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Linear Programming Duality

The Dual Problem

Consider the linear programming problem

$$\begin{aligned} &\min \langle c, x\rangle \\ &\text{subject to } Ax \geq b, \\ &\quad x \geq 0, \end{aligned} \tag{1}$$

with a matrix A of dimension $m \times n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

We formulate the Lagrangian:

$$L(x, \lambda) = \langle c, x\rangle + \langle \lambda, b - Ax\rangle = \langle c - A^T\lambda, x\rangle + \langle b, \lambda\rangle.$$

The dual function has the form

$$L_D(\lambda) = \langle b, \lambda\rangle + \inf_{x \geq 0} \langle c - A^T\lambda, x\rangle.$$

The infimum above is finite if and only if $c - A^T\lambda \geq 0$. In this case the infimum value of the product $\langle c - A^T\lambda, x\rangle$ over $x \geq 0$ is just zero. We conclude that

$$L_D(\lambda) = \begin{cases} \langle b, \lambda\rangle & \text{if } A^T\lambda \leq c, \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem has the form

$$\max_{\lambda \geq 0} L_D(\lambda)$$

Using the algebraic description of the dual function, we can write the dual problem as a linear programming problem:

$$\begin{aligned} &\max \langle b, \lambda\rangle \\ &\text{subject to } A^T\lambda \leq c, \\ &\quad \lambda \geq 0. \end{aligned} \tag{2}$$

A similar analysis can be carried out for a formulation of a linear programming problem with equality constraints.

It is instructive to consider the dual problem to the dual problem (2). We change the sign of the objective function to obtain a minimization problem and we use $u \in \mathbb{R}^n_+$ to denote the Lagrange multipliers associated with the constraints. We define the Lagrangian:

$$l(\lambda, u) = -\langle b, \lambda \rangle + \langle u, A^T \lambda - c \rangle = \langle Au - b, \lambda \rangle - \langle c, u \rangle.$$

The dual function can be calculated similarly to the previous case:

$$l_D(u) = \inf_{\lambda \geq 0} l(\lambda, u) = \begin{cases} -\langle c, u \rangle & \text{if } Au \geq b, \\ -\infty & \text{otherwise.} \end{cases}$$

We can thus write the dual problem as

$$\begin{aligned} \max \ & -\langle c, u \rangle \\ \text{subject to } & Au \geq b, \\ & u \geq 0. \end{aligned}$$

Changing the sign of the objective function again, we arrive at the primal problem (1).

Primal and dual pairs

Primal problem	*Dual problem*
$\min c^T x$	$\max b^T \lambda$
$Ax \geq b$	$A^T \lambda \leq c$
$x \geq 0$	$\lambda \geq 0$
$\min c^T x$	$\max b^T \lambda$
$Ax = b$	$A^T \lambda \leq c$
$x \geq 0$	λ unconstrained
$\min c^T x$	$\max b^T \lambda$
$Ax \geq b$	$A^T \lambda = c$
x unconstrained	$\lambda \geq 0$

Duality Theorem

The optimal value of the dual problem is always less than or equal to the optimal value of the primal problem. If any of these values is finite then they are equal.

Example: Non-arbitrage and state prices

We have n securities with present prices $c_1, \dots, c_n$. The securities can be either bought or sold *short*, that is, borrowed and sold for cash. Any amounts can be traded.

At some future time one of m states may occur. The price of security j in state i will be equal to a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$. At this time we will liquidate our holdings: the securities held will be sold, and the short positions will be covered by purchasing and returning the amounts borrowed.

An *arbitrage* is the existence of $\bar{x} \in \mathbb{R}^n$ such that $\langle c, \bar{x} \rangle < 0$ and $A\bar{x} \geq 0$. Such a portfolio $\bar{x}$ can be "purchased" with an immediate profit and then liquidated at the future point of time with no additional loss, whatever the state.

Denote by x_j the amount invested in security j. If $x_j < 0$ then short selling takes place. We formulate the problem of maximizing profits as a linear programming problem:

$$\begin{aligned} \min\ & c^T x \\ & Ax \geq 0 \\ & x \text{ unconstrained} \end{aligned}$$

If no arbitrage opportunities exist, the optimal value of this problem is 0.

The dual problem has the form

$$\begin{aligned} \max\ & 0^T \lambda \\ & A^T \lambda = c \\ & \lambda \geq 0 \end{aligned}$$

According to the duality theory, if the optimal value of the primal problem is 0, then the optimal value of the dual problem is 0. If we denote by $\hat{\lambda}$ the solution of the dual problem, we conclude that

$$A^T \hat{\lambda} = c, \qquad \hat{\lambda} \geq 0.$$

The vector $\hat{\lambda}$ is called in finance the vector of *state prices*.

Suppose the first security is cash, that is $c_1 = 1$, $a_{i1} = 1$, $i = 1, \dots, m$. The first equation of the system $A^T \hat{\lambda} = c$ reads

$$\hat{\lambda}_1 + \hat{\lambda}_2 + \dots + \hat{\lambda}_n = 1$$

We can then interpret the multipliers $\hat{\lambda}_i$ as implied probabilities of the states $i = 1, \ldots, m$. We conclude that the vector of current prices c is the expected value of the vectors of the future prices, with respect to these probabilities.

Example: Matrix Games

A two-player game is defined as follows. We have a matrix A of dimension $m \times n$. Player C chooses a column j of the matrix, that is, a number between 1 and n. Player R chooses a row of the matrix, a number i between 1 and m. None of them knows the opponent's decision. Then the decisions are revealed and the matrix entry a_{ij} in row i and column j is the amount that C pays R (if it is negative, R pays C).

It is difficult to decide on the course of action in such a game. The key concept that radically clarifies the problem of choosing the best play is that of a *mixed strategy*. A mixed strategy of Player C is a probability distribution x on the set of columns $\{1, \ldots, n\}$. In other words, we assume that Player C chooses her column at random, according to some probabilities x_j, $j = 1, \ldots, n$. The set of all possible mixed strategies is equal to

$$X_0 = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j = 1,\ x_j \geq 0,\ j = 1, \ldots, n\}.$$

If Player R chooses row i then the expected amount that C pays R equals

$$\sum_{j=1}^{n} a_{ij} x_j = \langle a_i, x \rangle,$$

with a_i denoting the ith row of A. To minimize the worst possible outcome, Player C solves the following problem:

$$\min_{x \in X_0} \max_{1 \leq i \leq m} \langle a_i, x \rangle. \tag{3}$$

We can reformulate it as a linear programming problem:

$$\begin{aligned}
\min_{v,x} \ & v \\
\text{s.t.} \ & \langle a_i, x \rangle - v \leq 0, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} x_j = 1, \\
& x_j \geq 0,\ j = 1, \ldots, n.
\end{aligned}$$

Denote by y the vector of Lagrange multipliers associated with the first group of constraints. We use the notation y instead of λ because of the specific interpretation of the multipliers in our context. The Lagrange multiplier associated with the last constraint is denoted by w.

The Lagrangian has the form

$$\begin{aligned} L(v,x,y,w) &= v + \sum_{i=1}^{m} y_i\big(\langle a_i, x\rangle - v\big) + w\big(1 - \sum_{j=1}^{n} x_j\big) \\ &= v\Big(1 - \sum_{i=1}^{m} y_i\Big) + \langle y, Ax\rangle + w\Big(1 - \sum_{j=1}^{n} x_j\Big). \end{aligned}$$

Let us calculate the dual function:

$$L_D(y,w) = \inf_{\substack{v\in\mathbb{R}\\ x\geq 0}} L(v,x,y,w) = w + \inf_{v\in\mathbb{R}} v\Big(1 - \sum_{i=1}^{m} y_i\Big) + \min_{x\geq 0}\Big(\langle y, Ax\rangle - w\sum_{j=1}^{n} x_j\Big).$$

We see that the infimum with respect to v is finite if and only if $\sum_{i=1}^{n} y_i = 1$. Thus y is an element of the set

$$Y_0 = \{y \in \mathbb{R}^m : \sum_{i=1}^{n} y_i = 1,\ y_i \geq 0,\ i = 1,\dots,m\}.$$

The dual function becomes

$$L_D(y,w) = \min_{x\geq 0}\Big(\langle y, Ax\rangle - w\sum_{j=1}^{n} x_j\Big) = \min_{x\geq 0}\Big(\langle A^T y, x\rangle - w\sum_{j=1}^{n} x_j\Big).$$

If we use a^j to denote the jth column of A, we can rewrite the dual function as follows:

$$L_D(y,w) = w + \min_{x\geq 0}\sum_{j=1}^{n} x_j\big(\langle a^j, y\rangle - w\big).$$

We see that for the finiteness of the dual function it is necessary that $\langle a^j, y\rangle \geq w$ for all j. The dual problem takes on the form

$$\begin{aligned} &\max_{y,w}\ w \\ &\text{s.t } \langle a^j, y\rangle \geq w, \quad j = 1,\dots,n, \\ &\sum_{i=1}^{n} y_i = 1, \\ &y \geq 0. \end{aligned} \tag{4}$$

A comparison with (3) reveals that the dual problem is the problem of finding the best mixed strategy of Player R.

From the duality theory it follows that the optimal values of both problems are identical. Their solutions, $\hat{x}$ and $\hat{y}$, form a saddle point of the payoff function:

$$\langle y, A\hat{x}\rangle \leq \langle \hat{y}, A\hat{x}\rangle \leq \langle \hat{y}, Ax\rangle \quad \text{for all} \quad x \in X_0,\ y \in Y_0.$$

It is the equilibrium of the game: if both players follow the mixed strategies $\hat{x}$ and $\hat{y}$, it is not profitable for any of them to deviate from their solutions.