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Optimization Models in Finance

Solutions to Homework 2

Problem 1.

Suppose $x \in X$ and $y \in X$. This means that

$$\sum_{j=1}^{n} |x_j| \leq 1, \qquad \text{and} \qquad \sum_{j=1}^{n} |y_j| \leq 1.$$

Let $0 \leq \alpha \leq 1$ and consider the point

$$z = \alpha x + (1-\alpha)y.$$

This means that $z_j = \alpha x_j + (1-\alpha)y_j$, $j = 1, \dots, n$. Observe that

$$|z_j| = |\alpha x_j + (1-\alpha)y_j| \leq |\alpha x_j| + |(1-\alpha)y_j| = \alpha|x_j| + (1-\alpha)|y_j|.$$

In the last equation we used the fact that both α and $1-\alpha$ are nonnegative. Adding these inequalities we obtain

$$\sum_{j=1}^{n} |x_j| \leq \sum_{j=1}^{n} \big(\alpha|x_j| + (1-\alpha)|y_j|\big) = \alpha \sum_{j=1}^{n} |x_j| + (1-\alpha) \sum_{j=1}^{n} |y_j| \leq 1.$$

This proves that $z \in X$. The set X is convex.

We shall show that the extreme points of X must be the unit vectors

$$e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e^2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad e^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

and their negatives $-e^1, \dots, -e^n$. Indeed, suppose that e^j, for some j, is not an extreme point. Then $e^j = \frac{1}{2}x + \frac{1}{2}y$ for some $x \in X$ and $y \in X$. In particular, the jth coordinate of e^j, which is equal to 1, must satisfy the equation:

$$1 = e^j_j = \frac{1}{2}x_j + \frac{1}{2}y_j.$$

As $x \in X$, it is necessary that $|x_j| \leq 1$. Similarly, $|y_j| \leq 1$. The only way to satisfy the above equation is to set $x_j = y_j = 1$. But then, to have $x \in X$ and $y \in X$, we must set $x_i = y_i = 0$, for all $i \neq j$. Consequently, e^j is an extreme point. In a similar way we show that $-e^j$ is an extreme point.

Now we have to prove that every extreme point of X must be of the specified form. To prove this, suppose that $z \in X$ is an extreme point of X. Consider the sets of indices:

$$A = \{j : z_j \geq 0\} \qquad \text{and} \qquad B = \{j : z_j < 0\}.$$

We have the equation

$$z = \sum_{j=1}^{n} z_j e^j = \sum_{j \in A} |z_j| e^j + \sum_{j \in B} |z_j|(-e^j)$$

Define $\alpha = 1 - \sum_{j=1}^{n} |z_j|$. Since $z \in X$, we have $\alpha \geq 0$. The last equation can be rewritten as follows:

$$z = \sum_{j \in A} |z_j| e^j + \sum_{j \in B} |z_j|(-e^j) + \alpha \cdot 0.$$

The last equation means that z is a convex combination of the vectors e^j for $j \in A$, $-e^j$ for $j \in B$, and the vector 0. If one of the coefficients of the convex combination is different than 0 or 1, the point z cannot be extreme.

Problem 2.

If x is not an extreme point of X, then $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$, with $x^1 \in X$, $x^2 \in X$, and $x^1 \neq x$, $x^2 \neq x$. If we remove the point x from the set X, the resulting set will not be convex, because x^1 and x^2 will remain in the set, but their convex combination x will not.

On the other hand, if x is extreme, it *cannot* be represented as a convex combination of points of $X \setminus \{x\}$. Therefore every convex combination of points of $X \setminus \{x\}$ will be in $X \setminus \{x\}$. Consequently, the set $X \setminus \{x\}$ is convex.

Problem 3.

(a) Take any x, y and $0 \leq \alpha \leq 1$. We have to verify the inequality

$$e^{\alpha x + (1-\alpha) y} \leq \alpha e^x + (1-\alpha) e^y.$$

Denote $z = \alpha x + (1-\alpha) y$, $d = y - x$. Then $x = z - (1-\alpha) d$, $y = z + \alpha d$. We have to check whether

$$e^z \leq \alpha e^{z-(1-\alpha)d} + (1-\alpha) e^{z+\alpha d}$$

Dividing both sides by e^z, we need to verify the inequality

$$1 \le \alpha e^{-(1-\alpha)d} + (1-\alpha)e^{\alpha d}.$$

From the elementary inequality $e^t \ge 1 + t$, the right hand side can be bounded as follows:

$$\alpha e^{-(1-\alpha)d} + (1-\alpha)e^{\alpha d} \ge \alpha\big[1 - (1-\alpha)d\big] + (1-\alpha)\big[1 + \alpha d\big] = 1,$$

as required.

(b) The function is a sum of convex functions $f_1(x_1) = x_1^2$, $f_2(x_1, x_2) = (x_1 - x_2)^2$, and $f_3(x_2) = x_2$.

Problem 4

(a) $X = \{x \in \mathbb{R}^n : \sum_{j=1}^n x_j \le C,\ x_j \ge 0,\ j = 1, \dots, n\}$. Its extreme points are the vectors

$$e^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e^1 = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e^2 = \begin{bmatrix} 0 \\ C \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad e^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ C \end{bmatrix}.$$

(b) $Y = \{y \in \mathbb{R}^m : y = Ax,\ x \in X\}$

(c) Consider an extreme point $y \in Y$. There exists $x \in X$ such that $y = Ax$. The point x is a convex combination of the extreme points $e^0, e^1, \dots, e^n$,

$$x = \alpha_0 e^0 + \alpha_1 e^1 + \cdots + \alpha_n e^n,$$

with $\alpha_j \ge 0$, $\sum_{j=0}^n \alpha_j = 1$. Therefore

$$y = \alpha_0 Ae^0 + \alpha_1 Ae^1 + \cdots + \alpha_n Ae^n.$$

Consider the set of indices $J = \{j : \alpha_j > 0\}$. We have

$$y = \sum_{j \in J} \alpha_j y^j, \qquad y^j = Ae^j, \qquad \sum_{j \in J} \alpha_j = 1, \qquad \alpha \ge 0,$$

that is, y is a convex combination of points $y^j \in Y$. We see that for y to be extreme it is necessary that all y^j, $j \in J$, be equal. Therefore $y = Ae^j$, for some $j \in J$.

(d) By Minkowski's theorem,

$$Y = \text{conv}\{Ae^0, Ae^1, \dots, Ae^n\},$$

i.e. , every $y \in Y$ is a convex combination of the extreme points Ae^j. By Caratheodory's theorem, y is a convex combination of no more than $m + 1$ of the points Ae^j.