26:711:564 **OPTIMIZATION MODELS IN FINANCE**
Professor Andrzej Ruszczyński

Currency Conversion

The Problem

We have currently cash reserves in 10 different currencies in the amounts given in the second column of the following table (in millions).

	Initial Position	Desired Position
EUR	70	60
USD	20	25
AUD	8	6
GBP	3	2
NZD	15	20
CAD	7	8
CHF	2	3
JPY	1,500	1,800
HKD	35	40
SGD	18	13

Our objective is to convert these amounts in order to have reserves at least equal to the amounts given in the last column of the table. The second table (included in the Excel spreadsheet currency_2019_data) contains the conversion rates (cross rates) as of September 1, 2019.

The numbers in this table should be read as follows: to buy 1 U.S. dollar for Euro requires € 0.9082 (the element in the second row and the first column), 1 U.S. dollar costs 1.48438 Australian dollars (AUD), etc. If we want to buy 1 Euro for U.S. dollars we have to pay $ 1.0988 (the element in the EUR row and USD column), etc.

Design the conversion plan to meet the desired goals and to maximize the U.S.$ value of the obtained positions.

The Model

Let us denote by n the number of currencies considered ($n = 10$ in our case). Let a_j, $j = 1, \ldots, n$ denote the initial positions and let b_i, $i = 1, \ldots, n$, denote the desired positions. Finally, we denote by r_{ij} the cost of currency i in currency j (the element in row i and column j of the rates table).

We introduce the **decision variables** x_{ij} representing the amounts of currency i purchased for currency j, $i, j = 1, \ldots, n$. For $i = j$ the variable x_{ii} represents the amount kept in currency i.

The decision variables have to satisfy the following equations and inequalities, called **constraints**:

$$\sum_{i=1}^{n} r_{ij} x_{ij} = a_j, \quad j = 1, \ldots, n \quad \text{(amount of currency } j \text{ spent)}$$

$$\sum_{j=1}^{n} x_{ij} \geq b_i, \quad i = 1, \ldots, n, \quad \text{(amount of currency } i \text{ obtained)}$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

In the first equation we assume that only the original amounts can be converted, not the amounts obtained from other conversions.

The set X of conversion plans x satisfying the constraints is called the **feasible set**.

The value of the obtained positions can be expressed as the dollar value the amounts obtained, that is

$$f(x) = \sum_{i=1}^{n} \frac{1}{r_{2i}} \sum_{j=1}^{n} x_{ij}.$$

It is called the **objective function**. The problem is to maximize the objective function over x in the feasible set X, that is to find a point $\hat{x} \in X$ (the **optimal solution**) such that

$$f(\hat{x}) \geq f(x) \quad \text{for all} \quad x \in X.$$

The problem above is an example of an **optimization problem**, in particular, a **linear programming problem**.

Linear Programming

A linear programming problem is an optimization problem in which the objective function and the constraint functions are linear functions of the decision variables $x \in \mathbb{R}^n$. The standard form of a linear programming problem is the following:

$$\begin{aligned} \min\ & \langle c, x \rangle \\ \text{subject to } & Ax = b, \\ & x \geq 0, \end{aligned}$$

with a cost vector $c \in \mathbb{R}^n$, a constraint matrix A of dimension $m \times n$, and a right hand side vector $b \in \mathbb{R}^m$. Every linear programming problem can be transformed to an equivalent standard form.

The role of the standard form is both theoretical and algorithmic. In applications, linear programming modeling languages and solvers accept a variety of formulations.

The theory of linear programming is necessary for the understanding of the models, application of off-the-shelf software, and correct interpretation of the results. It is also an important step towards more general optimization models: nonlinear, dynamic, and stochastic.