Homework 1—Advanced Probability

Due Monday, March 2

1.1 (10 points) Let $\{a_n, n \geq 1\}$ be any given enumeration of the set of all rational numbers and let $\{b_n, n \geq 1\}$ be a set of positive numbers such that $\sum_{n=1}^{+\infty} b_n < +\infty$. Let $f(x) = \sum_{n=1}^{+\infty} b_n \cdot \delta_{a_n}(x)$. Show that $f(-\infty) = 0$ and $f(+\infty) = \sum_{n=1}^{+\infty} b_n$.

1.2 (10 points) Prove that the trace of a B.F. $\mathscr{F}$ on any subset Δ of Ω is a B.F. Prove that the trace of $(\Omega, \mathscr{F}, \mathbb{P})$ on any Δ in $\mathscr{F}$ is a probability space, if $\mathbb{P}[\Delta] > 0$.

1.3 (10 points) Let $\mathscr{N}_0$ be the collection of all null sets in $(\Omega, \mathscr{F}, \mathbb{P})$. Then it is a monotone class that is closed with respect to the operation "$\backslash$".

1.4 (10 points) For any function X from Ω to $\Re^1$, the following are true: $X^{-1}(A^c) = (X^{-1}(A))^c$, $X^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha X^{-1}(A_\alpha)$, and $X^{-1}(\bigcap_\alpha A_\alpha) = \bigcap X^{-1}(A_\alpha)$.

1.5 (10 points) If $X \geq 0$ a.e. on Λ and $\int_\Lambda X d\mathbb{P} = 0$, then $X = 0$ a.e. on Λ.

1.6 (10 points) Show that two r.v.'s on $(\Omega, \mathscr{F})$ may be independent according to one p.m. $\mathbb{P}$ but not according to another.

1.7 (10 points) For measure spaces $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$, let $\mathscr{X} \times \mathscr{Y}$ be the B.F. generated by measurable rectangles $A \times B$ where $A \in \mathscr{X}$ and $B \in \mathscr{Y}$. According to Theorem 18.1 of Billingsley (1995), the section $E_x \equiv \{y : (x, y) \in E\}$ for any $x \in X$ and $E \in \mathscr{X} \times \mathscr{Y}$ is a member of $\mathscr{Y}$. Now suppose $\mathscr{L}$ is the class of sets E in $\mathscr{X} \times \mathscr{Y}$ for which $\nu(E_x)$ is an $\mathscr{X}$-measurable function of x. Show that $\mathscr{L}$ is a λ-system.

1.8 (10 points) Show that convergence a.e. implies convergence in probability.

1.9 (10 points) $X_n \to +\infty$ a.e. if and only if $\forall M > 0 : \mathbb{P}[X_n < M \text{ i.o.}] = 0$.

1.10 (10 points) Prove that $\mathbb{P}[\liminf_n E_n] \leq \liminf_n \mathbb{P}[E_n]$ and $\mathbb{P}[\limsup_n E_n] \leq \limsup_n \mathbb{P}[E_n]$.