Random Walks and Brownian Motion

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Random Walk

For $p = q = \frac{1}{2}$ define the outcomes of a the tosses of a coin as

$$X_j = \begin{cases} 1, & \text{if } \omega_j = H \\ -1, & \text{if } \omega_j = T \end{cases}$$

Define $M_0 = 0$ and

$$M_k = \sum_{j=1}^{k} X_j, k = 1, 2, \dots$$

The process $M_k, k = 0, 1, 2, \dots$ is a symmetric random walk. [1]

Independent Increments

A random walk (both symmetric and asymmetric) has independent increments.
An increment is defined as:

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

In addition, for a symmetric random walk, each increment has

$$\mathbb{E}[M_{k_{i+1}} - M_{k_i}] = 0$$

$$\mathbb{V}(M_{k_{i+1}} - M_{k_i}) = \sum_{j=k_i+1}^{k_{i+1}} \mathbb{V}(X_j) = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i$$

Martingale Property for Symmetric Random Walk

The argument that a symmetric random walk is as follows. Given nonnegative integers $k < l$, we have:

$$
\begin{aligned}
\mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[(M_l - M_k) + M_k|\mathcal{F}_k] \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k] \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k \\
&= \mathbb{E}[M_l - M_k] + M_k \\
&= M_k
\end{aligned}
$$

Therefore the symmetric random walk is a martingale.[2]

First Order Variation

The first order variation is determined as:

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

If the function $f(t)$ is a continuous differentiable function, then we can express the first order variation as:

$$FV_T(f) = \int_0^T |f'(t)| dt$$

Quadratic Variation

Definition 3.4.1: Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time T is

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$
[2]

Quadratic Variation for Random Walk

Let the function $f(t)$ simply be the random walk. This results in the formula for the quadratic variation of a random walk as being:

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k$$

This works for both symmetric and asymmetric random walks.

Scaled Symmetric Random Walk

For a fixed integer n, the scaled symmetric random walk is defined as

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

where nt is an integer. If nt is not an integer use linear interpolation to define the value.

SSRW cont.

For $0 = t_0 < t_1 < \cdots < t_m$ where each nt_j is an integer then the scaled symmetric random walk increments are independent. This can be seen as:

$$\begin{aligned} W^{(n)}(t_{j+1}) - W^{(n)}(t_j) &= \frac{1}{\sqrt{n}} M_{nt_{j+1}} - \frac{1}{\sqrt{n}} M_{nt_j} \\ &= \frac{1}{\sqrt{n}} (M_{nt_{j+1}} - M_{nt_j}) \\ &= \frac{1}{\sqrt{n}} \sum_{i=nt_j+1}^{nt_{j+1}} X_i \end{aligned}$$

SSRW cont.

In addition, if $0 \le s < t$ such that ns and nt are integers, we have

$$\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] = \mathbb{E}\left[\frac{1}{\sqrt{n}}\sum_{i=ns+1}^{nt} X_i\right]$$

$$= \frac{1}{\sqrt{n}}\sum_{i=ns+1}^{nt} \mathbb{E}[X_i] = 0$$

$$\mathbb{V}(W^{(n)}(t) - W^{(n)}(s)) = \mathbb{V}\left(\frac{1}{\sqrt{n}}\sum_{i=ns+1}^{nt} X_i\right)$$

$$= \frac{1}{n}\sum_{i=ns+1}^{nt} \mathbb{V}(X_i)$$

$$= t - s$$

SSRW cont.

The scaled symmetric random walk is a martingale as can be seen by looking at, for $0 \leq s < t$:

$$\begin{aligned}
\mathbb{E}[W^{(n)}(t)|\mathcal{F}(s)] &= \mathbb{E}[(W^{(n)}(t) - W^{(n)}(s)) + W^{(n)}(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W^{(n)}(t) - W^{(n)}(s)|\mathcal{F}(s)] + \mathbb{E}[W^{(n)}(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] + W^{(n)}(s) \\
&= W^{(n)}(s)
\end{aligned}$$

And the quadratic variation is:

$$\begin{aligned}
[W^{(n)}, W^{(n)}](t) &= \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right) \right]^2 \\
&= \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j \right]^2 = t
\end{aligned}$$

[2]

Normal Distribution

A normally distributed random variable is defined by the parameters μ, its mean, and σ^2, its variance. If X is a normally distributed random variable, it is denoted as $X \sim N(\mu, \sigma^2)$. The distribution of this normal random variable is given as:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

In addition it has the properties:

$$\mathbb{E}[X] = \mu$$

$$\mathbb{V}(X) = \sigma^2$$

and

$$\varphi_X(t) = \mathbb{E}[e^{xt}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Central Limit Theorem

Theorem 3.2.1: (Central Limit) Fix $t \geq 0$. As $n \to \infty$, the distribution of the symmetric scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.
Proof done using limits of moment generating functions.[2]

Brownian Motion

Definition 3.3.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous fuction $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t), t \geq 0$, is a Brownian Motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$

$$\mathbb{V}(W(t_{i+1}) - W(t_i)) = t_{i+1} - t_i$$

[2]

Quadratic Variation and Brownian Motion

Theorem 3.4.3: Let W be a Browninan motion. Then $[W, W](T) = T$ for all $T \geq 0$ almost surely.
The proof of this theorem involves showing convergence in $\mathcal{L}^2$ or mean square convergence. If there is mean square convergence, then there exists a subsequence that converges almost surely.
In addition, a consequence of this theorem is:

$$dW(t) \cdot dW(t) = dt$$

Cross Variation

If we determine the cross variation between Brownian motion and time, we see that

$$[W(t), t](t) = [t, W(t)](t) = 0$$

In addition, because time is a continuous differentiable process, we have the quadratic variation of time

$$[t, t](t) = 0$$

A result of this is that:

$$dW(t) \cdot dt = dt \cdot dW(t) = (dt)^2 = 0$$

Joint Normal Distribution

The joint normal distribution (or multivariate normal) of k different normal random variables $X_1, X_2, \ldots, X_k$ is denoted $N_k(\mu, \Sigma)$, where $\mu = (\mathbb{E}[X_1], \ldots, \mathbb{E}[X_k])$ and Σ is the covariance matrix (positive definite) with entries $a_{i,j} = Cov(X_i, X_j)$. This density is given as:

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} e^{-\frac{1}{2}\left((\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right)}$$

where $|\Sigma|$ is the determinant of the covariance matrix.

Because the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and normally distributed, the variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed. To determine the covariance matrix, we look at two Brownian motions at times $0 \leq s < t$:

$$\begin{aligned}
Cov(W(t), W(s)) &= \mathbb{E}[W(t)W(s)] - \mathbb{E}[W(t)]\mathbb{E}[W(s)] \\
&= \mathbb{E}[W(s)^2 + W(t)W(s) - W(s)^2] \\
&= \mathbb{E}[W(s)^2] + \mathbb{E}[W(s)(W(t) - W(s))] \\
&= \mathbb{V}(W(s)) + \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] \\
&= s
\end{aligned}$$

Alternate Characterization of Brownian Motion

Theorem 3.3.2: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. The following three properties are equivalent.

1. Definition 3.3.1
2. For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed with means equal to 0 and covariance matrix (Σ):

$$\Sigma = \begin{bmatrix} t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & \ldots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \ldots & t_m \end{bmatrix}$$

3. For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ have the joint moment-generating function

$$\varphi_{W(t_1),\ldots,W(t_m)}(u_1, u_2, \ldots, u_m) = \mathbb{E}[e^{u_m W(t_m) + u_{m-1} W(t_{m-1}) + \cdots + u_1 W(t_1)}]$$

[2]

Filtration for Brownian Motion

Definition 3.3.3: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t), t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t), t \geq 0$, satisfying:

1. For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.
2. For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable.
3. For $0 \leq t < u$,the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.

Let $\Delta(t), t \geq 0$, be a stochastic process. We say that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geq 0$ the random variable $\Delta(t)$ is $\mathcal{F}(t)$-measurable.[2]

Martingale Property of Brownian Motion

Theorem 3.3.4: Brownian motion is a martingale
Proof: Let $0 \leq s \leq t$ be given.

$$\begin{aligned}\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s)) + W(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s)] + W(s) \\ &= W(s)\end{aligned}$$

[2]

Theorem 3.2.2: As $n \to \infty$, the distribution of

$$S_n(t) = S(0)\left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+M_{nt})}\left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-M_{nt})}$$

converges to the distribution of

$$S(t) = S(0)e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$$

where $W(t)$ is a normal random variable with mean zero and variance t.[2]

Geometric Brownian Motion

For α and $\sigma > 0$ as constants, we define **geometric Brownian motion** as:

$$S(t) = S(0)e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

For a partition Π as typically defined, the **log returns** of the process is:

$$\log\left(\frac{S(t_{j+1})}{S(t_j)}\right) = \left(\alpha - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))$$

Convergence in Distribution

A sequence of random variables,$\{X_n\} = X_1, X_2, \ldots,$ is said to **converge in distribution** to a random variable Y if

$$\lim_{n \to \infty} F_{X_n}(x) = F_Y(x)$$

for every number $x \in \mathbb{R}$ where F is continuous and where the F functions are the cumulative distribution functions of the random variables.
We will denote this convergence as

$$X_n \xrightarrow{d} Y$$

Convergence in Probability

A sequence of random variables, $\{X_n\} = X_1, X_2, \ldots,$ is said to **converge in probability** to a random variable Y if $\forall \epsilon > 0$

$$\lim_{n \to \infty} P(|X_n - Y| \geq \epsilon) = 0$$

This is denoted:

$$X_n \xrightarrow{p} Y$$

Convergence in probability implies convergence in distribution, but not vice versa.

Almost Sure Convergence

A sequence of random variables,$\{X_n\}$, is said to **converge almost surely** to a random variable Y if

$$P\left(\lim_{n\to\infty} X_n = Y\right) = 1$$

This is denoted:

$$X_n \stackrel{a.s.}{\to} Y$$

We can denote this in the terms of our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as:

$$\mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = Y(\omega)\right) = 1$$

Almost sure convergence implies convergence in probability

Convergence in Mean

For a real number $r \geq 1$, the sequence $\{X_n\}$ **converges in the** r^{th} **mean** to the random variable Y if the r^{th} absolute moments $(\mathbb{E}[|X_n|^r], \mathbb{E}[|Y|^r])$ exist and

$$\lim_{n \to \infty} \mathbb{E}[|X_n - Y|^r] = 0$$

This is denoted:

$$X_n \xrightarrow{\mathcal{L}^r} Y$$

If $r = 1$ we say that X_n converges in mean to Y, and if $r = 2$ we say converges in mean square. Convergence in mean implies convergence in probability.

Almost Everywhere Convergence

Definition 1.4.3: Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued, Borel-measurable functions defined on $\mathbb{R}$. Let f be another real-valued, Borel-measurable function defined on $\mathbb{R}$. We say that $f_1, f_2, \ldots$ converges to f almost everywhere and write

$$\lim_{n\to\infty} f_n = f \text{ almost everywhere}$$

if the set of $x \in \mathbb{R}$ for which the sequence of numbers $f_1(x), f_2(x), \ldots$ does not have limit $f(x)$ is a set with Lebesgue measure zero.[2]

Monotone Convergence

Theorem 1.4.5: Let $\{X_n\}$ be a sequence of random variables converging almost surely to another random variable X. If

$$0 \leq X_1 \leq X_2 \leq X_3 \leq \dots \text{ almost surely,}$$

then

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$$

Let $f_1, f_2, \dots$ be a sequence of Borel-measurable function on $\mathbb{R}$ converging almost everywhere to a function f. If

$$0 \leq f_1 \leq f_2 \leq f_3 \leq \dots \text{ almost everywhere,}$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

[2]

Dominated Convergence

Theorem 1.4.9: Let $\{X_n\}$ be a sequence of random variables converging almost surely to a random variable X. If there is another variable Y such that $\mathbb{E}[Y] < \infty$ and $|X_n| \leq Y$ almost surely for every n, then

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$$

Let $f_1, f_2, \ldots$ be a sequence of Borel-measurable functions on $\mathbb{R}$ converging almost everywhere to a function f. If there is another function g such that $\int_{-\infty}^{\infty} g(x)dx < \infty$ and $|f_n| \leq g$ almost everywhere for every n, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

[2]

[1] S.E. Shreve. *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Number v. 1 in Springer Finance.

[2] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.