Homework 4.

Yifu He 190003956

Question 1.

Need to prove, $\rho_{ab} = = \frac{\text{Sharp Ratio}(A)}{\text{Sharp Ratio}(B)}$. let $E(\tilde{r}) = \mu$.

①. $SP_A = \frac{\mu_a - r_f}{\sigma_a}$ $SP_B = \frac{\mu_a - r_f}{\sigma_b}$

$\frac{SP_B}{SP_A} = \frac{\sigma_a(\mu_b - r_f)}{\sigma_b(\mu_a - r_f)}$ ①, when a is on efficient frontier,

use a ~~as~~ a market portfolio to construct b.

$E(\tilde{r}_b) = \mu_b = (1-\beta_b)\cdot r_f + \beta_b \cdot \mu_a$

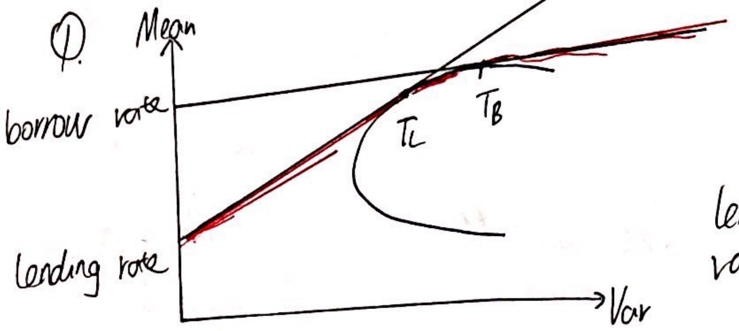
$= (1 - \frac{cov(a,b)}{\sigma_a^2})\cdot r_f + \frac{cov(a,b)}{\sigma_a^2}\mu_a$

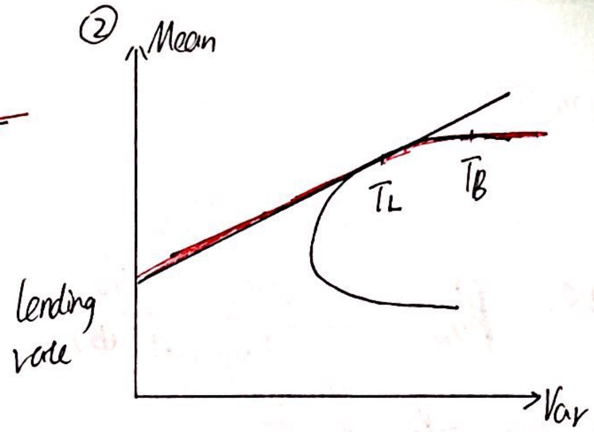
$\Rightarrow \mu_b - r_f = \frac{cov(a,b)}{\sigma_a^2}\cdot(\mu_a - r_f)$ ③,

take ③ into ①,

$\frac{SP_B}{SP_A} = \frac{\sigma_a \cancel{(\mu_b - r_f)}}{\sigma_b(\mu_a - r_f)} \cdot \frac{cov(a,b)\,(\mu_a - r_f)}{\sigma_a^2} = \frac{cov(a,b)}{\sigma_a \cdot \sigma_b} = \rho_{ab}$. finish

Question 2.

① 

② 

Question 3.

Proof 1. $cov(\tilde{r}_q, \tilde{r}_q)$. let Σ represent the covariance Matrix of ~~rp and rq~~ assets.

$cov(\tilde{r}_p, \tilde{r}_q) = W_p^T \Sigma W_q$

$Var(\tilde{r}_p) = W_p^T \cdot \Sigma W_p$.

$E[\tilde{r}_q] = (1-\beta_{pq}) E[\tilde{r}_{zc(p)}] + \beta_{pq} E[\tilde{r}_p]$ ①

$W_p = g + h \cdot E[\tilde{r}_p] = g + h \cdot E[\tilde{r}_q]$ ②

when $E[\tilde{r}_p] = E[\tilde{r}_q]$

$$\beta_{pq} = \frac{E[\tilde{r}_p] - E[\tilde{r}_{zc(p)}]}{E[\tilde{r}_q] - E[\tilde{r}_{zc(p)}]} = 1$$

$\therefore cov(\tilde{r}_q, \tilde{r}_p) = Var(\tilde{r}_p)$

Proof 2. $\rho = \dfrac{cov(\tilde{r}_p, \tilde{r}_q)}{\sigma_p, \sigma_q} = \dfrac{Var(\tilde{r}_p)}{\sigma_p \cdot \sigma_q}$

$\because$ P is on the efficient frontier, $E[\tilde{r}_p] = E[\tilde{r}_q]$

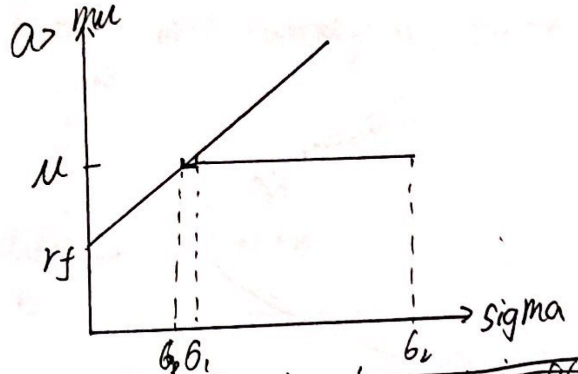
$\therefore \tilde{\sigma}_q > \tilde{\sigma}_p$

$\therefore \sigma_p \cdot \sigma_q > \sigma_p^2$

$\therefore \rho = \dfrac{Var(r_p)}{\sigma_p \cdot \sigma_q} \leq 1 \quad \therefore \rho \in [0, 1]$

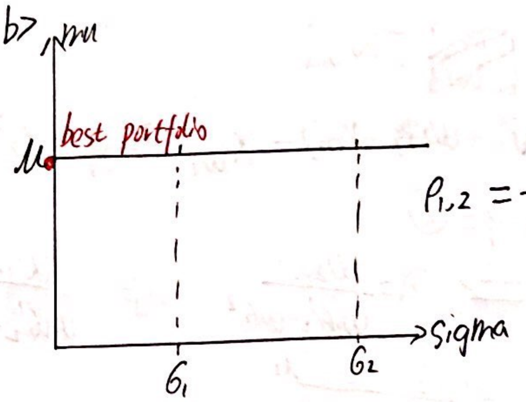
Question 4.

a>



In this senario. we ~~only invest in asset 1~~ can find a portfolio P has lower Var than 6_1.

b>



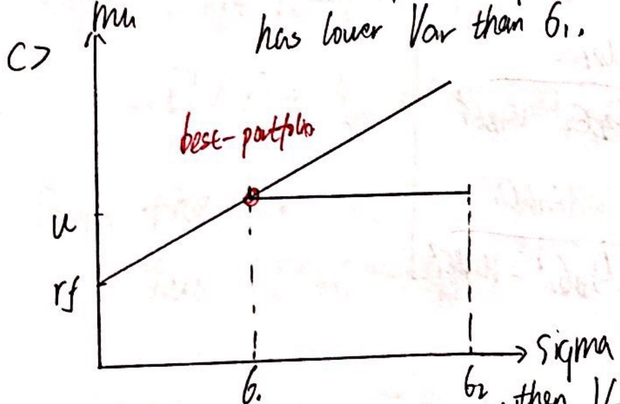
when $P=-1$, we can construct a portfolio with $Var(p) = W_1 6_1^2 + W_2 6_2^2 - 2W_1W_2 6_1 6_2$
$= (W_1 6_1 - W_2 6_2)^2$

Var(p) can be zero. and $u > r_f$.

We don't invest in risk-free asset.

This is one-fund seperation

c>



$P=1$ ~~when~~ then $Var(p) = (W_1 6_1 + W_2 6_2)^2$. thus, we only invest in Asset 1. we can use asset 1 and risk-free asse to construct the ~~risk free~~ efficiont frontier.

Question 5.

a> when $P=0$

$\vec{W} = (W_1, W_2)$. $COV(\tilde{r}_1, \tilde{r}_m) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 6_1^2 & 0 \\ 0 & 6_2^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$

$= W_1 6_1^2$

$COV(\tilde{r}_2, \tilde{r}_m) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 6_1^2 & 0 \\ 0 & 6_2^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = W_2 6_2^2$

$Var(\tilde{r}_m) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} 6_1^2 & 0 \\ 0 & 6_2^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = W_1^2 6_1^2 + W_2^2 6_2^2$

Investor A:

$\beta_{A_1} = \frac{COV(r_1, r_m)}{Var(\tilde{r}_m)} = \frac{W_{1A} 6_1^2}{W_{1A}^2 6_1^2 + W_{2A}^2 6_2^2}$, $\beta_{2A} = \frac{W_{2A} 6_2^2}{W_{1A}^2 6_1^2 + W_{2A}^2 6_2^2}$

Investor B:

$\beta_{B_1} = \frac{W_{1B} 6_1^2}{W_{1B}^2 6_1^2 + W_{2B}^2 6_2^2}$ $\beta_{2B} = \frac{W_{2B} 6_2^2}{W_{1B}^2 6_1^2 + W_{2B}^2 6_2^2}$

b7. Assume the zero-beta portfolio has the weight as

$W_{ZB} = \begin{bmatrix} \cancel{W_1} \\ \cancel{W_2} \end{bmatrix} [x\ y]$

$COV(W_1 r_1 + W_2 r_2, r_m) = x W_1 \sigma_1^2 + y W_2 \sigma_2^2 = 0$ ①

$x + y = $ ②

①② $\Rightarrow x = \dfrac{W_2\sigma_2^2}{W_2\sigma_2^2 - W_1\sigma_1^2}$, $y = \dfrac{-W_1\sigma_1^2}{W_2\sigma_2^2 - W_1\sigma_1^2}$

$E[r_{ZB}) = \dfrac{W_2\sigma_2^2 U_1 - W_1\sigma_1^2 U_2}{W_2\sigma_2^2 - W_1\sigma_1^2}$

when apply it for A.

$W_A = \begin{bmatrix} \dfrac{W_{2A}\sigma_2^2}{W_{2A}\sigma_2^2 - W_{1A}\sigma_1^2} \\ \dfrac{-W_{1A}\sigma_1^2}{W_{2A}\sigma_2^2 - W_{1A}\sigma_1^2} \end{bmatrix}$ $\qquad W_B = \begin{bmatrix} \dfrac{W_{2B}\sigma_2^2}{W_{2B}\sigma_2^2 - W_{1B}\sigma_1^2} \\ \dfrac{-W_{1B}\sigma_1^2}{W_{2B}\sigma_2^2 - W_{1B}\sigma_1^2} \end{bmatrix}$

$$E(r_1) = E[r_{ZC}] + \beta_1(E[r_m] - E[r_{ZC}])$$

$$= \frac{W_2 U_1\sigma_2^2 - W_1 U_2\sigma_1^2}{W_2\sigma_2^2 - W_1\sigma_1^2} + \frac{W_1\sigma_1^2}{W_1^2\sigma_1^2 + W_2^2\sigma_2^2}\left(W_1U_1 + W_2U_2 - \frac{W_2\sigma_2^2U_1 - W_1\sigma_1^2U_2}{W_2\sigma^2 - W_1\sigma_1^2}\right.$$

$$= \frac{W_2U_1\sigma_2^2 - W_1U_2\sigma_1^2}{W_2\sigma_2^2 - W_1\sigma_1^2} + \frac{W_1\sigma_1^2}{W_1^2\sigma_1^2 + W_2^2\sigma_2^2} \times \frac{(W_1^2\sigma_1^2 + W_2^2\sigma_2^2)\cdot(U_2 - U_1)}{W_2\sigma^2 - W_1\sigma_2^2}$$

$$= \frac{U_1(W_2\sigma_2^2 - W_1\sigma_1^2)}{W_2\sigma_2^2 - W_1\sigma_1^2} = U_1$$

Thus we prove their Expected return are the same.

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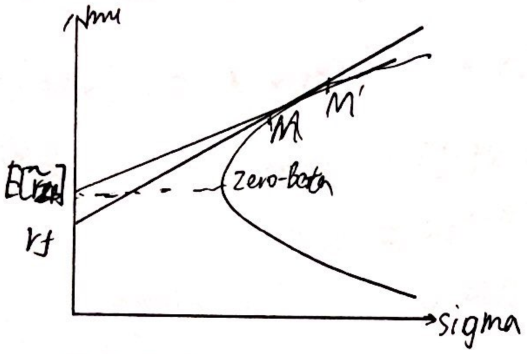
Question 6.

$r_f > 0$, ~~[illegible]~~ Borrowing is prohibited.



$E[\tilde{r}_q] = (1-\beta_{qm}) \cdot E[r_{ZB}] + \beta_{qm} \cdot E[\tilde{r}_m]$

$= E[\tilde{r}_{ZB}] + \beta_{qm} \cdot (E[\tilde{r}_m] - E[\tilde{r}_{ZB}])$

From the chart. we can find that r_m is on the efficient frontier.

thus, $E[\tilde{r}_m] > E[\tilde{r}_{ZB}]$

From the plot ~~E[r̃ZB] > E~~

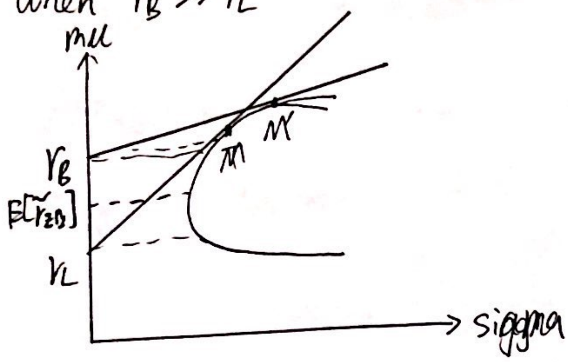
M' is on the right side of M.

Thus. $E[\tilde{r}_{M'}] > E[\tilde{r}_M]$

$E[\tilde{r}_{ZB}] > r_f$

Question 7.

when $r_B >> r_L$



From the chart, we can find that M is the Market Portfolio

$$E[\tilde{r}_q] = E[r_{zB}(m)] + \beta_{qm}(E[\tilde{r}_m] - E[\tilde{r}_{zB}])$$

From the plot. M' is on the right side of M'

thus. $E[\tilde{r}_m] > E[r_{zB}]$

and $r_B \geq E[\tilde{r}_{zB}] \geq r_L$.

Question 8.

$E[r_{market}] = 0.1 \times (-0.15) + 0.3 \times 0.05 + 0.4 \times 0.15 + 0.2 \times 0.2 = 0.1$

$E[r_x] = 0.1 \times -0.3 + 0.3 \times 0 + 0.15 \times 0.2 + 0.2 \times 0.5 = 0.15$

$Cov(r_x, r_{market}) = E[r_x \times r_m] - E[r_x] \cdot E[r_m]$

$= 0.1 \times (-0.15) \times (-0.3) + 0.3 \times 0.05 \times 0 + 0.4 \times 0.15 \times 0.2 + 0.2 \times 0.2 \times 0.5 - 0.1 \times 0.15$

$= 0.0215$

$E[r_m^2] = 0.02$

$Var(r_m) = E[r_m^2] - E^2[r_m] = 0.02 - (0.1)^2 = 0.01$

thus. $\beta_{xM} = \dfrac{cov(r_m, r_x)}{Var(r_m)} = \dfrac{0.0215}{0.01} = 2.15$

$E[r'_x] = r_f + \beta_{xm} \times (E(r_m) - r_f)$

$= 0.06 + 2.15 \times (0.1 - 0.06)$

$= 0.146$

$E[r_x] < E[r'_x]$.

Stock is misprice, it will move up.