Computational Methods in Finance

Please write a comprehensive report detailing your findings. Please attach any necessary supporting evidence as well as code to the report. For this exam you may use any programming language you may desire. Please submit all the source code used.

Problem 1 (25 points). Assume the risk neutral stock price follows a Geometric Brownian Motion

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Use one of the quadrature methods to calculate the no dividend European Call option with parameters $S_0 = 100, K = 100, \sigma = 0.2, r = 0.06, T = 1$. Specifically express and approximate the integral:

$$e^{-rT} E\left[(S_T - K)_+\right]$$

using the distribution of S_T. Compare your result with the value obtained using the Black-Sholes formula with the same parameters.

Problem 2 (10 points). Consider the Heston model:

$$dS_t = rS_t dt + \sqrt{Y_t} S_t dW_t \tag{1}$$

$$dY_t = \kappa(\bar{Y} - Y_t)dt + \nu\sqrt{Y_t} dZ_t \tag{2}$$

where $r = 0.01$, $\kappa = 2$, $\bar{Y} = 0.16$, $\nu = 1$ are parameters, $Y_0 = 0.17$, $S_0 = 40$. Please answer the following questions:

(a) What is the main reason why the Heston model can produce an implied volatility curve (smile)? Give parameter choices so that the Heston model reduces to the geometric Brownian motion.

(b) Apply expectations to the volatility process Y_t to obtain a deterministic equation in $y_t = \mathbf{E}[Y_t]$. Either solve this equation or approximate the solution and give a numerical answer to the value of $y_{\frac{1}{250}}$

Problem 3 (10 points). Here is the pseudo-code for the valuation of an European Call option using a multiplicative binomial tree. Please identify all the mistakes in the code below. Please indicate how to fix the mistakes.

```
Data: Parameters K, T, S, r, N, u, d
dt = T/N
p = (exp(r*dt)-d)/(u-d)
disc = exp(r*dt)
St[0] = S*d
for j = 1 to N do
 | St[j] = St[j-1]*u/d
end
for j = 0 to N do
 | C[j] = max (K-St[j], 0.0)
end
for i = (N-1) down to 0 do
 | for j = 0 to i do
 |  | C[j] = disc * ( (1-p) * C[j] + p * C[j+1])
 | end
end
European_call = C[0]
```

Problem 4 (40 points). Finite difference methods.

(a) Implement the Explicit Finite Difference method to price both European Call and Put options.

(b) Implement the Implicit Finite Difference method to price European Call and Put options.

(c) For both the Explicit and Implicit Finite Difference schemes estimate the numbers Δt, Δx as well as the total number N_j of points on the space grid x to obtain a desired error of $\varepsilon = 0.001$. *Hint.* You need to do this part in a theoretical way. Please use the convergence order as the actual error of the estimate.

(d) Consider $S_0 = 100, K = 100,\ T = 1$ year, $\sigma = 25\%, r = 6\%, \delta = 0.03$. Calculate and report the price for European Call and Put using both explicit and implicit FD methods and the number of steps that you calculated in the previous point (part c).

(e) Repeat part (c) of this problem but this time get the empirical number of iterations. Specifically, obtain the Black Scholes price for the data in (d), then do an iterative procedure to figure out the Δx, Δt, N, and N_j to obtain an accuracy of $\varepsilon = 0.001$.

(f) Using the parameters from part (d), plot on the same graph the implicit finite difference probabilities p_u, p_m, p_d as a function of σ, $\sigma \in \{0.05, 0.1, 0.15, ..., 0.6\}$. Write detailed comments on your observations.

(g) Implement the Crank-Nicolson Finite Difference method and price both European Call and Put options. Use the same parameters as in part (d) and the same number of steps in the grid. Put the results of the 3 methods (EFD, IFD, CNFD) side by side in a table and write your observations.

(h) Calculate the hedge sensitivities for the European call option using the Explicit Finite Difference method. You need to calculate Delta, Gamma, Theta, and Vega.

Problem 5 (15 points). We know that an option price under a certain stochastic model satisfies the following PDE:

$$\frac{\partial V}{\partial t} + 2\cos(S)\frac{\partial V}{\partial S} + 0.2S^{\frac{3}{2}}\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

Assume you have an equidistant grid with points of the form $(i, j) = (i\Delta t, j\Delta x)$, where $i \in \{1, 2, \ldots, N\}$ and $j \in \{-N_S, N_S\}$. Let $V_{i,j} = (i\Delta t, j\Delta x)$.

1. Discretize the derivatives and give the finite difference equation for an Explicit scheme. Use the notation introduced above.
2. What do you observe about the updating coefficients?
3. Can you use an SOR scheme instead of the Explicit equation in (1)? Explain? (yes/no without an explanation will earn 0).