Homework 2—Advanced Probability

Due Monday, April 20

2.1 (10 points) If $\{X_n\}$ is a sequence of independent and identically distributed r.v.'s not constant a.e., then $\mathbb{P}[X_n \text{ converges}] = 0$.

2.2 (10 points) Suppose $\lim_{n\to+\infty} \mathbb{P}[|X_n - X| > \epsilon] = 0$ for any $\epsilon > 0$ and $\mathbb{P}[X = x] = 0$. Show that $\mathbb{P}[\{X \le x\} \triangle \{X_n \le x\}] \to 0$.

2.3 (10 points) Let α be completely normal. Show that by looking at the expansion of α in some scale we can rediscover the complete works of Shakespeare from end to end without a single misprint or interruption.

2.4 (10 points) For any sequence of r.v.'s $\{X_n\}$, (a) $X_n \to 0$ a.e. would result in $S_n/n \to 0$ a.e. (b) $X_n \to 0$ in $\mathcal{L}^p$ would result in $S_n/n \to 0$ in $\mathcal{L}^p$ for $p \ge 1$.

2.5 (10 points) Let $\{X_n, n \ge 1\}$ be a sequence of independent, identically distributed r.v.'s; also, let τ be a positive integer-valued r.v. that is independent of the X_n's. Suppose that both τ and X_1 have finite second moments, then

$$\sigma^2(S_\tau) = \mathbb{E}[\tau]\sigma^2(X_1) + \sigma^2(\tau)(\mathbb{E}[X_1])^2.$$

2.6 (10 points) Let $\{X_n, n \ge 1\}$ be a sequence of independent, identically distributed r.v.'s; also for some finite l, we have $\sum_{k=1}^{l} p_k = 1$ where each $p_k \equiv \mathbb{P}[X_1 = k]$. Let $N_k(n)$ be the number of values of $j = 1, 2, ..., n$ such that $X_j = k$. Show that

$$\lim_{n\to+\infty} \frac{1}{n} \log\left(\prod_{k=1}^{l} p_k^{N_k(n)}\right) \quad \text{exists a.e.;}$$

in addition, find the limit.

2.7 (10 points) Suppose that $\sup_n \int f d\mu_n < +\infty$ for a nonnegative function f such that $f(x) \to +\infty$ as $x \to \pm\infty$. Show that $\{\mu_n\}$ is tight.

2.8 (10 points) Let f be the ch.f. of the p.m. μ. For each x_0, show that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx_0} f(t)dt = \mu(\{x_0\}).$$

2.9 (10 points) Show that the ch.f. for the standard normal Z is $f(t) = e^{-t^2/2}$.

2.10 (10 points) For a Poisson variable Y_λ such that $\mathbb{P}[Y_\lambda = n] = e^{-\lambda}\lambda^n/(n!)$ for $n = 0, 1, ...,$ show that $(Y_\lambda - \lambda)/\sqrt{\lambda} \Rightarrow Z$ the standard Normal as $\lambda \to +\infty$.