26:711:564 **OPTIMIZATION MODELS IN FINANCE**
Professor Andrzej Ruszczyński

Stochastic Dominance

1 Direct Forms of Stochastic Dominance

In the stochastic dominance approach, random outcomes (for example return rates) are compared by a point-wise comparison of some performance functions constructed from their distribution functions. For a real random variable V, its first performance function is defined as the cumulative distribution function of V:

$$F_1(V;\eta) = \mathbb{P}\{V \le \eta\} \quad \text{for } \eta \in \mathbb{R}.$$

A random return V is said to *stochastically dominate* another random return S in the first order, denoted $V \succeq_{(1)} S$, if

$$F_1(V;\eta) \le F_1(S;\eta) \quad \text{for all } \eta \in \mathbb{R}.$$

The second performance function F_2 is given by areas below the distribution function F,

$$F_2(V;\eta) = \int_{-\infty}^{\eta} F_1(V;\xi)\, d\xi \quad \text{for } \eta \in \mathbb{R},$$

and defines the relation of the *second-order stochastic dominance* (SSD). That is, random return V stochastically dominates S in the second order, denoted $V \succeq_{(2)} S$, if

$$F_2(V;\eta) \le F_2(S;\eta) \quad \text{for all } \eta \in \mathbb{R}.$$

The corresponding strict dominance relations $\succ_{(1)}$ and $\succ_{(2)}$ are defined as follows: $V \succ S$ if and only if $V \succeq S$, $S \not\succeq V$.

We can express the function $F_2(V;\cdot)$ as the expected shortfall: for each target value η we have

$$F_2(V;\eta) = \mathbb{E}\big[(\eta - V)_+\big], \tag{1}$$

where $(\eta - V)_+ = \max(\eta - V, 0)$. The function $F_{(2)}(V;\cdot)$ is continuous, convex, nonnegative and nondecreasing. It is well defined for all random variables V with finite expected value. Due to this representation, the second order stochastic dominance relation $V \succeq_{(2)} S$ can be equivalently characterized by the system of inequalities:

$$\mathbb{E}\big[(\eta - V)_+\big] \le \mathbb{E}\big[(\eta - S)_+\big] \quad \text{for all } \eta \in \mathbb{R}. \tag{2}$$

The stochastic dominance relations can be characterized using utility functions.

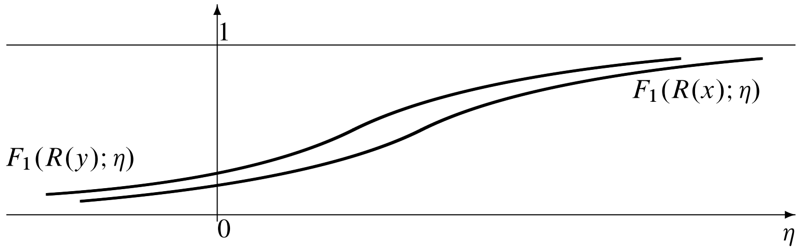

Figure 1: First degree stochastic dominance: $R(x) \succeq_{(1)} R(y)$.

(i) For any two random variables V, S the relation $V \succeq_{(1)} S$ holds if and only if for all nondecreasing functions $u(\cdot)$ defined on $\mathbb{R}$ we have

$$\mathbb{E}\big[u(V)\big] \geq \mathbb{E}\big[u(S)\big]. \tag{3}$$

(ii) For any two random variables V, S with finite expectations, the relation $V \succeq_{(2)} S$ holds if and only if (3) is satisfied for all nondecreasing concave functions $u(\cdot)$.

In the context of portfolio optimization, we consider stochastic dominance relations between random return rates. Thus, we say that portfolio x *dominates* portfolio y *in the first order*, if

$$F_1(R(x);\eta) \leq F_1(R(y);\eta) \quad \text{for all } \eta \in \mathbb{R}.$$

This is illustrated in Figure 1.

Similarly, we say that x *dominates* y *in the second order* ($R(x) \succeq_{(2)} R(y)$), if

$$F_2(R(x);\eta) \leq F_2(R(y);\eta) \quad \text{for all } \eta \in \mathbb{R}.$$

Recall that the individual return rates R_j have finite expected values and thus the function $F_2(R(x);\cdot)$ is well defined. The second order relation is illustrated in Figure 2.

Notice that the notion of stochastic dominance of second order assumes implicitly that larger outcomes are preferred, that is, it is suitable to use for comparison of incomes or investment returns. If we look at random losses or random cost, we prefer small realizations (outcomes). In that case, we should look at another comparison. We integrate the survival functions of the random variables as a counterpart of the integrated distribution functions. Requiring

$$\int_\eta^\infty P(Z > t)\,dt \geq \int_\eta^\infty P(V > \eta)\,dt \quad \text{for all } \eta \in \mathbb{R}$$

is equivalent to

$$\mathbb{E}(Z-\eta)_+ \geq \mathbb{E}(V-\eta)_+, \quad \text{for all } \eta \in \mathbb{R}. \tag{4}$$

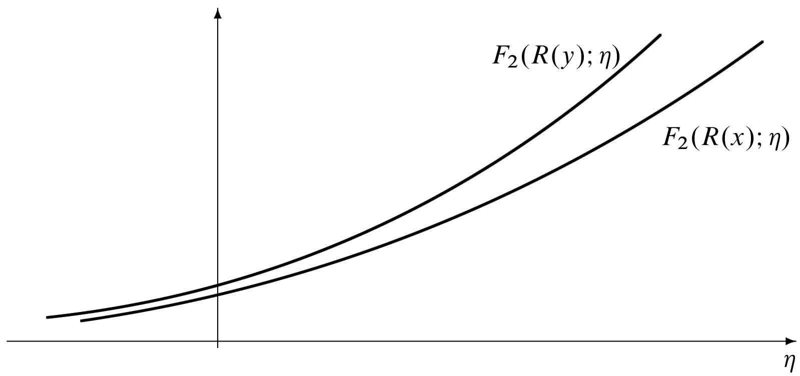

Figure 2: Second order dominance: $R(x) \succeq_{(2)} R(y)$.

We denote the integrated survival function is called the *excess function*; in actuarial sciences it is called sometimes *stop-loss-transform*. This function is non-increasing and convex. It is well known that relation (4) is equivalent to the *increasing convex order* between Z and V.

Definition 1 *A random variable V is stochastically smaller than a random variable Z with respect to the increasing convex order (denoted $V \preceq_{ico} Z$) if relation (4) holds for all $\eta \in \mathbb{R}$.*

Changing signs in (4), we obtain that (4) is equivalent to relation (2) for the variable $-Z$ and $-V$, i.e.,

$$V \preceq_{\text{ico}} Z \quad \Leftrightarrow \quad -V \succeq_{(2)} -Z. \tag{5}$$

2 Inverse Forms of Stochastic Dominance

For a real random variable V (for example, a random return rate), we define the inverse of the cumulative distribution function (the quntile function) $F_1(V;\cdot)$ as follows:

$$F_{(-1)}(V;p) = \inf\{\eta : F_1(V;\eta) \geq p\} \quad \text{for} \quad 0 < p < 1.$$

Given $p \in (0,1)$, the number $q = q(V;p)$ is called a *p-quantile* of the random variable V if

$$\mathbb{P}\{V < q\} \leq p \leq \mathbb{P}\{V \leq q\}.$$

For $p \in (0,1)$ the set of p-quantiles is a closed interval and $F_{(-1)}(V;p)$ represents its left end. Directly from the definition of the first order dominance, we see that

$$V \succeq_{(1)} S \quad \Leftrightarrow \quad F_{(-1)}(V;p) \geq F_{(-1)}(S;p) \quad \text{for all} \quad 0 < p < 1. \tag{6}$$

The first order dominance constraint can be interpreted as a continuum of probabilistic (chance) constraints.

Our analysis uses the *absolute Lorenz function* $F_{(-2)}(V;\cdot) : [0,1] \to \mathbb{R}$. It is defined as the cumulative quantile:

$$F_{(-2)}(V;p) = \int_0^p F_{(-1)}(V;t)\,dt \quad \text{for} \quad 0 < p \le 1, \tag{7}$$

$F_{(-2)}(V;0) = 0$. Similarly to $F_2(V;\cdot)$, the function $F_{(-2)}(V;\cdot)$ is well defined for any random variable V, which has a finite expected value. We notice that

$$F_{(-2)}(V;1) = \int_0^1 F_{(-1)}(V;t)\,dt = \mathbb{E}[V].$$

By construction, the Lorenz function is convex. Lorenz functions are commonly used in economics for income inequality.

It is well known that we may fully characterize the second order dominance relation by using the function $F_{(-2)}(V;\cdot)$:

$$V \succeq_{(2)} S \quad \Leftrightarrow \quad F_{(-2)}(V;p) \ge F_{(-2)}(S;p) \quad \text{for all} \quad 0 \le p \le 1. \tag{8}$$

Thus, we say that portfolio x *dominates* portfolio y *in the first order*, if

$$F_{(-1)}(R(x);p) \ge F_{(-1)}(R(y);p) \quad \text{for all } p \in (0,1).$$

This is illustrated in Figure 3.

Similarly, we say that x *dominates* y *in the second order* ($R(x) \succeq_{(2)} R(y)$), if

$$F_{(-2)}(R(x);p) \ge F_{(-2)}(R(y);p) \quad \text{for all } p \in [0,1]. \tag{9}$$

Recall that the individual return rates R_j have finite expected values and thus the function $F_{(-2)}(R(x);\cdot)$ is well defined. The second order relation is illustrated in Figure 4.

3 Relations to Value at Risk and Average Value at Risk

There are fundamental relations between the concepts of Value at Risk (VaR) and Average Value at Risk (AVaR) and the stochastic dominance constraints. The VaR constraint in the portfolio context is formulated as follows. We define the loss rate $L(x) = -R(x)$. We specify the maximum fraction ω_α of the initial capital allowed for risk exposure at risk level $\alpha \in (0,1)$, and we require that

$$\mathbb{P}\big[L(x) \le \omega_\alpha\big] \ge 1 - \alpha.$$

Denoting by $\mathrm{VaR}_\alpha(L(x))$ the $(1-\alpha)$-quantile of the random variable $L(x)$, we can equivalently formulate the VaR constraint as

$$\mathrm{VaR}_\alpha(L(x)) \le \omega_\alpha.$$

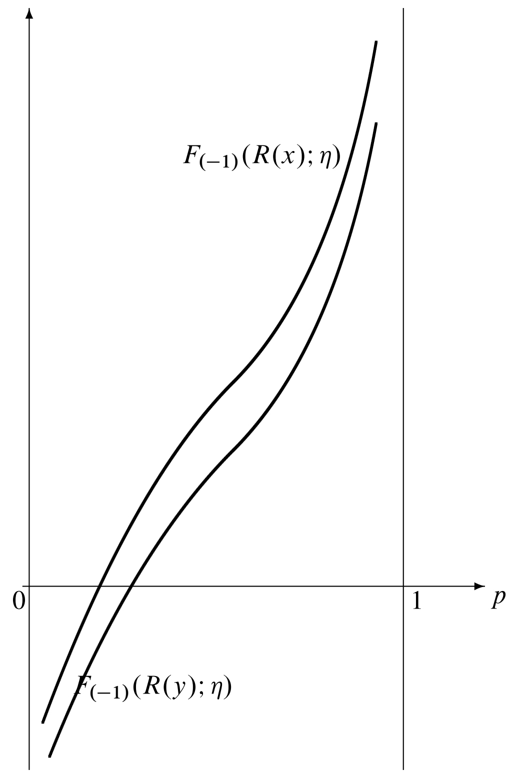

Figure 3: First degree stochastic dominance: $R(x) \succeq_{(1)} R(y)$ in the inverse form.

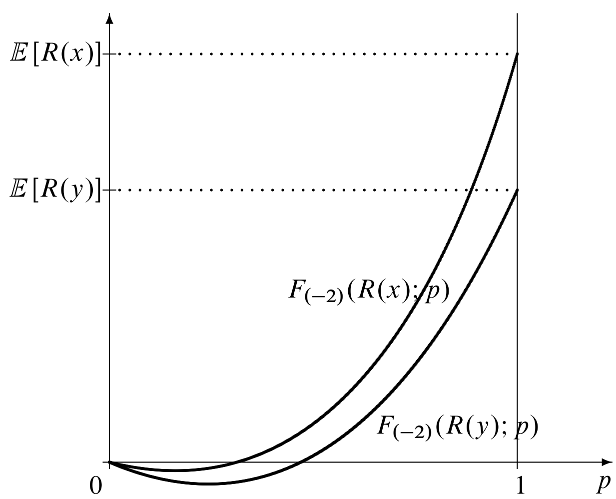

Figure 4: Second order dominance $R(x) \succeq_{(2)} R(y)$ in the inverse form.

The first order stochastic dominance relation between two portfolios is equivalent to the continuum of VaR constraints. Portfolio x dominates portfolio y in the first order, if

$$\mathrm{VaR}_\alpha(L(x)) \le \mathrm{VaR}_\alpha(L(y)) \ \text{ for all } \ \alpha \in (0,1).$$

The AVaR at level α is expressed as follows:

$$\mathrm{AVaR}_\alpha(L(x)) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(L(x))\, d\beta.$$

We note that

$$\mathrm{AVaR}_\alpha(L(x)) = -\frac{1}{\alpha} F_{(-2)}(R(x), \alpha). \tag{10}$$

Using (10) and (9) we conclude that the second order stochastic dominance relation for two portfolios x and y is equivalent to the continuum of AVaR inequalities:

$$R(x) \succeq_{(2)} R(y) \Leftrightarrow \mathrm{AVaR}_\alpha(L(x)) \le \mathrm{AVaR}_\alpha(L(y)) \text{ for all } \alpha \in (0,1]. \tag{11}$$

Assume that we compare the performance of a portfolio x with a random benchmark Y (e.g., an index return rate or another portfolio return rate) requiring $R(x) \succeq_{(2)} Y$. Then the fraction ω_α of the initial capital allowed for risk exposure at level α is given by the benchmark Y:

$$\omega_\alpha = \mathrm{AVaR}_\alpha(-Y), \quad \alpha \in (0,1].$$

Assume that Y has a discrete distribution with realizations y_k, $k = 1, \dots, K$. Then relation (2) is equivalent to

$$\mathbb{E}\big[(y_k - R(x))_+\big] \le \mathbb{E}\big[(y_k - Y)_+\big], \quad k = 1, \dots, K. \tag{12}$$