FM1 Notes, Spring 2020

Mean Variance Analysis

What is mean variance analysis?

Why do we want to do mean-variance analysis in the first place?

Two Justifications

Consider the Taylor-series expansion of the utility function:

$$u(\tilde{W}) = u(E(\tilde{W})) + u'(E(\tilde{W}))[\tilde{W} - E(\tilde{W})] + \frac{1}{2!}u''(E(\tilde{W}))[\tilde{W} - E(\tilde{W})]^2 + R_3$$

where $R_3 = \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(E(\tilde{W}))[\tilde{W} - E(\tilde{W})]^n$

Taking expectation of both sides, we have

$$E[u(\tilde{W})] = u(E(\tilde{W})) + \frac{1}{2}u''(E(\tilde{W}))\sigma^2(\tilde{W}) + E[R_3]$$

where $E[R_3] = \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(E(\tilde{W}))m^n(\tilde{W})$

where $m^n(\tilde{W}) \equiv E[\tilde{W} - E[\tilde{W}]]^n$ is the n$^{\text{th}}$ central moment of $\tilde{W}$.

In general expected utility cannot be solely defined over the mean $E[\tilde{W}]$ and the variance $\sigma^2(\tilde{W})$.

Special case one: quadratic utility.

$u^{(n)}(\tilde{W}) = 0$ for $n > 2$

$$\begin{cases} u(z) = z - \dfrac{b}{2} z^2, \ b > 0, \\ u' = 1 - bz, \\ u'' = -b, \\ u^{(n)} = 0, \ n > 2 \end{cases}$$

Then

$$\begin{aligned} E[u(\tilde{W})] &= u(E(\tilde{W})) + \frac{1}{2} u''(E(\tilde{W}))\sigma^2(\tilde{W}) \\ &= [E(\tilde{W}) - \frac{b}{2}[E(\tilde{W})]^2] + \frac{1}{2}(-b) * \sigma^2(\tilde{W}) \\ &= E(\tilde{W}) - \frac{b}{2}[\sigma^2(\tilde{W}) + (E(\tilde{W}))^2] \\ &= E[\tilde{W}] - \frac{b}{2} E(\tilde{W}^2) \end{aligned}$$

In other words, under the quadratic utility function, we can express the expected utility as a function of the mean and the variance only. But the quadratic utility function has its own problems.

Special case two: normal distribution of return.

$$\tilde{r} \sim N(E(\tilde{r}), Var(\tilde{r}))$$

Under normality, $E(R_3)$ can be expressed as a function of the first two moments, i.e.,

$$E[R_3] = f(E(\tilde{r}), Var(\tilde{r})) = f(\mu, \sigma^2)$$

Normal distributions are also stable under addition, i.e.,

$\tilde{r}_p = \sum_{i=1}^{n} w_i \tilde{r}_i$ is normally distributed if $\begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ . \\ \tilde{r}_n \end{bmatrix}$ follows a multivariate normal distribution.

However, the normality assumption has its own problems as well.

Assumptions on Asset Returns

Suppose that there are $N \geq 2$ risky assets traded, short selling is allowed, and

$$\sigma(\tilde{r}_i) < \infty, \qquad \forall i.$$

Assume that $\sum_{i=1}^{n} w_i \tilde{r}_i \neq 0$ for nontrivial w_i's, i.e., asset returns are linearly independent.

Also assume that

$$E[(\tilde{r} - E(\tilde{r}))(\tilde{r} - E(\tilde{r}))'] \equiv V \qquad \text{(non-singular)}$$

Note that V is positive definite if $W^T V \; W > 0, \;\; \forall W \neq 0$.

Because $W^T V\ W$ is the variance of a portfolio which is strictly positive, we conclude that V is positive definite.

Deriving the portfolio frontier

A portfolio is a frontier portfolio if it has the minimum variance among portfolios that have the same expected rate of return.

A portfolio p is a frontier portfolio if and only if W_p is the solution to the following quadratic programming program:

$$\min_{\{W\}} \frac{1}{2}(W^T V\ W)$$

$$\text{s.t.} \quad W^T e = E[\tilde{r}_p] \qquad \& \qquad W^T I = 1$$

$$\text{where} \quad I = \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \end{bmatrix}_{N\times 1} \qquad e = E[\tilde{r}] = E\begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ . \\ \tilde{r}_N \end{bmatrix}$$

Note that short sales are allowed, i.e., we allow $w_i < 0$ for some i. Therefore, the range of expected return on a feasible portfolio is unbounded.

$$\min_{\{W,\lambda,\gamma\}} L = \frac{1}{2} W^T V W + \lambda[E(\tilde{r}_p) - W^T e] + \gamma(1 - W^T I)$$

$$\frac{\partial L}{\partial W} = 0, \qquad V W_p - \lambda e - \gamma I = 0$$

$$\frac{\partial L}{\partial \lambda} = 0, \qquad E[\tilde{r}_p] - W_p{}^T e = 0$$

$$\frac{\partial L}{\partial \gamma} = 0, \qquad 1 - W_p{}^T I = 0$$

Some elementary and tedious algebra yields a unique set of portfolio weights for the frontier portfolio having the expected return of $E[\tilde{r}_p]$.

$$\begin{aligned} W_p &= \lambda(V^{-1}e) + \gamma(V^{-1}I) \\ &= \left[\frac{C \cdot E(\tilde{r}_p) - A}{D}\right](V^{-1}e) + \left[\frac{B - A \cdot E(\tilde{r}_p)}{D}\right](V^{-1}I) \\ &= \underset{N\times 1}{g} + \underset{N\times 1}{h} \cdot E(\tilde{r}_p) \end{aligned}$$

where

$$A = I^T V^{-1} e = e^T V^{-1} I$$

$$B = e^T V^{-1} e$$

$$C = I^T V^{-1} I$$

$$D = BC - A^2$$

B>0, C>0, D>0

$$\lambda = \frac{C \cdot E(\tilde{r}_p) - A}{D}$$

$$\gamma = \frac{B - A \cdot E(\tilde{r}_p)}{D}$$

$$\underset{N\times 1}{g} = \frac{1}{D}[B(V^{-1}I) - A(V^{-1}e)]$$

$$\underset{N\times 1}{h} = \frac{1}{D}[C(V^{-1}e) - A(V^{-1}I)]$$

Claim: The portfolio frontier can be generated by any two distinct frontier portfolios.

Proof: Let p_1 and p_2 be two distinct frontier portfolios. Let q be any frontier portfolio. We need to show that q can be expressed as a portfolio of p_1 and p_2.

Since $E(\tilde{r}_{p_1}) \neq E(\tilde{r}_{p_2})$

$$\exists\, \alpha \in R \quad \text{such that} \quad E(\tilde{r}_q) = \alpha E(\tilde{r}_{p_1}) + (1-\alpha)E(\tilde{r}_{p_2})$$

Now consider a portfolio of p_1 and p_2 with weights α and $(1-\alpha)$

$$\begin{aligned} \alpha W_{p_1} + (1-\alpha)W_{p_2} &= \alpha[g + h\cdot E(\tilde{r}_{p_1})] + (1-\alpha)[g + h\cdot E(\tilde{r}_{p_2})] \\ &= g + h[\alpha\cdot E(\tilde{r}_{p_1}) + (1-\alpha)E(\tilde{r}_{p_2})] \\ &= g + h\cdot E(\tilde{r}_q) \\ &= W_q \qquad\qquad QED. \end{aligned}$$

The covariance between any two frontier portfolios p and q

$$
\begin{aligned}
Cov(\tilde{r}_p, \tilde{r}_q) &= W_p{}^T V W_q \\
&= [g + h \cdot E(\tilde{r}_p)]^T V [g + h \cdot E(\tilde{r}_q)] \\
&= \frac{C}{D}[E(\tilde{r}_P) - \frac{A}{C}][E(\tilde{r}_q) - \frac{A}{C}] + \frac{1}{C}
\end{aligned}
$$

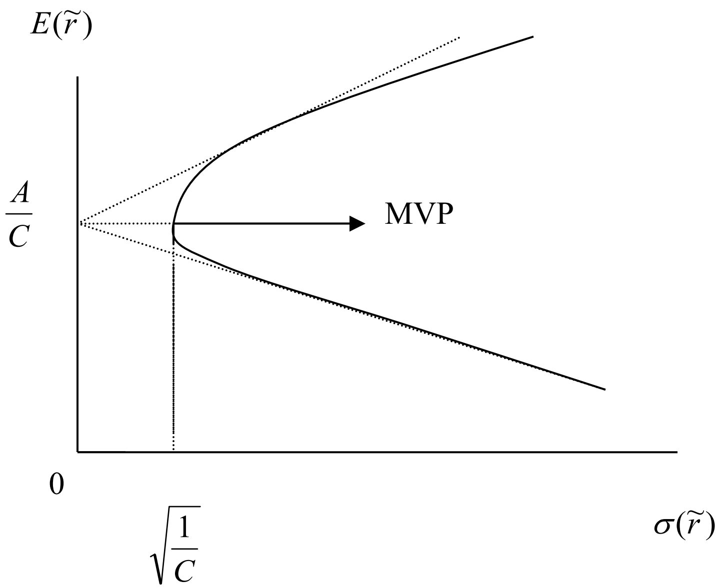
Using the definition of variance

$$\sigma^2(\tilde{r}_p) = Cov(\tilde{r}_p, \tilde{r}_p) = \frac{C}{D}[E(\tilde{r}_p) - \frac{A}{C}]^2 + \frac{1}{C},$$ which yields

$$\frac{\sigma^2(\tilde{r}_p)}{1/C} - \frac{[E(\tilde{r}_p) - \frac{A}{C}]^2}{D/C^2} = 1$$

This is a hyperbola in the σ-μ space with center $(0, \dfrac{A}{C})$ and asymptotes

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$$



We can rewrite the above relation as

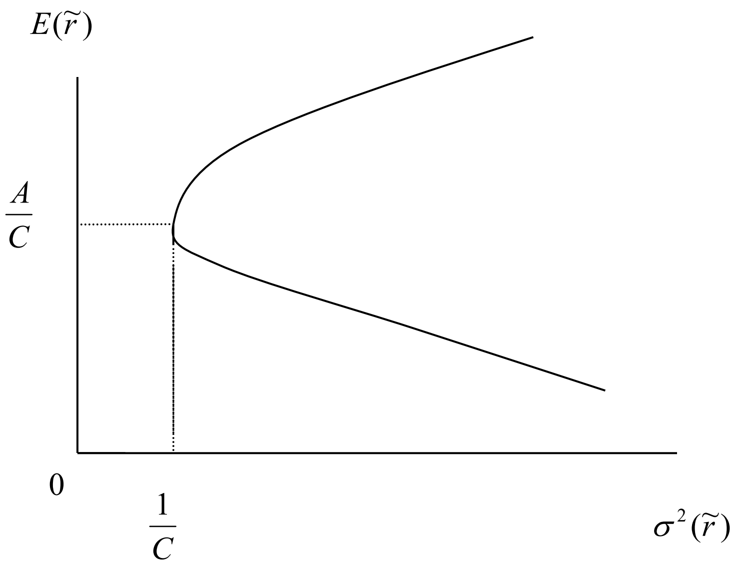
$$\sigma^2(\tilde{r}_p) = \frac{1}{C}[1 + \frac{C^2}{D}(E[\tilde{r}_p] - \frac{A}{C})^2]$$

$$= \frac{1}{C}\{1 + \frac{C^2}{D}[E^2(\tilde{r}_p) - 2\frac{A}{C}E(\tilde{r}_p) + \frac{A^2}{C^2}]\}$$

$$= \frac{1}{D}\{C \cdot E^2(\tilde{r}_p) - 2A \cdot E(\tilde{r}_p) + (\frac{A^2}{C} + \frac{D}{C})\}$$

$$= \frac{1}{D}\{C \cdot E^2(\tilde{r}_p) - 2A \cdot E(\tilde{r}_p) + B\}$$

which is a parabola in σ^2-μ space with vertex $(\frac{1}{C}, \frac{A}{C})$. The minimum variance portfolio is at $(\frac{1}{C}, \frac{A}{C})$.


$E(\tilde{r})$
$\frac{A}{C}$
0
$\frac{1}{C}$
$\sigma^2(\tilde{r})$

Claim: $Cov(\tilde{r}_p, \tilde{r}_{MVP}) = Var(\tilde{r}_{MVP})$,

where $\tilde{r}_p$ is any portfolio (not necessarily a frontier portfolio).

Proof: Consider the following portfolio

$$a\tilde{r}_p + (1-a)\tilde{r}_{MVP}$$

$$\min_{\{a\}} \quad \frac{1}{2} Var[a\tilde{r}_p + (1-a)\tilde{r}_{MVP}]$$

$$\min_{\{a\}} \frac{1}{2} a^2 \sigma^2(\tilde{r}_p) + a(1-a) Cov(\tilde{r}_{p,} \tilde{r}_{MVP}) + \frac{1}{2}(1-a)^2 \sigma^2(\tilde{r}_{MVP})$$

F.O.C.

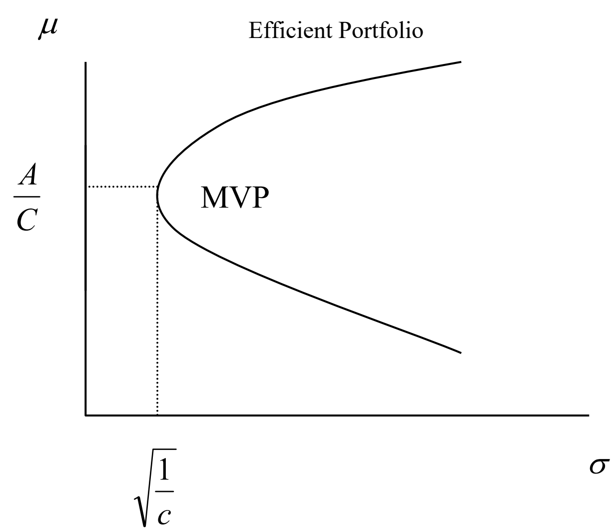
$$a\sigma^2(\tilde{r}_p) + (1-2a) Cov(\tilde{r}_{p,} \tilde{r}_{MVP}) - (1-a)\sigma^2(\tilde{r}_{MVP}) = 0$$

Since MVP is the minimum variance portfolio, $a=0$ must satisfy the FOC $\Rightarrow$

$$Cov(\tilde{r}_p, \tilde{r}_{MVP}) = \sigma^2(\tilde{r}_{MVP}) \qquad \text{QED.}$$

Efficient Portfolios

Those frontier portfolios with $E[\tilde{r}_p] > E[\tilde{r}_{MVP}] = \dfrac{A}{C}$ are called efficient portfolios.



Those frontier portfolios with $E[\tilde{r}_p] < E(\tilde{r}_{MVP}) = \dfrac{A}{C}$ are called inefficient portfolios.

Let W_i, $i = 1, 2, ..., m$, be any m frontier portfolios and

$$\alpha_i \in R, \quad \text{such that} \quad \sum_{i=1}^{m} \alpha_i = 1, \quad i = 1, 2, ..., m.$$

Denote the expected return of portfolio i by

$$E[\tilde{r}_i] \quad \text{for} \quad i = 1, 2, ..., m.$$

We have

$$\sum_{i=1}^{m} \alpha_i w_i = \sum_{i=1}^{m} \alpha_i [g + h \cdot E(\tilde{r}_i)] = g + h \cdot \sum_{i=1}^{m} \alpha_i E(\tilde{r}_i)$$

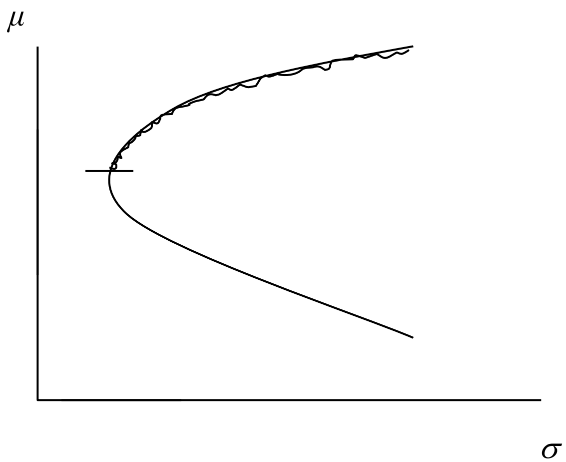
Therefore, any portfolio of frontier portfolios is a frontier portfolio.

If portfolios i = 1, 2, …, m are efficient portfolio and if

$$0 \le \alpha_i \le 1, \quad \forall i = 1, 2, ..., m. \text{ then}$$

$$\sum_{i=1}^{m} \alpha_i E(\tilde{r}_i) > \sum_{i=1}^{m} \alpha_i \frac{A}{C} = \frac{A}{C}$$

Formally, any convex combination of efficient portfolios will be an efficient portfolio. The set of efficient portfolios is thus a convex set.



Claim: For any frontier portfolio *p*, except for the MVP, there exists a unique frontier portfolio, denoted *zc(p)*, which has a zero covariance with portfolio *p*.

Proof: We set

$$Cov(\tilde{r}_p, \tilde{r}_{zc(p)}) = \frac{C}{D}[E(\tilde{r}_P) - \frac{A}{C}][E(\tilde{r}_{zc(p)}) - \frac{A}{C}] + \frac{1}{C}$$

$$= \frac{C}{D}\left\{[E(\tilde{r}_P) - \frac{A}{C}][E(\tilde{r}_{zc(p)}) - \frac{A}{C}] + \frac{D}{C^2}\right\} = 0$$

and solve for

$$E[\tilde{r}_{zc(p)}] = \frac{A}{C} - \frac{D/C^2}{E[\tilde{r}_p] - A/C}$$

which defines the portfolio *zc(p)* by the uniqueness of portfolio representation.

Recall that

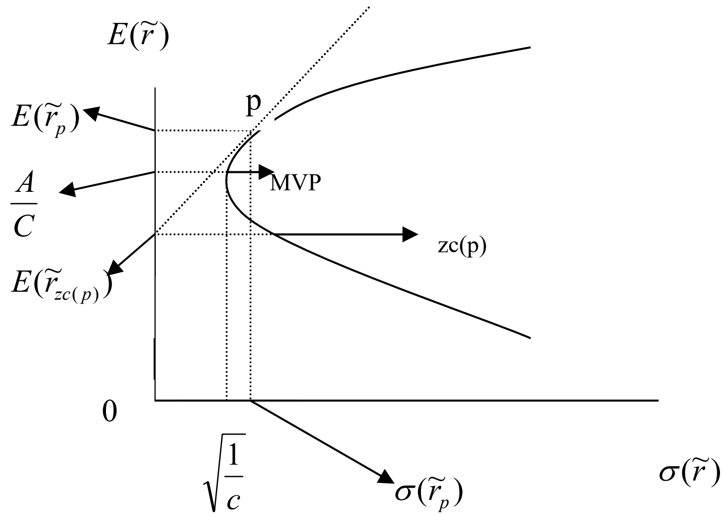
$$Cov(\tilde{r}_p, \tilde{r}_{MVP}) = \frac{1}{C} > 0.$$

Therefore, there does not exist a portfolio that has a zero covariance with the MVP.

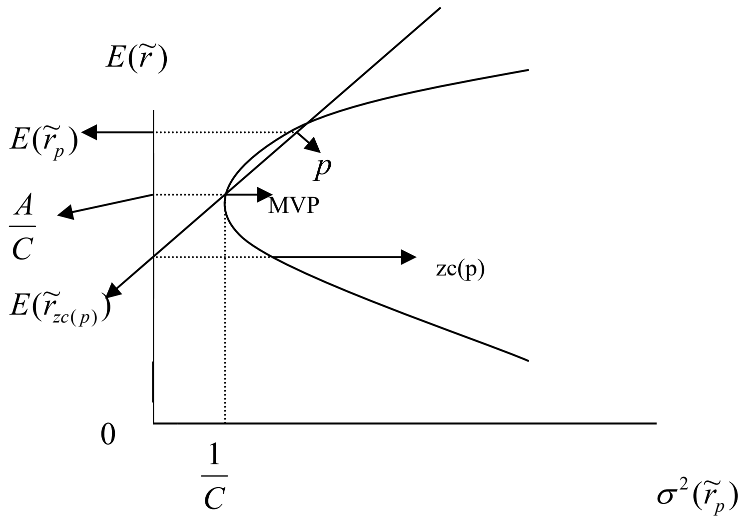
If p is an efficient portfolio, i.e., $E(\tilde{r}_p) > \frac{A}{C}$, then $E(\tilde{r}_{zc(p)}) < \frac{A}{C}$.

Thus, *zc(p)* is an inefficient portfolio and vice versa.

In the σ-μ space



In the σ^2-μ space



Let q be any portfolio (not necessarily a frontier portfolio)

Let p be a frontier portfolio other than the MVP.

$$Cov(\tilde{r}_p, \tilde{r}_q) = W_P^T V W_q$$
$$= \lambda e^T V^{-1} V W_q + \gamma I^T V^{-1} V W_q = \lambda e^T W_q + \gamma I^T W_q$$
$$= \lambda E[\tilde{r}_q] + \gamma$$

Substituting the definitions of λ and γ and after some algebra, we obtain

$$E(\tilde{r}_q) = (1 - \beta_{qp}) E[\tilde{r}_{zc(p)}] + \beta_{qp} E(\tilde{r}_p)$$
$$= E[\tilde{r}_{zc(p)}] + \beta_{qp}[E(\tilde{r}_p) - E(\tilde{r}_{zc(p)})]$$

where $\beta_{qp} = \dfrac{Cov(\tilde{r}_q, \tilde{r}_p)}{\sigma^2(\tilde{r}_p)}$

This says that, the expected return on any portfolio q (not necessarily a frontier portfolio) can be written as a linear combination of the expected returns on a frontier portfolio p and on its zero covariance portfolio $zc(p)$, with weights β_{qp} and $(1-\beta_{qp})$.

Regression implication

$$\tilde{r}_q = \beta_0 + \beta_1 \tilde{r}_{zc(p)} + \beta_2 \tilde{r}_p + \tilde{\varepsilon}_q$$

Restrictions on the regression equation

$$\beta_0 = 0, \;\; Cov(\tilde{r}_p, \tilde{\varepsilon}_q) = Cov(\tilde{r}_{zc(p)}, \tilde{\varepsilon}_q) = E(\tilde{\varepsilon}_q) = 0$$

Portfolio optimization with a risk-free asset

Let p be a frontier portfolio of all *(N+1)* assets.

Let W_p be N-vector portfolio weights of p on risky assets.

Note: $W_p^T I \neq 1,$ the entire portfolio is

$$\left.\begin{bmatrix} \left.\begin{matrix} w_1 \\ w_2 \\ \dots \\ w_N \end{matrix}\right\} \\ 1-\sum_{i=1}^{N} w_i \end{bmatrix}_{(N+1)\times 1}\right\} \begin{matrix} \text{on risky assets} \\ \to \text{ on risk-free asset} \end{matrix}$$

Choose W_p such that

$$\min_{\{W\}} \frac{1}{2} W^T V W$$

$$s.t. \quad W^T e + (1 - W^T I) r_f = E[\tilde{r}_p]$$

(Note: the other constraint is imbedded here.)

Form the Lagrangian

$$L = \frac{1}{2} W^T V W + \lambda \{ E(\tilde{r}_p) - W^T e - (1 - W^T I) r_f \}$$

$$\frac{\partial L}{\partial W} = 0 \qquad VW - \lambda(e - I r_f) = 0, \qquad (1)$$

$$\frac{\partial L}{\partial \lambda} = 0 \qquad E(\tilde{r}_p) - W^T e - (1 - W^T I) r_f = 0, \qquad (2)$$

From (1) $\quad W = \lambda V^{-1}(e - r_f I) = 0, \qquad (3)$

Substitute to (2)

$$E(\tilde{r}_p) - \lambda[e - r_f I]^T V^{-1} e - r_f + r_f \lambda (e - r_f I)^T V^{-1} I = 0,$$

Solving for λ

$$\lambda = \frac{E(\tilde{r}_p) - r_f}{(e - r_f I)^T V^{-1} (e - r_f I)}$$

Therefore

$$W_p = \lambda V^{-1}(e - r_f I) = \frac{[E(\tilde{r}_p) - r_f] V^{-1}}{(e - r_f I)^T V^{-1} (e - r_f I)} (e - r_f I)$$

$$= V^{-1}(e - r_f I) \frac{[E(\tilde{r}_p) - r_f]}{H}$$

where $H = (e - r_f I)^T V^{-1}(e - r_f I)$

$$= e^T V^{-1} e - 2r_f e^T V^{-1} I + r_f^{\ 2} I^T V^{-1} I$$

$$\equiv B - 2Ar_f + Cr_f^{\ 2}$$

Because V^{-1} is positive definite, $H>0$.

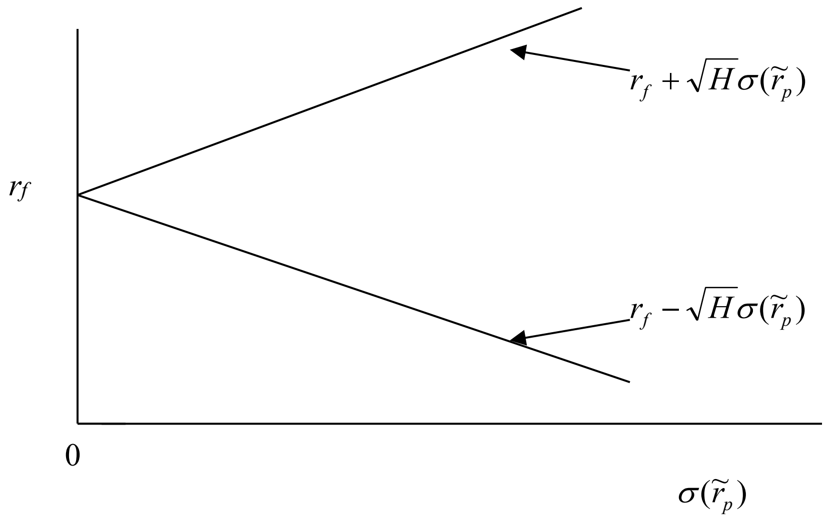
$$\sigma^2(\tilde{r}_p) = W_p^T V W_p = \frac{[E(\tilde{r}_p) - r_f](e - r_f I)^T V^{-1} V V^{-1}(e - r_f I)[E(\tilde{r}_p) - r_f]}{H \cdot H}$$

$$= \frac{[E(\tilde{r}_p) - r_f]^2 (e - r_f I)^T V^{-1}(e - r_f I)}{H^2} = \frac{[E(\tilde{r}_p) - r_f]^2}{H}$$

Because $H > 0$, we can write

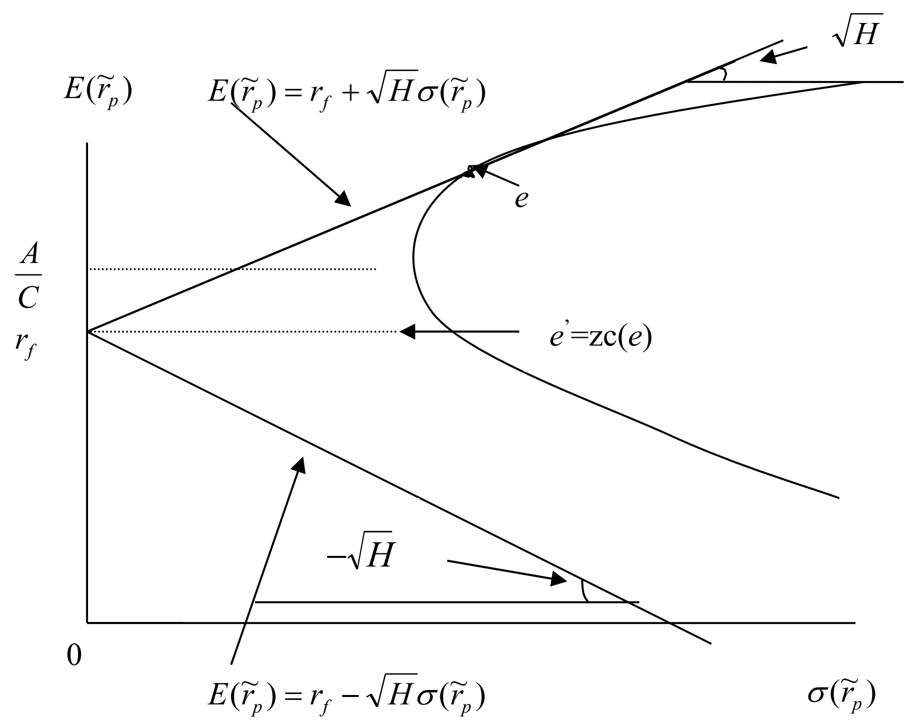
$$\sigma(\tilde{r}_p) = \begin{cases} \dfrac{E(\tilde{r}_p) - r_f}{\sqrt{H}} & \quad if \quad E(\tilde{r}_p) \geq r_f \\ (-)\dfrac{E(\tilde{r}_p) - r_f}{\sqrt{H}} & \quad if \quad E(\tilde{r}_p) < r_f \end{cases}$$

That is, the portfolio frontier of all assets is composed of two half-lines emanating from point $(0, r_f)$ in the σ-μ space with $\sqrt{H}$ and $-\sqrt{H}$ as slopes, respectively.


$r_f + \sqrt{H}\sigma(\tilde{r}_p)$
r_f
$r_f - \sqrt{H}\sigma(\tilde{r}_p)$
0
$\sigma(\tilde{r}_p)$

Special Cases

Case 1 $r_f < \dfrac{A}{C}$

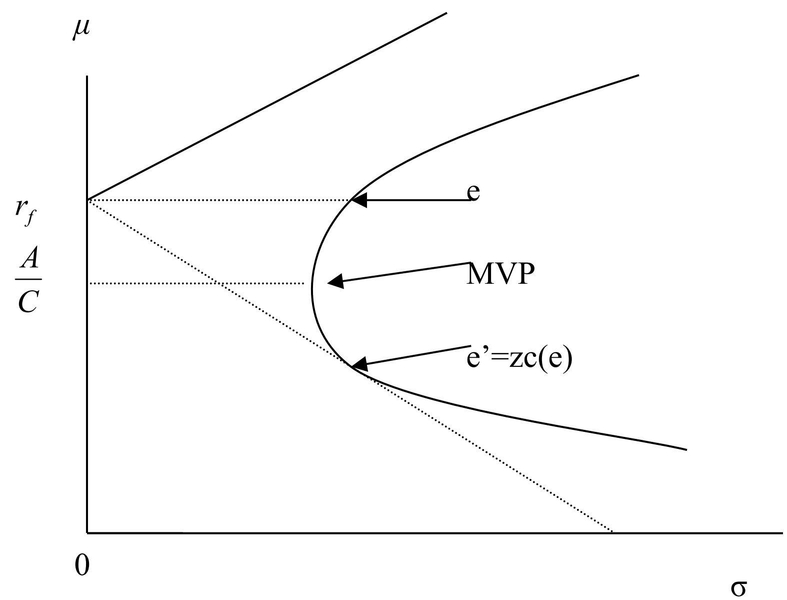


Remarks:

Point e is the tangent point of the half line $r_f + \sqrt{H}\sigma(\tilde{r}_p)$ and portfolio frontier of all risky assets.

Any portfolio on the line segment $r_f e$ is a convex combination of portfolio e and the riskless asset. Any portfolio on the half-line $r_f + \sqrt{H}\sigma(\tilde{r}_p)$ other than the line segment $r_f e$ involves short selling the risk-free asset at the rate r_f and investing the proceeds in the frontier portfolio e. Any portfolio on the half-line $r_f - \sqrt{H}\sigma(\tilde{r}_p)$ involves short selling the frontier portfolio e and investing the proceeds in the risk-free asset at the rate r_f.

Case 2 $\quad r_f > \dfrac{A}{C}$



Any portfolio on the half line $r_f + \sqrt{H}\sigma(\tilde{r}_p)$ involves short selling *e'* and investing the proceeds in the risk-free asset r_f. Any portfolio on the half-line $r_f - \sqrt{H}\sigma(\tilde{r}_p)$ involves a long position in the portfolio *e'*.

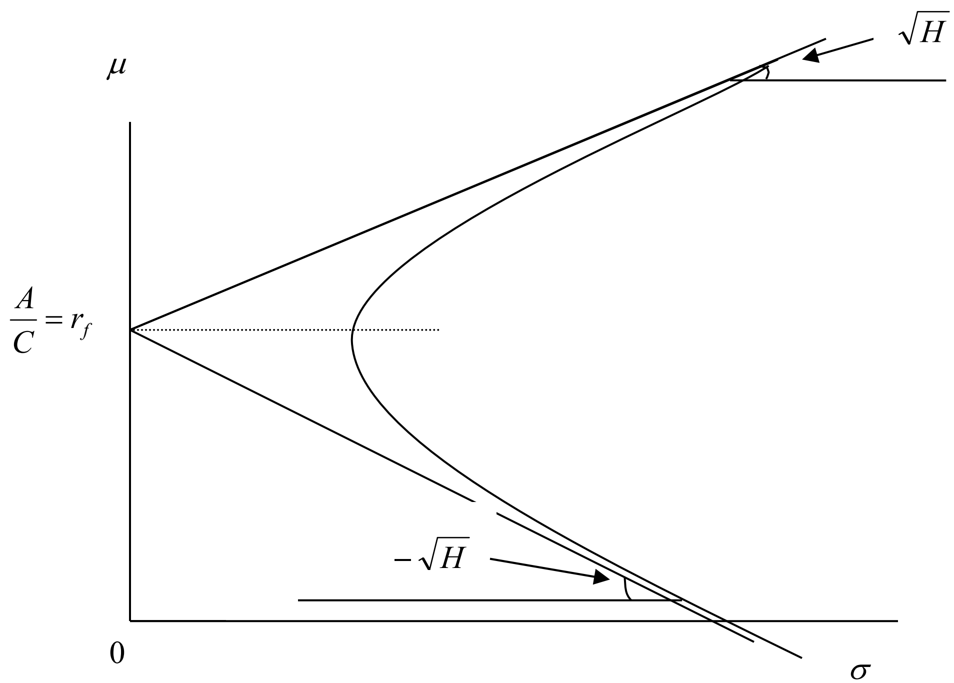
Case 3 $\quad r_f = \dfrac{A}{C}$

In this case

$$H = B - 2Ar_f + Cr_f^{\;2} = B - 2A\frac{A}{C} + C(\frac{A}{C})^2$$

$$= \frac{BC - A^2}{C} = \frac{D}{C} > 0$$

Recall that $E(\tilde{r}_p) = \dfrac{A}{C} \pm \sqrt{\dfrac{D}{C}} \cdot \sigma(\tilde{r}_p)$ are the two asymptotes of the portfolio frontier of risky assets. Therefore, the portfolio frontier of all assets is the area surrounded by the asymptotes. The curve never touches the half lines.



In this case, there is no tangency portfolio. Therefore, the portfolio frontier of all assets is not generated by the riskless asset and a portfolio on the frontier of risky assets. How is the portfolio frontier generated?

Now with a riskless asset

Let q – any portfolio (not necessarily frontier) with W_q the portfolio weights on risky assets.

p – frontier portfolio

W_p – weights on risky assets of portfolio p

Assume $E(\tilde{r}_p) \neq r_f$, after some algebra, we will obtain

$$E(\tilde{r}_q) = [1-\beta_{qp}]r_f + \beta_{qp}E(\tilde{r}_p)$$

$$\Rightarrow \qquad \tilde{r}_q = (1-\beta_{qp})r_f + \beta_{qp}\tilde{r}_p + \tilde{\varepsilon}_{qp}$$

where $\beta_{qp} \equiv \dfrac{\operatorname{cov}(\tilde{r}_q, \tilde{r}_p)}{\sigma^2(\tilde{r}_p)}$, $Cov(\tilde{r}_p, \tilde{\varepsilon}_{qp}) = E(\tilde{\varepsilon}_{qp}) = 0$ for any portfolio q and any frontier portfolio p other than the riskless asset.