Black-Scholes-Merton Model

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February 25, 2016

The Wealth Equation

Let X_0 be the initial wealth that an individual holds at time 0. If the individual purchases Δ_0 shares of stock with initial price S_0 (possible borrowing money at the risk-free interest rate r to do so) and then investing the surplus in a money market earning the risk-free interest rate. The value at time T of the portfolio (the position in the stock and the money market) is given by the equation:

$$X_T = \Delta_0 S_T + (X_0 - \Delta_0 S_0)(1 + r)$$

If we discretize the time interval, and allow the investor to change the position in the underlying by either investing or borrowing from the money market, we have the evolution of the wealth process X_n as:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n + \Delta_n S_n)$$

Note that the variable Δ_n is an adapted process.

To look at how we can extend this idea to a continuous, rather than discrete, portfolio valuation, we consider the following. Let the stock be modeled by geometric Brownian motion, and so we have the evolution of the stock price governed by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

If we have a money market with interest rate r and an adapted process such that at time t, the investor holds $\Delta(t)$ shares of stock and invests the remainder in the money market. The change in the portfolio will result in the change in the stock and the interest earnings.

$$dX(t) = \Delta(t)dS(t) + r\left(X(t) - \Delta(t)S(t)\right)dt$$

In the equation

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

the dynamics are

- average underlying rate of return
- risk premium for investing in the stock
- volatility term proportional to the size of the investment

The value of a European call option at expiration T is given as $c(T, S(T)) = (S(T) - K)_+$ where K is the value of the strike and $S(T)$ is the value of the stock at time T. This valuation depends on constants r, σ, and K. This leads, eventually, to the Black-Scholes-Merton differential equation:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x), \forall t \in [0,T), x \geq 0$$

with terminal condition

$$c(T,x) = (x-K)_+$$

Because of the domain $t, x \in [0, \infty), \mathbb{R}$ we need to observe the terminal condition at these extremes as well.

$$c(t, 0) = 0$$
$$\lim_{x \to \infty} c(t, x) = x$$

We then take all of this information and determine the solution to the differential equation with boundary conditions in the past slides.

Black-Scholes-Merton Formula

We have the solution as:

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), 0 \leq t < T, x > 0$$

where

$$d_{\pm}(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz$$

[1]

Forward Contract

The value of a forward contract is

$$f(t,x) = x - e^{-r(T-t)}K$$

If an agent sets up a static hedge (only takes a single position in the stock) at time 0, then we have:

$$f(0,S(0)) = S(0) - e^{-rT}K$$

The value of this strategy at time T will be:

$$f(T,S(T)) = S(T) - K$$

The value of a call option at expiration is

$$c(T, S(T)) = (S(T) - K)_+$$

The value of a put option at expiration is

$$p(T, S(T)) = (K - S(T))_+$$

If we form a portfolio by buying a call and shorting a put, the value of the portfolio will be

$$\begin{aligned} c(T, S(T)) - p(T, S(T)) &= (S(T) - K)_+ - (K - S(T))_+ \\ &= S(T) - K \\ &= f(T, S(T)) \end{aligned}$$

Because the two portfolios are equal at the end, they must be equal throughout. This lead us to the **put-call parity**

$$f(t,x) = x - e^{-r(T-t)}K = c(t,x) - p(t,x)$$

This allows us to determine the value of a put option without having to derive it.

$$\begin{aligned}
p(t,x) &= c(t,x) - x + e^{-r(T-t)}K \\
&= xN(d_+) - Ke^{-r(T-t)}N(d_-) - x + e^{-r(T-t)}K \\
&= -x(1 - N(d_+)) + Ke^{-r(T-t)}(1 - N(d_-)) \\
&= Ke^{-r(T-t)}N(-d_-) - xN(-d_+)
\end{aligned}$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.