Optimization Models in Finance
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Homework 4 - Solutions

Problem 1.
(i) The standard form is

$$\begin{aligned} \min \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 = 4, \\ & x_1 + x_2 - x_4 = 3, \\ & x_1 \geq 0,\ x_2 \geq 0.\ x_3 \geq 0.\ x_4 \geq 0. \end{aligned}$$

(ii) If we try $x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is a feasible basic solution.
If we try $x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is not a feasible basic solution, because $x_3 < 0$.
If we try $x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is a feasible basic solution.
If we try $x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is a feasible basic solution.
If we try $x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is not a feasible basic solution, because $x_4 < 0$.
If we try $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$ we obtain

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Bx_B = b, \qquad x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is not a feasible basic solution.
(iii) There are three feasible basic solutions

$$x^1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x^3 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad x^4 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

One of them must be optimal. We calculate the value of the objective function, and we conclude that the point x^4 is best. For this basic solution we have:

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \qquad c_B = \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B^T\pi = c_B, \qquad \pi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then the reduced costs are equal to:

$$\bar{c} = c - A^T\pi = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

As $\bar{c} \geq 0$, the optimality condition is satisfied.

Problem 2
The dual problem is

$$\begin{aligned} \max \quad & 4\lambda_1 + 3\lambda_2 \\ \text{subject to} \quad & \lambda_1 + \lambda_2 \leq 2, \\ & 2\lambda_1 + \lambda_2 \leq 1, \\ & \lambda_1 \geq 0, \;\; \lambda_2 \geq 0. \end{aligned}$$

Its solution is $\lambda = \pi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Observe that the optimal values of both problems coincide.

Problem 3
See the attached spreadsheet.