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Optimization Models in Finance

Solutions to Homework 6

Problem 1

With no loss of generality we assume that $\frac{u_1 c_1}{a_1} \geq \frac{u_2 c_2}{a_2} \geq \cdots \geq \frac{u_n c_n}{a_n}$. The Lagrangian has the form

$$L(x,\mu) = \sum_{j=1}^{n} c_j (x_j)^2 + \mu\Big(1 - \sum_{j=1}^{n} a_j x_j\Big) = \sum_{j=1}^{n} \big(c_j (x_j)^2 - \mu a_j x_j\big) + \mu.$$

Minimizing with respect to each $x_j \in [0, u_j]$ we obtain

$$x_j = \begin{cases} \frac{\mu a_j}{2c_j} & \text{if } 0 \leq \frac{\mu a_j}{2c_j} \leq u_j \\ u_j & \text{if } \frac{\mu a_j}{2c_j} > u_j \\ 0 & \text{if } \mu < 0. \end{cases} \tag{1}$$

Obviously, $\mu > 0$, and the assumed order of the variables implies that there exists k such that $x_j = \frac{\mu a_j}{2c_j}$ for $j = 1, \dots, k$ and $x_j = u_j$ for $j = k+1, \dots, n$. Let us temporarily fix k. We obtain

$$\sum_{j=1}^{n} x_j = \frac{\mu}{2} \sum_{j=1}^{k} \frac{a_j}{c_j} + \sum_{j=k+1}^{n} u_j = 1.$$

It follows that

$$\frac{\mu}{2} = \frac{1 - \sum_{j=k+1}^{n} u_j}{\sum_{j=1}^{k} \frac{a_j}{c_j}}. \tag{2}$$

Comparing to (1) we conclude that k is such that

$$\frac{u_k c_k}{a_k} \geq \frac{1 - \sum_{j=k+1}^{n} u_j}{\sum_{j=1}^{k} \frac{a_j}{c_j}} \geq \frac{u_{k+1} c_{k+1}}{a_{k+1}}. \tag{3}$$

After finding k we calculate μ from (2) and then x from (1).