$$E[f(W(t)) \mid \mathcal{F}(s)]$$

$$= E[f(\underbrace{W(t)-W(s)}_{\text{ind. of } \mathcal{F}(s)} + \underbrace{W(s)}_{\mathcal{F}(s)\text{-meas.}}) \mid \mathcal{F}(s)]$$

for x, a dummy variable,

$$E[f(\underbrace{W(t)-W(s)}_{\text{ind of } \mathcal{F}(s)} + \underbrace{x}_{d.v.}) \mid \mathcal{F}(s)]$$

$$= E[f(\underbrace{W(t)-W(s)}_{\sim N(0,t-s)} + x)]$$

$$= \int_{-\infty}^{\infty} f(w+x) \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-w^2}{2(t-s)}} dw$$

$$= g(x)$$

$$g(z) = E[f(W(t)-W(s)+z) \mid \mathcal{F}(s)]$$

$$g(W(s)) = E[f(W(t)-W(s)+W(s)) \mid \mathcal{F}(s)]$$

$$= E[f(W(t)) \mid \mathcal{F}(s)]$$

let $y = w + x$

$$\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-(y-x)^2}{2(t-s)}}}_{\text{prob. of transitioning from } x \text{ to } y \text{ in } t-s} f(y)\, dy$$

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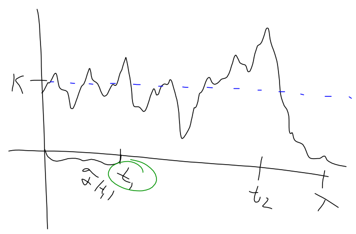
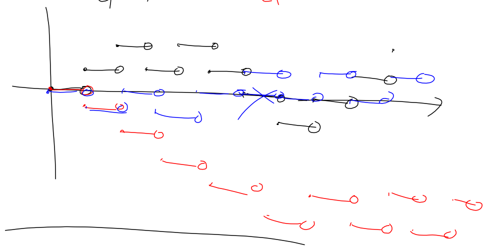


Call option $(S(t)-K)_+$

$\tau_1 = 1 \qquad \tau_1 = \infty \qquad \tau_1 = 5$



$E[Z(t) \mid \mathcal{F}(s)]$ for $s \le t$

$E[Z(t) - Z(s) + Z(s) \mid \mathcal{F}(s)]$

$= E\{e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} - e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} + e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} \mid \mathcal{F}(s)\}$

$= E\{e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} - e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} \mid \mathcal{F}(s)\}$

$+ E\{\underbrace{e^{\sigma W(s) - \frac{1}{2}\sigma^2 s}}_{\mathcal{F}(s)\text{-meas}} \mid \mathcal{F}(s)\}$

$e^{\sigma W(t)} + e^{\sigma W(s)} \sim ?$

$E\{e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} \mid \mathcal{F}(s)\}$

$= E\{e^{\sigma(W(t) - W(s) + W(s)) - \frac{1}{2}\sigma^2 t} \mid \mathcal{F}(s)\}$

$= e^{-\frac{1}{2}\sigma^2 t} E\{\underbrace{e^{\sigma(W(t) - W(s))}}_{\text{ind}} \underbrace{e^{\sigma W(s)}}_{\text{meas.}} \mid \mathcal{F}(s)\}$

$= e^{\sigma W(s) - \frac{1}{2}\sigma^2 t} E\{e^{\sigma(W(t) - W(s))} \mid \cancel{\mathcal{F}(s)}\}$

$= e^{\sigma W(s) - \frac{1}{2}\sigma^2 t} E\{e^{\sigma(W(t) - W(s))}\}$

for $Y \sim N(\mu, \sigma^2)$

$\varphi_Y(a) = E[e^{aY}] = e^{a\mu + \frac{1}{2}a^2\sigma^2}$

$a = \sigma$
$\mu = 0$
$\sigma^2 = t - s$

$= e^{\sigma W(s) - \frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2(t-s)} = e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} = Z(s)$

$W(t) - W(s) \to$ known dist

$e^{W(t)} - e^{W(s)} \to$ unknown

$(W(t) - W(s))^2 \to$ "known"

$W^2(t) - W^2(s) \to$ unknown

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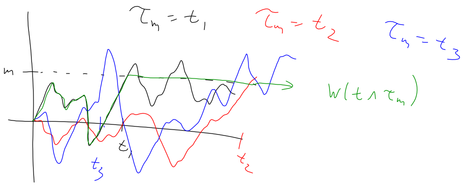

$Z(t \wedge \tau_m)$ where $(a \wedge b) = \min(a,b)$

$$E[W(t)] = E[W(t) \mid \mathcal{F}(0)] = W(0) = 0$$

$$E[Z(t)] = E[Z(t) \mid \mathcal{F}(0)] = Z(0) = e^{\sigma W(0) - \frac{1}{2}\sigma^2(0)} = 1$$

$$E[Z(t \wedge \tau_m)] = E[Z(t \wedge \tau_m) \mid \mathcal{F}(0)]$$
$$= E[Z(0 \wedge \tau_m)] = Z(0) = 1$$

$E[Z(t \wedge \tau_m)] = 1$ for <u>all</u> t

$$\lim_{t \to \infty} E[Z(t \wedge \tau_m)] = 1$$

$$\to E\left[\lim_{t \to \infty} Z(t \wedge \tau_m)\right] =$$
$$= E\left[\lim_{t \to \infty} e^{\sigma W(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)}\right]$$
$$= E\left[\underbrace{\lim_{t \to \infty} e^{\sigma W(t \wedge \tau_m)}}_{B} \underbrace{\lim_{t \to \infty} e^{-\frac{1}{2}\sigma^2(t \wedge \tau_m)}}_{A}\right]$$

A: $\lim_{t \to \infty} e^{-\frac{1}{2}\sigma^2(t \wedge \tau_m)} = \begin{cases} e^{-\frac{1}{2}\sigma^2 \tau_m} & , \tau_m < \infty \\ 0 & , \tau_m = \infty \end{cases}$

$$= e^{-\frac{1}{2}\sigma^2 \tau_m} \mathbb{1}_{\{\tau_m < \infty\}}$$

$$\mathbb{1}_{\{A\}} = \begin{cases} 1 & \text{if } A \\ 0 & \text{else} \end{cases}$$

$$E[\mathbb{1}_{\{A\}}] = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^c)$$
$$= \mathbb{P}(A)$$

B: $\lim_{t \to \infty} e^{\sigma W(t \wedge \tau_m)} : \begin{cases} e^{\sigma m} = e^{\sigma W(\tau_m)}, & \text{if } \tau_m < \infty \\ 0 \le \lim_{t \to \infty} e^{\sigma W(t)} \le e^{\sigma m}, & \text{if } \tau_m = \infty \end{cases}$

$$= E\left[e^{\sigma m - \frac{1}{2}\sigma^2 \tau_m} \mathbb{1}_{\{\tau_m < \infty\}}\right] = 1$$

$$\lim_{\sigma \downarrow 0} E\left[e^{\sigma m - \frac{1}{2}\sigma^2 \tau_m} \mathbb{1}_{\{\tau_m < \infty\}}\right] = 1$$

$$E[\mathbb{1}_{\{\tau_m < \infty\}}] = 1$$
$$\mathbb{P}(\tau_m < \infty) = 1$$

$$E\left[e^{\sigma m - \frac{1}{2}\sigma^2 \tau_m}\right] = 1$$

$$\tfrac{1}{2}\sigma^2 = \alpha \Rightarrow \sigma = \sqrt{2\alpha}$$

$$\Rightarrow E[e^{-\alpha \tau_m}] = e^{-|m|\sqrt{2\alpha}}$$

$$\frac{d}{d\alpha} E[e^{-\alpha \tau_m}] = \frac{d}{d\alpha} e^{-|m|\sqrt{2\alpha}}$$

$$E[-\tau_m e^{-\alpha \tau_m}] = -|m|\sqrt{2}\left(\tfrac{1}{2}\alpha^{-\frac{1}{2}}\right) e^{-|m|\sqrt{2\alpha}}$$

$$E[\tau_m e^{-\alpha \tau_m}] = \frac{|m|\sqrt{2}}{2\sqrt{\alpha}} e^{-|m|\sqrt{2\alpha}}$$

$\lim \alpha \downarrow 0$:

$$E[\tau_m] = \infty$$

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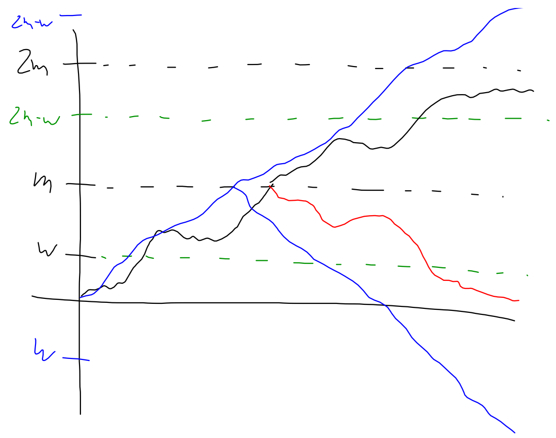


$m > 0$
$w < m$

let $w = m$

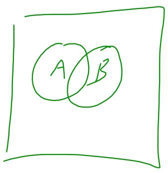
$$\mathbb{P}(\tau_m \le t,\ W(t) \le m) = \mathbb{P}(W(t) \ge m)$$

$$\mathbb{P}(\tau_m \le t,\ W(t) > m) = \mathbb{P}(W(t) \ge m)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \quad \text{if } A \cap B = \emptyset$$

$$\mathbb{P}(\{\tau_m \le t, W(t) \le m\} \cup \{\tau_m \le t, W(t) > m\})$$
$$= 2\,\mathbb{P}(W(t) \ge m)$$

$$\{A \cap B\} \cup \{A \cap B^c\} = A$$



$$\mathbb{P}(\tau_m \le t) = 2\,\mathbb{P}(W(t) \ge m)$$
$$= 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx$$

$$\mathbb{P}(\tau_m \le t) = 2\int_{|m|}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx$$

let $z = \frac{x-\mu}{\sigma} = \frac{x}{\sqrt{t}}$

$dz = \frac{dx}{\sqrt{t}}$

$$= 2\int_{\frac{|m|}{\sqrt{t}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$\frac{d}{dt}\mathbb{P}(\tau_m \le t) = -2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2t}}\right)\left(|m|\left(-\frac{1}{2}\right)t^{-3/2}\right)$$

$$f_{\tau_m}(t) = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}$$

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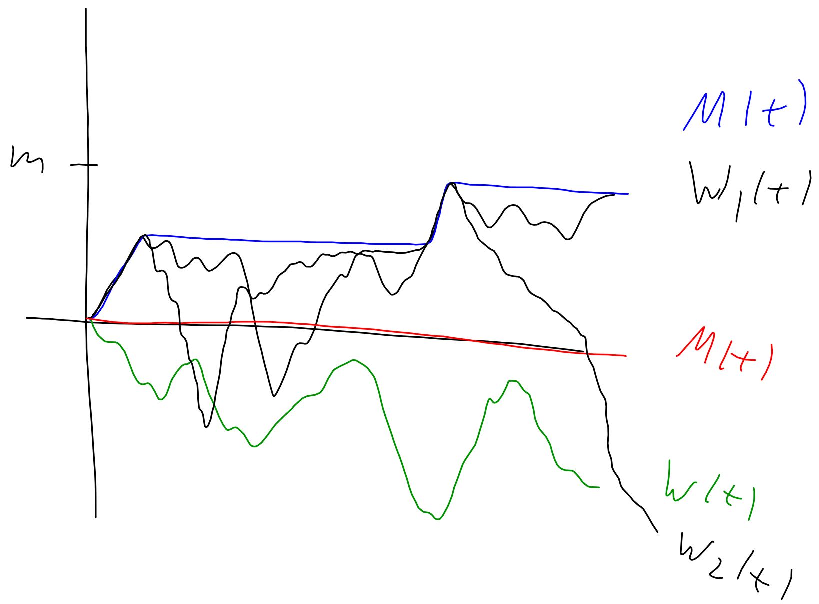

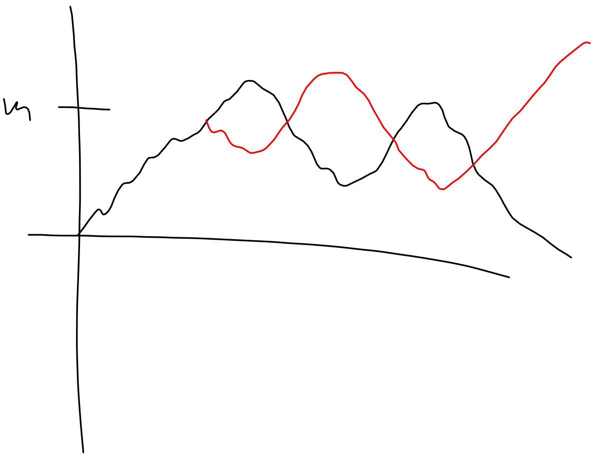

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