FE 610 Stochastic Calculus for Finance

May 4, 2017

Final Exam (Take-Home)

1. For a Brownian motion $W(t)$, define the process $L(t)$ as:

$$L(t) = \min_{0 \le u \le t} W(u)$$

   Determine the joint density $f_{L(t),W(t)}(\ell, w)$.

2. Determine whether the following processes are Martingales. ($W(t)$ is a Brownian motion)

   (a) $X(t) = e^{\int_0^t \cos(u) dW(u) - \frac{1}{2}\int_0^t (1-\sin^2(u)) du}$

   (b) $Y(t) = \sin(t^2 W(t))$

   (c) $Z(t) = W^5(t) - (10W^3(t) + 15tW(t))t$

3. Given a stock that follows Geometric Brownian Motion (GBM), you are interested in derivative security of the European variety that pays at maturity

$$V(T) = \sqrt{S(T)}$$

   Determine a pricing formula and hedging strategy for this derivative security at time t.

4. Let $W(t)$ be a Brownian motion and let $Q(t)$ be a compound Poisson process, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration $\mathcal{F}(t), t \geq 0$. Show that at every t, the random variable $W(t)$ and $Q(t)$ are independent.

5. For a stock $(S(t))$ that follows GBM and its maximum given by

$$Y(t) = \max_{0 \leq u \leq t} S(u),$$

prove that $[S, Y](t) = 0$.

6. Find the formula for a zero-coupon bond whose underlying interest rate follows the process

$$dR(t) = \alpha dt + \sigma d\widetilde{W}(t)$$

where α and σ are constants and $\widetilde{W}(t)$ is a Brownian motion.

7. For a multidimensional market model, we have 3 stocks, each with a stochastic differential

$$dS_i(t) = \alpha_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{3} \sigma_{ij} dW_j(t)$$

where $\alpha_i(t)$ is an adapted process and σ_{ij} is constant for all i, j. Does this market have a risk-neutral probability measure, and if so what is it (how do we get it)?

8. Solve the following using the direct definition of the Ito Integral.

$$I(t) = \int_0^t u dW(u) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \Delta_n(t_i)(W(t_{i+1}) - W(t_i))$$

By using the direct definition, I mean approximate the integrand with the simple function $\Delta_n(u)$ evaluated over the partition of $[0, t]$ given by $\Pi = \{t_0, t_1, \ldots, t_n\}$ and look at the limit. You may use Ito's formula to check your answer, but just using it to solve the integrand will result in 0 points.