Stochastic Calculus(Integrands)

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What is a Stochastic Integral

$$\int_0^T \Delta(t)dW(t) = ?$$

If we let $W(t), t \geq 0$ be a Brownian motion with respect to a filtration $\mathcal{F}(t), t \geq 0$ and $\Delta(t)$ be an adapted process such that it is $\mathcal{F}$-measurable, then we can define this expression.

Simple Functions

To do so, we begin with simple functions. A simple function in real analysis is defined to be a function that only takes finite values. We can think of this as:

$$f(x) = \sum_{k \in K} a_k \mathbb{I}_{\{x \in A_k\}}$$

Simple Process

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$ be a partition of the interval $[0, T]$
Let $\Delta(t)$ be constant in each interval $[t_j, t_{j+1})$, as such $\Delta(t)$ is a simple process.

Think of $\Delta(t)$ as the position taken in an underlying stock whose price is determined by the process $W(t)$. We can only change our position in the stock on the trading dates $t_0, t_1, \ldots, t_n$. As such the gain (or loss) of our portfolio at time t such that $t_k \leq t \leq t_{k+1}$ is given by the function:

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

As such, this is the same as the integral of the simple process $\Delta(t)$ and as such we now have a representation for:

$$I(t) = \int_0^t \Delta(u) dW(u)$$

This is known as the Ito integral.

Ito as Martingale

Theorem 4.2.1: The Ito integral $I(t)$ is a martingale.[1]
Proof: In order to prove it, let $0 \leq s \leq t \leq T$ be given and show that:

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$$

We will need to show it when s and t are in the same partition and when they are not. The case when they are in the same partition is much simpler.

Ito Isometry

Theorem 4.2.2: The Ito integral $I(t)$ satisfies

$$\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t \Delta^2(u)du\right]$$

[1]
This is the variance of the Ito integral, as because the Ito integral is a martingale and $I(0) = 0$, we have $\mathbb{E}[I(t)] = 0$

Theorem 4.2.3: The quadratic variation accumulated up to time t by the Ito Integral $I(t)$ is

$$[I, I](t) = \int_0^t \Delta^2(u) du$$

[1]
Note that this is not the same as the Isometry

Square Integrability

To expand the Ito integral to non-simple functions we need a couple of conditions. First, let $\Delta(t), t \geq 0$ be adapted to the filtration $\mathcal{F}(t), t \geq 0$. Second, the process $\Delta(t)$ must satisfy:

$$\mathbb{E}\left[\int_0^T \Delta^2(t)\right] < \infty$$

This is known as the **square-integrability condition**

Let $\Delta_n(t)$ be a sequence of simple processes, such that $\Delta_n(t) \to \Delta(t)$. By this convergence we mean:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt\right] = 0 \tag{1}$$

The Ito integral is then defined as

$$\int_0^t \Delta(u)dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)dW(u), 0 \leq t \leq T \tag{2}$$

Theorem 4.3.1 Let T be a positive constant and let $\Delta(t), 0 \leq t \leq T$, be an adapted stochastic process that satisfies (1). Then $I(t) = \int_0^t \Delta(u)dW(u)$ defined by (2) has the following properties.

1. **(Continuity)** As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.
2. **(Adaptivity)** For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.
3. **(Linearity)** If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)dW(u)$
4. **(Martingale)** $I(t)$ is a martingale
5. **(Ito Isometry)** $\mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t \Delta^2(u)du]$
6. **(Quadratic Variation)** $[I, I](t) = \int_0^t \Delta^2(u)du$

[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.