FM1 Notes, Spring 2020

Review of Expected Utility

An individual's preferences have an expected utility representation if there exists a function u such that random consumption $\tilde{x}$ is preferred to $\tilde{y}$ if and only if $E[u(\tilde{x})] \geq E[u(\tilde{y})]$.

For continuous distributions, we use the following notation

$$\int_{\Omega} u(x_{\omega})dp(\omega) \geq \int_{\Omega} u(y_{\omega})dp(\omega)$$

We assume

Transitivity: *if* $x \succ y$ *and* $y \succ z$ *then* $x \succ z$

Completeness: *either* $x \succ y$ *or* $y \succ x$

A preference relation is a binary relation that is transitive and complete.

Assume that individuals want to maximize expected utility.

An individual is said to be risk averse if he/she is unwilling to accept or is indifferent to any actuarially fair game.

A game is said to be actuarially fair when its expected payoff is zero, i.e.:

$$\begin{cases} h_1 > 0, & p \\ h_2 < 0, & (1-p) \end{cases}$$

$$ph_1 + (1-p)h_2 = 0$$

Risk aversion says that

$$u(W_0)\ (>) \geq\ p\,u(W_0 + h_1) + (1-p)\,u(W_0 + h_2)$$

where W_0 is the initial wealth level.

Optimal Portfolio Choice Problem

Consider a portfolio choice problem of a risk-averse individual who strictly prefers more to less. If he invests a_j dollars in the j^{th} risky asset and $W_0 - \sum_j a_j$ dollars in the risk free asset.

His uncertain end of period wealth is

$$\tilde{W} = (W_0 - \sum_j a_j)(1+r_f) + \sum_j a_j(1+\tilde{r}_j)$$

or

$$\tilde{W} = W_0(1+r_f) + \sum_j a_j(\tilde{r}_j - r_f)$$

The individual's choice problem is

$$\max_{\{a_j\}} E\{u[W_0(1+r_f) + \sum a_j(\tilde{r}_j - r_f)]\}$$

First order condition (F.O.C.):

$$E[u'(\tilde{W})(\tilde{r}_j - r_f)] = 0, \text{ for all j = 1, 2, …, n.}$$

Note that since u'(.) > 0, the probability of $(\tilde{r}_j - r_f) > 0$ must lie in (0, 1).

What this says is that there is a positive probability that stock j will do better than the risk-free asset, but there is also a positive probability that stock j can do worse. Therefore, there is no dominating investment strategy here, for otherwise the problem would become trivial.

Claim: An individual who is risk averse and strictly prefers more to less will undertake risky investment if and only if the expected rate of return on at least one risky asset is strictly larger than r_f.

Proof: To see this, we note the following. For an individual to invest \$0 in or even short sell the risky assets as an optimal choice, it is necessary that

$$E\{u'[W_0(1+r_f)](\tilde{r}_j - r_f)\} \leq 0, \quad \forall j$$

Equivalently,

$$u'[W_0(1+r_f)]\, E(\tilde{r}_j - r_f) \leq 0, \ \forall j$$

Because u'>0, we have

$$E(\tilde{r}_j - r_f) \leq 0, \ \forall j$$

Therefore, we conclude that

$$a_j \leq 0 \ \ \forall j \quad \text{only if} \quad E(\tilde{r}_j - r_f) \leq 0, \ \forall j$$

When one or more risky assets have strictly positive risk premiums, the individual will take part in some risky investment. In other words, keeping all the money in the risk-free asset is not rational.

Remark, when there is more than one risky asset,

$E(\tilde{r}_j - r_f) > 0$ does not necessarily imply that $a_j > 0$.

The Minimum Risk Premium

Consider one risky asset and one riskfree asset.

For an individual to invest all his wealth in the risky asset with return $\tilde{r}$, the following must be true

$$E\{u'[W_0(1+\tilde{r})](\tilde{r} - r_f)\} \geq 0$$

Take a Taylor-series expansion of the marginal utility function u' around the risk free rate r_f:

$$u'[W_0(1+\tilde{r})] = u'[W_0(1+r_f)] + u''[W_0(1+r_f)](\tilde{r} - r_f)W_0 + o(\tilde{r} - r_f)$$

Multiply both sides by $(\tilde{r} - r_f)$ and take expectation:

$$\begin{aligned}
&E\{u'[W_0(1+\tilde{r})](\tilde{r} - r_f)\} \\
&= u'[W_0(1+r_f)]E(\tilde{r} - r_f) + u''[W_0(1+r_f)]E(\tilde{r} - r_f)^2 W_0 + o[E(\tilde{r} - r_f)^2] \\
&\cong u'[W_0(1+r_f)]E(\tilde{r} - r_f) + u''[W_0(1+r_f)]E(\tilde{r} - r_f)^2 W_0
\end{aligned}$$

Rearranging terms yields

$$
\begin{aligned}
E(\tilde{r} - r_f) \; &\geq \frac{-u''}{u'} W_0 \, E(\tilde{r} - r_f)^2 \\
&\equiv \; R_A \, W_0 \; E(\tilde{r} - r_f)^2 \\
&\equiv \; R_R \; E(\tilde{r} - r_f)^2
\end{aligned}
$$

where $R_A = \dfrac{-u''}{u'}$ is called the Arrow-Pratt measure of absolute risk aversion and

$R_R = \dfrac{-u''}{u'} W_0 = R_A W_0$ is called the Arrow-Pratt measure of relative risk aversion.

Degree of absolute risk aversion

We define

$$\frac{dR_A(z)}{dz} \begin{matrix} > \\ = \\ < \end{matrix} 0, \ \forall \ z \quad \Rightarrow \quad \begin{matrix} \text{increasing} \\ \text{constant} \\ \text{decreasing} \end{matrix} \quad \text{absolute risk aversion.}$$

Claim:

$$\frac{dR_A(z)}{dz} \begin{matrix} > \\ = \\ < \end{matrix} 0, \ \forall \ z \quad \Rightarrow \quad \frac{da}{dW_0} \begin{matrix} < \\ = \\ > \end{matrix} 0, \ \forall W_0$$

Proof: We will do the proof only for the decreasing absolute risk aversion case. The other cases are similar and are left as an exercise.

At the optimum, we have (the FOC):

$$E[u'(\tilde{W})(\tilde{r} - r_f)] = 0$$

Using the implicit function theorem:

$$\frac{da}{dW_0} = \frac{E[u''(\tilde{W})(\tilde{r} - r_f)](1 + r_f)}{-E[u''(\tilde{W})(\tilde{r} - r_f)^2]}$$

where $\tilde{W} = W_0(1 + r_f) + a(\tilde{r} - r_f)$.

The denominator > 0 because u’’<0 by the assumption of risk aversion.

So the sign of $\dfrac{da}{dW_0}$ is the same as the sign of $E[u''(\tilde{W})(\tilde{r} - r_f)]$.

We need to show that $E[u''(\tilde{W})(\tilde{r} - r_f)] > 0$.

Under decreasing risk aversion,

When $\tilde{r} \geq r_f$, we have $\tilde{W} \geq \tilde{W}_0(1+r_f)$, implying $R_A(\tilde{W}) \leq R_A[W_0(1+r_f)]$.

When $\tilde{r} < r_f$, we have $\tilde{W} < \tilde{W}_0(1+r_f)$, implying $R_A(\tilde{W}) > R_A[W_0(1+r_f)]$.

Multiply both sides by $-u'(\tilde{W})(\tilde{r}-r_f)$ gives the following results:

when $\tilde{r} \geq r_f$, we have $u''(\tilde{W})(\tilde{r}-r_f) \geq -R_A[W_0(1+r_f)]u'(\tilde{W})(\tilde{r}-r_f)$, and

when $\tilde{r} < r_f$, we have $u''(\tilde{W})(\tilde{r}-r_f) > -R_A[W_0(1+r_f)]u'(\tilde{W})(\tilde{r}-r_f)$.

Combining the two cases yields

$$E[u''(\tilde{W})(\tilde{r}-r_f)] > -R_A[W_0(1+r_f)]E[u'(\tilde{W})(\tilde{r}-r_f)]$$

$$= -R_A[W_0(1+r_f)]\cdot 0 \qquad \text{(by the FOC)}$$

$$= 0$$

$$\Rightarrow \quad \frac{da}{dW_0} > 0. \qquad \square$$

Note that the property of decreasing risk aversion implies that the 3rd derivative of the utility function is positive u'''>0.

Relative risk aversion

Normalizing the measure of absolute risk aversion by wealth gives us a measure of relative risk aversion

$$R_R(z) \equiv R_A(z)\ Z$$

Claim: $\dfrac{dR_R(z)}{dz} \begin{array}{c} > \\ = \\ < \end{array} 0,\ \forall\ z \qquad \Rightarrow \quad \eta \equiv \dfrac{da}{dW_0}\dfrac{W_0}{a} \begin{array}{c} < \\ = \\ > \end{array} 1,\ \forall W_0.$

Proof: $\eta \equiv \dfrac{da}{dW_0}\dfrac{W_0}{a} = 1 + \dfrac{(da/dW_0)W_0 - a}{a}$

Substitute the expression of $\frac{da}{dW_0}$ from the last proof and rearrange terms, we obtain

$$\eta \equiv 1 + \frac{W_0(1+r_f)E[u''(\tilde{W})(\tilde{r}-r_f)] + aE[u''(\tilde{W})(\tilde{r}-r_f)^2]}{-aE[u''(\tilde{W})(\tilde{r}-r_f)^2]}$$

The numerator = $E[u''(\tilde{W})\tilde{W}(\tilde{r}-r_f)]$

[Recall that $\tilde{W} = W_0(1+r_f) + a(\tilde{r}-r_f)$]

Since the denominator >0,

Sign of $(\eta - 1)$ = sign of $E[u''(\tilde{W})\tilde{W}(\tilde{r} - r_f)]$

Under increasing RRA,

$$R_R(\tilde{W}) \underset{<}{\overset{>}{=}} R_R[W_0\,(1 + r_f)] \quad \text{if} \quad \tilde{r} \underset{<}{\overset{>}{=}} r_f$$

Multiply both sides by $-u'(\tilde{W})(\tilde{r} - r_f)$ yields

$$u''(\tilde{W})\tilde{W}(\tilde{r} - r_f) \leq -R_R[W_0(1 + r_f)]u'(\tilde{W})(\tilde{r} - r_f) \text{ in the event that } \tilde{r} \geq r_f$$

and

$$u''(\tilde{W})\tilde{W}(\tilde{r}-r_f) < -R_R[W_0(1+r_f)]u'(\tilde{W})(\tilde{r}-r_f) \text{ in the event that } \tilde{r} < r_f .$$

This implies that at the portfolio optimum,

$$E[u''(\tilde{W})\tilde{W}(\tilde{r}-r_f)] < -R_R[W_0(1+r_f)]E[u'(\tilde{W})(\tilde{r}-r_f)] = 0$$

$$\Rightarrow \quad \eta < 1. \quad \square$$

For DRRA and CRRA, show as an exercise.