X_0 is initial wealth

Δ_0 is the number of shares purchased

S_0 value of asset at time 0.

at time 0:

$$X_0 = (X_0 - \Delta_0 S_0) + \Delta_0 S_0$$

at time 1:

$$X_1 = (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_1$$

at time 2:

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$$

at time n+1:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

$$X_{n+1} - X_n = \Delta X$$

$$+\Delta_n S_n - \Delta_n S_n$$

$$\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) - X_n = \Delta X$$

$$\Delta_n (S_{n+1} - S_n) + (X_n - \Delta_n S_n) + r(X_n - \Delta_n S_n) - X_n + \Delta_n S_n = \Delta X$$

$$\Delta_n (S_{n+1} - S_n) + r(X_n - \Delta_n S_n) = \Delta X$$

$$\Rightarrow \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt = dX(t)$$

$$dX(t) = \Delta(t)\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big) + rX(t)\,dt - r\Delta(t)S(t)\,dt$$

$$= rX(t)\,dt + \Delta(t)S(t)(\alpha - r)\,dt + \Delta(t)S(t)\sigma\,dW(t)$$

$$f(t,x) = e^{-rt}x$$

$$f(t,S(t)) = e^{-rt}S(t)$$

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$$

$$d f(t,\underset{X(t)}{S(t)}) = f_t(t,\overset{X(t)}{S(t)})\,dt + f_x(t,\overset{X(t)}{S(t)})\,d\overset{X(t)}{S(t)}$$
$$+\frac{1}{2} f_{xx}(t,\overset{X(t)}{S(t)})\,d[\overset{X}{S},\overset{X}{S}](t)$$

$$f(t,x) = e^{-rt}x \qquad (dS(t))^2 = \sigma^2 S^2(t)\,dt$$
$$f_t = -re^{-rt}x$$
$$f_x = e^{-rt}$$
$$f_{xx} = 0$$

$$d(e^{-rt}S(t)) = -re^{-rt}S(t)\,dt$$
$$+e^{-rt}(\alpha S(t)\,dt + \sigma S(t)\,dW(t))$$
$$= e^{-rt}((\alpha-r)S(t)\,dt + \sigma S(t)\,dW(t))$$

$$f(t,x) \Rightarrow f(t,X(t)) = e^{-rt}X(t)$$

$$d(e^{-rt}X(t))$$
$$= -re^{-rt}X(t)\,dt + e^{-rt}dX(t)$$
$$= \underline{-re^{-rt}X(t)\,dt}$$
$$+e^{-rt}(\underline{rX(t)\,dt} + \Delta(t)S(t)(\alpha-r)\,dt$$
$$+\Delta(t)S(t)\sigma\,dW(t))$$
$$= e^{-rt}\Delta(t)((\alpha-r)S(t)\,dt + S(t)\sigma\,dW(t))$$
$$= \Delta(t)\,d(e^{-rt}S(t))$$

$c(t,S(t))$ value of a Euro Call option at time t with underlying asset price at time t of $S(t)$

$$d\,c(t,S(t)) = c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))\,d[S,S](t)$$

$$= c_t(t,S(t))dt + c_x(t,S(t))(\alpha S(t)dt + \sigma S(t)\,dW(t)) + \frac{1}{2}c_{xx}(t,S(t))\,\sigma^2 S^2(t)dt$$

$$= \left(c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\,c_{xx}(t,S(t))\right)dt + \sigma S(t)\,c_x(t,S(t))\,dW(t)$$

$$d\left(e^{-rt}c(t,S(t))\right) = -re^{-rt}c(t,S(t))dt + e^{-rt}d\left(c(t,S(t))\right)$$

$$= e^{-rt}\left(-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)\,c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\,c_{xx}(t,S(t))\right)dt + e^{-rt}\sigma S(t)\,c_x(t,S(t))\,dW(t)$$

$$e^{-rT}X(T) = e^{-rT}c(T,S(T))$$

$$\Rightarrow d(e^{-rt}X(t)) = d\left(e^{-rt}c(t,S(t))\right)$$

$$\int_0^T d(e^{-rt}X(t)) = \int_0^T d\left(e^{-rt}c(t,S(t))\right)$$

$$e^{-rT}X(T) - \underline{X(0)} = e^{-rT}c(T,S(T)) - \underline{c(0,S(0))}$$

$$X(0) = c(0,S(0))$$

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}c(t,S(t))\right)$$

$$\begin{array}{l|l} e^{-rt}\Delta(t)S(t)(\alpha-r)\,dt & e^{-rt}(\sim)\,dt \\ + e^{-rt}\Delta(t)\sigma S(t)\,dW(t) & + e^{-rt}(\sim)\,dW(t) \end{array}$$

$$Ax + By = Cx + Dy$$

$$\Rightarrow A = C$$
$$B = D$$

$\underline{dW(t):}$ $$e^{-rt}\Delta(t)\sigma S(t) = e^{-rt}\sigma S(t)c_x(t,S(t))$$

$$\Delta(t) = c_x(t,S(t))$$

$\underline{dt:}$

$$\underline{\Delta(t)}(\underline{\alpha}-r)\underline{S(t)} = -rc(t,S(t)) + c_t(t,S(t)) + \underline{\alpha S(t)c_x(t,S(t))} + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))$$

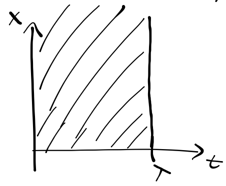
$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))$$

let $x = S(t)$

$$0 = -rc(t,x) + c_t(t,x) + rx\,c_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x)$$

Boundary Conditions for Euro Call

$C(T,x) = (x-K)_+$ $\quad (y)_+ = \max(0,y)$



$t \in [0,T]$

$x \in [0,\infty)$

$$\lim_{x\to 0} c(t,x) = 0$$

$$\lim_{x\to\infty} c(t,x) \approx x$$

$$c(t,x) = x N(d_+) - K e^{-r\tau} N(d_-)$$

$$d_\pm = \frac{\log\left(\frac{x}{K}\right) + \left(r \pm \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \qquad N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$\tau = T - t \qquad \frac{\partial d_\pm}{\partial x} = \frac{1}{x\sigma\sqrt{\tau}}$$

$$C_x(t,x) = ?$$

$$= N(d_+) + x N'(d_+)\frac{\partial d_+}{\partial x} - K e^{-r\tau} N'(d_-)\frac{\partial d_-}{\partial x}$$

$$= N(d_+) + x \frac{1}{\sqrt{2\pi}} e^{-\frac{d_+^2}{2}} \left(\frac{1}{x\sigma\sqrt{\tau}}\right) - \frac{K e^{-r\tau}}{x\sigma\sqrt{\tau}\sqrt{2\pi}} e^{-\frac{d_-^2}{2}}$$

$$= N(d_+) + \frac{1}{x\sigma\sqrt{2\pi\tau}}\left(x e^{-\frac{d_+^2}{2}} - K e^{-r\tau - \frac{d_-^2}{2}} \right)$$

$$d_- = d_+ - \sigma\sqrt{\tau}$$

$$-\frac{d_-^2}{2} = -\frac{d_+^2}{2} + d_+\sigma\sqrt{\tau} - \frac{\sigma^2\tau}{2}$$

$$= N(d_+) + \frac{1}{x e^{\frac{d_+^2}{2}} \sigma\sqrt{2\pi\tau}}\left(x - K e^{-r\tau - \frac{\sigma^2\tau}{2} + d_+\sigma\sqrt{\tau}} \right)$$

$$= N(d_+) + \sim\left(x - K e^{-r\tau - \frac{\sigma^2\tau}{2} + \log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau} \right)$$

$$= N(d_+) + \sim\left(x - K e^{\log\left(\frac{x}{K}\right)} \right)$$

$$= N(d_+)$$

$$C_{xx}(t,x) = N'(d_+)\frac{\partial d_+}{\partial x} = \frac{1}{x\sigma\sqrt{2\pi\tau}} e^{-\frac{d_+^2}{2}}$$

$$C_t(t,x) = -rKe^{-r\tau}N(d_-) - \frac{x\sigma}{2\sqrt{\tau}} N'(d_+)$$

$$rc = c_t + x r c_x + \frac{1}{2}\sigma^2 x^2 c_{xx}$$

$$= -rKe^{-r\tau}N(d_-) - \frac{x\sigma}{2\sqrt{\tau}} N'(d_+)$$

$$+ x r N(d_+) + \frac{1}{2}\sigma^2 x^2 N'(d_+)\frac{1}{x\sigma\sqrt{\tau}}$$

$$= -rKe^{-r\tau}N(d_-) + x r N(d_+)$$

$$- \frac{x\sigma}{2\sqrt{\tau}} N'(d_+) + \frac{1}{2}\sigma x N'(d_+)\frac{1}{\sqrt{\tau}}$$

$$= r\left(x N(d_+) - K e^{-r\tau} N(d_-)\right) = rc$$

$$\lim_{\tau \to 0} d_{\pm} = \frac{\log\left(\frac{x}{K}\right) + \left(r \pm \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}$$

if $x > K \Rightarrow +\infty$

$x < K \Rightarrow -\infty$

Portfolio A at 0:

buy one share of stock

& sell a zero-coupon bond that pays K at maturity $(e^{-rT}K)$

$f(0) = S(0) - e^{-rT}K$

$f(T) = S(T) - K$

$f(t, S(t)) = S(t) - e^{-r(T-t)}K$

Portfolio B:

buy one call & sell one put both with strike K

at t:

$c(t,x) - p(t,x)$

at T:

$$c(T,x) - p(T,x) = (x-K)_+ - (K-x)_+$$
$$= x - K$$

$c(T, S(T)) - p(T, S(T)) = S(T) - K$

$c(t,x) - p(t,x) = f(t,x)$

$c(t,x) - p(t,x) = x - e^{-r(T-t)}K$

$x N(d_+) - K e^{-r(T-t)} N(d_-) - x + e^{-r(T-t)} K = p(t,x)$

$K e^{-r(T-t)} (1 - N(d_-)) - x(1 - N(d_+)) = p(t,x)$

$K e^{-r(T-t)} N(-d_-) - x N(-d_+) = p(t,x)$