Final–Advanced Probability

Name:

ID:

Score:

1 (20 points) Let $\mathscr{F}$ be a field in Ω and μ a measure (satisfying $\mu(\emptyset) = 0$, monotonicity, and countable additivity) on it. For any $A \in \mathscr{T}$, define

$$\mu^*(A) = \inf \left\{ \sum_j \mu_j(B) | B_j \in \mathscr{F}_0 \text{ for all } j \text{ and } \bigcup_j B_j \supseteq A \right\}.$$

Prove that μ^* satisfies sub-additivity, namely, $\mu^* \left(\bigcup_j A_j \right) \leq \sum_j \mu^*(A_j)$.

2 (20 points) If $X_n \to X$, $Y_n \to Y$ both in probability, then $X_n + Y_n \to X + Y$ in probability.

3 (20 points) For any sequence of r.v.'s $\{X_n\}$, if $\mathbb{E}[X_n^2] \to 0$, then $(S_n - \mathbb{E}[S_n])/n \to 0$ in probability.

4 (20 points) Use means of the characteristic function, show that $a_n X_n + b_n \Rightarrow aX + b$ will follow from $X_n \Rightarrow X$, $a_n \to a$, and $b_n \to b$.

5 (20 points) For the Poisson process, show that for $0 < s < t$,

$$\mathbb{P}[N_s = k | N_t] = \frac{N_t!}{k!(N_t - k)!} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{N_t - k},$$

when $N_t \geq k$ and 0 otherwise.