Taylor series @ a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1! = 1} + \frac{f''(a)(x-a)^2}{2! = 2} + \frac{f'''(a)(x-a)^3}{3! = 6} + \frac{f^{iv}(a)(x-a)^4}{4! = 24} + \dots$$

let $x = x_{j+1}$ $a = x_j$

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2 + \frac{1}{6} f'''(x_j)(x_{j+1} - x)^3 + \dots$$

for $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$

$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} \left[f(W(t_{j+1})) - f(W(t_j)) \right]$$

$\Pi = \{0, 1, 2, T\}$

$$= \cancel{f(W(1))} - f(W(0))$$
$$+ \cancel{f(W(2))} - \cancel{f(W(1))}$$
$$+ f(W(T)) - \cancel{f(W(2))}$$

let $x_j = W(t_j)$ & $x_{j+1} = W(t_{j+1})$

$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} \left(f(W(t_{j+1})) - f(W(t_j)) \right)$$

$$= \sum_{j=0}^{n-1} f'(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))}_{dW(t)}$$

$$+ \frac{1}{2} \sum_{j=0}^{n-1} f''(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))^2}_{(dW(t))^2}$$

$$+ \frac{1}{6} \sum_{j=0}^{n-1} f'''(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))^3}_{(dW(t))^3}$$

$$+ \dots$$

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(u))\, dW(u)$$
$$+ \frac{1}{2} \int_0^T f''(W(u)) (dW(u))^2$$
$$\cancel{+ \frac{1}{6} \int_0^T f'''(W(u)) (dW(u))^3}$$
$$\cancel{+ \dots}$$

$$(dW(t))^2 = dt$$

$$(dW(t))^3 = dW(t)(dW(t))^2 = dW(t)\,dt = 0$$

2-D Taylon : $f(t,x)$

$$f(t_{j+1},x_{j+1}) = f(t_j,x_j) + f_t(t_j,x_j)(t_{j+1}-t_j)$$
$$+ f_x(t_j,x_j)(x_{j+1}-x_j) + \tfrac{1}{2} f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2$$
$$+ f_{xt}(t_j,x_j)(t_{j+1}-t_j)(x_{j+1}-x_j)$$
$$+ \tfrac{1}{2} f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2 + \tfrac{1}{6} \ldots$$

$$f(t_{j+1},W(t_{j+1})) - f(t_j,W(t_j))$$
$$= f_t(t_j,W(t_j))(t_{j+1}-t_j) + f_x(t_j,W(t_{j}))(W(t_{j+1})-W(t_j))$$
$$+ \tfrac{1}{2} f_{tt}(t_j,W(t_j))(t_{j+1}-t_j)^2 + \ldots$$
$$+ \tfrac{1}{6} \text{\_\_\_\_}$$

$$f(a,b) = a^2\cos(b)$$
$$f_a(a,b) = 2a\cos(b)$$
$$f_b(a,b) = -a^2\sin(b)$$
$$f_{bb}(a,b) = -a^2\cos(b)$$

$$f(t,3t) = t^2\cos(3t)$$
$$f_a(t,3t) = 2t\cos(3t)$$

$$f(T,W(T)) - f(0,W(0)) = \sum_{j=0}^{n-1} f_t(t_j,W(t_j))\underbrace{(t_{j+1}-t_j)}_{dt}$$
$$+ \sum_{j=0}^{n-1} f_x(t_j,W(t_j))\underbrace{(W(t_{j+1})-W(t_j))}_{dW(t)}$$
$$+ \tfrac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j,W(t_j))\underbrace{(t_{j+1}-t_j)^2}_{(dt)^2=0}$$
$$+ \sum_{j=0}^{n-1} f_{tx}(t_j,W(t_j))\underbrace{(t_{j+1}-t_j)}_{dt}\underbrace{(W(t_{j+1})-W(t_j))}_{dW(t)} = 0$$
$$+ \tfrac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j,W(t_j))\underbrace{(W(t_{j+1})-W(t_j))^2}_{(dW(t))^2=dt}$$
$$+ \tfrac{1}{6}\ldots = 0$$

$$f(T,W(T)) - f(0,W(0)) = \int_0^T f_t(u,W(u))\,du$$
$$+ \int_0^T f_x(u,W(u))\,dW(u) + \tfrac{1}{2}\int_0^T f_{xx}(u,W(u))\,du$$

$$\int_0^T W(u)\,dW(u) = \int_0^T f_x(u,W(u))\,dW(u)$$

$$\Rightarrow f_x(u,W(u)) = W(u)$$

$$\boxed{\begin{aligned} &f_x(t,x) = x \\ &f_{xx}(t,x) = 1 \\ &f(t,x) = \tfrac{x^2}{2} + g(t) + C \\ &f_t(t,x) = g'(t) \end{aligned}}$$

$$\frac{W^2(T)}{2} + g(T) + \cancel{C} - \underbrace{\frac{W^2(0)}{2}}_{=0} - g(0) - \cancel{C}$$
$$= \int_0^T g'(u)\,du + \int_0^T W(u)\,dW(u)$$
$$+ \tfrac{1}{2}\int_0^T du$$

$$\frac{W^2(T)}{2} + \underline{g(T)-g(0)} = \underline{g(T)-g(0)} + \int_0^T W(u)\,dW(u) + T/2$$

$$\int_0^T W(u)\,dW(u) = \frac{W^2(T)}{2} - T/\ldots$$

$$\int_0^T u^2 \cos(W(u))\,dW(u)$$

$f_x(t,x) = t^2 \cos x$

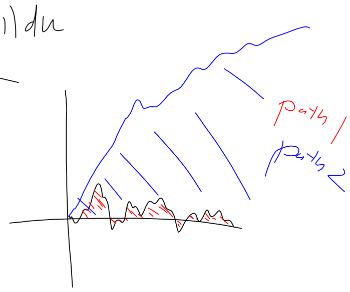
$f_{xx}(t,x) = -t^2 \sin x$

$f(t,x) = t^2 \sin x$

$f_t(t,x) = 2t\sin x$

$$= T^2 \sin(W(T)) - \int_0^T 2u \sin(W(u))\,du + \frac{1}{2}\int_0^T u^2 \sin(W(u))\,du$$

$$\int_0^T W(u)\,du$$



$$\int_0^T u\,du = \frac{T^2}{2}$$

$$S(t) = S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$dS(t) = ?$ $\Rightarrow$ $f(t, W(t)) = S(t)$?

$$f(t,W(t)) - f(0,W(0)) = \int_0^t f_t(u,W(u))\,du + \int_0^t f_x(u,W(u))\,dW(u) + \frac{1}{2}\int_0^t f_{xx}(u,W(u))\,du$$

$$df(t,W(t)) = f_t(t,W(t))dt + f_x(t,W(t))dW(t) + \frac{1}{2} f_{xx}(t,W(t))\, d[W,W](t)$$

$$S(t) = f(t,W(t)) = S(0)e^{(\alpha-\sigma^2/2)t + \sigma W(t)}$$

$f(t,x) = S(0)e^{(\alpha-\sigma^2/2)t + \sigma x}$

$f_t(t,x) = (\alpha - \sigma^2/2)\, f(t,x)$

$f_x(t,x) = \sigma f(t,x)$

$f_{xx}(t,x) = \sigma^2 f(t,x)$

$$dS(t) = (\alpha - \sigma^2/2)\,S(t)dt + \sigma S(t)\,dW(t) + \frac{1}{2}\sigma^2 S(t)\,(dW(t))^2$$

$$= \alpha S(t)dt + \sigma S(t)dW(t)$$

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma dW(t)$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)\,dW(t)$$

$$S(t) = S(0) + \int_0^t \alpha S(u)\,du + \int_0^t \sigma S(u)\,dW(u)$$

$$[X,X](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(X(t_{j+1}) - X(t_j) \right)^2$$

$$X(t_{j+1}) - X(t_j) = X(0) + \int_0^{t_{j+1}} \Delta(u)\,dW(u) + \int_0^{t_{j+1}} \Theta(u)\,du$$

$$- X(0) - \int_0^{t_j} \Delta(u)\,dW(u) - \int_0^{t_j} \theta(u)\,du$$

$$= \int_{t_j}^{t_{j+1}} \Delta(u)\,dW(u) + \int_{t_j}^{t_{j+1}} \Theta(u)\,du$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \Big(\Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big) + \theta(t_j)(t_{j+1} - t_j) \Big)^2$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \Big[\Delta^2(t_j) \underbrace{\big(W(t_{j+1}) - W(t_j)\big)^2}_{(dW(t))^2} + 2\Delta(t_j)\theta(t_j) \underbrace{(t_{j+1} - t_j)\big(W(t_{j+1}) - W(t_j)\big)}_{dt\, dW(t)}$$

$$+ \theta^2(t_j) \underbrace{(t_{j+1} - t_j)^2}_{(dt)^2} \Big]$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \Delta^2(t_j) \big(W(t_{j+1}) - W(t_j) \big)^2$$

$$= \int_0^t \Delta^2(u)\,du$$

$$d[X,X](t) = (dX(t))^2 = \Delta^2(t)\,dt$$

$f(t_{j+1}, x_{j+1}) = \_ \quad \_\_ \quad \_\_\_$

$dX(t) = \Delta(t)\,dW(t) + \theta(t)\,dt$

for $x_{j+1} = X(t_{j+1})$

$$f(T, X(T)) - f(0, X(0)) = \sum_{j=0}^{n-1} f_t(t_j, X(t_j))\underbrace{(t_{j+1} - t_j)}_{dt}$$

$$+ \sum_{j=0}^{n-1} f_x(t_j, X(t_j))\underbrace{(X(t_{j+1}) - X(t_j))}_{dX(t)}$$

$$+ \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j, X(t_j))\underbrace{(t_{j+1} - t_j)^2}_{dt^2 = 0}$$

$$+ \sum_{j=0}^{n-1} f_{tx}(t_j, X(t_j))\underbrace{(t_{j+1} - t_j)}_{dt}\underbrace{(X(t_{j+1}) - X(t_j))}_{dX(t)} = 0$$

$$+ \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j, X(t_j))\underbrace{(X(t_{j+1}) - X(t_j))^2}_{(dX(t))^2}$$

$$+ \frac{1}{6} \_\_\_\_\_\_\_ (dX(t))^3 \text{ for example}$$

$$dt\, dX(t) = dt\,(\Delta(t)\,dW(t) + \theta(t)\,dt)$$
$$= 0$$

$$(dX(t))^2 = (\Delta(t)\,dW(t) + \theta(t)\,dt)^2$$
$$= \Delta^2(t)\,dt$$

$$(dX(t))^3 = \Delta^2(t)\,dt\,(\Delta(t)\,dW(t) + \theta(t)\,dt) = 0$$

Is BM an Itô process?

$X(t) = W(t)$

$X(0) = 0$

$\theta(u) = 0$

$\Delta(u) = 1$

$$X(t) = X(0) + \int_0^t \Delta(u)\,dW(u) + \int_0^t \theta(u)\,du$$
$$= \int_0^t dW(u) = W(t)$$

$$S(t) = S(0) e^{\int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du + \int_0^t \sigma(u)\, dW(u)}$$

for $\alpha(u)$ & $\sigma(u)$ being adapted processes

Generalized GBM

$$\cancel{S(t) = f(t, W(t))}$$

$$X(t) = \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du + \int_0^t \sigma(u)\, dW(u)$$
$$+ \log S(0)$$

$$dX(t) = \left(\alpha(t) - \frac{\sigma^2(t)}{2}\right) dt + \sigma(t)\, dW(t)$$

$$(dX(t))^2 = \sigma^2(t)\, dt$$

$$S(t) = e^{X(t)} = f(t, X(t))$$

$$f(t,x) = e^x$$
$$f_t(t,x) = 0$$
$$f_x(t,x) = e^x$$
$$f_{xx}(t,x) = e^x$$

$$dS(t) = S(t)\, dX(t) + \frac{1}{2} S(t)\, d[X,X](t)$$

$$= S(t)\left(\alpha(t) - \frac{\sigma^2(t)}{2}\right) dt + S(t)\sigma(t)\, dW(t)$$
$$+ \frac{1}{2} S(t) \sigma^2(t)\, dt$$

$$= \alpha(t) S(t)\, dt + \sigma(t) S(t)\, dW(t)$$

$$\frac{dS(t)}{S(t)} = \alpha(t)\, dt + \sigma(t)\, dW(t)$$

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) + \sigma e^{-\beta t}X(t)$$

$$X(t) = \int_0^t e^{\beta s}\,dW(s)$$

$$dX(t) = e^{\beta t}\,dW(t)$$

$$\left(dX(t)\right)^2 = e^{2\beta t}\,dt$$

$$R(t) = g(t, X(t))$$

$$g(a,w) = e^{-\beta a}R(0) + \frac{\alpha}{\beta}\left(1-e^{-\beta a}\right) + \sigma e^{-\beta a}w$$

$$g_a = -\beta e^{-\beta a}R(0) + \frac{\alpha}{\beta}\left(\beta e^{-\beta a}\right) + -\beta\sigma e^{-\beta a}w$$

$$g_w = \sigma e^{-\beta a}$$

$$g_{ww} = 0$$

$$dR(t) = \left(-\beta e^{-\beta t}R(0) + \alpha e^{-\beta t} + -\beta\sigma e^{-\beta t}X(t)\right)dt$$

$$+ \sigma e^{-\beta t}\,dX(t) + \frac{1}{2}(0)$$

$$= -\beta\left(e^{-\beta t}R(0) - \frac{\alpha}{\beta}e^{-\beta t} + \sigma e^{-\beta t}X(t)\right)dt$$

$$+ \sigma e^{-\beta t}e^{\beta t}\,dW(t)$$

$$= -\beta\left(R(t) - \frac{\alpha}{\beta}\right)dt + \sigma\,dW(t)$$

$$= \left(\alpha - \beta R(t)\right)dt + \sigma\,dW(t)$$

$$E[R(t)] = E\left[e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s)\right]$$

$$= e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) + \sigma e^{-\beta t}E\left[\int_0^t e^{\beta s}\,dW(s)\right]$$

$$= e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)$$

$$\lim_{t\to\infty} E[R(t)] = \alpha/\beta$$

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}\,dW(t)$$

$$R(t) = R(0) + \int_0^t (\alpha - \beta R(u))du + \int_0^t \sigma\sqrt{R(u)}\,dW(u)$$

$$X(t) = e^{\beta t}R(t) = f(t, R(t)) \Rightarrow \quad f(t,x) = e^{\beta t}x$$

$$f_t = \beta e^{\beta t}x$$

$$f_x = e^{\beta t}$$

$$f_{xx} = 0$$

$$dX(t) = \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t)$$

$$= \underline{\beta e^{\beta t}R(t)dt} + e^{\beta t}(\alpha \underline{- \beta R(t)})dt + e^{\beta t}\sigma\sqrt{R(t)}\,dW(t)$$

$$= \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R(t)}\,dW(t)$$

$$d(e^{\beta t}R(t)) = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R(t)}\,dW(t)$$

$$e^{\beta t}R(t) - R(0) = \int_0^t \alpha e^{\beta u}du + \int_0^t \sigma e^{\beta u}\sqrt{R(u)}\,dW(u)$$

$$R(t) = e^{-\beta t}R(0) + e^{-\beta t}\int_0^t \alpha e^{\beta u}du + e^{-\beta t}\int_0^t \sigma e^{\beta u}\sqrt{R(u)}\,dW(u)$$

$$E[R(t)] = e^{-\beta t}R(0) + e^{-\beta t}\left[\frac{\alpha}{\beta}e^{\beta u}\Big|_0^t\right]$$

$$= e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

$$Var(R(t)) = E[R^2(t)] - E[R(t)]^2$$