Cosmology

A Note by chestnut

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Hello, this the notes of when I was studying cosmology, please take it if you needed. The reference books are:

1. *Modern Cosmology*, Scott Dodelson, Fabian Schmidt (Main)
2. *Cosmology*,

Thanks. I also have other notes in English or Chinese with latex. Or may be in the near future with Japneses, ;)

- Theoretical Mechanics (En.) done, in revising
- Quantum Mechanics (En.) done, in revising
- General Relativity (En.) to do, in learning
- Field Theory (En.) to do, to learn
- Electrodynamics (Ch.) doing, in reviewing
- Statistical Mechanics (Ch.) done, in revising
- Some self-solved notes of Japneses M.S. physical exam, want to do, in learning

Welcome to email me if you would like to have or discuss.

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Contents

PART

I

The Expansion of the Universe

Section 1

Intro: The concordance model of cosmology

Subsection 1.1

Scale factor

Introduce scale factor: a for expansion, physical distance = comoving distance $\times a$
Introduce stretching factor: z, observed wavelength = emit wavelength $\times 1/a_{\text{emit}}$

Definition 1 Redshift:

$$1+z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}}}{a_{\text{emit}}} = \frac{1}{a_{\text{emit}}}$$

Universe geometry: Euclidean, open, closed. history of universe: evolution of a with t, determined by energy density.

Definition 2 Hubble rate: measure how rapidly the scale factor changes.

$$H(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}$$

Hubble constant: $H_0 \equiv H(t=0) = \dot{a}$

$$H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = \frac{h}{0.98\times 10^{10}\,\text{years}} = 2.13\times 10^{-33}\,\frac{\text{eV}}{\hbar}h$$

$1\,\text{Mpc} = 3.856\times 10^{-24}\,\text{cm}$

- Euclidean and matter dominated, $a \approx t^{2/3}$

Friedmann equation:
$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{\text{cr}}-\rho(t_0)}{a^2(t)}\right]$$

Definition 3 Critical density:

$$\rho_{\text{cr}} \equiv \frac{3H_0^2}{8\pi G}$$

$\rho_{\text{cr}} = 1.88\,h^2\times 10^{-29}\,\mathrm{g\,cm^{-3}}$

Hubble found Distance galaxies are receding from us, i.e. redshifted. comoving motion $\dot{x}=0$

Law 1 Hubble-Lemaitre law: distance-redshift relation, the relative velocity is

$$v = \frac{\mathrm{d}}{\mathrm{d}t}(ax) = \dot{a}x = H_0 d \qquad (v \ll c) \tag{1.1}$$

Eq. (1.1) break down

Subsection 1.2

ΛCDM

Early universe hottter and denser, temperature 1MeV/k_B, no neutral atoms or bound nuclei. Then light forms (Big Bang Nuceosynthesis, BBN), -> CMB smooth -> not completely smooth -> inhomogeneities (universe structure)

Section 2

The expanding universe

$$\hbar = c = k_B = 1$$

Basic feature:

- Redshift of light
- Notion of distance
- Evolution of ρ
- Epoch of equality a_{eq}

Subsection 2.1

Expanding space

2.1.1 The metric

To describe a curved spacetime, general relativity

Definition 4 Metric: returens the actual physical distance between two infinitesimally close points in spacetime defined in some arbitrary coordinate system.

Proper-time interval

$$\mathrm{d}s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

For a Euclidean universe:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}$$

2.1.2 The geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\alpha}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0$$

Christoffel symbol:

$$\Gamma^\mu{}_{\alpha\beta}$$

$\Gamma^0{}_{00} = 0,$
$\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0,$
$\Gamma^0{}_{ij} = \delta_{ij}\dot{a}a,$
$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij}\frac{\dot{a}}{a}$

Four-dimensional energy-momentum vector: $P^\alpha = (E, \boldsymbol{P})$

$$P^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$$

Subsection 2.2

Distance

Total comoving distance

$$\chi(t) = \int_t^{t_0} \frac{\mathrm{d}t'}{a(t')} = \int_{a(t)}^{1} \frac{\mathrm{d}a'}{a'^2 H(a')} = \int_0^z \frac{\mathrm{d}z'}{H(z')}$$

Definition 5 Comoving horizon:

$$\eta(t) \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')}$$

t': Conformal time

Angular diameter distance:

$$d_A = \frac{l}{\theta}$$

l: physical size
θ: angle subtended

In Euclidean universe:

$$d_A^{\text{Euc}} = a\chi = \frac{\chi}{1+z}$$

Definition 6 Luminosity distance in a Euclidean expanding universe:

$$d_L^{\text{Euc}} \equiv \frac{\chi}{a}$$

SUBSECTION 2.3
Evolution of energy

Energy-momentum tensor in isotropic smooth universe:

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

$\mathcal{P}$: pressure

Law 2 Local energy and momentum conservation,

$$\nabla_\mu T^\mu{}_\nu \equiv 0$$

conservation law in an expanding universe,

$$\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}[3\rho + 3\mathcal{P}] = 0$$

Definition 7 Equation of state parameter

$$w_s \equiv \frac{\mathcal{P}_s}{\rho_s}$$

Evolution of any constituent of s

$$\rho_s \propto \exp\left\{-3\int^a \frac{\mathrm{d}a'}{a}[q + w_s(a')]\right\}$$
$$\overset{w_s=\text{const}}{\propto} a^{-3(1+w_s)}$$

SUBSECTION 2.4
Cosmic inventory

Definition 8 Density parameters

$$\Omega_s \equiv \frac{\rho_s(t_0)}{\rho_{\text{cr}}}$$

$$\rho_s(a) = \Omega_s \rho_{\text{cr}} a^{-3(1+w_s)}$$

c: cold dark matter
b: baryons
γ: photons
ν: neutrinos

Photons

Baryons

Dark matter

Neutrinos

Epoch of matter-radiation equality

$$a_{\text{eq}} =$$
$$1 + z_{\text{eq}} = 2.38 \times 10^4 \Omega_{\text{m}} h^2$$

Dark energy

SECTION 3

The fundamental equations of cosmology

SUBSECTION 3.1

Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.1}$$

Definition 9 Einstein tensor:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha}$$

Time-time component of the Einstein equations:

$R_{00} = -3\ddot{a}/a$
$R_{ij} = \delta_{ij}[2\dot{a}^2 + a\ddot{a}]$

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}$$

Lead to first Friedmann equations:

Ricci scalar: $R \equiv g_{\mu\nu} R_{\mu\nu}$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

FLRW metric and assuming a Euclidean universe,

$$\frac{H^2(t)}{H_0^2} = \frac{\rho(t)}{\rho_{\rm cr}} = \sum_{s={\rm r,m},\nu,{\rm DE}} \Omega_s [a(t)]^{-3(1+w_s)}$$

To calculate the evolution of the homogeneous universe.

$$\frac{H^2(t)}{H_0^2} = \sum_{s={\rm r,m},\nu,{\rm DE}} \Omega_s [a(t)]^{-3(1+w_s)} + \Omega_{\rm K}[a(t)]^{-2}$$

SUBSECTION 3.2

Boltzmann equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C[f]$$

Boltzmann equation in an expanding universe

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\cdot\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}$$

In homogeneous universe,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - Hp\frac{\partial f}{\partial p} \tag{3.2}$$

Eq. 3.2 is valid for all particles.

Collision term

SUBSECTION 3.3

General theory of small fluctuations

General theory of small fluctuations

3.3.1 Perturbed spacetime

Perturbed metric:

$$
\begin{aligned}
&g_{00}(\boldsymbol{x},t) = -1 - \Psi(\boldsymbol{x},t),\\
&g_{0i}(\boldsymbol{x},t) = 0,\\
&g_{ij}(\boldsymbol{x},t) = a^2(t)\delta_{ij}[1 + 2\Phi(\boldsymbol{x},t)].
\end{aligned}
$$

- Ψ: corresponds to Newtonian potential and governs the motion of slow-moving (non-relativistic) bodies;
- Φ: the perturbation to the spatial curvature

$a(t) \to$
$a(\boldsymbol{x},t) = q(t)\sqrt{1+\Phi(\boldsymbol{x},t)}$

3.3.2 The geodesic equation

Via perturbed metric,

$$P^\mu = \left[E(1-\Psi), p^i\frac{1-\Phi}{a}\right]$$

The change in the magnitude of the momentum of a particle as it moves through a perturbed FLRW universe,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\left[H + \dot{\Phi}\right]p - \frac{E}{a}\hat{p}^i\Psi_{,i}$$

Energy-momentum tensor in the perturbed universe for a single species with degeneracy factor g:

$$T^0{}_0(\boldsymbol{x},t) = -g$$

3.3.3 The collisionless Boltzmann equation for photons

$m = 0, E = p$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \left[H + \dot{\Phi} + \frac{1}{a}\hat{p}^i\Psi_{,i}\right]p\frac{\partial f}{\partial p} \tag{3.3}$$

3.3.4 The collisionless Boltzmann equation for massive particles

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \left[H + \dot{\Phi} + \frac{E}{ap}\hat{p}^i\Psi_{,i}\right]p\frac{\partial f}{\partial p} \tag{3.4}$$

PART II

Evolution of structure

Section 4

The origin of species

Section 5

Evolution of fluctuations: matter & radiation

Derive how matter, photons and neutrinos behave in a given expanding spacetime with perturbation.

Subsection 5.1

The collision Boltzmann equation for photons

5.1.1 Distribution function

$E = p$, distribution function of Bose-Einstein:

$$f(\boldsymbol{x}, p, \hat{\boldsymbol{p}}, t) = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\boldsymbol{x}, \hat{\boldsymbol{p}}, t)]}\right\}\right]^{-1}$$

- Θ: perturbation to the distribution function, temperature perturbation

Zeroth-order distribution function is set preciseby by the requirement that the collision term vanished.

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\dot{\Theta} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]$$

5.1.2 Collision term: Compton Scattering

Definition 10 Monopole:

$$\Theta_0(\boldsymbol{x}, t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega' \Theta(\hat{\boldsymbol{p}}', \boldsymbol{x}, t)$$

$$C[f(\boldsymbol{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_{\mathrm{T}}[\Theta_0 - \Theta(\hat{\boldsymbol{p}}) + \hat{\boldsymbol{p}} \cdot \boldsymbol{u}_{\mathrm{b}}]$$

monopole+dipole -> photon behave like a fluid.

5.1.3 The Boltzmann equation for phtoton

$\eta(t) \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')}$

$$\dot{\Theta} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_{\mathrm{T}}[\Theta_0 - \Theta + \hat{\boldsymbol{p}} \cdot \boldsymbol{u}_{\mathrm{b}}]$$

In terms of the conformal time,

Definition 11 The angle between the wavenumber $\boldsymbol{k}$ and the photon direction $\hat{\boldsymbol{p}}$ is denoted as μ:

$$\mu \equiv \frac{\boldsymbol{k} \cdot \hat{\boldsymbol{p}}}{k}$$

Definition 12 Optical depth

$$\tau(\eta) \equiv \int_\eta^{\eta_0} \mathrm{d}\eta' n_e \sigma_{\mathrm{T}} a$$

$$\tau' \equiv \frac{\mathrm{d}\tau}{\mathrm{d}\eta} = -n_e \sigma_{\mathrm{T}} a$$

In Fourier mode,

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'[\Theta_0 - \Theta + \mu u_b]$$

Subsection 5.2

The Boltzmann equation for cold dark matter

Collisionless,

Definition 13 Fluid velocity

$$u_{\mathrm{c}}^i \equiv \frac{1}{n_{\mathrm{c}}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\mathrm{c}} \frac{p\hat{p}^i}{E(p)}$$

continuity equation:

$$\frac{\partial \delta_{\mathrm{c}}}{\partial t} + \frac{1}{a}\frac{\partial u_{\mathrm{c}}^i}{\partial x^i} + 3\dot{\Phi} = 0$$

Euler equation:

$$\frac{\partial u_{\mathrm{c}}^j}{\partial t} + H u_{\mathrm{c}}^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0$$

In terms of conformal time η and in Fourier space,

$$\begin{aligned} \delta_{\mathrm{c}}' + iku_{\mathrm{c}} &= -3\Phi', \\ u_{\mathrm{c}}' + \frac{a'}{a} u_{\mathrm{c}} &= -ik\Psi, \end{aligned}$$

Subsection 5.3

The Boltzmann equation for baryon

Compton scattering.

Definition 14 Dipole

$$\Theta_1(k,\eta) \equiv i \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \Theta(\mu, k, \eta)$$

Subsection 5.4

The Boltzmann equation for newtrinos

Colissionless.

Summary

Equation:

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'\left[\Theta_0 - \Theta + \mu u_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\right]$$

And

$$\delta_{\mathrm{c}}' + iku_{\mathrm{c}} = -3\Phi', \tag{5.1}$$

$$u_{\mathrm{c}}' + \frac{a'}{a} u_{\mathrm{c}} = -ik\Psi, \tag{5.2}$$

$$\delta_{\mathrm{b}}' + iku_{\mathrm{b}} = -3\Phi', \tag{5.3}$$

$$u_{\mathrm{b}}' + \frac{a'}{a} u_{\mathrm{b}} = -ik\Psi + \frac{\tau'}{R}[u_{\mathrm{b}} + 3i\Theta_1], \tag{5.4}$$

$$\mathcal{N}' = \tag{5.5}$$

Section 6

Evolution of fluctuations: gravity

Non-gravitational to gravitational field

Subsection 6.1

Scalar vector-tensor decomposition

Generally, FLRW spacetime perturbed by a small ammount:

$$
\begin{aligned}
g_{00}(t,\boldsymbol{x}) &= -1 + h_{00}(t,\boldsymbol{x}),\\
g_{0i}(t,\boldsymbol{x}) &= a(t)h_{0i}(t,\boldsymbol{x}) = a(t)h_{i0}(t,\boldsymbol{x}),\\
g_{ij}(t,\boldsymbol{x}) &= a^2(t)[\delta_{ij} + h_{ij}(t,\boldsymbol{x})]
\end{aligned}
$$

Theorem 1 **Decomposition theorem:** perturbation of each type – scalar, vector, and tensor – evolve independently at linear order.

Subsection 6.2

From gauge to gauge

Consider a scalar field $\phi(x)$, separate into background and perturbation:

$$\phi(x) = \bar{\phi}(t) + \delta\phi(t,\boldsymbol{x})$$

Subsection 6.3

The Einstein equation for scalar perturbations

Conformal-Newtonian gauge,

6.3.1 Ricci tensor

$$
\begin{aligned}
\delta R = &-12\Psi\left(H^2 + \frac{\ddot{a}}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00}\\
&- 6H(\Psi, 0 - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}
\end{aligned}
$$

6.3.2 Two components of the Einstein equations

$$\delta G = -6H\Phi_{,0} + 6\Phi H^2 - 2\frac{k^2\Phi}{a^2}$$

Evolution equation for Ψ and Φ

$$k^2\Phi + 3\frac{a'}{a}\left(\Phi' - \Psi\frac{a'}{a}\right) = 4\pi G a^2[\rho_c\delta_c + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0]$$

Subsection 6.4

Tensor perturbations

Section 7

Initial conditions

Equations governing perturbation around a smooth background,

Subsection 7.1

The horizon problems and a solution

- $k \gg aH$, leave the horizon
- $k \ll aH$, at the end of inflation for all modes (can observe)

Subsection 7.2

Inflation

Scalar field energy-momentum tensor:

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - \delta^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]$$

Assume zero-order: homogeneous; first-order: perturbation;

$$\phi'' + 2aH\phi' + a^2V_{,\phi} = 0 \tag{7.1}$$

Subsection 7.3

Gravitational wave production

7.3.1 harmonic oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2x^2 = 0$$

7.3.2 Tensor perturbation

From 7.1,

$$v'' + \left(k^2 - \frac{a''}{a}\right)v = 0$$

Power spectrum of the primordial tensor perturbation

$$P_h = 16\pi G\frac{|v(k,\eta)|^2}{a^2}$$

Subsection 7.4

Salar perturbations

7.4.1 Salar field perturbations around an unexpected background

$$\delta\phi'' + 2aH\delta\phi' + k^2\delta\phi = 0$$

Find the spectrum of scalar perturbations emerging from inflation.

7.4.2 Super-horizon perturbation

Definition 15 | Curvature perturbation,

7.4.3 Spatially flat slicing

Subsection 7.5

The Einstein-Boltzmann equations at early times

Reduce the Boltzmann equations to when $k\eta \ll 1$,

$$\begin{aligned}\Theta_0' + \Phi' &= 0\\ \mathcal{N}_0' + \Phi' &= 0\\ \delta_c' &= -3\Phi'\\ \delta_b &= -3\Phi'\end{aligned}$$

Einstein equation to

$$3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) = 16\pi G a^2 \rho_r \Theta_{r,0}$$

Summary

Power spectrum of a gauge-invariant quantity

$$P_{\mathcal{R}}(k) = \left.\frac{2\pi G H^2}{\epsilon_{sr} k^3}\right|_{aH=k}$$

Or

$$P_{\mathcal{R}}(k) = \frac{2\pi}{k^3}\left.\frac{H^2}{m_{PI}^2 \epsilon_{sr}}\right|_{aH=k} \equiv 2\pi^2 \mathcal{A}_s k^{-3}\left(\frac{k}{k_p}\right)^{n_s-1}$$

- $\mathcal{A}_s$: variance of curvature perturbations in a logarithmic wave interval centered around the pivot scale k_p
- n_s: scalar spectral index

PART III

The Growth of Structure

Goal: a prediction for the linear matter power spectrum.

SECTION 8

Growth of structure: linear theory

SUBSECTION 8.1

Prelude

Three stages of evolution

$$\Phi(\boldsymbol{k}, a) = \frac{3}{5}\mathcal{R}(\boldsymbol{k}) \times \{\text{Transfer Function}(k)\} \times \{\text{Growth factor}(a)\}$$

matter density

$$\delta_{\mathrm{m}}(\boldsymbol{k}, a) = \frac{2}{5}\frac{k^2}{\Omega_{\mathrm{m}} H_0^2}\mathcal{R}\boldsymbol{k}T(k)D_+(a) \qquad (a > a_{\mathrm{late}},\, k \ll aH)$$

Important

Linear matter power spectrum at late times:

$$P_{\mathrm{L}}(k, a) = \frac{8\pi^2}{25}\frac{\mathcal{A}_s}{\Omega_m^2}D_+^2(a)T^2(k)\frac{k^{n_s}}{H_0^4 k_{\mathrm{p}}^{n_s - 1}} \tag{8.1}$$

$\mathcal{A}_s$:
D_+:

Closing the Boltzmann hierachy

SUBSECTION 8.2

Large-Scale

Section 9

The cosmic microwave background

Subsection 9.1

Overview

$$\Theta_0'' + k^2 c_s^2 \Theta_0 = F$$

Subsection 9.2

Large-scale anisotropies

Subsection 9.3

Acoustic oscillations

9.3.1 Tightly-coupled limit of the Boltzmann equations

$$\begin{aligned}\Theta_0'' + \frac{a'}{a}\frac{R}{1+R}\Theta_0' + k^2 c_s^2 \Theta_0 = F(k,\eta), \\ F(k,\eta) \equiv -\frac{k^2}{3}\Psi - \frac{a'}{a}\frac{R}{1+R}\Phi' - \Phi''\end{aligned}$$

9.3.2 Tightly-coupled solutions

$$\begin{aligned}\Theta_0(\boldsymbol{k},\eta) + \Phi(\boldsymbol{k},\eta) =& [\Theta_0(\boldsymbol{k},0) + \Phi(\boldsymbol{k},0)]\cos(kr_s) \\ &+ \frac{k}{\sqrt{3}}\int_0^\eta \mathrm{d}\tilde{\eta}[\Phi(\boldsymbol{k},\tilde{\eta}) - \Psi(\boldsymbol{k},\tilde{\eta})]\sin\left[k(r_s(\eta) - r_s(\tilde{\eta}))\right]\end{aligned}$$

$$\begin{aligned}\Theta_1(\boldsymbol{k},\eta) =& \frac{1}{\sqrt{3}}[\Theta_0(\boldsymbol{k},0) + \Phi(\boldsymbol{k},0)]\sin(kr_s) \\ &- \frac{k}{3}\int_0^\eta \mathrm{d}\tilde{\eta}[\Phi(\boldsymbol{k},\tilde{\eta}) - \Psi(\boldsymbol{k},\tilde{\eta})]\cos\left[k(r_s(\eta) - r_s(\tilde{\eta}))\right]\end{aligned}$$

Subsection 9.4

Diffusion damping

Subsection 9.5

Inhomogeneities to anisotropies

9.5.1 Free streaming

9.5.2 The angular power spectrum

Temperature perturbation

$$T(\boldsymbol{x},\hat{\boldsymbol{p}},\eta) = T(\eta)[1 + \Theta(\boldsymbol{x},\hat{\boldsymbol{p}},\eta)]$$

Expand the temperature perturbation in terms of spherical harmonics,

$$\Theta(\boldsymbol{x},\hat{\boldsymbol{p}},\eta) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}(\boldsymbol{x},\eta) Y_{lm}(\hat{\boldsymbol{p}})$$

l, m are conjugate to the real-space unit vector $\hat{\boldsymbol{p}}$ [1]

[1] *as* $\boldsymbol{k} \leftrightarrow \boldsymbol{x}$

SUBSECTION 9.6
The CMB power spectrum

SUBSECTION 9.7
Cosmological parameters

- Curvature parameter, $\Omega_\mathrm{K} \equiv 1 - \Omega\mathrm{m} - \Omega_\Lambda$, often set to zero in the concordance model
- Cosmological constant, parametrized by Ω_Λ
- Normalization of the primordial spectrum, $\mathcal{A}_s$
- Scalar spectral index, n_s
- Reionization, parametrized by the optical depth τ_rei to a redshift after recombination is completed
- Baryon density, $\Omega_\mathrm{b} h^2$
- CDM density, $\Omega_\mathrm{c} h^2$

SECTION 10

The polarized CMB

SUBSECTION 10.1

Acoustic oscillations

10.1.1 Tightly-coupled limit of the Boltzmann eq

10.1.2 Tightly-coupled solutoins

Definition 16 Sound horizon:

$$r_s(\eta) \equiv \int_0^\eta \mathrm{d}\tilde{\eta} c_s(\tilde{\eta})$$

SUBSECTION 10.2

Diffusion damping

SUBSECTION 10.3

Inhomogeneities to anisotropies

SECTION 11

Probes of structure: tracer

SUBSECTION 11.1

Galaxy clustering

11.1.1 Galaxy clustering

$$\chi_{\mathrm{fid}}(z) = \chi(z) + \delta\chi(z)$$

11.1.2 Redshift-space distortions

Redshift-space oversensity

$$\delta_{\mathrm{g,RSD}}(\boldsymbol{k}, t) = \left[b_1 + f\mu_k^2\right]\delta_{\mathrm{m}}\boldsymbol{k}$$

Galaxy power spectrum in redshift space

$$R_{\mathrm{g,RSD}}(k, \mu_k, \bar{z}) = P_{\mathrm{L}}(k, \bar{z})\left[b_1 + f\mu_k^2\right]^2 + P_N \tag{11.1}$$

11.1.3 BAO and Alcock-Paczyński

Observed galaxy power spectrum

$$R_{\mathrm{g,obs}}(\boldsymbol{k}_{\mathrm{obs}}, \bar{z}) = \left(P_{\mathrm{L}}(k, \bar{z})\left[b_1 + f\mu_k^2\right]^2\right)\Bigg|_{\boldsymbol{k}=\left([1+\alpha_\perp]k^1_{\mathrm{obs}},[1+\alpha_\perp]k^2_{\mathrm{obs}},[1+\alpha_\parallel]k^3_{\mathrm{obs}}\right)} + P_N \tag{11.2}$$

Section 12

Growth of structure: beyond linear theory

Section 13

Probes of structure: lensing

Section 14

Analysis and inference