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Q1. To prove $u^*(\bigcup_j A_j) \leq \sum_j u^*(A_j)$

Suppose $\forall \varepsilon > 0$, for every A_n,

we have. countable B-~~field~~ fielef. $B(n,m)\ m \in N$,

$$\sum_{m \in N} (B_{n,m}) < u^*(A_n) + \frac{\varepsilon}{2^n}$$

then. $(B_{n,m})_{(n,m) \in N^2}$ is $\bigcup_{m \in N} A_m$ countable B-field

We can conclude that

$$u^*(\bigcup_{j \in N} A_j) \leq \sum_{(n,m) \in N^2} u(B_{n,m}) \leq \sum_{m \in N} u^*(A_m) + \frac{\varepsilon}{2^n}, \quad \frac{\varepsilon}{2^n} \to 0.$$

Q2.

Proof:

Use $X_n \to X$, $Y_n \to Y$ in p.r. to prove $X_n + Y_n \to X + Y$ in p.r.

① $X_n \to X$ in p.r.

We have that

$\forall \varepsilon > 0$, we have

$\lim_{n\to\infty} P[|X_n - X| < \varepsilon] = 1$. $\langle a \rangle$

Similar to $Y_n - Y$ in p.r.

$\forall \epsilon > 0$, we have

$\lim_{n\to\infty} P[|X_n - X| < \varepsilon] = 1$. $\langle b \rangle$

② $|X_n - X| + |Y_n - Y| \geq |X_n - X + Y_n - Y|$ $\langle c \rangle$

according to $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\Rightarrow |X_n + Y_n - (X+Y)| \leq |X_n + X| + |Y_n - Y| \leq 2\varepsilon$

we have $\forall \delta_0 > 0$. $\exists\ \delta = 2\varepsilon > 0$.

$\lim_{n\to\infty} P[\ |X_n - X| + |Y_n - Y| \leq |X_n + X| + |Y_n - Y| \leq 2\varepsilon \leq \delta] = 1$.

thus, we can conclude that $X_n + Y_n \to X + Y$ in p.r.

Q3. ~~Sn~~ ①

For, r.v. $\{X_n\}$. we have $E[X_n^2] \to 0$. X_n converge in L^2.

$S_n = \sum_{i=1}^{n} X_i$

$$E[S_n^2] = \underbrace{\sum_{i=1}^{n} E[X_i^2]}_{I_1} + \underbrace{\sum_{1\le i<j\le n} 2E[X_i \cdot X_j]}_{I_2}$$

according to $ab \le \frac{a^2+b^2}{2}$

$$\Rightarrow I_2 \le \sum_{1\le i<j\le n} \left(E[X_i^2] + E[X_j^2]\right)$$

Use the condition of X_n converge in L^2.

We can conclude $E[S_n^2] \to 0$ converge in L^2

thus $E\left[\frac{S_n^2}{n^2}\right] \to 0$.

$$E\left[\left(\frac{S_n - E[S_n]}{n}\right)^2\right] = \frac{1}{n^2} E[(S_n - E[S_n])^2] = \frac{1}{n^2} E[S_n^2 - 2S_n E[S_n] + E^2[S_n]]$$

$$= \frac{1}{n^2}\left(E[S_n^2] - E^2[S_n]\right)$$

$$\le \frac{E[S_n^2]}{n^2} + E^2[S_n] \longrightarrow 0.$$

according to Chebyshev's inequality

$$\lim_{n\to+\infty} E\left[\left(\frac{S_n - E[S_n]}{n}\right)^2\right] = 0 \Rightarrow \lim_{n\to+\infty} E\left[\frac{S_n - E[S_n]}{n}\right] = 0.$$

Q4. Use Charasteristic function

$f(t) = E[e^{itX}]$.

$f_{aX+b}(t) = f_X(at)e^{itb} = E[e^{i(at)X}] \cdot e^{itb}$.

thus,

$f_{a_nX_n+b_n}(t) = E[e^{it(a_nt)X_n}]e^{itb_n}$

$\because b_n \to b$.

$\therefore f_{a_nX_n+b_n}(t) \to E[e^{i(a_nt)X_n}] \cdot e^{itb}$

$\because X_n \to X \; . \; a_n \to a$

$\therefore f_{a_nX_n+b_n}(t) \Rightarrow E[e^{i(a_nt)\cdot X}]e^{itb} \Rightarrow E[e^{i(at)X}]e^{itb}$

$f_n(t) \Rightarrow E[e^{iatX}]e^{itb} = f(t)$

conclude, $a_nX_n + b_n \to aX+b$.

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Q5.

For Poisson Distribution,

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$$

$$P[N_s=k \mid N_t=n]=\frac{P[N_s=k \text{ and } N_t=n]}{P[N_t=n]}$$

when $n>k$

$$P[N_t=n]=\frac{\lambda^n}{n!}e^{-\lambda}=\frac{t^n}{n!}e^{-t}$$

$$P[N_s=k \text{ and } N_t=n]=P[N_s=k]\times P[N_{t-s}=n-k]$$

~~$P[N_s=k$ and $N_t=n]=C_n^k\cdot\frac{s^k}{k!}e^{-s}\cdot\frac{t^{n-k}}{(n-k)!}e^{t-s}$~~

$$P[N_s=k \text{ and } N_t=n]=\frac{s^k}{k!}e^{-s}\cdot\frac{(t-s)^{n-k}}{n-k!}\cdot e^{-(t-s)}$$

$$\text{Thus } P[N_s=k \mid N_t=n]=\frac{\frac{s^k(t-s)^{n-k}}{k!(n-k)!}\cdot e^{-t}}{\frac{t^n}{n!}e^{-t}}=\frac{n!}{(k!)(n-k)!}\left(\frac{s}{t}\right)^k\left(\frac{t-s}{t}\right)^{n-k}$$

$$=\frac{n!}{k!(n-k)!}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

thus. ① when $0<s<t$, $k>n$.

$$P[N_s=k \mid N_t]=\frac{N_t!}{k!(N_t-k)!}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{N_t-k}$$

② and when $N_t<k$, it is impossible, $P[N_s=k \mid N_t]=0$