Optimization Models in Finance
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HOMEWORK 5 - SOLUTIONS

Problem 1

(a) $f(x) = \sum_{k=1}^{n} p_k(y_k - x)^2$. The function f is convex and differentiable, and thus the necessary and sufficient condition of optimality is $f'(\hat{x}) = 0$. Differentiating with respect to x we get

$$f'(x) = -2\sum_{k=1}^{n} p_k(y_k - x) = 2\Big(x - \sum_{k=1}^{n} p_k y_k\Big).$$

Therefore

$$\hat{x} = \sum_{k=1}^{n} p_k y_k = \mathbb{E}[Y].$$

(b) $f(x) = \sum_{k=1}^{n} p_k|y_k - x|$. The function is convex and nondifferentiable, but we can look at directional derivatives to identify the optimal point. Suppose, for simplicity, that the realizations of Y are ordered in such a way that

$$y_1 < y_2 < \cdots < y_n.$$

Define $f_k(x) = |y_k - x|$. We have

$$f_k'(x) = \begin{cases} 1 & \text{if } x > y_k \\ -1 & \text{if } x < y_k. \end{cases}$$

Therefore, at the points x which are not identical to any observation:

$$f'(x) = \sum_{k: y_k < x} p_k - \sum_{k: y_k > x} p_k = \mathbb{P}[Y < x] - \mathbb{P}[Y > x].$$

It can be zero only if both probabilities above are equal, that is, if there exists m such that

$$\sum_{k=1}^{m} p_k = \sum_{k=m+1}^{n} p_k = \frac{1}{2}.$$

In this case every point of the interval (y_m, y_{m+1}) is the solution. The ends are also solutions, as the analysis of the second case will show.
If x is equal to some y_m we must have

$$f'(x) \begin{cases} \le 0 & \text{for } x < y_m \\ \ge 0 & \text{for } x > y_m. \end{cases}$$

Therefore

$$\sum_{k=1}^{m} p_k \ge \frac{1}{2}, \qquad \sum_{k=m}^{n} p_k \ge \frac{1}{2}.$$

Thus, y_m is a median of Y.

We see that in both cases the set of solutions of the problem is the set of all medians of Y.

Linear Programming: The equivalent linear programming formulation is:

$$\begin{aligned} \min \ & \sum_{k=1}^{n} p_k v_k \\ \text{s.t. } & v_k \ge y_k - x, \quad k = 1, \dots, n \\ & v_k \ge x - y_k, \quad k = 1, \dots, n. \end{aligned}$$

We rearrange the constraints, to have all variables on the left:

$$\begin{aligned} \min \ & \sum_{k=1}^{n} p_k v_k \\ \text{s.t.} \quad & x + v_k \ge y_k, \quad k = 1, \dots, n \\ & -x + v_k \ge -y_k, \quad k = 1, \dots, n. \end{aligned}$$

We associate multipliers $\lambda_k \ge 0$ with the first group of constraints, and $\mu_k \ge 0$ with the second group. The dual problem has the form:

$$\begin{aligned} \max \ & \sum_{k=1}^{n} \big(y_k \lambda_k - y_k \mu_k \big) \\ \text{s.t. } & \sum_{k=1}^{n} \lambda_k - \mu_k = 0 \qquad \text{(the coefficient in front of } x) \\ & \lambda_k + \mu_k = p_k, \quad k = 1, \dots, m \qquad \text{(the coefficient in front of } v_k) \\ & \lambda \ge 0, \ \mu \ge 0. \end{aligned}$$

The variables in the primal problem are free, and thus the dual constraints are equations. We use the second group of constraints to eliminate the variables μ_k and we obtain the problem:

$$\begin{aligned} &\max 2\sum_{k=1}^{n} y_k\lambda_k \\ &\text{s.t. } 2\sum_{k=1}^{n}\lambda_k = 1, \\ &\quad 0\lambda_k \le p_k, \quad k = 1, \dots, m. \end{aligned}$$

Again, we suppose that $y_1 < y_2 < \cdots < y_n$. It is obvious that the solution of the last problem is to fill the λ_k's to the maximum p_k's starting from the last one (n), because their coefficients y_k are the largest. This can be done until the budget for the sum of the λ_k's, which is 1/2, is exhausted. This implies that m has to found, such that $\sum_{k=m+1}^{n} p_k \le 1/2$ and $\sum_{k=1}^{m} p_k > 1/2$. We obtain

$$\lambda_k = \begin{cases} 0 & \text{for } k = 1, \dots, m-1, \\ p_k & \text{for } k = m+1, \dots, n \\ \frac{1}{2} - \sum_{k=m+1}^{n} p_k & \text{for } k = m. \end{cases}$$

This determines the solution: the constraints $x + v_k \ge y_k$ are certainly active for $k \ge m+1$, which means that $x \le y_{m+1}$. If $\lambda_m > 0$ then also $x \le y_m$. The constraints $-x + v_k \le -y_k$ are active for $k = 1, \dots, m$, because $\mu_k > 0$. Thus, $x \ge y_m$.

(c) Proceeding exactly as in (b) we conclude that $\hat{x}$ is an α-quantile of Y: such a point that

$$\mathbb{P}\{Y \le x\} \ge \alpha, \qquad \mathbb{P}\{Y \ge x\} \ge 1 - \alpha.$$

All these observations extend to general random variables.

Problem 2

The condition that the lines cross at one point can be written as follows:

$$\tan\alpha_2 = \frac{1}{2}\big(\tan(\alpha_1) + \tan(\alpha_3)\big).$$

Using the approximation $\tan(\alpha) \approx \alpha$ for small angles, we get the constraint:

$$h(x) = \frac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2 = 0.$$

The problem is to minimize

$$f(x) = \sum_{i=1}^{3} (\alpha_i - \tilde{\alpha}_i)^2$$

subject to this constraint. Assigning multiplier μ to the constraint, we obtain optimality conditions

$$\begin{bmatrix} 2(\alpha_1 - \tilde{\alpha}_1) \\ 2(\alpha_2 - \tilde{\alpha}_2) \\ 2(\alpha_3 - \tilde{\alpha}_3) \end{bmatrix} + \mu \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This yields

$$\begin{aligned} \alpha_1 &= \tilde{\alpha}_1 - \frac{\mu}{4}, \\ \alpha_2 &= \tilde{\alpha}_2 + \frac{\mu}{2}, \\ \alpha_3 &= \tilde{\alpha}_3 - \frac{\mu}{4}. \end{aligned}$$

They have to satisfy the constraint, and thus

$$\frac{1}{2}(\tilde{\alpha_1} + \tilde{\alpha_3}) - \tilde{\alpha_2} - \frac{3\mu}{4} = 0.$$

The solution is: $\mu = 0.012$, $\alpha_1 = 0.08$, $\alpha_2 = 0.03$, $\alpha_3 = -0.02$. The distance to the ship equals

$$d \approx \frac{500}{\alpha_1 - \alpha_2} = \frac{500}{\alpha_2 - \alpha_3} = 10000.$$

Problem 3

Denote

$$\begin{aligned} f(x) &= (x_1 - 1)^2 + (x_2 + 2)^2 - x_1 x_2, \\ g_1(x) &= x_1 - x_2 - b, \\ g_2(x) &= -x_1. \end{aligned}$$

The problem has the form

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_1(x) \le 0, \\ &\quad\ g_2(x) \le 0. \end{aligned}$$

We introduce Lagrange multipliers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ associated with the constraints, and analyze the necessary conditions of optimality:

$$\begin{aligned} \nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0, \\ \lambda_1 g_1(x) = 0, \\ \lambda_2 g_2(x) = 0. \end{aligned}$$

After substituting the functions and the gradients, we get the system:

$$\begin{aligned} 2x_1 - 2 - x_2 + \lambda_1 - \lambda_2 = 0, \\ 2x_2 + 4 - x_1 + \lambda_1 = 0, \\ \lambda_1(x_1 - x_2 - b) = 0, \\ \lambda_2 x_1 = 0. \end{aligned}$$

We consider 4 cases.
Case 1: $\lambda_1 = 0$, $\lambda_2 = 0$. The first two equation simplify to:

$$\begin{aligned} 2x_1 - x_2 = 2 \\ -x_1 + 2x_2 = -4. \end{aligned}$$

Then $x_1 = 0$, $x_2 = -2$. This solution is valid for $b \geq 2$.
Case 2: $\lambda_1 > 0$, $\lambda_2 = 0$. From the first two equation we obtain:

$$\begin{aligned} 2x_1 - x_2 = 2 - \lambda_1 \\ -x_1 + 2x_2 = -4 - \lambda_1. \end{aligned}$$

The solution is $x_1 = -\lambda_1/3$, $x_2 = -2 + \lambda_1/3$. This solution is impossible, because we must have $x_1 \geq 0$.
Case 3: $\lambda_1 = 0$, $\lambda_2 > 0$. From the last equation we get $x_1 = 0$. From the second equation $x_2 = -2$. The first equation yields $\lambda_2 = -4$. Contradiction.
Case 4: $\lambda_1 > 0$, $\lambda_2 > 0$. The last equation gives $x_1 = 0$. The third equation then yields $x_2 = -b$. Calculating from the first two equations $\lambda_1 = 4 - 2b$ and $\lambda_2 = 2 - b$, we see that this solution is good for $b < 2$, because the multipliers must be positive.