Sample Final Exam Solutions

Problem 1

(a) Observe that $\mathbb{E}[X_1] = (e + e^{-1})/2 > 1 = X_0$. The process $\{X_n\}$ is not a supermartingale. If $\xi_0 = 1$ then $X_1 = e$. We obtain

$$\mathbb{E}\big[X_2 \big| X_1 = e\big] = X_1 e^{1-e} \frac{e + e^{-1}}{2} = X_1 \frac{e^{2-e} + e^{-e}}{2} < X_1.$$

It follows that the process $\{X_n\}$ is not a submartingale. It is neither.
(b)

$$\begin{aligned}
Z_n &= \min_{\substack{n \le \tau \le N \\ \tau-\text{stopping time}}} \mathbb{E}\big[X_\tau \bigm| \mathscr{F}_n\big] \\
&\le \min_{\substack{n+1 \le \tau \le N \\ \tau-\text{stopping time}}} \mathbb{E}\big[X_\tau \bigm| \mathscr{F}_n\big] \\
&= \min_{\substack{n+1 \le \tau \le N \\ \tau-\text{stopping time}}} \mathbb{E}\Big[\mathbb{E}\big[X_\tau \bigm| \mathscr{F}_{n+1}\big] \Big| \mathscr{F}_n\Big]
\end{aligned}$$

Consider the stopping time $\widetilde{\tau}$ in $[t+1, N]$ for which $\mathbb{E}\big[X_\tau \bigm| \mathscr{F}_{n+1}\big]$ is minimal. This is exactly the stopping time in the definition of

$$Z_{n+1} = \min_{\substack{n+1 \le \tau \le N \\ \tau-\text{stopping time}}} \mathbb{E}\big[X_\tau \bigm| \mathscr{F}_{n+1}\big] = \mathbb{E}\big[X_{\widetilde{\tau}} \bigm| \mathscr{F}_{n+1}\big].$$

We conclude that

$$Z_n \le \mathbb{E}\Big[\mathbb{E}\big[X_{\widetilde{\tau}} \bigm| \mathscr{F}_{n+1}\big] \Bigm| \mathscr{F}_n\Big] = \mathbb{E}\big[Z_{n+1} \mid \mathscr{F}_n\big],$$

that is, the process $\{Z_n\}$ is a submartingale.

Problem 2

It is convenient to write the integral as a sum of two parts

$$\begin{aligned}
I &= \int_1^2 \frac{1}{t^2} \left[W_1 + \int_1^t dW_s \right] dt \\
&= W_1 \int_1^2 \frac{1}{t^2}\, dt + \int_1^2 \frac{1}{t^2} \int_1^t dW_s\, dt = \frac{1}{2} W_1 + \int_1^2 \int_1^t \frac{1}{t^2}\, dW_s\, dt
\end{aligned}$$

The last integral can be calculated by changing the order of integration

$$I_2 = \int_1^2 \int_1^t \frac{1}{t^2}\, dW_s\, dt = \int_1^2 \int_s^2 \frac{1}{t^2}\, dt\, dW_s = \int_1^2 \left(\frac{1}{s} - \frac{1}{2} \right) dW_s$$

It follows that I_2 is a stochastic integral, it is independent of W_1, its expected value is 0, and its variance can be calculated from the Itô isometry:

$$\mathbb{E}\big[I_2^2\big] = \int_1^2 \Big(\frac{1}{s} - \frac{1}{2}\Big)^2 \, ds$$

We conclude that

$$\mathbb{E}\big[I\big] = 0, \qquad \mathbb{E}\big[I^2\big] = \frac{1}{4} + \int_1^2 \Big(\frac{1}{s} - \frac{1}{2}\Big)^2 \, ds$$

You may calculate the last integral by standard calculus for integrating rational functions.

Problem 3

We have $X_t = f(U_t, Y_t)$ with $f(u,y) = e^{u/y}$. Applying Itô calculus we obtain

$$dX_t = \frac{\partial f}{\partial u} dU_t + \frac{\partial f}{\partial y} dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(dU_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2 + \frac{\partial^2 f}{\partial u \partial y}(dU_t)(dY_t)$$

Observe that there is no fraction $\frac{1}{2}$ in front of the mixed derivative term, because the full formula has two identical terms with equal mixed derivatives $\frac{\partial^2 f}{\partial u \partial y}$ and $\frac{\partial^2 f}{\partial y \partial u}$. We can now simplify by noting that

$$\frac{\partial f}{\partial u} = \frac{X_t}{Y_t}, \quad \frac{\partial f}{\partial y} = -\frac{X_t U_t}{Y_t^2}, \quad \frac{\partial^2 f}{\partial u^2} = \frac{X_t}{Y_t^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{X_t U_t^2 + 2X_t U_t Y_t}{Y_t^4}, \quad \frac{\partial^2 f}{\partial u \partial y} = -\frac{X_t U_t + X_t Y_t}{Y_t^3}$$

and (symbolically)

$$(dU_t)^2 = \rho^2 \, dt, \quad (dY_t)^2 = \sigma^2 Y_t^2 \, dt, \quad (dU_t)(dY_t) = \sigma\rho Y_t \, dt.$$

Substituting, we obtain the stochastic differential equation for X_t. The process $\{X_t\}$ will be a submartingale, if the term with dt will always be nonnegative in this equation. We obtain the condition

$$v\frac{X_t}{Y_t} + \frac{\rho^2}{2}\frac{X_t}{Y_t^2} + \sigma^2 \frac{X_t U_t^2 + 2X_t U_t Y_t}{2Y_t^2} - \sigma\rho \frac{X_t U_t + X_t Y_t}{Y_t^2} \geq 0.$$

It cannot be satisfied for all values of u and y.

Problem 4

Method 1. In the equivalent martingale measure, the discounted stock price satisfies the equation

$$d\tilde{S}_t = \sigma \tilde{S}_t \, dX_t.$$

Thus

$$\tilde{S}_1 = S_0 \exp\big(\sigma X_1 - \frac{\sigma^2}{2}\big),$$
$$\tilde{S}_2 = \tilde{S}_1 \exp\big(\sigma(X_2 - X_1) - \frac{\sigma^2}{2}\big).$$

The option has value $S_2 - S_1$ if $S_1 \geq S_0$ and $S_2 \geq S_1$. The condition $S_1 \geq S_0$ is true, if

$$\exp\left(r + \sigma X_1 - \frac{\sigma^2}{2}\right) \geq 1$$

that is,

$$X_1 \geq \frac{\sigma}{2} - \frac{r}{\sigma},$$

where X_1 is a standard normal random variable.

Similarly, $S_2 \geq S_1$, if

$$Z \geq \frac{\sigma}{2} - \frac{r}{\sigma}$$

where $Z = X_2 - X_1$ is a standard normal random variable independent of X_1. The value of the option is

$$C_0 = \mathbb{E}^Q\left[\tilde{C}_2\right] = \frac{S_0}{2\pi} \int_{\frac{\sigma}{2}-\frac{r}{\sigma}}^{\infty} \int_{\frac{\sigma}{2}-\frac{r}{\sigma}}^{\infty} \exp\left(\sigma x - \frac{\sigma^2}{2}\right)\left[\exp\left(\sigma z - \frac{\sigma^2}{2}\right) - \exp(-r)\right]\exp\left(\frac{-x^2 - z^2}{2}\right) dz\, dx.$$

The rest is standard (but tedious) integration and can be omitted.

Method 2. At time 1, the option is related to the European call option with strike price S_1. We can denote its price by $F(S_1, 1)$ (given by the standard Black-Scholes formula). We know that

$$S_1 = S_0 \exp\left(r + \sigma X_1 - \frac{\sigma^2}{2}\right),$$

where X_1 is standard normal. As in Method 1, the condition $S_1 \geq S_0$ is true if $X_1 \geq \frac{\sigma}{2} - \frac{r}{\sigma}$. Therefore

$$C_0 = \int_{\frac{\sigma}{2}-\frac{r}{\sigma}}^{\infty} \exp(-r) F\left(S_0 \exp\left(r + \sigma x - \frac{\sigma^2}{2}\right), 1\right) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx$$

This method replaces the integral over z with the ready Black-Scholes formula.

Problem 5

As in problem 3, we get

$$\begin{aligned} dZ_t &= E_t\, dS_t + S_t\, dE_t + (dE_t)(dS_t) \\ &= Z_t\left(\mu\, dt + \sigma\, dW_t\right) + Z_t\left((a - E_t)\, dt + \gamma\, dW_t\right) + \sigma\gamma Z_t\, dt \\ &= Z_t\left(\mu + a - E_t + \sigma\gamma\right) dt + Z_t\left(\sigma + \gamma\right) dW_t. \end{aligned}$$