Sample Final Exam - Solutions

Problem 1

The Lagrangian has the form

$$\begin{aligned} L(x,\lambda) &= \sum_{j=1}^{n} \frac{a_j}{x_j} + \lambda\Big(\sum_{j=1}^{n} x_j - b\Big) \\ &= \sum_{j=1}^{n}\Big(\frac{a_j}{x_j} + \lambda x_j\Big) - \lambda b \end{aligned}$$

The dual function takes on the form:

$$L_D(\lambda) = \min_{0\le x\le u} L(x,\lambda) = -\lambda b + \sum_{j=1}^{n} \min_{0\le x_j\le u_j}\Big(\frac{a_j}{x_j} + \lambda x_j\Big)$$

Consider the function

$$\Phi_j(\lambda) = \min_{0\le x_j\le u_j}\Big(\frac{a_j}{x_j} + \lambda x_j\Big).$$

We obtain

$$x_j(\lambda) = \begin{cases} \sqrt{\frac{a_j}{\lambda}} & \text{if } \frac{a_j}{u_j^2} \le \lambda, \\ u_j & \text{otherwise.} \end{cases}$$

$$\Phi_j(\lambda) = \begin{cases} 2\sqrt{a_j\lambda} & \text{if } \frac{a_j}{u_j^2} \le \lambda, \\ \frac{a_j}{u_j} + \lambda u_j & \text{otherwise} \end{cases}$$

The dual problem is

$$\max_{\lambda\ge 0} -\lambda b + \sum_{j=1}^{n} \Phi_j(\lambda).$$

Problem 2

We denote by r the mean $\mathbb{E}[Z]$, by η the α-quantile (to be found) and by v_j the shortfall below η in scenario j. The primal problem is:

$$
\begin{aligned}
\min_{r,\eta,v} \quad & -\frac{1}{2}r - \frac{1}{2}\eta + \frac{1}{2\alpha}\sum_{j=1}^{n} p_j v_j \\
& r = \sum_{j=1}^{n} p_j z_j \\
& -\eta + v_j \geq -z_j, \quad j = 1, \ldots, n \\
& v_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

We assign multiplier μ to the first constraint, and multipliers $\lambda_j \geq 0$ to the second group of constraints. The dual problem has the form:

$$
\begin{aligned}
\max_{\mu,\lambda} \quad & \mu \sum_{j=1}^{n} p_j z_j - \sum_{j=1}^{n} \lambda_j z_j \\
& \mu = -\frac{1}{2}, \\
& \sum_{j=1}^{n} \lambda_j = \frac{1}{2}, \\
& 0 \leq \lambda_j \leq \frac{p_j}{2\alpha}, \quad j = 1, \ldots, n.
\end{aligned}
$$

Problem 3

(a)

$$
x = \frac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1}\mathbb{1}}.
$$

We first calculate $z = C^{-1}\mathbb{1}$ by solving the system of equations $Cz = \mathbb{1}$, that is

$$
\begin{aligned}
2z_1 + 0z_2 &= 1000 \\
0z_1 + z_2 &= 1000.
\end{aligned}
$$

This gives $z_1 = 500$, $z_2 = 1000$, and thus $x_{\text{MV}} = \left(\frac{1}{3}, \frac{2}{3}\right)$.

(b) The best portfolio with return rate 0.02 is $y = (1, 0)$ (the only one with this return rate).

(c) The set of all efficient portfolios has the form $x = (1-\alpha)x_{\text{MV}} + \alpha y$, that is

$$
\begin{aligned}
x_1 &= \frac{1}{3}(1-\alpha) + \alpha \\
x_2 &= \frac{2}{3}(1-\alpha),
\end{aligned}
$$

where $\alpha \geq 0$. The expected returns move from 0.0133 up.
(d) We calculate

$$\bar{r} = r - r_0\mathbb{1} = \begin{bmatrix} 0.015 \\ 0.005 \end{bmatrix}$$

and $z = C^{-1}\bar{r}$, that is

$$\begin{aligned} 2z_1 + 0z_2 &= 15 \\ 0z_1 + \ \ z_2 &= 5 \end{aligned}$$

This gives $z_1 = 7.5$ and $z_2 = 5$. Consequently, the market portfolio is $x_1^* = \frac{3}{5}$ and $x_2^* = \frac{2}{5}$.

Problem 4

The mean return is $\mu = 100000 * r^T x = 100000(0.02 * 1.5 - 0.01 * 0.5) = 100000 * 0.025 = 2500$.
The standard deviation equals: $\sigma = 100000 * \sqrt{x^T C x} = 6892$.
The return is normal with parameters μ and σ. The value at risk of the return is

$$\text{VaR}_\alpha = -\alpha\text{-quantile} = -(\mu - 1.65\sigma) = 8872.$$

The average value at risk is equal to

$$\begin{aligned} -\text{AVaR}_\alpha = E\big[X \mid X \leq -\text{VaR}_\alpha\big] &= \frac{1}{\alpha\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu - 1.65\sigma} x e^{-(x-\mu)^2/2\sigma^2}\, dx \\ &= \frac{1}{\alpha\sigma\sqrt{2\pi}} \int_{-\infty}^{-1.65\sigma} (y+\mu) e^{-y^2/2\sigma^2}\, dy \qquad \text{(substitute } y = x - \mu\text{)} \\ &= \frac{1}{\alpha\sigma\sqrt{2\pi}} \int_{-\infty}^{-1.65\sigma} \mu e^{-y^2/2\sigma^2}\, dy + \frac{1}{\alpha\sigma\sqrt{2\pi}} \int_{-\infty}^{-1.65\sigma} y e^{-y^2/2\sigma^2}\, dy \\ &= \mu - \frac{1}{\alpha\sigma\sqrt{2\pi}} \int_{1.65\sigma}^{\infty} y e^{-y^2/2\sigma^2}\, dy \\ &= \mu - \frac{\sigma}{\alpha\sqrt{2\pi}} \int_{1.36}^{\infty} e^{-z}\, dz \qquad \text{(substitute } z = y^2/2\sigma^2\text{)} \\ &= \mu - \frac{\sigma e^{-1.36}}{\alpha\sqrt{2\pi}} = -11614 \end{aligned}$$

Thus $\text{AVaR}_\alpha = 11614$.