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Q1.

(a). $f(x) = \sum_{k=1}^{n} p_k (y_k - x)^2$.

$f(x)$ is convex and differentiable, and thus the necessary and sufficient condition of optimal is $f'(x) = 0$.

$$f'(x) = \sum_{k=1}^{n} (2p_k y_k - 2x) \Rightarrow x = \sum_{k=1}^{n} p_k y_k = E[Y]$$

(b) $f(x) = \sum_{k=1}^{n} p_k |y_k - x|$ is convex and not differentiable.

$$f(x) = \begin{cases} y_k - x, & y_k > x \\ x - y_k, & x > y_k \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x > y_k \\ -1, & x < y_k \end{cases}$$

thus. $f'(x) = \sum_{x > y_k} p_k - \sum_{x < y_k} p_k = P[x > Y] - P[x < Y]$

① if there is a number m

$$\sum_{k=1}^{m} p_k = \sum_{k=n+1}^{n} p_k = \frac{1}{2}, \quad f'(x) = 0$$

② if there exists $y_m = x$, then $\begin{cases} f'(x) \geq 0, & x > y_m \\ f'(x) \leq 0, & x < y_m \end{cases}$. y_m is the median of Y.

the problem convert to.

$$\min \sum_{k=1}^{n} p_k v_k.$$

subject to: $v_k + x \geq y_k$

$-x + v_k \geq -y_k$.

convert it to standard form, let $\lambda_k, \mu_k \geq 0$.

$v_k = \lambda_k - \mu_k$

$\Rightarrow$ the dual problem become

$$\max \sum_{k=1}^{n} (y_k \lambda_k - y_k \mu_k) \quad \Rightarrow \quad \lambda_k = \begin{cases} 0 & k = 1, 2, \cdots, m-1 \\ p_k & k = m+1, \cdots, n \\ \frac{1}{2} - \sum_{k=m+1}^{n} p_k, & k = m, \end{cases}$$

s.t. $\sum_{k=1}^{n} \lambda_k - \mu_k = 0$.

$\lambda_k + \mu_k = p_k$

$\lambda \geq 0, \mu \geq 0$.

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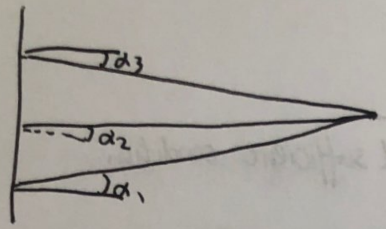
Problem 2.



$\tan\alpha_2 = \frac{1}{2}(\tan\alpha_1 + \tan\alpha_3)$

when α is small, $\tan\alpha \approx \alpha$, $\Rightarrow \alpha_2 = \frac{1}{2}(\alpha_1 + \alpha_3)$

so the problem becomes

$$\min: \sum_{i=1}^{3} (\alpha_i - \hat{\alpha}_i)^2$$

subject to:

$$2(\alpha_1 - \hat{\alpha}_1) + \frac{1}{2}\mu = 0$$
$$2(\alpha_2 - \hat{\alpha}_2) - \mu = 0$$
$$2(\alpha_3 - \hat{\alpha}_3) + \frac{1}{2}\mu = 0$$
$$\frac{1}{2}(\hat{\alpha}_1 + \hat{\alpha}_3) - \hat{\alpha}_2 - \frac{3}{4}\mu = 0$$

$$\Rightarrow \begin{cases} \alpha_1 = 0.08 \\ \alpha_2 = 0.03 \\ \alpha_3 = -0.02 \end{cases}$$

$\mu = 0.012$

solve the problem.

$$d \approx \frac{500}{0.05} = 10000$$

Question 3. ③

$f(x) = (x_1-1)^2 + (x_2+2)^2 - x_1 x_2$

$g_1(x) = x_1 - x_2 - b$

$g_2(x) = -x_1$

use Lagrange multipliers. $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

$$\begin{cases} \nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0 \\ \lambda_1 g_1(x) = 0 \\ \lambda_2 g_2(x) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} 2x_1 - 2 - x_2 + \lambda_1 - \lambda_2 = 0 \\ 2x_2 + 4 - x_1 + \lambda_1 = 0 \\ \lambda_1 (x_1 - x_2 - b) = 0 \\ \lambda_2 x_1 = 0 \end{cases}$$

case 1: $\lambda_1 = 0$, $\lambda_2 = 0 \Longrightarrow x_1 = 0$, $x_2 = -2$. this solution needs $b \geq 2$.

cas2: $\lambda_1 > 0$, $\lambda_2 = 0 \Longrightarrow x_1 = -\frac{\lambda_1}{3}$, $x_2 = -2 + \frac{\lambda_1}{3}$

Contradict with $x_1 > 0$.

case3: $\lambda_1 = 0$, $\lambda_2 > 0$, $\Longrightarrow x_1 = 0$, $x_2 = -2$.

$\lambda_2 = -4$ contradict with $\lambda_2 > 0$

case4: $\lambda_1 > 0$, $\lambda_2 > 0$.

$x_1 = 0$, $x_2 = -b$

$$\begin{cases} \lambda_1 = 4 - 2b > 0 \\ \lambda_2 = 2 - b > 0 \end{cases} \Longrightarrow \text{this solution needs } b < 2.$$

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