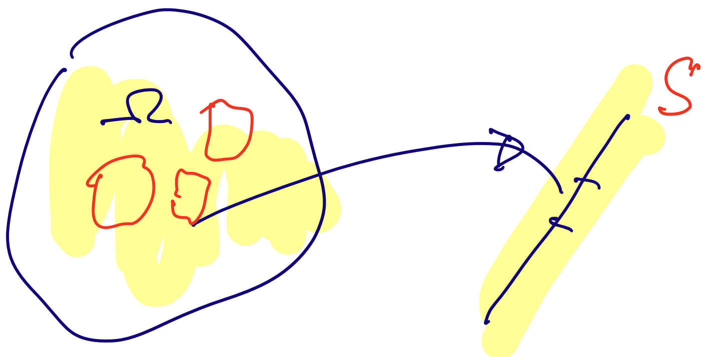
$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \underbrace{\sum_{n=0}^{\infty} \frac{[\lambda(1-p)]^n}{n!}}_{e^{\lambda(1-p)}}$$

$$P(X=k) = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

è ancora Poisson

ma di param. λp

$\lambda = n p_n$



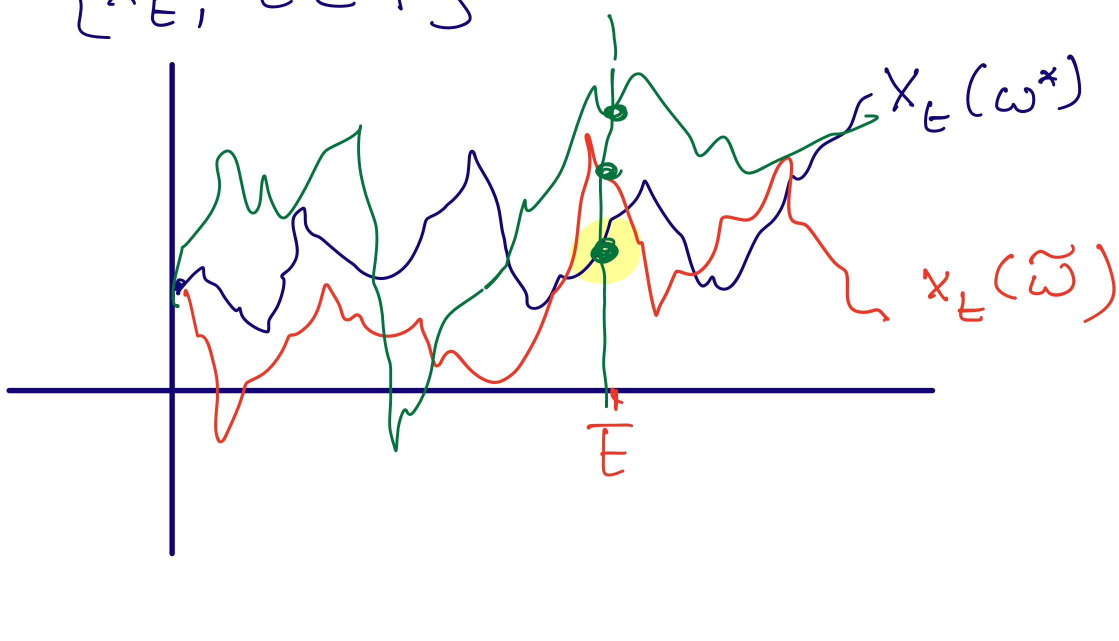
<u>Def</u> Una famiglia di v.a. definite tutte sullo stesso spazio di probabilità $(\Omega, \mathcal{A}, P)$ a valori su uno spazio misurabile S, indicizzate da un insieme ORDINATO T, $\{X_t, t \in T\}$ si dice PROCESSO STOCASTICO

OSS Le v.a. possono assumere valori discreti o continui

L'insieme indice può essere discreto o continuo

	tempo	spazio
processi	discreto	discreto
	discreto	continuo
	continuo	discreto
	continuo	continuo

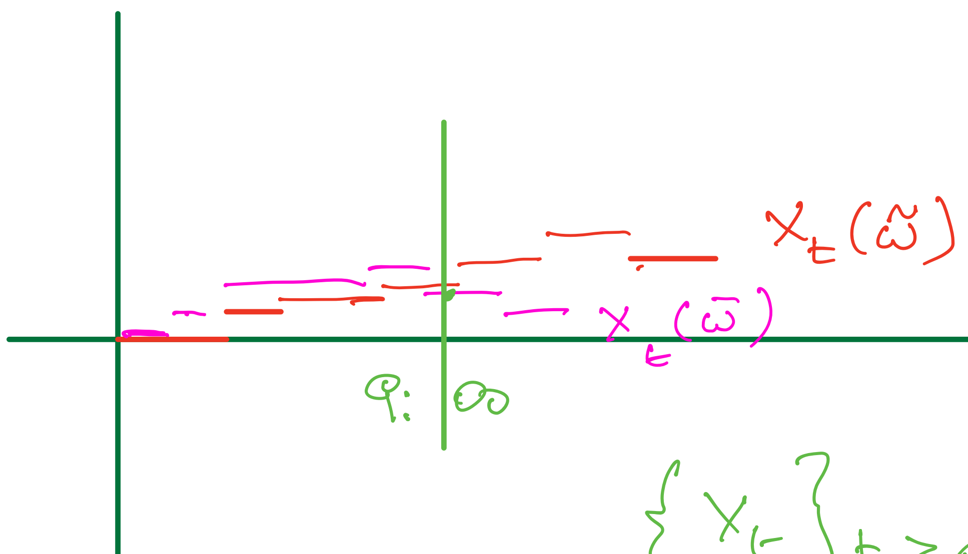
$\{X_t,\ t \in T\}$



Per $\omega = \omega^* (\circ \tilde{\omega})$ $\quad X_t(\omega^*)$ è una funz. del tempo

Per $t = \bar{t}$ $\quad X_{\bar{t}}(\omega)$ è una v. a.

$X_{\bar{t}}(\omega^*)$ è un numero



$\{X_t\}_{t \geq 0}$