Financial Econometrics
Lecture 6: Conditional Heteroscedastic Models

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Introduction

- What is asset volatility?
  - Conditional standard deviation of the asset returns.
- Why is volatility important?
  - Option (derivative) pricing, e.g., Black-Scholes formula
  - Risk management, e.g. value at risk (VaR)
  - Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
  - Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?

There are several versions of volatility, but conditional standard deviation is commonly used.

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
   - Realized volatility of daily log returns: use intraday highfrequency log returns.
   - Use daily high, low, and closing (log) prices, e.g. range = daily high - daily low.
2. Implied volatility of options data, e.g, VIX of CBOE. Figure 1.
3. Econometric modeling: use daily or monthly returns

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

How to calculate volatility?

Note: In most applications, volatility is annualized. This can easily be done by taking care of the data frequency. For instance, if we use daily returns in econometric modeling, then the annualized volatility (in the U.S.) is

$$\sigma_t^* = \sqrt{252}\sigma_t,$$

where σ_t is the estimated volatility derived from an employed model. If we use monthly returns, then the annualized volaitlity is

$$\sigma_t^* = \sqrt{12}\sigma_t,$$

where σ_t is the estimated volatility derived from the employed model for the monthly returns. Our discussion, however, continues to use σ_t for simplicity.

How to calculate volatility?

Basic idea of econometric modeling:
Shocks of asset returns are NOT serially correlated, but dependent, implying that the serial dependence in asset returns is nonlinear.
As shown by the ACF of returns and absolute returns of some assets we discussed so far.

Basic structure

$$r_t = \mu_t + a_t, \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i},$$

Volatility models are concerned with time-evolution of

$$\sigma_t^2 = \mathrm{Var}(r_t|F_{t-1}) = \mathrm{Var}(a_t|F_{t-1}),$$

the conditional variance of the return r_t.

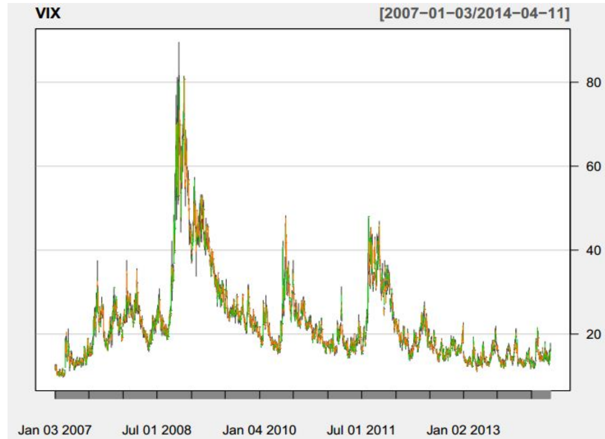

Figure 1: Time plot of the daily VIX index from January 3, 2007 to April 11, 2014.

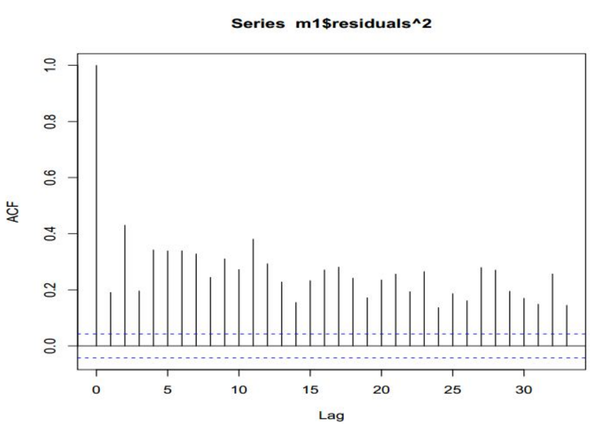

Figure 2: Sample ACF of the squared residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 13, 2015.

Univariate volatility models discussed

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models,
4. IGARCH models (used by RiskMetrics),

Univariate volatility models discussed

5. Exponential GARCH (EGARCH) model of Nelson (1991),
6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993),
7. Asymmetric power ARCH (APARCH) models of Ding, Granger and Engle (1994), [TGARCH and GJR models are special cases of APARCH models.]
8. Stochastic volatility (SV) models of Melino and Turnbull (1990),Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

ARCH model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2,$$

where $\{\epsilon_t\}$ is a sequence of iid r.v. with mean 0 and variance 1, $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$.

Distribution of ϵ_t:

- Standard normal
- Standardized Student-t
- Generalized error dist (ged)
- Their skewed counterparts.

Properties of ARCH models

Consider an ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

where $\alpha_0 > 0$ and $\alpha_1 \geq 0$.

1. $E(a_t) = 0$
2. $\mathrm{Var}(a_t) = \alpha_0/(1 - \alpha_1)$ if $0 < \alpha_1 < 1$
3. Under normality,

   $$m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)},$$

   provided $0 < \alpha_1^2 < 1/3$.

ARCH model

Advantages

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

Weaknesses

- Symmetric between positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

Building an ARCH Model

1. Modeling the mean effect and testing for ARCH effects
   H_o: no ARCH effects versus H_a : ARCH effects
   Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)
2. Order determination
   Use PACF of the squared residuals. (In practice, simply try some reasonable order).
3. Estimation: Conditional MLE
4. Model checking: Q-stat of standardized residuals and squared standardized residuals. Skewness & Kurtosis of standardized residuals.

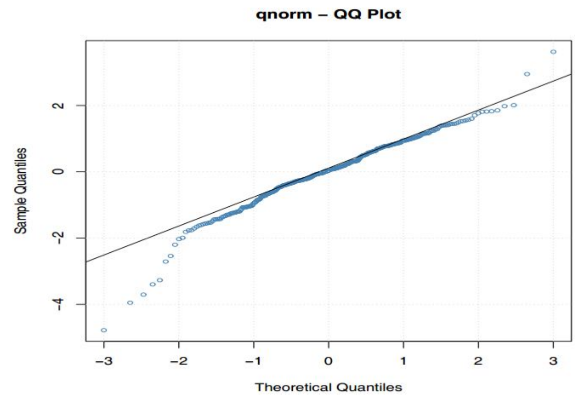

Figure 3: QQ-plot for standardized residuals of an ARCH(1) model with Gaussian innovations for monthly log returns of INTC stock: 1973 to 2003.

GARCH Model

$$a_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$, and $\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) < 1$.

- Re-parameterization: Let $\eta_t = a_t^2 - \sigma_t^2 . \{\eta_t\}$ un-correlated series. The GARCH model becomes:
  $$a_t^2 = \alpha_0 + \sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}.$$
  This is an ARMA form for the squared series a_t^2.
- Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.

GARCH Model

Focus on a GARCH(1,1) model
$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

- Weak stationarity: $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$.
- Volatility clusters
- Heavy tails: if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then
  $$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$
- For 1-step ahead forecast,
  $$\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2.$$

GARCH Model

For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1).$$

2-step ahead volatility forecast

$$\sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(1).$$

In general, we have

$$\sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(\ell - 1), \quad \ell > 1.$$

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial $1 - (\alpha_1 + \beta_1)B$.

GARCH Model

Example: Monthly excess returns of S&P 500 index starting from 1926 for 792 observations.
The fitted of a Gaussian AR(3) model

$$\tilde{r}_t = r_t - 0.0062$$
$$\tilde{r}_t = .089\tilde{r}_{t-1} - .024\tilde{r}_{t-1} - .123\tilde{r}_{t-3} + .007 + a_t,$$

$\hat{\sigma}_a^2 = 0.00333$.
For the GARCH effects, use a GARCH(1,1) model, we have a joint estimation:

$$r_t = 0.032r_{t-1} - 0.030r_{t-2} - 0.011r_{t-3} + 0.0077 + a_t$$
$$\sigma_t^2 = 7.98 \times 10^{-5} + .853\sigma_{t-1}^2 + 0.124a_{t-1}^2.$$

GARCH Model

- Implied unconditional variance of a_t is

$$\frac{0.0000798}{1 - 0.853 - 0.1243} = 0.00352$$

  close to the expected value. All AR coefficients are statistically insignificant.

- A simplified model:

$$r_t = 0.00745 + a_t, \ \sigma_t^2 = 8.06 \times 10^{-5} + .854\sigma_{t-1}^2 + .122a_{t-1}^2.$$

- Model checking:
  For $\tilde{a}_t : Q(10) = 11.22(0.34)$ and $Q(20) = 24.30(0.23)$.
  For $\tilde{a}_t^2 : Q(10) = 9.92(0.45)$ and $Q(20) = 16.75(0.67)$.

GARCH Model

Forecast: 1-step ahead forecast:

$$\sigma_h^2(1) = 0.00008 + 0.854\sigma_h^2 + 0.122a_h^2$$

Horizon	1	2	3	4	5	∞
Return	.0074	.0074	.0074	.0074	.0074	.0074
Volatility	.054	.054	.054	.054	.054	.059

Estimation of degrees of freedom:

$$r_t = 0.0085 + a_t, \qquad a_t = \sigma_t \epsilon_t, \qquad \epsilon_t \sim t_7$$

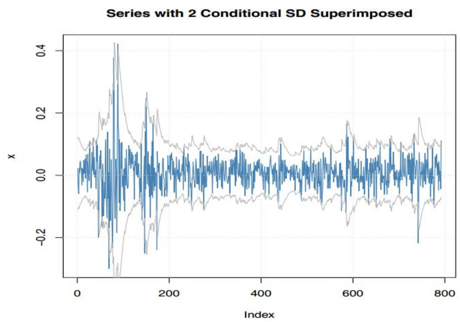

Figure 4: Monthly S&P 500 excess returns and fitted volatility

$$\sigma_t^2 = .000125 + .113a_{t-1}^2 + .842\sigma_{t-1}^2,$$

where the estimated degrees of freedom is 7.00.
Forecasting evaluation
Not easy to do; see Andersen and Bollerslev (1998).

IGARCH model

An IGARCH(1,1) model:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1-\beta_1)a_{t-1}^2.$$

For the monthly excess returns of the S&P 500 index, we have

$$r_t = .007 + a_t, \quad \sigma_t^2 = .0001 + .806\sigma_{t-1}^2 + .194a_{t-1}^2$$

For an IGARCH(1,1) model,

$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1,$$

where h is the forecast origin.
Effect of $\sigma_h^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0.
Special case: $\alpha_0 = 0$. Volatility forecasts become a constant.
This property is used in RiskMetrics to VaR calculation.