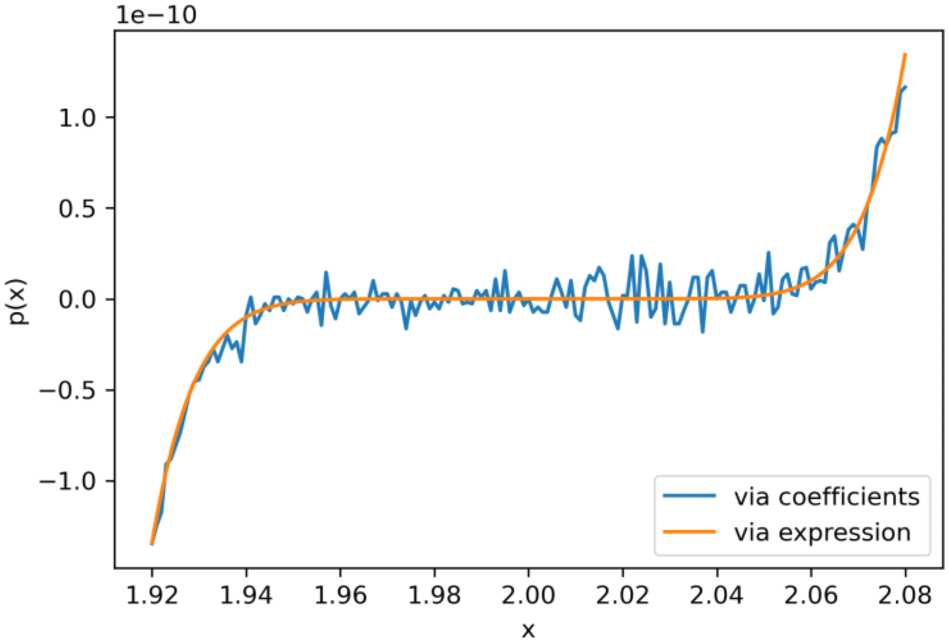
1.

Friday, January 26, 2024 7:00 PM

i. Plot $p(x)$ via its coefficients

ii. Plot $p(x)$ via the expression $p(x) = (x-2)^9$



iii Plotting via coefficients leads to loss of precision from subtraction and addition of numbers with different orders of magnitude while the expression has less operations that lose precision.

2.

Thursday, January 25, 2024 2:57 PM

i. Rearrange $\sqrt{x+1}-1$ to avoid cancellation for $x \simeq 0$

↳ The issue is subtracting two close values

$\sqrt{x+1}-1 \quad x \simeq 0$

$-1+\sqrt{x+1} \cdot \left(\frac{-1-\sqrt{x+1}}{-1-\sqrt{x+1}}\right)$

$\frac{1-(x+1)}{-1-\sqrt{x+1}} = \frac{-x}{-1-\sqrt{x+1}} = \frac{-x}{-(1+\sqrt{x+1})}$

$$\boxed{\sqrt{x+1}-1 = \frac{x}{1+\sqrt{x+1}}}$$

This rearrangement avoids subtraction of two close numbers and so avoids cancellation.

ii Rearrange $\sin(x)-\sin(y)$ to avoid cancellation for $x \simeq y$

↳ The issue is subtracting two close values

$\sin(x)-\sin(y)$ for $x \simeq y$

Assume $x = y+h$ with $h \approx 0$ known to full precision

Trig identity: $\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B)-\sin(A-B))$

$2\cos(A)\sin(B) = \sin(A+B)-\sin(A-B) = \sin(x)-\sin(y)$

① $A+B = x$

② $A-B = y$

③ $x = y+h$

Add ① & ②

$2A = x+y$

$A = \frac{x+y}{2}$

Backsubstitute for A into ①

$\frac{x+y}{2} + B = x$

$B = x - \frac{x+y}{2}$

$= \frac{2x-x-y}{2} = \frac{x-y}{2}$

Substitute $x-y = h$ into above equation

$B = \frac{h}{2}$

$$\boxed{\sin(x)-\sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{h}{2}\right)}$$

Because h is known and precise, this avoids cancellation because there is no subtraction of two close numbers that that reduces precision.

iii Rearrange $\frac{1-\cos(x)}{\sin(x)}$ to avoid cancellation for $x \simeq 0$

↳ The issue is $1-\cos(x)$

$\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$

$\frac{1-\cos(x)}{\sin(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)}$

$\frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))} = \frac{\sin^2(x)}{\sin(x)(1+\cos(x))} = \frac{\sin(x)}{1+\cos(x)}$ for $x \simeq 0$

$$\boxed{\frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)}}$$

This only uses addition and division and so avoids cancellation.

3.

Thursday, January 25, 2024 3:02 PM

3a. $P_2(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2$

$f'(x) = (1+x+x^3)(-\sin(x)) + \cos(x)(1+3x^2)$

$f''(x) = -x^3\cos(x) - 6x^2\sin(x) + 5x\cos(x) - 2\sin(x) - \cos(x)$

$f(0) = 1$

$f'(0) = 1$

$f''(0) = -1$

$P_2(x) = 1 + x - \frac{1}{2}x^2$

$P_2(0.5) = 1 + 0.5 - \frac{1}{2}(\frac{1}{2})^2$

$= 1.375$

$f(0.5) = (1 + 0.5 + 0.5^3)\cos(0.5)$

1.426071663

$\text{Error} = \frac{|f^{(3)}(c)|}{3!}(x)^3$

Where $0 < c < 0.5$ & $f^{(3)}(c)$ is the maximum value of $f^{(3)}(x)$ on $0 \le x \le 0.5$

To estimate this maximum for a "worst case" error value we will assume all sin(x) & cos(x) terms to be +1 or −1 so as to maximize $|f^{(3)}(c)|$ and take $c = 0.5$ for polynomial term

* Note that this estimate of sin(x) & cos(x) is lose and increases the estimate of $\max|f^{(3)}(x)|$ more than the actual max on the interval $0 \le x \le 0.5$. This inflates our error's upper bound

$f^3(x) = x^3\sin(x) + 9x^2\cos(x) + 17x\sin(x) + 3\cos(x) + \sin(x)$

Max of this for $0 < c < 0.5$

$= (0.5)^3 + 9(0.5)^2 + 17(0.5) + 3(0.5) + 1$

$= 13.375$

$\boxed{\text{Error} \le \left|\frac{13.375}{3!}(0.5)^3\right| = 0.2786458333}$ (via calculator)

Actual error $= 1.426071663 - 1.375$

$\boxed{\text{Actual error} = 0.051071663}$

The calculated upper bound is much greater than the actual error.

b. Upper bound error $= \frac{|f^{(3)}(c)|}{3!}(x)^3$

$|f(x) - P_2(x)| \le \frac{|f^{(3)}(c)|}{3!}(x)^3$

where c maximizes $|f^{(3)}(x)|$ on $0 < |c| < x$

$\text{Error} \le \left|\frac{f^{(3)}(c)}{6}x^3\right| = \left|\frac{x^3}{6}(c^3\sin(c) - 9c^2\cos(c) - 17c\sin(c) + 3\cos(c) + \sin(c))\right|$

To estimate this maximum for a "worst case" error value we will assume all sin(x) & cos(x) terms to be +1 or −1 so as to maximize $|f^{(3)}(c)|$ and c is the

$\boxed{\text{Error} \le \left|\frac{x^3}{6}(c^3 + 9c^2 + 17c + 3 + 1)\right|}$

For $c < 0$, we will change the sin(x) & cos(x) signs as needed to maximize this value such as $c^3\sin(c)$ going to $-c^3$ for $c < 0$.

For $c \ge 0$, the above equation holds

c. $\int_0^1 P_2(x)\,dx = \int_0^1 1 + x - \frac{1}{2}x^2\,dx$

$= \left[x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^1$

$= (1 + \frac{1}{2} - \frac{1}{6}) - (0)$

$\boxed{= \frac{4}{3} = 1.333}$

d. On [0,1], we can approximate our upper bound by using the bounds $|\cos(x)| \le 1$ & $|\sin(x)| \le 1$ where we take sin(x) & cos(x) to be ±1 so all signs are the same. The other terms are maximized by using $x = 1$.

Therefore:

$\text{Error} \le \int_0^1 \frac{x^3}{6}(x^3 + 9x^2 + 17x + 3 + 1)\,dx$

$\int_0^1 \frac{x^6}{6} + \frac{3}{2}x^5 + \frac{17}{6}x^4 + \frac{2}{3}x^3\,dx$

$\left[\frac{1}{42}x^7 + \frac{1}{4}x^6 + \frac{17}{30}x^5 + \frac{1}{6}x^4\right]_0^1$

$= \frac{1}{42} + \frac{1}{4} + \frac{17}{30} + \frac{1}{6}$

$\boxed{\text{Error} \le \frac{141}{140} \approx 1.007}$

4.

Thursday, January 25, 2024 3:19 PM

a. $ax^2+bx+c=0$, find the roots & relative errors if the square root can only be calculated with 3 correct decimals.

$a=1$, $b=-56$, $c=1$

$$\tilde{r}_1 = \frac{56+\sqrt{(-56)^2-4(1)(1)}}{2(1)}$$

$$= \frac{56+\sqrt{56^2-4}}{2}$$

$$= \frac{56+55.964}{2}$$

$$\boxed{\tilde{r}_1 = 55.982}$$

$$\tilde{r}_2 = \frac{56-\sqrt{(-56)^2-4(1)(1)}}{2(1)}$$

$$= \frac{56-55.964}{2}$$

$$\boxed{\tilde{r}_2 = 0.018}$$

$r_1 = 55.9821376$
r_2 0.0178628407 } from calculator

Relative errors:

$$\frac{|r_1-\tilde{r}_1|}{r_1} = 2.45\times10^{-6}$$

$$\frac{|r_2-\tilde{r}_2|}{r_2} = 7.68\times10^{-3}$$

b. Use $a(x-r_1)(x-r_2)=0$ to find a better approximation of the "bad" root by relating r_1 & r_2 to a, b, & c.

$a(x-r_1)(x-r_2)=0$

$$ax^2+bx+c = a(x-r_1)(x-r_2)$$
$$= a(x^2-r_2x-r_1x+r_1r_2)$$
$$= a(x^2-(r_1+r_2)x+r_1r_2)$$

relation 1: $b=-a(r_1+r_2)$
relation 2: $c=ar_1r_2$

Use good root, $\tilde{r}_1=r_1$ to solve

Relation 1:

$$-\frac{b}{a} = r_1+r_2$$
$$56 = 55.982+r_2$$
$$r_2 = 0.018$$

Relation 2:

$$\frac{c}{a} = r_1r_2$$
$$1 = (55.982)(r_2)$$
$$\boxed{r_2 = 0.0178628845}$$

Relation 2 results in a more accurate "bad" root.

5.

Thursday, January 25, 2024 4:47 PM

a. Find the upper bounds on absolute error $|\Delta y|$ and relative error $\frac{|\Delta y|}{|y|}$. When is rel. error large?

Absolute error:

$y = x_1 - x_2$

$\tilde{x}_1 = x_1 + \Delta x_1$

$\tilde{x}_2 = x_2 + \Delta x_2$

$\tilde{y} = y + \Delta y$

$|\Delta y| = |\tilde{y} - y|$

$= |\tilde{x}_1 - \tilde{x}_2 - (x_1 - x_2)|$

$= |x_1 + \Delta x_1 - (x_2 + \Delta x_2) - x_1 + x_2|$

$= |\cancel{x_1} + \Delta x_1 - \cancel{x_2} - \Delta x_2 - \cancel{x_1} + \cancel{x_2}|$

$= |\Delta x_1 - \Delta x_2|$

$|\Delta y| \le |\Delta x_1| + |\Delta x_2|$

Relative error:

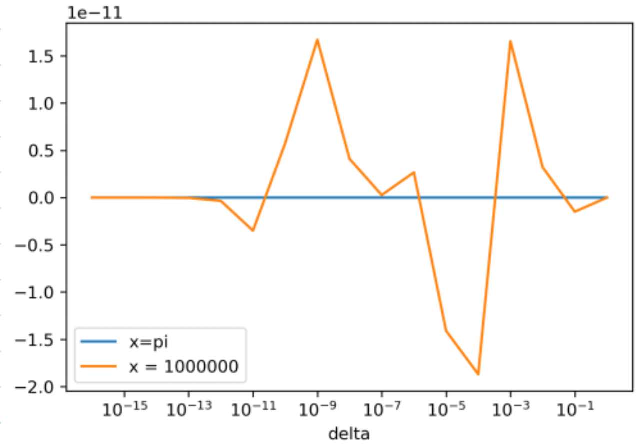
$\frac{|\Delta y|}{|y|} \le \frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}$

Relative error is largest when $|x_1 - x_2|$ is small and $|\Delta x_1|$ and/or $|\Delta x_2|$ are large.

b. $\cos(x+\delta) - \cos(x)$

$\cos(x+\delta) - \cos(x) = -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right)$

$= -2\sin\left(x + \frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$



c. $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi), \ \xi \in [x, x+\delta]$

Approximate $\cos(x+\delta) - \cos(x)$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$

Plugging into Taylor expansion

$\cos(x+\delta) - \cos(x) = -\delta\sin(x) \underbrace{- \frac{\delta^2}{2}\cos(\xi)}_{\text{error, so not plotted}}$