FE620 Pricing and Hedging

Lecture 8: The Black-Scholes-Merton Model

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The Stock Price Assumption

- Consider a stock whose price is S
- In a short period of time of length Δt, the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx \phi\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

where μ is expected return and σ is volatility

The Lognormal Property
(Equations 15.2 and 15.3, page 320)

- It follows from this assumption that

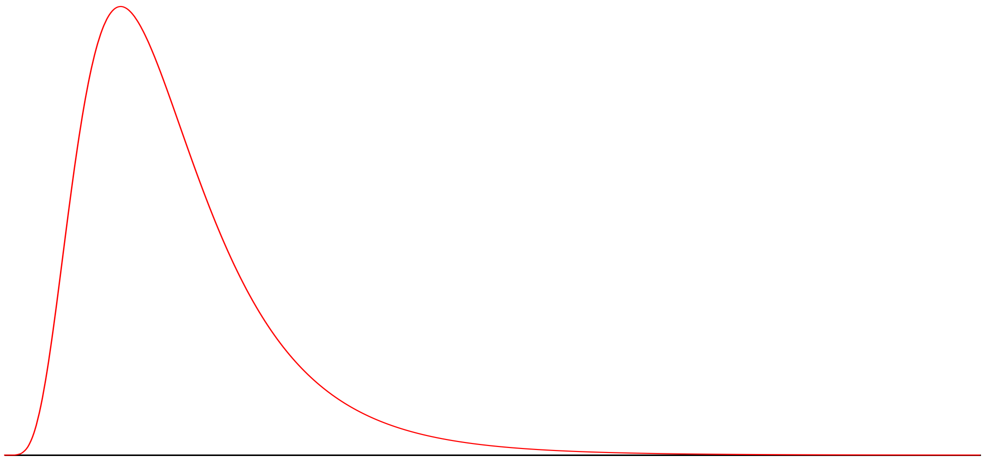
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0\ e^{\mu T}$$

$$\text{var}(S_T) = S_0{}^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Continuously Compounded Return (Equations 15.6 and 15.7, page 322)

If x is the realized continuously compounded return

$$S_T = S_0 \, e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$x \approx \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T} \right)$$

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is
  $\mu - \sigma^2/2$ not μ

  This is because

$$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

  are not the same

μ and $\mu - \sigma^2/2$

- μ is the expected return in a very short time, Δt, expressed with a compounding frequency of Δt
- $\mu - \sigma^2/2$ is the expected return in a long period of time expressed with continuous compounding (or, to a good approximation, with a compounding frequency of Δt)

Mutual Fund Returns (See Business Snapshot 15.1 on page 324)

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ
- The geometric mean of 12.4% is analogous to $\mu - \sigma^2/2$

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in a short time period time Δt is approximately

$$\sigma\sqrt{\Delta t}$$

- If a stock price is $50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data (page 324-326)

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years (e.g. for weekly data τ = 1/52)
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s, of the u_i´s
4. The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

Nature of Volatility (Business Snapshot 15.2, page 327)

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued
- It is assumed that there are 252 trading days in one year for most assets

Example

- Suppose it is April 1 and an option lasts to April 30 so that the number of days remaining is 30 calendar days or 22 trading days
- The time to maturity would be assumed to be 22/252 = 0.0873 years

The Concepts Underlying Black-Scholes-Merton

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

The Derivation of the Black-Scholes-Merton Differential Equation

$$\Delta S = \mu S \, \Delta t + \sigma S \, \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \, \Delta z$$

We set up a portfolio consisting of

-1: derivative

$+\dfrac{\partial f}{\partial S}$: shares

This gets rid of the dependence on Δz.

The Derivation of the Black-Scholes-Merton Differential Equation continued

The value of the portfolio, Π, is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The Derivation of the Black-Scholes-Merton Differential Equation continued

The return on the portfolio must be the risk - free rate. Hence

$$\Delta\Pi = r\,\Pi\Delta t$$

$$-\Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t$$

We substitute for Δf and ΔS in this equation to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is ƒ = $S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

The Black-Scholes-Merton Formulas for Options (See pages 333-334)

$$c = S_0 \, N(d_1) - K \, e^{-rT} N(d_2)$$

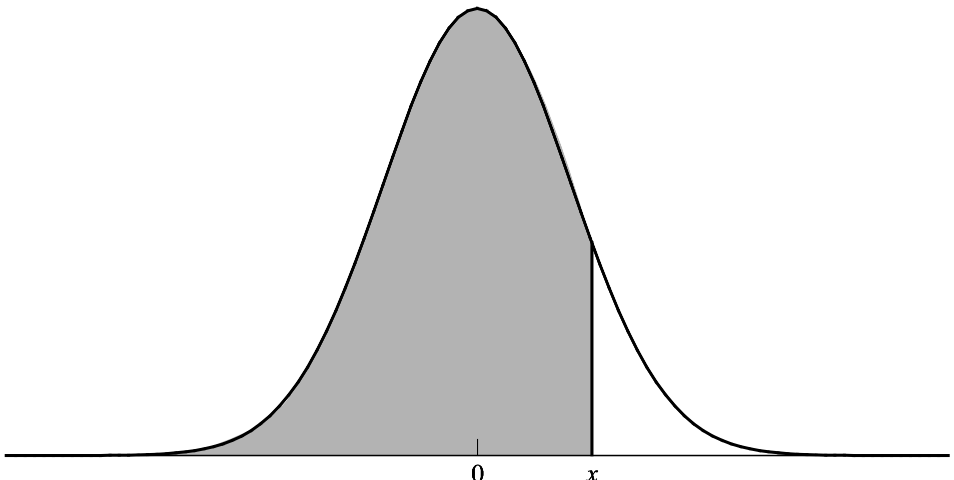
$$p = K \, e^{-rT} \, N(-d_2) - S_0 \, N(-d_1)$$

where $$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The N(x) Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x



- See tables at the end of the book

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$
- What happens as σ becomes very large?
- What happens as T becomes very large?

Understanding Black-Scholes

$$c = e^{-rT} N(d_2)\left(S_0 e^{rT} N\left(d_1\right) / N\left(d_2\right) - K\right)$$

$$
\begin{aligned}
e^{-rT}: &\quad \text{Present value factor} \\
N(d_2): &\quad \text{Probability of exercise} \\
S_0 e^{rT} N(d_1)/N(d_2): &\quad \text{Expected stock price in a risk-neutral world} \\
&\quad \text{if option is exercised} \\
K: &\quad \text{Strike price paid if option is exercised}
\end{aligned}
$$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S_0 e^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[S_0 e^{rT} - K] = S_0 - K e^{-rT}$$

Proving Black–Scholes–Merton Using Risk–Neutral Valuation (Appendix to Chapter 15)

$$c = e^{-rT}\int_K^\infty \max(S_T - K,\ 0)\,g(S_T)\,dS_T$$

where $g(S_T)$ is the probability density function for the lognormal distribution of S_T in a risk-neutral world. ln S_T is $\varphi(m, s^2)$ where

$$m = \ln S_0 + \left(r - \sigma^2/2\right)T \qquad s = \sigma\sqrt{T}$$

We substitute

$$Q = \frac{\ln S_T - m}{s}$$

so that

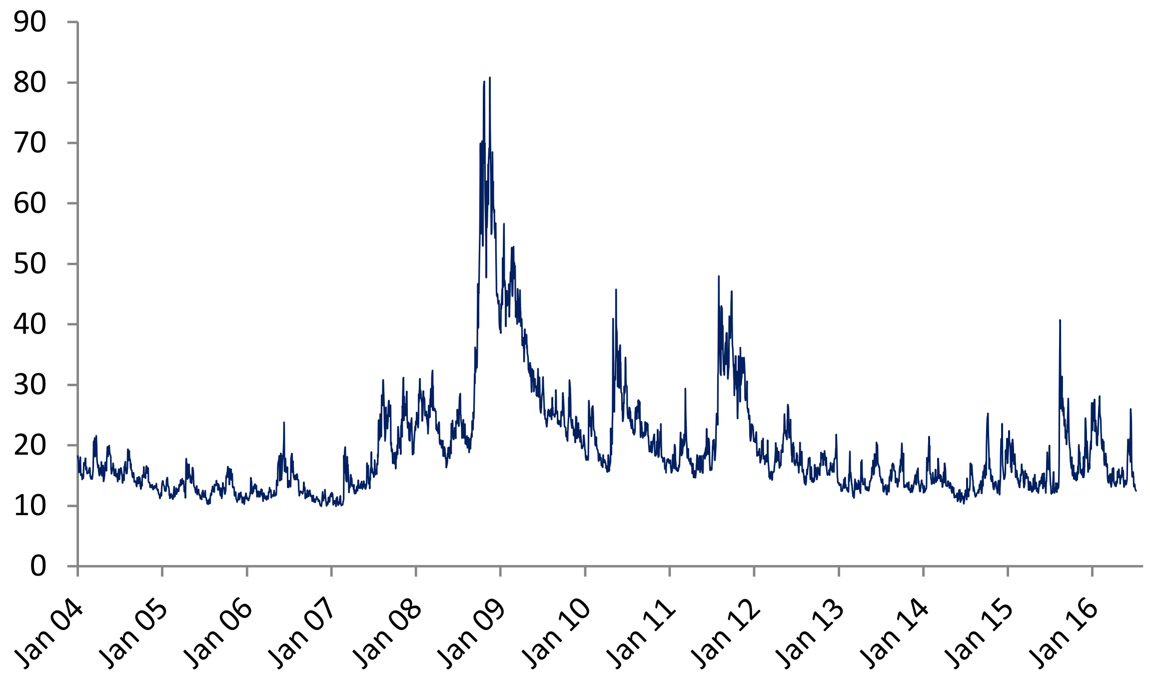
$$c = e^{-rT}\int_{(\ln K - m)/s}^\infty \max(e^{Qs+m} - K,\ 0)\,h(Q)\,dQ$$

where h is the probability density function for a standard normal. Evaluating the integral leads to the BSM result.

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes-Merton price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index



Chapter

An Issue of Warrants & Executive Stock Options

- When a regular call option is exercised the stock that is delivered must be purchased in the open market
- When a warrant or executive stock option is exercised new Treasury stock is issued by the company
- If little or no benefits are foreseen by the market, the stock price will reduce at the time the issue is announced.
- There is no further dilution (See Business Snapshot 15.3.)

The Impact of Dilution

- After the options have been issued it is not necessary to take account of dilution when they are valued
- Before they are issued we can calculate the cost of each option as $N/(N+M)$ times the price of a regular option with the same terms where N is the number of existing shares and M is the number of new shares that will be created if exercise takes place

Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The “dividend” should be the expected reduction in the stock price expected

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date
- Suppose dividend dates are at times t_1, t_2, …t_n. Early exercise is sometimes optimal at time t_i if the dividend at that time is greater than

$$K[1-e^{-r(t_{i+1}-t_i)}]$$

Black's Approximation for Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date