Stochastic Processes.

Stock process

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t \quad (\text{gBM}) \text{ under } Q$$

Option that is of European type

$$\text{Payoff} = \phi(S_T)$$

then ~~C(t,s)~~ $V(t,S)$ = value of option at time t when stock price is S

solves : $$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

$$V(T,S) = (S-K)_+$$

1. Elliptic equations (PDE's) $\quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial S^2} = 0$
2. Hyperbolic eq $\quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial S^2} = 0$ wave eq.
3. Parabolic PDE's : $\quad \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial S^2} = 0$ (Heat eq.)

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Implied volatility

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$C(S, t, K, r, \sigma) \longrightarrow$

S_t $\quad t = T - t \quad$ 0.01

annual numbers

$1 \text{ month} = \frac{1}{12}, \frac{30}{365}, \frac{25}{252}$

$C_{Bid} \quad C_{Ask}$

$$\frac{C_{Bid} + C_{Ask}}{2} = C(S, t, K, r, \sigma) = f(\sigma)$$

Find σ that is implied by the option price $\frac{C_{Ask} + C_{Bid}}{2}$

implied volatility

$f : \mathbb{R} \to \mathbb{R}$

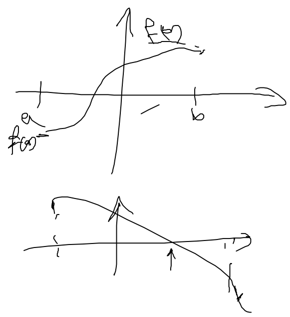
$f(x)$ find x s.t. $f(x) = y$

Bisection Method

$f(x) = B$ what is x

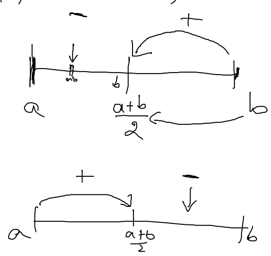
$BS(\sigma) = \frac{C_A + C_B}{2}$

A continuous function with a root in $[a,b]$ changes sign!



$f(a) \cdot f(b)$ is > 0 if they have same sign

$f(a)$. $f(\frac{a+b}{2})$ $f(b)$



start with a, b, $f(x)$

(*) if $\left(f(a) \cdot f\left(\frac{a+b}{2}\right) < 0\right)$ then $b = \frac{a+b}{2}$

if $\left(f(b)\, f\left(\frac{a+b}{2}\right) < 0\right)$ then $a = \frac{a+b}{2}$

(check is $|b-a| < \varepsilon$ if yes → report root as $\frac{a+b}{2}$

if not go back to (*)

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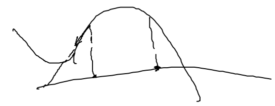
if $\left(f(a) \cdot f(b) > 0 \right)$ send message input wrong; BREAK;

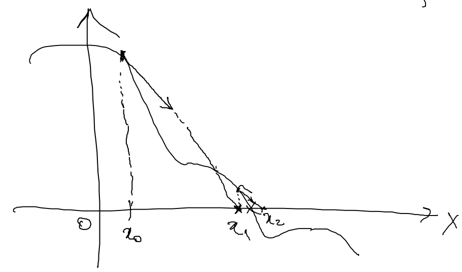
While $\left(|b-a| > \varepsilon \right)$ do

if $\left(f(a) f\left(\frac{a+b}{2}\right) < 0 \right)$ then $\left(b = \frac{a+b}{2} \right)$ else $\left(a = \frac{a+b}{2} \right)$

return $\frac{a+b}{2}$

Newton method

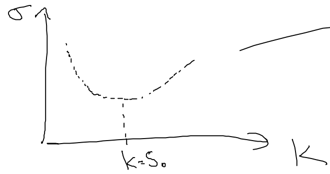


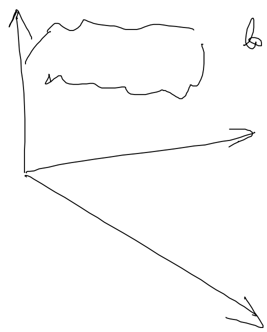


$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$dS_t = \sigma(S_t)\,S_t\,dW_t$$

f (option prices)





local volatility

$$\frac{\partial C}{\partial K}(K_0,T_0) \simeq \frac{C(K_1,T_0)-C(K_0,T_0)}{K_1-K_0}$$

$f : \mathbb{R} \to \mathbb{R}$

$f(x)$ $\qquad$ $f'(x)$ hard to calculate

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

approximate $\quad x_0 \quad x_1 \quad x_2$

$$f'(x_0) \simeq \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$f''(x_0) \simeq \frac{f'(x_1)-f'(x_0)}{x_1-x_0} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_1-x_0}$$

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