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Value at Risk

There are n investment opportunities, with random return rates $R_1, \ldots, R_n$ in the next year. We assume that the return rates have a joint normal probability distribution. We have a practically unlimited initial capital and our aim is to invest some of it in such a way that the expected value of our investment after a year is maximized, under the condition that the chance of losing more than some fixed amount $b > 0$ is smaller than α, where $\alpha \in (0, 1)$. Such a requirement is called the *Value at Risk* constraint.

Let $x_1, \ldots, x_n$ be the amounts invested in the n opportunities. The net increase of the value of our investment after a year is random and equals $R(x) = \sum_{i=1}^{n} R_i x_i$. Its expected value is linear in x:

$$\mathbb{E}\big[R(x)\big] = \sum_{i=1}^{n} r_i x_i,$$

with $r_i = \mathbb{E}\big[R_i\big]$. Our problem takes on the form:

$$\begin{aligned} &\max \sum_{i=1}^{n} r_i x_i \\ &\text{subject to } \mathbb{P}\Big\{\sum_{i=1}^{n} R_i x_i \geq -b\Big\} \geq 1-\alpha, \\ &x \geq 0. \end{aligned} \tag{1}$$

We do not impose here any constraint of the form $x_1 + \ldots + x_n = W_0$, where W_0 is the total invested amount. We also assume that $r \neq 0$, otherwise no improvement over $x = 0$ is possible.

Denote by C the covariance matrix of the joint distribution of the return rates. The distribution of the total profit (or loss) is normal, with the expected value $\langle r, x\rangle$, and the variance

$$\text{Var}\big[R(x)\big] = \mathbb{E}\Big(\sum_{i=1}^{n}(R_i - r_i)x_i\Big)^2 = \langle x, Cx\rangle.$$

We assume that C is positive definite. If $x \neq 0$ then the random variable

$$\frac{R(x) - \langle r, x\rangle}{\sqrt{\langle x, Cx\rangle}}$$

has the normal distribution with mean zero and variance one. Our probability constraint is therefore equivalent to the inequality

$$\frac{b + \langle r, x\rangle}{\sqrt{\langle x, Cx\rangle}} \geq z_\alpha,$$

where z_α is the $(1-\alpha)$-quantile of the standard normal random variable. If the risk level $\alpha \leq 1/2$ then $z_\alpha \geq 0$. Therefore the last constraint (after multiplying both sides by $\sqrt{\langle x, Cx\rangle}$) is equivalent to a constraint involving a convex function:

$$z_\alpha \sqrt{\langle x, Cx\rangle} - \langle r, x\rangle \leq b.$$

The convexity follows from the positive definiteness of C. If $x = 0$ the last constraint is satisfied as well, because $b > 0$. Consequently, we obtain the following convex optimization problem equivalent to problem (1):

$$\begin{aligned} &\min \ -\langle r, x\rangle \\ &\text{subject to } z_\alpha \sqrt{\langle x, Cx\rangle} - \langle r, x\rangle - b \leq 0, \\ &x \geq 0. \end{aligned} \tag{2}$$

It satisfies Slater's condition with $x_s = 0$.

To formulate the necessary (and sufficient) conditions of optimality, let $\lambda \geq 0$ be the Lagrange multiplier associated with the constraint. The optimality condition takes on the form

$$-(1 + \lambda)\bar{r} + \frac{\lambda z_\alpha C x}{\sqrt{\langle x, Cx\rangle}} \in -N_{\mathbb{R}^n_+}(\hat{x}).$$

Substituting the explicit form of the normal cone we conclude that

$$\begin{aligned} -(1 + \lambda) r + \frac{\lambda z_\alpha C x}{\sqrt{\langle x, Cx\rangle}} \geq 0, \\ \left\langle x, (1 + \lambda) r - \frac{\lambda z_\alpha C x}{\sqrt{\langle x, Cx\rangle}} \right\rangle = 0. \end{aligned} \tag{3}$$

If we ignore the nonnegativity constraint on x we can solve this system analytically. The optimality condition becomes

$$(1 + \lambda) r - \frac{\lambda z_\alpha C x}{\sqrt{\langle x, Cx\rangle}} = 0.$$

Since r is nonzero, $\lambda > 0$ and thus the inequality constraint must be satisfied as equality. From the last equation we deduce that the vectors Cx and r are colinear: there exists a scalar t such that $Cx = tr$. Substitution to the constraint (which is active) yields

$$t = b/\rho(z_\alpha - \rho), \quad \lambda = (z_\alpha/\rho - 1)^{-1}, \quad \text{with} \quad \rho = \sqrt{\langle r, C^{-1}\bar{r}\rangle}.$$

Note that C^{-1} is positive definite and hence $\langle r, C^{-1}r\rangle$ is positive. If $\rho = z_\alpha$, no solution to the system exists. If $\rho > z_\alpha$, the solution has $\lambda < 0$, and thus no optimal solution of the problem exists. In both cases the problem is unbounded.

If $\rho < z_\alpha$, the vector

$$\hat{x} = \frac{b}{\rho(z_\alpha - \rho)}\, C^{-1} r$$

is the solution to the problem without sign restrictions on x. If, in addition, $C^{-1}r \geq 0$, then the vector $\hat{x}$ solves our original problem.

If $C^{-1}r \ngeq 0$ the sign restrictions on x cannot be ignored. We have to find a subset I of decision variables, such that the problem restricted to this subset (with other variables set to zero), can be solved as above and its solution is nonnegative. Then we need to have

$$C_{(I)}^{-1} r_{(I)} \geq 0,$$

with $C_{(I)}$ denoting the quadratic submatrix of C corresponding to I, and $r_{(I)}$ is the subvector of r with components from I. Moreover, for the remaining variables the first relation of (3) must hold, and this requires tedious testing of many subsets I.

Numerical methods of convex optimization avoid examining all possible subsets of assets.