FM1 Notes, Spring 2020

Global Measure of Risk

Suppose that there are two investors. How do we compare their degrees of risk aversion?

Claim: If there are two individuals, *i* and *k*, with

$$R_A^i(z) \geq R_A^k(z), \quad \forall\ z$$

then individual *i* is willing to pay a higher risk premium than individual *k* to insure against a random loss. In this case, individual *i* is said to be more risk averse than individual *k*.

(In a sense, $R_A(\cdot)$ is also a global measure of risk.)

Proof: Suppose that there is a risky asset (e.g., a stock) and a risk-free asset. Consider the FOC for individual k

$$E[u_k^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)]=0$$

This is the necessary and sufficient condition for $E[\tilde{r}-r_f]$ to be the minimum risk premium that induces individual k to put all his wealth into the risky asset.

If we can show that

$$E[u_i{}'(W_0(1+\tilde{r}))(\tilde{r}-r_f)]\leq 0$$

then we are done. Why?

[

We shall make use of the following Lemma.

Lemma: The following statements are equivalent

(1) There exists a strictly increasing and concave function G, $u_i = G(u_k)$;

(2) $R_A^i(z) \geq R_A^k(z)$, $\forall z$.

The proof of the lemma is skipped.

]

We now show that if $u_i = G(u_k)$, the minimum risk premium required for individual i to invest all his wealth in the risky asset is higher than that for individual k.

We need to show that

$$E[u_i^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)]\leq 0$$

When $\tilde{r}-r_f \geq 0$, we have $W_0(1+\tilde{r}) \geq W_0(1+r_f)$, then

$$G'(u_k(W_0(1+\tilde{r})))u_k^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

$$\leq G'(u_k(W_0(1+r_f)))u_k^{'}(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

Similarly, when $\tilde{r}-r_f < 0$, we have

$$G'(u_k(W_0(1+\tilde{r})))u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

$$\leq G'(u_k(W_0(1+r_f)))u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)$$

Combining the above 2 relations yields,

$$\begin{aligned}
&E[u_i'(W_0(1+\tilde{r})(\tilde{r}-r_f)] \\
&= E[G'(u_k(W_0(1+\tilde{r})))u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)] \\
&\leq G'(u_k(W_0(1+r_f)))\underbrace{E[u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)]}_{=0} \\
&= 0
\end{aligned}$$

We conclude that the more risk averse an individual is, the larger the RP required for him to invest all his wealth in the risky asset.

Two Fund Monetary Separation

When there is more than one risky asset, in general we cannot say that the wealth elasticities of the demand for risky assets are greater than unity when an individual has decreasing RRA.

If an individual always chooses to hold the same portfolio of risky assets and only change the mix between that portfolio and the riskless asset for differing levels of initial wealth, then the comparative statics for the 2-asset case will be valid in a multiple-asset world. In such an event, the individual's optimal portfolios for different W_0's are always linear combinations of the riskless asset and a risky asset mutual fund. This is called <u>two-fund monetary separation</u>.

Utility functions exhibiting 2-fund monetary separation

Power utility $U' = (A+BZ)^c$

Negative exponential utility $U' = A \exp(BZ)$

Proof: Skipped.