Financial Econometrics
Lecture 5: More Volatility Models

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The GARCH-M model

$$r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where c is referred to as risk premium, which is expected to be positive.

Example: A GARCH(1,1)-M model for the monthly excess returns of S&P 500 index from January 1926 to December 1991. The fitted model is

$$r_t = 4.22\times 10^{-3} + 0.561\sigma_t^2 + a_t, \quad \sigma_t^2 = 0.814\times 10^{-5} + 0.122a_{t-1}^2 + .854\sigma_{t-1}^2.$$

Standard error of risk premium is 0.896 so that the estimate is not statistically significant at the usual 5% level.

The EGARCH model

- The idea (concept) of EGARCH model is useful. In practice, it is easier to use the TGARCH model.
- Asymmetry in responses to past positive and negative returns:

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)],$$

  with $E[g(\epsilon_t)] = 0$.
- To see asymmetry of $g(\epsilon_t)$, rewrite it as

$$g(\epsilon_t) = \begin{cases} (\theta+\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if} \quad \epsilon_t \geq 0, \\ (\theta-\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if} \quad \epsilon_t < 0 \end{cases}$$

The EGARCH model

An EGARCH(m, s) model:

$$a_t = \sigma_t\epsilon_t, \quad \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1})$$

Some features of EGARCH models:

- uses log trans. to relax the positiveness constraint
- asymmetric responses

Consider an EGARCH(1,1) model

$$a_t = \sigma_t\epsilon_t, \quad (1-\alpha B)\ln(\sigma_t^2) = (1-\alpha)\alpha_0 + g(\epsilon_{t-1}),$$

The EGARCH model

Under normality, $E(|\epsilon_t|) = \sqrt{2/\pi}$ and the model becomes

$$(1-\alpha B)\ln(\sigma_t^2) = \begin{cases} \alpha_* + (\theta+\gamma)\epsilon_{t-1} & \text{if} \quad \epsilon_{t-1} \geq 0, \\ \alpha_* + (\theta-\gamma)\epsilon_{t-1} & \text{if} \quad \epsilon_{t-1} < 0 \end{cases}$$

where $\alpha_* = (1-\alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma$.

This is a nonlinear fun. similar to that of the threshold AR model of Tong (1978, 1990).

Specifically, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha}\exp(\alpha_*)\begin{cases} \exp[(\theta+\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}} & \text{if} \quad a_{t-1} \geq 0, \\ \exp[(\theta-\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}} & \text{if} \quad a_{t-1} < 0 \end{cases}$$

The EGARCH model

The coefs $(\theta+\gamma)$ & $(\theta-\gamma)$ show the asymmetry in response to positive and negative a_{t-1}. The model is, therefore, nonlinear if $\theta \neq 0$. Thus, θ is referred to as the leverage parameter.

Focus on the function $g(\epsilon_{t-1})$. The leverage parameter θ shows the effect of the sign of a_{t-1} whereas γ denotes the magnitude effect.

Another example: Monthly log returns of IBM stock from January 1926 to December 1997 for 864 observations.
For textbook, an AR(1)-EGARCH(1,1) is obtained (RATS program):

$$r_t = 0.0105 + 0.092 r_{t-1} + a_t, \quad a_t = \sigma_t\epsilon_t$$
$$\ln(\sigma_t^2) = -5.496 + \frac{g(\epsilon_{t-1})}{1 - .856B},$$
$$g(\epsilon_{t-1}) = -.0795\epsilon_{t-1} + .2647[|\epsilon_{t-1}| - \sqrt{2/\pi}],$$

The EGARCH model

Model checking:

For $\tilde{a}_t : Q(10) = 6.31(0.71)$ and $Q(20) = 21.4(0.32)$
For $\tilde{a}_t^2 : Q(10) = 4.13(0.90)$ and $Q(20) = 15.93(0.66)$

Discussion:

Using $\sqrt{2/\pi} \approx 0.7979 \approx 0.8$, we obtain

$$\ln(\sigma_t^2) = -1.0 + 0.856\ln(\sigma_{t-1}^2) + \begin{cases} 0.1852\epsilon_{t-1} & \text{if} \quad \epsilon_{t-1} \geq 0, \\ -0.3442\epsilon_{t-1} & \text{if} \quad \epsilon_{t-1} < 0 \end{cases}$$

Taking anti-log transformation, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\times 0.856} e^{-1.001} \times \begin{cases} e^{0.1852\epsilon_{t-1}} & \text{if} \quad \epsilon_{t-1} \geq 0, \\ e^{-0.3442\epsilon_{t-1}} & \text{if} \quad \epsilon_{t-1} < 0 \end{cases}$$

The EGARCH model

For a standardized shock with magnitude 2, (i.e. two standard deviations), we have

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\exp[-0.3442 \times (-2)]}{\exp(0.1852 \times 2)} = e^{0.318} = 1.374.$$

Therefore, the impact of a negative shock of size two-standard deviations is about 37.4% higher than that of a positive shock of the same size.

Forecasting: some recursive formula available

Another parameterization of EGARCH models

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1\frac{|a_{t-1}| + \gamma_1 a_{t-1}}{\sigma_{t-1}} + \beta_1\ln(\sigma_{t-1}^2),$$

where γ_1 denotes the leverage effect.

The EGARCH model

Below, I re-analyze the IBM log returns by extending the data to December 2009. The sample size is 1008.

The fitted model is

$$
\begin{aligned}
r_t &= 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t \\
\ln(\sigma_t^2) &= -0.611 + \frac{0.231|a_{t-1}| - 0.250 a_{t-1}}{\sigma_{t-1}} + 0.92 \ln(\sigma_{t-1}^2)
\end{aligned}
$$

Since EGARCH and TGARCH (below) share similar objective and the latter is easier to estimate. We shall use TGARCH model.

The Threshold GARCH (TGARCH) or GJR Model

A TGARCH(s, m) or GJR(s, m) model is defined as

$$
r_t = \mu_t + a_t, \quad a_t = \sigma_t \epsilon_t \sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2,
$$

where N_{t-i} is an indicator variable such that

$$
N_{t-i} = \begin{cases} 1 & \text{if} \quad a_{t-i} < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

One expects γ_i to be positive so that prior negative returns have higher impact on the volatility.

The Asymmetric Power ARCH (APARCH) Model

This model was introduced by Ding, Engle and Granger (1993) as a general class of volatility models. The basic form is

$$
\begin{aligned}
r_t &= \mu_t + a_t, \quad a_t = \sigma_t \epsilon_t \quad \epsilon_t \sim D(0, 1) \\
\sigma_t^\delta &= \omega + \sum_{i=1}^{s} \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^\delta + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^\delta
\end{aligned}
$$

where δ is a non-negative real number. In particular, $\delta = 2$ gives rise to the TGARCH model and $\delta = 0$ corresponds to using $\log(\sigma_t)$.

Theoretically, one can use any power δ to obtain a model. In practice, two things deserve further consideration:

- First, δ will also affect the specification of the mean equation, i.e., model for μ_t.
- Second, it is hard to interpret δ, except for some special values such as 0, 1, 2.

The Asymmetric Power ARCH (APARCH) Model

For the percentage log returns of IBM stock from 1926 to 2009, the fitted GJR model is

$$
\begin{aligned}
r_t &= 1.20 + a_t, \quad a_t = \sigma_t \epsilon_t \quad \epsilon_t \sim t_{6.67}^* \\
\sigma_t^2 &= 3.99 + 0.105(|a_{t-1}| - 0.224 a_{t-1})^2 + .807 \sigma_{t-1}^2,
\end{aligned}
$$

where all estimates are significant, and model checking indicates that the fitted model is adequate.

Note that, we can obtain the model for the log returns as

$$
\begin{aligned}
r_t &= 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t \\
\sigma_t^2 &= 3.99 \times 10^{-4} + 0.105(|a_{t-1}| - 0.224 a_{t-1})^2 + .807 \sigma_{t-1}^2,
\end{aligned}
$$

The Asymmetric Power ARCH (APARCH) Model

The sample variance of the IBM log returns is about 0.005 and the empirical 2.5% percentile of the data is about -0.130. If we use these two quantities for σ_{t-1}^2 and a_{t-1}, respectively, then we have

$$\begin{aligned}\frac{\sigma_t^2(-)}{\sigma_t^2(+)} &= \frac{0.0004 + 0.105(0.130 + 0.224 \times 0.130)^2 + 0.807 \times 0.005}{0.0004 + 0.105(0.130 - 0.224 \times 0.130)^2 + 0.807 \times 0.005} \\ &= 1.849.\end{aligned}$$

In this particular case, the negative prior return has about 85% higher impact on the conditional variance.

Stochastic volatility model

A (simple) SV model is

$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha_1 B - \cdots - \alpha_m B^m) \ln(\sigma_t^2) = \alpha_0 + v_t$$

where ϵ_t's are iid $N(0,1)$, v_t's are iid $N(0, \sigma_v^2)$, $\{\epsilon_t\}$ and $\{v_t\}$ are independent.

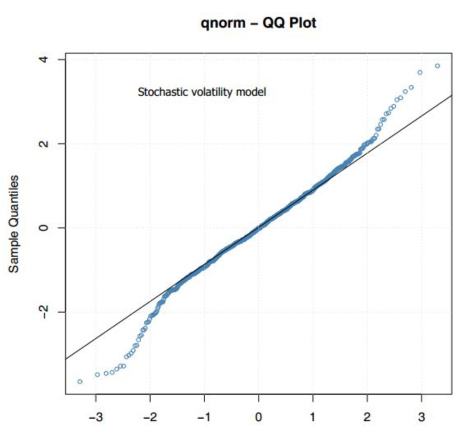

Figure 1: Normal probability plot for TGARCH(1,1) model fitted to monthly percentage log returns of IBM stock from 1926 to 2009.

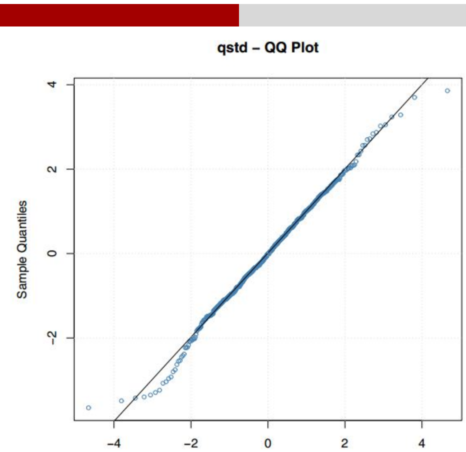

Figure 2: QQ plot for TGARCH(1,1) model fitted to monthly percentage log returns of IBM stock from 1926 to 2009.

Long-memory SV model

A simple LMSV is

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t = \sigma \exp(u_t/2), \quad (1-B)^d u_t = \eta_t$$

where $\sigma > 0$, ϵ_t's are iid $N(0,1)$, η_t's are iid $N(0, \sigma_\eta^2)$ and independent of ϵ_t, and $0 < d < 0.5$.

The model says:

$$\begin{aligned} \ln(a_t^2) &= \ln(\sigma^2) + u_t + \ln(\epsilon_t^2) \\ &= [\ln(\sigma^2) + E(\ln \epsilon_t^2)] + u_t + [\ln(\epsilon_t^2) - E(\ln \epsilon_t^2)] \\ &\equiv \mu + u_t + e_t. \end{aligned}$$

Thus, the $\ln(a_t^2)$ series is a Gaussian long-memory signal plus a non-Gaussian white noise; see Breidt, Crato and de Lima (1998).

Application:
See Examples 3.4 & 3.5 of the textbook.