PDE's and SDE's

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A stochastic differential equation is of the form

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

where $\beta(u, X(u))$ is called the drift and $\gamma(u, X(u))$ is called the diffusion of the process. Additionally, because it is a differential equation, a boundary condition is needed. This is given by the initial condition

$$X(t) = x, t \geq 0, x \in \mathcal{R}$$

The goal is then to find the process $X(T), T \geq t$ such that

$$X(t) = x$$

$$X(T) = X(t) + \int_t^T \beta(u, X(u))du + \int_t^T \gamma(u, X(u))dW(u)$$

Note that $X(T)$ is $\mathcal{F}(T)$-measurable.

One type of solvable SDE is the **one-dimensional linear stochastic differential equation**

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u)$$

where a, b, γ, and σ are nonrandom functions of time. We will look at some examples of this type of SDE and discuss their solutions.

Geometric Brownian Motion

Recall the formula

$$dS(u) = \alpha S(u)du + \sigma S(u)dW(u)$$

In this equation, $a(u) = 0, \gamma(u) = 0, b(u) = \alpha$, and $\sigma(u) = \sigma$. So the solution is found as:

$$S(T) = xe^{\sigma(W(T)-W(t))+\left(\alpha-\frac{1}{2}\sigma\right)(T-t)}$$

$$S(t) = x$$

Hull-White Interest Rate Model

$$dR(u) = (a(u) - b(u)R(u))du + \sigma(u)d\tilde{W}(u)$$

where a, b, and σ are nonrandom positive functions of time. To help with clarity, we use the dummy variable r instead of x when discussing interest rate processes. This can be solved with initial condition $R(t) = r$ as

$$R(T) = re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \int_t^T e^{-\int_t^T b(v)dv} \sigma(u)d\tilde{W}(u)$$

Cox-Ingersoll-Ross Interest Rate Model

Slightly different from the Hull-White model, we have

$$dR(u) = (a - bR(u))du + \sigma\sqrt{R(u)}d\tilde{W}(u)$$

where a, b, and σ are positive constants. For this model, unlike the previous one, it is impossible for the interest rate to drop below 0. There is no formula for $R(T)$, there is a unique solution given an initial condition.

Given the stochastic differential equation from the beginning of class,

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

let $0 \leq t \leq T$ be given and let $h(y)$ be a Borel-measurable function. Denote

$$g(t, x) = \mathbb{E}^{t,x} h(X(T))$$

as the expectation of $h(X(T))$ where $X(T)$ is a solution to the SDE with initial condtion $X(t) = x$.

Euler's Method

If we don't have a formula for the distribution of $X(T)$, there are a multiple of ways to approximate the solution. One of which is **Euler's Method** in which, you approximate small steps and use Monte-Carlo simulations to determine the expected values. To do so:

1. Choose a small step size δ such that

$$X(t+\delta) = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\epsilon_1$$

2. Work through the process setting

$$X(t+(i+1)\delta) = X(t+i\delta)+\beta(t+i\delta, X(t+i\delta))\delta+\gamma(t+i\delta, X(t+i\delta))\sqrt{\delta}\epsilon_{i-1}$$

3. Repeat first two steps taking the average of results to get $\mathbb{E}[X(T)$ (or $\mathbb{E}[h(X(T))])$

Theorem 6.3.1: Let $X(u), u \geq 0$, be a solution to the stochastic differential equation with initial condition given at time 0. Then, for $0 \leq t \leq T$,

$$\mathbb{E}[h(X(T))|\mathcal{F}(t)] = g(t, X(t))$$

[1]

Corollary 6.3.2: Solutions to stochastic differential equations are Markov processes.[1]

Theorem 6.4.1:(Feynman-Kac) Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = \mathbb{E}^{t,x}[h(X(T))]$$

Then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x), \forall x$$

[1]

Lemma 6.4.2: Let $X(u)$ be a solution to the stochastic differential equation with initial condition given at time 0. Let $h(y)$ be a Borel-measurable function, fix $T > 0$, and let $g(t, x)$ be given as in the previous theorem. Then the stochastic process

$$g(t, X(t)), 0 \leq t \leq T$$

is a martingale.[1]

General principle behind the proof of the Feynman-Kac is:

1. find the martingale
2. take the differential
3. set the dt term equal to zero

Theorem 6.4.3: (Discounted Feynman-Kac) Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

Let $h(y)$ be a Borel-measurable function and let r be a constant. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$f(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)}h(X(T))]$$

Then $f(t, x)$ satisfies the partial differential equation

$$f_t(t, x) + \beta(t, x)f_x(t, x) + \frac{1}{2}\gamma^2(t, x)f_{xx}(t, x) = rf(t, x)$$

and the terminal condition

$$f(T, x) = h(x), \forall x$$

[1]

Option Pricing Using GGBM

Given motion described as

$$dS(u) = rS(u)du + \sigma(u, S(u))S(u)d\tilde{W}(u)$$

This leads us to the PDE

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2(t,x)x^2v_{xx}(t,x) = rv(t,x)$$

If we solve this numerically, we determine option prices based on the parameters. We can also calculate the **implied volatility** based on the option price we determine. If the value of an option (for example a European call) based on this model is X then, the implied volatility is the value of the parameter σ^2 such that:

$$c_{BS}(t,x;r,\sigma^2,K) = X$$

For the simplest interest rate models, we will use the one factor equation:

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\tilde{W}(t)$$

The discount process is

$$D(t) = e^{-\int_0^t R(s)ds}$$

and the **money market account price process** is

$$\frac{1}{D(t)} = e^{\int_0^t R(s)ds}$$

This leads us to the formulas

$$dD(t) = -R(t)D(t)dt, d\left(\frac{1}{D(t)}\right) = \frac{R(t)}{D(t)}dt$$

Zero-Coupon Bond

A **zero-coupon bond** is a contract promising to pay a certain "face" amount, which we take to be 1, at a fixed maturity date T.[1]

For $0 \leq t \leq T$, the price of the bond ($B(t, T)$) will satisfy

$$D(t)B(t, T) = \tilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

Which gives us the formula

$$B(t, T) = \tilde{\mathbb{E}}[e^{-\int_t^T R(s)ds}|\mathcal{F}(t)]$$

We can now define the **yield** between times t and T as

$$Y(t, T) = -\frac{1}{T-t}\log B(t, T)$$

Because R is given by a stochastic differential equation, it must be Markov, and so

$$B(t,T) = f(t, R(t))$$

for some function $f(t,r)$. To find the partial differential equation for this function, we will need to find a martingale, determine its differential, and set its dt term equal to 0.

$$\begin{aligned} d(D(t)f(t,R(t))) &= f(t,R(t))dD(t) + D(t)df(t,R(t)) \\ &= D(t)\left[-Rfdt + f_t dt + f_r dR + \frac{1}{2} f_{rr} dRdR\right] \\ &= D(t)\left[-Rf + f_t + \beta f_r + \frac{1}{2}\gamma^2 f_{rr}\right] dt + D(t)\gamma f_r d\tilde{W} \end{aligned}$$

Hull-White Interest Rate Model

In this model, the evolution of the interest rate is given by

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t)$$

The PDE for the zero-coupon bond is then

$$f_t(t,r) + (a(t) - b(t)r)f_r(t,r) + \frac{1}{2}\sigma^2(t)f_{rr}(t,r) = rf(t,r)$$

This has an explicit formula solution as a function of the interest rate

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)}, 0 \leq t \leq T$$

$$A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T) \right) ds$$

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds$$

Cox-Ingersoll-Ross Interest Rate Model

The evolution for the interest rate in this model is given as

$$dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}d\tilde{W}(t)$$

Which has the PDE

$$f_t(t,r) + (a - br)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) = rf(t,r)$$

This solution will also be of the form

$$f(t,r) = e^{-rC(t,T)-A(t,T)}$$

but with different $A(t,T)$ and $C(t,T)$ from the previous problem.

Multidimensional Feynman-Kac

Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian Motion. Consider the two SDE's

$$\begin{aligned} dX_1(u) &= \beta_1(u, X_1(u), X_2(u))du + \gamma_{1,1}(u, X_1(u), X_2(u))dW_1(u) \\ &\quad + \gamma_{1,2}(u, X_1(u), X_2(u))dW_2(u) \\ dX_2(u) &= \beta_2(u, X_1(u), X_2(u))du + \gamma_{2,1}(u, X_1(u), X_2(u))dW_1(u) \\ &\quad + \gamma_{2,2}(u, X_1(u), X_2(u))dW_2(u) \end{aligned}$$

The solution to this pair of SDE's, starting at initial conditions $X_1(t) = x_1$ and $X_2(t) = x_2$, depends on initial time t and positions x_1 and x_2.[1]

Let $h(y_1, y_2)$ be given. Corresponding to the initial conditions t, x_1, x_2 where $0 \leq t \leq T$, we define

$$g(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2}[h(X_1(T), X_2(T)]$$

$$f(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2}\left[e^{-r(T-t)}h(X_1(T), X_2(T))\right]$$

Then

$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2}(\gamma_{1,1}^2 + \gamma_{1,2}^2) g_{x_1,x_1}$$
$$+ (\gamma_{1,1}\gamma_{2,1} + \gamma_{1,2}\gamma_{2,2}) g_{x_1,x_2} + \frac{1}{2}(\gamma_{2,1}^2 + \gamma_{2,2}^2) g_{x_2,x_2} = 0$$

$$f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2}(\gamma_{1,1}^2 + \gamma_{1,2}^2) f_{x_1,x_1}$$
$$+ (\gamma_{1,1}\gamma_{2,1} + \gamma_{1,2}\gamma_{2,2}) f_{x_1,x_2} + \frac{1}{2}(\gamma_{2,1}^2 + \gamma_{2,2}^2) f_{x_2,x_2} = rf$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.