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Optimality Conditions in Nonlinear Optimization

Let us consider the nonlinear optimization problem

$$\begin{aligned} \min\ & f(x) \\ & g_i(x) \le 0, \quad i = 1, \dots, m, \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{aligned} \tag{1}$$

We assume that the functions $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, p$, are continuously differentiable. We call such a nonlinear optimization problem *smooth*. The feasible set of this problem is denoted X.

Let $I^0(\hat{x})$ be the set of *active* inequality constraints at $\hat{x}$:

$$I^0(\hat{x}) = \{i : g_i(\hat{x}) = 0\}.$$

We say that problem (1) satisfies at $\hat{x}$ the *Mangasarian–Fromovitz condition* if

- the gradients $\nabla h_i(\hat{x})$, $i = 1, \dots, p$, are linearly independent; and
- there exists a direction d such that

$$\begin{aligned} \nabla g_i(\hat{x}), d\rangle &< 0 \quad i \in I^0(\hat{x}), \\ \nabla h_i(\hat{x}), d\rangle &= 0, \quad i = 1, \dots, p. \end{aligned}$$

Problem (1) is said to satisfy *Slater's condition*, if the functions g_i, $i = 1, \dots, m$ are convex, the functions h_i, $i = 1, \dots, p$ are affine, and there exists a feasible point x_{s} such that $g_i(x_{\mathrm{s}}) < 0$, $i = 1, \dots, m$.

In what follows we shall say that problem (1) *satisfies the constraint qualification condition*, if either the Mangasarian–Fromovitz or Slater's condition holds true.

Theorem

Let $\hat{x}$ be a local minimum of problem (1), where the functions f, g_i and h_i are continuously differentiable. Assume that at $\hat{x}$ the constraint qualification condition is satisfied. Then there exist multipliers $\hat{\lambda}_i \ge 0$, $i = 1, \dots, m$, and $\hat{\mu}_i \in \mathbb{R}$, $i = 1, \dots, p$, such that

$$\nabla f(\hat{x}) + \sum_{i=1}^{m} \hat{\lambda}_i \nabla g_i(\hat{x}) + \sum_{i=1}^{p} \hat{\mu}_i \nabla h_i(\hat{x}) = 0, \tag{2}$$

and

$$\hat{\lambda}_i g_i(\hat{x}) = 0, \quad i = 1, \dots, m. \tag{3}$$

Conversely, if the functions f and g_i are convex, and the functions h_i are affine, then every point $\hat{x} \in X$ satisfying these conditions for some $\lambda \ge 0$ and $\mu \in \mathbb{R}^p$ is an optimal solution of (1).

1 Example

Consider the following problem

$$\begin{aligned} \min \quad & (x_1-2)^2+(x_2-1)^2 \\ \text{subject to} \quad & x_1+x_2 \le 2. \end{aligned}$$

In this problem we have

$$\begin{aligned} f(x_1,x_2) &= (x_1-2)^2+(x_2-1)^2 \\ g(x_1,x_2) &= x_1+x_2-2. \end{aligned}$$

We calculate the gradients

$$\nabla f(x) = \begin{bmatrix} 2x_1-4 \\ 2x_2-2 \end{bmatrix}, \qquad \nabla g(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Conditions (2)–(3) take on the form:

$$\begin{bmatrix} 2x_1-4 \\ 2x_2-2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4}$$

$$\lambda(x_1+x_2-2) = 0. \tag{5}$$

Case 1: $\lambda = 0$.
It follows from (4) that $x_1 = 2$, $x_2 = 1$. This solution is not feasible and thus this case is not valid.
Case 2: $\lambda > 0$.
We solve (4) for x_1 and x_2 and get

$$x_1 = 2 - \frac{\lambda}{2}, \qquad x_2 = 1 - \frac{\lambda}{2}.$$

It follows from (5) that $x_1 + x_2 = 2$, and thus $\lambda = 0.5$. We conclude that $x_1 = 1.5$ and $x_2 = 0.5$.

This is a global minimum because both $f(\cdot)$ and $g(\cdot)$ are convex.