Fundamental Theorems, Dividend Paying Stocks, Futures and Forwards

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Theorem 5.4.1: Let T be a fixed positive time, and let $\Theta(t) = (\Theta_1(t), \dots, \Theta_d(t))$ be a d-dimensional adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u) du$$

and assume that

$$\mathbb{E}\left[\int_0^T \| \Theta(u) \|^2 Z^2(u) du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$, and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$ is a d-dimensional Brownian motion.[1]

Theorem 5.4.2: Let T be a fixed positive time, and assume that $\mathcal{F}(t), 0 \leq t \leq T$, is the filtration generated by the d-dimensional Brownian motion $W(t), 0 \leq t \leq T$. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration under $\mathbb{P}$. Then there is an adapted, d-dimensional process $\Gamma(u) = (\Gamma_1(u), \ldots, \Gamma_d(u)), 0 \leq t \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u), 0 \leq t \leq T$$

If, in addition, we assume the notation and assumptions of Theorem 5.4.1 and if $\widetilde{M}(t), 0 \leq t \leq T$, is a $\widetilde{\mathbb{P}}$-martingale, then there is an adapted, d-dimensional process $\widetilde{\Gamma}(u) = (\widetilde{\Gamma_1}(u), \ldots, \widetilde{\Gamma_d}(u))$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \cdot d\widetilde{W}(u), 0 \leq t \leq T$$

[1]

Multidimensional Market Model

Assume the existence of m stocks, each with a stochastic differential

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), i = 1, \ldots, m$$

In this representation, the σ_{ij}'s denote entries in the covariance matrix of a d-dimensional Brownian motion. Note that if $d = m$ and $\sigma_{ij} = 0$ for $i \neq j$ then each stock price is uncorrelated.

Definition 5.4.3: A probability measure $\tilde{\mathbb{P}}$ is said to be risk-neutral if

1. $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e. for every $A \in \mathcal{F}, \mathbb{P}(A) = 0$ if and only if $\tilde{\mathbb{P}}(A) = 0$)
2. under $\tilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, \ldots, m$

[1]

Lemma 5.4.5: Let $\tilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\tilde{\mathbb{P}}$, the discounted portfolio $D(t)X(t)$ is a martingale.[1]

Definition 5.4.6: An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$

$$\mathbb{P}(X(T) \geq 0) = 1, \mathbb{P}(X(T) > 0) > 0$$

[1]

Theorem 5.4.7: (First fundamental theorem of asset pricing) If a market model has a risk-neutral probability measure, then it does not admit arbitrage.[1]

Definition 5.4.8: A market model is complete if every derivative security can be hedged.[1]

Theorem 5.4.9: (Second fundamental theorem of asset pricing) Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.[1]

Continuously Paying Dividends

For a continuously dividend paying process, $A(t)$, we can express as the model of our stock price

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt$$

This would result in the portfolio dynamics given by

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + \Delta(t)A(t)S(t)dt + R(t)[X(t) - \Delta(t)S(t)]dt \\ &= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)\left[\Theta(t)dt + dW(t)\right] \end{aligned}$$

Lump Payments of Dividends

For given times $0 < t_1 < t_2 < \cdots < t_n < T$ where a payment is made and the payment at time t_j is given by $a_j S(t_{j-})$. This gives us the price of the stock at time t_j given by

$$S(t_j) = S(t_{j-}) - a_j S(t_{j-}) = (1 - a_j) S(t_{j-})$$

Despite these payments from the stock, the portfolio doesn't see jumps in its value (as the dividend is paying into the portfolio). This means that we still have the dynamic

$$dX(t) = R(t)X(t)dt + \Delta(t)\sigma(t)S(t)\left[\Theta(t)dt + dW(t)\right]$$

Definition 5.6.1: A forward contract is an agreement to pay a specified delivery price K at a delivery time T, where $0 \leq T \leq \bar{T}$, for the asset whose price at time t is $S(t)$. The T-forward price $For_S(t, T)$ of this asset at time t, where $0 \leq t \leq T \leq \bar{T}$, is the value of K that makes the forward contract have no-arbitrage price zero at time t.[1]

Theorem 5.6.2: Assume that zero-coupon bonds of all maturities can be traded. Then

$$For_S(t, T) = \frac{S(t)}{B(t, T)}, 0 \leq t \leq T \leq \bar{T}$$

[1]

Definition 5.6.4: The futures price of an asset whose value at time T is $S(T)$ is given by the formula

$$Fut_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)], 0 \le t \le T$$

A long position in the futures contract is an agreement to receive as a cash flow the changes in the futures price (which may be negative as well as positive) during the time the position is held. A short position in the futures contract receives the opposite cash flow.[1]

Theorem 5.6.5: The futures price is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, it satisfies $Fut_S(T,T) = S(T)$, and the value of a long (or a short) futures position to be held over an interval of time is always zero.[1]

Forward-Futures Spread

The **forward-futures spread** is given by:

$$For_S(0,T) - Fut_S(0,T) = \frac{1}{B(0,T)} \widetilde{Cov}(D(T), S(T))$$

if the interest rate is a constant, r, then the covariance will be 0 (as D(T) will be deterministic) and the forward price and the future price will agree.

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.