FM1 Notes, Spring 2020

Stochastic Dominance

Suppose that there two risky assets (or portfolios), how do we rank them? This is the topic to discuss.

We will introduce two concepts of stochastic dominance.

First Degree Stochastic Dominance

Risky asset A is said to first degree stochastically dominates risky asset B, denoted by $A \underset{FSD}{\geq} B$, if all individuals having utility functions in wealth that are increasing (but not necessarily risk averse) and continuous either prefer A to B or are indifferent between A and B.

Assume that $\tilde{r}_A$ and $\tilde{r}_B$ $\in [0,1]$. Let F_A (.) and F_B (.) denote their cumulative distribution functions (CDF's).

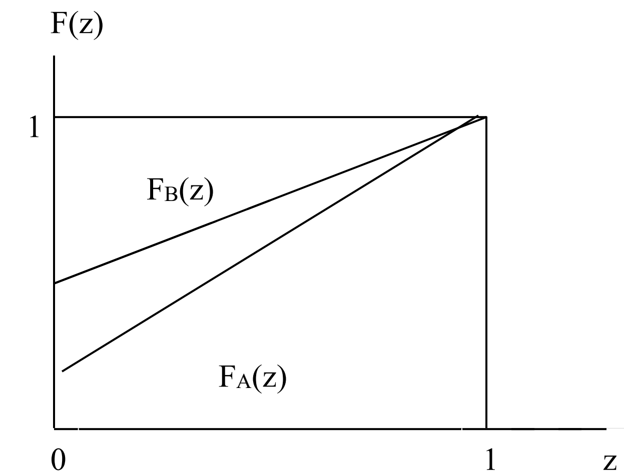
Suppose that

$$F_A(z) \leq F_B(z) \qquad \forall z \in [0,1]$$

Since F_A and F_B are CDF's, they are continuous from the right and

$$F_A(1) = F_B(1) = 1.$$

But, in general, $F_A(0) \neq F_B(0)$.



For any z, the probability that the rate of return on asset A is less than z is less than that for asset B.

$$\mathrm{P}\{\tilde{r}_A < z\} \leq \mathrm{P}\{\tilde{r}_B < z\}$$

Put it differently, $\mathrm{P}\{\tilde{r}_A \geq z\} \geq \mathrm{P}\{\tilde{r}_B \geq z\}$.

Let u(.) be any continuous and increasing utility function representing a non-satiable individual's preferences. Assume without loss of generality that $W_0 = 1$.

Claim: $F_A(z) \leq F_B(z),\ \forall z \in [0,1] \ \Leftrightarrow E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)]$.

Proof: skipped.

Claim: If $\tilde{r}_A \overset{d}{=} \tilde{r}_B + \tilde{\alpha},\ \tilde{\alpha} \geq 0,$ then $A \overset{FSD}{\geq} B$.

Proof: We have

$$E[u(1+\tilde{r}_A)] = E[u(1+\tilde{r}_B + \tilde{\alpha})] \geq E[u(1+\tilde{r}_B)]$$

because $\tilde{\alpha} \geq 0$ and u(.) is monotonic.

The converse is also true, i.e., if $A \underset{FSD}{\geq} B$,

then there exists an $\tilde{\alpha} \geq 0$, such that $\tilde{r}_A \overset{d}{=} \tilde{r}_B + \tilde{\alpha}$.

Combining, we have the following equivalences:

1. $A \underset{FSD}{\geq} B$;

2. $F_A(z) \leq F_B(z), \quad \forall z \in [0,1]$;

3. $\tilde{r}_A \overset{d}{=} \tilde{r}_B + \tilde{\alpha}, \quad \tilde{\alpha} \geq 0$.

From 3, we know that $E[\tilde{r}_A] \geq E[\tilde{r}_B]$ if $A \underset{FSD}{\geq} B$.

Is the converse also true?

no

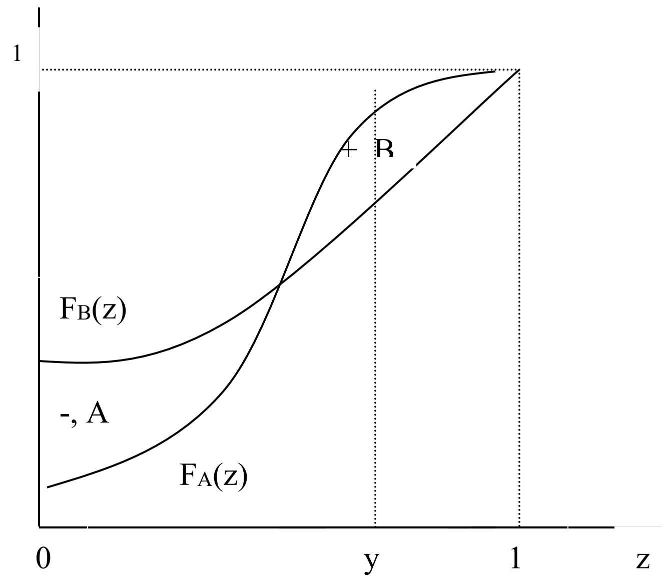
Second Degree Stochastic Dominance

Risky asset A is said to dominate risky asset B in the sense of second degree stochastic dominance, denoted by $A \underset{SSD}{\geq} B$, if all risk averse individuals prefer A to B.

Claim: $A \underset{SSD}{\geq} B \;\Leftrightarrow\; E[\tilde{r}_A] = E[\tilde{r}_B]$ and

$$s(y) \equiv \int_0^y [F_A(z) - F_B(z)]dz \leq 0 \quad \forall y \in [0,1]$$

Note: FSD does not allow $F_A(z)$ and $F_B(z)$ to cross. $F_B(z)$ has to be always above $F_A(z)$. But SSD allows the two to cross.



$A \geq B$

Proof: skipped.

Claim: $A \underset{SSD}{\geq} B \quad \Leftrightarrow \quad \tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$ with $E[\tilde{\varepsilon} \mid \tilde{r}_A] = 0$

Proof: (sufficiency)

Let u(.) be a concave function. Then

$$E[u(1+\tilde{r}_B)] = E[u(1+\tilde{r}_A+\tilde{\varepsilon})] = E\{E[u(1+\tilde{r}_A+\tilde{\varepsilon}) \mid \tilde{r}_A]\}$$

(using the law of iterative expectations, E[x]=E{E[x|y]}.)

By the conditional Jensen's inequalities

$$E[u(1+\tilde{r}_A+\tilde{\varepsilon}) \mid \tilde{r}_A] \leq u\{E[(1+\tilde{r}_A+\tilde{\varepsilon}) \mid \tilde{r}_A)\}$$

$$= u\{1+E(\tilde{r}_A \mid \tilde{r}_A)+\underbrace{E(\tilde{\varepsilon} \mid \tilde{r}_A)}_{=0}\}$$

$$= u(1+\tilde{r}_A)$$

So $E[u(1+\tilde{r}_B)] \leq E[u(1+\tilde{r}_A)]$.

Therefore we have 3 equivalent statements:

1. $A \underset{SSD}{\geq} B$

2. $E[\tilde{r}_A] = E[\tilde{r}_B]$ and $s(y) = \int_0^y [F_A(z) - F_B(z)]dz \leq 0 \quad \forall y \in [0,1]$

3. $\tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$, $E[\tilde{\varepsilon} \mid \tilde{r}_A] = 0$

Condition 3 implies that:

$var(\tilde{r}_B) \geq var(\tilde{r}_A)$

Therefore if $A \underset{SSD}{\geq} B$, it must be the case that $E[\tilde{r}_A] = E[\tilde{r}_B]$ and $Var(\tilde{r}_B) \geq Var(\tilde{r}_A)$.

The converse is not true.