FM1 Notes, Spring 2020

Two Fund Separation, CAPM and APT

Two Fund Separation

We know that all portfolios on the portfolio frontier can be generated by any two distinct frontier portfolios. Thus, if individuals prefer frontier portfolios, they can simply hold a linear combination of two frontier portfolios.

In this case, given any feasible portfolio, for an individual investor, there exists a portfolio of two mutual funds (also called separating portfolios) such that the investor prefers at least as much as the original portfolio. This is called <u>two (mutual) fund separation.</u>

When two fund separation obtains, any two distinct frontier portfolios can serve as separating portfolios. In particular, we can pick $p \neq mvp$ and $zc(p)$ to be the separating portfolios.

One fund separation: A vector of asset returns $\tilde{r}$ exhibits one fund separation if there exists a feasible portfolio α (also called the separating portfolio) such that every risk-averse individual prefers α to any other feasible portfolio.

The Market Portfolio

Let $W_0^i > 0$ be individual i's initial wealth (a scalar). Let w_{ij} be the proportion of the initial wealth invested in the j^{th} security by individual i.

The total wealth of the economy $W_{mo} = \sum_{i=1}^{I} W_0^i$, where I is the number of individuals in the economy. In equilibrium, the total wealth W_{mo} is equal to the total value of securities.

Let w_{mj} – proportion of total wealth contributed by the total value of the jth security. This is called the portfolio weights of the market portfolio.

For markets to clear, we must have

$$\sum_{i=1}^{I} w_{ij} W_0^i = w_{mj} W_{mo}$$

Dividing both sides by W_{mo}

$$\sum_{i=1}^{I} \left(\frac{W_0^i}{W_{mo}} \right) w_{ij} = w_{mj}$$

i.e., the market portfolio weights are a convex combination of the portfolio weights for individuals.

Claim: When two fund separation obtains, the market portfolio is a frontier portfolio.

Proof:

When two fund separation holds, the separating portfolios must be frontier portfolios and any two distinct frontier portfolios can be separating portfolios.

In this case, each individual will hold a linear combination of the two separating portfolios.

Any linear combination of frontier portfolios is also a frontier portfolio $\Rightarrow$ each individual holds a frontier portfolio.

In equilibrium, the market portfolio is a convex combination of individuals' portfolios. Thus, the market portfolio is a linear combination of frontier portfolios and is itself on the frontier.

QED.

Recall from the last class that if p is a frontier portfolio, with $p \neq mvp$ and let q be any feasible portfolio, we have the following linear relationship

$$E[\tilde{r}_q] = (1-\beta_{qp})E[\tilde{r}_{zc(p)}] + \beta_{qp}E[\tilde{r}_p]$$

When 2-fund separation obtains, the market portfolio is a frontier portfolio. If m is not the mvp, then we can have

$$E[\tilde{r}_q] = (1-\beta_{qm})E[\tilde{r}_{zc(m)}] + \beta_{qm}E[\tilde{r}_m]$$

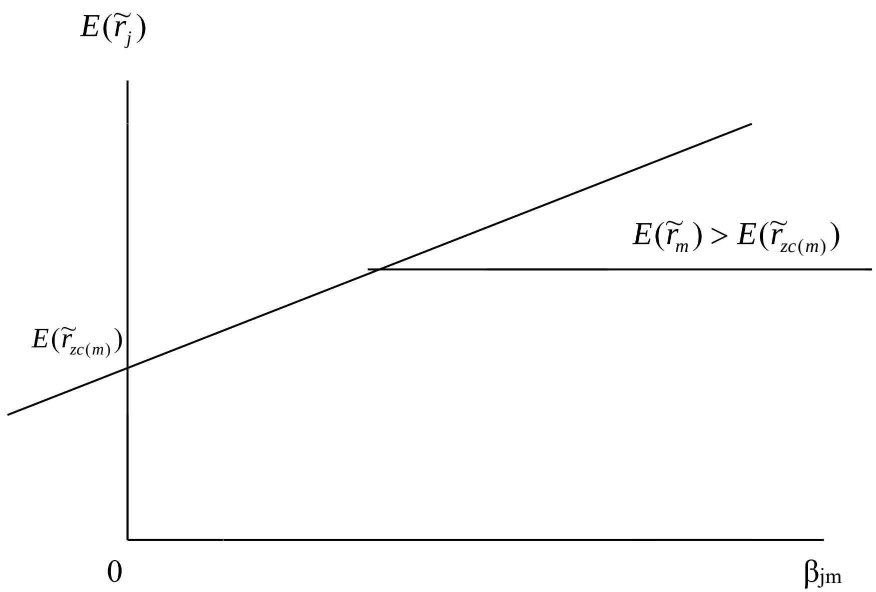
where the market portfolio's return is $\tilde{r}_m = \sum_{j=1}^{N} w_{mj} \tilde{r}_j$ and $\beta_{qm} = \dfrac{Cov(\tilde{r}_q, \tilde{r}_m)}{Var(\tilde{r}_m)}$

Since any risky asset *j* is itself a feasible portfolio, we have

$$E[\tilde{r}_j] = (1 - \beta_{jm}) E(\tilde{r}_{zc(m)}) + \beta_{jm} E(\tilde{r}_m) \quad \text{for j = 1, 2, …, N.}$$

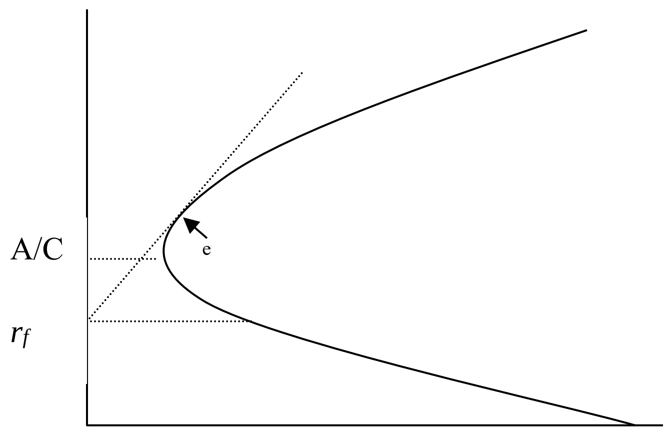
The security market line can be expressed as:

$$E[\tilde{r}_j] = E[\tilde{r}_{zc(m)}] + \beta_{jm} [E(\tilde{r}_m) - E(\tilde{r}_{zc(m)})]$$


$E(\tilde{r}_j)$
$E(\tilde{r}_m) > E(\tilde{r}_{zc(m)})$
$E(\tilde{r}_{zc(m)})$
0
β_{jm}

Asset pricing with a risk-free asset

Assume that $r_f < \frac{A}{C} \equiv E(\tilde{r}_{mvp})$, and let e be the tangency portfolio. We know that for any feasible portfolio q, $\tilde{r}_q = (1-\beta_{qe})\tilde{r}_f + \beta_{qe}\tilde{r}_e + \tilde{\varepsilon}_{qe}$, with $cov(\tilde{r}_e, \tilde{\varepsilon}_{qe}) = E(\tilde{\varepsilon}_{qe})$=0.



CAPM

Assume that two fund separation holds.

When $r_f \neq \frac{A}{C}$ and risky assets are in strictly positive supply, the tangency portfolio e must be the market portfolio of risky assets in equilibrium.

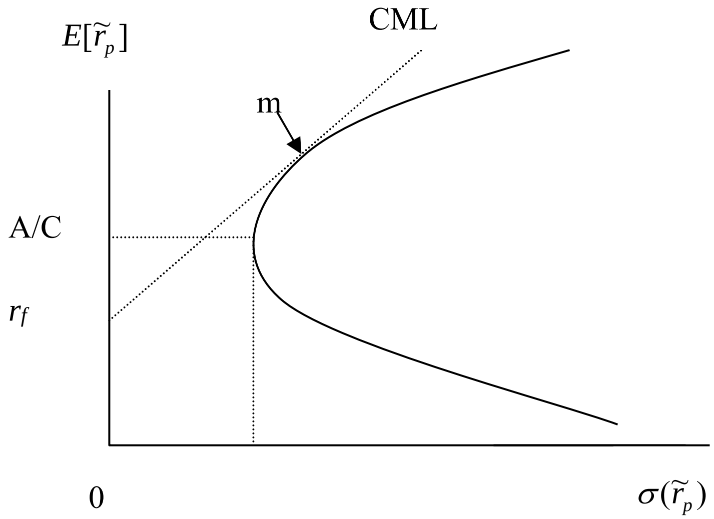
We have $E[\tilde{r}_q] - r_f = \beta_{qm}[E(\tilde{r}_m) - r_f]$ for any portfolio q in market equilibrium. This is the traditional CAPM.

When $r_f = \frac{A}{C}$, if there is an equilibrium, then we claim that the riskless asset is in strictly positive supply and the risky assets are in zero net supply.

To see this, when 2-fund separation holds, individuals hold frontier portfolios.

When $r_f = \dfrac{A}{C}$, an individual puts all his wealth into the riskless asset and holds a self-financing portfolio of risky assets. That is, an individual invests everything in the riskless asset and holds an arbitrage portfolio of risky assets $I^T W_p = 0$, where W_p is the portfolio weights of all risky assets.

For markets to clear, it is necessary that the riskless asset be in strictly positive supply and the risky assets be in zero net supply.



Remark 1: when there exists a riskless asset, an investor will never choose to hold a portfolio with return < r_f. The reason is we can always invest all money in r_f. So the conclusion holds even when borrowing is not allowed.

Remark 2: When riskless borrowing is strictly higher than riskless lending rate, $r_b > r_L$, we have $E(\tilde{r}_q) = E(\tilde{r}_{zc(m)}) + \beta_{qm}[E(\tilde{r}_m) - E(\tilde{r}_{zc(m)})]$ for any feasible portfolio and $E(\tilde{r}_m) - E(\tilde{r}_{zc(m)}) > 0$, and $\tilde{r}_b \geq E(\tilde{r}_{zc(m)}) \geq r_L$.

Alternative derivation of the CAPM

Relating market risk premium to investors' optimal portfolio position

Assume that: there is a risk free asset, with return r_f, and N risky assets, with returns $\{r_i\}_{i=1}^N$ which are multivariate normally distributed (required to apply the Stein's Lemma). Let $\tilde{W}_i$ be the optimally invested random time-1 wealth for person i.

From the FOC $\quad E[u_i^{'}(\tilde{W}_i)(\tilde{r}_j - r_f)] = 0, \qquad \forall i, j$

where $\tilde{W}_i = W_0^i[1 + \sum_{j=1}^{N} w_{ij}\tilde{r}_j + (1 - \sum_{j=1}^{N} w_{ij})r_f]$

$$= W_0^i[(1 + r_f) + \sum_{j=1}^{N} w_{ij}(\tilde{r}_j - r_f)]`$$

Using the definition of covariance, the FOC can be written as

$$E[u_i^{'}(\tilde{W}_i)]E[\tilde{r}_j - r_f] = -\mathrm{cov}[u_i^{'}(\tilde{W}_i), \tilde{r}_j]$$

Note: $\tilde{W}_i$ and $\tilde{r}_j$ are bi-variate normal.

[Stein's Lemma:
If $\tilde{x}$ and $\tilde{y}$ are bi-variate normal, then $Cov(g(\tilde{x}), \tilde{y}) = E[g'(\tilde{x})]Cov(\tilde{x}, \tilde{y})$, provided that g is differentiable and satisfies some regularity conditions.]

Assume that u_i is twice differentiable and apply Stein's Lemma

$$E[u'(\tilde{W}_i)]E[\tilde{r}_j - r_f] = -E[u_i^{''}(\tilde{W}_i)]Cov(\tilde{W}_i, \tilde{r}_j)$$

Define the i[th] investor's global absolute risk aversion

$$\theta_i \equiv \frac{-E[u_i''(\tilde{W}_i)]}{E[u_i'(\tilde{W}_i)]}$$

Dividing both sides by $E[u_i''(\tilde{W}_i)]$, and summing across i (assume I individuals)

$$(\sum_{i=1}^{I}\theta_i^{-1})E(\tilde{r}_j - r_f) = \sum_{i=1}^{I} Cov(\tilde{W}_i, \tilde{r}_j) = Cov(\sum_{i=1}^{I}\tilde{W}_i, \tilde{r}_j)$$

Now $\sum_{i=1}^{I}\tilde{W}_i = W_{mo}(1+\tilde{r}_m)$ is total wealth in the world in period 1.

Therefore substituting to the above yields $E[\tilde{r}_j - r_f] = W_{mo}(\sum_{i=1}^{I}\theta_i^{-1})^{-1}Cov(\tilde{r}_m, \tilde{r}_j)$,

where $\mathrm{I}(\sum_{i=1}^{I}\theta_i^{-1})^{-1}$ is the harmonic mean of individuals' global absolute risk aversion. $W_{mo}(\sum_{i=1}^{I}\theta_i^{-1})^{-1}$ can be interpreted as the aggregate relative risk aversion of the economy.

We therefore have

$$E[\tilde{r}_m - r_f] = W_{mo}(\sum_{i=1}^{I} \theta_i^{-1}) \cdot Var(\tilde{r}_m)$$

i.e., the risk premium on the market portfolio is proportional to the aggregate relative risk aversion of the economy.

$E[\tilde{r}_m - r_f] > 0$ when *u(.)* is increasing and strictly concave.

Direct substitution yields $E[\tilde{r}_j - r_f] = \dfrac{Cov(\tilde{r}_m, r_j)}{Var(\tilde{r}_m)} E(\tilde{r}_m - r_f)$. We therefore obtain the CAPM.

The Arbitrage Pricing Theory (APT)

Potential problems with the CAPM

1. Existence of transactions cost and other trading frictions
2. Existence of non-tradable assets, e.g., human capital
3. Market portfolio hard to identify

Motivations for multifactor APT model

Assume that asset return is driven by multiple risk factors and by an idiosyncratic component.

APT is a relative pricing model.

No assumption about investor preferences is made. It uses arbitrage pricing to restrict an asset's risk premium.

Assume the following return generating process:

$$\tilde{r}_i = a_i + \sum_{z=1}^{k} b_{iz} \tilde{f}_z + \tilde{\varepsilon}_i \tag{1}$$

where a_i is the expected return of asset i,

$$E(\tilde{\varepsilon}_i) = E(\tilde{f}_z) = E(\tilde{\varepsilon}_i \tilde{f}_z) = 0$$
$$E(\tilde{\varepsilon}_i \tilde{\varepsilon}_j) = 0, \quad i \neq j$$
$$E(\tilde{f}_x \tilde{f}_y) = 0, \quad x \neq y$$
$$E(\tilde{f}_z^{\,2}) = 1$$
$$E(\tilde{\varepsilon}_i^{\,2}) \equiv s_i^2 < S^2 < \infty$$
$$E(\tilde{r}_i \tilde{r}_j) \equiv \sigma_{ij}^2$$

Definition: Let a portfolio containing n assets be described by the vector of investment amounts in each of the n assets, $W^n \equiv \begin{bmatrix} W_1^n \\ W_2^n \\ \cdots \\ W_n^n \end{bmatrix}$.

Consider a sequence of these portfolios where n is increasing. An asymptotic arbitrage exists if the following conditions hold:

(A) The portfolio requires zero net investment

$$\sum_{i=1}^{n} W_i^n = 0$$

(B) The portfolio return becomes certain as n gets large

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} W_i^n W_j^n \sigma_{ij} = 0$$

(C) The portfolio's expected return is positive

$$\sum_{i=1}^{n} W_i^n a_i \geq \delta > 0.$$

Claim (APT): If no asymptotic arbitrage opportunities exist, then the expected return of asset i is described by the following linear relation

$$a_i = \lambda_0 + \sum_{z=1}^{k} \lambda_z b_{iz} + v_i$$

where λ_0 is a constant, λ_z is the risk premium for risk factor $\tilde{f}_z$ and

$$\sum_{i=1}^{n} v_i = 0 \qquad \text{(i)}$$

$$\sum_{i=1}^{n} b_{iz} v_i = 0, z = 1, 2, \ldots, k \qquad \text{(ii)}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\nu_i^2 = 0 \qquad \text{(iii)}$$

Note that if the economy contains a risk-free rate (implying $b_{iz} = 0$), the risk-free return will be approximated by λ_0.

Proof: For a given number of assets, n>k, consider the following cross-sectional regression

$$a_i = \lambda_0 + \sum_{z=1}^{k}\lambda_z b_{iz} + \nu_i$$

We know that (i) and (ii) can be satisfied by the properties of OLS regression.

To show that (iii) is also satisfied, consider a zero-net-investment arbitrage portfolio with the following investment amounts:

$$W_i = \frac{\nu_i}{\sqrt{n\sum_{i=1}^{n}\nu_i^2}}$$

The total arbitrage portfolio return is given by

$$
\begin{aligned}
&\tilde{r}_p \\
&= \sum_{i=1}^{n} W_i \tilde{r}_i \\
&= \frac{1}{\sqrt{n\sum_{i=1}^{n} \nu_i^2}} \sum_{i=1}^{n} (\nu_i \tilde{r}_i) \\
&= \frac{1}{\sqrt{n\sum_{i=1}^{n} \nu_i^2}} \sum_{i=1}^{n} [\nu_i (a_i + \sum_{z=1}^{k} b_{iz} \tilde{f}_z + \tilde{\varepsilon}_i)] \\
&= \frac{1}{\sqrt{n\sum_{i=1}^{n} \nu_i^2}} \sum_{i=1}^{n} [\nu_i (a_i + \tilde{\varepsilon}_i)]
\end{aligned}
$$

where we have used $\sum_{i=1}^{n} b_{iz} \nu_i = 0$ in the last equality.

We can calculate the mean and variance of the portfolio as follows

$$E(\tilde{r}_p) = \frac{1}{\sqrt{n\sum_{i=1}^{n} v_i^2}} \sum_{i=1}^{n} (v_i a_i)$$

since $E(\tilde{\varepsilon}_i) = 0$.

Substituting $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + v_i$, and using the fact that $\sum_{i=1}^{n} b_{iz} v_i = 0$ and $\sum_{i=1}^{n} v_i = 0$, we have

$$E(\tilde{r}_p)$$

$$= \frac{1}{\sqrt{n\sum_{i=1}^{n} v_i^2}}[\lambda_0 \sum_{i=1}^{n} v_i + \sum_{z=1}^{k} (\lambda_z \sum_{i=1}^{n} v_i b_{iz}) + \sum_{i=1}^{n} v_i^2]$$

$$= \frac{1}{\sqrt{n\sum_{i=1}^{n} v_i^2}} \sum_{i=1}^{n} v_i^2$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n} v_i^2}$$

To calculate the variance

$$\tilde{r}_p - E(\tilde{r}_p)$$

$$= \frac{1}{\sqrt{n\sum_{i=1}^{n} v_i^2}} \sum_{i=1}^{n} v_i \, \tilde{\varepsilon}_i$$

Using the fact that $E(\tilde{\varepsilon}_i\tilde{\varepsilon}_j) = 0, i \neq j$ and $E(\tilde{\varepsilon}_i^{\,2}) \equiv s_i^2 < S^2 < \infty$, we have

$$E[\tilde{r}_p - E(\tilde{r}_p)]^2$$

$$= \frac{\sum_{i=1}^{n} v_i^2 s_i^2}{n \sum_{i=1}^{n} v_i^2}$$

$$< \frac{S^2 \sum_{i=1}^{n} v_i^2}{n \sum_{i=1}^{n} v_i^2}$$

$$= \frac{S^2}{n} \to 0, \text{ as } n \to \infty$$

In other words, the portfolio return becomes certain. This implies that in the limit, the actual return equals the expected return

$$\lim_{n\to\infty} \tilde{r}_p = E(\tilde{r}_p) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} v_i^2}\,.$$

If there are no asymptotic arbitrage opportunities, this certain return on the portfolio must equal zero. Therefore, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} v_i^2 = 0$$

as required. QED.