Risk-Neutral Measure and Girsanov

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Outline

Theorem 1.6.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}[Z] = 1$. For $A \in \mathcal{F}$, define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.[2]

Equivalent Measure

Definition 1.6.3: Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree which sets in $\mathcal{F}$ have probability zero.[2]

Definition 1.6.5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbb{P}$, and let Z be an almost surely positive random variable that relates $\mathbb{P}$ and $\tilde{\mathbb{P}}$ via Theorem 1.6.1. Then Z is called the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

[2]

Let θ be a constant and let $X\ N(0,1)$ and define $Y = X + \theta$. Note that $\mathbb{E}[Y] = \theta$ and $Var(Y) = 1$. Define $Z(\omega) = e^{-\theta X(\omega) - \frac{1}{2}\theta^2}, \forall \omega \in \Omega$. We have

1. $Z(\omega) > 0$
2. $\mathbb{E}[Z] = 1$

$$
\begin{aligned}
\tilde{\mathbb{P}}(Y \leq b) &= \int_{(\omega: Y(\omega) \leq b)} Z(\omega) d\mathbb{P}(\omega) \\
&= \int_{\Omega} \mathbb{I}_{\{Y(\omega) \leq b\}} Z(\omega) d\mathbb{P}(\omega) \\
&= \int_{\Omega} \mathbb{I}_{\{X(\omega) \leq b-\theta\}} e^{-\theta X(\omega) - \frac{1}{2}\theta^2} d\mathbb{P}(\omega) \\
&= \int_{-\infty}^{b-\theta} \frac{1}{\sqrt{2\pi}} e^{-\theta x - \frac{1}{2}\theta^2} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\end{aligned}
$$

Radon-Nikodým Derivative Process

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}(t)$ for $0 \leq t \leq T$ where T is fixed. Further suppose that Z is an almost surely positive random variable with $\mathbb{E}[Z] = 1$. Define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$. This leads us to the definition of the **Radon-Nikodým derivative process:**

$$Z(t) = \mathbb{E}[Z|\mathcal{F}(t)], 0 \leq t \leq T$$

This process is a martingale as can be seen using iterated conditioning:

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}(t)]|\mathcal{F}(s)] = \mathbb{E}[Z|\mathcal{F}(s)] = Z(s)$$

Lemma 5.2.1: Let t satisfying $0 \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]$$

[2]

Proof:

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}(t)]] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}(t)]] = \mathbb{E}[YZ(t)]$$

Lemma 5.2.2: Let s and t satisfying $0 \leq s \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]$$

[2]

The proof of this lemma depends on using part of definition 2.3.1 which states

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega)d\mathbb{P}(\omega) = \int_A X(\omega)d\mathbb{P}(\omega), \text{ for all } A \in \mathcal{G}$$

or in our case

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]d\tilde{\mathbb{P}} = \int_A Yd\tilde{\mathbb{P}}, \text{ for all } A \in \mathcal{F}(s)$$

Girsanov

Theorem 5.2.3:(Girsanov, one dimension) Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$, the process $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion.[2]

Discount Process

Consider a stock process $(S(t))$ that follows the Generalized Geometric Brownian Motion given as

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \leq t \leq T$$

Which has the solution

$$S(t) = S_0 e^{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds}$$

In addition, we have an adapted interest rate process $R(t)$. The **discount process** is given as

$$D(t) = e^{-\int_0^t R(s)ds}$$

$$dD(t) = -R(t)D(t)dt$$

Discounted Stock Process

The discounted stock process is

$$D(t)S(t) = S_0 e^{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds}$$

and so its differential is

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \end{aligned}$$

where we define

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

which is the market price of risk

Risk-Neutral Measure

If we introduce the measure $\tilde{\mathbb{P}}$ and let $\Theta(t)$ be our adapted process in the Girsanov Theorem, we have

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t)$$

Note that under this measure, the discounted stock process is now a martingale. As such we call this measure $\tilde{\mathbb{P}}$ the **risk-neutral measure.** To see this just integrate both sides to get

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\tilde{W}(u)$$

and then take the expectation of each side

Portfolio Process Value under Risk-Neutral

Start with initial capital X_0 and adjust your portfolio at each time $0 \leq t \leq T$. The change of value of your portfolio at each time is given as

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)dt + \Delta(t)(\alpha(t) - R(t))S(t)dt + \Delta(t)\sigma(t)S(t)dW(t) \\ &= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)] \end{aligned}$$

Which leads us to

$$\begin{aligned} d(D(t)X(t)) &= \Delta(t)\sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \\ &= \Delta(t)d(D(t)S(t)) \\ &= \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t) \end{aligned}$$

Pricing Under the Measure

Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable. We need to know the value of $X(0)$ and the process $\Delta(t)$ in order to have $X(T) = V(T)$ almost surely. Once we have this property satisfied, because $D(t)X(t)$ is a martingale under $\tilde{\mathbb{P}}$ we also have

$$D(t)X(t) = \tilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}(t)] = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

This leads us to the **risk-neutral pricing formula** for the continuous time model

$$V(t) = \tilde{\mathbb{E}}[e^{-\int_t^T R(u)du} V(T)|\mathcal{F}(t)], 0 \leq t \leq T$$

Theorem 5.3.1: (Martingale representation, one dimension)
Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration. Then there is an adapted process $\Gamma(u), 0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u), 0 \leq t \leq T$$

Corollary 5.3.2: Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$, the process $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion.

Now let $\tilde{M}(t), 0 \leq t \leq T$, be a martingale under $\tilde{\mathbb{P}}$. Then there is an adapted process $\tilde{\Gamma}(u), 0 \leq u \leq T$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), 0 \leq t \leq T$$

[2]

From previously, we have

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

This discounted option price process is a martingale under the measure $\tilde{\mathbb{P}}$ as can be seen

$$\begin{aligned}\tilde{\mathbb{E}}[D(t)V(t)|\mathcal{F}(s)] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]|\mathcal{F}(s)] \\ &= \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(s)] \\ &= D(s)V(s)\end{aligned}$$

So there is a representation

$$D(t)V(t) = V(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), 0 \leq t \leq T$$

But we also have

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\tilde{W}(u), 0 \leq t \leq T$$

So in order to have $X(t) = V(t)$ for all $0 \leq t \leq T$ we set $X(0) = V(0)$ and

$$\Delta(t)\sigma(t)D(t)S(t) = \tilde{\Gamma}(t)$$

or equivalently

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{\sigma(t)D(t)S(t)}, 0 \leq t \leq T$$

Multidimensional Market Model

Assume the existence of m stocks, each with a stochastic differential

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), i = 1, \ldots, m$$

In this representation, the σ_{ij}'s denote entries in the covariance matrix of a d-dimensional Brownian motion. Note that if $d = m$ and $\sigma_{ij} = 0$ for $i \neq j$ then each stock price is uncorrelated.

Definition 5.4.3: A probability measure $\tilde{\mathbb{P}}$ is said to be risk-neutral if

1. $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e. for every $A \in \mathcal{F}, \mathbb{P}(A) = 0$ if and only if $\tilde{\mathbb{P}}(A) = 0$)
2. under $\tilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, \ldots, m$

[2]

Lemma 5.4.5: Let $\tilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\tilde{\mathbb{P}}$, the discounted portfolio $D(t)X(t)$ is a martingale.[2]

Definition 5.4.6: An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$

$$\mathbb{P}(X(T) \geq 0) = 1, \mathbb{P}(X(T) > 0) > 0$$

[2]

Theorem 5.4.7: (First fundamental theorem of asset pricing) If a market model has a risk-neutral probability measure, then it does not admit arbitrage.[2]

Definition 5.4.8: A market model is complete if every derivative security can be hedged.[2]

Theorem 5.4.9: (Second fundamental theorem of asset pricing) Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.[2]

[1] S.E. Shreve. *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Number v. 1 in Springer Finance.

[2] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.