Markov Processes, Passage Times and Reflection Principle

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Alternate Definition for Markov Property

A slightly different formulation for the Markov property that has the same result is:
For an adapted stochastic process, if the conditional probability distribution for future states depending on the entire history of the process is the same as the conditional probability distribution for future states depending only on the current value of the process, then the process is Markov.

This can be expressed easily for the random walk, which we already know is Markov. Given a random walk M_n as defined previously, we have for some integer x:

$$\mathbb{P}(M_{n+m} = x | M_0, M_1, \ldots, M_n) = \mathbb{P}(M_{n+m} = x | M_n)$$

Theorem 3.5.1: Let $W(t), t \geq 0$, be a Brownian motion and let $\mathcal{F}(t), t \geq 0$, be a filtration for this Brownian motion. Then $W(t), t \geq 0$, is a Markov process.[2]

To prove this, we must show that there exists a function g such that

$$\mathbb{E}[f(W(t))|\mathcal{F}(s)] = g(W(s))$$

for all functions f

Outline of Proof

$$\mathbb{E}[f(W(t))|\mathcal{F}(s)] = \mathbb{E}[f(W(t) - W(s) + W(s))|\mathcal{F}(s)]$$

$W(t) - W(s)$ is independent of $W(s)$ and $\mathcal{F}(s)$. If we let a dummy variable x take the place of $W(s)$, we have:

$$g(x) = \mathbb{E}[f(W(t) - W(s) + x)]$$

but since $W(t) - W(s)$ is a normally distributed random variable with mean 0 and variance $t - s$:

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(w + x) e^{-\frac{w^2}{2(t-s)}} dw$$

Transition Density

Let $\tau = t - s$ and $y = w + x$. This allows the previous formula to be expressed as:

$$g(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tau}} f(y) e^{-\frac{(y-x)^2}{2\tau}} dy$$

If we let $p(\tau, x, y)$ be the **transition density** for the Brownian Motion and set

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$$

then

$$g(x) = \int_{-\infty}^{\infty} f(y) p(\tau, x, y) dy$$

Stopping Times

Definition 4.3.1: In an N-period binomial model, a stopping time is a random variable τ that takes value $0, 1, \ldots, N$ or ∞ and satisfies the condition that if $\tau(\omega_1\omega_2\ldots\omega_n\omega_{n+1}\ldots\omega_N) = n$, then $\tau(\omega_1\omega_2\ldots\omega_n\omega'_{n+1}\ldots\omega'_N) = n$ for all $\omega'_{n+1}\ldots\omega'_N$.

Theorem 4.3.2: A martingale stopped at a stopping time is a martingale. A supermartingale (or submartingale) stopped at a stopping time is a supermartingale (or submartingale, respectively).[1]

First Passage Time

For a symmetric random walk M_n as defined previously, the first time that the random walk reaches level m denoted τ_m is defined as:

$$\tau_m = \min\{n : M_n = m\}$$

This time τ_m is called the **first passage time.** If the level m is never reached, we set $\tau_m = \infty$. This random variable τ_m is a stopping time which is finite almost surely, but has infinite expectation as we shall see.

Exponential Martingale

Let the process $Z(t)$ be defined as:

$$Z(t) = e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$$

Theorem 3.6.1: Let $W(t), t \geq 0$, be a Brownian motion with a filtration $\mathcal{F}(t), t \geq 0$, and let σ be a constant. The process $Z(t), t \geq 0$ is a martingale.[2]

To prove this we show that:

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = Z(s)$$

First Passage Time

For m being a real number, define the **first passage time to level** m

$$\tau_m = \min\{t \geq 0; W(t) = m\}$$

Using this and the property that a stopped martingale is still a martingale we have:

$$1 = Z(0) = \mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}[e^{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2 (t\wedge\tau_m)}]$$

Theorem 3.6.2: For $m \in \mathbb{R}$, the first passage time of Brownian motion to level m is finite almost surely, and the Laplace transform of its distribution is given by

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}$$

for all $\alpha > 0$
Note that in a Laplace transform ($F(s)$), for a function $f(t)$ defined on $[0, \infty)$,

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Reflection Principle: For a random walk, there are as many paths that end above a threshold(value,level) as there are that reach a threshold and then end below the threshold.

As a result, the probability that a first passage time to level m (τ_m) is less than or equal to some value n is given as:

$$\mathbb{P}(\tau_m \leq n) = \mathbb{P}(M_n = m) + 2\mathbb{P}(M_n > m)$$

[2]

We can see how this works for a simple example. Let $m = 1$ and $n = 3$ for a symmetric random walk. If we observe the paths, we can see that there are three paths that have $M_3 = 1$ and one path where $M_3 > 1$. Since each path for a symmetric random walk is equally likely, we have:

$$
\begin{aligned}
\mathbb{P}(\tau_1 \leq 3) &= \mathbb{P}(M_3 = 1) + 2\mathbb{P}(M_3 > 1) \\
&= \frac{3}{8} + 2\left(\frac{1}{8}\right) \\
&= \frac{5}{8}
\end{aligned}
$$

We can use a similar approach to determine probabilities using Brownian motion instead of random walks. The key idea to consider is that the reflection principle still holds. However, now the probability of ending exactly at the level m is 0 due to the continuous distribution. As such we are considering the paths that end above some level m and those that at one point were above m but now are at some level $w \leq m$.

Reflection Equality

Because of this reflection principle, for every path that has $\tau_m < t$, but $W(t) < m$, there is another path that ends with $W(t) > m$ that behaved exactly opposite after time τ_m. (If the first path moved upwards by .3 units at time t_1, then the second path moved downwards by .3 units at time t_1) This leads to the **reflection equality**:

$$\mathbb{P}\{\tau_m \leq t, W(t) \leq w\} = \mathbb{P}\{W(t) \geq 2m - w\}, w \leq m, m > 0$$

Theorem 3.7.1: For all $m \neq 0$, the random variable τ_m has cumulative distribution function

$$\mathbb{P}\{\tau_m \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, t \geq 0$$

and density

$$f_{\tau_m}(t) = \frac{d}{dt}\mathbb{P}\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, t \geq 0$$

[2]

Remark 3.7.2: From the previous result we have:

$$\begin{aligned}\mathbb{E}[e^{-\alpha\tau_m}] &= \int_0^\infty e^{-\alpha t} f_{\tau_m}(t)dt \\ &= \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\alpha t - \frac{m^2}{2t}} dt, \forall \alpha > 0\end{aligned}$$

But remember from Theorem 3.6.2, we have

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}$$

[2]

Maximum to Date

Define the **maximum to date** of a Brownian motion to be the process $M(t)$ defined as:

$$M(t) = \max_{0 \leq s \leq t} W(s)$$

Note that $M(t) \geq m$ iff $\tau_m \leq t$, and so we can rewrite the reflection equality as

$$\mathbb{P}\{M(t) \geq m, W(t) \leq w\} = \mathbb{P}\{W(t) \geq 2m - w\}, w \leq m, m > 0$$

Theorem 3.7.3: For $t > 0$, the joint density of $(M(t), W(t))$ is

$$f_{M(t),W(t)}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \leq m, m > 0$$

[2] The key to this proof is to look at the joint density function:

$$\mathbb{P}\{M(t) \geq m, W(t) \leq w\} = \int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)dydx$$

and equate it using the reflection equality.

Corollary 3.7.4: The conditional distribution of $M(t)$ given $W(t) = w$ is

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m-w)}{t}e^{-\frac{2m(m-w)}{t}}, w \leq m, m > 0$$

[2]

The key idea to this proof is to remember the formula for conditional distributions:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

[1] S.E. Shreve. *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Number v. 1 in Springer Finance.

[2] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.