Binary Search Trees

Compiled from internet

Number guessing game

I’m thinking of a number between 1 and n

You are trying to guess the answer

For each guess, I’ll tell you “correct”, “higher” or “lower”

Describe an algorithm that minimizes the number of guesses

Binary Search Trees

BST – A binary tree where a parent's value is greater than all values in the left subtree and less than or equal to all the values in the right subtree

$$leftTree(i) < i \text{ £ } rightTree(i)$$

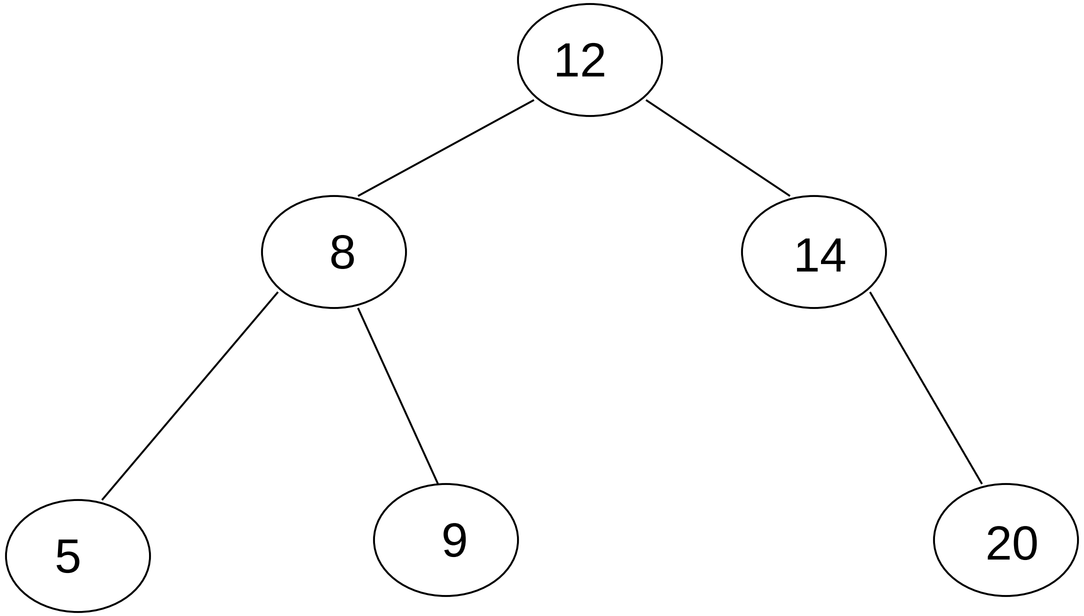
the left and right children are also binary trees

Why not?

$$leftTree(i) \text{ £ } i \text{ £ } rightTree(i)$$

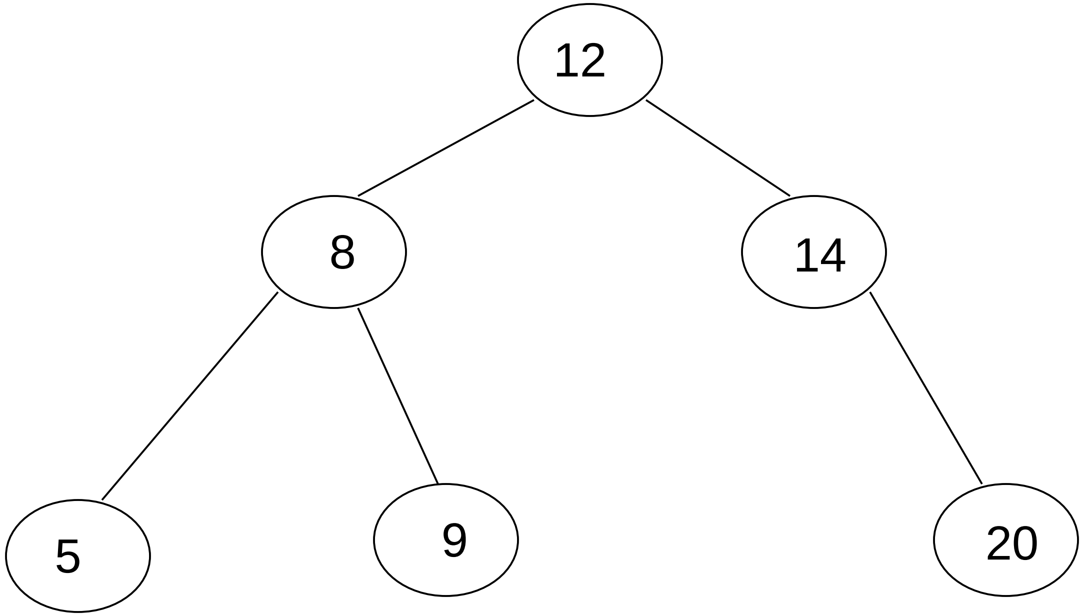
Can be implemented with with pointers or an array

Example



What else can we say?

$$left(i) < i \leq right(i)$$



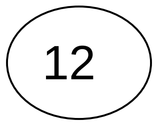
All elements to the left of a node are less than the node

All elements to the right of a node are greater than or equal to the node

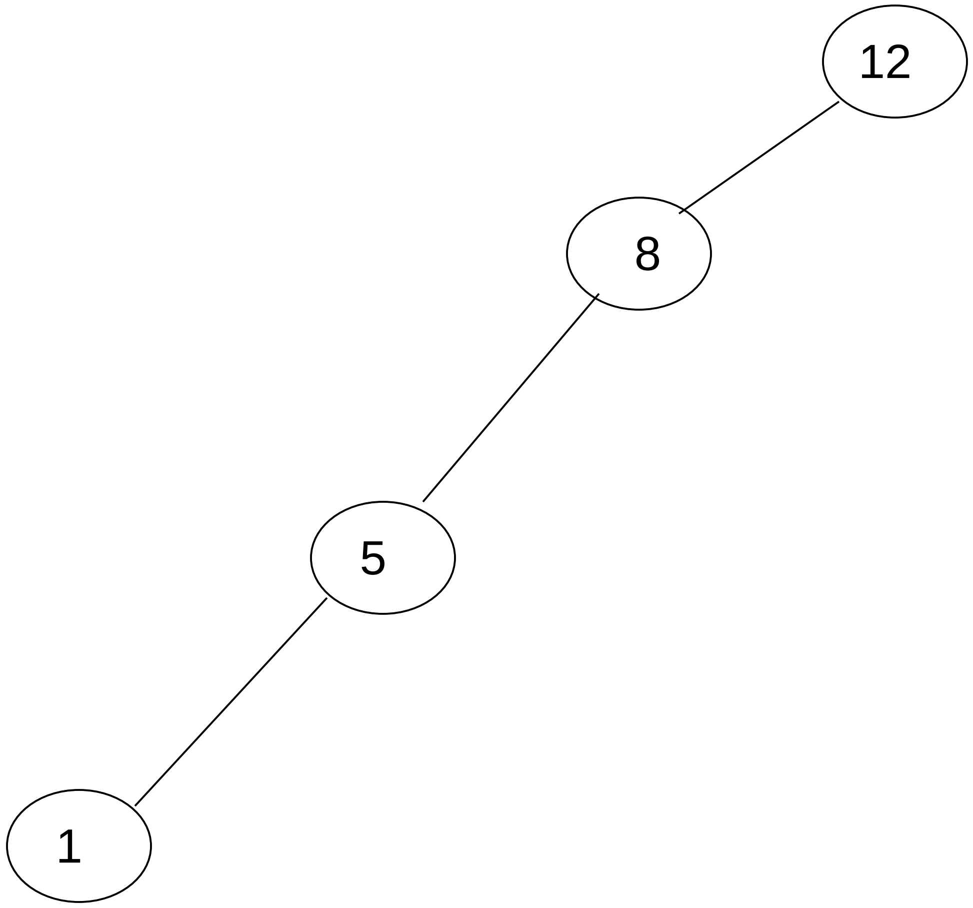
The smallest element is the left-most element

The largest element is the right-most element

Another example: the loner



Another example: the twig



Operations

Search(T,k) – Does value k exist in tree T

Insert(T,k) – Insert value k into tree T

Delete(T,x) – Delete node x from tree T

Minimum(T) – What is the smallest value in the tree?

Maximum(T) – What is the largest value in the tree?

Successor(T,x) – What is the next element in sorted order after x

Predecessor(T,x) – What is the previous element in sorted order of x

Median(T) – return the median of the values in tree T

Search

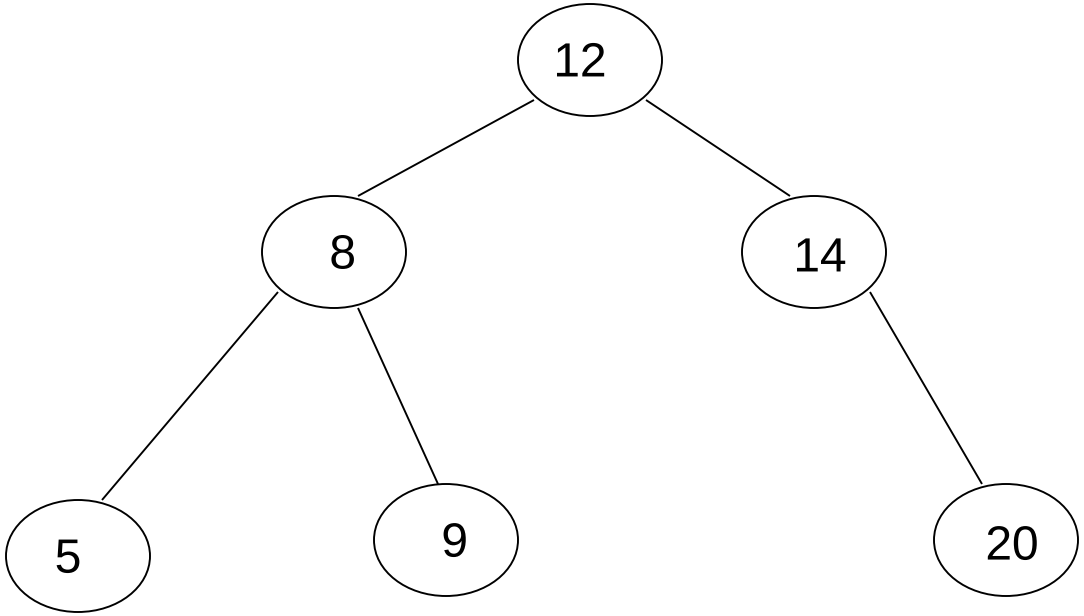
How do we find an element?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 9)

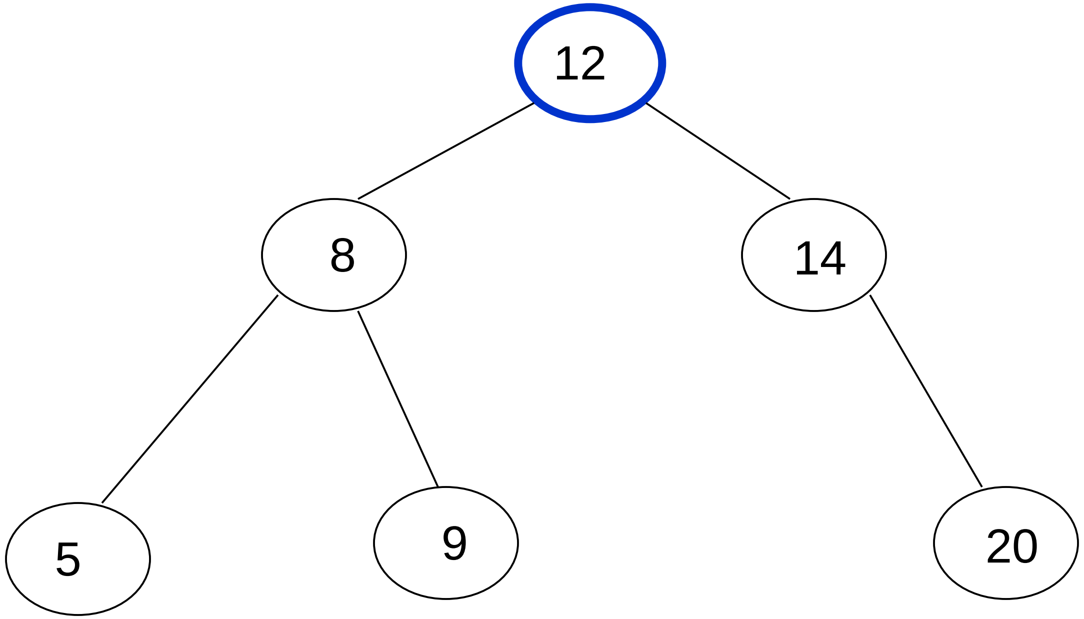


BSTSEARCH(x, k)
if $x = null$ or $k = x$
  return x
elseif $k < x$
  return BSTSEARCH(LEFT(x), k)
else
  return BSTSEARCH(RIGHT(x), k)

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 9)




```
BSTSearch(x, k)
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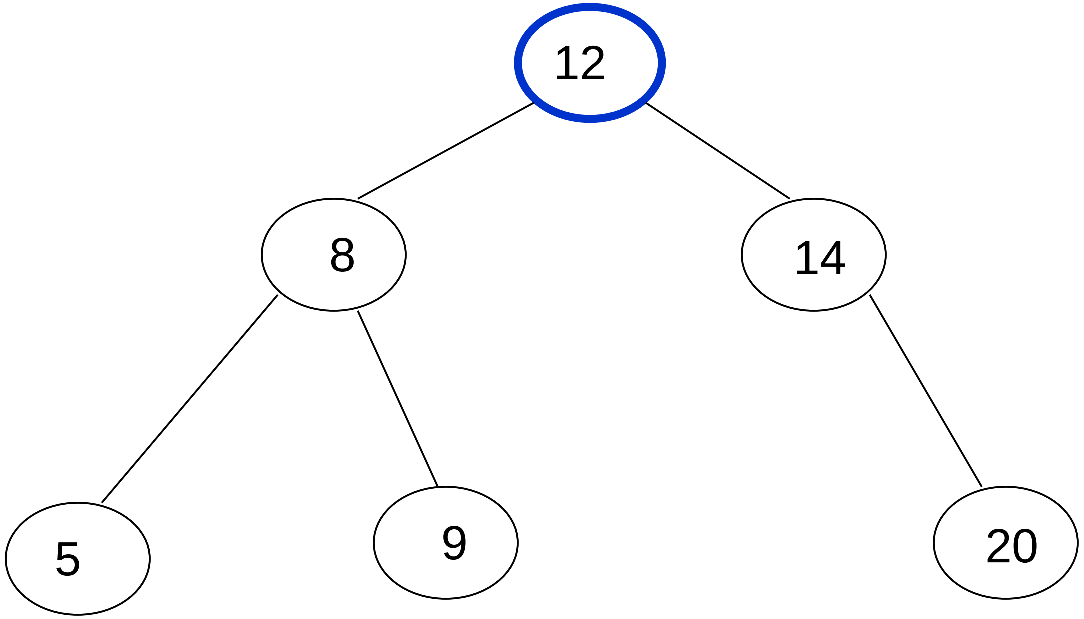
```

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 9)



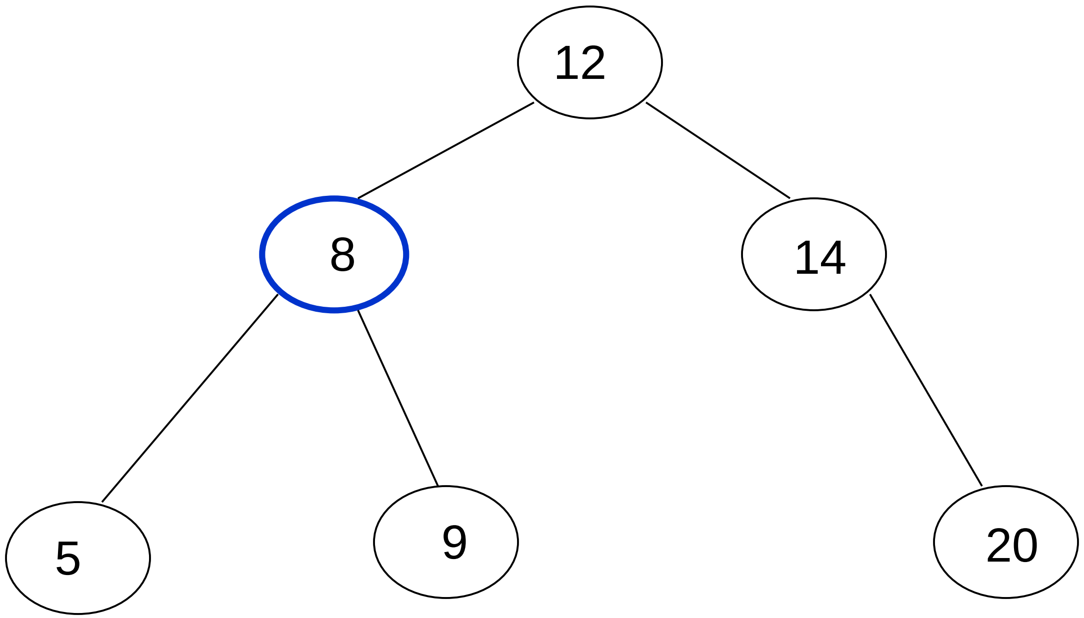
9 > 12?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

Finding an element

$$left(i) < i \le right(i)$$

Search(T, 9)

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BSTSearch(x, k)
if x = null or k = x
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elseif k < x
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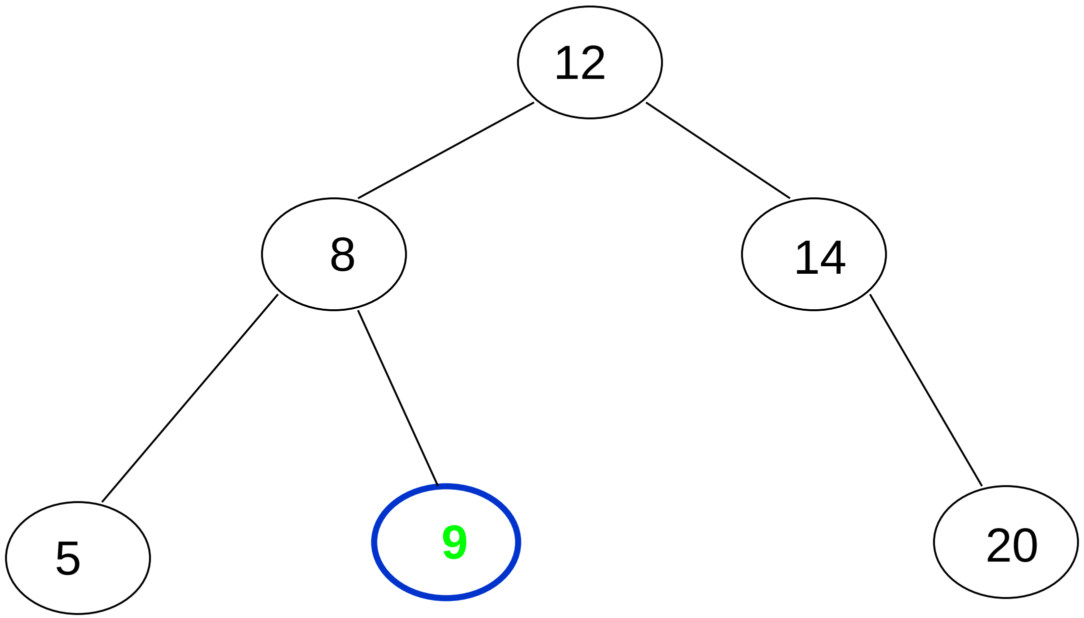
        return BSTSearch(Right(x), k)
```

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 9)



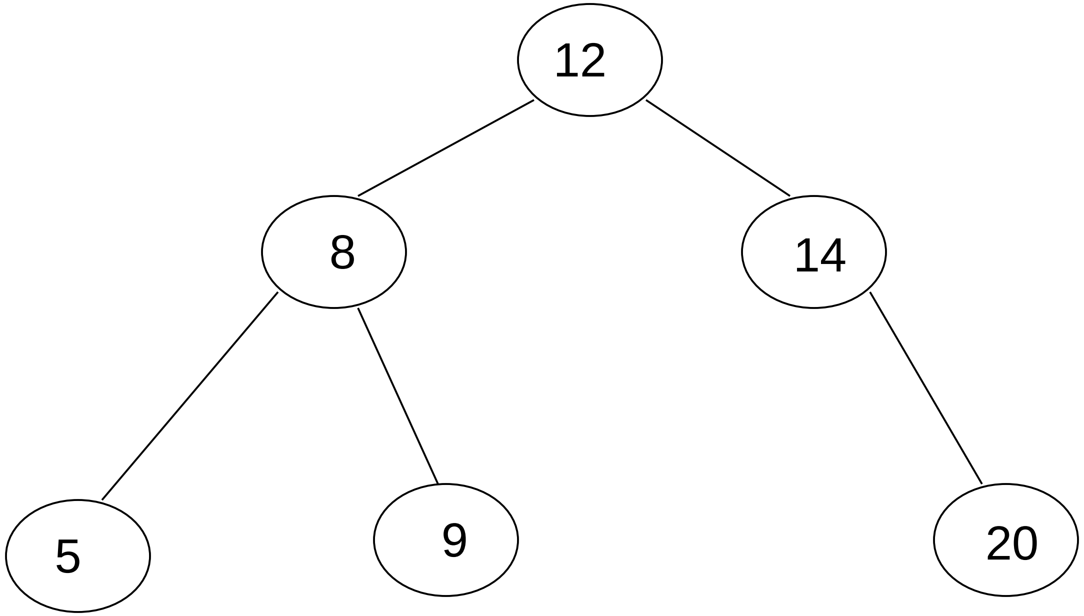
BSTSEARCH(x, k)

if $x = null$ or $k = x$
    return x
elseif $k < x$
    return BSTSEARCH(LEFT(x), k)
else
    return BSTSEARCH(RIGHT(x), k)

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 13)



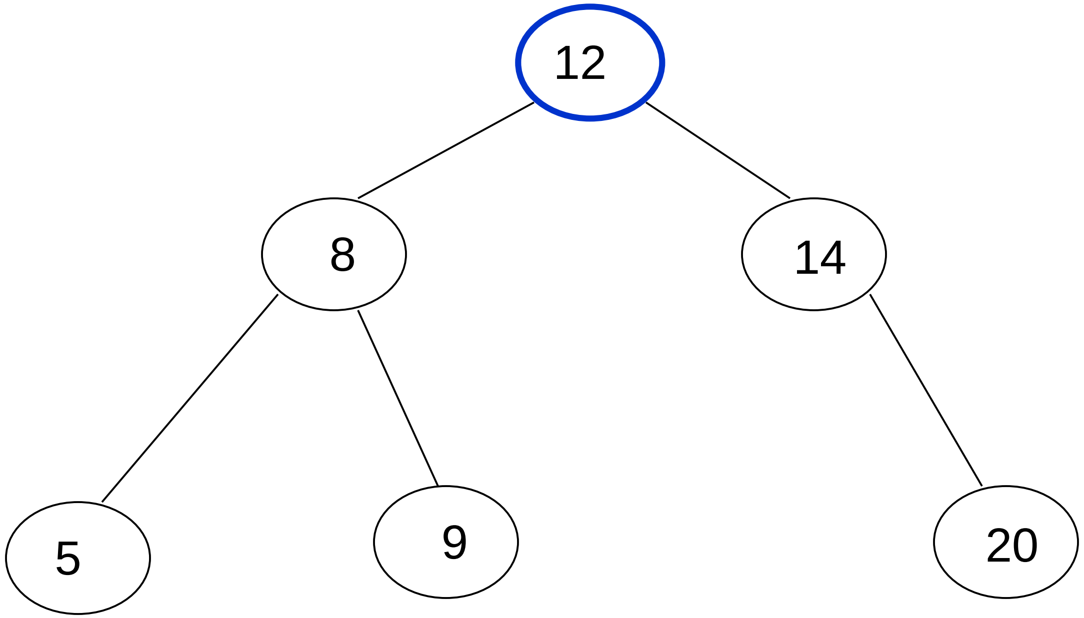
$\text{BSTSEARCH}(x, k)$

```
if x = null or k = x
        return x
elseif k < x
        return BSTSEARCH(LEFT(x), k)
else
        return BSTSEARCH(RIGHT(x), k)
```

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 13)



BSTSEARCH(x, k)

if $x = null$ or $k = x$
    return x
elseif $k < x$
    return BSTSEARCH(LEFT(x), k)
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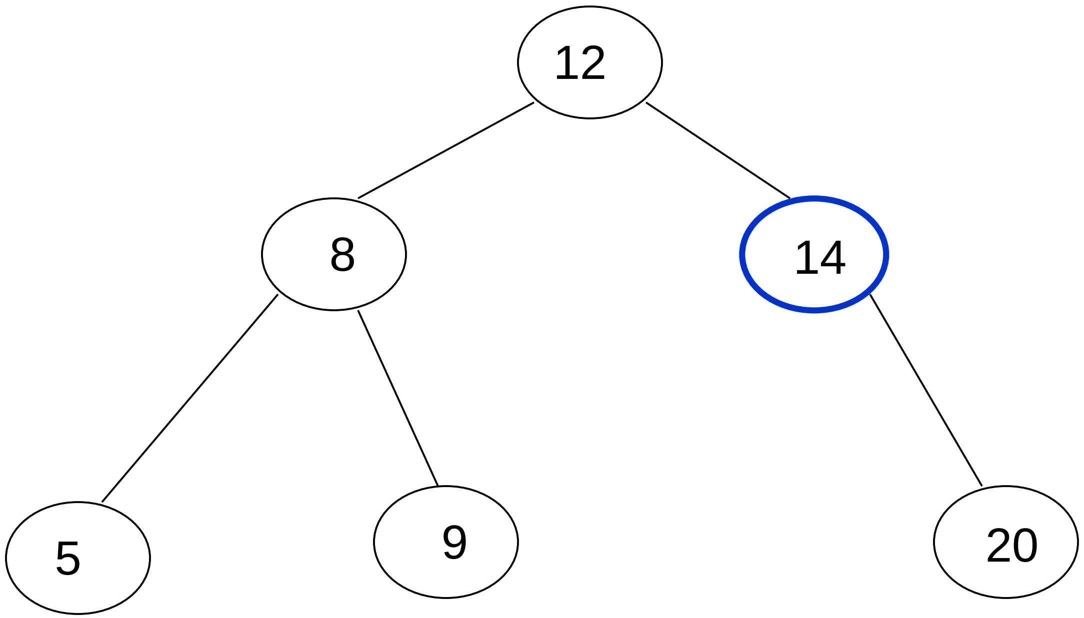
    return BSTSEARCH(RIGHT(x), k)

Finding an element

$$left(i) < i \leq right(i)$$

Search(T, 13)

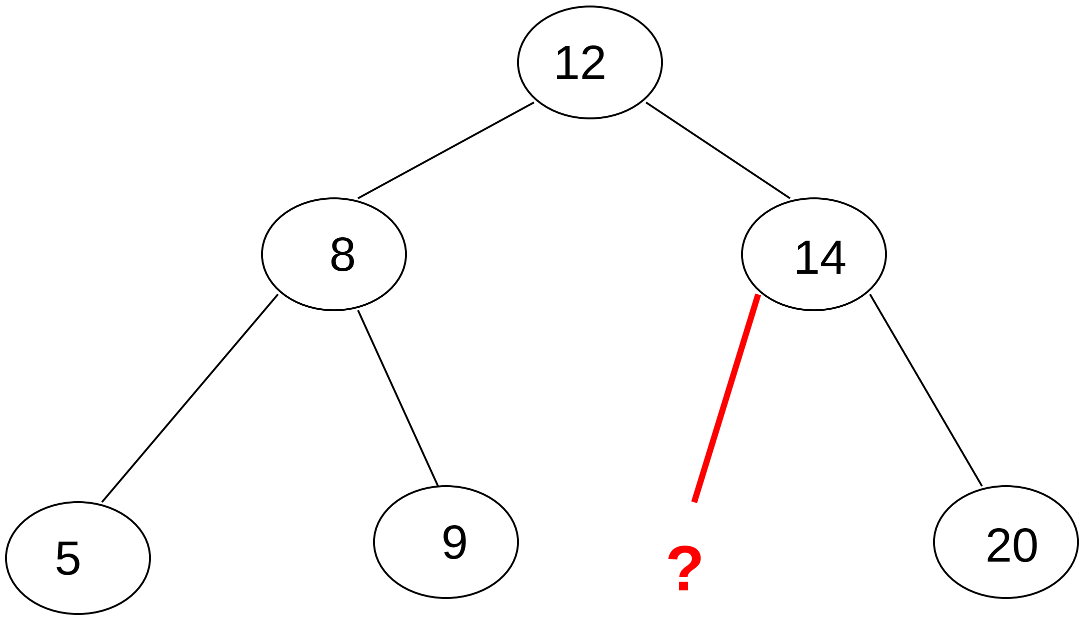



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

Finding an element

$$left(i) < i \le right(i)$$

Search(T, 13)




```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

Iterative search

```
IterativeBSTSearch(x, k)
    while x ≠ null and k ≠ x
        if k < x
            x ← Left(x)
        else
            x ← Right(x)
    return x
```

```
BSTSearch(x, k)
    if x = null or k = x
        return x
    elseif k < x
        return BSTSearch(Left(x), k)
    else
        return BSTSearch(Right(x), k)
```

Is BSTSearch correct?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

$$left(i) < i \leq right(i)$$

Running time of BST

Worst case?

- O(height of the tree)

Average case?

- O(height of the tree)

Best case?

- O(1)

Height of the tree

Worst case height?

- n-1
- "the twig"

Best case height?

- floor($\log_2 n$)
- complete (or near complete) binary tree

Average case height?

- Depends on two things:
  - the data
  - how we build the tree!

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Similar to search

```
IterativeBSTSearch(x, k)
while x ≠ null and k ≠ x
        if k < x
                x ← Left(x)
        else
                x ← Right(x)
return x
```

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Similar to search

Find the correct location in the tree

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

keeps track of the previous node we visited so when we fall off the tree, we know

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

add node onto the bottom of the tree

Correctness?

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

maintain BST property

Correctness

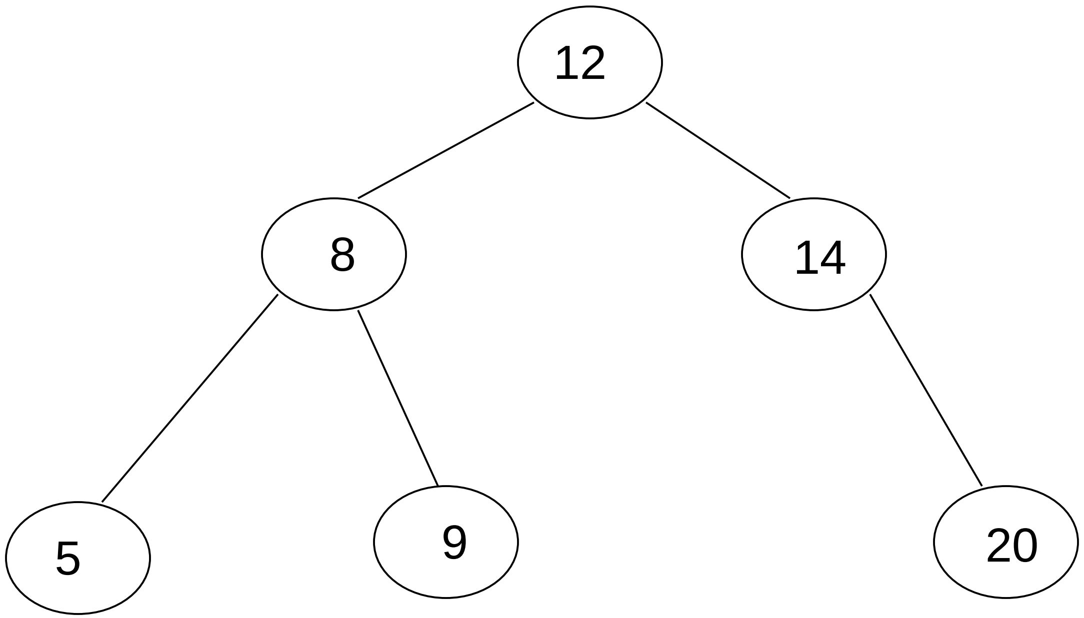
```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

What happens if it is a duplicate?

Inserting duplicate

$$left(i) < i \leq right(i)$$

Insert(T, 14)



Running time

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

O(height of the tree)

Running time

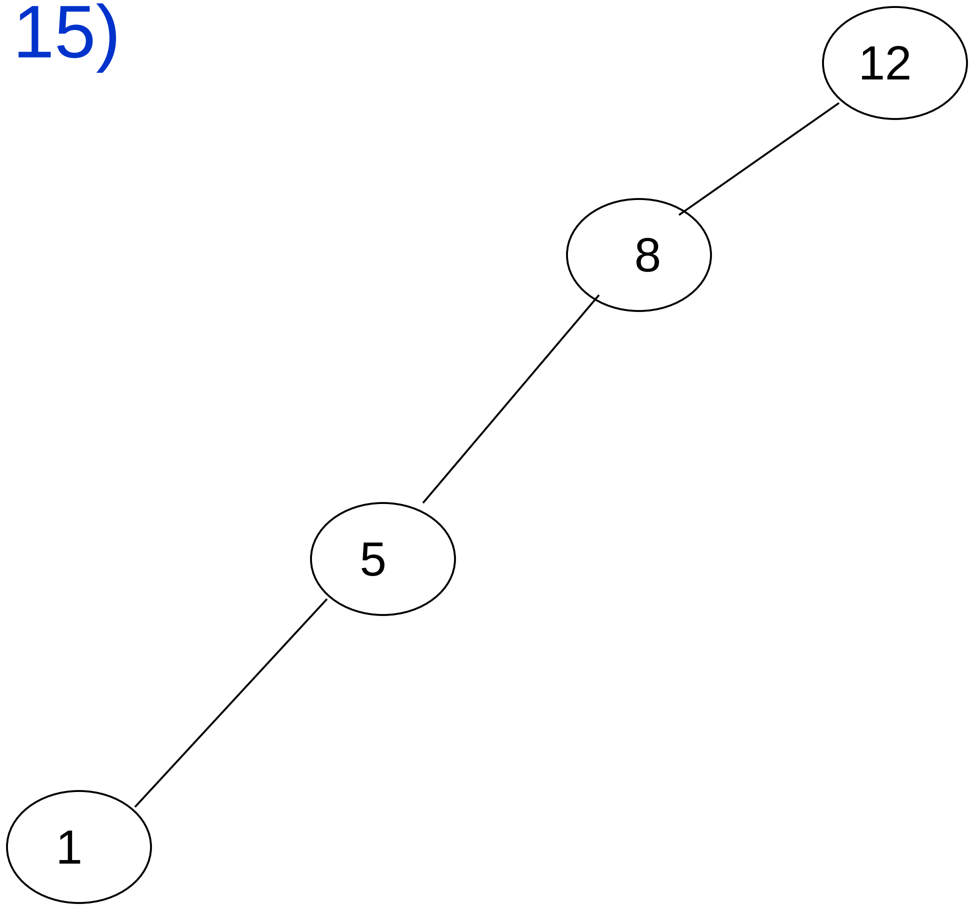
```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

O(height of the tree)

Why not
Θ(height of the tree)?

Running time

Insert(T, 15)



Height of the tree

Worst case: "the twig" – When will this happen?

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Height of the tree

Best case: "complete" – When will this happen?

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Height of the tree

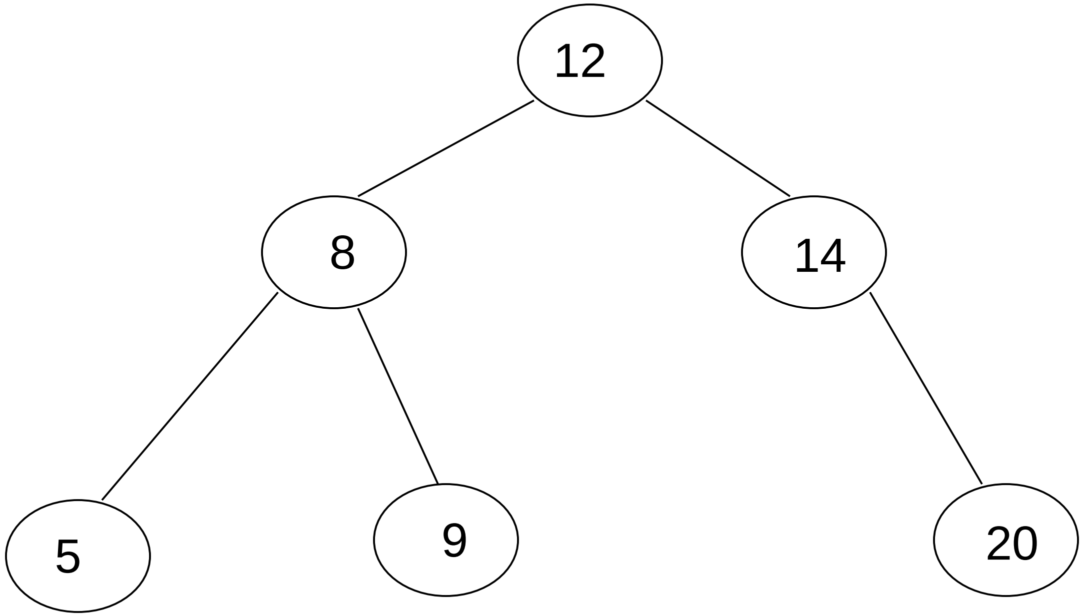
Average case for random data?

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Randomly inserted data into a BST generates a tree on average that is O(log n)

Visiting all nodes

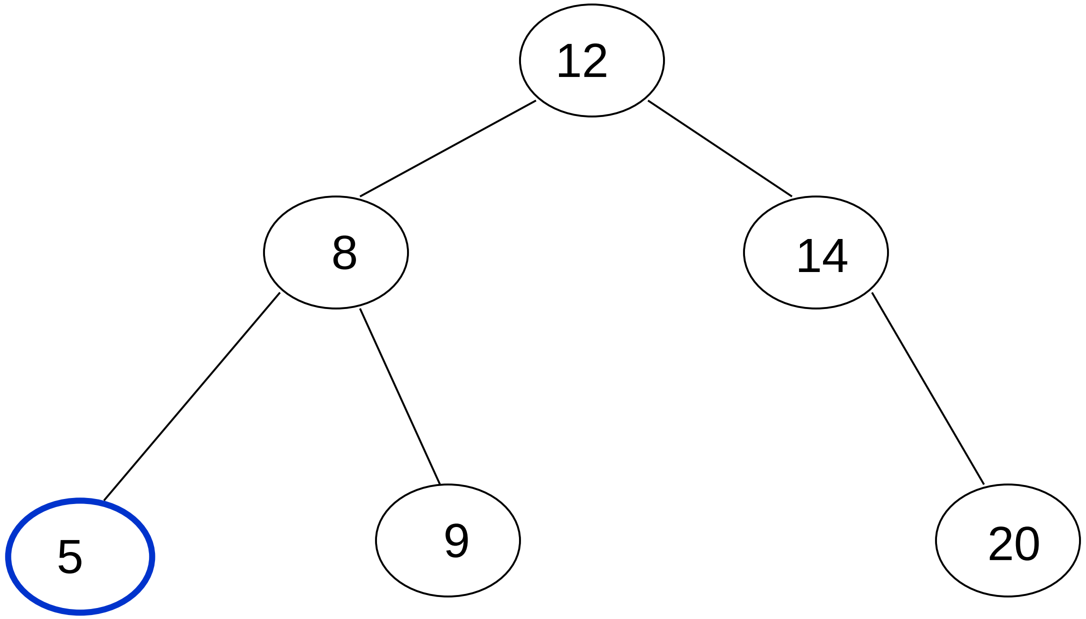
In sorted order



Visiting all nodes

In sorted order

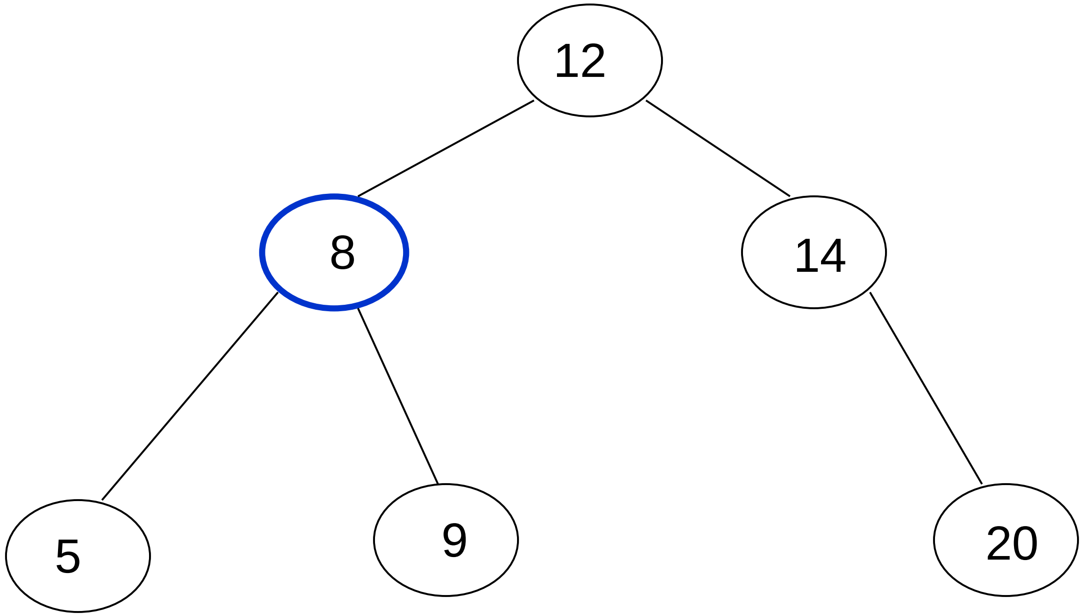
5



Visiting all nodes

In sorted order

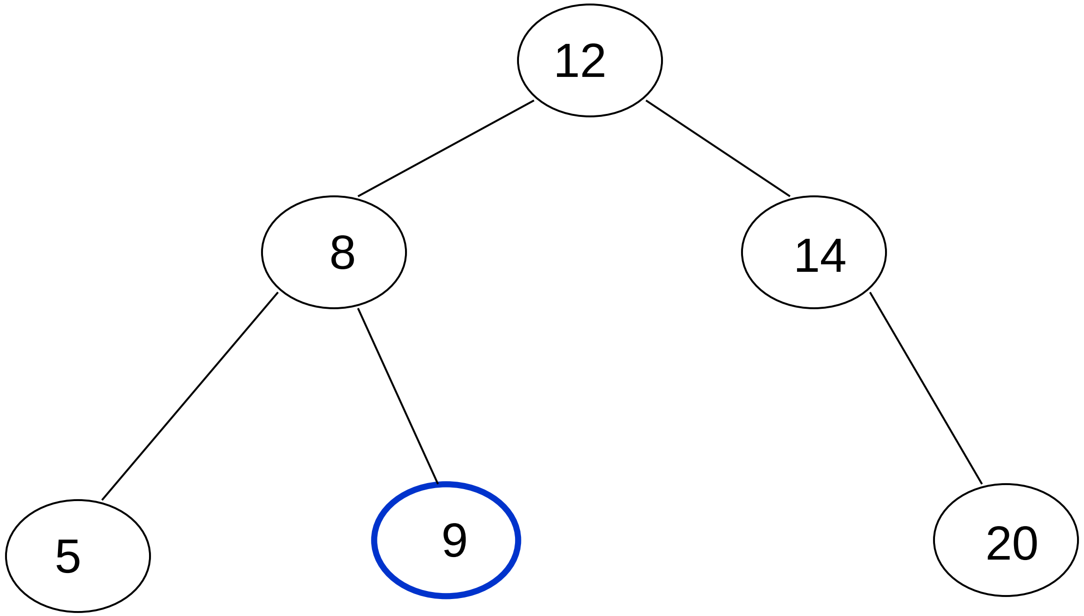
5, 8



Visiting all nodes

In sorted order

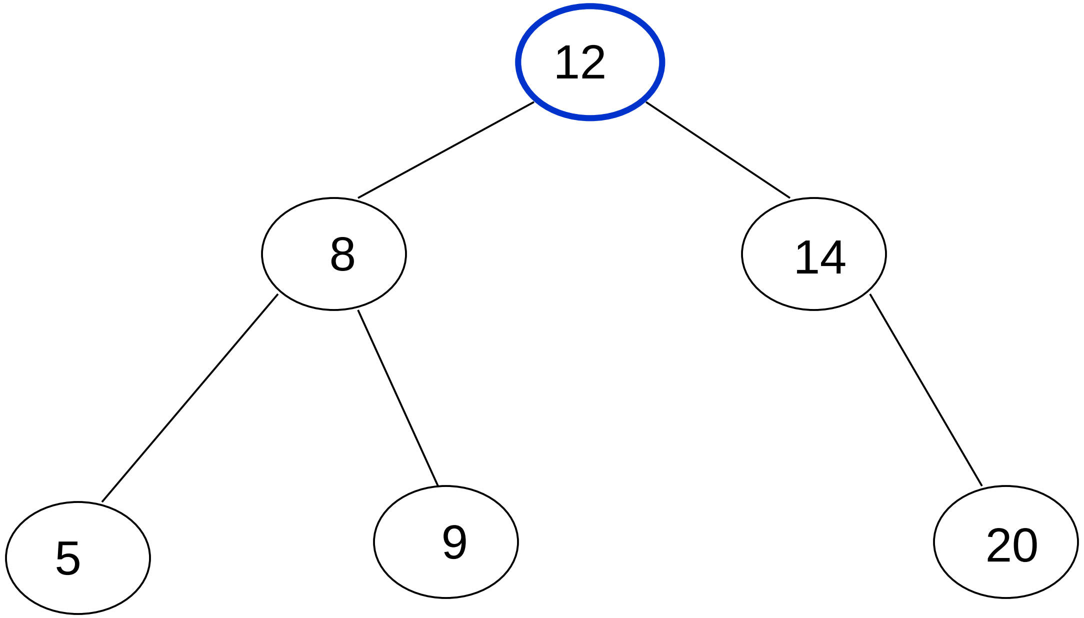
5, 8, 9



Visiting all nodes

In sorted order

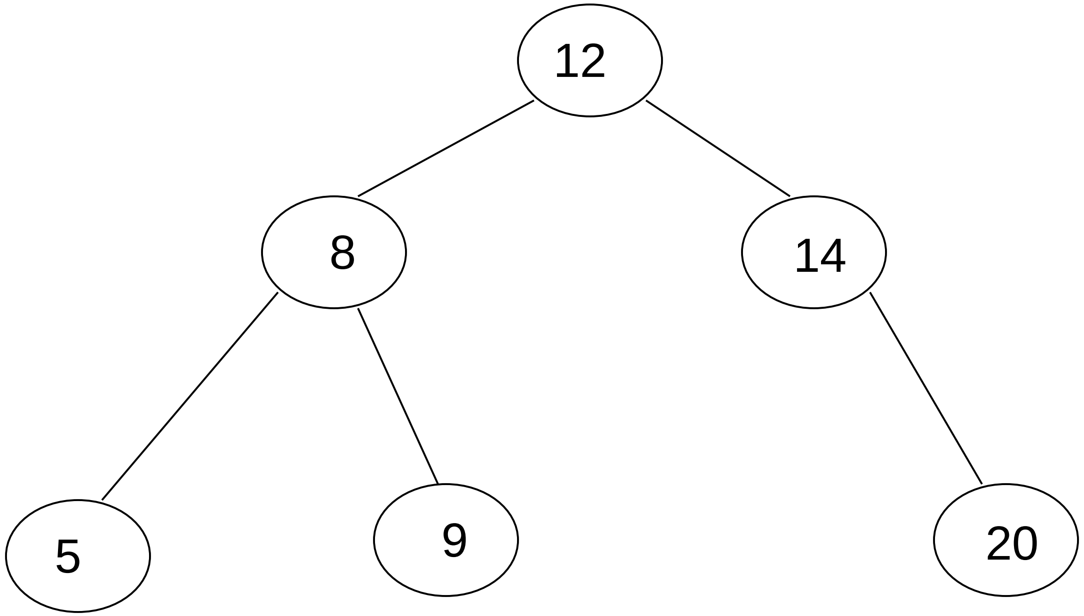
5, 8, 9, 12



Visiting all nodes

What's happening?

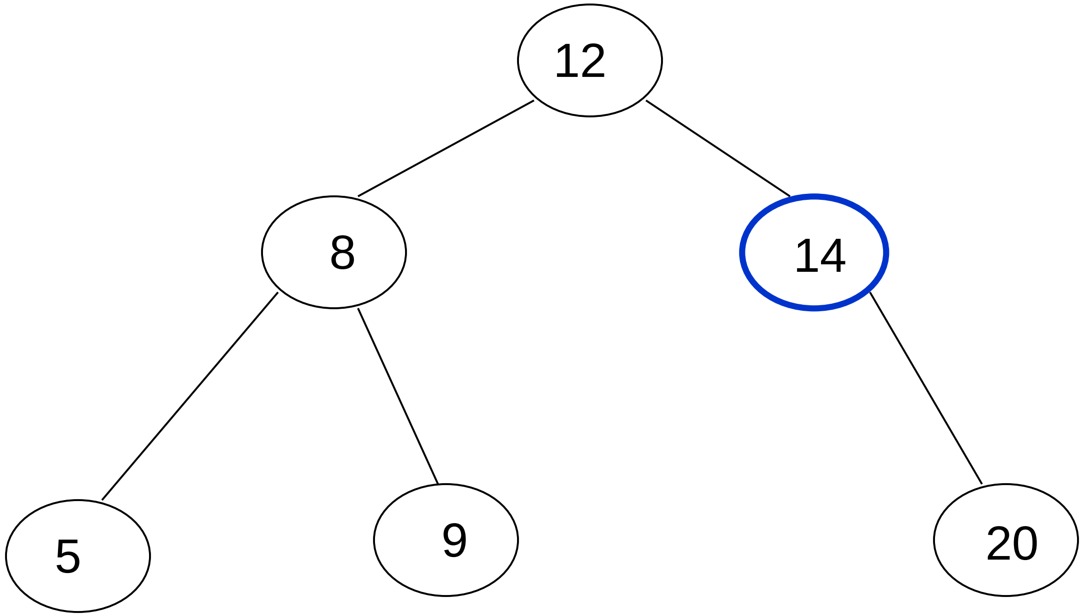
5, 8, 9, 12



Visiting all nodes

In sorted order

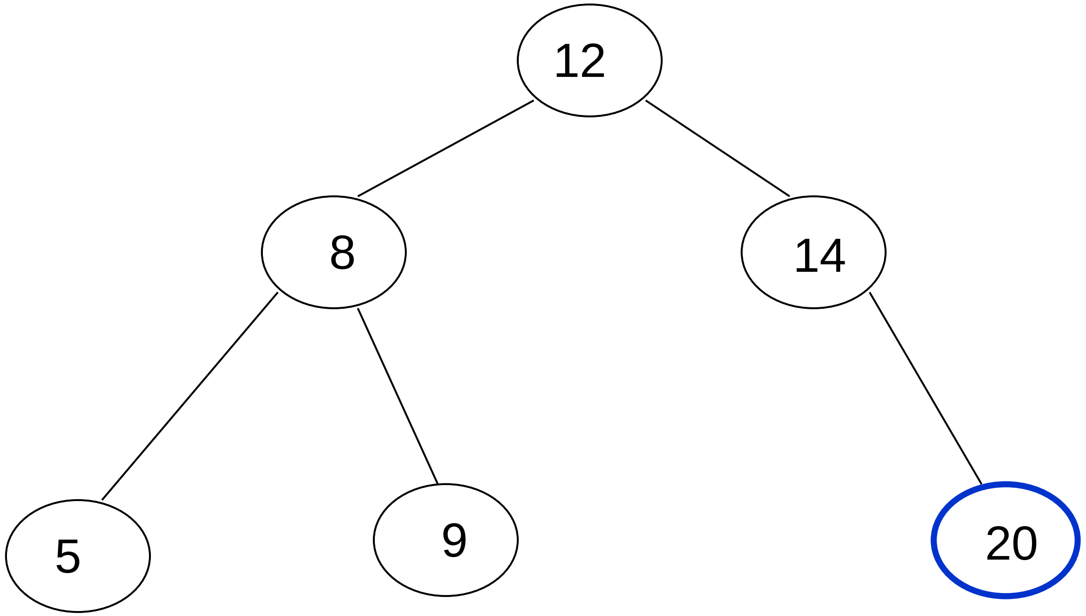
5, 8, 9, 12, 14



Visiting all nodes

In sorted order

5, 8, 9, 12, 14, 20



Visiting all nodes in order

```
INORDERTREEWALK(x)
if x ≠ null
        INORDERTREEWALK(LEFT(x))
        print x
        INORDERTREEWALK(RIGHT(x))
```

Visiting all nodes in order

```
InorderTreeWalk(x)
if x ≠ null
        InorderTreeWalk(Left(x))
        print x
        InorderTreeWalk(Right(x))
```

any operation

Is it correct?

```
InorderTreeWalk(x)
if x ≠ null
        InorderTreeWalk(Left(x))
        print x
        InorderTreeWalk(Right(x))
```

Does it print out all of the nodes in sorted order?

$$left(i) < i \leq right(i)$$

Running time?

```
INORDERTREEWALK(x)
if x ≠ null
        INORDERTREEWALK(LEFT(x))
        print x
        INORDERTREEWALK(RIGHT(x))
```

Recurrence relation:

- *j* nodes in the left subtree
- $n - j - 1$ in the right subtree

$$T(n) = T(j) + T(n - j - 1) + \Theta(1)$$

Or

- How much work is done for each call?
- How many calls?
- $\Theta(n)$

What about?

```
TreeWalk(x)
if x ≠ null
        print x
        TreeWalk(Left(x))
        TreeWalk(Right(x))
```

Preorder traversal

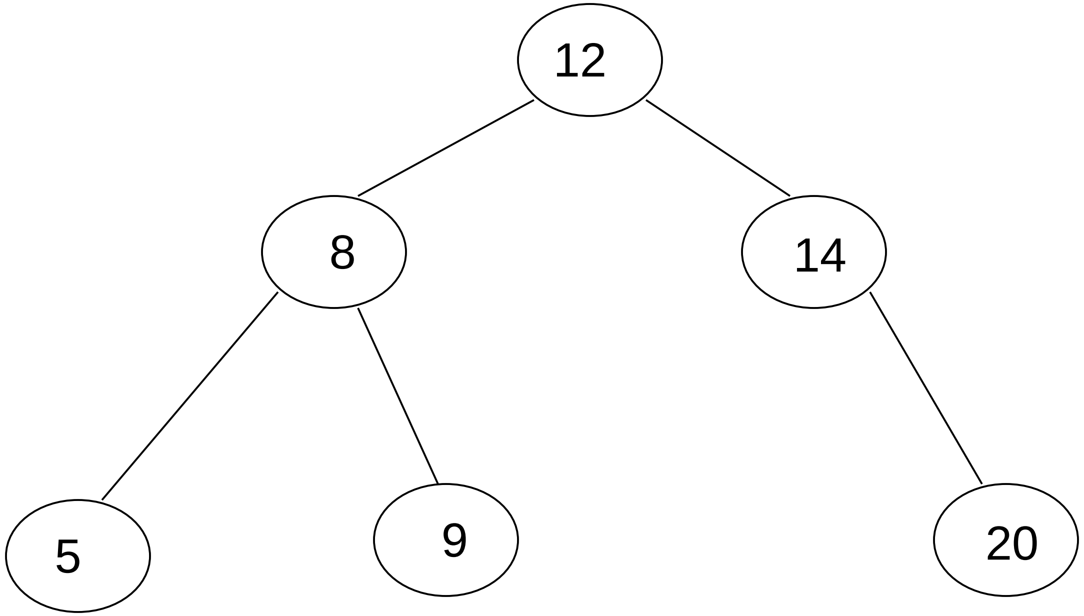
```
TreeWalk(x)
if x ≠ null
    print x
    TreeWalk(Left(x))
    TreeWalk(Right(x))
```

12, 8, 5, 9, 14, 20

How is this useful?

Tree copying: insert in to new tree in preorder

prefix notation: (2+3)*4 -> * + 2 3 4



What about?

TreeWalk(x)

if $x \neq null$

  TreeWalk(Left(x))

  TreeWalk(Right(x))

  print x

Postorder traversal

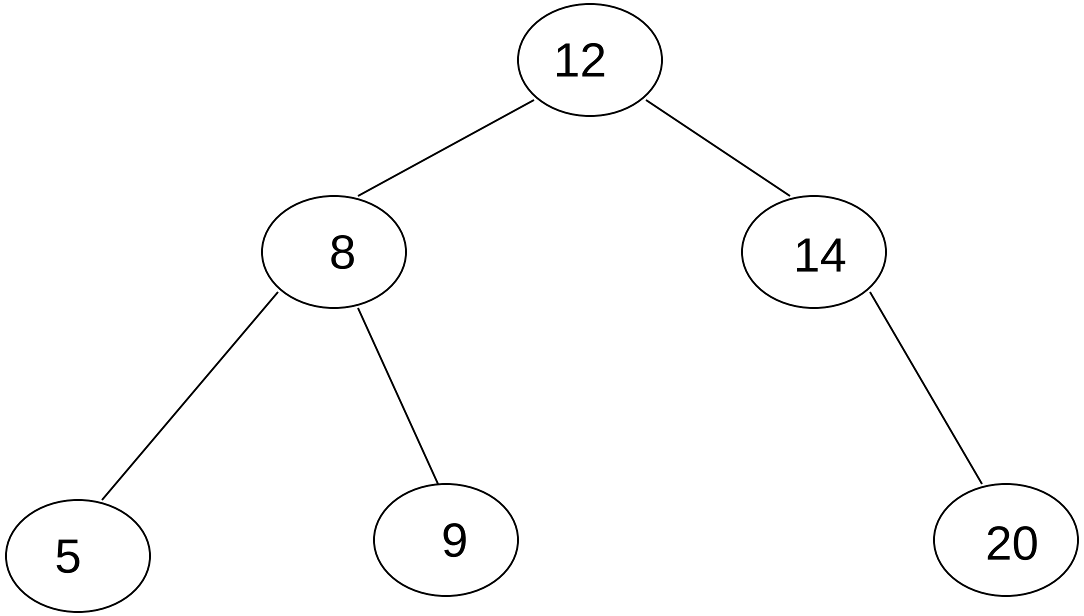
```
TREEWALK(x)
if x ≠ null
    TREEWALK(LEFT(x))
    TREEWALK(RIGHT(x))
    print x
```

5, 9, 8, 20, 14, 12

How is this useful?

postfix notation: (2+3)*4
-> 4 3 2 + *

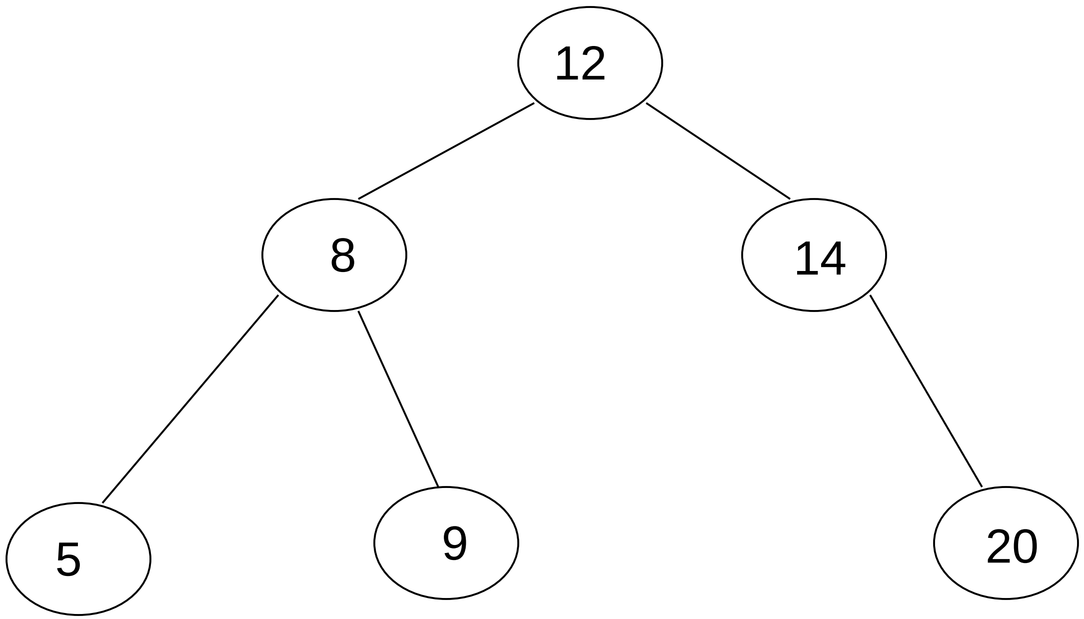
?



Min/Max

BSTMin(x)
```
if Left(x) = null
        return x
else
        return BSTMin(Left(x))
```

IterativeBSTMin(x)
```
while Left(x) ≠ null
        x ← Left(x)
return x
```



Running time of min/max?

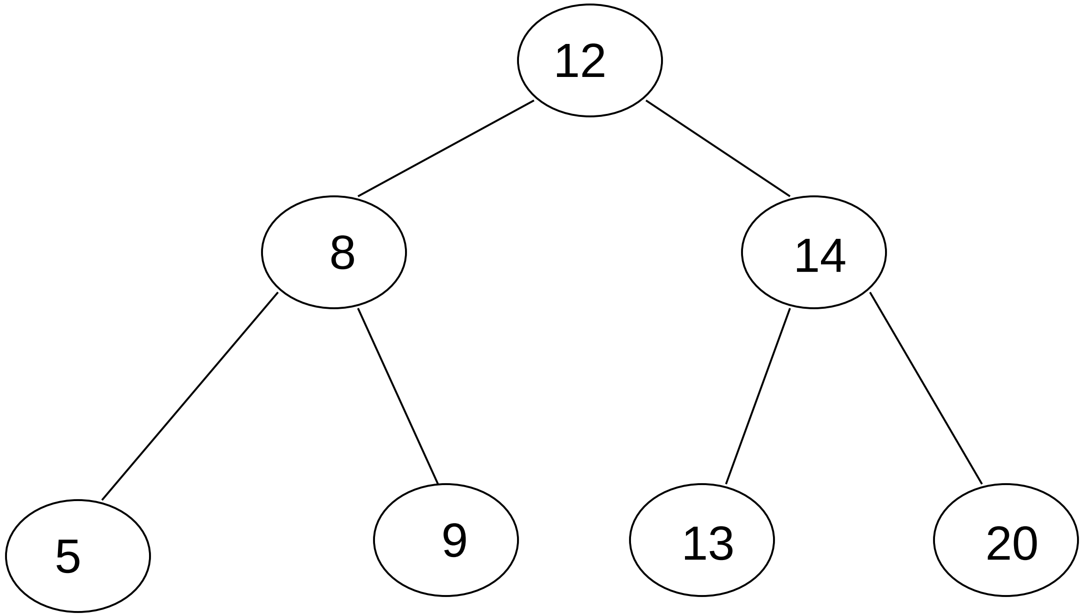
```
BSTMin(x)
if Left(x) = null
        return x
else
        return BSTMin(Left(x))
```

```
IterativeBSTMin(x)
while Left(x) ≠ null
        x ← Left(x)
return x
```

O(height of the tree)

Successor and predecessor

Predecessor(12)? 9

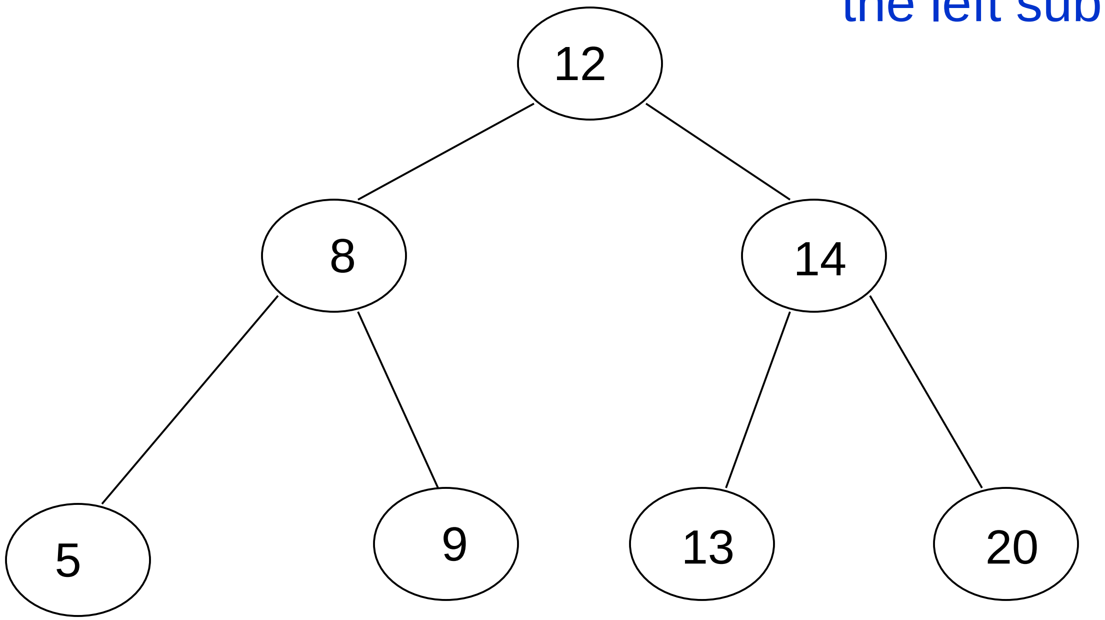


Successor and predecessor

Predecessor in general?

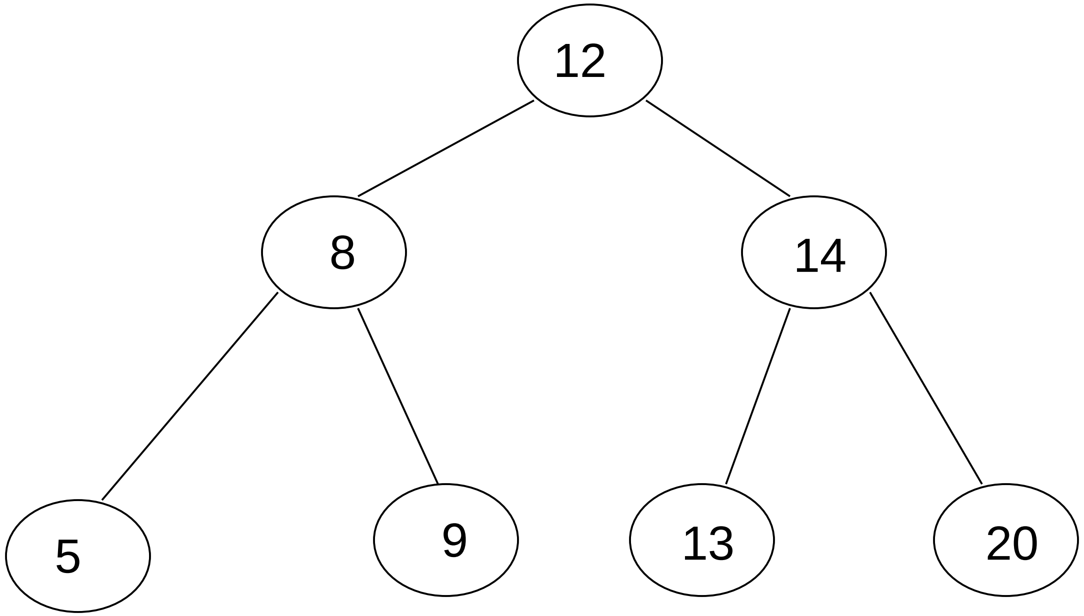
largest node of all those
smaller than this node

rightmost element of
the left subtree



Successor

Successor(12)? 13

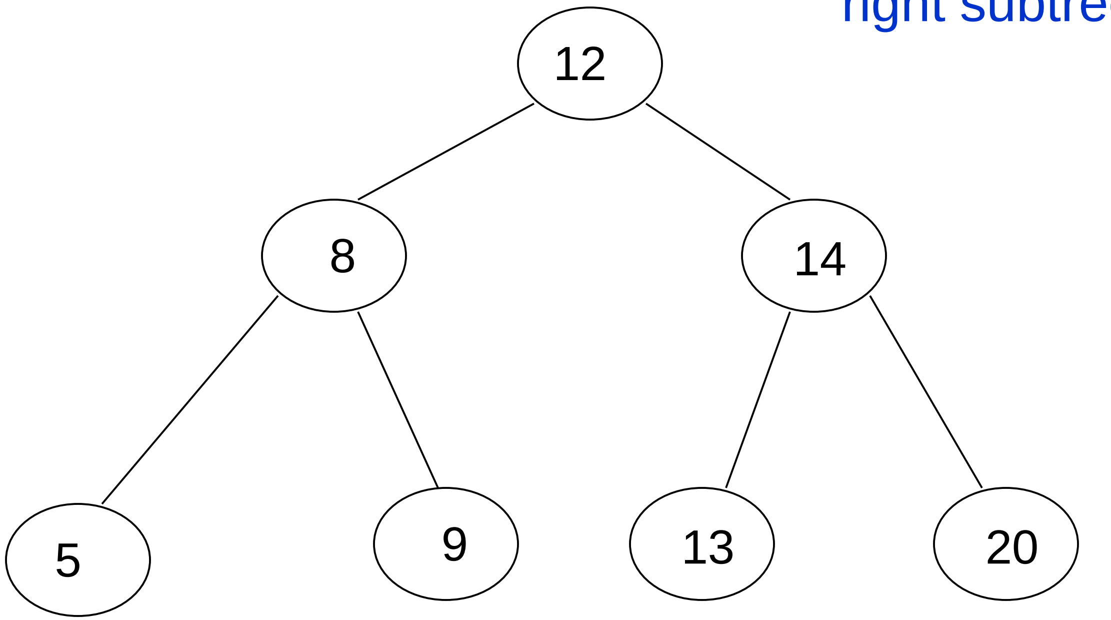


Successor

Successor in general?

smallest node of all those larger than this node

leftmost element of the right subtree

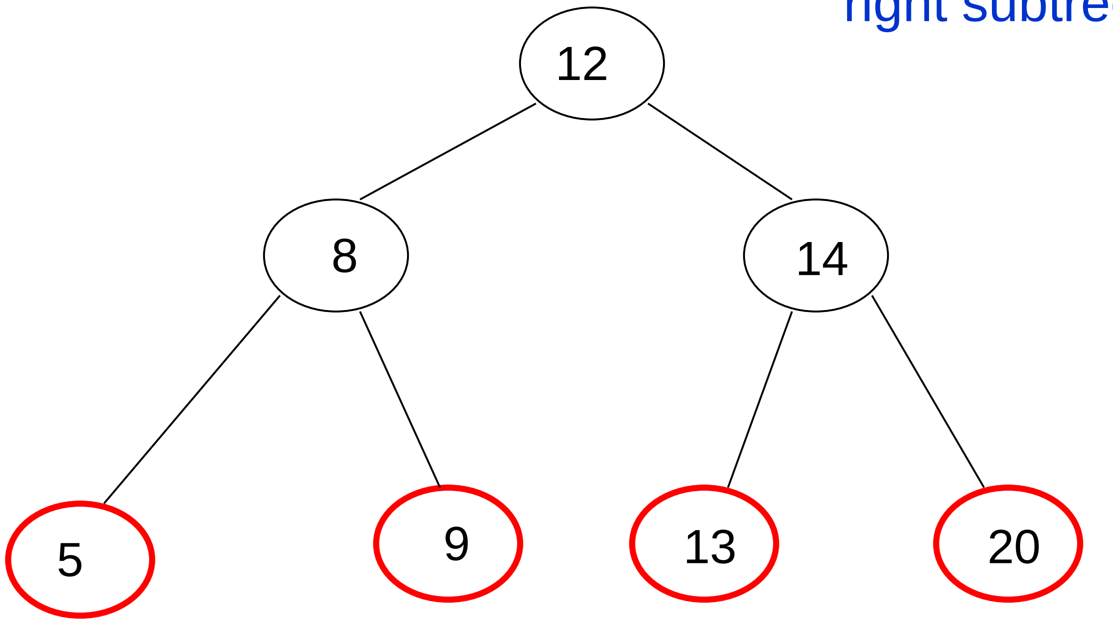


Successor

What if the node doesn't have a right subtree?

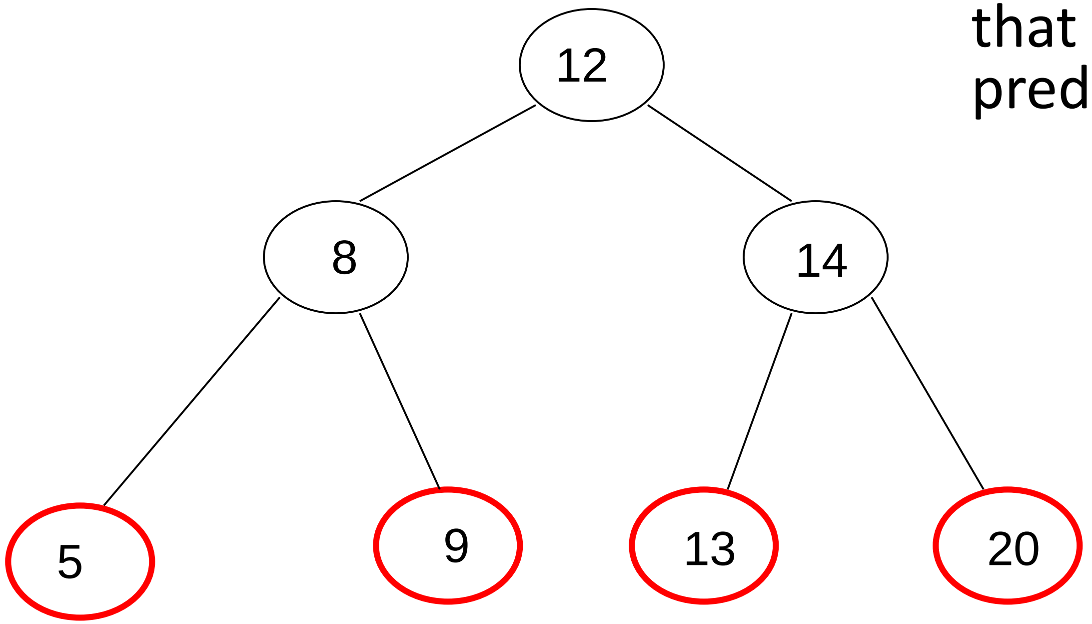
smallest node of all those larger than this node

leftmost element of the right subtree



Successor

What if the node doesn't have a right subtree?

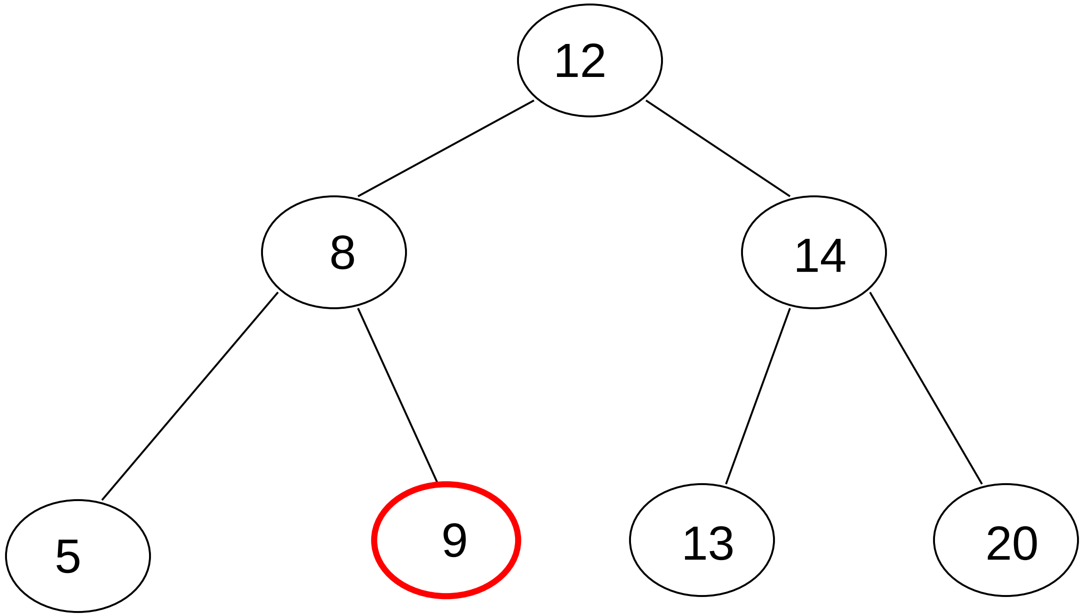


node is the largest

the successor is the node that has x as a predecessor

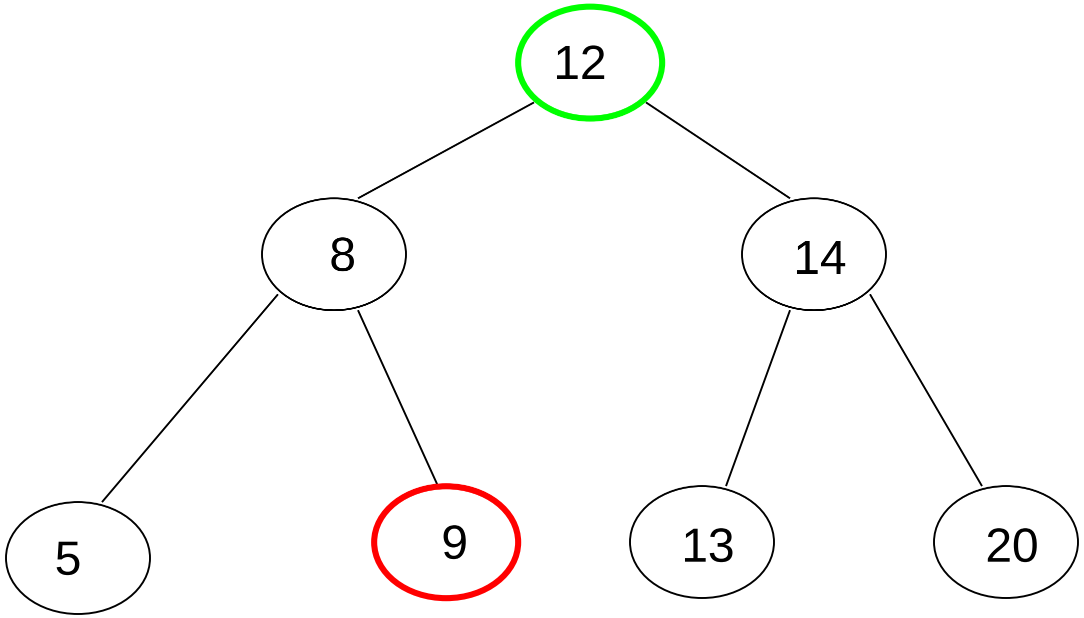
Successor

successor is the node
that has x as a
predecessor

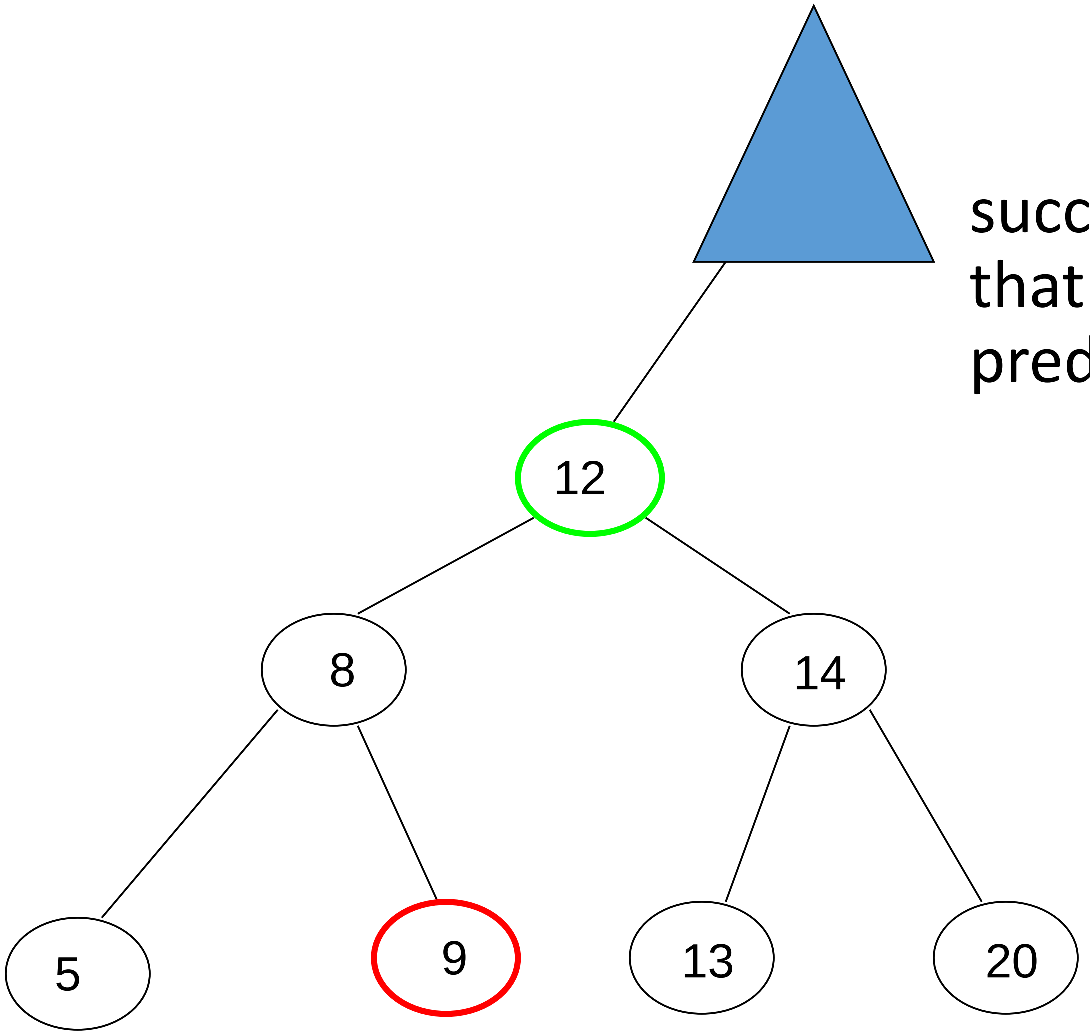


Successor

successor is the node that has x as a predecessor



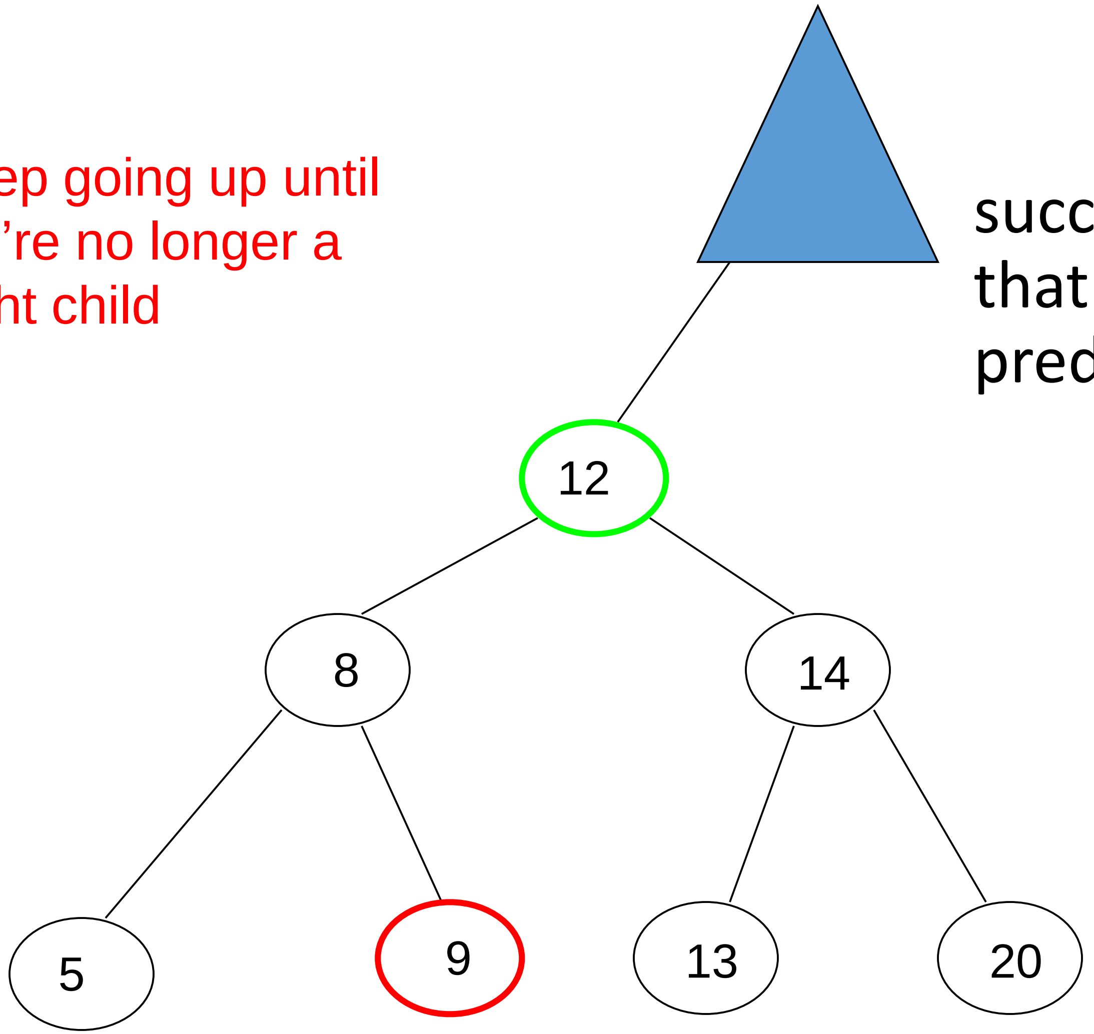
Successor



successor is the node that has x as a predecessor

Successor

keep going up until we're no longer a right child



successor is the node that has x as a predecessor

Successor

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

Successor

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

if we have a right subtree, return the smallest of the right subtree

Successor

SUCCESSOR(x)

if RIGHT(x) $\neq$ *null*

        return BSTMIN(RIGHT(x))

else

        $y \leftarrow$ PARENT(x)

        while $y \neq$ *null* **and** $x =$ RIGHT(y)

                $x \leftarrow y$

                $y \leftarrow$ PARENT(y)

return y

find the node that x is the predecessor of

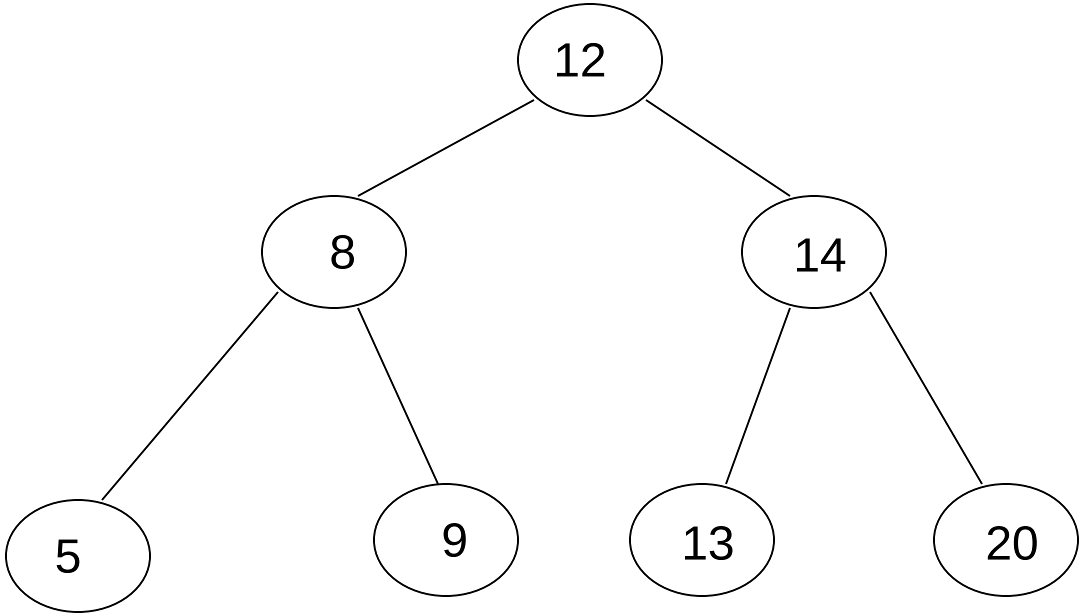
keep going up until we're no longer a right child

Successor running time

O(height of the tree)

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

Deletion

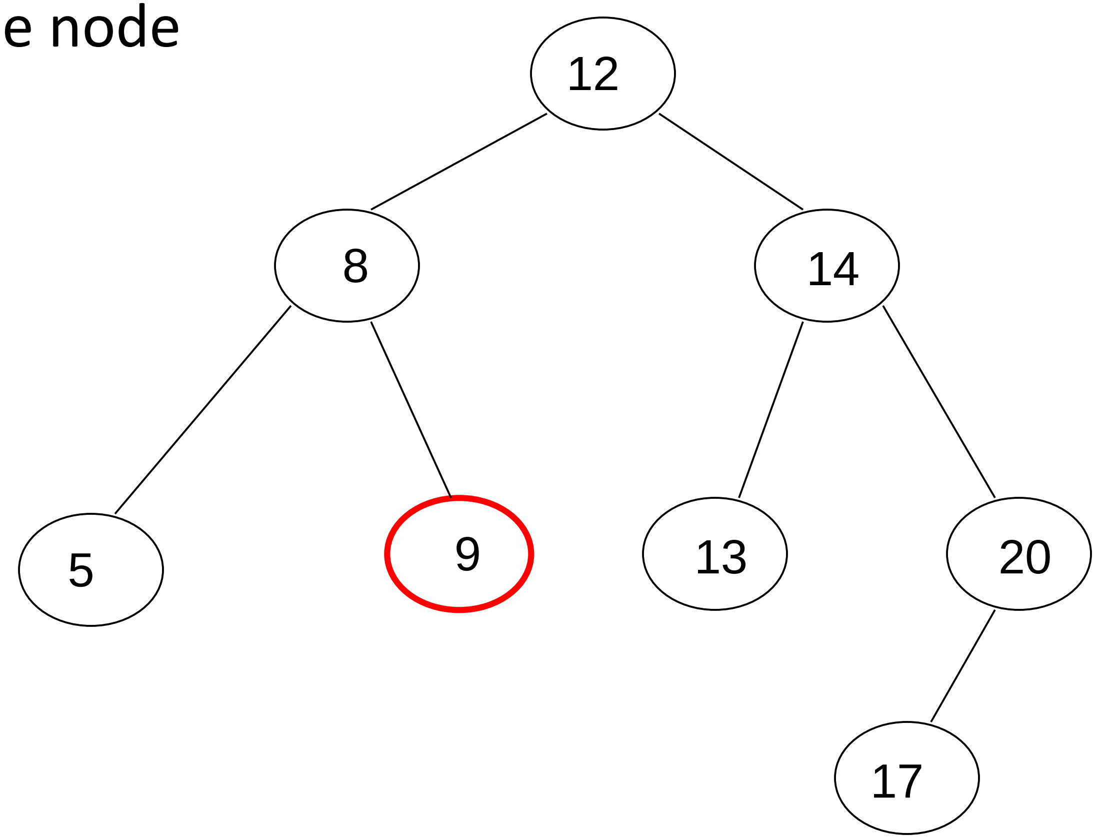


Three cases!

Deletion: case 1

No children

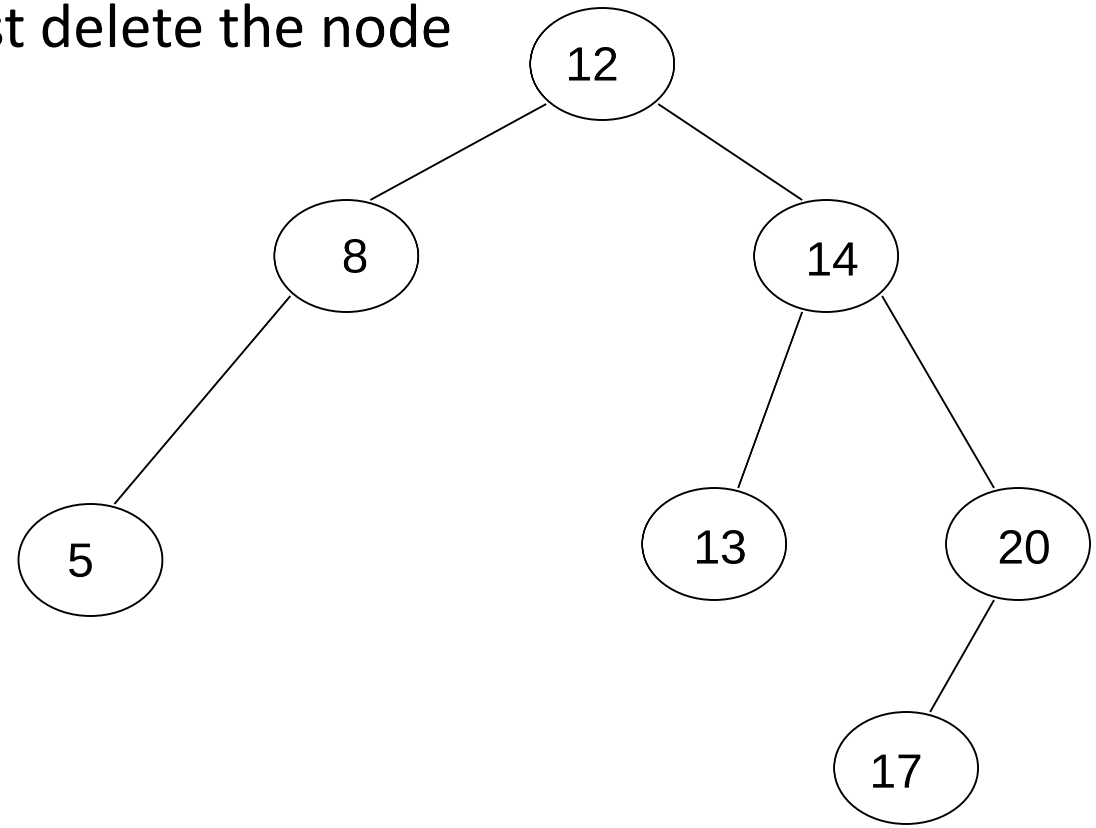
Just delete the node



Deletion: case 1

No children

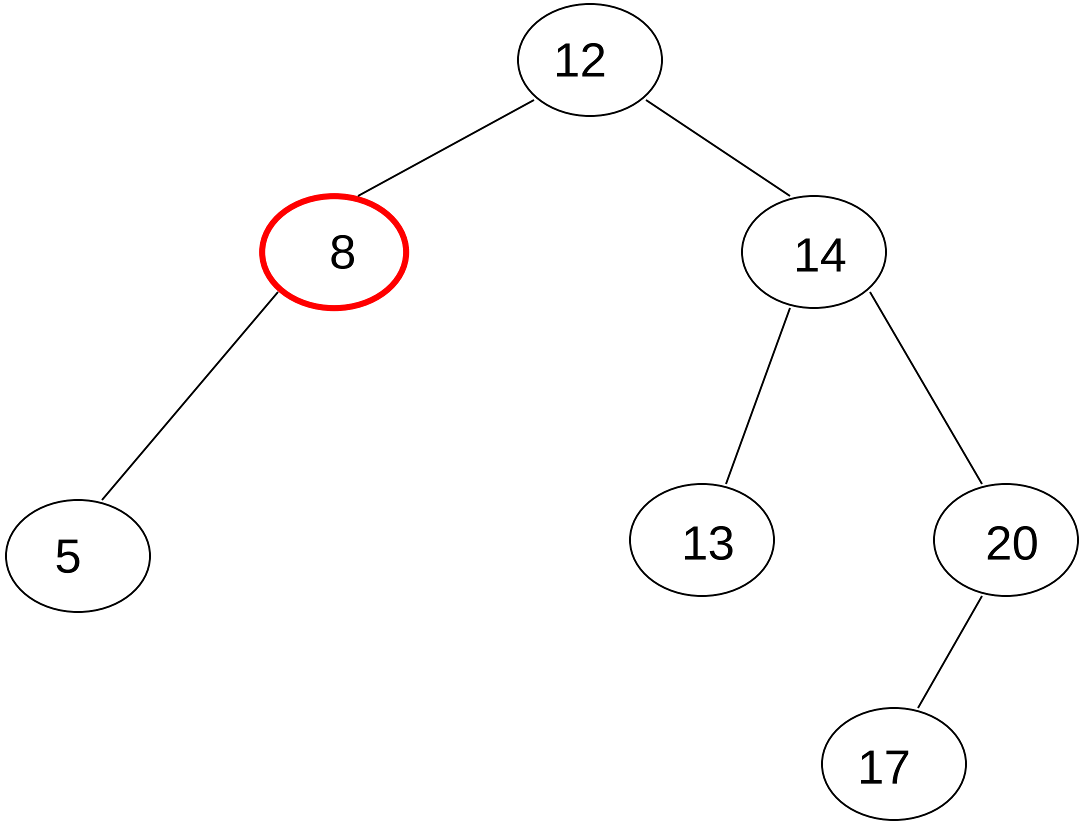
Just delete the node



Deletion: case 2

One child

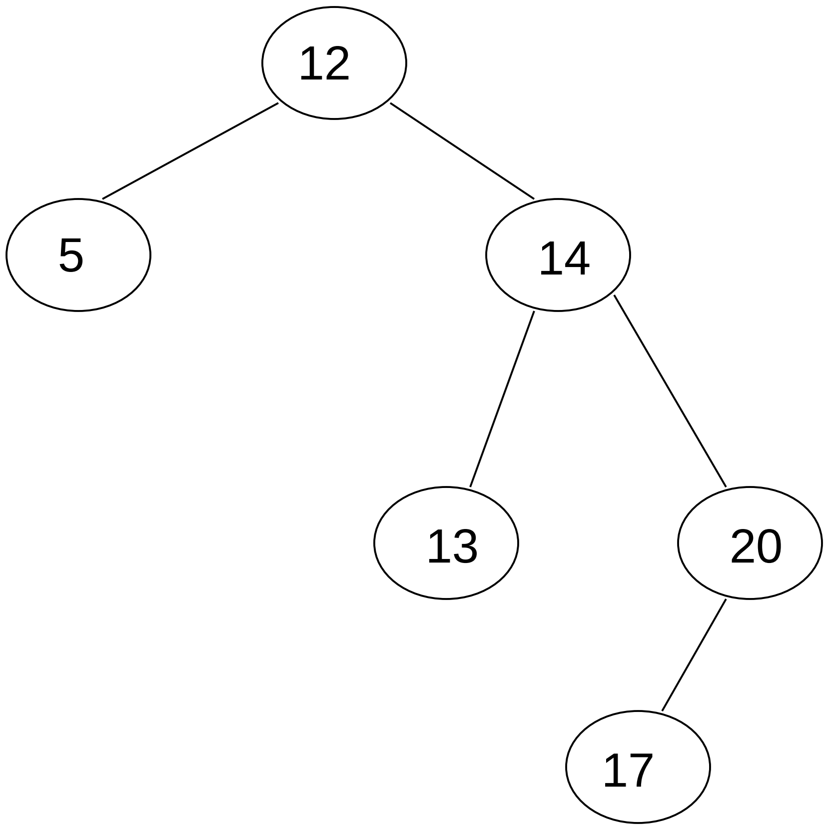
Splice out the node



Deletion: case 2

One child

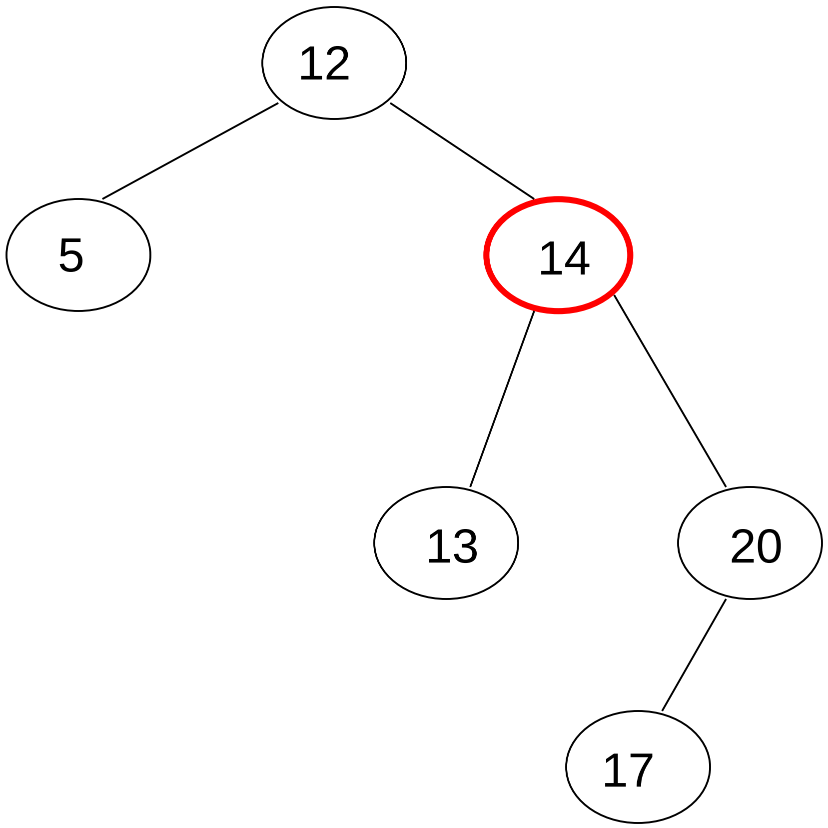
Splice out the node



Deletion: case 3

Two children

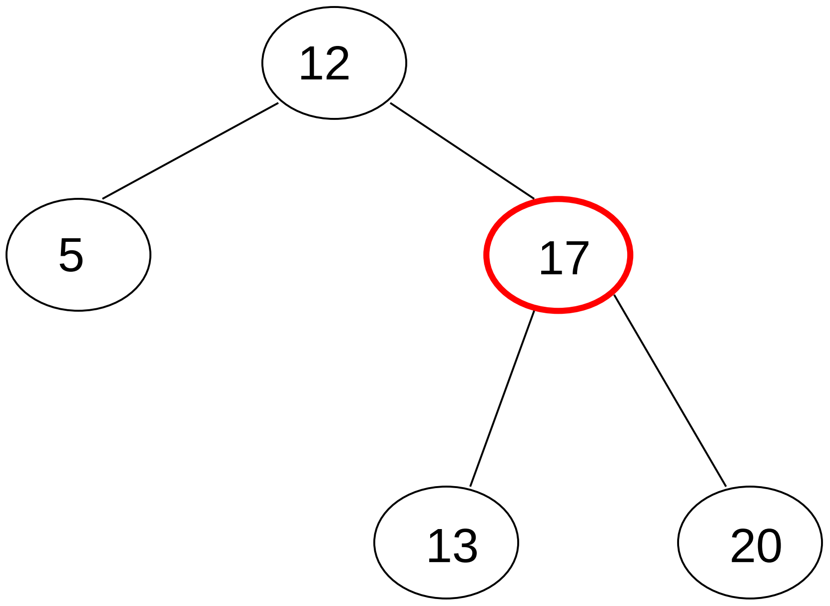
Replace x with it's successor



Deletion: case 3

Two children

Replace x with it’s successor



Deletion: case 3

Two children

Will we always have a successor?

Why successor?

- Case 1 or case 2 deletion
- Larger than the left subtree
- Less than or equal to right subtree

Height of the tree

Most of the operations take time
O(height of the tree)

We said trees built from random data have height O(log n), which is asymptotically tight

Two problems:

- We can't always insure random data
- What happens when we delete nodes and insert others after building a tree?

Balanced trees

Make sure that the trees remain balanced!

- Red-black trees
- AVL trees
- 2-3-4 trees
- …

B-trees