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Risk-adjusted probability measures in portfolio optimization with coherent measures of risk

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Abstract

We consider the problem of optimizing a portfolio of n assets, whose returns are described by a joint discrete distribution. We formulate the mean–risk model, using as risk functionals the semideviation, deviation from quantile, and spectral risk measures. Using the modern theory of measures of risk, we derive an equivalent representation of the portfolio problem as a zero-sum matrix game, and we provide ways to solve it by convex optimization techniques. In this way, we reconstruct new probability measures which constitute part of the saddle point of the game. These risk-adjusted measures always exist, irrespective of the completeness of the market. We provide an illustrative example, in which we derive these measures in a universe of 200 assets and we use them to evaluate the market portfolio and optimal risk-averse portfolios.

Keywords: Risk; Portfolio optimization

1. Introduction

The problem of optimizing a portfolio of finitely many assets is a classical problem in theoretical and computational finance. Since Markowitz [11], it is generally agreed that portfolio performance should be measured in two distinct dimensions: the *mean* $\mathbb{E}[R]$ of the portfolio return R and the *risk* $r[R]$, which measures the uncertainty of the return. In the mean–risk approach, we select from the universe of all possible portfolios those that are *efficient*: for a given value of the mean they minimize the risk or, equivalently, for a given value of risk they maximize the mean. Such an approach has many advantages: it allows one to formulate the problem as a parametric optimization problem, and it facilitates the trade-off analysis between mean and risk.

Markowitz used the variance of the return as the risk functional. It is easy to compute, and it reduces the portfolio selection problem to a parametric quadratic programming problem. One can, however, construct simple counterexamples that show the imperfection of the variance as the risk measure: it treats over-perfor-

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mance equally as under-performance, and more importantly its use may suggest a portfolio which is always outperformed by another portfolio.

By using the theory of stochastic dominance, Ogryczak and Ruszczyński [13–15] identified several important risk functionals, for which optimal portfolio returns are *stochastically non-dominated*, that is, no other feasible portfolio return may be better in terms of the second-order stochastic dominance relation. These are semideviations [8] and weighted deviations from quantiles. In the latter case, the resulting mean–risk optimization problem is equivalent to that of *conditional value at risk* optimization, discussed in [17,16,2]. Ruszczyński and Vanderbei [21] devise a parametric optimization method for solving such problems and apply it to a very large real-world problem.

Mean–risk portfolio problems can be also viewed from the perspective of the general theory of measures of risk, as initiated by Artzner et al. [3] and developed in multiple publications (see [7,5,10,18–20]). Our intention is to use this theory to derive dual characteristics of optimal risk-averse portfolio solutions. In this context, our work continues and develops the insights of [10]. In particular, we derive the equivalent risk-neutral measures implied by the mean–risk model and we provide constructive ways of obtaining these measures in typical portfolio optimization problems involving law invariant coherent measures of risk.

2. The portfolio optimization problem

Let $(\Omega, \mathscr{F}, P)$ be a probability space. We consider n assets, whose returns $R^j, j = 1,\ldots,n$, are integrable random variables. The random return vector of the assets is denoted by R, where $R \in \mathscr{L}_1^n(\Omega, \mathscr{F}, P)$.

A vector $z \in \mathbb{R}^n$ represents our asset allocation, with each z_j equal to the fraction of the capital invested in asset j. Clearly, the set of possible asset allocations can be defined as follows:

$$Z = \{z \in \mathbb{R}^n : z_1 + z_2 + \ldots + z_n = 1, \quad z_j \geqslant 0,\ j = 1, 2, \ldots, n\}. \tag{1}$$

Our analysis will not depend on the detailed way this set is defined; we shall only use the fact that Z is a convex compact set in $\mathbb{R}^n$. So, in some applications one may introduce the possibility of *short positions*, i.e., allow some z_j's to become negative. Limits on the exposure to particular assets or their groups may be introduced, by imposing upper bounds on the z_j's or on their partial sums. One can also bound the absolute differences between the z_j's and some reference investments $\bar{z}_j$, which may represent the existing portfolio, etc. All these modifications define some convex compact feasible sets, and are, therefore, covered by our approach.

Simple manipulation shows that the return of the portfolio is given by the random variable $R^T z$.

In order to define the abstract risk-averse portfolio optimization problem, we need to recall basic concepts of the theory of *coherent measures of risk* (see [5, Chapter 4] for a comprehensive treatment). We follow here the abstract construction proposed in [20].

An uncertain outcome is represented by a measurable function $X : \Omega \to \mathbb{R}$ (in our case X is the portfolio return $R^{\mathrm{T}} z$). We specify the vector space $\mathscr{X}$ of possible outcomes (in our case $\mathscr{X} = \mathscr{L}_1(\Omega, \mathscr{F}, P)$). A *coherent measure of risk* is a functional $\rho : \mathscr{X} \to \mathbb{R}$ satisfying the following axioms:

Convexity: $\rho(\alpha X + (1 - \alpha)Y) \leqslant \alpha\rho(X) + (1 - \alpha)\rho(Y)$, for all $X, Y \in \mathscr{X}$ and all $\alpha \in [0, 1]$;
Monotonicity: If $X, Y \in \mathscr{X}$ and $X(\omega) \leqslant Y(\omega)$ for all $\omega \in \Omega$, then $\rho(X) \geqslant \rho(Y)$;
Translation equivariance: If $a \in \mathbb{R}$ and $X \in \mathscr{X}$, then $\rho(X + a) = \rho(X) - a$;
Positive homogeneity: If $t > 0$ and $X \in \mathscr{X}$, then $\rho(tX) = t\rho(X)$.

Important examples of coherent measures of risk are obtained from mean–risk models:

$$\rho[X] = -\mathbb{E}[X] + \lambda r[X], \tag{2}$$

with a scalar parameter $\lambda > 0$ and with some risk functional $r : \mathscr{X} \to \mathbb{R}$ representing the variability of the outcome. In particular, we may set $r[X]$ to be the *semideviation* of order $p \geqslant 1$:

$$\sigma_p[X] = \mathbb{E}[(\mathbb{E}[X] - X)_+^p]^{1/p}, \tag{3}$$

or the *weighted mean deviation from quantile*:

$$r_\alpha[X] = \min_{\eta\in\mathbb{R}} \mathbb{E}\left[\max\left(\frac{1-\alpha}{\alpha}(\eta - X), X - \eta\right)\right], \quad \alpha \in (0,1). \tag{4}$$

It is well known that the optimal η in the problem above is any α-quantile of X.

In both cases, for every $\lambda \in [0,1]$, the resulting functional (2) satisfies the axioms of coherent measures of risk (see [20]).

Before proceeding to further analysis, let us mention an important relation of the *Average Value at Risk* (AVaR), also known as *expected shortfall* or *Conditional Value at Risk* [17,16,2], to the mean deviation from quantile $r_\alpha[\cdot]$:

$$\mathrm{AVaR}_\alpha(X) = -\max_{\eta\in\mathbb{R}}\left\{\eta - \frac{1}{\alpha}\mathbb{E}[(\eta - X)_+]\right\} = -\frac{1}{\alpha}\int_0^\alpha q_t(X)\mathrm{d}t = -\mathbb{E}[X] + r_\alpha[X]. \tag{5}$$

In (5) the symbol $q_t(X)$ denotes the t-quantile of the distribution of X. As $-q_t(X)$ is known in the financial literature as *Value at Risk*, denoted $\mathrm{VaR}_t(X)$, the second equation in (5) means that $\mathrm{AVaR}_\alpha(X)$ is indeed an average of $\mathrm{VaR}_t(X)$, for $0 < t \leqslant \alpha$. All these relations can be found in [15] and [6] (with obvious adjustments for the sign change of X). The last equation in (5) allows us to interpret $\mathrm{AVaR}_\alpha(X)$ as a mean–risk model.

The portfolio optimization problem with a coherent measure of risk can be now simply written as

$$\min_{z\in Z} \rho\left[R^{\mathrm{T}} z\right]. \tag{6}$$

Observe that the function $f(z) = \rho[R^{\mathrm{T}} z]$ is convex and finite-valued on $\mathbb{R}^n$. It is, therefore, continuous. As the set Z is compact, the minimum of $f(z)$ in Z is attained, that is, problem (6) has an optimal solution $\hat{z}$.

3. Optimality and duality

In order to develop optimality conditions for problem (6) we recall the representation theorem of convex risk functionals, first proved in [3], and refined and generalized in a series of papers [7,5,18–20]. The version we use here is a special case of Theorem 2 from [20].

We define the space $\mathscr{M}$ of regular countably additive signed measures μ on $(\Omega, \mathscr{F})$, which are absolutely continuous with respect to P, with densities $\frac{\mathrm{d}\mu}{\mathrm{d}P} \in \mathscr{L}_\infty(\Omega, \mathscr{F}, P)$. It is known that the space $\mathscr{M}$ is the topological dual of $\mathscr{X}$, when $\mathscr{X}$ is endowed with its strong (norm) topology. We endow the space $\mathscr{M}$ with the weak* topology.

In Theorem 1 below we use the symbol $\mathscr{P}$ to denote the set of probability measures in $\mathscr{M}$.

Theorem 1. *If $\rho[\cdot]$ is a lower semicontinuous coherent measure of risk, then there exists a convex and weakly* closed set $\mathscr{A} \subset \mathscr{P}$ such that*

$$\rho[X] = \sup_{\mu\in\mathscr{A}}\left(-\int_\Omega X(\omega)\mu(\mathrm{d}\omega)\right), \quad X \in \mathscr{X}. \tag{7}$$

If the probability space Ω is finite, with N elementary events $\omega_1, \ldots, \omega_N$ occurring with positive probabilities $p_1, \ldots, p_N$, the space $\mathscr{M}$ is simply the space of real N-tuples $(\mu_1, \ldots, \mu_N)$, that is, the space $\mathbb{R}^N$. The set $\mathscr{P}$ is the unit simplex in $\mathbb{R}^N$, the set $\mathscr{A}$ is a subset of the unit simplex and formula (7) takes on the form

$$\rho[X] = \sup_{\mu\in\mathscr{A}}\left(-\sum_{i=1}^N \mu_i X(\omega_i)\right).$$

The mean–risk models (2) with semideviations (3) and deviations from quantiles (4) satisfy the assumptions of Theorem 1 and enjoy the dual representation (7). The form of the set $\mathscr{A}$ in both cases has been developed in [20].

Owing to Theorem 1, the portfolio optimization problem can be written as a max–min problem:

$$\max_{z\in Z}\ \inf_{\mu\in\mathscr{A}} \int_\Omega R^{\mathrm{T}}(\omega) z \mu(\mathrm{d}\omega). \tag{8}$$

If the measure of risk $\rho[\cdot]$ is continuous, the "inf" operation can be replaced by the "min" operation. Indeed, it follows from representation (dual-representation) that $\mathscr{A} = -\partial\rho[0]$. For a continuous $\rho[\cdot]$, the subdifferential $\partial\rho[X]$ is bounded. Consequently, the set $\mathscr{A}$ is convex, weakly* closed, and bounded. It is, therefore, weakly* compact and the supremum in (7) is attained.

Furthermore, the "min" and the "max" operations can be interchanged, and we can prove the main optimality theorem.

Theorem 2. *Suppose $\rho[\cdot]$ is a continuous coherent measure of risk. A point $\hat{z}$ is an optimal solution of problem (6) if and only if there exists a convex and weakly* closed set $\hat{\mathscr{A}} \subset \mathscr{A}$ such that for every probability measure $\hat{\mu} \in \hat{\mathscr{A}}$ the point $\hat{z}$ is also the solution of the problem*

$$\max_{z\in Z} \int_\Omega R^{\mathrm{T}}(\omega) z \hat{\mu}(\mathrm{d}\omega). \tag{9}$$

Furthermore, the set $\hat{\mathscr{A}}$ is the set of solutions of the dual problem:

$$\min_{\mu\in\mathscr{A}} \max_{z\in Z} \int_\Omega R^{\mathrm{T}}(\omega) z \mu(\mathrm{d}\omega). \tag{10}$$

Proof. Our theorem can be deduced from Proposition 6.1 in [20], but owing to the simple structure of the portfolio problem, we can provide a very easy proof in our case. Consider the functional

$$F(z,\mu) = \int_\Omega R^{\mathrm{T}}(\omega) z \mu(\mathrm{d}\omega)$$

on the set $Z \times \mathscr{A}$. It is bi-linear and the sets Z and $\mathscr{A}$ are convex and compact (in the corresponding topologies on $\mathbb{R}^n$ and $\mathscr{M}$). By virtue of the von Neumann min–max theorem [4], the functional $F(\cdot,\cdot)$ has a saddle point $(\hat{z},\hat{\mu})$ on $Z \times \mathscr{A}$:

$$F(z,\hat{\mu}) \leqslant F(\hat{z},\hat{\mu}) \leqslant F(\hat{z},\mu), \quad \text{for all} \quad (z,\mu) \in Z \times \mathscr{A}.$$

This implies the assertions of the theorem. □

It follows that the optimal portfolio $\hat{z}$ optimizes the expected return with respect to the probability measure $\hat{\mu}$, as defined in (9). We shall call $\hat{\mu}$ the *risk-adjusted measure*. We also see from the dual problem (10) that it is the worst possible measure in the set $\mathscr{A}$.

Consider now the special case of the set Z defined in (1). As the functional $F(z,\mu)$ is linear in Z, all assets involved in the optimal portfolio with positive weights $\hat{z}_j$ must have equal expected returns with respect to the risk-adjusted measure $\hat{\mu}$. This can be easily seen from problem (9).

From now on we assume that the probability space Ω is finite, with N elementary events $\omega_1,\ldots,\omega_N$ occurring with probabilities $p_1,\ldots,p_N$. We use p to denote the vector in $\mathbb{R}^N$ with coordinates p_i, $i = 1,\ldots,N$.

We shall also write R for the matrix of asset returns; r_{ji} denotes the return of asset j in event i, where $j = 1,\ldots,n$ and $i = 1,\ldots,N$. With this notation, Rp is the vector of expected asset returns, $R^{\mathrm{T}}z$ is the vector of portfolio returns, and $p^{\mathrm{T}}R^{\mathrm{T}}z$ is the expected portfolio return. The measure μ will now be interpreted as a vector in $\mathbb{R}^N$.

In this notation, the portfolio problem (8) can be written in an equivalent form:

$$\max_{z\in Z} \min_{\mu\in\mathscr{A}} \langle \mu, R^{\mathrm{T}}z \rangle. \tag{11}$$

The dual problem has the form

$$\min_{\mu\in\mathscr{A}} \max_{z\in Z} \langle \mu, R^{\mathrm{T}}z \rangle. \tag{12}$$

It turns out that the problem can be interpreted as a matrix game with the payoff matrix R^{T}, and the mixed strategies of the players represented by the portfolio allocation z and by the measure μ. Finding the optimal asset allocation $\hat{z}$ and the risk-adjusted measure $\hat{\mu}$ is equivalent to finding a saddle point of this game, with the mixed strategies restricted to the sets Z and $\mathscr{A}$, respectively.

In the next two sections we show that in several important cases the solution of these problems can be easily obtained by linear programming, if the set Z is a convex polyhedron.

4. The mean–semideviation model

The absolute semideviation functional r of a random variable X is defined as

$$\sigma_1[X] = \mathbb{E}\max(\mathbb{E}[X] - X, 0). \tag{13}$$

The corresponding measure of risk takes on the form

$$\rho[X] = -\mathbb{E}[X] + \lambda\sigma_1[X],$$

with some fixed $\lambda \in [0, 1]$. For discrete distributions, we can identify X with a vector in $\mathbb{R}^N$ and write

$$\rho[X] = -\langle p, X\rangle + \lambda\sum_{i=1}^{N} p_i \max(\langle p, X\rangle - x_i, 0).$$

The set $\mathscr{A}$ for this measure has been described in [20]. In our notation and with discrete distributions it takes on the form

$$\mathscr{A} = -\partial\rho[0] = \{(1 - \lambda\langle g, \mathbb{1}\rangle)p + \lambda g : |g_i| \leqslant p_i,\ i = 1, \dots, N\}. \tag{14}$$

Here $\mathbb{1}$ denotes the vector in $\mathbb{R}^N$ having all coordinates equal to 1.

We now focus our attention on obtaining the optimal risk-adjusted probability measure $\hat{\mu}$ for the portfolio problem. Consider the portfolio optimization problem

$$\min_{z\in Z} -\langle p, R^{\mathrm{T}}z\rangle + \lambda\sum_{i=1}^{N} p_i \max\left(\langle p, R^{\mathrm{T}}z\rangle - \langle r_i, z\rangle, 0\right), \tag{15}$$

with $r_i \in \mathbb{R}^n$ representing the vector of returns of assets in event i.

Denoting the shortfalls $\max(\langle p, R^{\mathrm{T}}z\rangle - \langle r_i, z\rangle, 0)$ by s_i, we can rewrite (15) as a convex programming problem (see, e.g., [12,21]):

$$\begin{aligned} \text{Minimize}\quad & -\langle p, R^{\mathrm{T}}z\rangle + \lambda\langle p, s\rangle \\ \text{subject to}\quad & s_i \geqslant \langle p, R^{\mathrm{T}}z\rangle - \langle r_i, z\rangle, \quad i = 1, \dots, N, \\ & s \geqslant 0, \\ & z \in Z. \end{aligned} \tag{16}$$

Associate Lagrange multipliers ξ_i with the constraints (16). The lagrangian function is

$$\begin{aligned} L(z, s, \xi) &= -\langle p, R^{\mathrm{T}}z\rangle + \lambda\langle p, s\rangle + \sum_{i=1}^{N}\xi_i\left(\langle p, R^{\mathrm{T}}z\rangle - \langle r_i, z\rangle - s_i\right) \\ &= (\langle \xi, \mathbb{1}\rangle - 1)\langle p, R^{\mathrm{T}}z\rangle - \langle \xi, R^{\mathrm{T}}z\rangle + \langle \lambda p - \xi, s\rangle. \end{aligned} \tag{17}$$

The dual function is

$$L_D(\xi) = \inf_{\substack{z\in Z\\ s\geqslant 0}} L(z, s, \xi) = \min_{z\in Z}\left((\langle \xi, \mathbb{1}\rangle - 1)\langle p, R^{\mathrm{T}}z\rangle - \langle \xi, R^{\mathrm{T}}z\rangle\right) + \inf_{s\geqslant 0}\langle \lambda p - \xi, s\rangle.$$

The dual problem has the form

$$\max_{\xi\geqslant 0} L_D(\xi).$$

Our aim is to show that there is close relation of the above problem to the problem of finding the risk-adjusted measure $\hat{\mu}$. Observe that $L_D(\xi) = -\infty$, unless

$$\lambda p \geqslant \xi. \tag{18}$$

In this case

$$L_D(\xi) = \min_{z \in Z} \langle (\langle \xi, \mathbb{1} \rangle - 1)p - \xi, R^{\mathrm{T}} z \rangle.$$

We can now write the dual problem as follows:

$$\max_{0 \leqslant \xi \leqslant \lambda p} L_D(\xi). \tag{19}$$

Define

$$\mu = (1 - \langle \xi, \mathbb{1} \rangle)p + \xi. \tag{20}$$

Observe that $\langle \mu, \mathbb{1} \rangle = 1$. Moreover, due to (18),

$$\mu \geqslant (1 - \lambda \langle p, \mathbb{1} \rangle)p + \xi = (1 - \lambda)p + \xi \geqslant 0.$$

It follows that μ is a probability vector. Moreover, substituting $\xi = \lambda g$, we observe that μ is an element of the set $\mathscr{A}$ defined in (14). In this way we have established a one-to-one correspondence (20) between the feasible points ξ in the dual problem (19) and the elements of the set $\mathscr{A}$ in (14).

We can thus write the dual function as a function of μ:

$$\overline{L}_D(\mu) = \min_{z \in Z} \langle -\mu, R^{\mathrm{T}} z \rangle.$$

It follows that the convex programming dual problem (19) coincides with the game-theoretical dual, as defined in (12). In this way we have proved the following result.

Theorem 3. *Suppose $\rho[\cdot] = -\mathbb{E}[\cdot] + \lambda\sigma_1[\cdot]$ with $\lambda \in [0, 1]$. A vector $\hat{z}$ and a measure $\hat{\mu}$ constitute a saddle point of the game (11) if and only if the vector $\hat{z}$ is a solution of problem (16) and $\hat{\mu} = (1 - \langle \hat{\xi}, \mathbb{1} \rangle)p + \hat{\xi}$, where $\hat{\xi}$ is a solution of the dual problem (19).*

It follows that we can reconstruct risk-adjusted probabilities by solving the convex programming problem (16), recovering the Lagrange multipliers $\hat{\xi}$, and applying the transformation (20). If Z is a convex polyhedron, methods of linear programming can be employed.

5. The mean–deviation from quantile model

Consider the weighted deviation from α-quantile risk functional, as defined in (4). The corresponding coherent measure of risk takes on the form

$$\rho[X] = -\mathbb{E}[X] + \lambda r_\alpha[X],$$

where $\lambda \in [0, 1]$ is fixed. For discrete distributions, we identify X with a vector in $\mathbb{R}^N$ and write

$$\rho[X] = -\langle p, X \rangle + \lambda \min_{\eta \in \mathbb{R}} \sum_{i=1}^{N} p_i \max\left(\frac{1-\alpha}{\alpha}(\eta - x_i), x_i - \eta \right).$$

The set $\mathscr{A}$ for this measure has been described in [20]. In our notation and with discrete distributions it takes on the form

$$\mathscr{A} = -\partial\rho[0] = \left\{ (1-\lambda)p + \lambda g : 0 \leqslant g_i \leqslant \frac{p_i}{\alpha},\ i = 1, \ldots, N,\ \sum_{i=1}^{N} g_i = 1 \right\}. \tag{21}$$

It can be easily checked that for $\lambda \in [0, 1]$ the set $\mathscr{A}$ is indeed a set of probability measures.

We now focus our attention on obtaining the optimal risk-adjusted probability measure $\hat{\mu}$ for the portfolio problem:

$$\operatorname*{Minimize}_{z \in Z} -\langle p, R^{\mathrm{T}} z \rangle + \lambda \min_{\eta \in \mathbb{R}} \sum_{i=1}^{N} p_i \max\left(\frac{1-\alpha}{\alpha}(\eta - \langle r_i, z \rangle), \langle r_i, z \rangle - \eta \right). \tag{22}$$

Here $r_i \in \mathbb{R}^n$ represents the vector of asset returns in event i.

Denoting by u_i and v_i the positive and the negative parts of $\langle r_i, z\rangle - \eta$, respectively, we can reformulate problem (22) as a convex programming problem (see [12,21]):

$$\begin{aligned} \text{Minimize} \quad & -\langle p, R^{\mathrm{T}}z\rangle + \lambda \sum_{i=1}^{N} p_i \left(\frac{1-\alpha}{\alpha} v_i + u_i\right) \\ \text{subject to} \quad & \langle r_i, z\rangle - \eta = u_i - v_i, \; i = 1,\ldots,N, \\ & z \in Z, \\ & u_i, v_i \geqslant 0, \; i = 1,\ldots,N, \\ & \eta \in \mathbb{R}. \end{aligned} \tag{23}$$

Our further development follows the lines of the analysis in the preceding section. We associate Lagrange multipliers ξ_i with the constraints (23). The Lagrangian has the form

$$\begin{aligned} L(z,\eta,u,v,\xi) &= -\langle p, R^{\mathrm{T}}z\rangle + \lambda \sum_{i=1}^{N} p_i \left(\frac{1-\alpha}{\alpha} v_i + u_i\right) + \sum_{i=1}^{N} \xi_i(\langle r_i, z\rangle - \eta - u_i + v_i) \\ &= -\langle p-\xi, R^{\mathrm{T}}z\rangle - \eta\langle \xi, \mathbb{1}\rangle + \left\langle \frac{\lambda(1-\alpha)}{\alpha} p + \xi, v\right\rangle + \langle \lambda p - \xi, u\rangle. \end{aligned} \tag{24}$$

The dual function is

$$\begin{aligned} L_D(\xi) &= \inf_{\substack{z\in Z, \eta\in\mathbb{R} \\ u,v\geqslant 0}} L(z,\eta,u,v,\xi) \\ &= \min_{z\in Z} \left(-\langle p-\xi, R^{\mathrm{T}}z\rangle\right) - \sup_{\eta\in\mathbb{R}} \eta\langle \xi, \mathbb{1}\rangle + \inf_{v\geqslant 0} \left\langle \frac{\lambda(1-\alpha)}{\alpha} p + \xi, v\right\rangle + \inf_{u\geqslant 0} \langle \lambda p - \xi, u\rangle. \end{aligned}$$

The dual problem has the form

$$\max_{\xi\in\mathbb{R}^N} L_D(\xi).$$

Again, we shall show that there is close relation of the above problem to the problem of finding the risk-adjusted measure $\hat{\mu}$. Observe that $L_D(\xi) = -\infty$, unless

$$\langle \xi, \mathbb{1}\rangle = 0, \tag{25}$$

$$-\frac{\lambda(1-\alpha)}{\alpha} p \leqslant \xi \leqslant \lambda p. \tag{26}$$

In this case

$$L_D(\xi) = \min_{z\in Z}(-\langle p-\xi, R^{\mathrm{T}}z\rangle).$$

We can now write the dual problem as follows:

$$\max_{\xi\in\mathscr{U}} L_D(\xi), \tag{27}$$

with the set $\mathscr{U}$ given by (25)-(26). Define

$$\mu = p - \xi. \tag{28}$$

The conditions $\xi \in \mathscr{U}$ and $\lambda \in [0,1]$ imply that μ is a probability vector. Moreover, substituting $\xi = \lambda(p - g)$, we observe that μ is an element of the set $\mathscr{A}$ defined in (21). In this way we have again established a one-to-one correspondence between the feasible points ξ in the Lagrangian dual problem and the elements of the set $\mathscr{A}$ in (21).

We can thus write the Lagrangian dual function as a function of μ:

$$\overline{L}_D(\mu) = \min_{z\in Z}\langle -\mu, R^{\mathrm{T}}z\rangle.$$

It follows that the convex programming dual problem (27) coincides with the game-theoretical dual, as defined in (12). This can be summarized as follows.

Theorem 4. *Suppose $\rho[\cdot] = -\mathbb{E}[\cdot] + \lambda r_\alpha[\cdot]$ with $\lambda \in [0,1]$. A vector $\hat{z}$ and a measure $\hat{\mu}$ constitute a saddle point of the game (11) if and only if the vector $\hat{z}$ is a solution of problem (23) and $\hat{\mu} = p - \hat{\xi}$, where $\hat{\xi}$ is a solution of the dual problem (27).*

Again, we can reconstruct risk-adjusted probabilities by simply solving the convex programming problem (23), recovering the Lagrange multipliers $\hat{\xi}$, and applying the transformation (28). If Z is a convex polyhedron, methods of linear programming can be employed.

6. Law invariant coherent measures of risk

The fundamental result in the theory of law invariant measures is the Kusuoka theorem [9]: *For every lower semicontinuous law invariant coherent measure of risk $\rho[\cdot]$ on $\mathscr{L}_\infty(\Omega,\mathscr{F},P)$ with non-atomic $(\Omega,\mathscr{F},P)$ there exists a convex set $\mathscr{N}$ of probability measures on $[0,1]$ such that*

$$\rho[X] = \sup_{\nu\in\mathscr{N}} \int_0^1 \mathrm{AVaR}_\alpha[X]\nu(\mathrm{d}\alpha).$$

Using identity (5) we can rewrite $\rho[X]$ as follows:

$$\rho[X] = -\mathbb{E}[X] + \sup_{\nu\in\mathscr{N}} \int_0^1 r_\alpha[X]\nu(\mathrm{d}\alpha).$$

This means that every coherent law invariant measure of risk corresponds to a mean–risk model, with the risk functional

$$\varkappa_{\mathscr{N}}[X] = \sup_{\nu\in\mathscr{N}} \int_0^1 r_\alpha[X]\nu(\mathrm{d}\alpha). \tag{29}$$

In the case of discrete distributions, Kusuoka's theorem is not valid, in general, but formula (29) can be used to define a very broad class of mean–risk models. It is most convenient to present it in the case of equally likely elementary events, that is, with $p_i = 1/N$, $i = 1,\dots,N$. In this case the set $\mathscr{N}$ can be identified with convex set of probability vectors $\nu \in \mathbb{R}^N$. Formula (29) takes on the form

$$\varkappa_{\mathscr{N}}[X] = \sup_{\nu\in\mathscr{N}} \sum_{k=1}^N \nu_k r_{\alpha_k}[X], \tag{30}$$

with $\alpha_k = k/N$. This allows us to formulate the portfolio optimization problem

$$\operatorname*{Minimize}_{z\in Z} \; -\mathbb{E}[R^\mathrm{T}z] + \lambda\varkappa_{\mathscr{N}}[R^\mathrm{T}z]. \tag{31}$$

For every $\lambda \in [0,1]$ the objective of this problem is a law invariant coherent measure of risk. Of course, the constant λ may be simply absorbed into the definition of the set $\mathscr{N}$, but it is convenient to leave it in the model, to stress the analogy to the models discussed in the previous sections.

An easy to analyze case is that of a *spectral measure of risk*, for which the set $\mathscr{N}$ contains just one measure ν (see [1]). In this case, the portfolio optimization problem (31) can be formulated as a combination of problems (23):

$$\begin{aligned}
\text{Minimize}\quad & -\langle p, R^\mathrm{T}z\rangle + \lambda\sum_{k=1}^N \nu_k \sum_{i=1}^N p_i\left(\frac{1-\alpha_k}{\alpha_k}v_{ik} + u_{ik}\right)\\
\text{subject to}\quad & \langle r_i, z\rangle - \eta_k = u_{ik} - v_{ik},\quad i=1,\dots,N,\quad k=1,\dots,N,\\
& z\in Z,\\
& u_{ik}, v_{ik} \geqslant 0,\quad i=1,\dots,N,\quad k=1,\dots,N,\\
& \eta_k\in\mathbb{R},\quad k=1,\dots,N.
\end{aligned} \tag{32}$$

Denote by ξ_{ik} the Lagrange multipliers associated with the constraints (32). It is convenient to represent them as a matrix Ξ of dimension $N \times N$, and to write ξ_k for the kth column of Ξ.

In an analogous way to the considerations of the previous section, we can formulate the Lagrangian and the dual function. For the dual function to be finite, the following conditions have to be satisfied:

$$\Xi^{\mathrm{T}}\mathbb{1} = \mathbb{1}, \tag{33}$$

$$-\frac{\lambda v_k(1-\alpha_k)}{\alpha_k}p \leqslant \xi_k \leqslant v_k\lambda p. \tag{34}$$

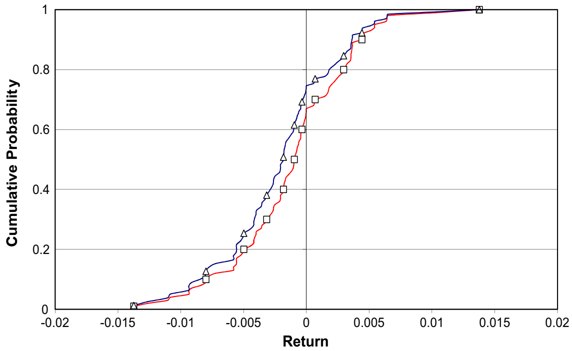

Fig. 1. Cumulative distribution curves of market portfolio for the semideviation measure.

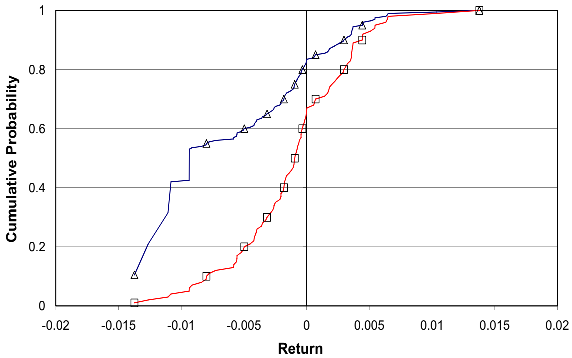

Fig. 2. Cumulative distribution curves of market portfolio for the deviation from quantile measure.

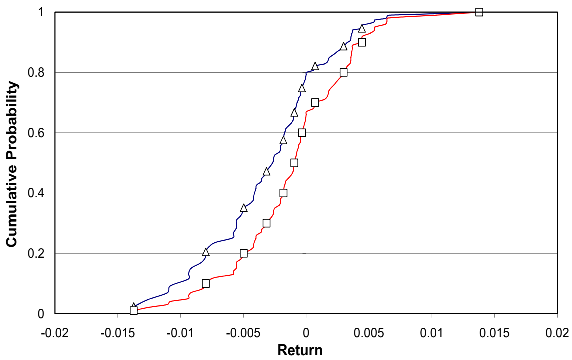

Fig. 3. Cumulative distribution curves of market portfolio for the spectral measure.

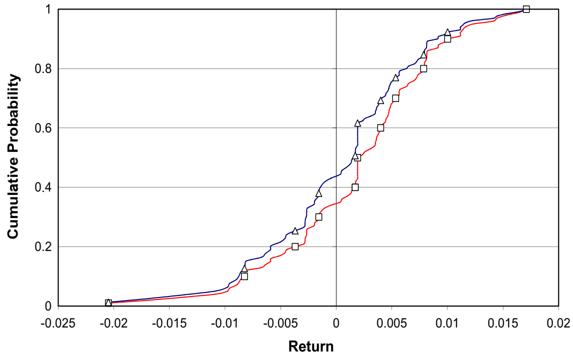

Fig. 4. The CDF curves corresponding to optimal portfolio returns for the semideviation measure.

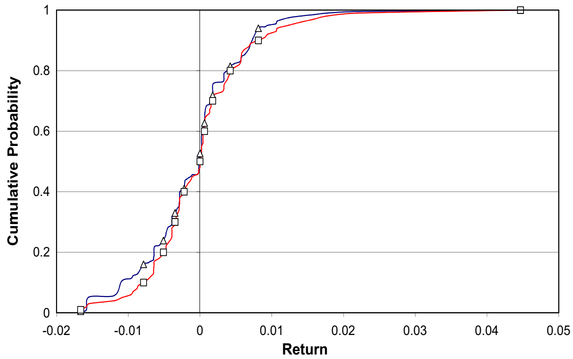

Fig. 5. The CDF curves corresponding to optimal portfolio returns for the deviation from quantile measure.

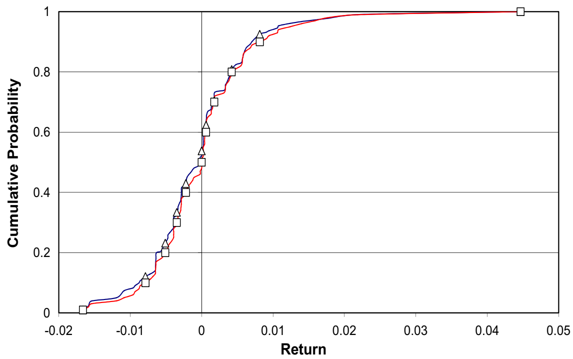

Fig. 6. The CDF curves corresponding to optimal portfolio returns for the spectral measure.

Under these conditions, the dual function can be written as follows:

$$L_D(\Xi) = \min_{z \in Z} \left(-\langle p - \Xi \mathbb{1}, R^{\mathrm{T}} z \rangle\right).$$

The dual problem takes on the form

$$\max_{\Xi \in \mathscr{U}} L_D(\xi), \tag{35}$$

with the set $\mathscr{U}$ given by (33)-(34).

By repeating the analysis of the previous section we obtain the following result.

Theorem 5. *Suppose* $\rho[\cdot] = -\mathbb{E}[\cdot] + \lambda\varkappa_{\mathcal{N}}[\cdot]$ *with* $\lambda \in [0,1]$ *and* $\mathcal{N} = \{v\}$. *A vector* $\hat{z}$ *and a measure* $\hat{\mu}$ *constitute a saddle point of the game* (11) *if and only if the vector* z *is a solution of problem* (32) *and* $\hat{\mu} = p - \hat{\Xi}\mathbb{1}$, *where* $\hat{\Xi}$ *is a solution of the dual problem* (35).

We can reconstruct risk-adjusted probabilities $\hat{\mu}$ by solving the convex programming problem (32) and recovering the Lagrange multipliers $\hat{\Xi}$. If Z is a convex polyhedron, (32) can be formulated as a linear programming problem.

7. Numerical Illustration

In this section, we find optimal portfolios and compute the risk-adjusted probability measures for mean–risk portfolio optimization problems based on the semideviation (problem (16)), mean deviation from quantile (problem (23)), and with a spectral measure of risk (problem (32)).

Each portfolio draws from a group of 200 assets taken from the S&P 500 index. Daily returns from the last 100 trading days are taken as equally likely scenarios.

In each case, we construct cumulative distribution functions (CDF) for the returns of for each portfolio: one using original probability measures and the other using risk-adjusted probability measures. These CDFs are plotted against each other.

We also compute the distribution of the market portfolio with respect to the original and the risk-adjusted probability measures.

The CDF for returns using original probability measures are also compared for optimal portfolios obtained for different values of the risk parameter λ.

In plotting the cumulative distribution curves for the original probability measures against the risk-adjusted curves, we strive to gain some insight into how the two probability measures are different. In particular, do the

Table 1
The numbers of assets with weights at least 0.001 and the ranges of their weights in different mean–risk models

Measure of risk	Number of assets	Range of positive weights
Mean–semideviation ($\lambda = 0.5$)	9	[0.0158, 0.3406]
Mean–deviation from quantile ($\lambda = 0.5$)	34	[0.0029, 0.1023]
Spectral measure ($\lambda = 0.5$)	26	[0.0010, 0.1787]
Mean–semideviation ($\lambda = 1$)	26	[0.0007, 0.1646]
Mean–deviation from quantile ($\lambda = 1$)	42	[0.0003, 0.1190]
Spectral measure ($\lambda = 1$)	37	[0.0013, 0.1077]

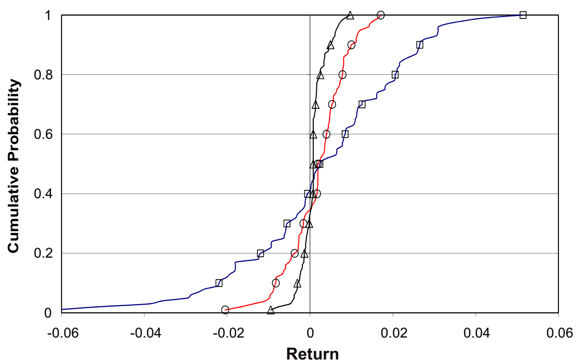

Fig. 7. The cumulative distribution curves corresponding to optimal portfolio returns, for different risk parameters λ, in the mean–semideviation model.

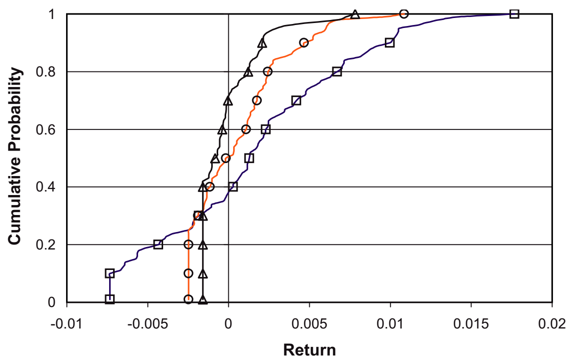

Fig. 8. The cumulative distribution curves corresponding to optimal portfolio returns, for different risk parameters λ, in the mean–deviation from quantile model.

risk-adjusted probability measures give a different perspective on the behavior of the portfolio returns to the original probability measures?

We begin with the case of the market portfolio. The market portfolio is defined here as the equally weighted index composed of all 200 assets. In this case, it is very easy to observe the differences in cumulative distributions between the two measures. The risk-parameter λ is set to 0.5.

The market portfolio is defined here as the equally weighted index composed of all 200 assets. The CDF for market portfolio returns is constructed, using the risk-adjusted probability measures calculated earlier. A cumulative Probability distribution for market returns, using the original probability measures, is also calculated. The CCDF is then plotted for both probability measures. The results are presented in Figs. 1–3, for the three measures of risk considered. On each of these figures, the left (blue)[1] curve represents the CDF with respect to the risk-adjusted probability measure, and the right (red) curve the CDF with respect to the original probability measure. In the case of market portfolio it is very easy to observe the differences in cumulative distributions between the two measures. In all these cases the risk-parameter λ is set to 0.5.

If we follow the data markers in Fig. 1, it is apparent that in the lower range of returns, $(-0.015, 0.005)$, the adjusted curve has greater cumulative probability values than the original curve. The risk-adjusted probability measures predict the lower half of returns will occur with higher frequency than the original probability measures. This reflects a more negative outlook on the return distribution from the risk-adjusted probability measures, than that predicted by the original probability measures. Figs. 2 and 3 also display a more negative outlook from the perspective of risk-adjusted probability measures, with the deviation from quantile example exhibiting most pessimistic outlook.

In Figs. 4–6 we present the CDF functions for the optimal portfolios, obtained from problems (16), (23), and (32). Again, in each case two functions are presented: one using the original probability measure, and the other using the risk-adjusted probability measure. In each figure the left (blue) curve represents the risk-adjusted probability measure, and the right (red) curve – the original probability measure. In all cases the risk-parameter λ is set to 0.5. No dramatic differences in these distribution functions can be observed. For the optimal risk-averse portfolios, the risk-adjusted probability measures have only a slightly more pessimistic outlook than original probability measures.

In summary, the risk-adjusted probability measures give a more pessimistic perspective of portfolio behavior than the original probability measures, for the market portfolio. Their perspective with respect to the optimal risk-averse portfolio is also more negative, but by a much smaller margin. This could reflect the fact that the optimal portfolio is robust, in the sense that the outcome looks similar even from a risk-averse perspective.

It is of interest to show that the robust performance can be achieved with relatively small numbers of assets. In Table 1 we summarize the characteristics of the optimal portfolios obtained from the three models. It is

[1] For interpretation of color in Figs. 1–9, the reader is referred to the web version of this article.

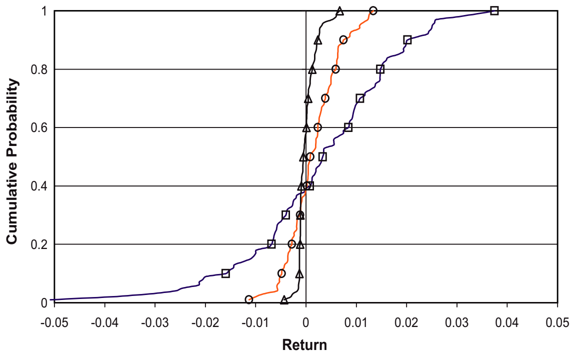

Fig. 9. The cumulative distribution curves corresponding to optimal portfolio returns, for different risk parameters λ, in the model with a spectral measure.

worth noting that substantial robustness of the returns can be achieved with relatively small numbers of assets in the portfolios.

The Cumulative Distribution curves, for different values of the risk-parameter λ were calculated, using original probability measures, for semideviation, deviation from quantile, and spectral measures. For each measure, the cumulative distribution curves were plotted and compared. The results are presented in Figs. 7–9. The coding on the figures is as follows: $\lambda = 0.1$ on the blue chart with rectangles, $\lambda = 0.5$ on the red chart with circles, and $\lambda = 1$ on the black chart with triangles. As anticipated, in each case the curves became wider in range as the risk-aversion parameter λ is decreased, while they became more narrow as λ is increased.

8. Conclusions

We formulated a mean–risk model for optimizing a portfolio of n assets with random returns, for different risk measures. Using convex analysis and Lagrangian duality, we formulated an equivalent representation of the portfolio problem as a zero-sum matrix game. The value of the game is the expected return with respect to new risk-adjusted probability measures. These measures can be derived from the Lagrange multipliers in the linear programming problems for portfolio selection.

We were interested to test how these new risk-adjusted probability measures might give a different perspective on the behavior of the portfolio returns. With this in mind, we graphed cumulative distribution for returns calculated from the original probability measures against that for the risk-adjusted probability measures. This was done for the market portfolio and for the optimal portfolios from the mean–risk models. In both cases, the risk-adjusted probability measures predicted a more negative outlook for returns. However, the distribution of the returns of our optimal solutions turned out to be less sensitive to the perturbation of the measure than that of the market portfolio.

In future research, we would like to extend the derivation and analysis of risk-adjusted probability measures to two-stage and multi-stage portfolio optimization problems. Another area of interest would be the valuation of financial derivatives using risk-adjusted probability measures obtained in this way, as our derivation does not hinge on the market completeness assumption.

Acknowledgements

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