Optimization Models in Finance (26:711:564)
(Prof. Andrzej Ruszczyński)

HOMEWORK 2 *(due Tuesday, September 17, 2019)*

Problem 1

Consider the set $\{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j| \leq 1\}$. Prove that it is convex. Find its extreme points. *Hint:* Consider the case of $n = 2$ first.

Problem 2

Prove that x is an extreme point of a convex set X if and only if $X \setminus \{x\}$ (the set X with the point x removed) is convex.

Problem 3

(a) Verify directly from the definition that the function of one variable $f(x) = e^x$ is convex.

(b) Prove that the function of two variables:

$$f(x_1,x_2) = x_1^2 + 3x_2^2 - 3x_1x_2 + 2x_1$$

is convex.

Problem 4 (required for PhD students, extra credit for other)

There are n mutual funds and m asset categories. Let a_{ij} be the fraction of the capital of fund j invested in category i. We have initial capital C and we want to invest all or part of it in these funds. We denote by x_j the amount invested in fund $j = 1,\dots,n$. No short selling is allowed, so x_j has to be nonnegative.

(a) Describe the set $X \subset \mathbb{R}^n$ of all possible amounts invested in these funds (fund portfolios). What are its extreme points?

(b) Describe the set $Y \subset \mathbb{R}^m$ of all possible amounts invested in this way in the m asset categories (asset portfolios).

(c) Show that if a point y is an extreme point of Y, it has the form $y = Ax$, where x is an extreme point of X.

(d) Suppose $y \in Y$ is an asset portfolio obtained by investing in some of the available funds. Prove that you can construct it by investing in no more than $m+1$ funds.