26:711:564 **OPTIMIZATION MODELS IN FINANCE**
Professor Andrzej Ruszczyński

Average Value at Risk

1 Value at Risk and Average Value at Risk

We return to our discussion of Value at Risk, but under the assumption that a prospect Z represents random "cost," that is, smaller values of Z are preferred. The Value at Risk at level $\alpha \in (0,1)$ of a random cost $Z \in \mathcal{Z}$ is defined as the $(1-\alpha)$-quantile of the distribution of Z:

$$\mathrm{V@R}^+_\alpha(Z) \triangleq \inf\{\eta : F_Z(\eta) \geq 1-\alpha\} = F_Z^{-1}(1-\alpha).$$

The Value at Risk has the monotonicity property:

$$Z \leq V \Longrightarrow \mathrm{V@R}^+_\alpha(Z) \leq \mathrm{V@R}^+_\alpha(V), \quad \forall\, \alpha \in (0,1).$$

It also has the translation property:

$$\mathrm{V@R}^+_\alpha(Z + c\mathbb{1}) = \mathrm{V@R}^+_\alpha(Z) + c, \quad \forall\, c \in \mathbb{R},\ \forall\, \alpha \in (0,1).$$

It is positively homogeneous:

$$\mathrm{V@R}^+_\alpha(\gamma Z) = \gamma\, \mathrm{V@R}^+_\alpha(Z), \quad \forall\, \gamma \geq 0,\ \forall\, \alpha \in (0,1).$$

However, it is not convex, as illustrated in the following example.

Example 1 *Suppose Z is a Bernoulli variable,*

$$Z = \begin{cases} 0 & \text{with probability } 1-p, \\ 1 & \text{with probability } p, \end{cases}$$

with $p \in (0,1)$, and V is independent of Z and has the same distribution as Z. For $p < \alpha < 1$ we have $\mathrm{V@R}^+_\alpha(Z) = \mathrm{V@R}^+_\alpha(V) = 0$. However, if $p < \alpha < 1-(1-p)^2$, we have

$$\mathrm{V@R}^+_\alpha\big(\lambda Z + (1-\lambda)V\big) > 0 = \lambda\, \mathrm{V@R}^+_\alpha(Z) + (1-\lambda)\, \mathrm{V@R}^+_\alpha(V),$$

for all $\lambda \in (0,1)$, which contradicts convexity.

If random prospects $Z \in \mathcal{Z}$ represent "profits," that is, their larger values are preferred, the Value at Risk is defined as

$$\mathrm{V@R}_\alpha^-(Z) \triangleq -\sup\{\eta : F_Z(\eta) \le \alpha\}.$$

It is clear that $\mathrm{V@R}_\alpha^-(Z) = \mathrm{V@R}_\alpha^+(-Z)$, and thus all considerations regarding the functional $\mathrm{V@R}_\alpha^+(\cdot)$ can be translated to case of the functional $\mathrm{V@R}_\alpha^-(\cdot)$.

In our further considerations we assume the prospect space $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, where $p \in [1, \infty]$. For integrable random variables we define the *Average Value at Risk* at level $\alpha \in (0, 1]$ as follows:

$$\mathrm{AV@R}_\alpha^+(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_\beta^+(Z)\, \mathrm{d}\beta. \tag{1}$$

Remark 2 *If the* $(1-\alpha)$*-quantile of* Z *satisfies the equation* $P[Z \ge \mathrm{V@R}_\alpha^+(Z)] = \alpha$*, we can change variables in* (1) *to obtain the equation*

$$\mathrm{AV@R}_\alpha^+(Z) = \frac{1}{\alpha}\int_{\mathrm{V@R}_\alpha^+(Z)}^\infty z\, dF_Z(z) = \mathbb{E}\big[Z \mid Z \ge \mathrm{V@R}_\alpha^+(Z)\big].$$

This is the reason why Average Value at Risk is also called the Conditional Value at Risk. However, the formula above is valid only under the assumption that the $(1-\alpha)$*-quantile of* Z *is unique.*

We can derive a useful extremal representation of $\mathrm{AV@R}_\alpha^+(Z)$.

Theorem 3 *For every* $\alpha \in [0, 1]$ *we have*

$$\mathrm{AV@R}_\alpha^+(Z) = \inf_{\eta \in \mathbb{R}} \left\{\eta + \frac{1}{\alpha}\mathbb{E}\big[(Z-\eta)_+\big]\right\}. \tag{2}$$

By construction, the function $\alpha \mapsto \mathrm{AV@R}_\alpha^+(Z)$ is nonincreasing. We can also define the function

$$\delta_\alpha^+(Z) \triangleq \mathrm{AV@R}_\alpha^+(Z) - \mathbb{E}[Z], \quad \alpha \in [0, 1].$$

From (2), after elementary manipulations we obtain the equation:

$$\delta_\alpha^+(Z) = \inf_{\eta \in \mathbb{R}} \mathbb{E}\Big[\max\Big(\eta - Z, \frac{1-\alpha}{\alpha}(Z-\eta)\Big)\Big] \ge 0.$$

The minimizer in the formula above, as well as in (2), exists and is equal to the $(1-\alpha)$-quantile of Z. Because of that, the function $\delta_\alpha^+(Z)$ is called the *weighted mean deviation from quantile*.

If $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ represents "profits," that is, larger values of Z are preferred, we define the Average Value at Risk as follows:

$$\mathrm{AV@R}^-_\alpha(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}^-_\beta(Z)\,\mathrm{d}\beta = -\frac{1}{\alpha}\int_0^\alpha F_Z^{-1}(\beta)\,\mathrm{d}\beta.$$

As the β-quantile of Z is uniquely defined for almost all $\beta \in (0,1)$, except for at most countable set, we could use the left quantile $F_Z^{-1}(\beta)$ in the right side of this equation. We also have the following equation

$$\mathrm{AV@R}^-_\alpha(Z) = \inf_{\eta\in\mathbb{R}} \Big\{ -\eta + \frac{1}{\alpha}\mathbb{E}\big[(\eta - Z)_+\big]\Big\}. \tag{3}$$

This relation can also be obtained by substituting $-Z$ for Z in (2).

2 The Case of Finitely Many Scenarios. Linear Programming Representation

Suppose Z represents "cost" and has finitely many realizations $z_1, z_2, \dots, z_K$ attained with probabilities $p_1, p_2, \dots, p_K$, where $p_1 + p_2 + \dots + p_K = 1$. In this case the extremal representation (2) takes on the following form:

$$\mathrm{AV@R}^+_\alpha(Z) = \min_{\eta\in\mathbb{R}} \Big\{\eta + \frac{1}{\alpha}\sum_{k=1}^K p_k(z_k - \eta)_+\Big\}. \tag{4}$$

We can convert the optimization problem on the right hand side to a linear programming problem in the following way. We introduce variables v_k, $k = 1, 2\dots, K$ and require that

$$\begin{aligned} v_k &\ge z_k - \eta, \quad && k = 1,2,\dots,K,\\ v_k &\ge 0, && k = 1,2,\dots,K. \end{aligned} \tag{5}$$

We can replace the expressions $(z_k - \eta)_+$ by v_k in (4), because the v_k's have the incentive to be small, and thus they will stop on one of the bounds in (5). We obtain the following linear programming problem:

$$\begin{aligned} \min_{\eta,v} \quad & \eta + \frac{1}{\alpha}\sum_{k=1}^K p_k v_k \\ & v_k \ge z_k - \eta, \quad k = 1,2,\dots,K,\\ & v_k \ge 0, \qquad\quad\; k = 1,2,\dots,K. \end{aligned} \tag{6}$$

The optimal value of the objective function of this problem is equal to $\mathrm{AV@R}^{+}_{\alpha}(Z)$. The optimal value of η in this problem is equal to $\mathrm{V@R}^{+}_{\alpha}(Z)$.

If the random variable Z represents profits or gains, the corresponding linear programming problem takes on the form:

$$\begin{aligned} \min_{\eta, v} \quad & -\eta + \frac{1}{\alpha} \sum_{k=1}^{K} p_k v_k \\ & v_k \geq \eta - z_k, \quad k = 1, 2, \ldots, K, \\ & v_k \geq 0, \qquad\quad\ \ k = 1, 2, \ldots, K. \end{aligned} \tag{7}$$

The optimal value of the objective function of this problem is equal to $\mathrm{AV@R}^{-}_{\alpha}(Z)$. The optimal value of $-\eta$ in this problem is equal to $\mathrm{V@R}^{-}_{\alpha}(Z)$.