Chapter 4

Analytical Solutions for the PDE's arising in Mathematical Finance

4.1 Introduction

Many problems in Mathematical Finance result in PDE's that need to be solved. The best way to solve these equations is to find the exact solution using analytical methods. This is however a hard task and only the simplest PDE's have exact analytical formulas for the solution. In this chapter, we talk about some PDE's that provide exact solution. We will also present transformation methods that may produce exact analytical solutions. When applied in practice these are in fact approximations as well but they are theoretical ones as opposed to the numerical approximations presented in later chapters.

We begin the chapter with a discussion of the 3 main types of PDE's and some important properties.

4.2 Useful definitions and types of PDE's

Partial differential equations (PDE's) describe the relationships among the derivatives of an unknown function with respect to different independent variables, such as time and position. PDE's are used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow and several others.

4.2.1 Types of PDE's (2-D)

A two dimensional linear PDE is of the form:

$$a\frac{\partial^2 U}{\partial t^2} + b\frac{\partial^2 U}{\partial x \partial t} + c\frac{\partial^2 U}{\partial x^2} + d\frac{\partial U}{\partial t} + e\frac{\partial U}{\partial x} + fU = g \tag{4.1}$$

where a, b, c, d, e, f and g are constants. The problem is to find the function U.

(4.1) can be rewritten as a polynomial in the form

$$P(\alpha, \beta) = a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + j. \tag{4.2}$$

The nature of the PDE (4.2) is determined by the properties of the polynomial $P(\alpha, \beta)$. Looking at the values of the discriminant i.e $\Delta = b^2 - 4ac$ of (4.2), we can classify the PDE (4.1) as follows:

1. If $\Delta < 0$, we have an elliptic equation. An example is the Laplace equations which is of the form

   $$\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2 U}{\partial x^2} = 0.$$

   The solutions of Laplace's equation are the harmonic functions, which have wide applications in electromagnetism, astronomy, and fluid dynamics, since they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials.

2. If $\Delta = 0$, we have the parabolic equation. An example is the diffusion (heat) equation i.e.

   $$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0.$$

   The diffusion equation arises in the modeling of a number of phenomena that describes the transport of any quantity that diffuses spreads out as a consequence of the spatial gradients in its concentration and is often used in financial mathematics in the modeling of options. The heat equation was used by Black and Scholes to model the behavior of the stock market.

3. If $\Delta > 0$, we have the hyperbolic equation. An example includes the wave equation i.e.

   $$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0.$$

   This type of equation appears often in wave propagation (acoustic, geophysical etc.)

Please note that the nature of the PDE is classified based on the coefficients a, b, c, d, e, f and g being constant. When dealing with variable coefficients (that is, when the coefficients depend on the state variables x and t), the character of the PDE may switch between these three basic types.

4.2.2 Boundary conditions (BC) for PDE's

PDE's are solved on a domain in $\mathbb{R}^2$. This domain could be bounded or unbounded. An example is $x, t \in [0, t] \times [0, \infty)$. The PDE's specified without boundary conditions have typically an infinite number of solutions. Once the behavior of the solution is specified on the boundaries the solution becomes unique.

There are two types of boundary conditions normally used in practice: the Dirichelet and the Newman type BC. The Dirichelet type conditions specify the actual value of the function on the boundary e.g., $U(T, x) = 4$. The Newman type BC specify the partial derivatives of the function on the boundaries e.g., $\frac{\partial U}{\partial t}(T, x) = 2x$.

There are certainly other type of boundary conditions, for instance the "mixed boundary condition" which is a weighted sum of the Dirichelet and Newman type BC. Nevertheless, the basic ones are the two mentioned.

In the next section, we introduce some notations that will be used throughout the chapter and the rest of the book. We also give a brief account of few functional spaces that are commonly used when studying the partial differential equations.

4.3 Functional Spaces useful for PDE's

For $d \geq 1$, let $\Omega \subset \mathbb{R}^d$ be an open set. $Q_T = \Omega \times (0, T)$ is called a parabolic domain for any $T > 0$. If the set Ω has a boundary, denoted $\partial\Omega$, then $\partial\Omega$ will be smooth. We shall use $x = (x_1, x_2, \ldots, x_d) \in \Omega$ and $t \in (0, T)$.

We denote with $C^{i,j}(Q_T)$ the space of all functions which are derivable i times in the first variable and j times in the second variable and both partial derivatives are continuous. We denote with $C_0^{i,j}(Q_T)$ the set of functions as above with additional property that they have compact support (i.e., they are zero outside a compact set in Q_T). The set of functions $C_0^{\infty}(Q_T)$ are infinitely derivable on the domain. These functions are also known as test functions.

The space $L^1_{loc}(Q_T)$ is the set of all functions that are locally integrable (i.e. integrable on every compact subset of its domain of definition Q_T).

A multi-index of nonnegative integers will be denoted by $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We will let k be a nonnegative integer and δ be a positive constant with $0 < \delta < 1$. Unless otherwise indicated, we will always use the standard Lebesgue integrals.

Definition 4.3.1. Let $u, v \in L^1_{loc}(Q_T)$. For a nonnegative integer ρ, we say v is the $\alpha\rho^{th}$ weak partial derivative of u of order $|\alpha| + \rho$, $D^{\alpha}\partial_t^{\rho} u = v$, provided that

$$\iint_{Q_T} u\, D^{\alpha}\partial_t^{\rho}\phi \;\; dx\, dt = (-1)^{|\alpha|+\rho} \iint_{Q_T} v\,\phi \;\; dx\, dt,$$

for any test function $\phi \in C_0^{\infty}(Q_T)$.

It can be shown that weak derivatives are unique up to a set of zero measure. In order to proceed we present several important definitions.

Definition 4.3.2. A function $u \in L^1_{loc}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f \partial_i \phi dx = -\int_\Omega g_i \phi dx \quad \text{for all} \quad \phi \in C_c^\infty(\Omega). \tag{4.3}$$

Definition 4.3.3. Let $1 \le p \le \infty$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We define the following Sobolev spaces

$$W_p^k(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), 1 \le |\alpha| \le k\}, \tag{4.4}$$

$$W_p^{2k,k}(Q_T) := \{u \in L^p(Q_T) \mid D^\alpha \partial_t^\rho u \in L^p(Q_T), 1 \le |\alpha| + 2\rho \le 2k\}. \tag{4.5}$$

The spaces above become Banach spaces if we endow them with the respective norms

$$|u|_{W_p^k(\Omega)} = \sum_{0 \le |\alpha| \le k} |D^\alpha u|_{L^p(\Omega)}, \tag{4.6}$$

$$|u|_{W_p^{2k,k}(Q_T)} = \sum_{0 \le |\alpha| + 2\rho \le 2k} |D^\alpha \partial_t^\rho u|_{L^p(Q_T)}. \tag{4.7}$$

For more on the theory of Sobolev spaces, we refer the reader to [2].

Next, we discuss spaces with classical derivatives, known as Hölder spaces. We will follow the notation and definitions given in the books [128] and [203]. We define $C^k_{loc}(\Omega)$ to be the set of all real-valued functions $u = u(x)$ with continuous classical derivatives $D^\alpha u$ in Ω, where $0 \le |\alpha| \le k$. Next, we set

$$|u|_{0;\Omega} = [u]_{0;\Omega} = \sup_\Omega |u|,$$
$$[u]_{k;\Omega} = \max_{|\alpha|=k} |D^\alpha u|_{0;\Omega}.$$

Definition 4.3.4. The space $C^k(\Omega)$ is the set of all functions $u \in C^k_{loc}(\Omega)$ such that the norm

$$|u|_{k;\Omega} = \sum_{j=0}^k [u]_{j;\Omega}$$

is finite. With this norm, it can be shown that $C^k(\Omega)$ is a Banach space.

If the seminorm

$$[u]_{\delta;\Omega} = \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|u(x) - u(y)|}{|x-y|^\delta}$$

is finite, then we say the real-valued function u is *Hölder continuous in Ω with exponent δ*. For a k-times differentiable function, we will set

$$[u]_{k+\delta;\Omega} = \max_{|\alpha|=k} [D^\alpha u]_{\delta;\Omega}.$$

Definition 4.3.5. The Hölder space $C^{k+\delta}(\overline{\Omega})$ is the set of all functions $u \in C^k(\Omega)$ such that the norm

$$|u|_{k+\delta;\Omega} = |u|_{k;\Omega} + [u]_{k+\delta;\Omega}$$

is finite. With this norm, it can be shown that $C^{k+\delta}(\overline{\Omega})$ is a Banach space.

For any two points $P_1 = (x_1, t_1)$, $P_2 = (x_2, y_2) \in Q_T$, we define the parabolic distance between them as

$$d(P_1, P_2) = \left(|x_1 - x_2|^2 + |t_1 - t_2|\right)^{1/2}.$$

For a real-valued function $u = u(x,t)$ on Q_T, let us define the semi-norm

$$[u]_{\delta,\delta/2;Q_T} = \sup_{\substack{P_1,P_2\in Q_T \\ P_1\neq P_2}} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{d^\delta(P_1,P_2)}.$$

If this semi-norm is finite for some u, then we say u is *Hölder continuous with exponent δ*. The maximum norm of u is given by

$$|u|_{0;Q_T} = \sup_{(x,t)\in Q_T} |u(x,t)|.$$

Definition 4.3.6. The space $C^{\delta,\delta/2}\left(\overline{Q}_T\right)$ is the set of all functions $u \in Q_T$ such that the norm

$$|u|_{\delta,\delta/2;Q_T} = |u|_{0;Q_T} + [u]_{\delta,\delta/2;Q_T}$$

is finite. Furthermore, we define

$$C^{2k+\delta,k+\delta/2}\left(\overline{Q}_T\right) = \{u : D^\alpha \partial_t^\rho u \in C^{\delta,\delta/2}\left(\overline{Q}_T\right), 0 \le |\alpha| + 2\rho \le 2k\}.$$

We define a semi-norm on $C^{2k+\delta,k+\delta/2}\left(\overline{Q}_T\right)$ by

$$[u]_{2k+\delta,k+\delta/2;Q_T} = \sum_{|\alpha|+2\rho=2k} [D^\alpha \partial_t^\rho u]_{\delta,\delta/2;Q_T},$$

and a norm by

$$|u|_{2k+\delta,k+\delta/2;Q_T} = \sum_{0\le|\alpha|+2\rho\le 2k} |D^\alpha \partial_t^\rho u|_{\delta,\delta/2;Q_T}.$$

Using this norm, it can be shown that $C^{2k+\delta,k+\delta/2}\left(\overline{Q}_T\right)$ is a Banach space.

Next we discuss a classical solution method to PDEs using the method of separation of variables.

4.4 Separation of variables

Solutions to many partial differential equations (PDEs) can be obtained using the technique known as separation of variables. This solution technique is based on the fact that if $f(x)$ and $g(t)$ are functions of independent variables x and t respectively and if $f(x) = g(t)$ then there must be a constant λ for which $f(x) = \lambda$ and $g(t) = \lambda$. Applying the method of separation of variables enables one to obtain two ordinary differential equations. To illustrate the method of separation of variables for an initial-boundary value problem, we consider the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{4.8}$$

$$u(0,t) = u(L,t) = 0 \quad \text{for all} \quad t$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

where L is a constant and $f(x)$, $g(x)$ are given functions. In order to solve (4.8) using the method of separation of variables, we assume a solution of the form:

$$u(x,t) = f(x)g(t) \tag{4.9}$$

By inserting (4.9) into the wave equation, we obtain:

$$f(x)g''(t) = f(x)''g(t) \tag{4.10}$$

and dividing by the product $f(x)g(t)$ (4.10) becomes:

$$\frac{g''(t)}{g(t)} = \frac{f(x)''}{f(x)} \tag{4.11}$$

Since the left hand side of (4.11) is a function of x alone and the right hand side is a function of t alone, both expressions must be independent of x and t. Therefore,

$$\frac{g''(t)}{g(t)} = \frac{f(x)''}{f(x)} = \lambda. \tag{4.12}$$

for a suitable $\lambda \in \mathbb{R}$. Thus, we obtain

$$g''(t) - \lambda g(t) = 0, \tag{4.13}$$

$$f''(x) - \lambda f(x) = 0. \tag{4.14}$$

We allow λ to take any value and then show that only certain values allow you to satisfy the boundary conditions. This gives us three distinct types of solutions that are restricted by the initial and boundary conditions.

Case 1: $\lambda = 0$
In this case, equations (4.13) and (4.14) become:

$$g'' = 0 \quad \Longrightarrow \quad g(t) = At + B,$$

and

$$f'' = 0 \quad \Longrightarrow \quad f(x) = Cx + D,$$

for constants A, B, C and D. However, the boundary conditions in (4.8) imply that:

$$C = 0 \quad \text{and} \quad D = 0.$$

Thus, the only solution, with $\lambda = 0$, it is trivial solution.
Case 2: $\lambda = p^2$
When λ is positive, the separable equations are

$$g'' - p^2 g = 0, \tag{4.15}$$

and

$$f'' - p^2 f = 0. \tag{4.16}$$

Equations (4.15) and (4.16) have constant coefficients and so can be solved. The solution to (4.16), is

$$f(x) = Ae^{px} + Be^{-px}. \tag{4.17}$$

To satisfy the boundary conditions in x we find that the only solution is the trivial solution with

$$A = 0 \quad \text{and} \quad B = 0.$$

Case 3: $\lambda = -p^2$
The case when the separation constant, λ, is negative is the interesting case and generates a non-trivial solution. The two equations are now

$$g'' + p^2 g = 0, \tag{4.18}$$

and

$$f'' + p^2 f = 0. \tag{4.19}$$

The solution to (4.19) is

$$f(x) = A\cos px + B\sin px. \tag{4.20}$$

The boundary condition at $x = 0$ implies that

$$u(0, t) = f(0)g(t) = 0 \quad \Longrightarrow \quad f(0) = 0, \tag{4.21}$$

and using this in (4.21) gives

$$A = 0.$$

The condition at $x = L$ gives

$$f(L) = B\sin pL = 0. \tag{4.22}$$

From (4.22), if $B = 0$ we have the trivial solution, but there is a non-trivial solution if

$$\sin pL = 0 \implies pL = n\pi \implies p = \frac{n\pi}{L}, \tag{4.23}$$

where n is an integer. The solution for (4.21) follows in a similar manner and we obtain:

$$g(t) = C\cos pt + D\sin pt, \tag{4.24}$$

where $p = \frac{n\pi}{L}$. Thus, a solution for $u(x,t)$ is

$$u(x,t) = \sin\frac{n\pi}{L}x\left(C\cos\frac{n\pi}{L}t + D\sin\frac{n\pi}{L}t\right). \tag{4.25}$$

where C and D are constants. The solution of (4.25) satisfy the initial conditions:

$$u(x,0) = C\sin\frac{n\pi}{L}x$$

and

$$\frac{\partial u}{\partial t}(x,0) = D\frac{n\pi}{L}\sin\frac{n\pi}{L}x.$$

In order to obtain the general solution, we consider the superposition of solutions of the form (4.25) and obtain:

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left(C_n\cos\frac{n\pi}{L}t + D_n\sin\frac{n\pi}{L}t\right). \tag{4.26}$$

with initial conditions:

$$u(x,0) = f(x), \text{ for } 0 \le x \le L$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x), \text{ for } 0 \le x \le L.$$

The first initial condition

$$u(x,0) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi}{L}x = f(x),$$

is satisfied by choosing the coefficients of C_n to be

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x dx,$$

the Fourier sine coefficients for $f(x)$. The second initial condition

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} D_n\frac{n\pi}{L}\sin\frac{n\pi}{L}x = g(x),$$

is satisfied by choosing the coefficients of D_n to be

$$D_n = \frac{2}{n\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx,$$

the Fourier sine coefficient for $g(x)$. Inserting the values of C_n and D_n into (4.26) gives the complete solution to the wave equation.

Next, we discuss a transformation methodology that is very useful in finance. The Laplace transform is a very important tool in finance since it can produce quasi-analytical formulas delivering real time option pricing [95].

4.5 Moment generating functions (M.G.F) (Laplace Transform)

The kth moment of a distribution is the expected value of the kth power of the random variable. Generally, moments are hard to compute. Sometimes this calculation may be simplified using a moment generating function.

Definition 4.5.1 (Moment Generating Function)**.** The moment generating function of a random variable X with probability density function $f(x)$ is a function $M : \mathbb{R} \to [0, \infty)$ given by

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Definition 4.5.2 (Laplace transform)**.** If f is a given positive function the Laplace transform of f is:

$$\mathcal{L}f(t) = \int_{-\infty}^{\infty} e^{-tx} f(x) dx$$

for all t for which the integral exists and is finite.

We note that the two definitions are very much related. In fact we can write: $M_X(t) = \int_{-\infty}^{\infty} e^{tX} f(x) dx = \mathcal{L}f_-(t) 1_{[0,\infty)}(t) + \mathcal{L}f_+(t) 1_{[-\infty,0]}(t)$ where

$$f_-(t) = \begin{cases} f(-x), & x > 0 \\ 0, & \text{Otherwise.} \end{cases}$$

$$f_+(t) = \begin{cases} f(x), & x > 0 \\ 0, & \text{Otherwise.} \end{cases}$$

Therefore is we know the Laplace transform of the density function we may find the M.G.F. of the variable with that density.

We can calculate moments of random variables using M.G.F as follows:

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k,$$

using the exponential function expansion. Then the moments are

$$E(X^k) = M_X^{(k)}(0)$$

i.e., the k-th derivative of the M.G.F. calculated at 0.

We remark that if two random variable X and Y are independent then we can write:

$$M_{X+Y}(t) = M_X(t) + M_Y(t).$$

This simplifies greatly calculating M.G.F. for sums (and averages) of independent random variables.

Some properties of the moment generating functions are presented below:

1. If X and Y have the same M.G.F they have the same distribution.
2. If $X_1, \dots, X_n$, is a sequence of random variable with M.G.F $M_{X_1}(t), \dots, M_{X_n}(t)$, and
$$M_{X_n}(t) \to M(t)$$
for all t then $X_n \to X$ in distribution and M.G.F of X is $M(t)$.

For more properties please consult [72].

Theorem 4.5.1 (Inversion theorem)**.** *If f has a Laplace transform $\mathcal{L}f(t)$:*

$$\mathcal{L}f(t) = \int_0^t e^{-tx} f(x) dx$$

then we must have:

$$f(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} e^{tx} \mathcal{L}f(t) dt$$

where C is any constant greater than the real part of all irregularities of $\mathcal{L}f(t)$.

4.5.1 Numeric Inversion for Laplace Transform

In general, finding the inverse Laplace transform theoretically is complicated unless we are dealing with simple functions.

Numerically, it seems like a simple problem. We can potentially approximate the integral using standard quadrature methods. Specifically:

$$\mathcal{L}f(t) = \sum_{i=1}^{n} w_i e^{-t\zeta_i f(\zeta_i)}$$

where ζ_i are quadrature nodes and w_i are weights.

The idea is to use $t_1, \dots, t_n$ and write a system of n equations:

$$\mathcal{L}f(t_j) = \sum_{i=1}^{n} w_i e^{-t_j \zeta_i f(\zeta_i)}. \tag{4.27}$$

However, the system (4.27) thus formed is ill conditioned (unstable) in $\mathbb{R}$. Specifically, small changes in values of $\mathcal{L}f(t_j)$ lead to very different solutions. The main reason for this behavior is taht the exponential term in (4.27) dominates the values of the function. Thus, this method generally fails.

The proper way is to use complex integrals. By the inversion theorem we have to calculate:

$$f(x) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{tx}\mathcal{L}f(t)dt. \tag{4.28}$$

In this instance, we take a contour and perform a complex integration. We set $t = C + iu$ which implies that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{Cx}e^{iux}\mathcal{L}f(C+iu)du \tag{4.29}$$

where $e^{iux} = \cos(ux) + i\sin(ux)$.

Transforming (4.29) to $[0,\infty)$, we get:

$$f(x) = \frac{2e^{Cx}}{\pi}\int_{0}^{\infty} Re(\mathcal{L}f(C+iu))\cos(ux)dx \tag{4.30}$$

and

$$f(x) = -\frac{2e^{Cx}}{\pi}\int_{0}^{\infty} Im(\mathcal{L}f(C+iu))\sin(ux)dx \tag{4.31}$$

The next step is to approximate these integrals.

4.5.2 Fourier series approximation method

The main idea of the Fourier series approximation method is to apply the trapezoidal rule (see chapter 1 section 1.11) to equation (4.30) and obtain:

$$f(x) \simeq \frac{\Delta e^{Cx}}{\pi}Re(\mathcal{L}f(C)) + \frac{2\Delta e^{Cx}}{\pi}\sum_{k=1}^{\infty} Re(\mathcal{L}f(C+ik\Delta))\cos(k\Delta x) \tag{4.32}$$

where Δ is the step in the variable $u(u = k\Delta)$. If we set $\Delta = \frac{\pi}{2x}$ and $C = \frac{A}{2x}$ (for some constant A), then we have that $\frac{\Delta}{\pi} = \frac{1}{2x}$ and $Cx = \frac{A}{2}$. This implies that $k\Delta x = k\frac{\pi}{2}$, and therefore $\cos(k\Delta x) = 0$ if k is odd. The equation (4.32) becomes:

$$f(x) \simeq \frac{\Delta e^{A/2}}{2x}Re(\mathcal{L}f(\frac{A}{2x})) + \frac{e^{A/2}}{x}\sum_{k=1}^{\infty}(-1)^k Re\left(\mathcal{L}f\left(\frac{A+2k\pi i}{2x}\right)\right) \tag{4.33}$$

Generally, if f is a probability density function, $C = 0$. We can show that the discretization error if $|f(x)| < M$ is bounded by:

$$|f(x) - f_\Delta(x)| < M\frac{e^{-A}}{1-e^{-A}} \simeq Me^{-A}$$

therefore A should be large. In practice, A is chosen to be approximately 18.4. Next we briefly describe a methodology for calculating the infinite series in equation (4.33). The idea is to apply the so called Euler's algorithm. The idea of the algorithm to use explicitly the first n terms and then a weighted average of next m terms as follows:

$$f_\Delta(x) \simeq E(x,n,m) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} \Delta_{n+k}(x)$$

where

$$\Delta_n(x) = \frac{\Delta e^{A/2}}{2x} Re\left(\mathcal{L}f\left(\frac{A}{2x}\right)\right) + \frac{e^{A/2}}{x}\sum_{k=1}^{\infty}(-1)^k Re\left(\mathcal{L}f\left(\frac{A+2k\pi i}{2x}\right)\right).$$

In order for this to be effective, the term

$$a_k = Re\left(\mathcal{L}f\left(\frac{A+2k\pi i}{2x}\right)\right)$$

needs to satisfy the following 3 properties:

1. Has to be the same sign for all k
2. Has to be monotone in k
3. The higher order differences $(-1)^m \Delta^m a_{n+k}$ are monotone.

However, during practical implementation, we rarely check these properties. Usually, $E(x,n,m)$ is within 10^{-13} of the integral for $n = 38$ and $m = 11$. The test algorithm for series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges to $-\ln 2$. Please see [1] for more details.

Next we present some properties of the Laplace transform methodology. This transformation is well known for its applications to differential equations, especially ordinary differential equations. Laplace transforms can also be used to compute option prices.

We recall that the Laplace transform function $\mathcal{L}f$ of a function $f(x)$ is defined by:

$$\mathcal{L}f(s) = \int_0^\infty f(x)e^{-sx}dx,$$

for all numbers s for which the integral converges.

For example if $f(x) = e^{3x}$, then its Laplace transform $\mathcal{L}f(s)$ is determined by:

$$\begin{aligned}\mathcal{L}f(s) &= \int_0^\infty e^{3x}e^{-sx}dx\\ &= \int_0^\infty e^{(3-s)x}dx.\end{aligned}$$

Clearly $\mathcal{L}f(s)$ is undefined for $s \leq 3$ and $\mathcal{L}f(s) = \frac{1}{s-3}$, when $s > 2$.

We list some properties of the Laplace transform.

1. Given a function $y(t)$ with Laplace transform $\mathcal{L}(y)$, the Laplace transform of dy/dt is given by:
$$\mathcal{L}(\frac{dy}{dt}) = s\mathcal{L}(y) - y(0).$$
2. Given functions f and g and a constant c,
$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g),$$
and
$$\mathcal{L}(cf) = c\mathcal{L}(f).$$

Next we define inverse Laplace transform. The notation for this inverse transformation is $\mathcal{L}^{-1}$, that is
$$\mathcal{L}^{-1}(F) = f,$$
if and only if
$$\mathcal{L}(f) = F.$$
The inverse Laplace transform is also a linear operator.

Example 4.5.1. Compute the inverse Laplace transform of
$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{4}{(s-1)(s+1)}\right].$$
By linearity we have
$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{4}{(s-1)(s+1)}\right].$$
But we observe $\mathcal{L}(e^t) = \frac{1}{s-1}$. Hence
$$\mathcal{L}^{-1}(\frac{1}{s-1}) = e^t.$$
By the partial fraction decomposition $\frac{4}{(s-1)(s+1)} = \frac{2}{s-1} - \frac{2}{s+1}$. Thus,
$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{4}{(s-1)(s+1)}\right] &= \mathcal{L}^{-1}\left[\frac{2}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{2}{s+1}\right]\\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]\\ &= 2e^t - 2e^{-t}.\end{aligned}$$
Hence
$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{4}{(s-1)(s+1)}\right] = e^t - (2e^t - 2e^{-t}) = -e^t + 2e^{-t}.$$

Example 4.5.2. Consider the initial value problem

$$\frac{dy}{dt} = y - 4e^{-t}, \quad y(0) = 1.$$

Taking the Laplace transform of both sides we obtain

$$\mathcal{L}(\frac{dy}{dt}) = \mathcal{L}(y - 4e^{-t}).$$

Therefore, using the properties of Laplace transform we obtain,

$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y) - 4\mathcal{L}(e^{-t}).$$

This implies (using $y(0) = 1$),

$$s\mathcal{L}(y) - 1 = \mathcal{L}(y) - 4\mathcal{L}(e^{-t}).$$

Now $\mathcal{L}(e^{at}) = \frac{1}{(s-a)}$. Therefore the previous equation becomes,

$$s\mathcal{L}(y) - 1 = \mathcal{L}(y) - \frac{4}{s+1}.$$

Hence

$$\mathcal{L}(y) = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}.$$

Thus

$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{4}{(s-1)(s+1)}\right].$$

This is solved in Example 4.5.1. We obtained

$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{4}{(s-1)(s+1)}\right] = e^t - (2e^t - 2e^{-t}) = -e^t + 2e^{-t}.$$

4.6 Application of the Laplace transform to the Black-Scholes PDE

We recall the forward time form of the Black-Scholes PDE,

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = rF$$

with the boundary condition $F(T, S) = \Phi(S)$ (this is $(S-K)_+$) for a Call option for example).

Define $\zeta = (T-t)\frac{\sigma^2}{2}$, $x = \ln S$ and f such that $F(t, S) = f(\zeta, x)$, then using this change of variables the Black-Scholes PDE becomes:

$$-\frac{\partial f}{\partial \zeta} + \left(\frac{r}{\sigma^2/2} - 1\right)\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} - \frac{r}{\sigma^2/2} f = 0 \tag{4.34}$$

where $f(0,x) = F(T, e^x) = \Phi(e^x)$ (which is $(e^x - K)_+$) for a Call option).

Applying Laplace transform to $f(\zeta, x)$ in ζ, we obtain:

$$\begin{aligned}
(\mathcal{L}f)(u,x) &= \int_0^\infty e^{-u\zeta} f(\zeta,x)d\zeta \\
(\mathcal{L}\frac{\partial f}{\partial \zeta}(\zeta,x))(u,x) &= u\mathcal{L}f(u,x) - f(0,x) \\
(\mathcal{L}\frac{\partial f}{\partial \zeta}(\zeta,x))(u,x) &= \int_0^\infty e^{-u\zeta}\frac{\partial f}{\partial \zeta}(\zeta,x))d\zeta = \frac{\partial}{\partial x}\int_0^\infty e^{-u\zeta} f(\zeta,x)d\zeta
\end{aligned}$$

since $\mathcal{L}(\frac{\partial f}{\partial x})(u,x) = \frac{\partial \mathcal{L}f}{\partial x}(u,x)$ and similarly, $\mathcal{L}(\frac{\partial^2 f}{\partial x^2})(u,x) = \frac{\partial^2 \mathcal{L}f}{\partial x^2}(u,x)$.

For simplicity of notations we use $\mathcal{L}f(u,x) = \hat{f}(u,x)$, and the equation(4.34) reduces to:

$$-\left(u\hat{f} - \Phi(e^x)\right) + \left(\frac{r}{\sigma^2/2} - 1\right)\frac{\partial \hat{f}}{\partial x} + \frac{\partial^2 \hat{f}}{\partial x^2} - \frac{r}{\sigma^2/2}\hat{f} = 0. \tag{4.35}$$

We observe that (4.35) is now a second order ordinary differential equation (ODE). Let denote $m = \frac{r}{\sigma^2/2}$ and for the Call option $\Phi(e^x) = (e^x - e^k)_+$ where $K = e^k$ then ODE (4.35) becomes:

$$\frac{\partial^2 \hat{f}}{\partial x^2} + (m-1)\frac{\partial \hat{f}}{\partial x} - (m+u)\hat{f} + (e^x - e^k)_+ = 0. \tag{4.36}$$

The boundary values for equation (4.36) are obtained by applying the Laplace transform as follows:

$$\begin{aligned}
&\hat{f}(u,x) \to \mathcal{L}(e^z - e^{mt}e^k) = \frac{e^x}{u} - \frac{e^k}{u+m} \quad \text{as } x \to \infty \\
&\hat{f}(u,x) \to \mathcal{L}(0) = 0 \quad \text{as } x \to \infty
\end{aligned}$$

In order to solve the ODE (4.36) for $\hat{f}$, we multiply through the equation with an integration factor $e^{-\alpha x}$ where $\alpha = \frac{1-m}{2}$ and specifically choose $\hat{g}$ such that $\hat{f}(u,x) = e^{\alpha x}\hat{g}(u,x)$. Thus the first derivative will cancel out and obtain:

$$\frac{\partial^2 \hat{g}}{\partial x^2} - (b+u)\hat{g} + e^{-\alpha x}(e^x - e^k)_+ = 0 \tag{4.37}$$

where $b = \alpha^2 + m = \frac{(m-1)^2}{4} + m$. We solve (4.37) on $x > k$ and on $x \le k$ to get:

$$\hat{g}(u,x) = \begin{cases} \frac{e^{-(\alpha-1)x}}{u} - \frac{e^{-\alpha x + k}}{u+m} + h_1(u,x)A_1 + h_2(u,x)A_2 \text{ if } x > k \\ h_1(u,x)B_1 + h_2(u,x)B_2 \text{ if } x < k \end{cases}$$

where $h_1(u,x) = e^{-\sqrt{b+ux}}$, $h_2(u,x) = e^{\sqrt{b+ux}}$ and A_1, A_2, B_1, B_2 are constants to be determined from the boundary conditions. We have that

$$\lim_{x\to\infty} e^{\alpha x} h_1(u,x) = \lim_{x\to\infty} e^{\alpha x-\sqrt{b+ux}} = 0 \text{ when } u > 0$$

and also

$$\lim_{x\to\infty} e^{\alpha x} h_2(u,x) = \infty \text{ when } u > 0.$$

We must have $A_2 = 0$, otherwise the solution will explode i.e. it will diverge. Similarly when $x < k$ the same reasoning for $x \to -\infty$ gives $B_1 = 0$. Therefore

$$\hat{g}(u,x) = \begin{cases} \frac{e^{-(\alpha-1)x}}{u} - \frac{e^{-\alpha x+k}}{u+m} + h_1(u,x)A_1 \text{ if } x > k \\ h_2(u,x)B_2 \text{ if } x < k \end{cases}$$

Using the relation

$$\lim_{\substack{x\to k\\ x>k}} \hat{f}(u,x) = \lim_{\substack{x\to k\\ x<k}} \hat{f}(u,x),$$

$$\lim_{\substack{x\to k\\ x>k}} \frac{\partial \hat{f}}{\partial x} = \lim_{\substack{x\to k\\ x<k}} \frac{\partial \hat{f}}{\partial x}$$

and after performing some algebra, we get:

$$A_1(u) = \frac{e^{(1-a+\sqrt{b+u})k}(u-(a-1+\sqrt{b+u})m)}{2u\sqrt{b+u}(u+m)}$$

$$B_2(u) = \frac{e^{(1-a-\sqrt{b+u})k}(u-(a-1-\sqrt{b+u})m)}{2u\sqrt{b+u}(u+m)}$$

Thus,

$$\hat{f}(u,x) = e^{\alpha x}\left(\frac{e^{-(\alpha-1)x}}{u} - \frac{e^{-\alpha x+k}}{u+m}\right)1_{z>k} + \frac{e^{\sqrt{b+u}(x-k)+(1-a)k}(u-(a-1+\sqrt{b+u}\ \mathrm{sgn}\ (x-k))m)}{2u\sqrt{b+u}(u+m)}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases},$$

and the indicator function $1_{z>k}$ is one if the condition is true and 0 otherwise.

Finally, in order to obtain the solution for the original PDE we need to numerically invert this exact form of the Laplace transform.

Finite difference methods and Monte Carlo methods can also be used to approximate a PDE. These methods will be discussed in chapters 7 and 9 respectively. In those two chapters, we will also discuss how to apply the methods to option pricing.

4.7 Problems

1. (a) Find the solution to the following PDE:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = 0$$

$$\frac{\partial u}{\partial t}(x, 0) = f(x)$$

where L is a constant and f is a given function.

(b) Find the solution when $f(x) = \sin x$.

2. Find the solution to the following PDE:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = u(\pi, t) = 0$$

$$u(x, 0) = 0$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x)$$

3. Find the solution to the Laplace equation in R^3:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

assuming that:

$$u(x, y, z) = f(x)g(y)h(z).$$

4. Solve the following initial value problem using Laplace transform:

(a)

$$\frac{dy}{dt} = -y + e^{-3t}, \quad y(0) = 2.$$

(b)

$$\frac{dy}{dt} + 11y = 3, \quad y(0) = -2.$$

(c)

$$\frac{dy}{dt} + 2y = 2e^{7t}, \quad y(0) = 0$$

5. Explain what is a linear second order differential equation, and a non-linear second order differential equation, give examples.

6. Solve the following boundary value problem:

$$y''(x) = f(x), \quad y(0) = 0, \quad y(1) = 0.$$

Hence solve $y''(x) = x^2$ subject to the same boundary conditions.