Yifu He 190003956 Homework 2. ①

Q1.

If $\{X_n\}$ is a sequence of independant and identical distributed r.v.'s. not constant a.e.
then $\mathbb{P}[X_n \text{ converge}] = 0$.

Proof: $\{X_n\}$ iid. r.v. not constant a.e. $\Rightarrow \nexists X$. $P[X_n \to X] = 0$.

Use Borel-cantelli lemma

①. $\sum_{n=1}^{+\infty} P[E_n] < +\infty$. $P[\limsup_{+\infty} E_n]$

$= P[\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} E_m]$

$= P(w \in \Omega \mid w \text{ belongs to i.o. of } E_n) = 0$

② E_n's are independant

$\sum_{n=1}^{+\infty} P(E_n) = +\infty$

$P(\limsup E_n) = P(\bigcap_{n\geq 1}^{+\infty} \bigcup_{m\geq n}^{+\infty} E_m) = P[E_n \text{ i.o.}] = 1$

$P(X_n(w)) = 0$, $P(X_n \text{ diverge}) = 1$

$P(X_n < a \text{ and } X_n > b. \text{ i.o.}) = 1$

X_n is not constant. a.e. $a < b$.

$P(X_n < a) > 0$, $P(X_n > b) > 0$.

$\sum_{n=1}^{+\infty} P(X_n > a) = +\infty$

Use Borel-Cantelli ②

$P(X_n < a \text{ i.o.}) = 1$

Q2.

For. $\forall E. F,\ E \Delta F = (E\setminus F) \cup (F\setminus E)$

$\{X \le x\} \Delta \{X_n \ge x\} = \{X \le x \text{ and } X_n > x\} \cup (X > x \text{ and } X_n \le x\}$

$P[|X_n - X| > \varepsilon] \to 0$. $X_n \to X$ in p.r.

Suppose $A_n = \{X \le x \text{ and } X_n > x\}$

$A_n(\varepsilon) = \{X \le x \text{ and } X_n > x+\varepsilon\} \subseteq \{|X_n - x| > \varepsilon\}$

$A_n(\varepsilon) \nearrow A_n$ $\quad A_n(\frac{1}{k})$ uniform in n.

$A_1(\frac{1}{1}),\ A_1(\frac{1}{2}),\ \cdots \longrightarrow A_1$

$A_2(\frac{1}{1}),\ A_2(\frac{1}{2}),\ \cdots \longrightarrow A_2$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0$

We need to prove

$P[A_1] - P[A_1(\frac{1}{k})] \le f(k)$ independant of n

$f(k) \downarrow 0 \quad k \to +\infty$

$P(A_n) - P(A_n(\varepsilon)) = P(x \le X_n \le x+\varepsilon \text{ and } X \le x)$

$\because$ We suppose $P(X=x) = 0$

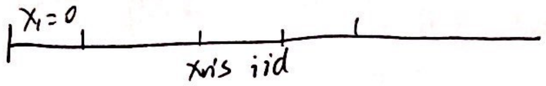
$P(|X_n - X| > \varepsilon) \to 0$

$\therefore P(A_n) - P(A_n(\varepsilon)) \le P(x \le X_n < x+\varepsilon)$

$\xrightarrow{n \to +\infty} P(x < X < x+\varepsilon) \to 0$

Q3.

Suppose L long as complete works of Shakespear.



$$P(X_i=1)=\frac{1}{2^L}>0.$$

$$\frac{X_1+X_2+\cdots+X_n}{n}\longrightarrow\frac{1}{2^L} \quad n \text{ is large enough}$$

$$X_1+X_2+\cdots+X_n\geq 1 \quad \text{for sure}$$

Q4. $X_n\to 0$, $\frac{S_n}{n}\to 0$. a.e.

$$X_n\to 0, \quad \frac{X_1+X_2+\cdots+X_n}{n}\to 0$$

$\forall\varepsilon>0$ $\quad \exists N$, $\forall n>N$, $|X_n|<\varepsilon$

$\forall\varepsilon>0$, $\quad M>N\vee\left(\frac{X_1+X_2+\cdots X_n}{\varepsilon}\right)$

$\forall n>N$ $\quad \left|\frac{X_1+\cdots+X_N}{n}\right|<\frac{|X_1+X_2+\cdots+X_N|}{M}<\varepsilon$

$$\frac{|X_{N+1}+\cdots+X_n|}{n}<\frac{\varepsilon}{2} \qquad \left|\frac{S_n}{n}\right|<\varepsilon$$

Minkowski's inequality

$$\|X+Y\|_p\leq\|X\|_p+\|Y\|_p$$

$$\|X\|_p=\left(\int|X|^p d\mu\right)^{\frac{1}{p}} \quad L^p$$

$$L^p=\{X:\|X\|_p<+\infty\}$$

$$\therefore \|X_1+\cdots+X_n\|_p\leq\|X_1\|_p+\cdots+\|X_n\|_p$$

$$\|\alpha X\|_p=\alpha\cdot\|X\|_p$$

$$\left\|\frac{S_n}{n}\right\|_p=\frac{1}{n}\|S\|_p=\frac{1}{n}\|X_1+\cdots+X_n\|_p$$
$$\leq\frac{1}{n}(\|X_1\|+\cdots+\|X_n\|)\to 0.$$

Q5. $\{X_n\}$. i.i.d. $m = E[X_i]$

$a = Std[X_i]$ ⑥

$Std[X_i] = \sqrt{Var(X_i)}$

$$
\begin{aligned}
\sigma^2 &= Var(X_i) \\
&= E[(X_i - E(X_i))^2] \\
&= E[X_i^2] - E^2(X_i)
\end{aligned}
$$

τ : $P_n = E[\tau = n]$

$\sum_{n=0}^{+\infty} P_n = 1$

$$
\begin{aligned}
S_\tau &= X_1 + X_2 + \cdots + X_\tau \\
&= X_1 + X_2 + \cdots + X_n, \quad \tau = n
\end{aligned}
$$

$$\sigma^2(S_\tau) = E(\tau)\sigma^2(X_1) + \sigma^2(\tau)(E(X_1))^2$$

$$\sigma^2(S_\tau) = E(S_\tau^2) - E^2(S_\tau)$$

$$E(S_\tau) = \sum P_n E(S_n) = \sum P_n \, n \cdot m$$

$$
\begin{aligned}
E[S_\tau^2] &= \sum P_n E[S_n^2] \\
&= \sum P_n E[X_1^2 + X_2^2 + \cdots + X_n^2 + X_1X_2 + \cdots X_1X_n + X_2X_1 + \cdots + X_nX_1 + \cdots + X_nX_{n-1}] \\
&= \sum P_n (n \cdot E[X_1^2] + n(n-1)E[X_1X_2]) \\
&= \cancel{\sum_n P_n (nm^2 +} \\
&\quad \sum_n (nm^2 + n\sigma^2 + n(n-1)m^2) \cdot P_n \\
&= m^2 \sum_n P_n n^2 + \sigma^2 \sum_n P_n \cdot n
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2(S_\tau) &= E[S_\tau^2] - E^2(S_\tau) \\
&= \sigma^2 \sum_n P_n n + m^2 \left(\sum_n P_n n^2 - \left(\sum_n P_n n\right)^2\right) \\
&= E(\tau)\sigma^2(X_1) + \sigma^2(\tau) \cdot E^2(X_1)
\end{aligned}
$$

Q6. $\log\left(\prod_{k=1}^{L} P_k^{N_k(n)}\right) = \log\left(\prod_{k=1}^{L} P_k^{\sum_{m=1}^{n} \mathbb{I}(X_m=k)}\right)$ $\qquad N_k(n) = \sum_{m=1}^{n} \mathbb{I}(X_m=k)$

$\cancel{N_k(n) = \sum_{m=1}^{n} \mathbb{I}(X_m=k)}$

$$= \sum_{k=1}^{L} \sum_{m=1}^{n} \mathbb{I}(X_m=k) \cdot \log(P_k)$$

$$= \sum_{m=1}^{n} \sum_{k=1}^{L} \mathbb{I}(X_m=k) \cdot \log(P_k)$$

let $y_m = \sum_{k=1}^{L} \mathbb{I}(X_m=k)\log(P_k) = \begin{cases} \log(P_1) & P_1 \\ \log P_2 & P_2 \\ \vdots & \\ \log P_L & P_L \end{cases}$

$E[y_m] = \sum_{k=1}^{L} P_n \log(P_k)$

$Var[y_m] = \sum_{k=1}^{L} P_n (\log(P_k))^2 - \left(\sum_{k=1}^{L} P_n (\log P_n)\right)^2$

$\frac{\sum_{m=1}^{n} y_m}{n} \longrightarrow \sum_{k=1}^{L} P_k (\log(P_k))$ a.e.

Q7. $\{\mu_n\}$ is tight.

$\forall \varepsilon>0, \ \exists (a,b]$

$\forall n \quad \mu_n((a,b]) > 1-\varepsilon$

$\exists \varepsilon>0, \ \forall (a,b] \quad \exists n \ \text{s.t.} \ \mu((a,b]) \le 1-\varepsilon$

$\mu_n((-\infty,a] \cup (b,+\infty)) > \varepsilon$

Choose any $M>0$, $f(x) \to +\infty$, $x \to \pm\infty$

there are a and b. such that

$f(x) \geq \frac{M}{\varepsilon}$, when $x \le a$, or $x > b$

$\exists n$, $\int f \, d\mu_n \geq \int_{(-\infty,a] \cup (b,+\infty)} f \, d\mu_n$

$\geq \frac{M}{\varepsilon} \mu_n((-\infty,a] \cup (b,+\infty)) f d\mu_n \geq \frac{M}{\varepsilon} \cdot \varepsilon = M$

~~$\geq \frac{M}{\varepsilon} \mu_n((-\infty,a] \cup (b,+\infty))$~~

Q.8. proof: $\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx_0} f(t) \, dt = \mu(\{x_0\})$

$\frac{1}{2T} \int_{-T}^{T} e^{-itx_0} dt \int_{R'} e^{itx} \mu(dx)$

$= \frac{1}{2T} \int_{R'} \mu(dx) \cdot \int_{-T}^{T} e^{it(x-x_0)} dt$

$= \int_{R'} \mu(dx) \times I(T, x, x_0)$

$\frac{1}{T} \int_0^T \cos(t(x-x_0)) \, dt$

$= \frac{\sin(T(x-x_0))}{T(x-x_0)}$

$I(T, x, x_0) = \frac{1}{2T} \int_{-T}^{T} e^{it(x-x_0)} dt$

$= \begin{cases} 1, & x = x_0 \\ \frac{\sin(T(x-x_0))}{T(x-x_0)} & x \neq x_0 \end{cases}$

$\lim_{T\to+\infty} \int_{R'} \mu(dx) I(T, x, x_0)$

$= \mu(\{x_0\}) + \int_{R' \setminus \{x_0\}} \mu(dx) \lim_{T\to+\infty} I(T, x, x_0)$

$= \mu(\{x_0\})$

Q9. $f(t)=E[e^{itX}]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{itx}\cdot e^{-\frac{x^2}{2}}dx$

$$=e^{-\frac{t^2}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-it)^2}{2}}dx$$

$$=e^{-\frac{t^2}{2}}\cdot\int_{-\infty-it}^{+\infty-it} e^{\frac{s^2}{2}}ds$$

$$=e^{-\frac{t^2}{2}}$$

Q10.

$$\frac{X_\lambda-\lambda}{\sqrt{\lambda}}\Longrightarrow Z\sim N(0,1)$$

proof $f_\lambda(t)\xrightarrow{\lambda\to+\infty}e^{-\frac{t^2}{2}}$

$$f_{Y_\lambda}(t)=E[e^{itY_\lambda}]=\sum_{n=0}^{+\infty}e^{itn}\cdot e^{-\lambda}\frac{\lambda^n}{n!}$$

$$=e^{-\lambda}\cdot\sum_{n=0}^{+\infty}\frac{(\lambda e^{it})^n}{n!}$$

$$=e^{-\lambda(e^{it}-1)}$$

$$f_\lambda(t)=e^{-i\sqrt{\lambda}t}f_{Y_\lambda}\left(\frac{t}{\sqrt{\lambda}}\right)$$

$$=e^{\lambda(e^{\frac{it}{\sqrt{\lambda}}}-1)-i\sqrt{\lambda}t}$$

$$=\exp(\lambda\cdot e^{\frac{it}{\sqrt{\lambda}}}-i\sqrt{\lambda}t-\lambda)$$

$$\lambda e^{\frac{it}{\sqrt{\lambda}}}-i\sqrt{\lambda}t-\lambda=\lambda\left(1+\frac{it}{\sqrt{\lambda}}-\frac{t^2}{2\lambda}\right)-i\sqrt{\lambda}t-\lambda+o\left(\frac{1}{\lambda}\right)$$

$$=-\frac{t^2}{2}+o\left(\frac{1}{\lambda}\right)\to-\frac{t^2}{2}$$

thus. $\exp(\lambda e^{\frac{it}{\sqrt{\lambda}}}-i\sqrt{\lambda}t-\lambda)\to e^{-\frac{t^2}{2}}$