Review Problems for the Final Exam (May 5, 2020)

Problem 1

Discrete-time stochastic processes $\{X_n\}$, and $\{Z_n\}$ are defined by the equations

$$\begin{aligned} X_{n+1} &= X_n \exp(1 - X_n + \xi_n), \quad n = 0,1,2,\ldots,N, \qquad X_0 = 1, \\ Z_n &= \min_{\substack{n \le \tau \le N \\ \tau-\text{stopping time}}} \mathbb{E}\big[X_\tau \mid \mathscr{F}_n\big], \quad n = 0,1,2,\ldots,N, \end{aligned}$$

where the random variables $\{\xi_n\}$ are independent, with $P[\xi_n = 1] = P[\xi_n = -1] = 1/2$. For each of these processes verify whether it is a martingale, supermartingale, submartingale, or neither.

Problem 2

Suppose $\{W_t\}$ is a standard Brownian motion. Evaluate the mean and the variance of the integral

$$I = \int_1^2 \frac{W_t}{t^2}\, dt.$$

Problem 3

Suppose $\{W_t\}$ is a standard Brownian motion and the processes $\{X_t\}$, $\{Y_t\}$, and $\{U_t\}$ are defined by the equations

$$\begin{aligned} X_t &= e^{U_t/Y_t}, \\ dY_t &= \sigma Y_t\, dW_t, \qquad Y_0 = 1, \\ dU_t &= v\, dt + \rho\, dW_t, \qquad U_0 = 1, \end{aligned}$$

where $\sigma > 0,\ v > 0,\ \rho > 0$.

(a) Derive the stochastic differential equation satisfied by the process $\{X_t\}$.

(b) When is the process $\{X_t\}$ a submartingale?

Problem 4

Suppose the stock price evolves according to the differential equation

$$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t,$$

where $\{W_t\}$ is a standard Brownian motion, $S_0 > 0$, $\mu > 0$, $\sigma > 0$. The bond price evolves according to the equation

$$dB_t = rB_t \, dt, \quad B_0 = 1,$$

where the riskless interest rate $r > 0$ is fixed. At time $T = 2$ the following option is available: we can buy one share of the stock for the price S_1, provided that $S_1 \geq S_0$. If $S_1 < S_0$, the option is not available. Find the price of the option at price $t = 0$.

Problem 5

Suppose the stock price in sterling pounds evolves according to the differential equation

$$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t,$$

where $\{W_t\}$ is a standard Brownian motion, $S_0 > 0$, $\mu > 0$, $\sigma > 0$. The dollar-pound exchange rate (the dollar price of one pound) follows the stochastic differential equation

$$dE_t = E_t(a - E_t) \, dt + \gamma E_t \, dW_t, \qquad E_0 > 0,$$

where $a > 0$, $\gamma > 0$. Derive the stochastic differential equation for the stock price in dollars: $Z_t = S_t E_t$.