Multivariable Stochastic Calculus

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Definition 4.6.1: A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t), \ldots, W_d(t))$$

with the following properties.

1. Each $W_i(t)$ is a one-dimensional Brownian motion
2. If $i \neq j$, then the processes $W_i(t)$ and $W_j(t)$ are independent
3. **(Information Accumulates)** For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$
4. **(Adaptivity)** For each $t \geq 0$, the random vector $W(t)$ is $\mathcal{F}(t)$-measurable
5. **(Independence of future increments)** For $0 \leq t < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}(t)$

[1]

Because of the nature of Brownian motion, we have

$$[W_i, W_i](t) = t$$

which can be expressed as

$$dW_i(t)dW_i(t) = dt$$

But, as we shall see, if $i \neq j$ then by the independence of the Brownian motions

$$dW_i(t)dW_j(t) = 0$$

2-dimensional Itô Process

Let $W(t)$ be a 2-dimensional Brownian motion. Recall that we have the form for an Itô process as

$$X(t) = X(0) + \int_0^t \Delta(u) dW(u) + \int_0^t \Theta(u) du$$

If we rewrite this process letting $W(t) = (W_1(t), W_2(t))$ and $\Delta(t) = (\sigma_1(t), \sigma_2(t))$ where both functions are adapted processes, we have

$$X(t) = X(0) + \int_0^t \sigma_1(u) dW_1(u) + \int_0^t \sigma_2(u) dW_2(u) + \int_0^t \Theta(u) du$$

Let $X(t)$ and $Y(t)$ be Itô processes. This means we now have

$$X(t) = X(0) + \int_0^t \sigma_{1,1}(u)dW_1(u) + \int_0^t \sigma_{1,2}(u)dW_2(u) + \int_0^t \Theta_1(u)du$$

$$Y(t) = Y(0) + \int_0^t \sigma_{2,1}(u)dW_1(u) + \int_0^t \sigma_{2,2}(u)dW_2(u) + \int_0^t \Theta_2(u)du$$

where Θ_i and $\sigma_{i,j}$ are adapted process for all i and j. We can express this in differential notations as

$$dX(t) = \Theta_1(t)dt + \sigma_{1,1}dW_1(t) + \sigma_{1,2}dW_2(t)$$

$$dY(t) = \Theta_2(t)dt + \sigma_{2,1}dW_1(t) + \sigma_{2,2}dW_2(t)$$

Looking at the quadratic variation for these Itô processes, we have

$$[X, X](t) = \int_0^t (\sigma_{1,1}^2(u) + \sigma_{1,2}^2(u))du$$

or in differential form

$$dX(t)dX(t) = (\sigma_{1,1}^2(t) + \sigma_{1,2}^2(t))dt$$

Similarly we have

$$dY(t)dY(t) = (\sigma_{2,1}^2(t) + \sigma_{2,2}^2(t))dt$$

$$dX(t)dY(t) = \left(\sigma_{1,1}(t)\sigma_{2,1}(t) + \sigma_{1,2}(t)\sigma_{2,2}(t)\right) dt$$

Two-dimensional Itô formula

Theorem 4.6.2: Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} are defined and are continuous. Let $X(t)$ and $Y(t)$ be Itô processes as discussed. The two-dimensional Itô formula in differential form is

$$\begin{aligned} df\left(t, X(t), Y(t)\right) &= f_t(t, X(t), Y(t))dt \\ &+ f_x(t, X(t), Y(t))dX(t) + f_y(t, X(t), Y(t))dY(t) \\ &+ \frac{1}{2} f_{xx}(t, X(t), Y(t))dX(t)dX(t) \\ &+ f_{xy}(t, X(t), Y(t))dX(t)dY(t) \\ &+ \frac{1}{2} f_{yy}(t, X(t), Y(t))dY(t)dY(t) \end{aligned}$$

[1]

This form can be written much more succinctly. If we suppress the arguments, we have

$$df(t, X(t), Y(t)) = f_t dt + f_x dX + f_y dY + \frac{1}{2} f_{xx} dX dX + f_{xy} dX dY + \frac{1}{2} f_{yy} dY dY$$

If you compare this form to equation (4.6.10) from your book, you can see how much simpler the differential form is to express compared to the integral form.

Itô Product Rule

Corollary 4.6.3: Let $X(t)$ and $Y(t)$ be Itô processes. Then

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

[1] Proof: Use the differential form from the previous slide, letting $f(t,x,y) = xy$, so $f_t = 0$, $f_x = y$, $f_y = x$ and all second order partial derivatives are 0 except for $f_{xy} = 1$

Lévy, one dimension

Theorem 4.6.4: Let $M(t), t \geq 0$, be a martingale relative to a filtration, $\mathcal{F}(t), t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.[1]

Note that this theorem doesn't mention normality. The key to the proof is to use

$$df(t, M(t)) = f_t(t, M(t)dt + f_x(t, M(t))dM(t) + \frac{1}{2}f_{xx}(t, M(t))dt$$

which holds from the condition $[M, M](t) = t$

Lévy, Two Dimensions

Theorem 4.6.5: Let $M_1(t)$ and $M_2(t), t \geq 0$, be martingales relative to a filtration $\mathcal{F}(t), t \geq 0$. Assume that for $i = 1, 2$ we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1, M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.[1]
To prove this theorem, use the one-dimensional Lévy theorem and an approach similar to its proof as well as the two-dimensional Itô formula.

Correlated Stock Prices

Example 4.6.6: Suppose

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 dW_1(t)$$

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2[\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)]$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions, $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < 1$ are constants.

Can we express this instead in terms of correlated Brownian Motions?

Definition 4.7.1: A Gaussian process $X(t), t \geq 0$, is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.[1]

Jointly normally distributed variables are defined through their means and covariances. Let $m(t) = \mathbb{E}[X(t)]$ and $c(s,t) = \mathbb{E}[(X(s) - m(s))(X(t) - m(t))]$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.