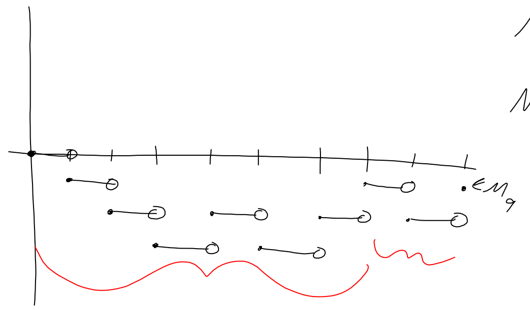


$M_9 = -1$

$M_9 - M_7$

ind of M_7

$$M_n = \sum_{i=1}^{n} X_i$$

$$M_9 - M_7 = \sum_{i=1}^{9} X_i - \sum_{i=1}^{7} X_i = X_8 + X_9$$

$E[M_k - M_\ell]$ for $\ell < k$

$$E[X_j] = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\omega)$$
$$= X(H)\,\mathbb{P}(H) + X(T)\,\mathbb{P}(T)$$
$$= 1\left(\tfrac{1}{2}\right) + -1\left(\tfrac{1}{2}\right) = 0$$

$$Var(X_j) = E[X_j^2] - \underbrace{E[X_j]^2}_{=0}$$

$$E[X_j^2] = \sum_{\omega \in \Omega} X_j^2(\omega)\, \mathbb{P}(\omega)$$
$$= 1^2\left(\tfrac{1}{2}\right) + (-1)^2\left(\tfrac{1}{2}\right) = 1$$

$Var(X_j) = 1$

$$E[M_k - M_\ell] = E\left[\sum_{j=\ell+1}^{k} X_j\right]$$
$$= \sum_{j=\ell+1}^{k} E[X_j] = 0$$

$Var(M_k - M_\ell)$

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$$

$$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + 2\alpha\beta\, Cov(X,Y) + \beta^2 Var(Y)$$

if $Cov(X,Y) = 0$

$$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y)$$

$$Var(M_k - M_\ell) = \sum_{j=\ell+1}^{k} Var(X_j)$$
$$= k - \ell$$

$$E[M_k \mid \mathcal{F}(\ell)] = M_\ell \ ? \qquad \text{for } \ell < k$$

$$E[M_k \mid \mathcal{F}(\ell)] = E[\underbrace{M_k - M_\ell}_{\text{ind of } \mathcal{F}(\ell)} + \underbrace{M_\ell}_{\mathcal{F}(\ell)\text{-measurable}} \mid \mathcal{F}(\ell)]$$

$$= E[M_k - M_\ell \mid \mathcal{F}(\ell)] + E[M_\ell \mid \mathcal{F}(\ell)]$$

$$= E[M_k - M_\ell \mid \mathcal{F}(\ell)] + M_\ell \underbrace{E[1 \mid \mathcal{F}(\ell)]}_{=1}$$

$$= E[M_k - M_\ell] + M_\ell$$

$$= M_\ell$$

$$E[f(M_k) \mid \underline{\mathcal{F}(\ell)}] = g(M_\ell) \qquad \text{for } \ell < k$$

$$E[f(\underbrace{M_k - M_\ell}_{\text{ind.}} + \underbrace{M_\ell}_{\text{meas.}}) \mid \mathcal{F}(\ell)]$$

Let x be dummy variable, i.e. $f(x) = x^2$

$$E[f(M_k - M_\ell + x) \mid \mathcal{F}(\ell)]$$

$$= E[f(M_k - M_\ell + x)]$$

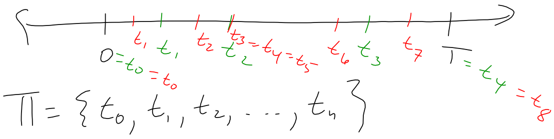
$$= \sum_{i=0}^{k-\ell} \underbrace{\binom{k-\ell}{i} \frac{1}{2}^{i} \frac{1}{2}^{k-\ell-i}}_{\mathbb{P}\text{ getting } i \text{ heads}} f(2i - (k-\ell) + x)$$

$$= g(x)$$

$$g(7) = E[f(M_k - M_\ell + 7) \mid \mathcal{F}(\ell)]$$

$$g(M_\ell) = E[f(M_k - M_\ell + M_\ell) \mid \mathcal{F}(\ell)]$$

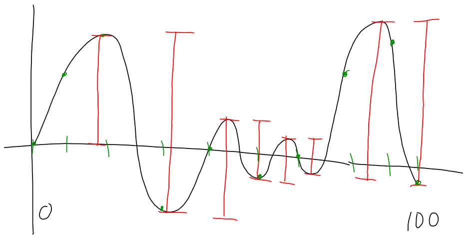
$$= E[f(M_k) \mid \mathcal{F}(\ell)]$$

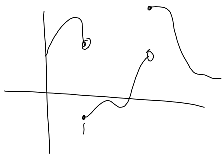


$$\Pi = \{t_0, t_1, t_2, \dots, t_n\}$$

$$\begin{cases} t_0 \le t_1 \le t_2 \le \dots \le t_n \\ t_0 = 0 \\ t_n = T \end{cases}$$

$$\|\Pi\| = \max_{0 \le k \le n-1} (t_{k+1} - t_k)$$





MVT: for (x_j, x_{j+1}) $\exists\, x_j^* \in [x_j, x_{j+1}]$

s.t. $f'(x_j^*) = \dfrac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}$

assuming f is cont. & diff

$$f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$$

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum |f'(t_j^*)|(t_{j+1} - t_j)$$

$$= \int_0^T |f'(t)|\,dt$$

2nd order variation = quadratic variation

$$[f,f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(f(t_{j+1}) - f(t_j)\right)^2$$

if f is cont. & diff

$$[f,f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} (f'(t_j^*))^2 (t_{j+1} - t_j)^2$$

$$\le \lim_{\|\Pi\| \to 0} \left(\|\Pi\| \sum_{j=0}^{n-1} (f'(t_j^*))^2 (t_{j+1} - t_j) \right)$$

$X = X$ $\qquad (t_{j+1} - t_j) = (t_{j+1} - t_j)$

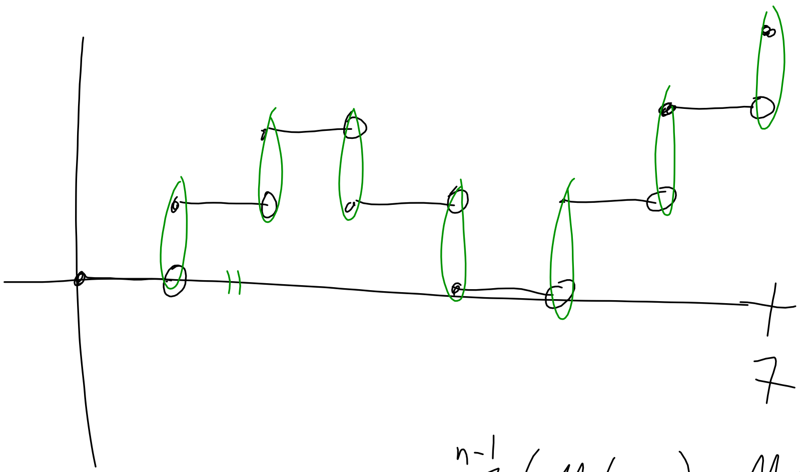
$a \ge b$ $\qquad \|\Pi\| \ge (t_{j+1} - t_j)$

$ax \ge bx$ $\qquad \|\Pi\| (t_{j+1} - t_j) \ge (t_{j+1} - t_j)^2$

$$= \underbrace{\lim_{\|\Pi\| \to 0} (\|\Pi\|)}_{0} \cdot \underbrace{\int_0^T (f'(t_j))^2\,dt}_{< \infty}$$

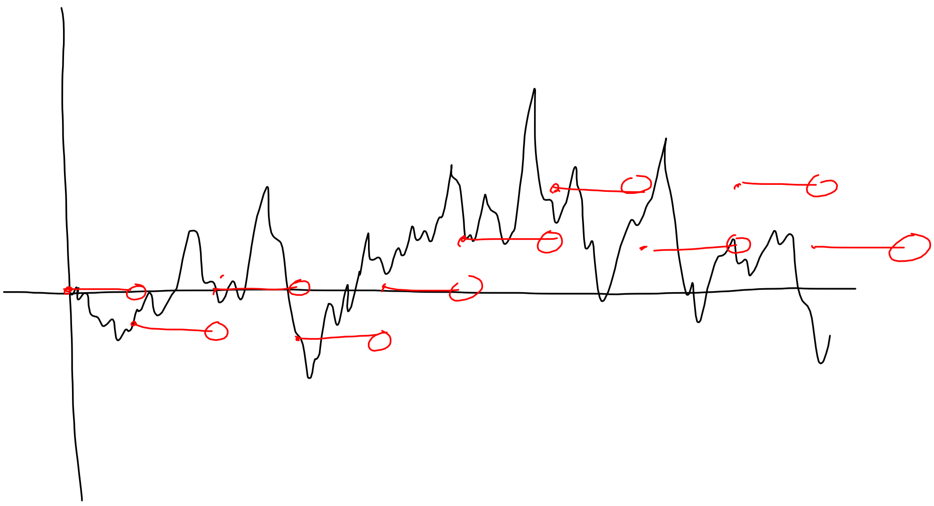
$$= 0$$

$$[M,M](7) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(M(t_{j+1}) - M(t_j) \right)^2$$



$$[M,M](7) = \sum_{j=0}^{n-1} \left(M(t_{j+1}) - M(t_j) \right)^2$$

$$= \sum_{j=0}^{n-1} \left(X_{t_{j+1}} \right)^2 = \sum_{j=1}^{n} X_j^2 = 7$$



$$\lim_{n \to \infty} W^{(n)}(t) \to W(t)$$

$$\text{WTS: } \lim_{n\to\infty} \underbrace{E\left[e^{uW^{(n)}(t)}\right]}_{\varphi_{W^{(n)}(t)}(u)} = \underbrace{e^{\frac{1}{2}u^2 t}}_{\text{MGF of } N(0,t)}$$

$$\varphi_{W^{(n)}(t)}(u) = E\left[e^{uW^{(n)}(t)}\right] = E\left[e^{u\frac{1}{\sqrt{n}}\sum_{j=1}^{nt} X_j}\right]$$

$$= E\left[e^{\sum_{j=1}^{nt}\frac{uX_j}{\sqrt{n}}}\right]$$

$$= \prod_{j=1}^{nt} E\left[e^{\frac{uX_j}{\sqrt{n}}}\right]$$

$$= \prod_{j=1}^{nt}\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)$$

$$= \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^{nt}$$

$$\lim_{n\to\infty}\left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^{nt} = e^{\frac{1}{2}u^2 t} \quad \text{WTS:}$$

$$\lim_{n\to\infty} \log\left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^{nt} = \underline{\frac{1}{2}u^2 t}$$

let $n = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{n}}$ & $\lim_{n\to\infty} = \lim_{x\to 0^+}$

$$t \cdot \lim_{x\to 0^+} \frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}$$

$$\overset{L}{=} t \cdot \lim_{x\to 0^+} \frac{\left(\frac{1}{\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}}\right)\left(\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}\right)}{2x}$$

$$= t \cdot \lim_{x\to 0^+} \frac{\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}}{2x}$$

$$= \frac{ut}{2} \lim_{x\to 0^+} \frac{e^{ux} - e^{-ux}}{2x}$$

$$\overset{L}{=} \frac{ut}{2} \cdot \lim_{x\to 0^+} \frac{ue^{ux} + ue^{-ux}}{2}$$

$$= \frac{u^2 t}{2} \cdot \lim_{x\to 0^+} \frac{e^{ux} + e^{-ux}}{2} = \frac{u^2 t}{2}$$

$$X_n \xrightarrow{\mathcal{L}^2} Y \Rightarrow \lim_{n\to\infty} E[X_n] = E[Y]$$
$$\lim_{n\to\infty} E[X_n^2] = E[Y^2]$$

$$Y = T \qquad E[Y] = T \qquad Var(Y) = 0$$

for $\Pi = \{t_0, t_1, t_2, \dots, t_n\}$
$$Q_\Pi = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$$

$$\lim_{\|\Pi\|\to 0} E[Q_\Pi] = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} E[(W(t_{j+1}) - W(t_j))^2]$$

$$Var(X) = E[X^2] - E[X]^2$$
$$Var(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j$$
$$E[(W(t_{j+1}) - W(t_j))^2] = t_{j+1} - t_j + E[W(t_{j+1}) - W(t_j)]^2$$
$$= t_{j+1} - t_j$$

$$= \lim_{\|\Pi\|\to 0} \underbrace{\sum_{j=0}^{n-1} (t_{j+1} - t_j)}_{T}$$

$$= T$$

$$Var(Q_\Pi) = Var\left(\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right)$$
$$= \sum_{j=0}^{n-1} Var((W(t_{j+1}) - W(t_j))^2)$$

$$Var((W(t_{j+1}) - W(t_j))^2) = E[(W(t_{j+1}) - W(t_j))^4] - \underbrace{E[(W(t_{j+1}) - W(t_j))^2]^2}$$
$$= 3(t_{j+1} - t_j)^2 \quad - (t_{j+1} - t_j)^2$$
$$= 2(t_{j+1} - t_j)^2$$

$$Var(Q_\Pi) = \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2$$

$$\lim_{\|\Pi\|\to 0} Var(Q_\Pi) = 2 \underbrace{\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2} = 0$$

$$[W,W](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

$$d[W,W](t) = \lim_{\|\Pi_2\|\to 0} \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

$$- \lim_{\|\Pi_1\|\to 0} \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

where $\Pi_1 = \{t_0 = 0, t_1, t_2, \dots, t_n = t\}$

$\Pi_2 = \{t_0 = 0, t_1, t_2, \dots, t_n = t + \Delta t\}$

$$d[W,W](t) = \Big(\underbrace{W(t+\Delta t) - W(t)}_{dW(t)}\Big)^2 = \big(dW(t)\big)^2$$

$$[W,W](t) = t \Rightarrow d[W,W](t) = dt$$

$$S(t) = S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$$\log\left(\frac{S(t_{i+1})}{S(t_i)}\right) = \log\left(\frac{S(0)\, e^{(\alpha-\sigma^2/2)t_{i+1} + \sigma W(t_{i+1})}}{S(0)\, e^{(\alpha-\sigma^2/2)t_i + \sigma W(t_i)}}\right)$$

$$= (\alpha - \sigma^2/2)(t_{i+1} - t_i) + \sigma\left(W(t_{j+1}) - W(t_j)\right)$$

if you observe "often" from T_1 to T_2

$$\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2 = (\alpha - \sigma^2/2)^2 (t_{i+1} - t_i)^2 + 2\sigma\left(\alpha - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i)\left(W(t_{i+1}) - W(t_i)\right)$$

$$+ \sigma^2\left(W(t_{i+1}) - W(t_i)\right)^2$$

$$\sum_{j=0}^{M-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2 = \left(\alpha - \frac{\sigma^2}{2}\right)^2 \sum_{j=0}^{M-1} \underbrace{(t_{j+1} - t_j)^2}_{dt^2 = 0}$$

$$+ 2\left(\alpha - \frac{\sigma^2}{2}\right)\sigma \sum_{j=0}^{M-1} \underbrace{(t_{j+1} - t_j)}_{dt}\underbrace{\left(W(t_{j+1}) - W(t_j)\right)}_{dW(t)} = 0$$

$$+ \sigma^2 \sum_{j=0}^{M-1} \underbrace{\left(W(t_{j+1}) - W(t_j)\right)^2}_{dW(t)^2 = dt}$$

$$= \sigma^2 \underbrace{\sum_{j=0}^{M-1} \left(W(t_{j+1}) - W(t_j)\right)^2}_{T_2 - T_1}$$

$$\sigma^2 \approx \frac{1}{T_2 - T_1} \sum_{j=0}^{M-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2$$