Lezione 3

Es.

$X \sim \text{Poisson}(\lambda_1)$ $\qquad$ $Y \sim \text{Poisson}(\lambda_2)$

Indipendenti

$Z = X + Y$ $\qquad$ $P(Z = z) = ?$

$P(X + Y = z)$

v.a $\quad$ v.a

$N \sim \text{Poisson}(\lambda)$

$$P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

OSS

$X + 3 = z$

$$P(X + 3 = z) = P(X = z - 3) = e^{-\lambda_1} \frac{\lambda_1^{z-3}}{(z-3)!} \qquad z = 3, 4, \dots$$

$$P(X + Y = n) = \sum_{k=0}^{\infty} P(X + Y = n \mid Y = k) P(Y = k)$$

$$= \sum_{k=0}^{\cancel{\infty}\; n} P(X + k = n \mid Y = k) P(Y = k)$$

$$= \sum_{k=0}^{\cancel{\infty}\; n} P(X = n - k \mid Y = k) P(Y = k)$$

$$\underset{\substack{\text{sono}\\ \text{indip}}}{\overset{X \text{ e } Y}{=}} \sum_{k=0}^{\cancel{\infty}\, n} \underbrace{P(X=n-k)}\ \underbrace{P(Y=k)}$$

$$= \sum_{k=0}^{\cancel{\infty}\, n} \frac{\lambda_1^{n-k}}{(n-k)!} e^{-\lambda_1} \frac{\lambda_2^{k}}{k!} e^{-\lambda_2}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \boxed{\frac{n!}{k!(n-k)!}} \lambda_1^{n-k} \lambda_2^{k}$$

$$\frac{n!}{k!(n-k)!} \to \binom{n}{k}$$

$$\rightsquigarrow (\lambda_1+\lambda_2)^n$$

$$\frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n \sim \text{Poisson}(\lambda_1+\lambda_2)$$

Ripensiamo alle distribuzioni condizionali:

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

OSS.

1. $\sum_x P(X=x \mid Y=y) = \frac{\overbrace{\sum_x P(X=x, Y=y)}^{P(Y=y)}}{P(Y=y)}$

$= 1$

2. $\mathbb{P}(X=x \mid Y=y) \geq 0$

↳ é una distribuzione

Quando avevo $X \sim P(X=x)$ $x = \dots$

↳ $\mathbb{E}X = \sum x P(X=x)$

Ora ho le distribuz. condiziona-male

$$P(X=x \mid Y=y)$$

↳ posso definire il valore atteso

$$\mathbb{E}[X \mid Y=y] = \sum_x x \, P(X=x \mid Y=y)$$
$$= \varphi(y)$$

Es.

$$\mathbb{E}[X \mid Y=y_1] = \varphi(y_1)$$
$$\mathbb{E}[X \mid Y=y_2] = \varphi(y_2)$$

Dopo: $\mathbb{E}[X \mid Y] = \varphi(Y)$ è una v.a.

Esempio

$X \sim \text{Poisson}(\lambda_1) \qquad Y \sim \text{Poisson}(\lambda_2)$

$$\mathbb{E}[X \mid X+Y=n] = ?$$

$$\underbrace{P(X=x \mid X+Y=n)}_{\text{è una dist.}} =$$

$$\frac{P(X=x,\ X+Y=n)}{P(X+Y=n)} =$$

$$= \frac{P(X=x,\ Y=n-x)}{P(X+Y=n)} =$$

$$= \frac{P(X=x)\,P(Y=m-x)}{P(X+Y=m)}$$

$$\frac{\lambda_1^{x} e^{-\lambda_1}}{x!}$$

$$\underbrace{P(X+Y=m)}_{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^m}{m!}}$$

$$= \frac{\dfrac{\lambda_1^{x}\cancel{e^{-\lambda_1}}}{x!}\;\dfrac{\lambda_2^{(m-x)}\cancel{e^{-\lambda_2}}}{(m-x)!}}{\cancel{e^{-(\lambda_1+\lambda_2)}}\dfrac{(\lambda_1+\lambda_2)^m}{m!}} =$$

$$= \binom{m}{x}\Big(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{p}\Big)^{x}\Big(\underbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}_{1-p}\Big)^{m-x}$$

$$\sim Bi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}, m\right)$$

$M \sim Bi(m, p)$

$$\mathbb{E}(M) = mp$$

$$\mathbb{E}[X \mid X+Y=m] = m\frac{\lambda_1}{\lambda_1+\lambda_2} = \varphi(m)$$

Es. X, Y i.i.d. $Bi(n,p)$

$$P(X=k \mid X+Y=m) =$$

$k \leq \min(n,m)$

$$= P(X=k \mid X+Y=m) = \frac{P(X=k)\,P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{k} \cancel{p^{k}} \cancel{(1-p)^{n-k}} \binom{n}{m-k} \cancel{p^{m-k}} \cancel{(1-p)^{n-m+k}}}{\binom{2n}{m} \cancel{p^{m}} \cancel{(1-p)^{2n-m}}}$$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$ è ipergeom.

$$\mathbb{E}[X \mid X+Y=m] = \frac{m\cancel{n}}{2\cancel{n}} = \frac{m}{2}$$

$= \varphi(m)$

Esempio Ci siano $n+m$ prove indip. con probab. di successo p. Voglio il n° atteso di successi in n prove, sapendo che in tutto ci sono stati k successi

Sol.

$$Y = \{ n° \text{ totale di successi} \}$$

$$X_i = \begin{cases} 1 & i\text{-ma prova è un suc.} \\ 0 \end{cases}$$

Voglio

$$\mathbb{E}\left[\sum_{1}^{n} X_i \mid Y = k \right] =$$

$$= \sum_{1}^{n} \underbrace{\mathbb{E}\left[X_i \mid Y = k \right]}_{P(X=1 \mid Y=k) = \frac{k}{n+m}} = n \frac{k}{n+m} = \varphi(k)$$

Se $m = n$ $k = m$ $\rightarrow$ esercizio prec.

CONDIZIONAMENTO PER V.A. CONTINUE

(X,Y) v.a. continue

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} P(X \le x, Y \le y)$$

$$P(X \le x \mid Y = y) = ?$$

Problema $P(Y=y) = 0$??

↳ non posso considerare

$$P(X \le x \mid Y=y) = \frac{P(X \le x, Y=y)}{\underbrace{P(Y=y)}_{=0!}}$$

$$\lim_{h \to 0} P(X \le x \mid Y \in (y, y+h)) =$$

$$P(X \le x \mid Y=y)$$

$$= \lim_{h \to 0} \frac{P(X \le x, Y \in (y, y+h))}{P(Y \in (y, y+h))} =$$

$$= \lim_{h \to 0} \frac{\int_{-\infty}^{x} \int_{y}^{y+h} f_{xy}(u,v)\,du\,dv}{\int_{y}^{y+h} f_y(v)\,dv} \overset{\text{Hôpital}}{=}$$

$$= \lim_{h \to 0} \frac{\int_{-\infty}^{x} f_{xy}(u, y+h)\,du}{f_y(y+h)} = \phi$$

Formule di derivate di funzioni integrali

$$\left[\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} f(u,x)\,du = \beta'(x) f(\beta(x), x) - \alpha'(x) f(\alpha(x), x) + \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(u,x)\,du \right.$$

$$\phi = \frac{\int_{-\infty}^{x} f_{xy}(u,y)\,du}{f_y(y)} =$$

$$= \int_{-\infty}^{x} \frac{f_{XY}(u,y)}{f_Y(y)}\,du$$

funzione di densità di probab. condizionale di X dato y

$f_{X|Y}(u|y)$

$$P(X \leq x \mid Y = y) = \int_{-\infty}^{x} f_{X|Y}(u|y)\,du$$

OSS.

1. $f_{X|Y}(x|y) \geq 0$

2. $\int_{-\infty}^{+\infty} f_{X|Y}(x|y)\,dx = 1$

$f_{X|Y}(x|y)$ è una funz. d.d.p.

$\hookrightarrow$ posso definire

$$E[X \mid Y=y] = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y) \, dx$$

$$= \varphi(y)$$

Esempio

$$f_{X,Y}(x,y) = \begin{cases} 6xy(2-x-y) & x \in (0,1) \\ & y \in (0,1) \\ 0 & \text{altrove.} \end{cases}$$

$$E[X \mid Y=y] = ?$$

$$f_{X|Y}(x|y) = \frac{6xy(2-x-y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^1 6xy(2-x-y) \, dx =$$

$$= y(4-3y)$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{6\,x\cancel{y}(2-x-y)}{\cancel{y}(4-3y)} & x \in (0,1) \\ & y \in (0,1) \\ 0 & \text{~~a~~ altrimenti} \end{cases}$$

$$E[X \mid Y = y] = \int_0^1 x \, \frac{6\,x(2-x-y)}{(4-3y)} \, dx$$

$$= \frac{5-4y}{8-6y} = \varphi(y) \qquad y \in [0,1]$$

Esercizio

$$f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy} & y \in (0,2) \\ & x \in \mathbb{R}^+ \\ 0 & \text{altrimenti.} \end{cases}$$

$$\mathbb{E}[e^{X/2} \mid Y = 1]$$

$$f_{X|Y}(x \mid y=1) = \frac{f_{XY}(x,y)\big|_{y=1}}{f_Y(y)\big|_{y=1}} =$$

$$\frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{\frac{1}{2}e^{-x}}{\int_0^{\infty} \frac{1}{2} e^{-x} dx} =$$

$$e^{-x} \quad x \in (0, \infty)$$

$$\mathbb{E}[e^{X/2} \mid Y=1] =$$

$$\int_0^{\infty} e^{-x} \cdot e^{x/2} dx = 2$$

$\uparrow$

numero

$$\boxed{\begin{aligned} &\mathbb{E}[X] = \text{numero} \\ &\mathbb{E}[X \mid Y=y] = \varphi(y) \\ &\mathbb{E}[X \mid Y] = \underbrace{\varphi(Y)}_{\text{v. a.}} \end{aligned}}$$

Caso Y discreto

$$\underbrace{\mathbb{E}[X \mid Y]}_{\varphi(Y)} = \begin{cases} \mathbb{E}[X \mid Y=y_1] & P(Y=y_1) \\ \mathbb{E}[X \mid Y=y_2] & P(Y=y_2) \\ \vdots & \\ \mathbb{E}[X \mid Y=y_n] & P(Y=y_n) \end{cases}$$

$$X = \begin{cases} X = x_1 & P(X=x_1) \\ \vdots & \vdots \\ X = x_n & P(X=x_n) \end{cases}$$

$E[X|Y]$ è una v.a.

$\mathbb{E}[\mathbb{E}[X|Y]]$

<u>Teorema</u> (della doppia attesa)

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}(X|Y)]$$

Dim

$$\mathbb{E}[\mathbb{E}[X|Y]] = \sum_y \mathbb{E}[X|Y=y] P(Y=y)$$

$$= \sum_y \sum_x x \underbrace{P(X=x|Y=y)P(Y=y)}_{P(X=x,\, Y=y)}$$

$$= \sum_x x \underbrace{\sum_y P(X=x, Y=y)}_{P(X=x)}$$

$$= \mathbb{E}[X]$$

Esempio

$N \sim Geo(p) \qquad E(N) = ?$

$$Y = \begin{cases} 0 & \text{I esito è ins.} \\ 1 & \text{" \quad " suc.} \end{cases}$$

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}(N \mid Y)] =$$

$$\underbrace{\mathbb{E}[N \mid Y=1]}_{1} \underbrace{P(Y=1)}_{p} + \underbrace{E[N \mid Y=0]}_{1+EN} \underbrace{P(Y=0)}_{1-p}$$

$$EN = p + (1+EN)(1-p)$$

$$EN = 1/p$$

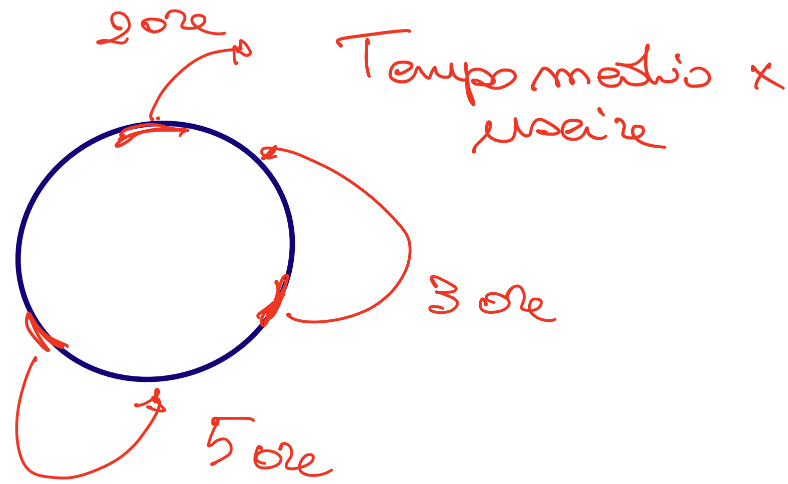
Esempio

Un minatore intrappolato in una stanza con 3 porte

- I porta conduce fuori in 2 ore

- II porta   riporta nella stanza
                dopo 3 ore

III porta   ... dopo 5 ore



$Y_i = i$ $i = 1, 2, 3$

↑ porta scelta

X : tempo per uscire

$$EX = E[E[X|Y]]$$

$$E[X|Y=1] = 2$$

$$E[X|Y=2] = 3 + EX$$

$$E[X \mid Y=3] = 5 + EX$$

$$EX = \frac{1}{3}\left[2 + 3 + EX + 5 + EX\right]$$

$$EX = 10$$

$$E[X] = E[E(X \mid Y)] =$$

$$E[X \mid Y=1]\,\underbrace{P(Y=1)}_{1/3} + \ldots E[X \mid Y=3]\,P(Y$$

Diseg. di Markov $\quad EX < \infty$

$$P(X \geq a) \leq \frac{EX}{a}$$

Diseg. Cebicev $\quad EX = \mu$

$\quad Var\,X = \sigma^2$

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Legge dei grandi numeri

$\{X_i\}$ ind. ident. dist.

$EX_i = \mu$

$\hookrightarrow \dfrac{\sum_{1}^{n} X_i}{n} \xrightarrow{n \to \infty} \mu$ LLN

Teorema del limite centrale

$\{X_i\}$ i.i.d

$$\frac{\sum^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{\text{legge}} Z \sim \mathcal{N}(0,1)$$