Rutgers Business School--Newark & New Brunswick
MQF 22:839:571, Financial Modeling I, Spring 2020
Assignment IV
Assigned 3/27/20, Due 4/11/20

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1. Suppose that there are n risky stocks whose returns are multivariate normally distributed. Let $E(\tilde{r})$ denote the *n*x*1* vector of expected returns and $E[(\tilde{r}-E(\tilde{r}))(\tilde{r}-E(\tilde{r}))'] \equiv V$ denote the *n*x*n* covariance matrix. There is a risk-free asset whose return is r_f. Let *a* be a frontier portfolio with mean $E(\tilde{r}_a)$ and standard deviation σ_a. Let *b* be any other portfolio (not necessarily a frontier portfolio) with mean $E(\tilde{r}_b)$ and standard deviation σ_b. Show that the correlation between portfolios *a* and *b* equals portfolio *b*'s Sharpe ratio divided by portfolio *a*'s Sharpe ratio.
(Note: The Sharpe ratio of a portfolio *i* is defined as $\frac{E(\tilde{r}_i)-r_f}{\sigma_i}$.)

2. Suppose that there are *N* risky assets, but the risk-free borrowing rate is higher than the risk-free lending rate. Show graphically the portfolio frontier of all assets. Next show graphically the portfolio frontier of all assets when risk-less borrowing is prohibited.

3. Suppose that *p* is a frontier portfolio and *q* is any portfolio, such that $E(\tilde{r}_q)=E(\tilde{r}_p)$. Show that $\mathrm{cov}(\tilde{r}_q,\tilde{r}_p)=\mathrm{var}(\tilde{r}_p)$ and that $\rho(\tilde{r}_q,\tilde{r}_p)\in(0,1]$, where $\rho(\tilde{r}_q,\tilde{r}_p)$ is the correlation coefficient.

4. Suppose that there are two risky assets and one risk-free asset with

$$\tilde{r} \equiv \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix} \sim N\left(\mu \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 \end{bmatrix} \right)$$

$r_f<\mu,\ 0<\sigma_1<\sigma_2$, where $\rho_{1,2}$ is the correlation coefficient between risky assets 1 and 2, and r_f is the risk-free rate.

(a) Assume that there is no restriction on short sale of risky assets or on risk-free borrowing. Also assume that $-1<\rho_{1,2}<1$. Show that $\{\tilde{r},r_f\}$ exhibits two-fund separation.
(b) Assume that both short sale of risky assets and risk-free borrowing are prohibited. Further assume that $\rho_{1,2}=-1$. Show that there is one-fund separation.
(c) Assume that both short sale of risky assets and risk-free borrowing are prohibited. Further assume that $\rho_{1,2}=1$. Show that there is two-fund separation.

5. Suppose that there are only two risky assets, 1 and 2, in the economy and the mean-variance frontier is constructed from these two risky assets. No risk-free asset is available. Furthermore, we have

$$\tilde{r} \equiv \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 \end{bmatrix} \right)$$

where $\mu_1 \neq \mu_2$. Without loss of generality, you can assume zero correlation between the two assets, i.e., $\rho_{1,2} = 0$. Note also that the normality assumption is not necessary for this problem.

(a) Suppose that investor A believes that the "market portfolio" is

$$W_A = \begin{bmatrix} W_{1A} \\ W_{2A} \end{bmatrix}$$

while investor B believes that the "market portfolio" is

$$W_B = \begin{bmatrix} W_{1B} \\ W_{2B} \end{bmatrix}$$

and $W_A \neq W_B$. Given these facts, what beta value will each investor calculate for assets 1 and 2?

(b) Compute the zero-beta portfolio and construct the security market line for each investor. Verify that the expected return of asset 1 computed by both investors is the same.

6. Suppose that there is a risk-free asset in strictly positive supply and investors prefer to hold efficient frontier portfolios. Borrowing at the risk-free rate is prohibited. Show that

$$\begin{aligned}
&E(\tilde{r}_q) = E(\tilde{r}_{zc(m)}) + \beta_{qm}\{E(\tilde{r}_m) - E(\tilde{r}_{zc(m)})\} \\
&E(\tilde{r}_m) > E(\tilde{r}_{zc(m)}) \quad \text{and} \\
&E(\tilde{r}_{zc(m)}) \geq r_f \ .
\end{aligned}$$

7. Suppose that investors would like to hold efficient portfolios and that the risk-free borrowing rate is strictly higher than the risk-free lending rate, i.e., $r_B > r_L$. Show that

$$\begin{aligned}
&E(\tilde{r}_q) = E(\tilde{r}_{zc(m)}) + \beta_{qm}\{E(\tilde{r}_m) - E(\tilde{r}_{zc(m)})\} \\
&E(\tilde{r}_m) > E(\tilde{r}_{zc(m)}) \quad \text{and} \\
&r_B \geq E(\tilde{r}_{zc(m)}) \geq r_L \ .
\end{aligned}$$

8. The probability distributions of the market portfolio and stock X are given as follows:

state	probability	market portfolio return	stock x return
1	0.10	-0.15	-0.30
2	0.30	0.05	0.00
3	0.40	0.15	0.20
4	0.20	0.20	0.50

The risk free rate is 6%. Assume that the market portfolio is an efficient portfolio and the CAPM holds. Is stock X currently correctly priced? If not, how do you expect its price to move?