Quiz 2.

Name: Yifu He

ID: 190003956.

Score.

Q1. For any sequence $\{X_n\}$. the convergence of S_n/n to 0 in probability would lead to that X_n/n in probability as well.

Proof: $S_n = X_1 + X_2 + \cdots + X_n$

$= \sum_{i=1}^{n} X_i$,

Use the information of. $\frac{S_n}{n}$ coverge to 0 in p.r.

That's to say. iff.. $\forall \delta > 0$,

$\lim_{n\to\infty} P[|\frac{S_n}{n} - 0| \le \delta] = 1$ ①

②

$\frac{X_n}{n} = \frac{S_n - S_{n-1}}{n} < |\frac{S_n - S_{n-1}}{n}| < |\frac{S_n}{n}| + |\frac{S_{n-1}}{n}|$

We need to prove, $\frac{X_n}{n}$ converge to 0 in p.r. too.

Then for ~~every~~ $\forall$ $\varepsilon > 0$,

we need to prove $\lim_{n\to\infty} P[|\frac{X_n}{n} - 0| \le \varepsilon] = 1$.

Use the information of. ①. for any δ, we have $\varepsilon = 2\delta$.

thus. $\lim_{n\to\infty} P[\frac{X_n}{n} < \underset{\delta}{|\frac{S_n}{n}|} + \underset{\delta}{|\frac{S_{n-1}}{n}|} < \underset{\varepsilon}{\underset{\shortparallel}{2\delta}}] = 1$

proof finished.

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Q2. Let $\{X_n, n\geq 1\}$ be a sequence of r.v. $S_n = X_1+X_2+\cdots+X_n$, and for any t, define $N(t)$ such that $S_{N(t)} \leq t \leq S_{N(t)+1}$. Show that.

$$E[N(t)] = \sum_{n=1}^{+\infty} P[S_n \leq t]$$

Proof: $S_n = X_1 + X_2 + \cdots X_n$

$$= \sum_{i=1}^{n} X_i$$

then $S_{N(t)} = \sum_{i=1}^{N(t)} X_i$

the relationship between $P[N(t) \geq n]$, $P[S_n \leq t]$ is that

$$P[N(t) \geq n] = P[S_n \leq t]$$

$$E[N(t)] = 1 \cdot P[N(t)=1] + 2 \cdot P[N(t)=2] + 3 \cdot P[N(t)=3] + \cdots + {+\infty} P[N(t)={+\infty}]$$

$$= P[N(t)\geq 1] + P[N(t)\geq 2] + \cdots + P[N(t) \geq {+\infty}]$$

$$= \sum_{i=1}^{+\infty} P[N(t) \geq n]$$

$$= \sum_{i=1}^{+\infty} P[S_n \leq t]$$

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Q3. Find the nth iterated convolution of an exponential distribution

Proof: the density function of exponential ~~function~~ distribution with λ. is

$$f_\lambda(x;\lambda)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0.\\ 0, & x<0\end{cases}$$

when $n=2$,

$$f_2(x;\lambda)=\int_{-\infty}^{\cancel{+\infty}} f_1(x-y)\cdot f_1(y)dy$$

$$=\int_{-\infty}^{\cancel{+\infty}} \lambda\cdot e^{-\lambda(x-y)}\lambda e^{-\lambda(y)}\,dy$$

$$=\int_{-\infty}^{\cancel{+\infty}} \lambda^2 e^{-\lambda x\cancel{+}\lambda y}\cdot e^{-\lambda y}\,dy$$

$$=\int_{-\infty}^{\cancel{+\infty}} \lambda^2 e^{-\lambda x}dy$$

$$=\cancel{\lambda^2 e^{-\lambda x}\int_{-\infty}^{+\infty} dy}=\lambda^2 e^{-\lambda x}\int_0^x dy=\lambda^2\cdot xe^{-\lambda x}=\frac{\lambda^2}{(2-1)!}\cdot x^{2-1}\cdot e^{-\lambda x}$$ ①

when $n=k$, $k\geq 2$ suppose $f_k(x;\lambda)=\frac{1}{(k-1)!}\lambda^k\cdot x^{k-1}\cdot e^{-\lambda x}$ ②

when $n=k+1$,

$$f_{k+1}(x;\lambda)=\int_0^x f_1(x-y)\cdot f_k(y)dy$$ ③

$$=\int_0^x \lambda e^{-\lambda(x-y)}\cdot\frac{1}{(k-1)!}\lambda^k\cdot y^{k-1}\cdot e^{-\lambda y}\,dy$$

$$=\frac{\lambda^{k+1}\cdot e^{-\lambda x}}{(k-1)!}\int_0^x y^{k-1}e^{\lambda y-\lambda y}\,dy=\frac{\lambda^{k+1}}{(k-1)!}\cdot e^{-\lambda x}\cdot\int_0^x y^{k-1}dy$$

$$=\frac{\lambda^{k+1}}{(k-1)!}e^{-\lambda x}\cdot\frac{y^k}{k}\Big|_0^x$$

$$=\frac{\lambda^{k+1}}{k!}\cdot e^{-\lambda x}\cdot y^k$$

according to ①②③. using Mathematical Induction

We conclude $f_n(x;\lambda)=\frac{1}{(n-1)!}\lambda^n\cdot x^{n-1}\cdot e^{-\lambda x}$

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