Homework 2

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Problem 1

Solution

1a Show that $(1+x)^n = 1 + nx + o(x)$
$\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x}$
Use a Taylor expansion of $(1+x)^n$
$= \lim_{x\to 0} \frac{1+nx+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n-1-nx}{x}$
$= \lim_{x\to 0} \binom{n}{2}x + \binom{n}{3}x^2 \ldots + \binom{n}{n}x^{n-1}$
$= 0$ So $(1+x)^n - 1 - nx$ converges at least as quickly as x.

1b
Show that $xsin(\sqrt{x}) = O(x^{\frac{3}{2}})$
$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
$xsin\sqrt{x} = x^{\frac{3}{2}} - \frac{1}{6}x^{\frac{5}{2}} + \frac{1}{120}x^{\frac{7}{2}} + \ldots$
$\lim_{x\to 0} xsin\sqrt{x} = x^{\frac{3}{2}} - \frac{1}{6}x^{\frac{5}{2}} + \frac{1}{120}x^{\frac{7}{2}} + \ldots$
$= 1$
So $xsin(\sqrt{x})$ converges as fast as $x^{\frac{3}{2}}$

1c
$\lim_{x\to\infty} \frac{e^{-t}}{t^{-2}}$

L'hopitals
$= \lim_{x\to\infty} \frac{2t}{e^t}$
$= 0$
So e^{-t} converges at least as fast as $\frac{1}{t^2}$

1d
Show that $\int_0^\epsilon e^{-x^2}dx = O\epsilon$
$\lim_{x\to\infty} \frac{\int_0^\epsilon e^{-\epsilon^2}}{\epsilon}$ L'hopitals
$\lim_{x\to\infty} \frac{e^{-\epsilon^2}}{1}$
$= 1$

Problem 2

Solution

2a

$Ax = b$
$\widetilde{b} = b + \Delta b$
$\widetilde{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$
$\widetilde{x} = A^{-1}\widetilde{b}$
$= A^{-1}(b + \Delta b)$
$= A^{-1}b + A^{-1}\Delta b$
$x = A^{-1}b$
$\Delta x = \widetilde{x} - x$
$\Delta x = A^{-1}b + A^{-1}\Delta b - AA^{-1}b$
$\Delta x = A^{-1}\Delta b$

2b

Condition Number: 1999999944.5415618
(Computed using np.linalg.cond)

2c

When Δb_1 and Δb_2 are of magnitude 10^{-5} but are different values, the relative error is high. When $\Delta b_1 = 2e - 5$ and $\Delta b_2 = 5e - 5$

$$x = \begin{bmatrix} 30001 \\ -29999 \end{bmatrix}$$

This is far from our actual answer,

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

relative error is

$$RelativeError = \begin{bmatrix} 29999.9991734 \\ 29999.99913335 \end{bmatrix}$$

A high condition number corresponds to a high relative error even with relatively small perturbations. The perturbations in the original inputs lead to much larger output perturbations.

When the perturbations are the same value,

$$x = \begin{bmatrix} 1.00002003 \\ 1.00002003 \end{bmatrix}$$

$$RelativeError = \begin{bmatrix} 2.00271606e - 05 \\ 2.00271606e - 05 \end{bmatrix}$$

Different value perturbations seem more realistic, as they can come from realistic noise, measurement errors, or other real-world issues that result in relatively random perturbations.

Problem 3

Solution

3a
$f(x) = e^x - 1$
$f'(x) = e^x$
$\kappa_f(x) = |\frac{xe^x}{e^x-1}|$
$\kappa_f(x)$ is ill-conditioned at $x = 0$

3b Is the algorithm stable?
Because the algorithm "breaks" at $x = 0$, the algorithm is unstable.

3c True value: 10^{-10}
Computed value: $1.000000082740371 \times 10^{-9}$ This correctly gives 8 digits, which is expected because the algorithm only breaks at $x = 0$ rather than near $x = 0$.

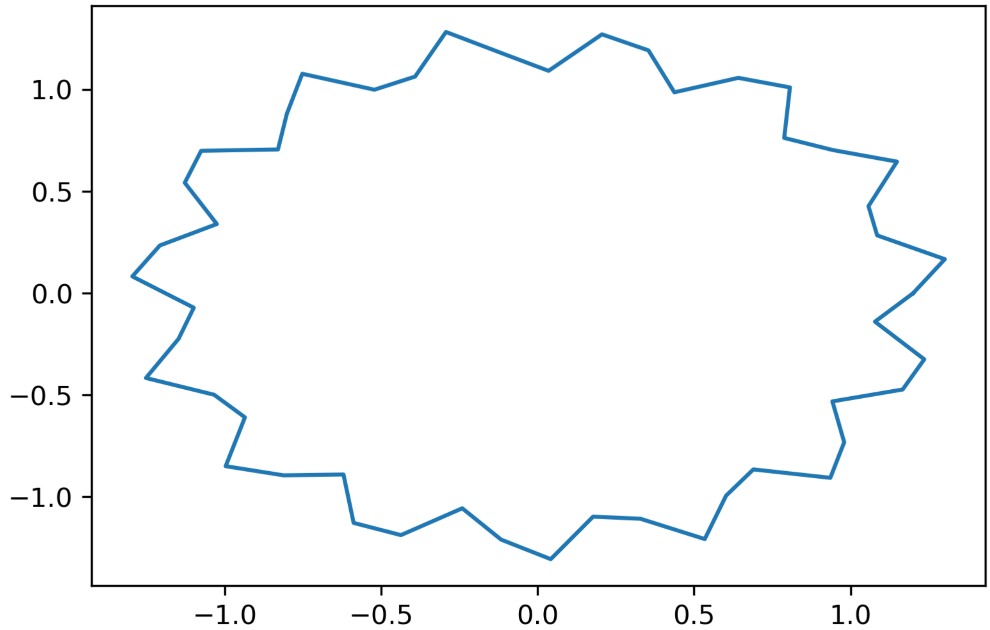
3d
Taylor series approximation: $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

Problem 4

Solution

4a
$S = -20.050835454935093$
(See code on github)

4b
Part 1: $R = 1.2, \delta r = 0.1, f = 15, p = 0$



Part 2: $R = i, \delta r = 0.05, f = 2 + i$ for the i^{th} curve. p is a uniformly distributed random number.