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Mean–Variance Portfolio Models

1 The Portfolio Problem

Suppose we have assets $1, 2, \ldots, n$ with random return rates $R_1, R_2, \ldots, R_n$, and a capital K to invest. Denoting by $z_1, z_2, \ldots, z_n$ the amounts invested in the corresponding assets, we can easily compute the random return rate of the portfolio:

$$R(z) = \frac{1}{K}\big[(1 + R_1)z_1 + (1 + R_2)z_2 + \cdots + (1 + R_n)z_n - K\big].$$

Observing that $z_1 + z_2 + \cdots + z_n = K$, we can simplify the above expression as follows:

$$R(z) = \frac{1}{K}\big[R_1 z_1 + R_2 z_2 + \cdots + R_n z_n\big].$$

We conclude that the return rate of the portfolio depends on the *fractions*:

$$x_1 = \frac{z_1}{K}, \quad x_2 = \frac{z_2}{K}, \quad \ldots x_n = \frac{z_n}{K},$$

and can be simply written as follows:

$$R(x) = R_1 x_1 + R_2 x_2 + \cdots + R_n x_n.$$

The fractions satisfy the equation

$$x_1 + x_2 + \cdots + x_n = 1.$$

If we restrict our investment to *long positions*, then $x_1 \geq 0$, $x_2 \geq 0$, …, $x_n \geq 0$, and the portfolio return rate is a convex combination of the return rates of the assets. In the following material we shall consider *both* cases: with investment fractions restricted to be non-negative, and without this restriction.

Our objective is to make the return rate $R(x)$ as large as possible. As $R(x)$ is a random variable, a naive and straightforward approach would be to maximize the expected value $\mathbb{E}\big[R(x)\big]$. Denoting

$$r_j = \mathbb{E}\big[R_j\big], \quad j = 1, 2, \ldots, n,$$

we can express the expected return rate as follows:

$$\mathbb{E}\big[R(x)\big] = \sum_{j=1}^{n} r_j x_j.$$

We obtain the problem (with short-selling forbidden)

$$\begin{aligned} \max \ & \sum_{j=1}^{n} r_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned}$$

It is a linear programming problem, whose solution is obvious: invest only in the asset(s) with the highest expected rate of return. This contradicts the practice of investment, where diversification of investments is a major issue.

The main reason is that the expected value does not reflect the preferences of the investors. We need to introduce an additional measure, which will reflect the uncertainty of the returns. Many possibilities exist to define such measure; in this lecture we focus on a particular measure: the *variance* of $R(x)$, which is commonly used in practice. Consequently, we shall characterize the random return rate $R(x)$ by *two* objectives:

- The *mean* $\mathbb{E}\big[R(x)\big]$; and
- The *risk* $\mathbb{V}\text{ar}\big[R(x)\big]$

Our goal is to have the mean high and the risk low.

Both objectives cannot be optimized simultaneously, and a compromise is necessary. One way to look for such compromise is to *minimize the variance*, while requiring the expected return rate to be equal to some fixed value m. By varying m, we can explore the *efficient frontier*, that is, the set of portfolios, at which simultaneous improvement of both objectives is not possible.

2 The Minimum Variance Portfolio

Let us start from finding the portfolio, at which our second objective, the variance, has the smallest possible value. To this end, we need to calculate the variance of $R(x)$. Let us introduce the covariances of the return rates of the assets:

$$c_{ij} = \text{cov}(R_i, R_j) = \mathbb{E}\big[(R_i - r_i)(R_j - r_j)\big], \quad , ij, j = 1, 2, \ldots, n.$$

They form the *covariance matrix*:

$$C = \begin{bmatrix} c_{11} & c_{11} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

The matrix C is positive semidefinite. It is also convenient to introduce the vectors:

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \quad \text{and} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}.$$

Then

$$C = \mathbb{E}\big[(R-r)(R-r)^T\big],$$

and

$$\mathbb{E}[R(x)] = r^T x.$$

The variance of the portfolio return rate can be calculated as follows:

$$\begin{aligned} \mathbb{V}\text{ar}\big[R(x)\big] &= \mathbb{E}\big[(R(x) - \mathbb{E}[R(x)])^2\big] = \mathbb{E}\big[(R^T x - \mathbb{E}[R^T x])^2\big] = \mathbb{E}\big[(R^T x - r^T x)^2\big] \\ &= \mathbb{E}\big[((R-r)^T x)^2\big] = \mathbb{E}\big[x^T (R-r)(R-r)^T x\big] \\ &= x^T \mathbb{E}\big[(R-r)(R-r)^T\big] x = x^T C x. \end{aligned}$$

Introducing the vector

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

we can formulate the problem of finding the minimum variance portfolio as follows:

$$\begin{aligned} &\min\ x^T C x \\ &\text{s.t.}\ \mathbb{1}^T x = 1. \end{aligned}$$

We allow short positions.

The problem has a closed-form solution. Introducing the Lagrange multiplier μ associated with the constraint, we write the optimality condition:

$$2Cx - \mu \mathbb{1} = 0.$$

If the matrix C is nonsingular, we obtain the solution:

$$x = \frac{\mu}{2} C^{-1} \mathbb{1}.$$

The normalization condition $\mathbb{1}^T x = 1$ yields the minimum variance portfolio

$$x_{\text{MV}} = \frac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1} \mathbb{1}}. \tag{1}$$

If no short-selling condition $x \geq 0$ is included, the solution cannot be found in closed form; numerical methods of optimization are needed.

3 Risky Assets. The Two-Fund Theorem

Denote by m_0 the expected return rate of the minimum variance portfolio:

$$m_0 = r^T x_{\text{MV}}.$$

If we choose $m > m_0$, it will not be possible to achieve the return rate $\mathbb{E}\big[R(x)\big] = m$ with the variance equal to the variance of the minimum-variance portfolio. To find the smallest variance possible in this case, we formulate the following problem:

$$\begin{aligned} &\min\ x^T C x \\ &\text{s.t. } r^T x = m, \\ &\quad\ \ \mathbb{1}^T x = 1. \end{aligned}$$

We allow short positions.

We associate multipliers λ and μ with the constraints, and formulate the optimality conditions

$$2Cx - \lambda r - \mu \mathbb{1} = 0.$$

Thus

$$x = \frac{1}{2} C^{-1}(\lambda r + \mu \mathbb{1}). \tag{2}$$

After substitution to the constraints we obtain the system of two linear equations with two unknowns:

$$\begin{aligned} r^T C^{-1} r \lambda + r^T C^{-1} \mathbb{1} \mu &= 2m, \\ \mathbb{1}^T C^{-1} r \lambda + \mathbb{1}^T C^{-1} \mathbb{1} \mu &= 2. \end{aligned} \tag{3}$$

If the vector r is not a multiple of the vector $\mathbb{1}$, that is, if the expected return rates of the assets are not all equal, then the matrix A of this system,

$$A = \begin{bmatrix} r^T C^{-1} r & r^T C^{-1} \mathbb{1} \\ \mathbb{1}^T C^{-1} r & \mathbb{1}^T C^{-1} \mathbb{1} \end{bmatrix},$$

is nonsingular. It is obviously symmetric. The solution of the system is

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = A^{-1} \begin{bmatrix} 2m \\ 2 \end{bmatrix}.$$

It follows that both λ and μ are affine functions of m. Thus, the formula (2) defines x as an affine function of m. Clearly, for $m = m_0$ we shall have $x = x_{\text{MV}}$. It follows that the solution of the problem has the following form

$$x(m) = x_{\text{MV}} + d(m - m_0), \tag{4}$$

where the vector d can be calculated from the solution of the system (3). Formula (4) defines a straight line in the x-space. Every efficient portfolio is located on this line, and is thus a linear combination of just two different portfolios on this line. One of them may be chosen to be the minimum-variance portfolio x_{MV}, the other may be the solution of the problem for an arbitrary $m = m_1 > m_0$. We conclude that every efficient x can be written as

$$x = (1 - \alpha)x_{\text{MV}} + \alpha x(m_1)$$

with some $\alpha \in \mathbb{R}$. This property is called the *Two-Fund Theorem*.

4 Introduction of a Riskless Asset. The One-Fund Theorem

The condition that the matrix C was nonsingular implied that all returns have non-zero variances, i.e., all assets in our problem were *risky* and their returns were linearly independent (as random variables). Suppose we have one additional *riskless* asset, with the return rate r_0. Let the fractions invested in the assets be $x_0, x_1, \dots, x_n$, with the index 0 representing the riskless asset. If $x_0 \neq 1$, we may consider the implied portfolio y of risky assets only, with

$$y_j = \frac{x_j}{1 - x_0}, \quad j = 1, 2, \dots, n.$$

The return rate of the portfolio x can be written as follows:

$$R(x) = r_0 x_0 + R_1 x_1 + \cdots + R_n x_n = x_0 r_0 + (1 - x_0) R(y).$$

Then the expected return rate of the portfolio x has the form

$$\mu(x) = \mathbb{E}[R(x)] = x_0 r_0 + (1 - x_0)\mathbb{E}\big[R(y)\big] = x_0 r_0 + (1 - x_0) r^T y.$$

The variance the return rate $R(x)$ is the variance of the random part $(1 - x_0)R(y)$ and equals

$$\sigma^2(x) = \mathbb{V}\text{ar}[R(x)] = (1 - x_0)^2 \mathbb{V}\text{ar}[R(y)] = (1 - x_0)^2 \sigma^2(y).$$

It follows that

$$\begin{bmatrix} \sigma(x) \\ \mu(x) \end{bmatrix} = x_0 \begin{bmatrix} 0 \\ r_0 \end{bmatrix} + (1 - x_0) \begin{bmatrix} \sigma(y) \\ \mu(y) \end{bmatrix}.$$

By varying x_0 we can associate with a fixed risky portfolio y the entire family of portfolios x. The standard deviation $\sigma(x)$ and the mean $\mu(x)$ depend linearly on x_0. In order for these portfolios to be efficient, the slope of this line in the standard deviation – mean graph, must be the highest. The slope can be calculated as follows

$$\tan(\theta) = \frac{\mu(y) - r_0}{\sigma(y)}. \tag{5}$$

Our objective is to find a risky portfolio y such that this slope is maximized.

It is convenient to consider the vector $\bar{r}$ of excess return rates, with

$$\bar{r}_j = r_j - r_0, \quad j = 1, 2, \ldots, n.$$

As y is a portfolio, $\mathbb{1}^T y = 1$, and the numerator of (5) can be written as follows:

$$\mu(y) - r_0 = r^T y - r_0 = r^T y - r_0 \mathbb{1}^T y = (r - r_0 \mathbb{1})^T y = \bar{r}^T y.$$

The denominator of (5) is equal to

$$\sigma(y) = \sqrt{\mathbb{V}\text{ar}[R(y)]} = \sqrt{y^T C y}.$$

We obtain the problem

$$\max \frac{\bar{r}^T y}{\sqrt{y^T C y}}$$
$$\text{s.t. } \mathbb{1}^T y = 1.$$

Observe that the objective function has identical values for all y on a line $y = t\hat{y}$, $t > 0$. Thus, we may ignore the constraint, find the unconstrained maximum of the objective function, and normalize the result to represent a portfolio.

The rest is a simple calculation. Equating the gradient of the objective function to zero, we obtain the equation

$$\frac{\bar{r}\sqrt{y^T C y} - \bar{r}^T y \frac{C y}{\sqrt{y^T C y}}}{y^T C y} = 0.$$

It follows that

$$\frac{y^T C y}{\bar{r}^T y} \bar{r} = C y. \tag{6}$$

It makes sense only to consider portfolios y with higher expected return rate than r_0, and thus $\bar{r}^T y > 0$. Denote

$$\alpha = \frac{y^T C y}{\bar{r}^T y}.$$

It follows from (6) that

$$y = \alpha C^{-1} \bar{r}.$$

The value of α can be now calculated from the normalization condition $\mathbb{1}^T y = 1$. We obtain

$$\alpha = \frac{1}{\mathbb{1}^T C^{-1} \bar{r}}.$$

This yields the final result:

$$\hat{y} = \frac{C^{-1}\bar{r}}{\mathbb{1}^T C^{-1}\bar{r}}.$$

It is the maximizer of the slope, provided $\mathbb{1}^T C^{-1}\bar{r} > 0$. The last condition can be easily deciphered, by relating it to the minimum variance portfolio (1):

$$\mathbb{1}^T C^{-1}\bar{r} = \bar{r}^T C^{-1}\mathbb{1} = \mathbb{1}^T C^{-1}\mathbb{1}\bar{r}^T x_{\mathrm{MV}}.$$

The solution is correct, if the minimum variance portfolio has the expected return rate exceeding the riskless rate r_0.

Summing up, all efficient portfolios can be obtained by combining the best risky portfolio $\hat{y}$ and the riskless asset. This is the *One-Fund Theorem*; the risky portfolio $\hat{y}$ is called the *master fund*.