Poisson Processes and Jump Diffusion

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Outline

Exponential Random Variables

Let τ be a random variable with pdf

$$f_\tau(t) = \begin{cases} \lambda e^{-\lambda t}, t \geq 0 \\ 0, t < 0 \end{cases}$$

We have $\mathbb{E}[\tau] = \frac{1}{\lambda}$ and cdf

$$F_\tau(t) = \mathbb{P}(\tau \leq t) = 1 - e^{-\lambda t}, t \geq 0$$

This is a memoryless random variable, or

$$\mathbb{P}(\tau > t) = \mathbb{P}(\tau > t + s | \tau > s)$$

Construct a sequence of i.i.d. exponential random variables $\{\tau_i\}$ with parameter λ. The "jump" times can then be defined as

$$S_n = \sum_{i=1}^{n} \tau_i$$

We have a resulting Poisson process that we call $N(t)$ that counts the number of jumps before a given time

$$N(t) = \begin{cases} 0, 0 \leq t < S_1 \\ 1, S_1 \leq t < S_2 \\ \vdots \\ n, S_n \leq t < S_{n+1} \\ \vdots \end{cases}$$

Lemma 11.2.2 The Poisson process $N(t)$ with intensity λ has the distribution

$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, \ldots$$

[1]

Theorem 11.2.3: Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < \cdots < t_n$ be given. Then the increments

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

are stationary and independent, and

$$\mathbb{P}(N(t_{j+1}) - N(t_j) = k) = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}, k = 0, 1, \ldots$$

[1]

Poisson Increments

As a result of Theorem 11.2.3, we have the following results about our Poisson increments for $0 \leq s < t$

$$\mathbb{P}(N(t) - N(s) = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}, k = 0, 1, \ldots$$

$$\mathbb{E}[N(t) - N(s)] = \lambda(t-s)$$

$$\mathbb{V}[N(t) - N(s)] = \lambda(t-s)$$

Martingale Property

Theorem 11.2.4: Let $N(t)$ be a Poisson process with intensity λ. We define the compensated Poisson process

$$M(t) = N(t) - \lambda t$$

Then $M(t)$ is a martingale.[1] Proof: Let $0 \leq s < t$ be given.

$$\begin{aligned}
\mathbb{E}[M(t)|\mathcal{F}(s)] &= \mathbb{E}[M(t) - M(s)|\mathcal{F}(s)] + \mathbb{E}[M(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[N(t) - N(s) - \lambda(t - s)|\mathcal{F}(s)] + M(s) \\
&= \mathbb{E}[N(t) - N(s)] - \lambda(t - s) + M(s) \\
&= M(s)
\end{aligned}$$

Compound Poisson Process

Let $N(t)$ be a Poisson process with intensity λ, and let $\{Y_i\}$ be a sequence of i.i.d. random variables with $\mathbb{E}[Y_i] = \beta$ that are independent of both each other and of the Poisson process. We define the **compound Poisson process** as

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0$$

Using this definition and previous results we have

$$\mathbb{E}[Q(t)] = \beta\lambda t$$

Theorem 11.3.1: Let $Q(t)$ be the compound Poisson process. Then the compensated Poisson process

$$Q(t) - \beta\lambda t$$

is a martingale.[1]

Theorem 11.3.2: Let $Q(t)$ be a compound Poisson process and let $0 = t_0 < t_1 < \cdots < t_n$ be given. The increments

$$Q(t_1) - Q(t_0), Q(t_2) - Q(t_1), \ldots, Q(t_n) - Q(t_{n-1})$$

are independent and stationary. In particular, the distribution of $Q(t_j) - Q(t_{j-1})$ is the same as the distribution of $Q(t_j - t_{j-1})$[1]

Definition 11.4.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}(t), t \geq 0$, be a filtration on this space. We say that a Brownian motion $W(t)$ is a Brownian motion relative to this filtration if $W(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. Similarly, we say that a Poisson process N is a Poisson process relative to this filtration if $N(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $N(u) - N(t)$ is independent of $\mathcal{F}(t)$. Finally, we say that a compound Poisson process Q is a compound Poisson process relative to this filtration if $Q(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $Q(u) - Q(t)$ is independent of $\mathcal{F}(t)$.[1]

We need to define the stochastic integral

$$\int_0^t \Phi(s)dX(s)$$

where X can have jumps. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which $\mathcal{F}(t), t \geq 0$, is a filtration. All processes are adapted to this filtration from the previous definition and will be right-continuous and can be expressed as

$$X(t) = X(0) + I(t) + R(t) + J(t)$$

A **pure jump process** $J(t)$ is an adapted, right-continuous process with $J(0) = 0$ and $J(t) = \lim_{s \downarrow t} J(s)$

Definition 11.4.2: A process $X(t)$ of the form in the previous slide, with Itô integral part $I(t)$, Riemann integral part $R(t)$, and pure jump part $J(t)$ will be called a jump process. The continuous part of this process is $X^c(t) = X(0) + I(t) + R(t)$.[1]

Definition 11.4.3: Let $X(t)$ be a jump process of the previous form and let $\Phi(s)$ be an adapted process. The stochastic integral of Φ with respect to X is defined to be

$$\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\Theta(s)ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s)$$

or in differential notation,

$$\begin{aligned}\Phi(t)dX(t) &= \Phi(t)dI(t) + \Phi(t)dR(t) + \Phi(t)dJ(t)\\ &= \Phi(t)dX^c(t) + \Phi(t)dJ(t)\end{aligned}$$

[1]

Theorem 11.4.5: Assume that the jump process $X(s)$ is a martingale, the integrand $\Phi(s)$ is left-continuous and adapted, and

$$\mathbb{E}\left[\int_0^t \Gamma^2(s)\Phi^2(s)ds\right] < \infty, \forall t \geq 0$$

Then the stochastic integral $\int_0^t \Phi(s)dX(s)$ is also a martingale.

Quadratic Variation

For a partition Π, we denote, as usual, the sample quadratic variation of a process as:

$$Q_{\Pi}(X) = \sum_{j=0}^{n-1} (X(t_{j+1}) - X(t_j))^2$$

And then to get the actual quadratic variation of the process $X(t)$, we let $||\Pi|| \to 0$ to get $[X, X](t)$. For the processes $X_1(t)$ and $X_2(t)$ we define the sample cross variation as

$$C_{\Pi}(X) = \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j))$$

Theorem 11.4.7: Let $X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$ be a jump process, where $I_1(t) = \int_0^t \Gamma_1(s)dW(s)$, $R_1(t) = \int_0^t \Theta(s)ds$, and $J_1(t)$ is a right continuous pure jump process. Then $X_1^c(t) = X_1(0) + I_1(t) + R_1(t)$ and

$$\begin{aligned}[X_1, X_1](T) &= [X_1^c, X_1^c](T) + [J_1, J_1](T) \\ &= \int_0^T \Gamma_1^2(s)ds + \sum_{0<s\leq T} (\Delta J_1(s))^2\end{aligned}$$

Define $X_2(t)$ similarly to $X_1(t)$.

$$\begin{aligned}[X_1, X_2](T) &= [X_1^c, X_2^c](T) + [J_1, J_2](T) \\ &= \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s)\end{aligned}$$

[1]

Ito Formula for One Jump Process

Theorem 11.5.1: Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. Then

$$f(X(t)) = f(0, X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) + \sum_{0<s\leq T}[f(X(s)) - f(X(s-))]$$

Corollary 11.5.3: Let $W(t)$ be a Brownian motion and let $N(t)$ be a Poisson process with intensity $\lambda > 0$, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to same filtration $\mathcal{F}(t), t \geq 0$. Then the processes $W(t)$ and $N(t)$ are independent.

Two Dimensional Ito Formula

Theorem 11.5.4: Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and continuous. Then

$$\begin{aligned} f(t, X_1(t), X_2(t)) = {} & f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))ds \\ & + \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) \\ & + \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))dX_1^c(s)dX_1^c(s) \\ & + \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))dX_1^c(s)dX_2^c(s) \\ & + \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))dX_2^c(s)dX_2^c(s) \\ & + \sum_{0<s\leq t} [f(s, X_1(s), X_2(s)) - f(s-, X_1(s-), X_2(s-))] \end{aligned}$$

[1]

Ito Product Rule for Jump Processes

Corollary 11.5.5: Let $X_1(t)$ and $X_2(t)$ be jump processes, Then

$$
\begin{aligned}
X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) \\
&\quad + [X_1^c, X_2^c](t) + \sum_{0<s\leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\
&= X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) \\
&\quad + [X_1, X_2](t)
\end{aligned}
$$

[1]

Corollary 11.5.6: Let $X(t)$ be a jump process. The **Doleans-Dade exponential** of X is defined to be the process

$$Z^X(t) = e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0 < s \leq t} (1 + \Delta X(s))$$

This process is the solution to the stochastic differential equation

$$dZ^X(t) = Z^X(t-)dX(t)$$

with initial condition $Z^X(0) = 1$, which in integral form is

$$Z^X(t) = 1 + \int_0^t Z^X(s-)dX(s)$$

[1]

Let $\tilde{\lambda}$ be a positive number and define:

$$Z(t) = e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$$

Fix a time $T > 0$, and we will use $Z(T)$ to change to measure $\tilde{\mathbb{P}}$ under which $N(t)$ has intensity $\tilde{\lambda}$ instead of λ.

Lemma 11.6.1: The process $Z(t)$ satisfies:

$$dZ(t) = \frac{\tilde{\lambda}-\lambda}{\lambda} Z(t_{-})dM(t)$$

In particular, Z(t) is a martingale under $\mathbb{P}$ and $\mathbb{E}[Z(t)] = 1$ for all t.[1]

Theorem 11.6.2 (Change of Poisson Intensity): Under the probability measure $\tilde{\mathbb{P}}$, the process $N(t)$, $0 \leq t \leq T$, is Poisson with intensity $\tilde{\lambda}$.[1]

Define:

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda_m})t} \left(\frac{\tilde{\lambda_m}}{\lambda_m} \right)^{N_m(t)}$$

$$Z(t) = \prod_{m=1}^{M} Z_m(t)$$

Theorem 11.6.4: The process $Z(t)$ is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all t.

Theorem 11.6.5: Under $\widetilde{\mathbb{P}}$, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, and $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with

$$\widetilde{\mathbb{P}}\{Y_i = y_m\} = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$$

[1]

Lemma 11.6.6: For $Z(t)$ given by

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

this process is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.

Define:

$$Z_1(t) = e^{-\int_o^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du}$$

$$Z_2(t) = e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

$$Z(t) = Z_1(t)Z_2(t)$$

Lemma 11.6.8: The process $Z(t)$ is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.[1]

Theorem 11.6.9: Under the probability measure $\widetilde{\mathbb{P}}$, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds$$

is a Brownian motion, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$ and independent, identically distributed jump sizes having density $\tilde{f}(y)$, and the processes $\widetilde{W}(t)$ and $Q(t)$ are independent.[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.