FE620 Pricing and Hedging

Lecture 7: Binomial Trees

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A Simple Binomial Model

- A stock price is currently \$20
- In 3 months it will be either \$22 or \$18

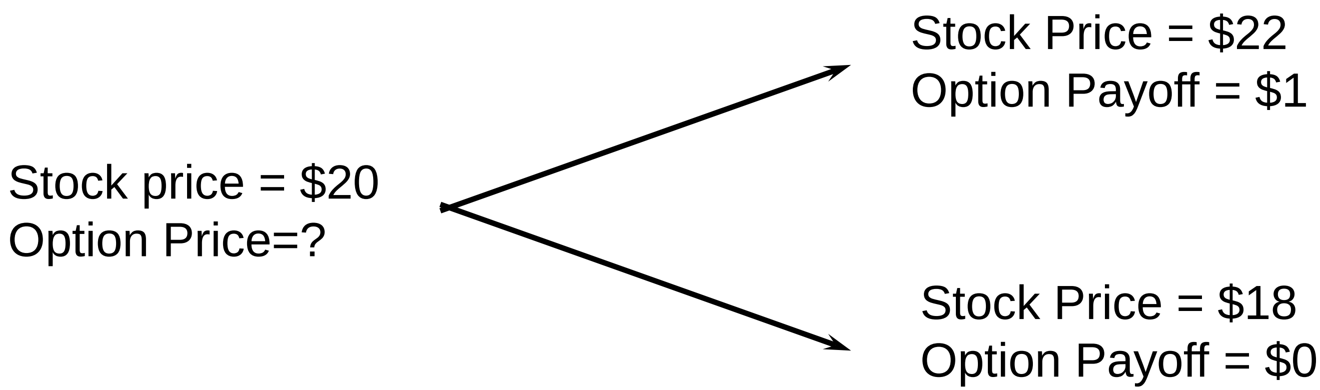
Stock Price = \$22

Stock price = \$20

Stock Price = \$18

A Call Option (Figure 13.1, page 275)

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are

$22\Delta - 1$

18Δ

- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 4%)

- The riskless portfolio is:
  long 0.25 shares
  short 1 call option
- The value of the portfolio in 3 months is
  $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is
  $4.5 e^{-0.04 \times 0.25} = 4.455$

Valuing the Option

- The portfolio that is

  long 0.25 shares

  short 1 option

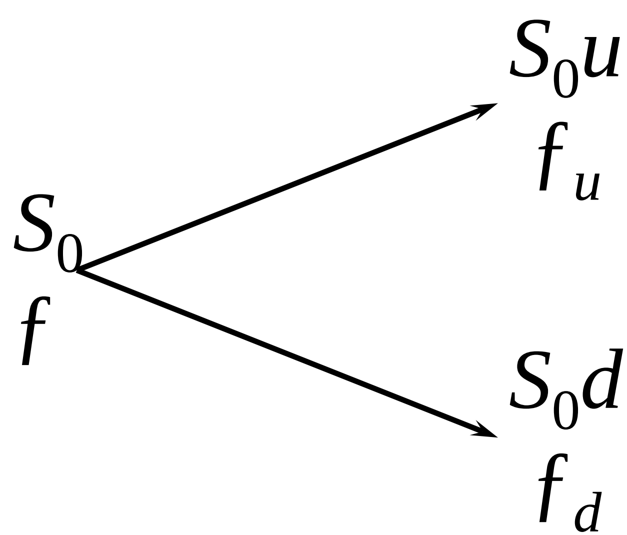
  is worth 4.455
- The value of the shares is

  5.000 (= 0.25 × 20)
- The value of the option is therefore

  5.000 – 4.455 = 0.545

Generalization (Figure 13.2, page 276)

A derivative lasts for time T and is dependent on a stock



Generalization (continued)

- Value of a portfolio that is long Δ shares and short 1 derivative:

$$S_0 u\Delta - f_u$$

$$S_0 d\Delta - f_d$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization (continued)

- Value of the portfolio at time T is $S_0 u\Delta - f_u$
- Value of the portfolio today is $(S_0 u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

Generalization
(continued)

Substituting for Δ we obtain

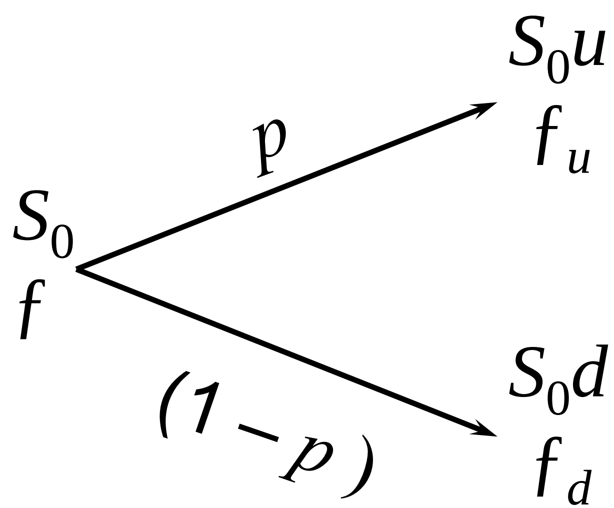
$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

p as a Probability

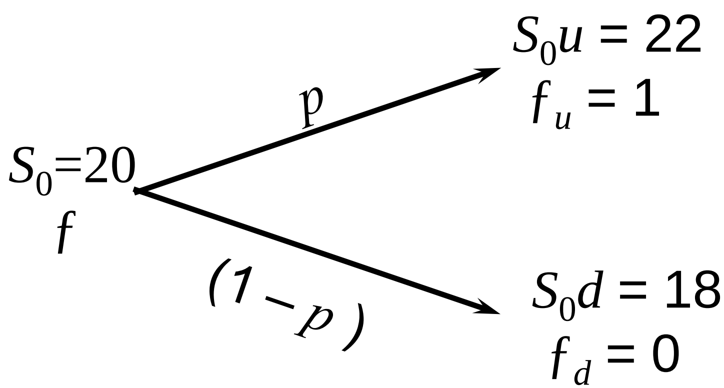
- It is natural to interpret p and 1-p as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

- When the probability of an up and down movements are p and 1-p the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Original Example Revisited



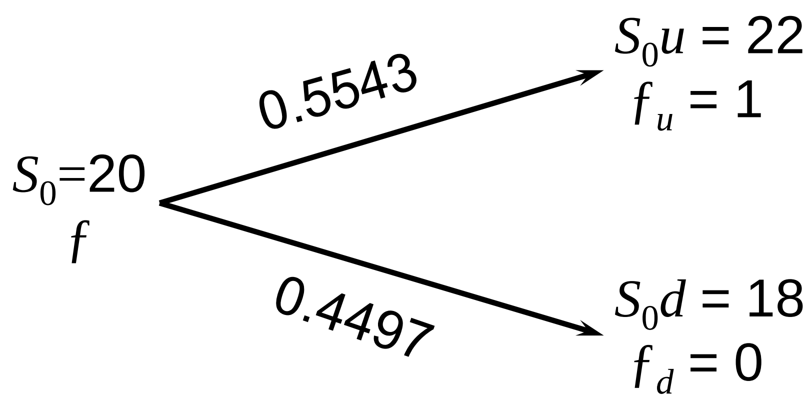
p is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.04\times0.25} = 22p + 18(1-p)$ so that $p = 0.5503$

Alternatively:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.04\times0.25} - 0.9}{1.1 - 0.9} = 0.5543$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$$e^{-0.04\times 0.25}(0.5543\times 1 + 0.4497\times 0)$$

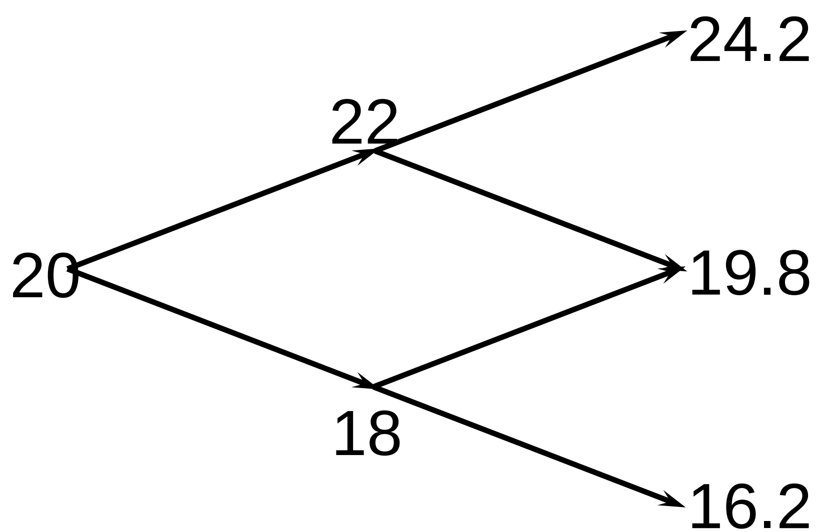
$$= 0.545$$

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

A Two-Step Example

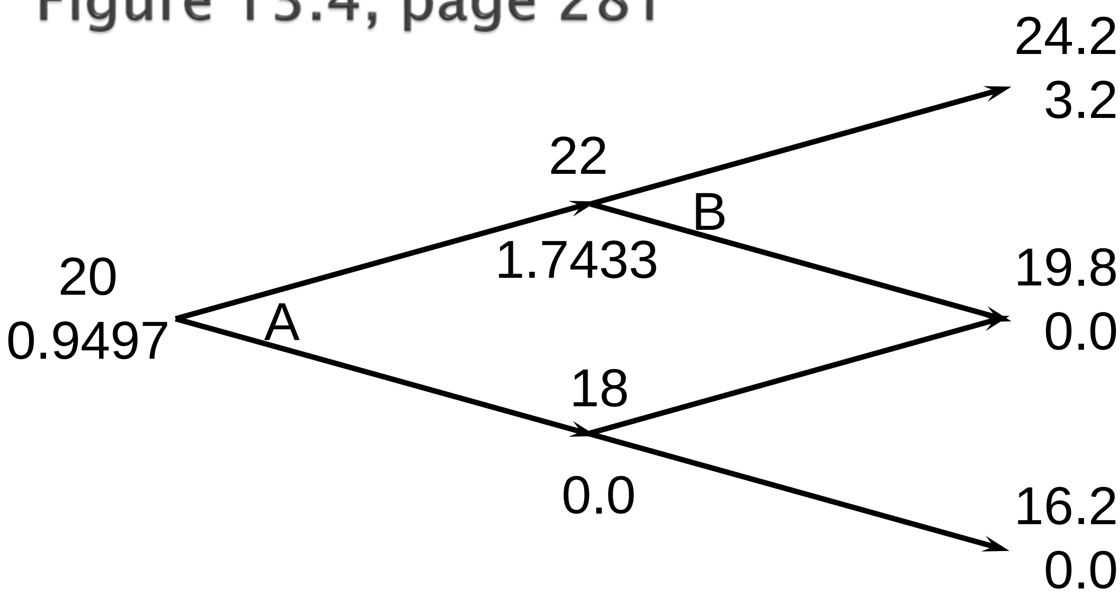
Figure 13.3, page 281



- K=21, r = 4%
- Each time step is 3 months

Valuing a Call Option

Figure 13.4, page 281



Value at node B

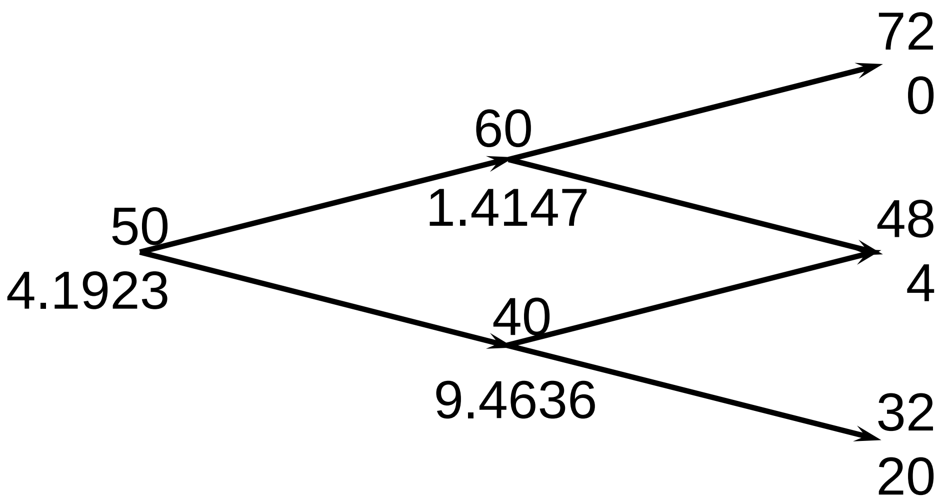
$= e^{-0.04\times0.25}(0.5503\times3.2 + 0.4497\times0) = 1.7433$

Value at node A

$= e^{-0.04\times0.25}(0.5503\times1.7433 + 0.4497\times0) = 0.9497$

A Put Option Example

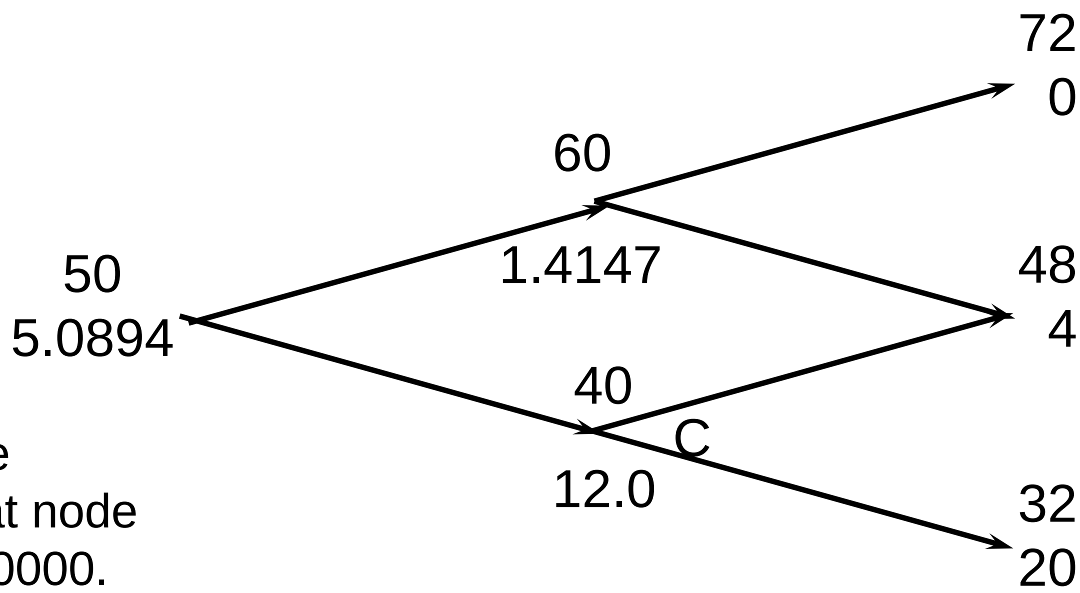
Figure 13.7, page 284



K = 52, time step =1yr

r = 5%, u =1.2, d = 0.8, p = 0.6282

What Happens When the Put Option is American (Figure 13.8, page 285)



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

Girsanov's Theorem

- Volatility is the same in the real world and the risk-neutral world
- We can therefore measure volatility in the real world and use it to build a tree for the an asset in the risk-neutral world

Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

Proving Black-Scholes-Merton from Binomial Trees (Appendix to Chapter 13)

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K,\, 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j}$$

Proving Black–Scholes–Merton from Binomial Trees continued

- The expression for U_1 can be written

$$U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} \left(p^*\right)^j \left(1-p^*\right)^{n-j} = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

where

$$p^* = \frac{pu}{pu + (1-p)d}$$

- Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution