Exotic Options

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June 27, 2016

Define:

$$\widehat{W}(t) = \alpha t + \widetilde{W}(t), 0 \le t \le T$$

$$\widehat{M}(T) = \max_{0 \le t \le T} \widehat{W}(t)$$

Theorem 7.2.1: The joint density under $\widetilde{\mathbb{P}}$ of the pair $(\widehat{M}(T), \widehat{W}(T))$ is

$$\tilde{f}_{\widehat{M}(T),\widehat{W}(T)}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, w \le m, m \ge 0$$

and is zero for other values of m and w[1]

Corollary 7.2.2: We have

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \leq m\} = N\left(\frac{m - \alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), m \geq 0$$

and the density under $\widetilde{\mathbb{P}}$ of the random variable $\widehat{M}(T)$ is

$$\tilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), m \geq 0$$

and is zero for $m < 0$.[1]

Given a process $\widehat{W}(t)$ and a risky asset given by

$$S(t) = S(0)e^{\sigma\widetilde{W}(t)+(r-\frac{1}{2}\sigma^2)t} = S(0)e^{\sigma\widehat{W}(t)}$$

with $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$. Using our process $\widehat{M}(t)$ we then have

$$\max_{0\leq t\leq T} S(t) = S(0)e^{\sigma\widehat{M}(t)}$$

This option will have pay off given by

$$V(T) = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)_+ \mathbb{I}_{\{S(0)e^{\sigma\widehat{M}(T)}\leq B\}}$$

Theorem 7.3.1: Let $v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t) = x$. Then $v(t, x)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

in the rectangle $\{(t, x); 0 \leq t < T, 0 \leq x \leq B\}$ and satisfies the boundary conditions

$$\begin{aligned} v(t, 0) &= 0, 0 \leq t \leq T, \\ v(t, B) &= 0, 0 \leq t < T, \\ v(T, x) &= (x - K)_+, 0 \leq x \leq B \end{aligned}$$

[1]

Lemma 7.3.2: We have

$$V(t) = v(t, S(t)), 0 \leq t \leq \rho$$

In particular, $e^{-rt}v(t, S(t))$ is a $\widetilde{\mathbb{P}}$-martingale up to time ρ, or, put another way, the stopped process

$$e^{-r(t \wedge \rho)}v(t \wedge \rho, S(t \wedge \rho)), 0 \leq t \leq T$$

is a martingale under $\widetilde{\mathbb{P}}$.[1]

$$
\begin{aligned}
v(t,x) = & x\left[N\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\right] \\
& - e^{-rt}K\left[N\left(\delta_-\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau,\frac{x}{B}\right)\right)\right] \\
& - B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau,\frac{B}{x}\right)\right)\right] \\
& e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right], \\
& \text{for } 0 \le t < T, 0 < x \le B
\end{aligned}
$$

Let

$$Y(t) = \max_{0 \le u \le t} S(u) = S(0) e^{\sigma \widehat{M}(t)}$$

So the lookback option has payoff

$$V(T) = Y(T) - S(T)$$

This leads us to the valuation:

$$V(t) = \widetilde{\mathbb{E}} \left[e^{-r(T-t)} (Y(T) - S(T)) | \mathcal{F}(t) \right]$$

Because the two-dimensional process $(S(t), Y(t))$ is Markov there exists a function:

$$V(t) = v(t, S(t), Y(t))$$

Theorem 7.4.1: Let $v(t, x, y)$ denote the price at time t of the floating strike lookback option under the assumption that $S(t) = x$ and $Y(t) = y$. Then $v(t, x, y)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t,x,y) + rxv_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y)$$

in the region $\{(t,x,y); 0 \leq t < T, 0 \leq x \leq y\}$ and satisfies the boundary conditions

$$\begin{aligned} v(t,0,y) &= e^{-r(T-t)}y, 0 \leq t \leq T, y \geq 0 \\ v_y(t,y,y) &= 0, 0 \leq t \leq T, y > 0 \\ v(T,x,y) &= y - x, 0 \leq x \leq y \end{aligned}$$

[1]

$$v(t,x,y) = \left(1+\frac{\sigma^2}{2r}\right) xN\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) + e^{-r\tau} yN\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right)$$
$$-\frac{\sigma^2}{2r}e^{-r\tau}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} xN\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) - x$$
$$\text{for } 0 \le t < T, 0 < x \le y$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.