FM1 Notes, Spring 2020
Consumption-Based Asset Pricing

We now move to a multi-period portfolio choice problem and study the asset pricing implications.

I. Basic assumptions

1. one good, non-storable;
2. n production units (firms). WLOG, we can make n=1 (this is not essential);
3. production is costless, random and exogenous (endowment economy);
4. homogenous consumers/investors.

Objective: to determine the relationship between these exogenously determined productivity changes and market-determined movements in asset prices.

II. The economy

The representative investor's objective is

$$\max_{\{c_{t+j}\}} E_t \left\{ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right\}$$

where β is the discount factor. This utility function is a so-called time-separable utility function.

Output vector (n production units)

$$y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{nt} \end{bmatrix}$$

Without loss of generality, we can assume that n=1.

Output is perishable so that feasible consumption levels must satisfy

$$0 \le c_t \le \sum_{i=1}^{n} y_{it}$$

Assume that production is exogenous. No resources are utilized and there is no way to offset the output of any good at any time. This is a so-called endowment economy.

Each production unit has outstanding one equity share perfectly divisible which entitles its owner as of the beginning of period t to all of the unit's output in period t.

Shares are traded after payment of real dividends at a competitively determined price

$$p_t = \begin{bmatrix} p_{1t} \\ p_{2t} \\ \vdots \\ p_{nt} \end{bmatrix}$$

Let

$$z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \\ \vdots \\ z_{nt} \end{bmatrix}$$

be the beginning of period share holdings.

In equilibrium, we should have

$$c_t = \sum_{i=1}^{n} y_{it}\,,\ z_t = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \forall t$$

All information on the current and future physical state of the economy is summarized in the current output vector, y.

So equilibrium price p(.) should be some fixed function of the state of the economy $p_t = p(y_t)$.

Rational expectations assumption:

The market clearing price function implied by consumer behavior = price function on which consumer decisions are based.

III. Optimization

Budget constraint

$$\underset{\text{(allocation)}}{c_t + p(y_t)\,x_t} \quad = \quad \underset{\text{(source)}}{[y_t + p(y_t)]z_t} \quad \equiv \quad \theta_t$$

$$[y_{t+1} + p(y_{t+1})]x_t \equiv \theta_{t+1} \qquad \text{(next period source)}$$

Value function

y – summarizes the state of the economy and can be treated as the state vector

θ - wealth at time t (resource)

$$V(\theta_t, y_t) = \max_{\{c_t, x_t\}} \left\{ u(c_t) + \beta E_t V(\theta_{t+1}, y_{t+1}) \right\}$$

Using the budget constraint, we have

$$V(\theta_t, y_t) = \max_{\{c_t, x_t\}} \left\{ u(\theta_t - p(y_t) x_t) + \beta E_t V(\theta_{t+1}, y_{t+1}) \right\}$$

FOC

$$x_t : \quad -u'(c_t) p(y_t) + \beta E_t \left\{ V_1(\theta_{t+1}, y_{t+1})[y_{t+1} + p(y_{t+1})] \right\} = 0$$

$$\theta_t : \quad \mathrm{V}_1(\theta_t, y_t) = u'(c_t)$$

Combining the above two equations yields

$$p(y_t) = \beta E_t \left\{ \frac{V_1(\theta_{t+1}, y_{t+1})}{u'(c)} [y_{t+1} + p(y_{t+1})] \right\}$$

$$= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} [y_{t+1} + p(y_{t+1})] \right\}$$

or,

$$u'(c_t) p(y_t) = \beta E_t \left\{ u'(c_{t+1}) [y_{t+1} + p(y_{t+1})] \right\}$$

We can iterate the above equation to yield

$$u'(c_t) p(y_t) = E_t \sum_{j=1}^{\infty} \left\{ \beta^j u'(c_{t+j}) y_{t+j} \right\}$$

or,

$$p(y_t) = E_t \sum_{j=1}^{\infty} \left\{ \beta^j \frac{u'(c_{t+j})}{u'(c_t)} y_{t+j} \right\}$$

provided that

$$\lim_{j \to \infty} E_t \left\{ \beta^j u'(c_{t+j}) p(y_{t+j}) \right\} = 0.$$

This is called the transversality condition or the no-bubble condition.

IV. Some implications

1. Linear utility

In this case
$$\frac{u'(c_{t+1})}{u'(c_t)} = 1$$

Therefore
$$p(y_t) = \beta E_t \left\{ y_{t+1} + p(y_{t+1}) \right\}$$

Iterating this equation yields
$$p(y_t) = \sum_{j=1}^{\infty} \left\{ \beta^j E_t(y_{t+j}) \right\}$$

provided that
$$\lim_{j \to \infty} E_t \left\{ \beta^j p(y_{t+j}) \right\} = 0 .$$

This is again called the transversality condition or the no-bubble condition.

2. The Equity premium puzzle

Summary Statistics Based on U.S. Annual Data for the Past Century, All in Real Terms

Sample Means			
r_s	0.070		
r_f	0.010		
$\Delta c_t / c_{t-1}$	0.018		
Sample Variance-Covariance Matrix			
	r_s	r_f	c_t / c_{t-1}
r_s	0.02740		
r_f	0.00104	0.00308	
c_t / c_{t-1}	0.00219	-0.00019	0.00127

Recall the pricing equation

$$p_t = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} [y_{t+1} + p_{t+1}] \right\} \equiv E_t \left[m_{t+1} x_{t+1} \right]$$

or,

$$1 = E_t \left[m_{t+1} R_{t+1} \right]$$

where $m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$ is the so-called *pricing kernel* or *stochastic discount factor* (SDF), $x_{t+1} \equiv y_{t+1} + p_{t+1}$ is the total payoff, and $R_{t+1} \equiv \frac{y_{t+1} + p_{t+1}}{p_t}$ is the *gross* return rate of the asset. (Note: $r_t = R_t - 1$ is the net return).

Consider a one-period discount bond which pays \$1 when the bond matures. Then

$$\frac{1}{R_{f,t}} = p_t = E_t\left[m_{t+1} \cdot 1\right] = E_t(m_{t+1})$$

Let $R_t^e \equiv R_t - R_{f,t}$ be the excess return of equity over the risk-free asset. The asset pricing equation implies that

$$E_t\left\{m_{t+1} \cdot R_{t+1}^e\right\} = 0$$

Using the property of covariance, we have

$$0 = E_t(m_{t+1})E_t(R_{t+1}^e) + \sigma_{m,R} = E_t(m_{t+1})E_t(R_{t+1}^e) + \rho_{m,R}\sigma_m\sigma_R$$

Therefore

$$\sigma_m = \frac{E_t(m_{t+1})E_t(R_{t+1}^e)}{-\rho_{m,R}\sigma_R} \geq \frac{E_t(m_{t+1})E_t(R_{t+1}^e)}{\sigma_R}$$

$$\frac{\sigma_m}{E_t(m_{t+1})} \geq \frac{E_t(R_{t+1}^e)}{\sigma_R} \quad \text{(Sharpe ratio)}$$

Consider the CRRA utility function

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

In this case, $m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$

Empirically, $E(R^e) \approx 6\%$, $\sigma_R \approx 16\%$, $E(m) = \frac{1}{R_f} \approx 0.99$, implying that the theoretical value of the volatility of the price kernel must be $\sigma_m \geq 37\%$.

However, with log utility ($\gamma = 1$), we find empirically that $\sigma_m \approx 3.6\%$. We need a high γ to generate the volatility.

From the asset pricing restriction, we have

$$0 = E_t(m_{t+1}) E_t(R^e_{t+1}) + \sigma_{m,R}$$

We also have

$$E_t(R_{t+1}^e) = (-)\frac{\sigma_{m,R}}{E_t(m_{t+1})}$$

Under CRRA utility, $m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, we have

$$E_t(R_{t+1}^e) = (-)\frac{\text{cov}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}, R_{t+1}^e\right]}{E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]} = (-)\frac{\text{cov}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}, R_{t+1}^e\right]}{E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}$$

The equity premium $E(R^e) \approx 6\%$, which is large. In order to generate this large premium, we need a large covariance between stock return and the price kernel. However, empirically consumption is too "smooth", so we need a large γ. In practice, no "reasonable" γ is able to generate such a large equity premium. This is the so-called *equity premium puzzle*.

3. The risk-free rate puzzle

Is there a problem when γ is high? Recall that

$$\frac{1}{R_{f,t}} = E_t(m_{t+1}) = E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]$$

Assuming a log-normal distribution of consumption, we have

$$\frac{1}{R_{f,t}} = E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = \beta E_t\left[\exp\left(-\gamma \ln\frac{c_{t+1}}{c_t}\right)\right] = \beta \exp\left(-\gamma\mu_{c,t} + \frac{1}{2}\gamma^2\sigma_{c,t}^2\right)$$

Therefore

$$r_{f,t} \equiv \ln(R_{f,t}) = -\ln(\beta) + \gamma\,\mu_{c,t} - \frac{1}{2}\gamma^2\sigma_{c,t}^2$$

Empirically, $\mu_c \approx 1.8\%$, $\sigma_c \approx 3.6\%$. If we set $\beta = 0.99$ and $\gamma = 10$, we have

$$r_f = -\ln(0.99) + 10 * 0.018 - \frac{1}{2}(10^2)(0.036^2) = 0.1253 = 12.53\%$$

Too high! This is the so-called *risk-free rate puzzle*.