Itô's Lemma and Applications

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Differential Form

If $f(x)$ is a differentiable function, we would like to discuss how to find the derivative of $f(W(t))$. If $W(t)$ were a differentiable process, then using the chain rule we would have the result

$$df(W(t)) = f'(W(t))W'(t)dt = f'(W(t))dW(t)$$

But, as we have repeatedly proven and discussed, $W(t)$ has non-zero quadratic variation, and as a result we have the expression

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt$$

Integral Form

If we take the Itô formula in differential form from the previous slide and integrate both sides, we have the **Itô formula in integral form**:

$$\int_0^t df(W(u)) = \int_0^t f'(W(u))dW(u) + \frac{1}{2}\int_0^t f''(W(u))du$$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))dW(u) + \frac{1}{2}\int_0^t f''(W(u))du$$

Notice that the terms on the right hand of the equation are just an Itô integral discussed last week and a Lebesgue integral.

Itô Formula for Brownian Motion

Theorem 4.4.1: (Itô formula for Brownian motion) Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,

$$\begin{aligned} f(T, W(T)) =& f(0, W(0)) + \int_0^T f_t(t, W(t))dt \\ &+ \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt \end{aligned}$$

[1]

Recall from last week's discussion that we determined:

$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$$

We can confirm this result using Itô's formula using $f(x) = \frac{1}{2}x^2$

$$f(W(T)) = f(W(0)) + \int_0^T f_x(W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(W(t))dt$$

$$\frac{1}{2}W^2(T) = \int_0^T W(t)dW(t) + \frac{1}{2}\int_0^T dt$$

$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$$

Itô Processes

Definition 4.4.3: Let $W(t), t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes. [1]

Lemma 4.4.4: The quadratic variation of the Itô process from the previous definition is

$$[X, X](t) = \int_0^t \Delta^2(u)du$$

[1]

Integral With Respect to an Itô Process

Definition 4.4.5: Let $X(t), t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $\Gamma(t), t \geq 0$, be an adapted process. We define the integral with respect to an Itô process

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Gamma(u)\Theta(u)du$$

[1]

Theorem 4.4.6: (Itô formula for an Itô process) Let $X(t), t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,

$$\begin{aligned} f(T,X(T)) &= f(0,X(0)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))dX(t) \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t,X(t))d[X,X](t) \\ &= f(0,X(0)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))\Delta(t)dW(t) \\ &\quad + \int_0^T f_x(t,X(t))\Theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\Delta^2(t)dt \end{aligned}$$

[1]

We can express the results of the previous theorem in two different ways that can make utilizing the results easier. The first is

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))dX(t)dX(t)$$

and the second, using only dt and $dW(t)$ is

$$\begin{aligned} df(t, X(t)) &= f_t(t, X(t))dt + f_x(t, X(t))\Delta(t)dW(t) \\ &\quad + f_x(t, X(t))\Theta(t)dt + \frac{1}{2} f_{xx}(t, X(t))\Delta^2(t)dt \end{aligned}$$

Generalized Geometric Brownian Motion

Example 4.4.8 Let $W(t), t \geq 0$ be a Brownian motion, let $\mathcal{F}(t), t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds$$

Then

$$dX(t) = \sigma(t)dW(t) + (\alpha(t) - \frac{1}{2}\sigma^2(t))dt$$

and

$$dX(t)dX(t) = \sigma^2(t)dW(t)dW(t) = \sigma^2(t)dt$$

Let

$$S(t) = S(0)e^{X(t)}$$

Show that

$$\frac{dS(t)}{S(t)} = \alpha(t)dt + \sigma(t)dW(t)$$

Theorem 4.4.9: (Itô integral of a deterministic integrand)
Let $W(s), s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s)dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)ds$.[1]

Vasicek

Let $W(t), t \geq 0$, be a Brownian motion. The Vasicek interest rate model is:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

for α, β, and σ as positive constants. Verify the closed form solution of this stochastic differential equation is:

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW(s)$$

Cox-Ingersoll-Ross

Let $W(t), t \geq 0$, be a Brownian motion. The CIR interest rate model is:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t)$$

for α, β, and σ positive constants. This model does not have a closed form solution to verify. However, we can determine the mean and variance of $R(t)$ in order to better understand its distribution.

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.