Heston Model

S_t = price

$$dS_t = \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t$$

$$dV_t = k(\theta - V_t)\, dt + \sigma\sqrt{V_t}\, dZ_t$$

W_t, Z_t 2 BM's
correlated
coef. ρ

$$E(dW_t\, dZ_t) = \rho\, dt$$

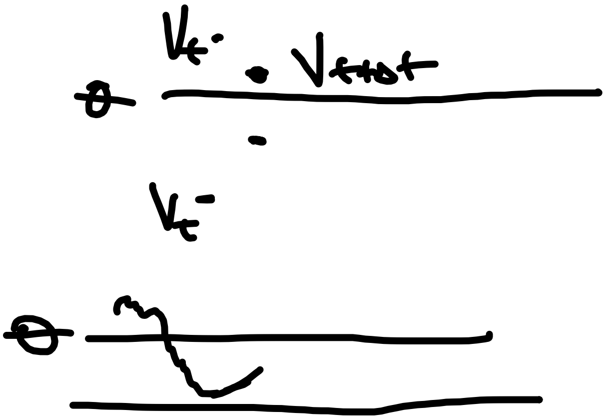
$k > 0$ speed of mean reversion
θ mean reversion parameter
σ = volatility of variance

$\frac{dS_t}{S_t} = \ldots$
$X_t = \log S_t$
Ito eqn for X_t

CIR model (Cox Ingersoll Ross)

$$dV_t = k(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dZ_t$$

$$\Delta t \to V_{t+\Delta t} - V_t = \underline{k(\theta - V_t)\Delta t} + \underline{\sigma\sqrt{V_t}\,\Delta Z_t}$$



$$dr_t = k(\theta - r_t)\,dt + \sigma\,dZ_t$$

Vasiček model

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dZ_t$$

For CIR model please read lecture notes.

$$dV_t = k(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dZ_t$$

calculate $E(V_t)$

$E(V_t^2)$

$$V_t - V_0 = \int_0^t k(\theta - V_s)\,ds + \int_0^t \sigma\sqrt{V_s}\,dZ_s$$

$$E(V_t) - E(V_0) = E\left[\int_0^t k(\theta - V_s)\,ds\right] + 0$$

$$E(V_t) - E(V_0) = \int_0^t E(k(\theta - V_s))\,ds$$

$E[V_t] = \mu_t$

$$\mu_t - \mu_0 = \int_0^t k(\theta - \mu_s)\,ds$$

$$\frac{d\mu_t}{dt} = k(\theta - \mu_t)$$

$$d\mu_t + k\mu_t\,dt = k\theta\,dt \;\Big|\; e^{kt}$$

..... etc

Hint for V_t^2 use Itô lemma first to get V_t^2 dynamics

$$E(V_t \mid V_0) = \theta + (V_0 - \theta)e^{-\kappa(t-0)}$$

Heston PDE

$C(S, v, t)$ = price of an option at time t
Stock value $= S$
Variance value of $V_t = v$

$x = \ln S$

$u(x, v, t) = C(e^x, v, t)$

$$\frac{\partial u}{\partial t} + \left(r - \frac{1}{2}v\right)\frac{\partial u}{\partial x} + \frac{1}{2}v\frac{\partial^2 u}{\partial x^2} + \rho\sigma v\frac{\partial^2 u}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 u}{\partial v^2}$$
$$- ru + \left(\kappa(\theta - v)\right)\frac{\partial u}{\partial v} = 0$$

SABR

$$dS_t = \alpha_t S_t^{\beta} dW_t$$

$$d\alpha_t = \nu \alpha_t dZ_t$$

ρ corr coef between BM's -

Option price is given by plugging into the B-S formula a volatility value σ_{SABR}.

Option Sensitivities

$C(S, t, K, r, \sigma)$

$$\delta = \frac{\partial C}{\partial S} =$$

$$\delta = \frac{\partial C}{\partial S}(S_0) = \frac{C(S_0+\Delta S) - C(S_0)}{\Delta S}$$

Quadrature methods

Methodology to approximate a definite integral

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

price of an European type derivative is:

$$F(t, X_t) = E\left[e^{-\int_t^T r(s)ds} F(T, X_T) \,\middle|\, \mathcal{F}_t \right]$$

X_t $\quad p(s, x, t, y)$ = density of r.v. $X_t \mid X_s = x$

$$F(t, \underset{\substack{\shortparallel \\ x}}{\cancel{X_t}}) = \int_{-\infty}^{\infty} e^{-\int_t^T r(s)ds} F(t, y)\, p(s, x, t, y)\, dy$$

Problem: Consider $I(f) = \int_A f(x)\,dx$

Def: A quadrature rule of order n is an expression:

$$I_n(f) = \sum_{i=1}^{n} w_i^n f(x_i^n)$$

w_i^n = weights
x_i^n = quadrature nodes

$$I_n(f) \longrightarrow I(f) \text{ as } n \to \infty$$

Basic rule of constructing a quadrature rule is

- approximate f using some interpolating function p_n (polynomials) s.t.

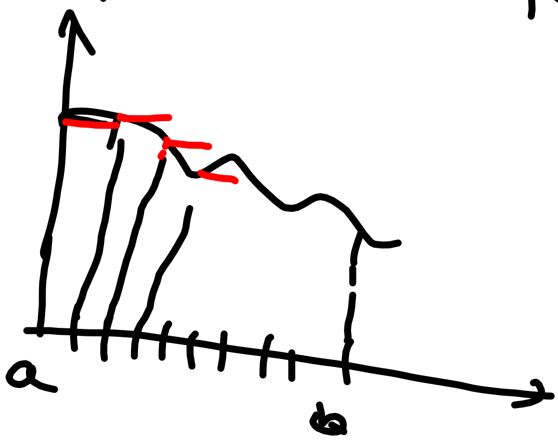
$$p_n(x_i^n) = f(x_i^n) \text{ for all } x_i^n$$

- Integrate p_n and return $I(p_n)$ as I_n approx

Rectangle rule (order 0)

Idea: approximate f with piecewise constant fct.

$$\int_a^b f(x)\,dx$$



$x_0 = a$

$x_1 = a + \frac{b-a}{n}$

$x_2 = a + 2\frac{b-a}{n}$

- -

$x_n = a + n\frac{b-a}{n} = b$

$h = \frac{b-a}{n}$

$$I_n(f) = \sum_{i=1}^{n} h f(x_{i-1}^{n})$$

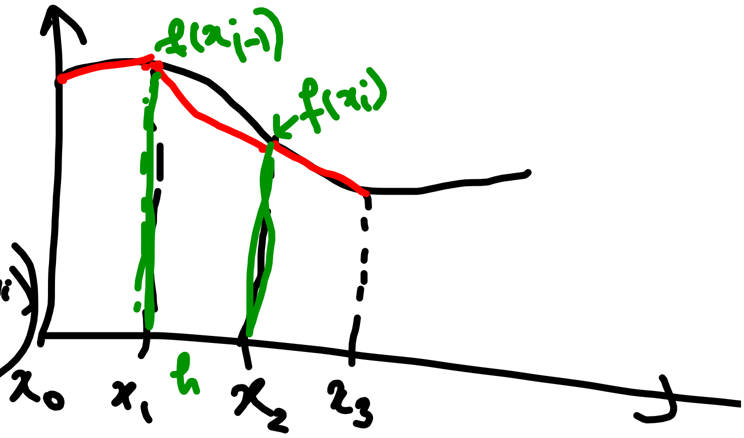
1) If $a=\infty$ or $b=\infty$ we need a modification

2) there is nothing special about left hand point

<u>Midpoint rule</u>

$$I_n(f) = \sum_{i=1}^{n} h\, f\left(\frac{x_i+x_{i-1}}{2}\right)$$

Trapezoid rule



$$I_n(f) = \sum_{i=1}^{n} h \cdot \left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)$$

$$= h\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right)$$

Simpson's Rule

Use a quadratic polynomial

Construct a 2^{nd} order polynomial so that

$f(x_{i-1}) = P(x_{i-1})$

$f(x_i) = P(x_i)$

$f(x_{i+1}) = P(x_{i+1})$

Lagrange interpolating polynomials.

given $(x_1, y_1)\ (x_2, y_2)\ \ldots\ (x_k, y_k)$

$x_1 \neq x_2 \ldots \neq x_k$

$$l_j(x) = \prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{x - x_i}{x_j - x_i} = \frac{x - x_1}{x_j - x_1} \cdots \frac{x - x_{j-1}}{x_j - x_{j-1}} \cdot \frac{x - x_{j+1}}{x_j - x_{j+1}} \cdots \frac{x - x_k}{x_j - x_k}$$

$l_j(x_i) = 0$ for all $i \neq j$

$l_j(x_j) = 1$

$$L(x) = \sum_{j=1}^{k} y_j\, l_j(x) = y_1 l_1(x) + \cdots + y_k l_k(x)$$

Our problem

$$P(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} \frac{x - x_{i+1}}{x_{i-1} - x_{i+1}} + - \quad -$$

Next integrate $\int_{x_{i-1}}^{x_{i+1}} P(x)\,dx$

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx \simeq \frac{x_{i+1} - x_{i-1}}{6}\left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right)$$

Divide interval in $x_0, x_1 \quad - \quad .. x_n$

use $x_{i-1}, \frac{x_{i-1}+x_i}{2}, x_i$

$$I(f) = \frac{h}{6} \sum_{i=1}^{n} \left(f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i) \right)$$