Midterm

Friday, October 20, 2017 4:51 PM

1. For a positive constant r we have the process $X(t)$ defined as:

$$X(t) = \cos(\pi r^3) + W^2(t) + \int_0^t W(u)du$$

(a) Find $[X, X](t)$

$$(a)\quad dX(t) = W(t)dt + 2W(t)dW(t) + 1$$

$$d[X,X](t) = W(t)dt + 4W(t)dt + 2W(t)dW(t) + W(t)dt + 2W(t)dW(t)$$

$$= 6W(t)dt + 4W(t)dW(t)$$

$$\int d[X,X](t) = [X,X](t) - [X,X](0) = \int_0^t 6W(u)\,du + \int_0^t 4W(u)dW(u)$$

$$\therefore [X,X](t) = \int_0^t 6W(u)\,du + \int_0^t 4W(u)dW(u)$$

Correct:

$$f(t, W(t)) = W^2(t),\ f_t = 0,\ f_w = 2W(t),\ f_{ww} = 2$$

$$W^2(t) = W^2(0) + \int_0^t 2W(u)\,dW(u) + \frac{1}{2}\int_0^t 2\,dt$$

$$= 2\int_0^t W(u)\,dW(u) + t$$

$$X(t) = \cos(\pi r^3) + \int_0^t 2W(u)\,dW(u) + \int_0^t W(u)+1\,du$$

So, $X(t)$ is an Ito process

$$[X,X](t) = \int_0^t (2W(u))^2 du = \int_0^t 4W^2(u)du$$

(b) Find $\mathbb{E}[X, X](t)$

$$E[X,X](t) = E\left[\int_0^t 4W^2(u)\,du\right]$$

$$= \int_0^4 4E[W^2(u)]\,du$$

$$= \int_0^t 4u\,du$$

$$= 2u^2\Big|_0^t$$

$$= 2u^2 \Big|_0^t$$
$$= 2t^2$$

2. For a process

$$Z(t) = \int_0^t \Delta_1(u)du + \int_0^t W(u)dW(u)$$

where $\Delta_1(t)$ is an adapted process, find:

$$d(\log(Z(t)))$$

$$d(\log(Z(t))) = \frac{d(Z(t))}{Z(t)} \qquad \times$$
$$dZ(t) = \Delta_1(t)dt + W(t)dW(t)$$
$$\therefore d(\log(Z(t)))$$
$$= \left(\int_0^t \Delta_1(u)du + \int_0^t W(u)dW(u)\right)^{-1}(\Delta_1(t)dt + W(t)dW(t))$$

Correct:

$Z(t)$ is an Ito process

$$f(t, Z(t)) = \log(Z(t)), \quad f_t = 0, \quad f_x = \frac{1}{Z(t)}, \quad f_{xx} = \frac{-1}{Z(t)^2}$$
$$dZ(t) = \Delta_1(t)dt + W(t)dW(t)$$
$$dZ(t)dZ(t) = W^2(t)dt$$
$$d(\log(Z(t))) = \frac{1}{Z(t)}dZ(t) + \frac{1}{2}\frac{-1}{Z^2(t)}dZ(t)dZ(t)$$
$$= \frac{1}{Z(t)}(\Delta_1(t)dt + W(t)dW(t)) - \frac{1}{2Z^2(t)}(W^2(t)dt)$$
$$= \left(\frac{\Delta_1(t)}{Z(t)} - \frac{W^2(t)}{2Z^2(t)}\right)dt + \frac{W(t)}{Z(t)}dW(t)$$

3. Find the following integral(s): (express in their simplest form i.e. non-stochastic integrals)

(a)

$$\int_0^t 4W^3(s)dW(s)$$

Let $I(t)=\int_0^t 4W^3(s)dW(s)$

$f_x = 4x^3$, $f = x^4 + g(t) + C$, $f_t = g'(t)$, $f_{xx} = 12x^2$

$W^4(t)+g(t)+C - W(0) - g(0) - C =$

$\int_0^t g'(u)du + \int_0^t 4W^3(s)dW(s) + \frac{1}{2}\int_0^t 12W^2(s)ds$

$\int_0^t 4W^3(s)dW(s) = W^4(t)+g(t)-g(0)-(g(t)-g(0)) - \frac{1}{2}\int_0^t 12W^2(s)ds$

$= W^4(t) - 6\int_0^t W^2(s)ds$

4. For a process $X(t)$ governed by the dynamic:

$$dX(t) = X(t)dW(t)$$

with $X(0) = 2$. Determine:

$$\mathbb{E}[e^{-t}X^2(t)]$$

$f(t,W(t)) = e^{-t}X^2(t)$, $f_t = -e^{-t}X^2(t)$, $f_x = e^{-t}\cdot 2\cdot X(t)$, $f_{xx} = e^{-t}2$

$dX(t)dX(t) = X^2(t)dt$

$d(e^{-t}X^2(t)) = -e^{-t}X^2(t)dt + 2e^{-t}X(t)dX(t) + \frac{1}{2}\cdot 2e^{-t}dX(t)dX(t)$

$= e^{-t}(-X^2(t)dt + 2X(t)(X(t)dW(t)) + X^2(t)dt)$

$= 2e^{-t}X^2(t)dW(t)$

$\int_0^t d(e^{-u}X^2(u)) = e^{-t}X^2(t) - e^{0}X^2(0) = \int_0^t 2e^{-u}X^2(u)dW(u)$

$E[e^{-t}X^2(t)] = 4 + E[\int_0^t 2e^{-u}X^2(u)dW(u)] = 4$

1. For $\Omega = \{a,b,c,d\}$, with $\mathbb{P}(a) = 4\mathbb{P}(b) = 2\mathbb{P}(c) = 3\mathbb{P}(d)$. Define the random variables:

$$X(\omega) = \begin{cases} 5 & \omega \in \{a,b\} \\ -2 & \omega \in \{c,d\} \end{cases}$$

$$Y(\omega) = \begin{cases} 3 & \omega \in \{a,c\} \\ 4 & \omega \in \{b,d\} \end{cases}$$

Express:

(a) $\mathbb{E}[X|Y]$

$P(a)+P(b)+P(c)+P(d) = 1$

$\therefore P(a) = \frac{12}{25}$, $P(b) = \frac{3}{25}$, $P(c) = \frac{6}{25}$, $P(d) = \frac{4}{25}$

$E[X|Y] =$

$\frac{2}{3}\times 5 + \frac{1}{3}\times(-2) + \frac{3}{7}\times 5 + \frac{4}{7}\times(-2) = \frac{10}{3} - \frac{2}{3} + \frac{15}{7} - \frac{8}{7} = \frac{8}{3} + \frac{7}{7} = \frac{11}{3}$

(b) $\mathbb{E}[Y|X]$

i) when $X=5$

$$E[Y|X] = \frac{4}{5}\times 3 + \frac{1}{5}\times 4 = \frac{16}{5}$$

ii) when $X=-2$

$$E[Y|X] = \frac{3}{5}\times 3 + \frac{2}{5}\times 4 = \frac{17}{5}$$

2. For a process

$$Z(t) = \log(\pi) + \int_0^t W^3(u)du + \int_0^t \cos(u)dW(u)$$

Find:

(a)

$$d(e^{Z(t)})$$

$Z(t)$ is an Ito process

$f(t, Z(t)) = e^{Z(t)}$, $f_t = 0$, $f_x = e^{Z(t)}$, $f_{xx}\ e^{Z(t)}$

$dZ(t) = W^3(t)dt + \cos t\, dW(t)$

$dZ(t)dZ(t) = \cos^2 t\, dt$

$$d(e^{Z(t)}) = e^{Z(t)}dZ(t) + \frac{1}{2}e^{Z(t)}dZ(t)dZ(t)$$

$$= e^{Z(t)}(W^3(t)dt + \cos t\, dW(t) + \frac{1}{2}\cos^2 t\, dt)$$

$$= e^{Zt}((W^3(t) + \frac{1}{2}\cos^2 t)dt + \cos t\, dW(t))$$

(b)

$$[Z, Z](t)$$

$[Z,Z](t) = \int_0^t \cos^2 u\, du$ calculate this

$$= \int_0^t \frac{1}{2}du + \int_0^t \cos 2u\, du$$

$$= \frac{1}{2}t + \frac{1}{2}\sin 2u \Big|_0^t$$

$$= \frac{1}{2}t + \frac{1}{2}\sin 2t$$

3. Simplify the following integral: (express in its simplest form i.e. non-stochastic integrals)

$$\int_0^t 4ts^2W^2(s)dW(s)$$

$$\int_0^T 4Tt^2W^2(t)dW(t)$$

$$f_x(t,W(t)) = 4Tt^2W^2(t)$$

$$f(t,W(t)) = \frac{4}{3}Tt^2W^3(t)$$

$$f_{xx} = 8Tt^2W(t)$$

$$f_t = \frac{8}{3}TtW^3(t)$$

$$\int f_x(t,W(t))dW(t) = 4t\int_0^t s^2W^2(s)dW(s)$$

$$f(t,x) = 4ts^2\frac{x^3}{3},\quad f_t = 4t\cdot 2s\cdot\frac{x^3}{3},\quad f_{xx} = 8ts^2x$$

$$\frac{4}{3}ts^2W^3(s) = 0 + \int_0^t \frac{8}{3}tsW^3(s)ds + \int_0^t 4ts^2W^2(s)dW(s) + \frac{1}{2}\int_0^t 8ts^2W(s)ds$$

$$\therefore \int_0^t 4ts^2W^2(s)dW(s) = \frac{4}{3}ts^2W^3(s) - \int_0^t \frac{8}{3}tsW^3(s)ds - \frac{1}{2}\int_0^t 8ts^2W(s)ds \quad \times$$

$$df(t,W(t)) = \frac{8}{3}TtW^3(t)dt + 4Tt^2W^2(t)dW(t) + 4Tt^2W(t)dt$$

$$\int_0^T df = \frac{4}{3}T^3W^3(T) = \int_0^T\left(\frac{8}{3}TtW^3(t) + 4Tt^2W(t)\right)dt + \int_0^T 4Tt^2W^2(t)dW(t)$$

$$\therefore \int_0^t 4ts^2W^2(s)dW(s)$$

$$= \frac{4}{3}t^3W^3(t) - \int_0^t\left(\frac{8}{3}tsW^3(s) + 4ts^2W(s)\right)ds$$

4. For a process $X(t)$ governed by the dynamic:

$$dX(t) = X(t)dt + \sqrt{X(t)}dW(t)$$

with $X(0) = 2$.

(a) Determine:

$$\mathbb{E}[X(t)]$$

(Hint: investigate the process $Y(t) = e^{-t}X(t)$)

$$f(t,X(t)) = e^{-t}X(t),\quad f_t = -e^{-t}X(t),\quad f_x = e^{-t},\quad f_{xx} = 0$$

$$d(e^{-t}X(t)) = -e^{-t}X(t)dt + e^{-t}dX(t)$$

$$= -e^{-t}X(t)dt + e^{-t}\left(X(t)dt + \sqrt{X(t)}dW(t)\right)$$

$$= \sqrt{X(t)}dW(t)$$

$$\int_0^t d(e^{-u}X(u)) = e^{-t}X(t) - X(0) = \int_0^t \sqrt{X(u)}dW(u)$$

$$\Rightarrow X(t) = 2e^t + e^t\int_0^t \sqrt{X(u)}dW(u)$$

$$E[X(t)] = 2e^t + e^tE\left[\int_0^t \sqrt{X(u)}dW(u)\right]$$

$$= 2e^t$$

(b) Is $X(t)$ a martingale?

No, if $X(t)$ is a martingale $E[X(t)|F(0)] = E[X(t)] = 0 \neq 2e^t$

1. Let $W(t)$ be a Brownian motion with respect to a filtration $\mathcal{F}(t), t \geq 0$, and $I(t)$ is the Ito integral. What is the best descriptor of each process; martingale, submartingale, supermartingale, or none of the above?

(a) $X(t) = 3I(t) - \frac{t}{3}$

$$E[X(t)|F(s)] = X(s)$$

$$E[X(t)|F(s)] = X(s)$$

$$= E[3I(t) - \frac{t}{3} | F(s)]$$

$$= E[3\int_0^t \Delta(u)dW(u) | F(s)] - \frac{t}{3}$$

$$= 3E[\int_0^t \Delta(u)dW(u) | F(s)] - \frac{t}{3}$$

$$= 3I(s) - \frac{t}{3} < 3I(s) - \frac{s}{3}$$

So, $X(t)$ is a submartingale.

(b) $Y(t) = \frac{W(t)+I(t)}{2}$

$$E[Y(t)|F(s)]$$

$$= E[\frac{W(t)+I(t)}{2} | F(s)]$$

$$= \frac{1}{2}E[W(t) - W(s) + W(s) + I(t) - I(s) + I(s) | F(s)]$$

$$= \frac{1}{2}\left(E[W(t)-W(s)|F(s)] + W(s) + E[I(t)-I(s)|F(s)] + I(s) \right)$$

$$= \frac{1}{2}(0 + W(s) + I(s) + 0)$$

$$= \frac{W(s)+I(s)}{2} = Y(s)$$

So, $Y(t)$ is a martingale.

(c) $Z(t) = (W(t)+t)^2$

$$E[Z(t)|F(s)] = E[(W(t)+t)^2 | F(s)]$$

$$= E[W^2(t) + 2tW(t) + t^2 | F(s)]$$

$$= E[W^2(t)|F(s)] + 2tE[W(t)|F(s)] + t^2$$

$$= E[W^2(t) - W^2(s) + W^2(s) | F(s)] + 2tW(s) + t^2$$

$$= E[(W(t)-W(s))(W(t)+W(s)) | F(s)] + W^2(s) + 2tW(s) + t^2$$

$$= E[W(t)-W(s)|F(s)]\,E[W(t)+W(s)|F(s)] + W^2(s) + 2tW(s) + t^2$$

$$= 0 + W^2(s) + 2tW(s) + t^2 \neq W^2(s) + 2sW(s) + s^2 = Z(s)$$

So, $Z(t)$ is not a martingale nor ....

2. Define a random walk through the tossing of a fair coin and the rolling of a fair six-sided die. Define the random variable X through the outcome of the die and the random variable Y through the outcome of the coin. Let Q_n be a random walk with $Q_0 = 0$ and

$$X_j = \begin{cases} 1, & \omega_{1,j} \in \{1,5\} \\ 3, & \omega_{1,j} \in \{2,3,4,6\} \end{cases}$$

$$Y_j = \begin{cases} X_j, & \omega_{2,j} = H \\ -X_j, & \omega_{2,j} = T \end{cases}$$

$$Q_n = \sum_{j=1}^{n} Y_j$$

where $\omega_{1,j}$ is the outcome of the die on the j^{th} toss and $\omega_{2,j}$ is the outcome of the coin on the j^{th} toss.

(a) Prove whether or not the process Q_n is a martingale

$E[Q_n | F(s)]$ (where $1 < s < n$)

$= E[Q_n - Q_s + Q_s | F(s)]$

$= E[Q_n - Q_s | F(s)] + E[Q_s | F(s)]$

$= E[\sum_{j=1}^{n} Y_j - \sum_{j=1}^{s} Y_j | F(s)] + Q_s$

$= \sum_{s+1}^{n} E[Y_j] + Q_s$

$= \sum_{s+1}^{n} (\frac{1}{2}X_j - \frac{1}{2}X_j) + Q_s = Q_s$

consider j as time t, $X_j = X(t)$, $X(t)$ is a random variable

So, Q_n is a martingale.

And $X(t)$ has not a certain value but a range of values.

$= \sum_{s+1}^{n} (\frac{1}{2}(\frac{1}{3}\times 1 + \frac{2}{3}\times 3) - \frac{1}{2}(\frac{1}{3}\times 1 + \frac{2}{3}\times 3)) + Q_s$

(b) Prove whether or not the process Q_n is Markov.

$E[f(Q_n - Q_s + Q_s) | F(s)] = E[f(Q_n) | F(s)]$

Let x be a dummy variable

$E[f(Q_n - Q_s + x) | F(s)]$

$=$

(c) What is $\sigma(X_1)$? -

$\sigma(X_1) = \{\{1,5\}, \{2,4,3,6\}, \phi, \Omega\}$

3. Given the stochastic integral

$$I(t) = \int_0^t u\,dW(u)$$

(a) Define a sequence of adapted simple processes $\Delta_n(u)$ such that, for this integral,

$$\lim_{n\to\infty} \Delta_n(u) = \Delta(u)$$

$\Delta(u) = u$

$$\Delta_n(u) = \begin{cases} 0 & 0 \leq u < \frac{t}{n} \\ \frac{t}{n} & \frac{t}{n} \leq u < \frac{2t}{n} \\ \frac{2t}{n} & \frac{2t}{n} \leq u < \frac{3t}{n} \\ & \vdots \\ \frac{(n-1)t}{n} & \frac{(n-1)t}{n} \leq u < \frac{nt}{n} \end{cases}$$

(b) Use the simple processes to express the Ito integral $I(t)$. (bonus: 5pts What is this equal to?)

$I(t) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \Delta_n(t_j)(W(t_{j+1}) - W(t_j)) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{jt}{n}\left(W\left(\frac{(j+1)t}{n}\right) - W\left(\frac{jt}{n}\right)\right)$

$f_x = t$, $f(t, W(t)) = tW(t)$, $f_{xx} = 0$, $f_t = W(t)$

$d(tW(t)) = t\,dW(t) + W(t)dt$

$\int_0^T d(tW(t)) = TW(T) = \int_0^T t\,dW(t) + \int_0^T W(t)\,dt$

$\Rightarrow \int_0^T t\,dW(t) = TW(T) - \int_0^T W(t)dt$

$\therefore \int_0^t u\,dW(u) = tW(t) - \int_0^t W(u)\,du$

4. Consider the Scaled Asymmetric Random Walk, $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$ where $M_{nt} = \sum_{j=1}^{nt} X_j$ and define

$$X_j = \begin{cases} 2 & \omega_j = H \\ -1 & \omega_j = T \end{cases}.$$

let $\mathbb{P}(X_j = 2) = \frac{1}{3}$ and $\mathbb{P}(X_j = -1) = \frac{2}{3}$. Assume nt is a positive integer. Is $W^{(n)}(t)$ a Martingale, Markov, or both?

a) $E[W^{(n)}(t) \mid \mathcal{F}(s)]$

$= E[W^{(n)}(t) - W^{(n)}(s) + W^{(n)}(s) \mid \mathcal{F}(s)]$

$= E[W^{(n)}(t) - W^{(n)}(s) \mid \mathcal{F}(s)] + E[W^{(n)}(s) \mid \mathcal{F}(s)]$

$= E[W^{(n)}(t) - W^{(n)}(s)] \quad + W^{(n)}(s)$

$= E[\frac{1}{\sqrt{n}} \sum_{j=ns+1}^{nt} X_j] + W^{(n)}(s)$

$= \frac{1}{\sqrt{n}} \sum_{j=ns+1}^{nt} E[X_j] + W^{(n)}(s)$

$= \frac{1}{\sqrt{n}} \sum (\frac{1}{3} \times 2 - 1 \times \frac{2}{3}) + W^{(n)}(s) = W^{(n)}(s)$

So, $W^{(n)}(s)$ is a martingale.

b) $E[f(W^{(n)}(t)) \mid \mathcal{F}(s)] = E[f(W^{(n)}(t) - W^{(n)}(s) + W^{(n)}(s)) \mid \mathcal{F}(s)]$

Let x be a dummy variable

$E[f(W^{(n)}(t) - W^{(n)}(s) + x) \mid \mathcal{F}(s)]$

$=$

2. Let $M(t)$ be a Martingale with respect to the filtration $\mathcal{F}(t)$ and probability measure $\mathbb{P}$. Let $\delta(t)$ be an adapted process. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of the interval $[0, T]$. Define:

$$G(t) = \sum_{i=0}^{n-1} \{\delta(t_i)(M(t_{i+1}) - M(t_i))\} + \delta(t_n)(M(t) - M(t_n))$$

Determine:

$$\mathbb{E}[G(t_3) \mid \mathcal{F}(t_1)]$$

$= E[\delta(t_0)(M(t_1) - M(t_0)) + \delta(t_1)(M(t_2) - M(t_1)) + \delta(t_2)(M(t_3) - M(t_2))$

$+\delta(t_3)(M(t_3)-M(t_3))\,|\,F(t_1)]$

$= E[\delta(t_0)(M(t_1)-M(t_0))\,|\,F(t_1)] + E[\delta(t_1)(M(t_2)-M(t_1))\,|\,F(t_1)]$

$+ E[\delta(t_2)(M(t_3)-M(t_2))\,|\,F(t_1)]$

$= \delta(t_0)(M(t_1)-M(t_0)) + \delta(t_1)\{E[M(t_2)|F(t_1)] - M(t_1)\}$

$+ E[E[\delta(t_2)(M(t_3)-M(t_2))\,|\,F(t_2)]\,|\,F(t_1)]$

$= G(t_1) + \delta(t_1)\{M(t_1)-M(t_1)\} + E[\delta(t_2)\{E[M(t_3)|F(t_2)] - E[M(t_2)|F(t_2)]\}\,|\,F(t_1)]$

$= G(t_1) + 0 + E[\delta(t_2)\{M(t_2)-M(t_2)\}]$

$= G(t_1)$

3. Let $Z(t) = W(s)(W(t) - W(s))$ for $s \leq t$, and $W(t)$ being a Brownian motion. Calculate:

$$\mathbb{E}[Z^2(t)|\mathcal{F}(s)] \text{ and } \mathbb{E}[Z^2(t)]$$

What can we determine about the relationship between $Z(t)$ and $\mathcal{F}(s)$ from these results?

$E[Z^2(t)\,|\,F(s)]$

$= E[W^2(s)(W^2(t) - 2W(t)W(s) + W^2(s))\,|\,F(s)]$

$= W^2(s)\{E[W^2(t)|F(s)] - 2W(s)E[W(t)|F(s)] + E[W^2(s)|F(s)]\}$ X

$= W^2(s)\{$

$= E[W^2(s)(W(t)-W(s))^2\,|\,F(s)]$

$= W^2(s)\,E[(W(t)-W(s))^2]$

$= W^2(s)(t-s)$

$E[Z^2(t)]$

$= E[W^2(s)(W(t)-W(s))^2]$

$= E[W^2(s)]\,E[(W(t)-W(s))^2]$

$= s(t-s)$

$E[Z(t)|F(s)]$
$= E[W(s)(W(t)-W(s))|F(s)]$
$= W(s)E[W(t)-W(s)]$
$= W(s)\cdot 0 = 0$
So $Z(t)$ is indepent of $F(s)$. X

$Z(t) = W(s)(W(t)-W(s))$
and $F(s) \sim \delta(W(s))$, but $W(t)-W(s)$ is independent of $F(s)$
So, $Z(t)$ is not independent of $F(s)$ nor $F(s)$-measurable.

5. Let $f(t)$ be a nonrandom and continuously differentiable function and $W(t)$ be a Brownian Motion with respect to the filtration $\mathcal{F}(t)$.

Compute the quadratic variation of :

$$X(t) = f(t)W(t) - \int_0^t f(s)W(s)ds$$

$dX(t) = f'(t)W(t)dt + f(t)dW(t) - f(t)W(t)dt$
$= W(t)(f'(t)-f(t))dt + f(t)dW(t)$
$(dX(t))(dX(t)) = f^2(t)dt$
$\therefore [X(t), X(t)](t) = f^2(t)$ X
$d[X,X](t) = d(X(t))d(X(t)) = f^2(t)dt$
$\int_0^t d[X,X](u) = \int_0^t f^2(u)du$
$\Rightarrow [X,X](t) - [X,X](0) = \int_0^t f^2(u)du$
$\Rightarrow [X,X](t) = \int_0^t f^2(u)du$

4. Let $W(t)$ be a Brownian Motion with respect to the filtration $\mathcal{F}(t)$ and consider the Geometric Brownian Motion:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad S(0) = S_0 \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

where μ and σ are some given real constants.

(a) Express $S(t)$ in closed form. (You must show how you derive this, you cannot just use the fact that it is GBM, hint: consider $f(x) = \log x$),

(a) Set $f(t, S(t)) = \log S(t)$, $f_t = 0$, $f_x = \frac{1}{S(t)}$, $f_{xx} = -\frac{1}{S^2(t)}$

So $df(t, S(t)) = f_t dt + f_x dS(t) + \frac{1}{2} f_{xx} dS(t) dS(t)$

$$= \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{1}{S^2(t)}(\sigma^2 S^2(t) dt)$$

$$= \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt$$

$$= (\mu - \frac{1}{2}\sigma^2) dt + \sigma dW(t)$$

$$\int_0^t df(u, S(u)) = \int_0^t d\log S(u) = \int_0^t (\mu - \frac{1}{2}\sigma^2) du + \int_0^t \sigma dW(u)$$

$$\Rightarrow \log S(t) - \log S(0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)$$

$$\Rightarrow \frac{S(t)}{S(0)} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

$$\Rightarrow S(t) = S(0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

(b) Compute $\mathbb{E}[S(t)|\mathcal{F}(s)]$ for $s \leq t$

$$E[S(t)|F(s)] = E[S(0) e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma W(t)} | F(s)]$$

$$= S(0) e^{(\mu-\frac{1}{2}\sigma^2)t} E[e^{\sigma W(t)} | F(s)]$$

$$= S(0) e^{(\mu-\frac{1}{2}\sigma^2)t} e^{\sigma W(s)} \quad \times$$

$$E[S(t)|F(s)] = E[\frac{S(t)}{S(s)} S(s) | F(s)]$$

$$= E[\frac{S(t)}{S(s)} | F(s)] S(s)$$

$$= S(s) E[e^{(\mu-\frac{1}{2}\sigma^2)(t-s) + \sigma(W(t)-W(s))} | F(s)]$$

$$= S(s) e^{(\mu-\frac{1}{2}\sigma^2)(t-s)} E[e^{\sigma(W(t)-W(s))}]$$

$$= S(s)\, e^{(\mu-\frac{1}{2}\sigma^2)(t-s)}\, E\left[e^{\sigma(W(t)-W(s))}\right]$$
$$= S(s)\, e^{(\mu-\frac{1}{2}\sigma^2)(t-s)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, e^{\sigma x}\, e^{-\frac{x^2}{2(t-s)}}\, dx$$

$$\because \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, e^{\sigma x}\, e^{-\frac{x^2}{2(t-s)}}\, dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{x^2-2\sigma(t-s)x+\sigma^2(t-s)^2}{2(t-s)}} \cdot e^{\frac{\sigma^2(t-s)}{2}}\, dx$$
$$= 1 \cdot e^{\frac{\sigma^2(t-s)}{2}}$$

$$\therefore E[S(t)|F(s)] = S(s)\, e^{(\mu-\frac{1}{2}\sigma^2)(t-s)+\frac{1}{2}\sigma^2(t-s)} = S(s)\, e^{\mu(t-s)}$$

(c) Let $\mu = 0$. Compute

$$C(x, K, T) := \mathbb{E}\Big[\big(S(T) - K\big)_+ \Big| S(0) = x\Big].$$

$$C(x,K,T) = E\left[\left(x e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W(T)} - K\right)_+\right] = E\left[\left(x e^{-\frac{1}{2}\sigma^2 T+\sigma W(T)} - K\right)_+\right]$$
$$= \int_{-\infty}^{+} \left(x e^{-\frac{1}{2}\sigma^2 T+\sigma y} - K\right)_+ \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{y^2}{2T}}\, dy$$

$$\because \left(x e^{-\frac{1}{2}\sigma^2 T+\sigma y} - K\right)_+ = \begin{cases} x e^{-\frac{1}{2}\sigma^2 T+\sigma y} - K & y \geqslant \frac{\log\frac{K}{x}+\frac{1}{2}\sigma^2 T}{\sigma} \\ 0 & y < \frac{\log\frac{K}{x}+\frac{1}{2}\sigma^2 T}{\sigma} \end{cases}$$

$$\therefore C(x,K,T) = \int_{\frac{\log\frac{K}{x}+\frac{1}{2}\sigma^2 T}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\sigma^2 T+\sigma y-\frac{y^2}{2T}} \cdot x\, dy - \int_{\frac{\log\frac{K}{x}+\frac{1}{2}\sigma^2 T}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} K e^{-\frac{y^2}{2T}}\, dy$$

let $d_1 = \frac{\log\frac{K}{x}+\frac{1}{2}\sigma^2 T}{\sigma}$, $y = m\sqrt{T}$

$$\therefore C(x,K,T) = \int_{d_1}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot x \cdot e^{-\frac{\sigma^2 T^2-2T\sigma y+y^2}{2T}}\, dy - \int_{\frac{d_1}{\sqrt{T}}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} K e^{-\frac{m^2}{2}} \sqrt{T}\, dm$$
$$= \int_{d_1}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} x \cdot e^{-\frac{(y-\sigma T)^2}{2T}}\, dy - \sqrt{T} K N\left(\frac{d_1}{\sqrt{T}}\right)$$

let $\frac{y-\sigma T}{\sqrt{T}} = m \Rightarrow \sqrt{T}m + \sigma T = y$

$$\therefore C(x,K,T) = \int_{\frac{d_1-\sigma T}{\sqrt{T}}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot x\, e^{-\frac{m^2}{2}} \sqrt{T}\, dm - \sqrt{T} K N\left(\frac{d_1}{\sqrt{T}}\right)$$
$$= x\sqrt{T}\, N\left(\frac{d_1-\sigma T}{\sqrt{T}}\right) - \sqrt{T} K N\left(\frac{d_1}{\sqrt{T}}\right)$$