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Coherent Risk Measures

1 Definition and Properties

Let $(\Omega, \mathcal{F})$ be a sample space, equipped with the sigma algebra $\mathcal{F}$, on which considered uncertain outcomes (random functions $Z = Z(\omega)$) are defined. By a *risk measure* we understand a function $\rho(Z)$ which maps Z into the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. In order to make this concept precise we need to define a space $\mathcal{Z}$ of allowable random functions $Z(\omega)$ for which $\rho(Z)$ is defined. It seems that a natural choice of $\mathcal{Z}$ will be the space of all $\mathcal{F}$-measurable functions $Z : \Omega \to \mathbb{R}$. However, typically, this space is too large for development of a meaningful theory. Unless stated otherwise, we deal in this chapter with spaces $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, where $p \in [1, +\infty)$. By assuming that $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$, we assume that random variable $Z(\omega)$ has a finite p-th order moment with respect to the reference probability measure P. Also by considering function ρ to be defined on the space $\mathcal{L}_p(\Omega, \mathcal{F}, P)$, it is implicitly assumed that actually ρ is defined on classes of functions which can differ on sets of P-measure zero, i.e., $\rho(Z) = \rho(Z')$ if $P\{\omega : Z(\omega) \neq Z'(\omega)\} = 0$.

We assume throughout this chapter that risk measures $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ are *proper*. That is, $\rho(Z) > -\infty$ for all $Z \in \mathcal{Z}$ and the domain

$$\mathrm{dom}(\rho) := \{Z \in \mathcal{Z} : \rho(Z) < +\infty\}$$

is nonempty. We consider the following axioms associated with a risk measure ρ. For $Z, Z' \in \mathcal{Z}$ we denote by $Z \succeq Z'$ the pointwise partial order meaning $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. We also assume that the smaller the realizations of Z, the better; for example Z may represent a random **cost**.

(R1) *Convexity:*

$$\rho(tZ + (1-t)Z') \leq t\rho(Z) + (1-t)\rho(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(R2) *Monotonicity:* If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \geq \rho(Z')$.

(R3) *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(R4) *Positive Homogeneity:* If $t > 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

It is said that a risk measure ρ is *coherent* if it satisfies the above conditions (R1)–(R4). An example of a coherent risk measure is the Average Value-at-Risk $\rho(Z) := \mathsf{AVaR}^+_\alpha(Z)$. It is natural to assume in this example that Z has a finite first order moment, i.e., to use $\mathcal{Z} := \mathcal{L}_1(\Omega, \mathcal{F}, P)$. For such space $\mathcal{Z}$ in this example, $\rho(Z)$ is finite (real valued) for all $Z \in \mathcal{Z}$.

If the random outcome represents a **reward**, i.e., larger realizations of Z are preferred, we can define a risk measure $\varrho(Z) = \rho(-Z)$, where ρ satisfies axioms (R1)–(R4). In this case, the function ϱ also satisfies (R1) and (R4). The axioms (R2) and (R3) change to:

(R2a) *Monotonicity:* If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\varrho(Z) \le \varrho(Z')$.

(R3a) *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\varrho(Z + a) = \varrho(Z) - a$.

All our considerations regarding risk measures satisfying (R1)–(R4) have their obvious counterparts for risk measures satisfying (R1)-(R2a)-(R3a)-(R4).

With each space $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ is associated its dual space $\mathcal{Z}^* := \mathcal{L}_q(\Omega, \mathcal{F}, P)$, where $q \in (1, +\infty]$ is such that $1/p + 1/q = 1$. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ their scalar product is defined as

$$\langle \zeta, Z \rangle := \int_\Omega \zeta(\omega) Z(\omega) dP(\omega). \tag{1}$$

Recall that the conjugate function $\rho^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ of a risk measure ρ is defined as

$$\rho^*(\zeta) := \sup_{Z \in \mathcal{Z}} \big\{ \langle \zeta, Z \rangle - \rho(Z) \big\}, \tag{2}$$

and the conjugate of ρ^* (the biconjugate function) as

$$\rho^{**}(Z) := \sup_{\zeta \in \mathcal{Z}^*} \big\{ \langle \zeta, Z \rangle - \rho^*(\zeta) \big\}. \tag{3}$$

By the Fenchel-Moreau Theorem we have that if $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is convex, proper and lower semicontinuous, then $\rho^{**} = \rho$, i.e., $\rho(\cdot)$ has the representation

$$\rho(Z) = \sup_{\zeta \in \mathcal{Z}^*} \big\{ \langle \zeta, Z \rangle - \rho^*(\zeta) \big\}, \quad \forall Z \in \mathcal{Z}. \tag{4}$$

Conversely, if the representation (4) holds for some proper function $\rho^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$, then ρ is convex, proper and lower semicontinuous. Note that if ρ is convex, proper and lower semicontinuous, then its conjugate function ρ^* is also proper. Clearly, we can write the representation (4) in the following equivalent form

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \big\{ \langle \zeta, Z \rangle - \rho^*(\zeta) \big\}, \quad \forall Z \in \mathcal{Z}, \tag{5}$$

where $\mathfrak{A} := \mathrm{dom}(\rho^*)$ is the domain of the conjugate function ρ^*.

The following basic duality result for convex risk measures is a direct consequence of the Fenchel-Moreau Theorem.

Theorem 1.1 *Suppose that $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is convex, proper and lower semicontinuous. Then the representation* (5) *holds with* $\mathfrak{A} := \mathrm{dom}(\rho^*)$. *Moreover, we have that:* (i) *Condition* (R2) *holds iff every $\zeta \in \mathfrak{A}$ is nonnegative, i.e., $\zeta(\omega) \geq 0$ for a.e. $\omega \in \Omega$;* (ii) *Condition* (R3) *holds iff $\int_\Omega \zeta dP = 1$ for every $\zeta \in \mathfrak{A}$;* (iii) *Condition* (R4) *holds iff $\rho(\cdot)$ is the support function of the set $\mathfrak{A}$, i.e., can be represented in the form*

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle, \quad \forall Z \in \mathcal{Z}. \tag{6}$$

It follows from the above theorem that if ρ is a risk measure satisfying conditions (R1)–(R3) and is proper and lower semicontinuous, then the representation (5) holds with $\mathfrak{A}$ being a subset of the set of probability density functions,

$$\mathfrak{P} := \left\{ \zeta \in \mathcal{Z}^* : \int_\Omega \zeta(\omega) dP(\omega) = 1, \ \zeta \succeq 0 \right\}. \tag{7}$$

If, moreover, ρ is positively homogeneous (i.e., condition (R4) holds), then its conjugate ρ^* is the indicator function of a convex set $\mathfrak{A} \subset \mathcal{Z}^*$, and $\mathfrak{A}$ is equal to the subdifferential $\partial\rho(0)$ of ρ at $0 \in \mathcal{Z}$. Furthermore, $\rho(0) = 0$ and hence by the definition of $\partial\rho(0)$ we have that

$$\mathfrak{A} = \{ \zeta \in \mathfrak{P} : \langle \zeta, Z \rangle \leq \rho(Z), \ \ \forall \, Z \in \mathcal{Z} \}. \tag{8}$$

The set $\mathfrak{A}$ is weakly* closed. Recall that if the space $\mathcal{Z}$, and hence $\mathcal{Z}^*$, is reflexive, then a convex subset of $\mathcal{Z}^*$ is closed in the weak* topology of $\mathcal{Z}^*$ iff it is closed in the strong (norm) topology of $\mathcal{Z}^*$. If ρ is positively homogeneous and continuous, then $\mathfrak{A} = \partial\rho(0)$ is a *bounded* (and weakly* compact) subset of $\mathcal{Z}^*$.

We have that if ρ is a *coherent* risk measure, then the corresponding set $\mathfrak{A}$ is a set of probability density functions. Consequently, for any $\zeta \in \mathfrak{A}$ we can view $\langle \zeta, Z \rangle$ as the expectation $\mathbb{E}_\zeta[Z]$ taken with respect to the probability measure ζdP, defined by the density ζ. Consequently representation (6) can be written in the form

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \mathbb{E}_\zeta[Z], \quad \forall Z \in \mathcal{Z}. \tag{9}$$

Definition of a risk measure ρ depends on a particular choice of the corresponding space $\mathcal{Z}$. In many cases there is a natural choice of $\mathcal{Z}$ which ensures that $\rho(Z)$ is finite valued for all $Z \in \mathcal{Z}$. We have the following result which shows that for real valued convex and monotone risk measures, the assumption of lower semicontinuity in Theorem 1.1 holds automatically.

Proposition 1.2 *Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, with $p \in [1, +\infty]$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a real valued risk measure satisfying conditions* (R1) *and* (R2). *Then ρ is continuous and subdifferentiable on $\mathcal{Z}$.*

Theorem 1.1 together with Proposition 1.2 imply the following basic duality result.

Theorem 1.3 *Let $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$, where $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ with $p \in [1, +\infty)$. Then ρ is a real valued coherent risk measure iff there exists a convex bounded and weakly* closed set $\mathfrak{A} \subset \mathfrak{P}$ such that the representation* (6) *holds.*

Of course, the analysis simplifies considerably if the space Ω is finite, say $\Omega := \{\omega_1, \dots, \omega_K\}$ equipped with sigma algebra of all subsets of Ω and respective (positive) probabilities $p_1, \dots, p_K$. Then every function $Z : \Omega \to \mathbb{R}$ is measurable and the space $\mathcal{Z}$ of all such functions can be identified with $\mathbb{R}^K$ by identifying $Z \in \mathcal{Z}$ with the vector $(Z(\omega_1), \dots, Z(\omega_K)) \in \mathbb{R}^K$. The dual of the space $\mathbb{R}^K$ can be identified with itself by using the standard scalar product in $\mathbb{R}^K$, and the set $\mathfrak{P}$ becomes

$$\mathfrak{P} = \left\{ \zeta \in \mathbb{R}^K : \sum_{k=1}^{K} p_k \zeta_k = 1,\ \zeta \geq 0 \right\}. \tag{10}$$

The above set $\mathfrak{P}$ forms a convex bounded subset of $\mathbb{R}^K$, and hence the set $\mathfrak{A}$ is also bounded.

2 Portfolio Problem

We have n assets with random return rates $R_1, \dots, R_n$. We assume that the n-dimensional random vector $R = \begin{bmatrix} R_1 & R_2 & \dots R_n \end{bmatrix}^T$ has K realizations $R^1, R^2, \dots, R^K$, attained with probabilities $p_1, p_2, \dots, p_K$.

The capital to invest is fixed and equals S. For a portfolio $x = \begin{bmatrix} x_1 & x_2 & \dots x_n \end{bmatrix}^T$, the profit is random and has the form

$$Z(x) = R^T x.$$

Our problem has the form

$$\begin{aligned} &\min \varrho[Z(x)] \\ &\text{ s.t. } x_1 + x_2 + \dots x_n = S, \\ &\qquad x \in X_0, \end{aligned} \tag{11}$$

where X_0 represents additional constraints on the amounts invested (for example, no short-selling, limits on individual investments, etc.). The function $\varrho[\cdot]$ is a coherent measure of risk.

3 Average Value at Risk

Consider the extremal representation of the Average Value at Risk:

$$\text{AV@R}_\alpha^-[Z] = \min_{\eta\in\mathbb{R}} \left\{ -\eta + \frac{1}{\alpha}\mathbb{E}\big[(\eta - Z)_+\big] \right\}. \tag{12}$$

For a random variable Z with has K realizations $z_1, z_2, \ldots, z_K$, attained with probabilities $p_1, p_2, \ldots, p_K$, this representation takes on the form of a linear programming problem:

$$\begin{aligned} \text{AV@R}_\alpha^-[Z] = \min_{\eta,v} & \left\{ -\eta + \frac{1}{\alpha}\sum_{k=1}^K p_k v_k \right\} \\ \text{s.t.} \quad & v_k \geq \eta - z_k, \quad v_k \geq 0, \quad k = 1,2,\ldots,K. \end{aligned}$$

This representation can be integrated to (11) to obtain the problem

$$\begin{aligned} \min_x \text{AV@R}_\alpha^-[Z(x)] = \min_{\eta,v,x} & \left\{ -\eta + \frac{1}{\alpha}\sum_{k=1}^K p_k v_k \right\} \\ \text{s.t.} \quad & v_k \geq \eta - z_k, \quad v_k \geq 0, \quad k = 1,2,\ldots,K \\ & z_k = \sum_{j=1}^n R_j^k x_j \\ & x_1 + x_2 + \ldots x_n = S, \\ & x \in X_0. \end{aligned}$$

In a more general formulation, we may combine $\text{AV@R}_\alpha^-[Z(x)]$ and the expected value to form the risk measure

$$\varrho[Z(x)] = -(1-c)\mathcal{E}[Z(x)] + c\,\text{AV@R}_\alpha^-[Z(x)],$$

where $c \in [0,1]$. The linear programming formulation is similar:

$$\begin{aligned} \min_x \varrho[Z(x)] = \min_{\eta,v,x} & \left\{ -(1-c)\sum_{k=1}^K p_k z_k + c\Big(-\eta + \frac{1}{\alpha}\sum_{k=1}^K p_k v_k \Big) \right\} \\ \text{s.t.} \quad & v_k \geq \eta - z_k, \quad v_k \geq 0, \quad k = 1,2,\ldots,K \\ & z_k = \sum_{j=1}^n R_j^k x_j \\ & x_1 + x_2 + \ldots x_n = S, \\ & x \in X_0. \end{aligned}$$

This problem is usually solved for many values of $c \in [0,1]$ to obtain points on the efficient frontier in the mean–risk plane, where risk is represented by $\delta[Z] = \text{AV@R}_\alpha^-[Z] + \mathcal{E}[Z]$.

4 Mean–Semideviation

Consider the mean–semideviation measure of risk:

$$\varrho[Z] = \mathcal{E}[Z] + c\sigma_1^-[Z] = \mathcal{E}[Z] + c\mathcal{E}\Big\{\big(Z - \mathcal{E}[Z]\big)_+\Big\}, \quad c \in [0, 1]. \tag{13}$$

For a random variable Z with has K realizations $z_1, z_2, \dots, z_K$, attained with probabilities $p_1, p_2, \dots, p_K$, this measure of risk takes on the form of a linear programming problem:

$$\begin{aligned}
\varrho[Z] = \min_{\mu,v} \ & \Big\{ -\mu + c \sum_{k=1}^{K} p_k v_k \Big\} \\
\text{s.t.} \quad & v_k \geq \mu - z_k, \quad v_k \geq 0, \quad k = 1, 2, \dots, K \\
& \mu = \sum_{k=1}^{K} p_k z_k.
\end{aligned}$$

This representation can be integrated to (11) to obtain the problem

$$\begin{aligned}
\min_x \varrho[Z(x)] = \min_{\mu,v,x} \ & \Big\{ -\mu + c \sum_{k=1}^{K} p_k v_k \Big\} \\
\text{s.t.} \quad & v_k \geq \mu - z_k, \quad v_k \geq 0, \quad k = 1, 2, \dots, K \\
& \mu = \sum_{k=1}^{K} p_k z_k \\
& z_k = \sum_{j=1}^{n} R_j^k x_j \\
& x_1 + x_2 + \dots x_n = S, \\
& x \in X_0.
\end{aligned}$$

This problem is usually solved for many values of $c \in [0, 1]$ to obtain points on the efficient frontier in the mean–risk plane, where risk is represented by $\sigma_1^-[Z]$.