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Q4.1.

Proof: suppose, $0 \le s \le t \le T$. for the purpose of proving $I(t)$ is a martingale is to prove $E[I(t)|F(s)] = I(s)$

case 1. t and s are in the same subinterval. $[t_k, t_{k+1}]$

$$I(s) = \sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(s) - M(t_k)] \quad ①$$

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)] \quad ②$$

for $E[I(t)|F(s)] = E[I(t) - I(s)|F(s)] + E[I(s)|F(s)]$ ($I(t) - I(s)$ is independent of $F(s)$. $I(s)$ is $F(s)$-measurable)

$= \Delta(t_k) E[M(t) - M(s)] + I(s)$ ($\Delta(t_k)$ is $F(s)$-measurable)

$= I(s)$ ($E[M(t) - M(s)] = 0.$)

case. 2. t and s are in the different subinterval $[t_k, t_{k+1}], [t_n, t_{n+1}]$ $k<n$

$$I(t) = \underbrace{\sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)]}_{① = V_1} + \underbrace{\Delta(t_k)[M(t_{k+1}) - M(t_k)]}_{② = V_2} + \underbrace{\sum_{i=k+1}^{n-1} \Delta(t_i)[M(t_{i+1}) - M(t_i)]}_{③ = V_3} + \underbrace{\Delta(t_n)[M(t) - M(t_n)]}_{④ = V_4}$$

$E[I(t)|F(s)] = E[V_1|F(s)] + E[V_2|F(s)] + E[V_3|F(s)] + E[V_4|F(s)]$

for ①. ① is $F(s)$-measurable. $\Rightarrow E[V_1|F(s)] = V_1$

for ③, ④ ③, ④ are independent of $F(s)$. $i > s$.

$E[V_3|F(s)] = \sum_{i=k+1}^{n-1} E[E[(\Delta t_i)(M_{t_{i+1}} - M_{t_i}) | F(t_i)] | F(s)]$

$= \sum_{i=k+1}^{n-1} E[\Delta(t_i) E[M_{t_{i+1}} - M_{t_i}] | F(s)] = \sum_{i=k+1}^{n-1} E[\Delta t_i \cdot 0 | F(s)] = 0.$

$E[V_4|F(s)] = E[E[\Delta(t_n)(M(t) - M(t_n)) | F(t_n)] | F(s)] = E[\Delta t_n \cdot 0 | F(s)] = 0.$

for ②. $E[\Delta(t_k)(M_{t_{k+1}} - M(s) + M(s) - M_{t_k}) | F(s)] = \Delta t_k (E[M_{t_{k+1}} - M(s) | F(s)] + M(s) - M_{t_k}) = \Delta t_k (M_s - M_{t_k})$

$\therefore ① + ② + ③ + ④ = \sum_{j=0}^{k-1} \Delta(t_j)[M_{t_{j+1}} - M_{t_j}] + \Delta t_k (M_s - M_{t_k}) = I(s)$ concluded!

Q4.3. ②

(i). s, t is the point in the subinterval $[s,t)$

So, $I(t)-I(s) = \Delta(s)(W(t)-W(s))$
$= W(s)(W(t)-W(s))$

$W(s)$ is $F(s)$-measurable, not independent of $F(s)$

(ii). use hint to prove.

let $X = \frac{E[(I(t)-I(s))^4]}{D^2[I(t)-I(s)]}$ ∵ $I(s)$ is in the same interval. $W(s)$ is independent of $W(t)-W(s)$

∴ $E\,I(t)-I(s) = E(W(s)(W(t)-W(s)))$
$= E[W(s)] \cdot E[W(t)-W(s)]$

$= \frac{E[W(s)]^4 \cdot E[W(t)-W(s)]^4}{D^2[I(t)-I(s)]}$

$W(s) \sim N(0,s)$. $W(t)-W(s) \sim N(0,t-s)$. $E[W(s)]^4 = 3s$. $E[W(t)-W(s)]^4 = 3(t-s)$

$D^2[I(t)-I(s)] = (E(I(t)-I(s))^2 - E^2(I(t)-I(s)))^2 = E^2(I(t)-I(s))^2 = (E(W(s))^2 \cdot E(W(t)-W(s))^2)^2$
$= (D(W(s)) \cdot D(W(t)-W(s)))^2$
$= s^2 (t-s)^2$

$X = \frac{9s(t-s)}{s^2(t-s)^2} = \frac{9}{s(t-s)} \neq 3.$

∴ $I(t)-I(s)$ is not Normally Distribution.

(iii) ~~$E\,I(t)-I(s)$~~ $E(I(t)|F(s)) = E(I(t)-I(s)|F(s)) + E[I(s)|F(s)]$
$= E(W(s)(W(t)-W(s))|F(s)) + E[W(0)(W(s)-W(0))|F(s)]$
$= W(s) \cdot E(W(t)-W(s)) + I(s)$
$= 0 + I(s) = I(s).$

besides, $I(s) = W(0)(W(s)-W(0)) = 0.$

(iv) defined by Itô Integral. $I(t) = 0 + W(s)(W(t)-W(s))$

right. $I^2(s) = [W(0)(W(s)-W(0))]^2 = 0.$ left, $E(I^2(t)|F(s)) = E[W^2(s) \cdot (W(t)-W(s))^2 | F(s)] = W^2(s) \cdot (t-s)$

$\int_0^s \Delta^2(u)du = \int_0^s 0\,du = 0.$ $E[\int_0^t \Delta^2(u)du | F(s)] = \int_0^s \Delta^2(u)du + E[\int_s^t \Delta^2(u)du | F(s)]$
$= 0 + \int_s^t W^2(s)du = W^2(s)(t-s)$

$I^2(s) - \int_0^s \Delta^2(u)du = 0.$ ∴ $E[I^2(t) - \int_0^t \Delta^2(u)du | F(s)] = 0.$

in a sum $E[I^2(t) - \int_0^t \Delta^2(u)du | F(s)] = I^2(s) - \int_0^s \Delta^2(u)du$

Q4.5

(i) ~~$S(t) = S(0) e^{X(t)} = S(0) e^{\int_0^t \sigma(s) dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s)) ds}$~~

~~$\log_e S(t) = \log_e S(0) + \int_0^t \sigma(s) dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s)) ds$~~

$\log S(t)$ means $\log_e S(t)$. $\ln S(t)$, $S(t)$ is an

$f(S(t)) = \ln S(t)$

$$df(S(t)) = \underbrace{f'(S(t)) dS(t)}_{①} + \underbrace{\frac{1}{2} f''(S(t)) . dS(t) . dS(t)}_{②} + \underbrace{\text{higher order term.}}_{③}$$

any term in ③ : $dS(t) . dS(t) = \alpha^2(t) S^2(t) dt . dt + \alpha(t) . \sigma(t) . S^2(t) dW(t) dt + \sigma^2(t) . S^2(t) . dW(t) . dW(t)$
$= 0 + 0 + \sigma^2(t) . S^2(t) . dt = \sigma^2(t) S^2(t) dt$

$\therefore$ ③ must either contain $dW(t) . dt = 0$ or $dt . dt = 0$

$\therefore$ ③ $= 0$.

② + ① $= \alpha(t) . dt + \sigma(t) . dW(t) + -\frac{1}{2S(t)^2} . \sigma^2(t) . S^2(t) dt = \sigma(t) dW(t) + (\alpha(t) - \frac{1}{2}\sigma^2(t)) dt$

in a sum $d(\ln S(t)) = \sigma(t) dW(t) + (\alpha(t) - \frac{1}{2}\sigma^2(t)) dt$

(ii). integrate $d(\ln S(t))$, $\ln S(u) \Big|_0^t = \int_0^t \sigma(u) dW(u) + (\alpha(u) - \frac{1}{2}\sigma^2(u)) du$

$$\ln S(t) = \ln S(0) + \int_0^t \sigma(u) dW(u) + \int_0^t \alpha(u) - \frac{1}{2}\sigma^2(u) du$$

$$S(t) = S(0) . e^{\int_0^t \sigma(u) dW(u) + \int_0^t \alpha(u) - \frac{1}{2}\sigma^2(u) du}.$$

get function (4.4.26)

Q4.7. $f(x) = x^4$ ④

(i) $dW^4_t = f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t)) \cdot dt$

$= 4W^3(t) \cdot dW(t) + 6W^2(t) \cdot dt$

$W^4(u)\Big|_0^T = \int_0^T 4W^3(u)\,dW(u) + \int_0^T 6W^2(u)\,du$

$W^4(T) = W^4(0) + \int_0^T 4W^3(u)\,dW(u) + \int_0^T 6W^2(u)\,du$

$= 4\int_0^T W^3(u)\,dW(u) + 6\int_0^T W^2(u)\,du$

(ii). $E(W^4(T)) = 4E\left[\int_0^T W^3(u)\,dW(u)\right] + 6E\left[\int_0^T W^2(u)\,du\right]$

$= 4 \cdot 0 + 6\int_0^T E[W^2(u)]\,du$

$= 6 \cdot \int_0^T u\,du = 3T^2$

Itô. Integral is a martingale.
$\int_0^T W^3(u)\,dW(u)$ is an Itô Integral.
whose expectation is $I(0) = 0$.

(iii) similar to i.

$dW^6_t = 6W^5(t)\,dt + 15W^4(t)\,dt$

$W^6(T) = 6\int_0^T W^5(u)\,dW(u) + 15\int_0^T W^4(u)\,du.$

similar to ii

$E\,W^6(T) = 0 + 15\int_0^T E[W^4(u)]\,du = 15 \cdot T^3.$

Q.4.8. $f(t, R(t)) = e^{\beta t} \cdot R(t)$

(i). $d(e^{\beta t} R(t)) = \beta \cdot e^{\beta t} \cdot R(t)\,dt + e^{\beta t} dR(t) = \beta \cdot e^{\beta t} R(t)\,dt + e^{\beta t}[(\alpha - \beta R(t))dt + \sigma\, dW(t)]$

$f_t(t, R(t)) = \beta e^{\beta t} \cdot R(t)$ $\qquad = e^{\beta t} \cdot (\alpha\, dt + \sigma\, dW(t))$

$f_{R(t)}(t, R(t)) = e^{\beta t}$

$f_{R(t)R(t)}(t, R(t)) = 0$ $\qquad$ thus. $d(e^{\beta t} \cdot R(t)) = e^{\beta t} \cdot (\alpha \cdot dt + \sigma \cdot dW(t))$

(ii). $e^{\beta t} \cdot R(t) \big|_0^T = \int_0^T e^{\beta t}(\alpha \cdot dt) + e^{\beta t} \cdot \sigma \cdot dW(t)$

$e^{\beta T} \cdot R(T) - R(0) = \frac{\alpha}{\beta}(e^{\beta T} - 1) + \sigma \int_0^T e^{\beta t}\, dW(t)$

$R(T) = \frac{\alpha}{\beta}(1 - e^{-\beta T}) + \sigma \int_0^T e^{\beta(t-T)}\, dW(t) + R(0)$

Q. addition

a). $X_t = f(t, W(t)) = e^{tW(t)}$

$f_t = W(t) e^{tW(t)}$ $\qquad$ ~~$dX(t) = \int W(t) \cdot e^{tW(t)} dt + \int t \cdot e^{tW(t)} dW(t) + \int t^2 e^{tW(t)} dt$~~

$f_{(x)} = t e^{tW(t)}$ $\qquad$ $dX(t) = W(t) e^{t \cdot W(t)} dt + t \cdot e^{tW(t)} dW(t) + \frac{1}{2} t^2 \cdot e^{tW(t)} dt$

$f_{xx} = t^2 e^{t \cdot W(t)}$

b). when $X_t = e^{rt} f(W(t))$. let $g(t, W(t)) = X_t$

$g_t(t, W(t)) = r \cdot e^{rt} f(W(t))$ ~~$+ e^{rt} f'$~~

$g_x(t, W(t)) = e^{rt} f'(W(t))$

$g_{xx}(t, W(t)) = e^{rt} f''(W(t)) = \lambda e^{rt} f(W(t))$

$dX(t) = r \cdot e^{rt} f(W(t))\,dt + e^{rt} f'(W(t)) \cdot dW(t) + \lambda e^{rt} f(W(t))$

$X(t) = X(0) + \int_0^t r e^{ru} f(W(u))\,du + \int_0^t e^{ru} f'(W(u))\,dW(u) + \int \lambda e^{ru} f(W(u))\,du \times \frac{1}{2}$

$X(t) = 1 + \int_0^t (r + \frac{\lambda}{2}) e^{ru} \cdot f(W(u))\,du + \int_0^t e^{ru} f'(W(u))\,dW(u)$.

$E[X(t)] = 1 + E[\int_0^t (r + \frac{\lambda}{2}) e^{ru} f(W(u))\,du]$

$E[\frac{X_t}{e^{rt}}] = e^{-rt} + e^{-rt} E[\int_0^t (r + \frac{\lambda}{2}) e^{ru} f(W(u))\,du]$

when, $r + \frac{\lambda}{2} = 0$, $-r = \frac{\lambda}{2}$

$E[f(W(t))] = e^{\frac{\lambda}{2} t} + 0$

$= e^{\frac{\lambda}{2} t}$.

Recommend. 如何理解非随机过程 ⑥

Q4.2

(i): for $I(t)-I(s)$, because $\Delta(t)$ is a nonrandom simple process, we know $\Delta(t_k)$ at time t_k, when $t_k > t_i$.

so we can divide the interval $[t_s, t_{s+1})$ into $[t_s, S)$ and $[S, t_{s+1})$. suppose $S\in[t_s, t_{s+1})$

thus. $I(s)$ also equals to.

$$I(s) = \lim_{i\to 0}\sum_{j=0}^{k-1}\Delta(t_j+i)\left[W(t_{j+1}) - W(t_j)\right] + \Delta(t_k+i)\left[W(s) - W(t_k)\right],$$

We use the new defination of $I(s)$.

$$I(t) - I(s) = \lim_{i\to 0}\left(\sum_{j=s}^{t-1}\Delta(j+i)\left[W(t_{j+1}) - W(t_j)\right] + \Delta(t_t+i)\left[W(t) - W(t_t)\right]\right)$$

① in every part $\lim_{i\to 0}\left(\Delta(j+i)\left[W(t_{j+1}) - W(t_j)\right]\right)$ (when $j>s$) is independent of $F(s)$.

② in the part $\lim_{i\to 0}\Delta(s+i)\left[W(t_{s+1}) - W(s)\right]$, $\Delta(s+i)$ and $W(t_{s+1}) - W(s)$ are both independent of $F(s)$.

In a sum, $I(t) - I(s)$ is independent of $F(s)$

(ii). proved by (i). $I(t) - I(s)$ is independent of $F(s)$,

So. for every subinterval $[t_k, t_l)$. t_k, t_l is the partition point and $t_k < t_l$

$I(t_l) - I(t_k)$ is independent of $F(t_k)$. $\Rightarrow D(X+Y) = D(X) + D(Y)$

thus. ~~$I(t) - I(s) = \sum_{i=s}$~~ Π is a partition between s and t, $\{x_0 = s, x_1, x_2 \cdots x_n = t\}$

$$I(t) - I(s) = \sum_{i=0}^{n-1}\left(I(x_{i+1}) - I(x_i)\right)$$

for each $I(x_{i+1}) - I(x_i) =$ ~~$\Delta(t)[W(x_{i+1}) - W(x_i)]$~~ $\Delta(x_i)\left(W(x_{i+1}) - W(x_i)\right) \sim N\left(0, \Delta(x_i)^2 \cdot (x_{i+1} - x_i)\right)$

(正态分布相加?)

$$I(t) - I(s) \sim N\left(0, \sum_{i=0}^{n-1}\Delta(x_i)^2 (x_{i+1} - x_i)\right)$$

$\because \Delta(x_i)$ is a constant $\therefore \Delta(x_i)^2$ is a constant.

$$\therefore \Delta(x_i)^2 (x_{i+1} - x_i) = \int_i^{i+1}\Delta^2(u)\,du$$

$$\therefore I(t) - I(s) \sim N\left(0, \int_s^t \Delta^2(u)\,du\right)$$

Q.4.6.

S(t) is a geometric Brownian Motion

$$dS(t) = \alpha(t) S(t)\,dt + \sigma(t) S(t)\,dW(t) .$$

$$\cancel{dS(t)\cdot dS(t) = \alpha(\alpha S(t)dt + \sigma S(t)\cdot dW(t))}\,dt + \cancel{\sigma(\alpha S(t)dt + \sigma S(t) dW(t))\,dW(t)}.$$

$$\cancel{= 0 + 0 + 0 + \sigma^2 S(t)\,dt}$$

$$dS(t)\cdot dS(t) = 0 + 0 + 0 + \sigma^2(t) S^2(t)\,dt$$

$$= \sigma^2(t) S^2(t)\,dt .$$

$$\because f(x) = x^p ,\quad f'(x) = p\,x^{p-1} ,\quad f''(x) = p(p-1)\,x^{p-2}$$

$$df(S) = p\cdot S(t)^{p-1}\cdot dS(t) + \frac{p(p-1)}{2}\,S(t)^{p-2}\,dS(t)\cdot dS(t)$$

$$= p\cdot S(t)^{p-1}\,(\alpha S(t)\,dt + \sigma S(t)\,dW(t)) + \frac{p(p-1)}{2}\cdot \sigma^2 S(t)^p\,dt$$

$$= p\cdot S(t)^p (\alpha\,dt + \sigma\,dW(t)) + \frac{p}{2}(p-1)\,\sigma^2 S(t)^p\,dt$$

$$= \left(p\alpha\cdot S(t)^p + \frac{p(p-1)}{2}\sigma^2 S(t)^p\right)dt + p\cdot\sigma S(t)^p\cdot dW(t)$$