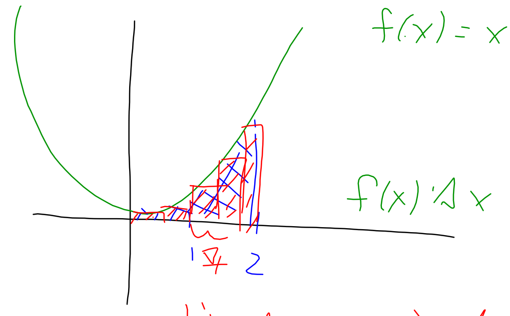
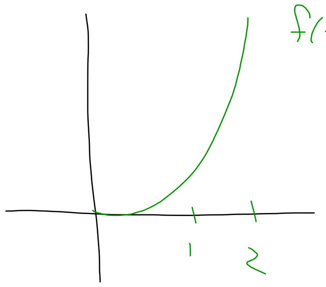
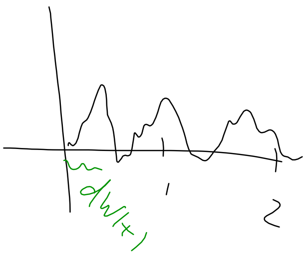
$$\int_0^2 x^2\,dx$$



$$\int_0^2 t^2\,dW(t)$$

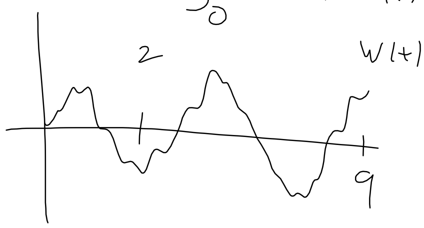


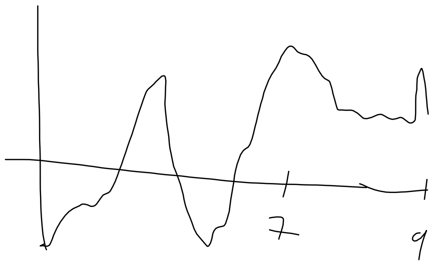


$$\int_0^9 W(t)\,dW(t) \leftarrow \text{known}$$

unknown

$$\int_0^9 W(t+2)\,dW(t)$$





$[0,T]$, $T = 9$

$\Pi = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$

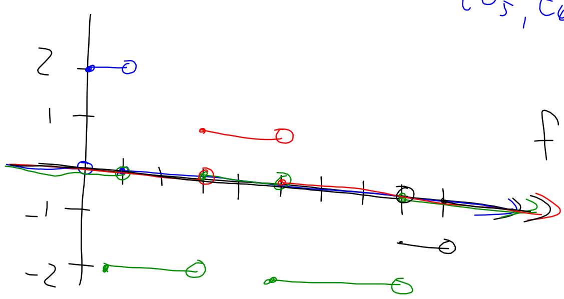
$t_0 = 0,\ t_1 = 1,\ t_2 = 3,\ t_3 = 4,\ t_4 = 5,\ t_5 = 8,\ t_6 = 9$

$a_1 = 2$ $\qquad A_1 = [t_0, t_1)$

$a_2 = -2$ $\qquad A_2 = [t_1, t_2) \cup [t_4, t_5)$

$a_3 = 1$ $\qquad A_3 = [t_2, t_4)$

$a_4 = -1$ $\qquad A_4 = [t_5, t_6)$



$$f(x) = \sum_{i=1}^{4} a_i \mathbb{1}_{\{x \in A_i\}}$$

$i=1$
$i=2$
$i=3$
$i=4$

Portfolio: Asset, whose value is $W(t)$

$\Delta(t)$ is # shares we own, only trade at beginning of the day

day 0: $t_0 \le t < t_1$

position $\Delta(t_0)$ at cost $W(0)\Delta(t_0)$

$I(t) \Rightarrow$ "the value of the portfolio @ time t"

$$I(t) = \Delta(t_0)\big(W(t) - W(t_0)\big)$$

Day 1: $t_1 \le t < t_2$

position $\Delta(t_1)$

$$I(t) = \Delta(t_1)\big(W(t) - W(t_1)\big) + \Delta(t_0)\big(W(t_1) - W(t_0)\big)$$

Day 2: $t_2 \le t < t_3$

$$I(t) = \Delta(t_0)\big(W(t_1) - W(t_0)\big) + \Delta(t_1)\big(W(t_2) - W(t_1)\big) + \Delta(t_2)\big(W(t) - W(t_2)\big)$$

Day n: $t_n \le t < t_{n+1}$

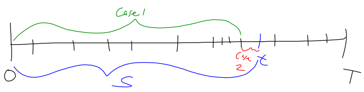
$$I(t) = \sum_{i=0}^{n-1} \Delta(t_i)\big(\underbrace{W(t_{i+1}) - W(t_i)}_{\approx dW(t)}\big) + \Delta(t_n)\big(\underbrace{W(t) - W(t_n)}\big)$$

---

back to day 2: $t_2 \le t < t_3$

$$I(t) = \int_0^t \Delta(u)\,dW(u) = \int_{t_0}^{t_1} \Delta(u)\,dW(u) + \int_{t_1}^{t_2} \Delta(u)\,dW(u) + \int_{t_2}^{t} \Delta(u)\,dW(u)$$

$$= \Delta(t_0)\underbrace{\int_{t_0}^{t_1} dW(u)}_{W(t_1)-W(t_0)} + \Delta(t_1)\underbrace{\int_{t_1}^{t_2} dW(u)}_{W(t_2)-W(t_1)} + \Delta(t_2)\underbrace{\int_{t_2}^{t} dW(u)}_{W(t)-W(t_2)}$$



for $s \le t$ & $0 \le s, t \le T$

Case 1: $t \in [t_K, t_{K+1})$ $\quad s \in [t_\ell, t_{\ell+1})$ such that $\ell < K$

$$I(t) = \sum_{j=0}^{K-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_K)(W(t) - W(t_K))$$

$$= \underbrace{\sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))}_{A} + \underbrace{\Delta(t_\ell)(W(t_{\ell+1}) - W(t_\ell))}_{B}$$

$$+ \underbrace{\sum_{j=\ell+1}^{K-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))}_{C} + \underbrace{\Delta(t_K)(W(t) - W(t_K))}_{D}$$

$$E[I(t) \mid \mathcal{F}(s)] = E[A + B + C + D \mid \mathcal{F}(s)]$$

$$E[A \mid \mathcal{F}(s)] : E\left[\sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(s)\right]$$

$$= \sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))$$

$$E[B \mid \mathcal{F}(s)] = E[\Delta(t_\ell)(W(t_{\ell+1}) - W(t_\ell)) \mid \mathcal{F}(s)]$$

$$\cancel{*} = \Delta(t_\ell) E[W(t_{\ell+1}) - W(t_\ell) \mid \mathcal{F}(s)]$$

$$= \Delta(t_\ell) E[\underbrace{W(t_{\ell+1}) - W(s)}_{\text{ind.}} + \underbrace{W(s) - W(t_\ell)}_{\text{meas.}} \mid \mathcal{F}(s)]$$

$$= \Delta(t_\ell)\left(E[W(t_{\ell+1}) - W(s) \mid \mathcal{F}(s)] + E[W(s) - W(t_\ell) \mid \mathcal{F}(s)]\right)$$

$$= \Delta(t_\ell)\left(E[W(t_{\ell+1}) - W(s)] + W(s) - W(t_\ell)\right)$$

$$= \Delta(t_\ell)(W(s) - W(t_\ell))$$

$$= \Delta(t_\ell)\left(E[W(t_{\ell+1}) \mid \mathcal{F}(s)] - W(t_\ell)\right)$$

$$= \Delta(t_\ell)(W(s) - W(t_\ell))$$

$$C: \ E\left[\sum_{j=\ell+1}^{K-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(s)\right]$$

let $\ell+1 \le j \le K-1$

$\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for $s < t$

$$E[\Delta(t_j)(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(s)]$$

$$= E[E[\Delta(t_j)(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(t_j)] \mid \mathcal{F}(s)]$$

for $\mathcal{H} \le \mathcal{G}$: $E[E[X \mid \mathcal{G}] \mid \mathcal{H}] = E[X \mid \mathcal{H}]$

$$= E[\Delta(t_j) E[W(t_{j+1}) - W(t_j) \mid \cancel{\mathcal{F}(t_j)}] \mid \mathcal{F}(s)]$$

$$= E[\Delta(t_j)(E[W(t_{j+1}) - W(t_j)] \mid \mathcal{F}(s)] = 0$$

so $E[C \mid \mathcal{F}(s)] = 0$

$$D: \ E[\Delta(t_K)(W(t) - W(t_K)) \mid \mathcal{F}(s)]$$

$$= E[E[\Delta(t_K)(W(t) - W(t_K)) \mid \mathcal{F}(t_K)] \mid \mathcal{F}(s)]$$

$$= 0$$

$$E[I(t) \mid \mathcal{F}(s)] = \sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_\ell)(W(s) - W(t_\ell))$$

$$= I(s)$$

Case 2: $s, t \in [t_K, t_{K+1})$, $s \le t$

$$E[I(t) \mid \mathcal{F}(s)] = E\left[\sum_{j=0}^{K-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(s)\right]$$

$$+ E[\Delta(t_K)(W(t) - W(t_K)) \mid \mathcal{F}(s)]$$

$$= \sum_{j=0}^{K-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \underbrace{E[\Delta(t_K)(W(t) - W(t_K)) \mid \mathcal{F}(s)]}$$

$$\Delta(t_K) E[W(t) - W(t_K) \mid \mathcal{F}(s)]$$

$$\Delta(t_K)\{E[W(t) \mid \mathcal{F}(s)] - W(t_K)\}$$

$$\Delta(t_K)(W(s) - W(t_K))$$

$$= I(s)$$

$$D_k = W(t) - W(t_k)$$

$$\text{"} = W(t_{i+1}) - W(t_i) \quad \text{for } 0 \le i \le k-1$$

$$I(t) = \sum_{j=0}^{k} \Delta(t_j) D_j$$

$$I^2(t) = \sum_{j=0}^{k} \Delta^2(t_j) D_j^2 + 2 \sum_{0 \le i < j \le k} \Delta(t_i)\Delta(t_j) D_i D_j$$

$D_j \sim$ ind $\mathcal{F}(t_j)$

$\Delta(t_i)\Delta(t_j) D_i \sim \mathcal{F}(t_j)$-meas.

$$E[I^2(t)] = E\left[\sum_{j=0}^{k} \Delta^2(t_j) D_j^2\right] + 2E\left[\sum_{0 \le i < j \le k} D_i D_j \Delta(t_i)\Delta(t_j)\right]$$

$$= \sum_{j=0}^{k} E[\Delta^2(t_j) D_j^2] + 2 \sum_{0 \le i < j \le k} \underbrace{E[\Delta(t_i)\Delta(t_j) D_i D_j]}_{E[\Delta(t_i)\Delta(t_j) D_i] \underbrace{E[D_j]}_{=0}}$$

$$= \sum_{j=0}^{k} E[\Delta^2(t_j)] \underbrace{E[D_j^2]}_{t_{j+1} - t_j}$$

$$= \sum_{j=0}^{k} E[\Delta^2(t_j)(t_{j+1} - t_j)]$$

$$= E\left[\underbrace{\sum_{j=0}^{k} \Delta^2(t_j)(t_{j+1} - t_j)}_{\int_0^t \Delta^2(u)\,du}\right]$$

$$E[I^2(t)] = E\left[\int_0^t \Delta^2(u)\,du\right]$$

$$[I,I](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(I(t_{j+1}) - I(t_j) \right)^2$$

let $\Pi_0 = \{ t_0 = s_0, s_1, s_2, \dots, s_n = t_1 \}$

$\vdots$

$\Pi_j = \{ t_j = s_0, s_1, \dots, s_n = t_{j+1} \}$

$$[I,I](t) = \sum_{k=0}^{n-1} \left(\lim_{\|\Pi_k\|\to 0} \sum_{i=0}^{\ell-1} \left(I(s_{i+1}) - I(s_i) \right)^2 \right)$$

$$\lim_{\|\Pi_k\|\to 0} \sum_{i=0}^{\ell-1} \left(I(s_{i+1}) - I(s_i) \right)^2$$

for $s \in [t_k, t_{k+1})$: $I(s) = \sum_{j=0}^{k-1} \Delta(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta(t_k)\left(W(s) - W(t_k)\right)$

$$I(s_{i+1}) - I(s_i) = \Delta(t_k)\left(W(s_{i+1}) - W(s_i) \right)$$

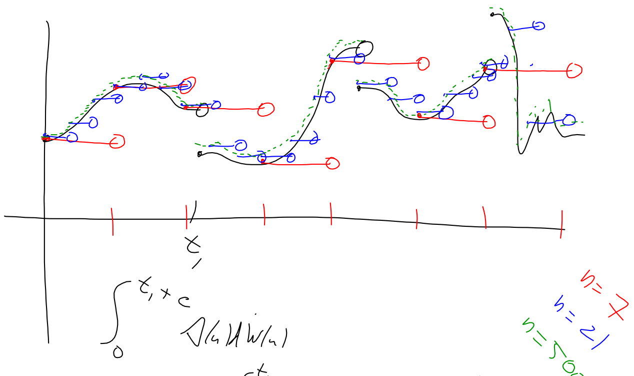
$$\lim_{\|\Pi_k\|\to 0} \sum_{i=0}^{\ell-1} \Delta^2(t_k) \left(W(s_{i+1}) - W(s_i) \right)^2$$

$$= \Delta^2(t_k) \lim_{\|\Pi_k\|\to 0} \sum_{i=0}^{\ell-1} \left(W(s_{i+1}) - W(s_i) \right)^2$$

$$= \Delta^2(t_k) \left(t_{k+1} - t_k \right)$$

$$[I,I](t) = \sum_{k=0}^{n-1} \Delta^2(t_k) \left(t_{k+1} - t_k \right)$$

$$= \int_0^t \Delta^2(u)\,du$$



$$\int_0^{t_1 + c} \Delta(u)\,dW(u)$$

$$= \int_0^{t_1} \Delta(u)\,dW(u) + \underbrace{\int_{t_1}^{t_1+c} \Delta(u)\,dW(u)}_{\lim_{c\to t_1^+}}$$

$$\int_{t_1}^{t_1+c} \Delta(u)\,dW(u) \approx \Delta(t_1)\left(W(t_1 + c) - W(t_1) \right)$$

$$\lim_{c\to t_1^+} \Delta(t_1)\left(W(t_1+c) - W(t_1) \right)$$

$$= 0$$

$$\int_0^t W(u)\,dW(u) = \cancel{\frac{W^2(t)}{2}}$$

$$\int_0^t W(u)\,dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)\,dW(u)$$

$$\Delta_n(u) = \begin{cases} 0 & 0 \le u < t/n \\ W(t/n) & t/n \le u < 2t/n \\ W(2t/n) & 2t/n \le u < 3t/n \\ \vdots & \\ W\left(\frac{(n-1)t}{n}\right) & \frac{(n-1)t}{n} \le u < \frac{nt}{n} = t \end{cases}$$

$$\int_0^t \Delta_n(u)\,dW(u) = \sum_{j=0}^{n-1} W\left(\frac{jt}{n}\right)\left(W\left(\frac{(j+1)t}{n}\right) - W\left(\frac{jt}{n}\right)\right)$$

let $W\left(\frac{jt}{n}\right) = W_j$

$$\int_0^t \Delta_n(u)\,dW(u) = \sum_{j=0}^{n-1} W_j\left(W_{j+1} - W_j\right)$$

$$\frac{1}{2}\sum_{j=0}^{n-1}\left(W_{j+1} - W_j\right)^2 = \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \sum_{j=0}^{n-1} W_{j+1}W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2$$

$$= \frac{1}{2}\sum_{k=\phi}^{n} W_k^2 - \sum_{j=0}^{n-1} W_{j+1}W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2$$

$$= \frac{1}{2}W_n^2 + \underbrace{\frac{1}{2}\sum_{k=0}^{n-1} W_k^2 + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2}_{} - \sum_{j=0}^{n-1} W_{j+1}W_j$$

$$= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}W_j$$

$$= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j\left(W_j - W_{j+1}\right)$$

$$\sum_{j=0}^{n-1} W_j\left(W_{j+1} - W_j\right) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}\left(W_{j+1} - W_j\right)^2$$

$$\int_0^t \Delta_n(u)\,dW(u) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}\left(W_{j+1} - W_j\right)^2$$

$$\lim_{n\to\infty}\int_0^t \Delta_n(u)\,dW(u) = \int_0^t W(u)\,dW(u)$$

$$\int_0^t W(u)\,dW(u) = \lim_{n\to\infty}\left(\frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}\left(W_{j+1} - W_j\right)^2\right)$$

$$= \frac{1}{2}W^2(t) - \frac{1}{2}\underbrace{\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(W_{j+1} - W_j\right)^2}_{t}$$

$$= \frac{W^2(t)}{2} - t/2$$