Financial Econometrics
Lecture 4: Analysis of Multiple Financial Time Series with Applications

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2019

Introduction

We shall focus on two series (i.e., the bivariate case) Time series:

$$\mathbf{X}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

Data: $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$.

Some examples:

(a) U.S. quarterly GDP and unemployment rate series.

(b) The daily closing prices of oil related ETFs, e.g. oil services holdings (OIH) and energy select section SPDR (XLE); and, for more than 2 series.

(c) Quarterly GDP grow rates of Canada, United Kingdom, and United States.

Introduction

Why consider two series jointly?

(a) Obtain the relationship between the series

(b) improve the accuracy of forecasts (use more information).

Some background:

Weak stationarity: Both

$$E(\mathbf{X}_t) = \begin{bmatrix} E(x_{1t}) \\ E(x_{2t}) \end{bmatrix} = \mu, \quad \textit{and}$$

$$\mathrm{Cov}(\mathbf{X}_t, \mathbf{X}_{t-j}) = \begin{bmatrix} \mathrm{Cov}(x_{1t}, x_{1,t-\ell}) & \mathrm{Cov}(x_{1t}, x_{2,t-\ell}) \\ \mathrm{Cov}(x_{2t}, x_{1,t-\ell}) & \mathrm{Cov}(x_{2t}, x_{2,t-\ell}) \end{bmatrix} = \mathbf{\Gamma}_j$$

are time invariant.

Auto-covariance matrix

Lag-ℓ

$$\begin{aligned}\mathbf{\Gamma}_\ell &= E[(\mathbf{X}_t - \mu)(\mathbf{X}_{t-\ell} - \mu)'] \\ &= \begin{bmatrix} E(x_{1t} - \mu_1)(x_{1,t-\ell} - \mu_1) & E(x_{1t} - \mu_1)(x_{2,t-\ell} - \mu_2) \\ E(x_{2t} - \mu_2)(x_{1,t-\ell} - \mu_1) & E(x_{2t} - \mu_2)(x_{2,t-\ell} - \mu_2) \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{11}(\ell) & \Gamma_{12}(\ell) \\ \Gamma_{21}(\ell) & \Gamma_{22}(\ell) \end{bmatrix}.\end{aligned}$$

Not symmetric if $\ell \neq 0$. Consider $\mathbf{\Gamma}_1$:

- $\Gamma_{12}(1) = \mathrm{Cov}(x_{1t}, x_{2,t-1})(x_{1t}$ depends on past $x_2t)$
- $\Gamma_{21}(1) = \mathrm{Cov}(x_{2t}, x_{1,t-1})(x_{2t}$ depends on past $x_1t)$

Let the diagonal matrix $\mathbf{D}$ be

$$\mathbf{D} = \begin{bmatrix} \mathrm{std}(x_{1t}) & 0 \\ 0 & \mathrm{std}(x_{2t}) \end{bmatrix} = \begin{bmatrix} \sqrt{\Gamma_{11}(0)} & 0 \\ 0 & \sqrt{\Gamma_{22}(0)} \end{bmatrix}.$$

Cross-Correlation matrix

$$\rho_\ell = \mathbf{D}^{-1}\Gamma_\ell\mathbf{D}^{-1}$$

Thus, $\rho_{ij}(\ell)$ is the cross-correlation between x_{it} and $x_{j,t-\ell}$.
From stationarity:

$$\mathbf{\Gamma}_\ell = \mathbf{\Gamma}'_{-\ell}, \qquad \rho_\ell = \rho'_{-\ell}.$$

For instance, $\mathrm{Cor}(x_{1t}, x_{2,t-1}) = \mathrm{Cor}(x_{2t}, x_{1,t+1})$.

Testing for serial dependence

Multivariate version of Ljung-Box $Q(m)$ statistics available.
$H_0 : \rho_1 = \cdots = \rho_m = \mathbf{0}$ vs. $H_a : \rho_i \neq \mathbf{0}$ for some i. The test statistic is

$$Q_2(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} tr(\hat{\mathbf{\Gamma}}'_\ell \hat{\mathbf{\Gamma}}_0^{-1} \hat{\mathbf{\Gamma}}_\ell \hat{\mathbf{\Gamma}}_0^{-1})$$

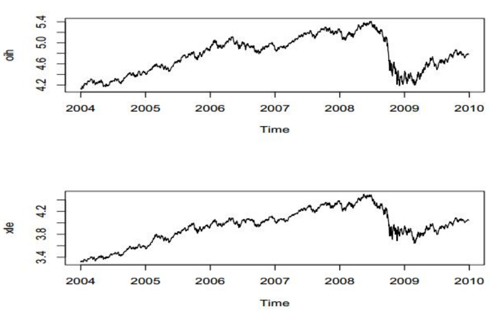

Figure: Daily log prices of OIH and XLE funds from January 2004 to December 2009

Vector Autoregressive Models(VAR)

VAR(1) model for two return series:

$$\begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} r_{1,t-1} \\ r_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix},$$

where $\mathbf{a}_t = (a_{1t}, a_{2t})'$ is a sequence of iid bivariate normal random vectors with mean zero and covariance matrix

$$\mathrm{Cov}(\mathbf{a}_t) = \sum = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

where $\sigma_{12} = \sigma_{21}$.
Rewrite the model as

$$r_{1t} = \phi_{10} + \phi_{11} r_{1,t-1} + \phi_{12} r_{2,t-1} + a_{1t}$$
$$r_{2t} = \phi_{20} + \phi_{21} r_{1,t-1} + \phi_{22} r_{2,t-1} + a_{1t}$$

Thus, ϕ_{11} and ϕ_{12} denotes the dependence of r_{1t} on the past returns $r_{1,t-1}$ and $r_{2,t-1}$, respectively.

Unidirectional dependence

For the VAR(1) model, if $\phi_{12} = 0$, but $\phi_{21} \neq 0$, then

- r_{1t} does not depend on $r_{2,t-1}$, but
- r_{2t} depends on $r_{1,t-1}$,

implying that knowing $r_{1,t-1}$ is helpful in predicting r_{2t}, but $r_{2,t-1}$ is not helpful in forecasting r_{1t}.

Here $\{r_{1t}\}$ is an *input*, $\{r_{2t}\}$ is the *output* variable. This is an example of **Granger** causality relation.

If $\sigma_{12} = 0$, then r_{1t} and r_{2t} are not concurrently correlated.

Unidirectional dependence

Stationarity condition: Generalization of 1-dimensional case Write the VAR(1) model as

$$\mathbf{r}_t = \phi_0 + \mathbf{\Phi}_{\mathbf{r}_{t-1}} + \mathbf{a}_t.$$

$\{\mathbf{r}_t\}$ is stationary if zeros of the polynomial $|\mathbf{I} - \mathbf{\Phi}_x|$ are greater than 1 in modulus. Equivalently, if solutions of $|\mathbf{I} - \mathbf{\Phi}_x| = 0$ are all greater than 1 in modulus.

Unidirectional dependence

Mean of $\mathbf{r}_t$ satisfies

$$(\mathbf{I} - \mathbf{\Phi})\mu = \phi_0, \quad \text{or}$$

$$\mu = (\mathbf{I} - \mathbf{\Phi})^{-1}\phi_0$$

if the inverse exists.
Covariance matrices of VAR(1) models:

$$\mathrm{Cov}(\mathbf{r}_t) = \sum_{i=0}^{\infty} \Phi^i \sum (\Phi^i)',$$

so that

$$\Gamma_\ell = \Phi\Gamma_{\ell-1}$$

for $\ell > 0$.
Can be generalized to higher order models.

Building VAR models

- Order selection: use AIC or BIC or a stepwise χ^2 test Eq. (8.18). See Section 8.2.4, pp 405-406.
  For instance, test VAR(1) vs VAR(2).
- Estimation: use ordinary least-squares method
- Model checking: similar to the univariate case
- Forecasting: similar to the univariate case

Simple AR models are sufficient to model asset returns.

Building VAR models

Program note: Commands for VAR modeling

- VARorder: compute various information criteria for a vector time series
- VAR: estimate a VAR model
- refVAR: refine an estimated VAR model by fixing insignificant estimates to zero
- MTSdiag: model checking
- VARpred: predict a fitted VAR model.

Co-integration

Basic ideas

- x_{1t} and x_{2t} are unit-root nonstationary
- a linear combination of x_{1t} and x_{2t} is unit-root stationary

That is, x_{1t} and x_{2t} share a single unit root!

Why is it of interest?

- Stationary series is *mean reverting*.
- Long term forecasts of the “linear” combination converge to a mean value, implying that the long-term forecasts of x_{1t} and x_{2t} must be linearly related.
- This mean-reverting property has many applications. For instance, pairs trading in finance.

Co-integration

Example: Consider the exchange-traded funds (ETF) of U.S. Real Estate. We focus on the iShares Dow Jones (IYR) and Vanguard REIT fund (VNQ) from October 2004 to May 2007. The daily adjusted prices of the two funds are shown in Figure 2.

- What can be said about the two prices? Is there any arbitrage opportunity between the two funds?
- The two series all have a unit root (based on ADF test). Are they co-integrated?

Figure: Daily prices of IYR and VNQ from October 2004 to May 2007

Co-integration test

Several tests available, e.g. Johansen's test (Johansen, 1988).

Basic idea

Consider a univariate AR(2) model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t.$$

Let $\Delta x_t = x_t - x_{t-1}$.
Subtract x_{t-1} from both sides and rearrange terms to obtain

$$\Delta x_t = \gamma x_{t-1} + \phi_1^* \Delta x_{t-1} + a_t,$$

Co-integration test

where $\phi_1^* = -\phi_2$ and $\gamma = \phi_2 + \phi_1 - 1$.
(Derivation involves simple algebra.)

- x_t is unit-root nonstationary if and only if $\gamma = 0$.
- Testing that x_t has a unit root is equivalent to testing that $\gamma = 0$. in the above model.
- The idea applies to general AR(p) models.
- Turn to the VAR(p) case. The original model is

$$\mathbf{X}_t = \Phi_1 \mathbf{X}_{t-1} + \cdots + \Phi_p \mathbf{X}_{t-p} + \mathbf{a}_t.$$

Co-integration test

Let $\mathbf{Y}_t = \mathbf{X}_t - \mathbf{X}_{t-1}$.
Subtracting $\mathbf{X}_{t-1}$ from both sides and re-grouping of the coefficient matrices, we can rewrite the model as

$$\mathbf{Y}_t = \mathbf{\Pi} \mathbf{X}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \mathbf{Y}_{t-i} + \mathbf{a}_t,$$

where

$$\begin{aligned}
\Phi_{p-1}^* &= -\Phi_p \\
\mathbf{\Phi}_{p-2}^* &= -\mathbf{\Phi}_{p-1} - \mathbf{\Phi}_p \\
\vdots &= \vdots \\
\mathbf{\Phi}_1^* &= -\mathbf{\Phi}_2 - \cdots - \mathbf{\Phi}_p \\
\mathbf{\Pi} &= \mathbf{\Phi}_p + \cdots + \mathbf{\Phi}_1 - \mathbf{I}.
\end{aligned}$$

This is the $Error - Correction$ Model (ECM).

Co-integration test

There are m linear combinations of $\mathbf{X}_t$ that are unit-root stationary. If Π has rank m, then

$$\mathbf{\Pi} = \alpha\beta$$

where α is a $k \times m$ and β is a $m \times k$ full-rank matrix.
$\mathbf{Z}_t = \beta \mathbf{X}_t$ is unit-root stationary.
β is the co-integrating vector.

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Co-integration test

Discussion

- ECM formulation is useful
- Co-integration tests have some weaknesses, e.g. robustness
- Co-integration overlooks the effect of scale of the series

Package: The package **urca** of **R** can be used to perform co-integration test.

Pairs trading

Reference: *Pairs Trading : Quantitative Methods and Analysis* by Ganapathy Vidyamurthy, Wiley, 2004.

Motivation: General idea of trading is to sell overvalued securities and buy undervalued ones. But the *true* value of the security is hard to determine in practice. Pairs trading attempts to resolve this difficulty by using *relative pricing*. Basically, if two securities have similar characteristics, then the prices of both securities must be more or less the same. Here the true price is not important.

Statistical term: The prices behave like random-walk processes, but a linear combination of them is stationary, hence, the linear combination is mean-reversting. Deviations from the mean lead to trading opportunities.

Pairs trading

Theory in Finance: Arbitrage Pricing Theory (APT): If two securities have exactly the same risk factor exposures, then the expected returns of the two securities for a given time period are the same. [The key here is that the returns must be the same for all times.]

More details: Consider two stocks: Stock 1 and Stock 2. Let pit be the log price of Stock i at time t. It is reasonable to assume that the time series $\{p_{1t}\}$ and $\{p_{2t}\}$ contain a unit root when they are analyzed individually.

Assume that the two log-price series are co-integrated, that is, there exists a linear combination $c_1p_{1t} - c_2p_{2t}$ that is stationary. Dividing the linear combination by c_1, we have

$$w_t = p_{1t} - \gamma p_{2t},$$

which is stationary.

Pairs trading

The stationarity implies that wt is mean-reverting. Now, form the portfolio Z by buying 1 share of Stock 1 and selling short on γ shares of Stock 2. The return of the portfolio for a given period h is

$$\begin{aligned} r(h) &= (p_{1,t+h} - p_{1,t}) - \gamma(p_{2,t+h} - p_{2,t}) \\ &= p_{1,t+h} - \gamma p_{2,t+h} - (p_{1,t} - \gamma p_{2,t}) \\ &= w_{t+h} - w_t \end{aligned}$$

which is the increment of the stationary series $\{w_t\}$ from t to $t+h$. Since w_t is stationary, we have obtained a direct link of the portfolio to a stationary time series whose forecasts we can predict.
Assume that $E(w_t) = \mu$. Select a threshold δ.

Pairs trading

A trading strategy:

- Buy Stock 1 and short γ shares of Stock 2 when the $w_t = \mu - \delta$.
- Unwind the position, i.e. sell Stock 1 and buy γ shares of Stock 2, when $w_{t+h} = \mu + \delta$.

Profit: $r(h) = w_{t+h} - w_t = 2\delta$.

Some practical considerations:

- The threshold δ is chosen so that the profit out-weights the costs of two trading. In high frequency, δ must be greater than *trading slippage*, which is the same linear combination of bid-ask spreads 11 of the two stock, i.e. bid-ask spread of Stock 1 + $\gamma\times$ (bid-askspread) of Stock 2.

Pairs trading

- Speed of mean-reverting of w_t plays an important role as h is directly related to the speed of mean-reverting.
- There are many ways available to search for co-integrating pairs of stocks. For example, via fundamentals, risk factors, etc.
- For unit-root and co-integration tests, see the textbook and references therein.

  Example: Consider the daily adjusted closing stock prices of BHP Billiton Limited of Australia and Vale S.A. of Brazil. These are two natural resources companies. Both stocks are also listed in the New York Stock Exchange with tick symbols BHP and Vale, respectively. The sample period is from July 1, 2002 to March 31, 2006.
- How to estimate γ?
- Speed of mean reverting? (zero-crossing concept)

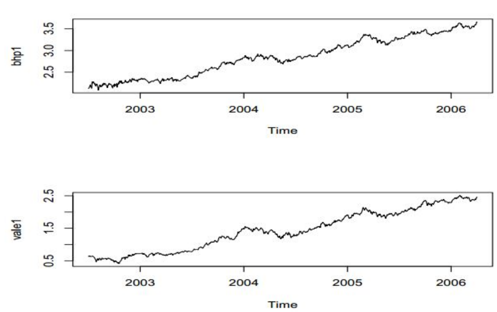

Figure: Daily log prices of BHP and VALE from July 1, 2002 to March 31, 2006.

Multivariate Volatility Models

How do the correlations between asset returns change over time?

Forcus on two series (Bivariate)
Two asset return series:

$$\mathbf{r}_t = \begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix}.$$

Data: $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_T$.

Basic concept
Let F_{t-1} denote the information available at time $t-1$.
Partition the return as

$$\mathbf{r}_t = \boldsymbol{\mu}_t + \boldsymbol{a_t}, \quad \boldsymbol{a_t} = \sum_t^{1/2} \boldsymbol{\epsilon}_t$$

Multivariate Volatility Models

where $\boldsymbol{\mu}_t = E(\mathbf{r}_t|F_{t-1})$ is the predictable component, and

$$\mathrm{Cov}(\boldsymbol{a_t}|F_{t-1}) = \sum_{\boldsymbol{t}} = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{bmatrix},$$

$\{\boldsymbol{\epsilon}_t\}$ are iid 2-dimensional random vectors with mean zero and identity covariance matrix.

Multivariate volatility modeling

See Chapter 10 of the textbook, Study time evolution of $\{\sum_{\boldsymbol{t}}\}$.

$\sum_{\boldsymbol{t}}$ is symmetric, i.e. $\sigma_{12,t} = \sigma_{21,t}$
There are 3 variables in $\sum_{\boldsymbol{t}}$.
For k asset returns, $\sum_{\boldsymbol{t}}$ has $k(k+1)/2$ variables.

Requirement
$\sum_{\boldsymbol{t}}$ must be positive definite for all t,

$$\sigma_{11,t} > 0, \quad \sigma_{22,t} > 0 \quad \sigma_{11,t}\sigma_{22,t} - \sigma_{12,t}^2 > 0.$$

The time-varying correlation between r_{1t} and r_{2t} is

$$\rho_{12,t} = \frac{\sigma_{12,t}}{\sqrt{\sigma_{11,t}\sigma_{22,t}}}$$

. **Some complications**

- Positiveness requirement is not easy to meet
- Too many series to consider

Some simple models available

- Exponentially weighted covariance
- Use univariate approach, e.g. $\mathrm{Cov}(X,Y) = \frac{\mathrm{Var}(x+y) - \mathrm{Var}(x-y)}{4}$.
- BEKK model
- Dynamic conditional correlation (DCC) models

Exponentially weighted model

$$\sum_{\boldsymbol{t}} = (1-\lambda)\boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1} + \lambda \sum_{\boldsymbol{t-1}},$$

where $0 < \lambda < 1$. That is,

$$\sum_{\boldsymbol{t}} = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1}\boldsymbol{a}_{t-i}\boldsymbol{a}'_{t-i}.$$

R command ¡EWMAvol¿ of the ¡MTS¿ package can be used.

BEKK model

BEKK model of Engle and Kroner (1995) Simple BEKK(1,1) model

$$\sum_{\boldsymbol{t}} = \boldsymbol{A}_0\boldsymbol{A}'_{\boldsymbol{0}} + \boldsymbol{A}_1(\boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1})\boldsymbol{A}'_{\boldsymbol{1}} + \boldsymbol{B}_1\sum_{\boldsymbol{t-1}}\boldsymbol{B}'_{\boldsymbol{1}}$$

where $\boldsymbol{A}_0$ is a lower triangular matrix, $\boldsymbol{A}_{\boldsymbol{1}}$ and $\boldsymbol{B}_1$ are square matrices without restrictions.
Pros: positive definite
Cons: Many parameters, dynamic relations require further study
Estimation: BEKK11 command in MTS package can be used for $k = 2$ and 3 only.

DCC mdoels

DCC mdoels: A two-step process

- Marginal models: Use univariate volatility model for individual return series
- Use DCC model for the time-evolution of conditional correlation Specifically, the volatility matrix can be written as

  $$\sum_{\boldsymbol{t}} = \mathbf{V}_t\mathbf{R}_t\mathbf{V}_t,$$

  where $\mathbf{V}_t$ is a diagonal matrix of volatilities for individual return series and $\mathbf{R}_t$ is the conditional correlation matrix. That is,

  $$\mathbf{V}_t = \text{diag}\{v_{1t}, v_{2t}, \ldots, v_{kt}\} \quad \mathbf{R}_t = [\rho_{ij,t}]$$

  where $\rho_{ij,t}$ is the correlation between ith and jth return series.

DCC mdoels

1. Engle (2002):

   $$\mathbf{Q}_t = (1-\theta_1-\theta_2)\mathbf{R}_0 + \theta_1\mathbf{Q}_{t-1} + \theta_2\mathbf{a}_{t-1}\mathbf{a}'_{t-1},$$

   $$\mathbf{R}_t = \mathbf{q}_t^{-1}\mathbf{Q}_t\mathbf{q}_t^{-1},$$

   where
   $0 \le \theta_i$ and $\theta_1 + \theta_2 < 1, \mathbf{q}_t = \text{diag}\{\sqrt{Q_{11,t}}, \sqrt{Q_{22,t}}, \ldots, \sqrt{Q_{kk,t}}\}$ and $\mathbf{R}_0$ is the sample correlation matrix.

2. Tse and Tsui (2002):

   $$\mathbf{R}_t = (1-\theta_1-\theta_2)\mathbf{R}_0 + \theta_1\mathbf{R}_{t-1} + \theta_2\psi_{t-1},$$

   where $0 \le \theta_i$ and $\theta_1 + \theta_2 < 1$, and ψ_{t-1} is the sample correlation matrix of $\{\mathbf{a}_{t-1}, \mathbf{a}_{t-2}, \ldots, \mathbf{a}_{t-m}\}$ for a pre-specified positive integer m, e.g. $m = 3$.

Discussion

1. DCC model is extremely simple with two parameters
2. On the other hand, model checking tends to reject the DCC models.

R commands of the MTS package for DCC modeling:

1. dccPre: fit individual GARCH models (standardized return series is included in the output)
2. dccFit: estimate a DCC model for the standardized return series
3. MCHdiag: model checking of multivariate volatility models.

If time permits, demonstration will be given in class.