FE 610 Stochastic Calculus for Finance Final Webcampus

August 18, 2018

- There are 4 problems, worth a total of 100 points.
- Showcase your work: providing just the answer will result in a minimum of points.
- For the duration of this exam, you should assume that $W(t)$ is Brownian Motion

1. Define
$$X(t) = e^{\alpha W(t)}$$
and
$$Y(t) = \beta W(t) - 2\alpha\beta t$$
for $\alpha, \beta \in \mathbb{R}$. Define $Z(t) = X(t)Y(t)$ and determine:
   (a) $dZ(t)$
   (b) $[Z, Z](t)$
   (c) The conditions for α and β to ensure that the process $Z(t)$ is a martingale.

2. The Ornstein–Uhlenbeck process is defined by the following stochastic differential equation:
$$dX(t) = -\alpha X(t)dt + \sigma dW(t), \quad X(0) = c \in \mathbb{R}$$

(a) Compute the mean $\mathbb{E}[X(t)]$

(b) Compute the variance $\mathbb{V}(X(t))$.

3. True or False (you must provide reason for why you believe your answer)

   (a) All adapted stochastic processes are Martingales.

   (b) Given the stochastic differential equation:

   $$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u$$

   Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

   $$f(t, x) = \mathbb{E}^{t,x}[h(X(T))]$$

   Then $f(t, x)$ is a martingale.

4. Assume that a stock process $S(t)$ follows Geometric Brownian Motion with a constant risk-free interest rate r. We will create a new instrument whose payoff at maturity T is given by:

   $$V(T) = S^3(T) - K$$

   Determine the value of this instrument at any time $t \in [0, T]$.