Yifu (Jason) He 10442277 HW 1.

12 /15

Q1.6.

(i). $Ee^{uX} = \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 - (2\mu x + 2\sigma^2 u x) + (\mu+\sigma^2 u)^2 - (\mu+\sigma^2 u)^2 + \mu^2]}$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 u)]^2}{2\sigma^2} + \frac{2\sigma^2(u\cdot\mu + \sigma^4 u^2 \cdot \frac{1}{2})}{2\sigma^2}} dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 u)]^2}{2\sigma^2}}$

$= e^{u\cdot\mu + \frac{1}{2}\sigma^2 u^2} \cdot \frac{1}{\sigma\cdot\sqrt{2\pi}} \cdot \int e^{-\frac{1}{2\sigma^2}\cdot[x-(\mu+\sigma^2 u)]^2} dx.$

let. $\frac{x-(\mu+\sigma^2 u)}{\sqrt{2}\cdot\sigma} = t$, $x = \sqrt{2}\cdot\sigma\cdot t + (\mu+\sigma^2 u)$

$Ee^{uX} = e^{u\cdot\mu + \frac{1}{2}\sigma^2 u^2} \cdot \frac{1}{\sigma\cdot\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-t^2} \cdot d(\sqrt{2}\sigma\cdot t + \mu + \sigma^2 u)$

$= e^{u\cdot\mu + \frac{1}{2}\sigma^2 u^2} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt.$

$\because \int_{-\infty}^{\infty} e^{-t^2} \geq 0$

$\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\left(\int_{-\infty}^{\infty} e^{-t^2} dt\right)^2} = \sqrt{\int_{-\infty}^{\infty} e^{-t^2} dt \cdot \int_{-\infty}^{\infty} e^{-y^2} dy} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t^2+y^2)} dt\cdot dy} = \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-\rho^2}\cdot\rho d\rho\cdot d\theta}$
$= \sqrt{\pi}$

No need to calculate this part, but it's fine

$\therefore Ee^{uX} = e^{u\cdot\mu + \frac{1}{2}\sigma^2\cdot u^2} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = e^{u\cdot\mu + \frac{1}{2}\sigma^2\cdot u^2}$

(ii). $E\varphi(X) = Ee^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}$, $\varphi(EX) = e^{u\cdot\mu}$

let $g(x) = e^x$.

$g'(x) = (e^x)' = e^x > 0,$

$\because$ ~~$e^{u\mu+\frac{1}{2}}$~~ $u\cdot\mu + \frac{1}{2}u^2\sigma^2 \geq u\cdot\mu$

$\therefore e^{u\cdot\mu + \frac{1}{2}u^2\sigma^2} \geq e^{u\cdot\mu}$, $E\varphi(X) \geq \varphi(EX)$

Q1.10.

12

(i). the probability density function of $P(\omega)$.

$$f(\omega) = \frac{1}{1-0} = 1.$$

$$\tilde{P}(\Omega) = \int_\Omega Z(\omega) dP(\omega) = \int_0^{\frac{1}{2}} 0 \cdot 1 d\omega + \int_{\frac{1}{2}}^1 \cdot 2 \cdot 1 d\omega = 1.$$

(Countable. additivity):

$$\tilde{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \int_{\bigcup_{n=1}^{\infty} A_n} Z(\omega) dP(\omega) = \lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} A_i} Z(\omega) d P(\omega) = \sum_{i=1}^{\infty} \int_{A_i} Z(\omega) dP(\omega) = \sum_{i=1}^{\infty} \tilde{P}(A_i)$$

(ii) $P(A) = 0.$

$$\tilde{P}(A) = \int_A Z(\omega) \cdot dP(\omega) = \int_{A\cap[0,\frac{1}{2}]} 0 \cdot dp + \int_{A\cap[\frac{1}{2},1]} 2 \, dp = 2 \cdot P(A\cap[\tfrac{1}{2},1]) = 0.$$

(iii). when. $\omega \in [0, \frac{1}{2})$.

$$\tilde{P}(A) = \int_A \cdot 0 \cdot dP(\omega) = 0,$$

but, $P(A) = \int_A \cdot 1 \, dP(\omega) = \omega|_A \neq 0$. so $\tilde{P}$ and $P$ are not equivalent.

12

Q.1.11.

(i). $\frac{1}{\varepsilon}P\{X\in B(x,\varepsilon)\}=\frac{1}{\varepsilon}\cdot\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du$, let $F(x)$ be the distribution function of $X$

$P\{X\in(x,\varepsilon)\}=F(x+\frac{\varepsilon}{2})-F(x-\frac{\varepsilon}{2})$

Use the knowledge of Taylor series expension and Peano Remainder

$$F(x+\tfrac{\varepsilon}{2})=F(x)+F'(x)(x+\tfrac{\varepsilon}{2}-x)+o(x+\tfrac{\varepsilon}{2}-x)$$

$$\approx F(x)+\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}\cdot\frac{\varepsilon}{2}$$ very accurate to use Taylor here

similarly, $F(x-\frac{\varepsilon}{2})=F(x)+F'(x)(x-\frac{\varepsilon}{2}-x)+o(x-\frac{\varepsilon}{2}-x)$

$$\approx F(x)+\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot(-\tfrac{\varepsilon}{2})$$

$\therefore\frac{1}{\varepsilon}P\{X\in(x,\varepsilon)\}\approx\left[F(x)+\frac{1}{\sqrt{2\pi}}\cdot\frac{\varepsilon}{2}e^{-\frac{x^2}{2}}-(F(x)-\frac{1}{\sqrt{2\pi}}\cdot\frac{\varepsilon}{2}\cdot e^{-\frac{x^2}{2}})\right]\frac{1}{\varepsilon}\approx\frac{1}{\sqrt{2\pi}}e^{-\frac{X^2(\bar\omega)}{2}}$

(ii) ~~$\frac{1}{\varepsilon}\tilde{P}\{Y\in B(y,\varepsilon)\}=\frac{1}{\varepsilon}\int_{y-\frac{\varepsilon}{2}}^{y+\frac{\varepsilon}{2}}Z(\omega)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du$, the situation is similar to (i)~~

$\frac{1}{\varepsilon}\tilde{P}\{Y\in B(y,\varepsilon)\}=\frac{1}{\varepsilon}\left(P\{Y\le y+\frac{\varepsilon}{2}\}-P\{Y<y-\frac{\varepsilon}{2}\}\right)$

$=\frac{1}{\varepsilon}[F(y+\frac{\varepsilon}{2})-F(y-\frac{\varepsilon}{2})]$

suppose. $F(y)$ be the distribution function of $Y$

$f(y)$ be the probability density function of $Y$ under $\tilde{P}$

Similarly to (i). $F(y+\frac{\varepsilon}{2})=F(y)+F'(y)\cdot\frac{\varepsilon}{2}+o(\frac{\varepsilon}{2})$

$F(y-\frac{\varepsilon}{2})=F(y)+F'(y)\cdot(-\frac{\varepsilon}{2})+o(\frac{\varepsilon}{2})$

$\therefore\frac{1}{\varepsilon}(\tilde{P})\{Y\in B(y,\varepsilon)\}\approx\frac{1}{\varepsilon}\cdot F'(y)\cdot(\frac{\varepsilon}{2}+\frac{\varepsilon}{2})=F'(y)=f(y)\Rightarrow f(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2(\bar\omega)}{2}}$

thus, $Y$ is a standard normal random variable under $\tilde{P}$

(iii). $X\sim N(0,1)$, $Y=X+\theta$, $\Rightarrow Y\sim N(\theta,1)$

$\because y=x+\theta$, $\Rightarrow x=y-\theta$

$\therefore$ ~~$\{Y\in B(y,\varepsilon)\}=\{X+\theta\in B(y,\varepsilon)\}=\{X+\theta\in B(x+\theta,\varepsilon)\}$~~

$\{X\in B(x,\varepsilon)\}=\{X+\theta\in B(x+\theta,\varepsilon)\}=\{Y\in B(y,\varepsilon)\}$

$\{X\in B(x,\varepsilon)\}$ and $\{Y\in B(y,\varepsilon)\}$ are the same set.

(iv): known from (i), (ii).

$$\frac{\tilde{P}(A(\bar\omega,\varepsilon))}{P(A(\bar\omega,\varepsilon))}\approx\frac{\frac{\varepsilon}{\sqrt{2\pi}}e^{-\frac{Y^2(\bar\omega)}{2}}}{\frac{\varepsilon}{\sqrt{2\pi}}e^{-\frac{X^2(\bar\omega)}{2}}}=e^{-\frac{1}{2}(Y^2(\bar\omega)-X^2(\bar\omega))}=e^{-\frac{1}{2}(\theta^2+2\theta X(\bar\omega))}=e^{-\frac{1}{2}\theta^2-\theta X(\bar\omega)}$$

Q2.2.

(i). ~~the~~

when, $S_2=4$, the 2 coin toss $=HT, TH$,

$6(X)=\{\phi, \Omega_2, \{HT, TH\}, \{HH, TT\}\}$

(ii). $6(S_1)=\{\phi, \Omega_2, \{HH, HT\}, \{TH, TT\}\}$

(iii). To prove. $6(X)$ and $6(S_1)$ are independent,

we should prove. $P(A\cap B)=P(A)\cdot P(B)$. $\forall A\in 6(X)$. $B\in 6(S_1)$

$\tilde{P}(\{HT, TH\}\cap\{TH, TT\})=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=\tilde{P}(\{HT, TH\})\cdot\tilde{P}(\{TH, TT\})$

$\tilde{P}(\{HT, TH\}\cap\{HH, HT\})=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=\tilde{P}(\{HT, TH\})\cdot\tilde{P}(\{HH, HT\})$

$\tilde{P}(\{HH, TT\}\cap\{TH, TT\})=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=\tilde{P}(\{HH, TT\})\cdot\tilde{P}(\{TH, TT\})$

$\tilde{P}(\{HH, TT\}\cap\{HH, HT\})=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=\tilde{P}(\{HH, TT\})\cdot\tilde{P}(\{HH, HT\})$

......

So. $6(X)$ and $6(S_1)$ are independent under $\tilde{P}$

(iv). suppose. $6(X)$ and $6(S_1)$ are independent under $P$

$P(\{HT, TH\})\cdot P(\{TH, TT\})=\frac{4}{9}\cdot\frac{3}{9}=\frac{12}{81}=\frac{4}{27}$

$P(\{HT, TH\}\cap\{TH, TT\})=\frac{2}{9}$, $P(\{HT, TH\}\cap\{TH, TT\})\neq P(\{HT, TH\})\,P(\{TH, TT\})$

$\therefore$ the hypothese is wrong. $6(X)$ and $6(S_1)$ are not independent under $\tilde{P}$

(v). $X$ and $S_1$ are not independent under $P$. (proved by (iv))

if $X=1$.

$P\{S_1=8\mid X=1\}=\dfrac{\frac{2}{9}}{\frac{4}{9}}=\frac{1}{2}$. $P\{S_1=2\mid X=1\}=\dfrac{\frac{2}{9}}{\frac{4}{9}}=\frac{1}{2}$

So, we change the estimate of distribution in $S_1$.

Q2.4.

(i) $Ee^{uX+vY} = Ee^{uX+vX\cdot Z}$

$= E\{e^{uX+vXZ} \mid Z=1\} + E\{e^{uX+vXZ} \mid Z=-1\}$

$= \frac{1}{2}(Ee^{(u+v)X}) + \frac{1}{2}(Ee^{(u-v)X})$

for $Ee^{(u+v)x} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2(u+v)x)} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[x-(u+v)]^2+\frac{(u+v)^2}{2}} dx = e^{\frac{(u+v)^2}{2}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-u-v)^2}{2}} dx$

$= e^{\frac{(u+v)^2}{2}}$

similarly. $Ee^{(u-v)x} = e^{\frac{(u-v)^2}{2}}$.

$\therefore Ee^{uX+vY} = \frac{1}{2}(e^{\frac{u^2+v^2+2uv}{2}} + e^{\frac{u^2+v^2-2uv}{2}}) = e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv}+e^{-uv}}{2}$

(ii). put $u=0$ in $Ee^{uX+vY} = e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv}+e^{-uv}}{2}$

$E\cdot e^{vY} = e^{\frac{v^2}{2}} \cdot \frac{1+1}{2} = e^{\frac{v^2}{2}}$

(iii). $Ee^{uX} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2+ux} = \int_{-\infty}^{\infty} e^{\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-u)^2} dx = e^{\frac{u^2}{2}}$

$\because Ee^{uX} \cdot Ee^{vY} = e^{\frac{u^2}{2}+\frac{v^2}{2}} \neq Ee^{uX+vY}$.

$\therefore$ X and Y are not independent.

Q 2.8.

12

(i). $X(\omega) = \begin{cases} 1, & \omega \in \{a,b\} \\ -1, & \omega \in \{c,d\} \end{cases}$

$\sigma(X) = \{\phi, \Omega, \{a,b\}, \{c,d\}\}$

(ii) $Y(\omega) = \begin{cases} 1, & \omega \in \{a,c\} \\ -1, & \omega \in \{b,d\} \end{cases}$

$E[Y|X=1] = 1 \times \frac{\frac{1}{6}}{\frac{1}{2}} + (-1) \times \frac{\frac{1}{3}}{\frac{1}{2}} = -\frac{1}{3} = \int_{\{a,b\}} Y(\omega)\, dP(\omega) = 1 \times \frac{1}{3} - 1 \times \frac{2}{3} = -\frac{1}{3}$

$E[Y|X](X=1) = \frac{EY}{P(X=1)}$

$E[Y|X=-1] = 1 \times \frac{\frac{1}{4}}{\frac{1}{2}} + (-1)\frac{\frac{1}{4}}{\frac{1}{2}} = 0 = \int_{\{c,d\}} Y(\omega)\, dP(\omega) = 0$

$\Rightarrow \int_\Omega Y(\omega) P(\omega) = 1 \times (\frac{1}{6} + \frac{1}{4}) + (-1)(\frac{1}{3} + \frac{1}{4}) = -\frac{1}{6} = \int_\Omega E[Y|X]\, dP(\omega) = -\frac{1}{3} \times (\frac{1}{2}) + 0 \times (\frac{1}{2}) = -\frac{1}{6}.$

(iii) $Z(\omega) = \begin{cases} 2, & \omega \in \{a\} \\ 0, & \omega \in \{b,c\} \\ -2, & \omega \in \{d\} \end{cases}$

$E[Z|X=1] = 2 \times \frac{\frac{1}{6}}{\frac{1}{2}} + 0 \times \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} = \int_{\{a,b\}} Z(\omega)\, dP(\omega) = 2 \times \frac{1}{3} + 0 \times \frac{2}{3} = \frac{2}{3}$

$E[Z|X=-1] = -2 \times \frac{\frac{1}{4}}{\frac{1}{2}} + 0 \times \frac{\frac{1}{4}}{\frac{1}{2}} = -1 = \int_{\{c,d\}} Z(\omega)\, dP(\omega) = -2 \times \frac{1}{2} + 0 \times \frac{1}{2} = -1$

$\Rightarrow \int_\Omega Z(\omega) P(\omega) = 2 \times \frac{1}{6} + 0 \times (\frac{1}{3} + \frac{1}{4}) + (-2) \times \frac{1}{4} = -\frac{1}{6} = \int_\Omega E[Z|X]\, dP(\omega) = \frac{2}{3} \times \frac{1}{2} + (-1) \times \frac{1}{2} = -\frac{1}{6}$

(iv). $E[Z|X] - E[Y|X]$

$= E[(X+Y)|X] - E[Y|X]$

$= E[X|X] + E[Y|X] - E[X|X]$ (Linearity).

$= E[X|X]$

$= X$ (Taking out what is known)

good

16

Q.2.8

to prove $Y_2$ and $X$ are uncorrelated, is to prove $\text{cov}(Y_2, X) = 0$

$$\text{cov}(X, Y_2) = E[X \cdot Y_2] - E[X] \cdot E[Y_2]$$
$$= E[X(Y - E[Y|X])] - E[X] \cdot E[(Y - E[Y|X])]$$
$$= \underbrace{E(XY) - E[X \cdot E[Y|X]]}_{} - \underbrace{(E[X] \cdot E[Y] - (E[X] \cdot E[E[Y|X]]))}_{}$$

$= EX \cdot EY - EX \cdot EY$ (Iterated conditioning)

$= 0$

$$= E(XY) - E[E[XY|X]] \quad \text{(taking out what is known)}$$
$$= E(XY) - E(XY) \quad \text{(Iterated conditioning)}$$
$$= 0$$

so, $\text{cov}(X, Y_2) = 0$.

very good

suppose $Z$ is a random variable under $\sigma(X)$-measure.

$$\text{cov}(Z, Y_2) = E(Z \cdot Y_2) - EZ \cdot EY_2$$
$$= E(ZY) - E[Z \cdot E[Y|X]] - (EZ \cdot EY - EZ \cdot E[E[Y|X]])$$

(similar to first question)

$$= E(ZY) - E(ZY) - (EZ \cdot EY - EZ \cdot EY)$$
$$= 0 - 0$$
$$= 0$$

Addition HW1.

12

8

Consider random walk $M_n=\sum_{j=1}^{n} X_j$ with steps generated by the process determined by flipping a coin with $P(w_j=H)=p$ and $P(w_j=T)=q$, with each step evaluated by the process $X_j=\begin{cases} 2 & ;\text{ for } w_j=H \\ -1 & ;\text{ for } w_j=T \end{cases}$

A. Define $M_0=0$, Find values for $p, q$ that $E[M_1]=M_0$.

B. Given that $M_{300}=60$, using probabilities from the first part, determine $E[M_{302} \mid M_{300}=60]$

Answer:

A. $M_1=X_1$, $E[M_1]=2p+(-1)\cdot q$, and $p+q=1$

let $E[M_1]=M_0=0$. $\Longrightarrow p=\frac{1}{3}$, $q=\frac{2}{3}$

B. $E[M_{302} \mid M_{300}=60] = \cancel{62}\ \cancel{p^2 \cdot M_{302}}$

$\cancel{M}$

$64 \cdot P\{X_{301}=2 \cap X_{302}=2\} + 61 \cdot (P\{X_{301}=2 \cap X_{302}=-1\} + P\{X_{301}=-1 \cap X_{302}=2\})$

$+ 58 \cdot P\{X_{301}=-1 \cap X_{302}=-1\}$

$= 64 \cdot \frac{1}{3} \cdot \frac{1}{3} + 61 \cdot (\frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3}) + 58 \cdot \frac{2}{3} \times \frac{2}{3}$

$= \frac{64}{9} + \frac{244}{9} + \frac{232}{9} = \frac{540}{9} = 60.$

Q1.2.

(i). for the set $A$, we define a set $B$.

$B = \{w = w_1 w_2 \cdots w_n \cdots ;\}$. Let, the sequence of $B$. $w^{(B)}$: the element in every sequent $w^{(B)}_{(i)}$

define. $w^{(B)}_{i} = w^{(A)}_{2i-1}$, we get a $\begin{bmatrix} \text{injection} \\ \text{mapping} \end{bmatrix}$ of $A$ to $B$,

So, every sequence in $A$ has an one-to-one correspondence relationship to every sequece in $B$. (bijective).

$\because$ set $B$ is ~~infinite~~ uncountable infinite

$\therefore$ set $A$ is uncountable infinite

(ii). suppose $A = \{w_1 w_1' \cdot w_2 w_2' \cdots w_n w_n' \cdots\}$

$$P(A) = \lim_{n\to\infty} (p^2 + (1-p)^2)^n$$

$$p^2 + (1-p)^2 = 1 - 2p + 2p^2 = (p - \tfrac{1}{2})^2 + \tfrac{1}{2}$$

$\because 0 < p < 1$

$\therefore \frac{1}{2} < p^2 + (1-p)^2 < 1$

$\therefore P(A) = \lim_{n\to\infty} (p^2 + (1-p)^2)^n = 0.$

Q1.8.

(i) $Y_n = \dfrac{e^{tX}-e^{s_nX}}{t-s_n}$.

$$\lim_{n\to\infty} EY_n = \lim_{n\to\infty}\int_\Omega Y_n dp = \int_\Omega \lim_{n\to\infty} Y_n dp = E[\lim_{n\to\infty} Y_n]$$

$$E[\lim_{n\to\infty} Y_n] = E[\lim_{s_n\to t} \frac{e^{tX}-e^{s_nX}}{t-s_n}] = E[\frac{1}{t-s_n}\cdot(t-s_n)X(w)e^{\theta(w)X(w)}] \text{ use } (1.9.1).$$

$$= E[X(w)\cdot e^{\theta(w)X(w)}]$$

$\because$ ~~w∈[s,t]~~ $w$ is between $s_n$ and $t$, when $s_n\to t$, $\theta\to t$

$$\therefore \lim_{n\to\infty} EY_n = E[Xe^{tX}]$$

~~$\because$ for $|Y_n| = |\frac{e^{tX}-e^{s_nX}}{t-s_n}|$~~

(ii). $Y_n = \dfrac{e^{tX}-e^{s_nX}}{t-s_n} = \dfrac{t-s_n}{t-s_n}\cdot X(w)e^{\theta(w)X(w)}$ use (1.9.1)

$$= X(w)\cdot e^{\theta(w)X(w)}$$

$\because \theta \le t$

$\therefore X(w)e^{\theta(w)\cdot X(w)} < X(w)e^{tX(w)}$ ①

known by the Question "$E[Xe^{tX}]<\infty$ for every $t\in R$". ②.

with ①,②. We can use the Dominated Convergence Theory. (i)

known from. notation (1.3.1).

$$\int_\Omega X(w)dP(w) = \int_\Omega X^+(w)dP(w) - \int_\Omega X^-(w)dP(w) = \int_\Omega X^+(w)dP(w) + \int_\Omega |X(w)|dP(w)$$

$$\therefore |Y_n| = |X(w)e^{\theta_n X(w)}| < |X(w)e^{tX(w)}|.$$

$|Y_n|$ also meet the condition of ① ②.

We can use Dominated Covergence Theory to prove

$$E[X(w)e^{tX(w)}] = \lim_{s_n\to t} E[|Y_n|] = \lim_{s_n\to t}[\frac{e^{tX}-e^{sX}}{t-s}] = \varphi(x)$$

Q2.10. $\int_A g(x)\,dp = \int_A g(x) f_X(x)\,dx$

$$= \int_A \int_B \frac{y \cdot f_{X,Y}(x,y)}{f_X(x)}\,dy\, f_X(x)\,dx$$

$$= \int_A \int_B y \cdot f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_B y \int_A f_{X,Y}(x,y)\,dx \cdot dy$$

$$= \int_B y\, f_Y(y)\,dy$$

$\because Y = X(\omega) \in B$. and. $A = \{\omega \in \Omega;\ X(\omega) \in B\}$

$\therefore \int_A g(x)\,dp = \int_A Y\,dP$