HW 2. Yifu He 10442277 ①

Q.3.2.

Proof: suppose $0 \le s \le t$,

to show $W^2(t)-t$ is a martingale, is to prove

$E[W^2(t)-t \mid F(s)] = W^2(s)-s$

then, we begin to prove.

$E[W^2(t)-t \mid F_s] = E[(W(t)-W(s))^2 - t \mid F(s)]$

$= E[\underline{(W(t)-W(s))^2} - \underline{2W(s)\cdot(W(t)-W(s))} + W(s)^2 - t \mid F(s)]$

$E[(Wt-W(s))^2] = E[(Wt-W(s))^2] - E^2(W(t)-W(s))$

$= D[Wt-W(s)]$

$= t-s.$

$\because W(s)$ and $W(t)-W(s)$ are independent.

$\therefore E[W(s)\cdot(W(t)-W(s)) \mid F_s]$

$= W(s)\cdot E[W(t)-W(s)]$

$= W(s)\cdot 0 = 0$

$W(s)$ is s-measurable

$E[W^2(s) \mid F(s)] = W^2(s)$

and $W(t)-W(s)$ are independent of $W(s)$

$E[(Wt-W(s))^2 \mid F(s)] = E[W(t)-W(s)]$

$= t-s$

in a sum,

$E[W^2t - t \mid F_s] = t-s-0+W^2(s)-t = W^2(s)-s.$

.3.3. let. $X \sim N(\mu, \sigma^2)$

to prove $\dfrac{E[(X-\mu)^4]}{\sigma^4} = 3$

$$\varphi_X(k) = E[e^{X\cdot k}] = e^{\mu\cdot k + \frac{1}{2}k^2\cdot\sigma^2}, \quad X-\mu \sim N(0, \sigma^2)$$

$$\cancel{\varphi'_X(k) = E[} \quad \varphi_{X-\mu}(k) = E[e^{(X-\mu)\cdot k}] = e^{\frac{1}{2}k^2\sigma^2}$$

$$\varphi'_{X-\mu}(k) = E[(X-\mu)e^{(X-\mu)k}] = k\sigma^2\cdot e^{\frac{1}{2}k^2\sigma^2}$$

$$\varphi''_{(X-\mu)}(k) = E[(X-\mu)^2 e^{(X-\mu)k}] = (\sigma^2 + k^2\sigma^4)\cdot e^{\frac{1}{2}k^2\sigma^2}$$

$$\varphi'''_{(X-\mu)}(k) = E[(X-\mu)^3 e^{(X-\mu)k}] = (2k\sigma^4 + k\sigma^4 + k^3\sigma^6)\, e^{\frac{1}{2}k^2\sigma^2} = (3k\sigma^4 + k^3\sigma^6)\cdot e^{\frac{1}{2}k^2\sigma^2}$$

$$\varphi''''_{(X-\mu)}(k) = E[(X-\mu)^4 e^{(X-\mu)k}] = (3\sigma^4 + 3k^2\sigma^6 + 3k^2\sigma^6 + k^4\sigma^8)\cdot e^{\frac{1}{2}k^2\sigma^2}$$

$$= (3\sigma^4 + 6k^2\sigma^6 + k^4\sigma^8)\cdot e^{\frac{1}{2}k^2\sigma^2}$$

let $k=0$.

$$E[(X-\mu)^4] = 3\cdot\sigma^4$$

Q3.4. 布朗运动的其他变差 ③

(i) proved by theorem 3.4.3.

$[W,W](T)=T$ is almost surely true

~~Q_{II}~~

$\boxed{\text{suppose.}}$ there exists a path whose Q_I ~~less~~ ~~are~~ isn't infinite, suppose it's T'

for $\sum_{j=0}^{n-1}(W(t_{j+1})-W_{t_j})^2 \le \max_{0\le k\le n-1}|W_{t_{k+1}}-W_{t_k}|\cdot\sum_{j=0}^{n-1}|W_{t_{j+1}}-W_{t_j}|$

$\lim_{\|\Pi\|\to 0} Q_{II} \le \lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}|W_{t_{k+1}}-W_{t_k}|\cdot Q_I$

Because W is continuous. so. $\max_{0\le k\le n-1}|W_{t_{k+1}}-W_{t_k}|$ has the limit 0 as $\|\Pi\|$

$T\le 0\cdot T'$, that's false. we cannot prove the ~~sup~~ hypothesis.

So. Q_I approaches ∞.

(ii). similar as (i)

$Q_{III}=\sum_{j=0}^{n-1}|W_{t_{j+1}}-W_{t_j}|^3 \le \max_{0\le k\le n-1}|W_{t_{j+1}}-W_{t_j}|\cdot\sum_{j=0}^{n-1}(W_{t_{j+1}}-W_{t_j})^2$

$\lim_{\|\Pi\|\to 0}$

$= \max_{0\le k\le n-1}|W_{t_{j+1}}-W_{t_j}|\cdot T$

$= 0\cdot T = 0.$

∴ So Q_{III} approaches 0. for almost every path of W.

Q3.6

(i) ~~the~~ to prove the Markov property of $X(t)$

$$E[f(X(t))|F(s)] = E[f(X(t) - X(s) + X(s))|F(s)]$$

let dummy variable $w = X(s)$, let w be a constant

$\therefore E[f(X(t))|F(s)] = E[f(X(t)-X(s)+w)|F(s)]$ ($X(t)-X(s)$ is independent of $X(s)$, $X(s)$ is s-measurable)

$$= E[f(X(t)-X(s)+w)]$$

$\because X(t)-X(s) = W(t)-W(s) + (t-s)\cdot\mu \sim N((t-s)\mu, t-s)$, let $y = X(t)-X(s)+W$ then $X(t)-X(s) = y-w$

$$\therefore = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} f(X(t)-X(s)+w)\, e^{-\frac{(X(t)-X(s)-\mu(t-s))^2}{2(t-s)}}\, dy$$

$$= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} f(y)\, e^{-\frac{(y-w-\mu(t-s))^2}{2(t-s)}}\, dy = g(w), \quad w \text{ is } X(s)$$

rewrite. $\tau = t-s$, $p(\tau, w, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-w-\mu\tau)^2}{2\tau}}$

~~(ii) to prove the Markov property of $S(t)$ similar to (i)~~

~~$E[f(S(t))|F(s)] = E[f((S(t)-S(s))+S(s))|F(s)]$, let $w = S(s)$. w is a constant~~

~~$= E[f((S(t)-S(s))+w)]$ $S(t)-S(s)$ and $S(s)$ are independent, $S(s)$ is s-measurable.~~

(ii) next page.

⑤

(ii). $S(t) = S(0) \cdot e^{\sigma W(t) + \nu t}$

$= S(0) e^{\sigma (W(t) + \frac{\nu}{\sigma} t)}$

similar to (i). $\mu = \frac{\nu}{\sigma}$. $X(t) = W(t) + \frac{\nu}{\sigma} t$.

$S(t) = S(0) \cdot e^{\sigma \cdot X(t)}|_{\mu = \frac{\nu}{\sigma}}$

$\therefore$ in this situation $\underbrace{f(y) \text{ in (i)}}$ is composite function $f(g(y))$ in (ii) where $g(y) = S(t)$.

$\therefore$ ~~So~~ we can use the conclusion of (i).

let. $y = X(t) - X(s) + W$

$X(t) - X(s) = y - W$.

$$E[f(S(t)) \mid F(s)] = E[f(S_0 \cdot e^{\sigma X(t)}) \mid F(s)].$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(S_0 e^{\sigma y}) e^{-\frac{1}{2(t-s)}(y - W - \mu(t-s))^2} dy. \quad \text{(conclusion of (i))}$$

let $z = S_0 e^{\sigma y}$

$dz = S_0 \sigma \cdot e^{\sigma y}$ ①

$y = \ln \frac{z}{S_0} \cdot \frac{1}{\sigma}$ ②

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{0}^{\infty} f(z) \cdot e^{-\frac{1}{2(t-s)\sigma^2} \cdot (\frac{1}{\sigma}\ln\frac{z}{S_0} - \frac{1}{\sigma}\ln\frac{S_s}{S_0} - \frac{\nu}{\sigma}(t-s))^2} dz \cdot \frac{1}{\sigma \cdot z} \quad \text{(put ①② in)}$$

$$= \frac{1}{\sigma z\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(z) e^{-\frac{(\ln\frac{z}{x} - \nu(t-s))^2}{2\sigma^2(t-s)}} dz = g(S_s)$$

rewrite. $\tau = t - s$.

$$p(\tau, x, y) = \frac{1}{\sigma y\sqrt{2\pi\tau}} e^{-\frac{(\ln\frac{y}{x} - \nu\tau)^2}{2\sigma^2\tau}}$$

Additional Question

a). $\because$ W. B are brownian Motion $\boxed{No.}$

$\therefore$ let $A = \frac{1}{2}(W+B)$.

according to the definition.

(i). $A(0) = \frac{1}{2}(W(0)+B(0)) = 0$. , $A(t) = \frac{1}{2}(W(t)+B(t)) > 0$.

(ii) for each increments.

$\Delta(t_j) = A(t_{j+1}) - A(t_j) = \frac{1}{2}(W_{t_{j+1}} - W_{t_j}) + \frac{1}{2}(B_{t_{j+1}} - B_{t_j})$, $\Delta_{t_{j+1}} = \frac{1}{2}(W_{t_{j+2}} - W_{t_{j+1}}) + \frac{1}{2}(B_{t_{j+1}} - B_{t_j})$ . . . . .

$\because$ ~~$(W_{t_{j+1}} - W_{t_j}) + (B_{t_{j+1}} - B_{t_j})$~~ every increments of B. and W are independent

$\therefore$ every increments of A are independents.

(iii) $E[A(t_{j+1}) - A(t_j)] = \frac{1}{2}E[W_{t_{j+1}} - W_{t_j}] + \frac{1}{2}E[B_{t_{j+1}} - B_{t_j}] = 0$.

$Var[A_{t_{j+1}} - A_{t_j}] = Var[\frac{1}{2}(W_{t_{j+1}} - W_{t_j})] + Var[\frac{1}{2}(B_{t_{j+1}} - B_{t_j})] = \frac{1}{2}(t_{j+1} - t_j)$. A_t is not Brownian . Motion

b). X. Y are martingales. $\boxed{Yes}$

suppose $0 \le s \le t$. and $A(t) = \frac{1}{2}(X(t) + Y(t))$

$E[X(t) | F(s)] = X(s)$. $E[Y(t) | F(s)] = Y(s)$.

thus. $E[A(t) | F(s)] = E[\frac{1}{2}X(t) | F(s)] + E[\frac{1}{2}Y(t) | F(s)] = \frac{1}{2}(X(s) + Y(s)) = A(s)$

c) suppose, ~~Q_{II}~~ $Q_{II} = [X, X](T) = A$. $0 < A < \infty$, $\boxed{No}$.

Q_I is the first order variation.

$$Q_{II} = \sum_{k=0}^{n-1} \left| X(t_{k+1}) - X(t_k) \right|^2 \le \max_{0 \le k \le n-1} |X(t_{k+1}) - X(t_k)| \cdot \sum_{k=0}^{n-1} |X(t_{k+1}) - X(t_k)|$$

i). if X is continuous, $\max_{0 \le k \le n-1} |X(t_{k+1}) - X(t_k)|$ has the limit 0.

thus, $A = 0 \cdot Q_I$. $Q_I = \infty$,

ii). if X isn't continous. that's false.

the counter example is $f(x) = \begin{cases} 0, & x \in [0,1] \\ 1, & x \in [1,2]. \end{cases}$

Bonus Question. 3.5. ⑥

$$E[e^{-rT}(S(T)-K)^+] = S[e^{-rT}\cdot S(T)-K \mid S(T)-K \geq 0]$$

$$\because S_T - K > 0 \longrightarrow S(0)\cdot e^{(r-\frac{1}{2}\sigma^2)T+\sigma W(T)} - K > 0$$

$$\sigma W(T) > \frac{K}{S(0)\cdot e^{(r-\frac{1}{2}\sigma^2)T}}$$

$$W(T) > \frac{1}{\sigma}\left(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T\right)$$

$$= d_1$$

$$\therefore E[e^{-rT}(S(T)-K)^+] = \int_{d_1}^{+\infty} (S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x} - K)\cdot e^{-\frac{x^2}{2T}} \frac{1}{\sqrt{2\pi T}}\, dx$$

$$= \frac{S(0)}{\sqrt{2\pi T}}\int_{d_1}^{\infty} e^{-\frac{(\frac{x}{\sqrt{T}}-\sigma\sqrt{T})^2}{2}}\, dx - \frac{K\cdot e^{-rT}}{\sqrt{2\pi T}}\int_{d_1}^{+\infty} e^{-\frac{(\frac{w}{\sqrt{T}})^2}{2}}\, dw$$

$$\frac{d_1}{\sqrt{T}} - \sigma\sqrt{T} = \frac{-1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + (r+\frac{1}{2}\sigma^2)T\right] = -d_+[T, S(0)],$$

$$\frac{-d_1}{\sqrt{T}} = \frac{1}{\sqrt{\sigma T}}\left[\ln\frac{S(0)}{K} + (r-\frac{1}{2}\sigma^2)T\right] = d_-[T, S(0)]$$

$$\therefore E[e^{-rT}(S(T)-K)^+] = S(0)N(d_+(T,S(0))) - Ke^{-rT}N(d_-[T,S(0)])$$