Homework 3

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Problem 1

Find a solution near $(x, y) = (1, 1)$ to the nonlinear set of equations

$$f(x, y) = 3x^2 - y^2 = 0$$
$$g(x, y) = 3xy^2 - x^3 - 1 = 0$$

Solution

(a) Iterate on this system numerically using the iteration scheme

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \frac{1}{6} & \frac{1}{18} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix} \text{ for } n = 1, 2, \ldots$$

starting with $x_0 = y_0 = 1$ and check how well it converges.

Using the above iteration scheme, with

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } TOL = 1 \times 10^{-8}$$

The root, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8660254 \end{bmatrix}$, is found in **56** iterations.

(b) Provide some motivation for the particular choice of 2×2 matrix in the equation above.

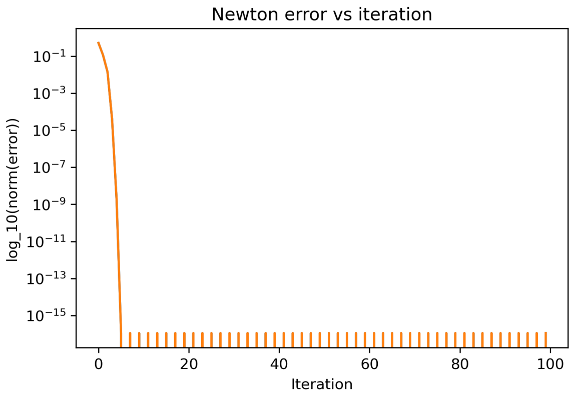
For

$$F(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

The particular matrix used in part 1a is the inverse of the Jacobian of $F(x_0, y_0)$ for the given x_0 and y_0, making this iteration scheme the Lazy Newton method.

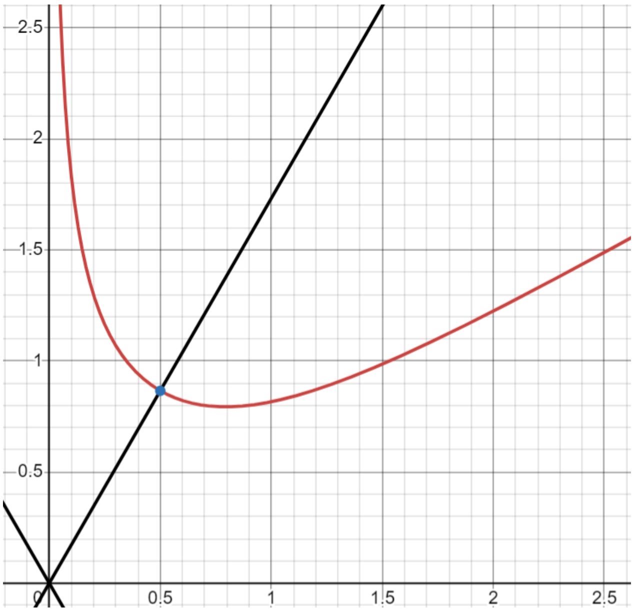
(c) Iterate on (0.1) using Newton's method, using the same starting approximation $x_0 = y_0 = 1$ and check how well this converges.

Newton's method found the same rate in 7 iterations to find the same value with the same tolerance of 1×10^{-8}, however, after 7 iterations, Newton's method oscillated between the final found value and a value very close to the final value found by the method in 1a. This can be seen in the below plot of $log_{10}(error)$ vs iterations below.



As seen in the two plots above, Newton converges quadratically to an oscillating solution while Lazy Newton (or the method used in 1a) converges about linearly.

(d) From the graph we observe that the root occurs at $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2} \approx 0.8660254$:



We can prove this analytically by plugging in these values $(x, y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. When we evaluate $f(x, y)$ and $g(x, y)$ at these values we see

$$f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 3\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = 0$$

$$g\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^3 - 1 = 0$$

Problem 2

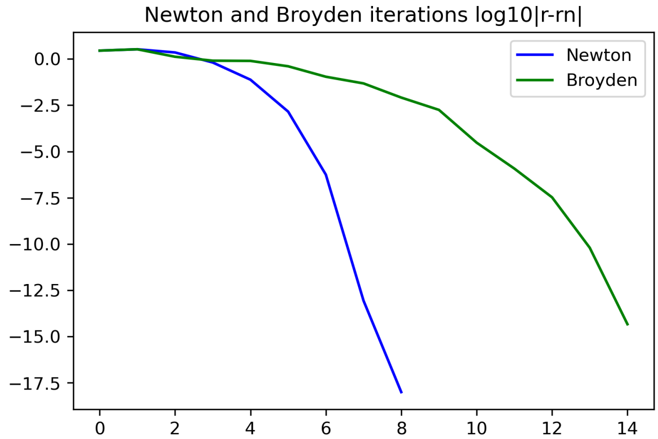
The nonlinear system

$$f(x, y) = x^2 + y^2 - 4 = 0$$
$$g(x, y) = e^x + y - 1 = 0$$

has two real solutions. Use Newton, Lazy Newton, and Broyden with different initial guesses to find the roots and compare the other two methods to Newton's method.

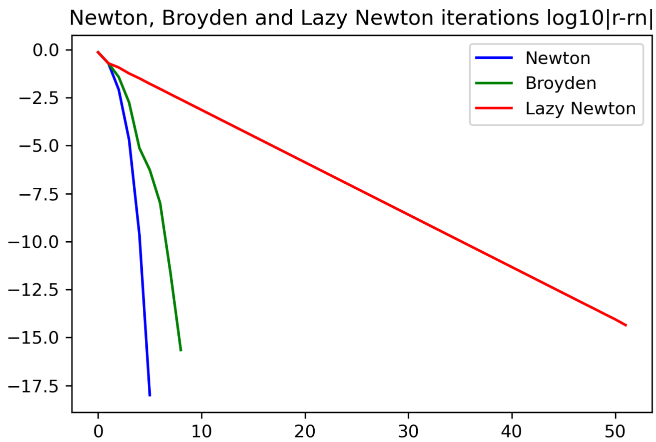
Solution

i. $x = 1, y = 1$
Lazy Newton did not converge, but Broyden did. This is because for Lazy Newton, the given initial conditions make the `lu_solve` returns a matrix with either `inf` or `NaNs`



The above figure shows the $log_{10}(|r - r_n|)$ vs iteration for both Newton and Broyden. Here you can see that Newton converges faster than Broyden, although both converge superlinearly.

ii. $x = 1, y = 1$



In this case, Lazy Newton fails to converge within 100 iterations and appears to converge linearly, while Newton and Broyden both converge superlinearly. Like in part i, Broyden still converges slower than Newton as can be show in the graph of $log_{10}(|r - r_n|)$ vs iteration below.

iii. $x = 0, y = 0$
In this case, all methods failed to converge unless $B0 = I$ (the identity matrix), because $J(x_0, y_0)$ is singular.

Problem 3

Consider the nonlinear system

$x + cos(xyz) - 1 = 0, (1-x)^{\frac{1}{4}} + y + 0.05z^2 - 0.15z - 1 \quad = 0, -x^2 - 0.1y^2 + 0.01y + z - 1 = 0$

Test the following techniques for approximating the solution to the nonlinear system to within 10^{-6}:

- Newton's method
- Steepest descent method
- First steepest descent method with a stopping tolerance of 5×10^{-2}. Use the result of this as the initial guess for Newton's method. Which technique converges fastest and explain why.

Solution

Sorry Alex, my poor time management let me down :(

Problem 4

Given the data, write $p(x)$ using the Lagrange polynomial basis for this problem.

Solution

Lagrange polynomial basis:

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3 x) + y_4 l_4(x)$$

$$\begin{aligned} p(x) = -125 \frac{(x-2)(x-3)(x-5)(x-8)}{(-2)(-3)(-5)(-8)} - 27 \frac{(x)(x-3)(x-5)(x-8)}{(2-0)(2-3)(2-5)(2-8)} \\ -8 \frac{(x)(x-2)(x-5)(x-8)}{(3-0)(3-2)(3-5)(3-8)} + 27 \frac{(x)(x-2)(x-3)(x-5)}{(8-0)(8-2)(8-3)(8-5)} \end{aligned}$$

$$\begin{aligned} p(x) = -\frac{125}{240} \big((x-2)(x-3)(x-5)(x-8)\big) - \frac{27}{36} \big(x(x-3)(x-5)(x-8)\big) \\ -\frac{8}{30} \big(x(x-2)(x-5)(x-8)\big) + \frac{27}{720} \big((x)(x-2)(x-3)(x-5)\big) \end{aligned}$$

Newton Interpolation:

$$p(x)) = 27x(x-2)(x-3)(x-5) - 8x(x-2) - 27x - 125$$