Sample Midterm Exam - Solutions

Problem 1

Solution. The set is the intersection of five level sets:

$$\begin{aligned} M_1 &= \{x : g_1(x) \le 6\}, \\ M_2 &= \{x : g_2(x) \le 4\}, \\ M_3 &= \{x : g_3(x) \le 0\}, \\ M_4 &= \{x : g_4(x) \le 0\}, \\ M_5 &= \{x : g_5(x) \le 0\}. \end{aligned}$$

with $g_1(x) = (x_1)^2 + 2(x_2)^2 + (x_3)^2 + x_1x_2 - x_1x_3 + 2x_2x_3$, $g_2(x) = 3x_1 + (x_2)^2 - x_3$, $g_3(x) = -x_1$, $g_4(x) = -x_2$, $g_5(x) = -x_3$.

We verify that the function $g_1(x) = (x_1)^2 + 2(x_2)^2 + (x_3)^2 + x_1x_2 - x_1x_3 + 2x_2x_3$ is convex. Its Hessian equals

$$\nabla^2 g_1(x) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 6 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Indeed, its principal minors are positive:

$$2 > 0, \qquad \begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} = 10 > 0, \qquad \begin{vmatrix} 2 & 1 & -1 \\ 1 & 6 & 2 \\ -1 & 2 & 2 \end{vmatrix} = 4 > 0.$$

The function $g_2(x) = 3x_1 + (x_2)^2 - x_3$ is a sum of convex function, and is, therefore, convex. The other three functions: $g_3(x) = -x_1$, $g_4(x) = -x_2$, $g_5(x) = -x_3$ are linear. Each set M_i is convex, as a level set of a convex function. Their intersection is convex as well.

Problem 2

Primal Problem a

We introduce the following variables:
x_1 - number of shares of XYZ to buy,
x_2 - number of call options to buy,
x_3 - number of call options to sell,
x_4 - number of bonds to buy.

It is also convenient to introduce the auxiliary variables:
v_1 - the value of the portfolio in scenario 1,
v_2 - the value of the portfolio in scenario 2,
v_3 - the value of the portfolio in scenario 3.

The option has value \$ 500 in scenario 1, \$ 2500 in scenario 2, and \$ 0 in scenario 3.

Constraints:

$$
\begin{aligned}
&20x_1 + 1000x_2 - 1000x_3 + 90x_4 \leq 20000 \quad \text{(budget)}\\
&x_2 \leq 50 \qquad \text{(limit on the number of options)}\\
&x_3 \leq 50 \qquad \text{(limit on the number of options)}\\
&-20x_1 - 500x_2 + 500x_3 - 100x_4 + v_1 = 0 \qquad \text{(value in scenario 1)}\\
&-40x_1 - 2500x_2 + 2500x_3 - 100x_4 + v_2 = 0 \qquad \text{(value in scenario 2)}\\
&-12x_1 - 0x_2 + 0x_3 - 100x_4 + v_3 = 0 \qquad \text{(value in scenario 3)}\\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.
\end{aligned}
$$

Objective function:

$$\max\ \frac{1}{3}\big(v_1 + v_2 + v_3\big).$$

Dual Problem a

We introduce *dual variables* associated with the constraints, in the order, in which the constraints appear in the primal problem. The variables associated with the "$\leq$" constraints are nonnegative, the variables associated with the "$=$" constraints are not restricted in sign:
$\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$,
μ_1 - unrestricted in sign, μ_2 - unrestricted in sign, μ_3 - unrestricted in sign.

The *objective function* coefficients are the right hand sides of the constraints:

$$\min 2000 * \lambda_1 + 50 * \lambda_2 + 50 * \lambda_3 + 0 * \mu_1 + 0 * \mu_2 + 0 * \mu_3.$$

The dual constraints are formulated by transposing the original constraint matrix. The first four constraints are inequalities "$\leq$", because the corresponding variables x_1, x_2, x_3, x_4 are required to be nonnegative. The right hand sides are 0, because x_1, x_2, x_3, x_4 do not occur in the objective function:

$$\begin{aligned}
20 * \lambda_1 + 0 * \lambda_2 + 0 * \lambda_3 - 20 * \mu_1 - 40 * \mu_2 - 12 * \mu_3 &\geq 0, \\
1000 * \lambda_1 + 1 * \lambda_2 + 0 * \lambda_3 - 500 * \mu_1 - 2500 * \mu_2 - 0 * \mu_3 &\geq 0, \\
-1000 * \lambda_1 + 0 * \lambda_2 + 1 * \lambda_3 + 500 * \mu_1 + 2500 * \mu_2 + 0 * \mu_3 &\geq 0, \\
90 * \lambda_1 + 0 * \lambda_2 + 0 * \lambda_3 - 100 * \mu_1 - 100 * \mu_2 - 100 * \mu_3 &\geq 0.
\end{aligned}$$

The remaining three constraints are equations, because v_1, v_2, v_3 are not restricted to be nonnegative. Their right hand sides are all 1/3 - the coefficients of the objective function:

$$\begin{aligned}
\mu_1 &= 1/3 \\
\mu_2 &= 1/3 \\
\mu_3 &= 1/3
\end{aligned}$$

These constraints are very simple, because μ_1 occurs only in the fourth row of the primal problem, etc. We can simplify the dual problem by plugging the values of the μs to the other constraints, and obtain a problem in λs only.

All different ways to formulate the dual problem lead to identical formulations (after eliminating some variables).

Primal Problem b

We introduce the following variables:
x_1 - number of shares of XYZ to buy,
x_2 - number of call options to buy,
x_3 - number of call options to sell,
x_4 - number of bonds to buy.

It is also convenient to introduce the auxiliary variables:
v_1 - the value of the portfolio in scenario 1,
v_2 - the value of the portfolio in scenario 2,
v_3 - the value of the portfolio in scenario 3.

The option has value \$ 500 in scenario 1, \$ 2500 in scenario 2, and \$ 0 in scenario 3.

Constraints:

$$
\begin{aligned}
&20x_1 + 1000x_2 - 1000x_3 + 90x_4 \leq 20000 \quad \text{(budget)}\\
&x_2 \leq 50 \qquad \text{(limit on the number of options)}\\
&x_3 \leq 50 \qquad \text{(limit on the number of options)}\\
&-20x_1 - 500x_2 + 500x_3 - 100x_4 + v_1 = 0 \qquad \text{(value in scenario 1)}\\
&-40x_1 - 2500x_2 + 2500x_3 - 100x_4 + v_2 = 0 \qquad \text{(value in scenario 2)}\\
&-12x_1 - 0x_2 + 0x_3 - 100x_4 + v_3 = 0 \qquad \text{(value in scenario 3)}\\
&-v_1 \leq -22000 \qquad \text{(values in scenarios ...)}\\
&-v_2 \leq -22000 \qquad \text{(... must be ...)}\\
&-v_3 \leq -22000 \qquad \text{(... at least 22000)}\\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.
\end{aligned}
$$

Objective function:

$$\max\ \frac{1}{3}\big(v_1 + v_2 + v_3\big).$$

Dual Problem b

Similarly to 1a we introduce *dual variables* associated with the constraints:
$\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$,
μ_1 - unrestricted in sign, μ_2 - unrestricted in sign, μ_3 - unrestricted in sign,
and three new variables for the new constraints:
$\psi_1 \geq 0$, $\psi_2 \geq 0$, $\psi_3 \geq 0$.

The *objective function* coefficients are the right hand sides of the constraints:

$$\min \; 2000 * \lambda_1 + 50 * \lambda_2 + 50 * \lambda_3 - 2200(\psi_1 + \psi_2 + \psi_3).$$

The dual constraints are formed similarly to case 1a, but with three new dual variables. As they are associated with the last three constraints featuring only v_1, v_2, v_3, they do not appear in the first 4 dual constraints:

$$\begin{aligned}
20 * \lambda_1 + 0 * \lambda_2 + 0 * \lambda_3 - 20 * \mu_1 - 40 * \mu_2 - 12 * \mu_3 &\geq 0, \\
1000 * \lambda_1 + 1 * \lambda_2 + 0 * \lambda_3 - 500 * \mu_1 - 2500 * \mu_2 - 0 * \mu_3 &\geq 0, \\
-1000 * \lambda_1 + 0 * \lambda_2 + 1 * \lambda_3 + 500 * \mu_1 + 2500 * \mu_2 + 0 * \mu_3 &\geq 0, \\
90 * \lambda_1 + 0 * \lambda_2 + 0 * \lambda_3 - 100 * \mu_1 - 100 * \mu_2 - 100 * \mu_3 &\geq 0.
\end{aligned}$$

The ψ's appear only here:

$$\begin{aligned}
\mu_1 - \psi_1 &= 1/3 \\
\mu_2 - \psi_1 &= 1/3 \\
\mu_3 - \psi_1 &= 1/3
\end{aligned}$$

We could also write these constraints in the "$\leq$" form, because we now know that v_1, v_2, v_3 are nonnegative.

Problem 3

Solution. We define the functions:

$$\begin{aligned} f(x) &= (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 - x_1x_2 - x_1x_3 + 2x_2x_3 \\ g_1(x) &= -x_1, \\ g_2(x) &= -x_2, \\ g_3(x) &= -x_3, \\ h(x) &= 1 - x_1 - x_2 - x_3. \end{aligned}$$

The optimality conditions have the form:

$$\begin{bmatrix} 2x_1 - x_2 - x_3 \\ 4x_2 - x_1 + 2x_3 \\ 6x_3 - x_1 + 2x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\lambda_1 x_1 = 0, \qquad \lambda_2 x_2 = 0, \qquad \lambda_3 x_3 = 0,$$

$$\lambda_1 \geq 0, \qquad \lambda_2 \geq 0, \qquad \lambda_3 \geq 0.$$

We consider 8 cases:
Case 1: $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$
Case 2: $\lambda_1 > 0,\ \lambda_2 = 0,\ \lambda_3 = 0$
Case 3: $\lambda_1 = 0,\ \lambda_2 > 0,\ \lambda_3 = 0$
Case 4: $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 > 0$
Case 5: $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 = 0$
Case 6: $\lambda_1 > 0,\ \lambda_2 = 0,\ \lambda_3 > 0$
Case 7: $\lambda_1 = 0,\ \lambda_2 > 0,\ \lambda_3 > 0$
Case 8: $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0.$

Out of these, Case 1 is the only correct one; the other lead to contradiction. The solution is $x_1 = 0.590909745$, $x_2 = 0.272727519$, $x_3 = 0.136363736$.