2/26/20

Suggested Solutions for HW2

1.
(1) Easy. Just take derivatives.

(2) We set the expected utilities with and without buying the insurance equal

$$-\frac{p}{W_0+y}-\frac{1-p}{W_0}=-\frac{1}{W_0+y-c} \tag{1}$$

Simple algebra yields

$$W_0=\frac{(y-c)(1-p)y}{c-(1-p)y} \tag{2}$$

A reasonable insurance premium must satisfy the following conditions

$$(1-p)y<c<y \tag{3}$$

The first inequality guarantees that the premium can cover the expected loss while the second inequality guarantees that the investor will voluntarily purchase the insurance.

It is interesting to see that

$$\frac{dW_0}{dc}<0,\frac{dW_0}{dy}>0,\frac{dW_0}{dp}<0 \tag{4}$$

These comparative statistics make sense given that the investor has constant relative risk aversion and decreasing absolute risk aversion.

It is interesting to rewrite (2) as

$$c=\frac{(W_0+y)(1-p)y}{W_0+(1-p)y} \tag{5}$$

This is the maximum premium the investor with initial wealth W_0 is willing to pay.

We can also show that

$$\frac{dc}{dW_0}<0,\frac{dc}{dy}>0,\frac{dc}{dp}<0 \tag{6}$$

Again, these make sense for the decreasing absolute risk aversion preference.

Suppose the investor has the negative exponential utility function

$$u(w)=-\exp(-w)$$

In this case, we will not easily get analytical expressions like (2) and (5). However, by the implicit function theorem, it is easy to show that

$$\frac{dc}{dW_0}=0 \tag{7}$$

This is intuitively justified because the investor has constant absolute risk aversion.

2. The first part is easy. I will leave it to you.

To show that the converse is not true, we need to construct a counter example.

Let $u(z) = -\exp(-z)$

$$\tilde{r}_A = \begin{cases} 1, & p = 1/10 \\ 0, & p = 9/10 \end{cases}$$

$$\tilde{r}_B = 0.09$$

It is easy to see that $E(\tilde{r}_A) = 0.1 > 0.09 = E(\tilde{r}_B)$.

However,

$$\begin{aligned} E[u(\tilde{r}_A)] &= -\frac{1}{10}\exp(-1) - \frac{9}{10}\exp(-0) \\ &= -0.94 \\ E[u(\tilde{r}_B)] &= -\exp(-0.09) = -0.91 \end{aligned}$$

$$E[u(\tilde{r}_A)] < E[u(\tilde{r}_B)]$$

contradicting $A \geq^{FSD} B$.

QED.

3. Again the first part is easy and is left to you.

For the second part, the idea of the proof is similar to that in the FSD case.

Consider $u(z) = \ln(z)$

$$\tilde{r}_A = \begin{cases} 2, & p=1/5 \\ 12, & p=4/5 \end{cases}$$

$$\tilde{r}_B = \begin{cases} 22, & p=1/5 \\ 7, & p=4/5 \end{cases}$$

$$E(\tilde{r}_A) = E(\tilde{r}_B) = 10$$

$$\text{var}(\tilde{r}_A) = (2-10)^2\frac{1}{5} + (12-10)^2\frac{4}{5} = 16$$

$$\text{var}(\tilde{r}_B) = (22-10)^2\frac{1}{5} + (7-10)^2\frac{4}{5} = 36$$

$$\text{var}(\tilde{r}_A) < \text{var}(\tilde{r}_B)$$

$$E[u(\tilde{r}_A)] = \frac{1}{5}\ln(2) + \frac{4}{5}\ln(12) = 2.13$$

$$E[u(\tilde{r}_B)] = \frac{1}{5}\ln(22) + \frac{4}{5}\ln(7) = 2.17$$

$$E[u(\tilde{r}_A)] < E[u(\tilde{r}_B)]$$

which is a contradiction.

QED.

Remarks: while the proof appears technical, the intuition is very important. Why does this investor prefer Stock B to Stock A, even though both stocks have the same mean return, and Stock B's variance more than doubles that of Stock A? Higher momentums of stock returns clearly play an important role in the investor's decision making. In particular, skewness affects expected utility in a crucial way.

4. It is enough to consider a two-asset case. The case with n assets is a generalization.

Consider the equally weighted portfolio $\tilde{r}_A \equiv \dfrac{\tilde{r}_1 + \tilde{r}_2}{2}$. Since this is the minimum variance portfolio, by the necessary condition for SSD, it is the only candidate dominating portfolio. We have to show that this is also a sufficient condition.

Since $\tilde{r}_1$ and $\tilde{r}_2$ are iid, we can claim by symmetry that

$$E(\tilde{r}_1 \mid \frac{\tilde{r}_1 + \tilde{r}_2}{2}) = E(\tilde{r}_2 \mid \frac{\tilde{r}_1 + \tilde{r}_2}{2})$$

(Note: we need both independence and identical distributions in order to claim the above.)

But $E(\tilde{r}_1 \mid \dfrac{\tilde{r}_1 + \tilde{r}_2}{2}) + E(\tilde{r}_2 \mid \dfrac{\tilde{r}_1 + \tilde{r}_2}{2}) = E[(\tilde{r}_1 + \tilde{r}_2) \mid \dfrac{\tilde{r}_1 + \tilde{r}_2}{2}] = (\tilde{r}_1 + \tilde{r}_2)$

we therefore have

$$E(\tilde{r}_1 \mid \frac{\tilde{r}_1 + \tilde{r}_2}{2}) = E(\tilde{r}_2 \mid \frac{\tilde{r}_1 + \tilde{r}_2}{2}) = \frac{\tilde{r}_1 + \tilde{r}_2}{2}$$

We claim that $\tilde{r}_A \equiv \dfrac{\tilde{r}_1 + \tilde{r}_2}{2}$ is the dominating portfolio.

Let $\tilde{r}_B \equiv \lambda \tilde{r}_1 + (1-\lambda)\tilde{r}_2$ be an arbitrary portfolio. We let $\tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$.

Then $E(\tilde{\varepsilon} \mid \tilde{r}_A) = E(\tilde{r}_B - \tilde{r}_A \mid \tilde{r}_A) = E(\tilde{r}_B \mid \tilde{r}_A) - \tilde{r}_A = [\lambda \tilde{r}_A + (1-\lambda)\tilde{r}_A] - \tilde{r}_A = 0$

which says that $A \geq^{SSD} B$.

QED.

5. Consider the FOC

$$E[u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)]=0$$

This is a necessary & sufficient condition for $E[\tilde{r}-r_f]$ to be the minimum risk premium that induces individual i to put all his wealth into the risky asset.

If we can show that

$$E[u_k\,'(W_0(1+\tilde{r}))(\tilde{r}-r_f)]\geq 0$$

then we are done. Why?

[Recall the Lemma proved in class. We can easily prove a similar Lemma below:

The following statements are equivalent

(1) There exists a strictly increasing and convex function G, $u_k = G(u_i)$;

(2) $R_A^i(z)\geq R_A^k(z)$, $\forall z$.
We will use this Lemma to show the results.]

We now show that if $u_k = G(u_i)$, the minimum risk premium required for individual k to invest all his wealth in the risky asset is lower than that for individual i.

When $\tilde{r}-r_f\geq 0$, we have $W_0(1+\tilde{r})\geq W_0(1+r_f)$, then

$$G\,'(u_i(W_0(1+\tilde{r})))u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

$$\geq G\,'(u_i(W_0(1+r_f)))u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

Similarly, when $\tilde{r}-r_f<0$, we have $W_0(1+\tilde{r})<W_0(1+r_f)$, then

$$G\,'(u_i(W_0(1+\tilde{r})))u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

$$\geq G\,'(u_i(W_0(1+r_f)))u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

Combining the above 2 relations yields,

$$
\begin{aligned}
&E[u_k^{'}(W_0(1+\tilde{r})(\tilde{r}-r_f)] \\
&= E[G'(u_i(W_0(1+\tilde{r})))u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)] \\
&\geq G'(u_i(W_0(1+r_f)))\underbrace{E[u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)]}_{=0} \\
&= 0
\end{aligned}
$$

We conclude that investor k who is less risk averse requires a lower risk premium for him to invest all his wealth in the risky asset.

6.
(1) If there is limited liability, the share holders' payoffs are

	project 1				project 2	
probability		pay off		probability		pay off
0.2		0		0.4		0
0.6		0		0.2		0
0.2		0		0.4		4,000

Clearly project 2 dominates project 1 in both first-degree and second-degree sense.

(2) If there is no limited liability, then the share holders' payoffs are

	project 1				project 2	
probability		pay off		probability		pay off
0.2		-2,000		0.4		-6,000
0.6		-1,000		0.2		-1,000
0.2		0		0.4		4,000
expected value		-1,000				-1,000

In this case, shareholders have the obligation to make debt payments from their personal wealth when a project is losing money. Both projects have the same expected value, but have different risks. In general we are not able to demonstrate stochastic dominance of these projects.

7. If returns are normal, then utility maximization is equivalent to mean-variance optimization.

Case 1: B second degree stochastically dominates A. Nothing can be said about first-degree dominance.
Case 2: A first degree stochastically dominates B. A also second-degree dominates B.
Case 3: Nothing can be said about stochastic dominance in either order in this case.

If returns are not normal, in general we can't say anything about stochastic dominance in all three cases.