Optimization Models in Finance
(Prof. Andrzej Ruszczyński)

HOMEWORK 5 *(due Tuesday, October 8, 2019)*

Problem 1

We have a discrete real random variable Y with realizations y_k, $k = 1, \dots, n$, attained with probabilities $p_k > 0$, $\sum_{k=1}^{n} p_k = 1$.

(a) Find the minimum of the function $f(x) = \mathbb{E}\big[(Y - x)^2\big]$.
The symbol $\mathbb{E}$ denotes the expected value of a random variable.

(b) Find the minimum of the function $f(x) = \mathbb{E}\big[|Y - x|\big]$.

(c) **(Doctoral Students)** For $\alpha \in (0, 1)$ find the minimum of the function
$f(x) = \mathbb{E}\big[\max\big(\alpha(Y - x), (1 - \alpha)(x - Y)\big)\big]$.

To illustrate your general results, calculate the solutions for the random variable Y attaining values -3, -2, 0, 2, 5, 6, 9, 10, 11, and 20, with probabilities 0.05, 0.1, 0.05, 0.15, 0.15, 0.05, 0.1, 0.2, 0.05, and 0.1, respectively. In (c) consider $\alpha = 0.1$ and $\alpha = 0.3$.

Hint: In cases (b) and (c) the functions are not differentiable. Consider equivalent linear programming formulations of these problems and notice the key feature of the solution.

Problem 2

A ship on sea is observed from three stations located on the coast line 500 meters apart (see the figure on the next page). At each station i the angle α_i between the line to the ship and the normal to the coast line is measured (see Figure 1 on the next page). The observed angles (which are subject to measurement errors) are

$$\tilde{\alpha}_1 = 0.084, \quad \tilde{\alpha}_2 = 0.024, \quad \tilde{\alpha}_3 = -0.018$$

(in radians). Correct these results in a consistent way, so that the lines given by the angles cross at one point, and the sum of the squares of the differences of the errors is minimized. Then determine the location of the ship.

Hint: Use the approximation $\tan\alpha \approx \alpha$ for small angles and express the condition that the lines cross at one point as a linear constraint on the angles.

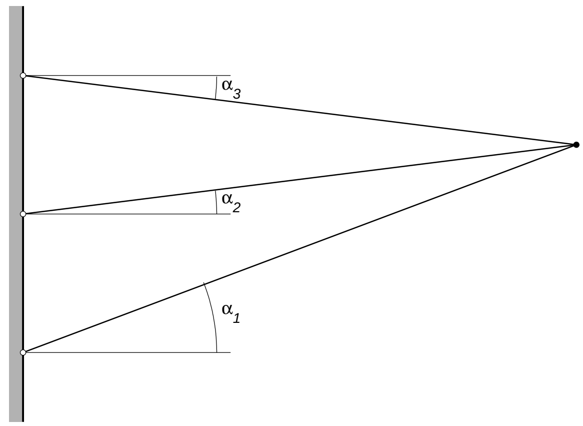

Figure 1: Three stations observe the ship.

Problem 3
Consider the problem:

$$\begin{aligned} \text{min} \quad & (x_1 - 1)^2 + (x_2 + 2)^2 - x_1 x_2 \\ \text{subject to} \quad & x_1 - x_2 \le b, \\ & x_1 \ge 0. \end{aligned}$$

How do the solution $\hat{x}(b)$ and the optimal value $f(\hat{x}(b))$ depend on the parameter b? Calculate the derivative of the objective function,

$$\frac{df(\hat{x}(b))}{db},$$

and relate it to the value of the Lagrange multiplier.