Financial Econometrics
Lecture 1: Introduction

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Processes considered

- return series (e.g., ch. 1, 2, 5)
- volatility processes (e.g., ch. 3, 4, 10, 12)
- continuous-time processes (ch. 6)
- extreme events (ch. 7)
- multivariate series (ch. 8, 9, 10)

Likelihood function (for self study)
Finally, it pays to study the likelihood function of returns $\{r_1, \ldots, r_T\}$ discussed in Chapter 1.

Processes considered

Basic concept:
Joint dist = Conditional dist × Marginal dist, i.e.

$$f(x, y) = f(x|y)f(y)$$

For two consecutive returns r_1 and r_2, we have

$$f(r_2, r_1) = f(r_2|r_1)f(r_1).$$

For three returns r_1, r_2 and r_3, by repeated application,

$$\begin{aligned} f(r_3, r_2, r_1) &= f(r_3|r_2, r_1)f(r_2, r_1) \\ &= f(r_3|r_2, r_1)f(r_2|r_1)f(r_1). \end{aligned}$$

Processes considered

In general, we have

$$\begin{aligned} & f(r_T, r_{T-1}, \ldots, r_2, r_1) \\ &= f(r_T|r_{T-1}, \ldots, r_1)f(r_{T-1}, \ldots, r_1) \\ &= f(r_T|r_{T-1}, \ldots, r_1)f(r_{T-1}|r_{t-2}, \ldots, r_1)f(r_{T-2}, \ldots, r_1) \\ &= \vdots \\ &= \left[\prod_{t=2}^{T} f(r_t|r_{t-1}, \ldots, r_1)\right] f(r_1), \end{aligned}$$

where $\prod_{t=2}^{T}$ denotes product.

Processes considered

If $r_t | r_{t-1}, \ldots, r_1$ is normal with mean μ_t and variance σ_t^2, then likelihood function becomes

$$f(r_T, r_{T-1}, \ldots, r_1) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[\frac{-(r_t - \mu_t)^2}{2\sigma_t^2}\right] f(r_1).$$

For simplicity, if $f(r_1)$ is ignored, then the likelihood function becomes

$$f(r_T, r_{T-1}, \ldots, r_1) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[\frac{-(r_t - \mu_t)^2}{2\sigma_t^2}\right].$$

This is the *conditional* likelihood function of the returns under normality. Other dists, e.g. Student-t, can be used to handle heavy tails.

Model specification

- μ_t: discussed in Chapter 2
- σ_t^2: Chapters 3 and 4.

Quantifying dependence: Consider two variables X and Y.

- Correlation coefficient:

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\mathrm{std}(X)\mathrm{std}(Y)}.$$

- Kendall's tau: Let $(\tilde{X}, \tilde{Y})$ be a random copy of (X, Y).

$$\begin{aligned}\rho_\tau &= P[(X - \tilde{X})(Y - \tilde{Y}) > 0] - P[(X - \tilde{X})(Y - \tilde{Y}) < 0] \\ &= E[\mathrm{sign}[(X - \tilde{X})(Y - \tilde{Y})]].\end{aligned}$$

  This measure quantifies the probability of *concordant* over *discordant*. Here concordant means $(X - \tilde{X})(Y - \tilde{Y}) > 0$. For spherical distributions, e.g., normal, $\rho_\tau = \frac{2}{\pi}\sin^{-1}(\rho)$.

Model specification

- Spearman's rho: rank correlation. Let $F_x(x)$ and $F_y(y)$ be the cumulative distribution function of X and Y .

$$\rho_s = \rho(F_x(X), F_y(Y)).$$

  That is, the correlation coefficient of probability-transformed variables. It is just the correlation coefficient of the **ranks** of the data.

  Why do we consider different measures of dependence?

- Correlation coefficient encounters problems when the distributions are not normal (spherical, in general). This is particularly relevant in risk management.
- Correlation coefficient focuses no linear dependence and is not robust to outliers.
- The actual range of the correlation coefficient can be much smaller than $[-1, 1]$.

Takeaway

1. Understand the summary statistics of asset returns
2. Understand various definitions of returns & their relationships
3. Learn basic characteristics of FTS
4. Learn the basic **R** functions. (See Rcommands-lec1.txt on the course web.)

Linear Time Series (TS) Models

Financial TS: collection of a financial measurement over time Example: log return r_t
Data: $\{r_1, r_2, \ldots, r_T\}$ (T data points)
Purpose: What is the information contained in $\{r_t\}$?

Basic concepts

- Stationary:
  - Strict: distributions are time-invariant
  - Weak: first 2 moments are time-invariant

  What does weak stationarity mean in practice?
  Past: time plot of $\{r_t\}$ varies around a fixed level within a finite range!
  Future: the first 2 moments of future r_t are the same as those of the data so that meaningful inferences can be made.

Linear Time Series (TS) Models

- Mean (or expectation) of returns: $\mu = E(r_t)$
- Variance (variability) of returns: $\text{Var}(r_t) = E[(r_t - \mu)^2]$
- Sample mean and sample variance are used to estimate the mean and variance of returns.

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t \quad \& \quad \text{Var}(r_t) = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2$$

- Test $H_o : \mu = 0$ vs $H_a : \mu \neq 0$. Compute

$$t = \frac{\bar{r}}{\text{std}(\bar{r})} = \frac{\bar{r}}{\sqrt{\text{Var}(r_t)/T}}$$

  Compare t ratio with $N(0, 1)$ dist.
  Decision rule: Reject H_o of zero mean if $|t| > Z_{\alpha/2}$ or p-value is less than α.

Linear Time Series (TS) Models

- Lag-k autocovariance:

$$\gamma_k = \text{Cov}(r_t, r_{t-k}) = E[(r_t - \mu)(r_{t-k} - \mu)].$$

- Serial (or auto-) correlations:

$$\rho_\ell = \frac{\text{Cov}(r_t, r_{t-\ell})}{\text{Var}(r_t)}$$

  Note: $\rho_0 = 1$ and $\rho_k = \rho_{-k}$ for $k \neq 0$. Why? Existence of serial correlations implies that the return is predictable, indicating market inefficiency.
- Sample autocorrelation function (ACF)

$$\hat{\rho}_\ell = \frac{\sum_{t=1}^{T-\ell}(r_t - \bar{r})(r_{t+\ell} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2},$$

  where $\bar{r}$ is the sample mean & T is the sample size.

Linear Time Series (TS) Models

- Test zero serial correlations (market efficiency)
  - Individual test: for example,
    $H_o : \rho_1 = 0$ vs $H_a : \rho_1 \neq 0$

$$t = \frac{\hat{\rho}_1}{\sqrt{1/T}} = \sqrt{T}\hat{\rho}_1$$

    Asym. $N(0, 1)$.
    Decision rule: Reject H_o if $|t| > Z_{\alpha/2}$ or p-value less than α.
  - Joint test (Ljung-Box statistics):
    $H_o : \rho_1 = \cdots = \rho_m = 0$ vs $H_a : \rho_i \neq 0$

$$Q(m) = T(T+2)\sum_{\ell=1}^{m}\frac{\hat{\rho}_\ell^2}{T-\ell}$$

    Asym. chi-squared dist with m degrees of freedom.
    Decision rule: Reject H_o if $Q(m) > \chi_m^2(\alpha)$ or p-value is less than α.

Linear Time Series (TS) Models

- Sources of serial correlations in financial TS
  - Nonsynchronous trading (ch. 5)
  - Bid-ask bounce (ch. 5)
  - Risk premium, etc. (ch. 3)

  Thus, significant sample ACF does not necessarily imply market inefficiency.

  Example: Monthly returns of IBM stock from 1926 to 1997.
- $R_t : Q(5) = 5.4(0.37)$ and $Q(10) = 14.1(0.17)$
- $r_t : Q(5) = 5.8(0.33)$ and $Q(10) = 13.7(0.19)$

  Remark: What is p-value? How to use it?
  Implication: Monthly IBM stock returns do not have significant serial correlations.

Linear Time Series (TS) Models

Example: Monthly returns of CRSP value-weighted index from 1926 to 1997.

- $R_t : Q(5) = 27.8$ and $Q(10) = 36.0$
- $r_t : Q(5) = 26.9$ and $Q(10) = 32.7$

All highly significant. Implication: there exist significant serial correlations in the value-weighted index returns. (Nonsynchronous trading might explain the existence of the serial correlations, among other reasons.) Similar result is also found in equal-weighted index returns.

Back-shift (lag) operator

A useful notation in TS analysis.

- Definition: $B_{r_t} = r_{t-1}$ or $L_{r_t} = r_{t-1}$
- $B^2_{r_t} = B(B_{r_t}) = B_{r_{t-1}} = r_{t-2}$.

B (or L) means time shift! B_{r_t} is the value of the series at time $t-1$.
Suppose that the daily log returns are

Date	1	2	3	4
r_t	0.017	-0.005	-0.014	0.021

Answer the following questions:

- $r_2 =$
- $B_{r_3} =$
- $B^2_{r_5} =$

Question: What is B_2?
What are the important statistics in practice?
Conditional quantities, not unconditional

Back-shift (lag) operator

A proper perspective: at a time point t

- Available data: $\{r_1, r_2, \ldots, r_{t-1}\} \equiv F_{t-1}$
- The return is decomposed into two parts as

$$\begin{aligned} r_t &= \text{predictable part} + \text{not predictable part} \\ &= \text{function of elements of } F_{t-1} + a_t \end{aligned}$$

In other words, given information F_{t-1}

$$\begin{aligned} r_t &= \mu_t + a_t \\ &= E(r_t|F_{t-1}) + \sigma_t \epsilon_t \end{aligned}$$

Back-shift (lag) operator

- μ_t: conditional mean of r_t
- a_t: shock or innovation at time t
- ϵ_t: an iid sequence with mean zero and variance 1
- σ_t: conditional standard deviation (commonly called volatility in finance)

Traditional TS modeling is concerned with μ_t:
Model for μ_t: **mean equation**
Volatility modeling concerns σ_t.
Model for σ_t^2: **volatility equation**

Univariate TS analysis serves two purposes

- a model for μ_t
- understanding models for σ_t^2: properties, forecasting, etc.

Linear time series: r_t is linear if

- the predictable part is a linear function of F_{t-1}
- $\{a_t\}$ are independent and have the same dist. (iid)

Mathematically, it means r_t can be written as

$$r_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i},$$

where μ is a constant, $\psi_0 = 1$ and $\{a_t\}$ is an iid sequence with mean zero and well-defined distribution.
In the economic literature, a_t is the *shock* (or *innovation*) at time t and $\{\psi_i\}$ are the *impulse* responses of r_t.
White noise: iid sequence (with finite variance), which is the building block of linear TS models.
White noise is not predictable, but has zero mean and finite variance.

Univariate linear time series models

1. autoregressive (AR) models
2. moving-average (MA) models
3. mixed ARMA models
4. seasonal models
5. regression models with time series errors
6. fractionally differenced models (long-memory)

Example Quarterly growth rate of U.S. real gross national product (GNP), seasonally adjusted, from the second quarter of 1947 to the first quarter of 1991.

Univariate linear time series models

An AR(3) model for the data is

$$r_t = 0.005 + 0.35 r_{t-1} + 0.18 r_{t-2} - 0.14 r_{t-3} + a_t, \quad \hat{\sigma}_a = 0.01,$$

where $\{a_t\}$ denotes a white noise with variance σ_a^2. Given r_n, r_{n-1} & r_{n-2}, we can predict r_{n+1} as

$$\hat{r}_{n+1} = 0.005 + 0.35 r_n + 0.18 r_{n-1} - 0.14 r_{n-2}.$$

Other implications of the model?

In this course, we use *statistical methods* to find models that fit the data well for making inference, e.g. prediction. On the other hand, there exists economic theory that leads to time-series models for economic variables. For instance, consider the $real\ business - cycle\ theory$ in macroeconomics. Under some simplifying assumptions, one can show that $\ln(Y_t)$, where Y_t is the output (GDP), follows an AR(2) model. See $Advanced\ Macroeconomics$ by David Romer (2006, 3rd, pp. 190).

Univariate linear time series models

Example: Monthly simple return of Center for Research in Security Prices (CRSP) equal-weighted index

$$R_t = 0.013 + a_t + 0.178a_{t-1} - 0.13a_{t-3} + 0.135a_{t-9}, \quad \hat{\sigma}_a = 0.073$$

Checking: $Q(10) = 11.4(0.122)$ for the residual series a_t.
Implications of the model?
Statistical significance vs economic significance.

In this course, we shall discuss some reasons for the observed serial dependence in index returns. See, for example, Chapter 5 on nonsynchronous trading.

Important properties of a model

- Stationarity condition
- Basic properties: mean, variance, serial dependence
- Empirical model building: specification, estimation, & checking
- Forecasting