∞-Harmonic Functions On SG

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The Infinity Laplacian Δ_∞

- The infinity Laplacian is constructed differently on different domains.

Definitions

- **Continuous Domains**:

$$\Delta_\infty u(x) = \left(\frac{\nabla u(x)}{||\nabla u(x)||}\right)^\top D^2 u(x) \left(\frac{\nabla u(x)}{||\nabla u(x)||}\right),$$

  where $D^2 f$ is the Hessian matrix of f.

- **Graph Domains**:

$$\Delta_\infty u(x) = \frac{1}{2}\left(\max_{y\in N(x)} u(y) + \min_{y\in N(x)} u(y)\right) - u(x)$$

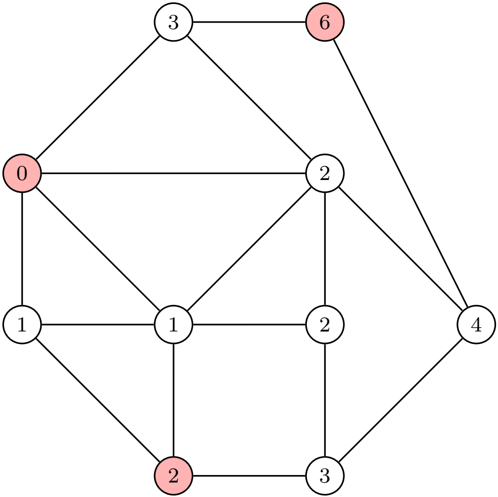
- **Fractal Domains**: ??

Constructing Δ_∞ on SG

- There is currently no construction for Δ_∞ on SG. We would like to base a construction on a limit of graph approximations.
- This requires a better understanding of the graph infinity Laplacian.
- We can try to first understand ∞-harmonic functions on a graph:

$$
\begin{aligned}
0 &= \Delta_\infty u(x) \\
0 &= \frac{1}{2}\left(\max_{y \in N(x)} u(y) + \min_{y \in N(x)} u(y)\right) - u(x) \\
u(x) &= \boxed{\frac{1}{2}\left(\max_{y \in N(x)} u(y) + \min_{y \in N(x)} u(y)\right)}
\end{aligned}
$$

- This condition holds everywhere in a graph but at boundary nodes, where values are prescribed and fixed.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∞-Harmonic Functions on SG

- As is standard, let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be the sequence of graph approximations to SG.
- V_i is the vertex set of Γ_i.
- For fixed boundary data, let $u_n : \Gamma_n \to \mathbb{R}$ be the (discrete) ∞-harmonic function on Γ_n (with graph ∞-Laplacian $\Delta_\infty^{(n)}$).
- Define $u : SG \to \mathbb{R}$ to be ∞-harmonic on SG if

$$u(x) = \lim_{n\to\infty} u_n(x), \quad \Delta_\infty^{(n)} u_n = 0$$

for all n such that $x \in \Gamma_n$.

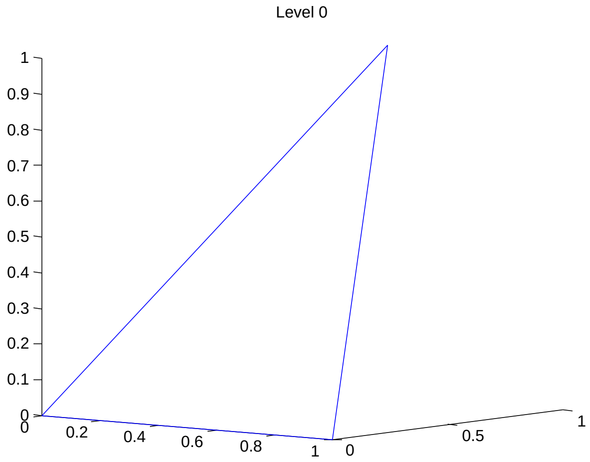
Existence of ∞-Harmonic functions

- The goal of this talk is to develop the machinery necessary to prove that the limit defined in the previous slide exists.
- In particular, we will show that, for any boundary data, there exists an N such that

$$u_n(x)|_{V_1} = u_m(x)|_{V_1}$$

  for all $m, n > N$.
- This result implies that the same is true for any $V_j, j \geq 1$.

Examples: The Symmetric Case



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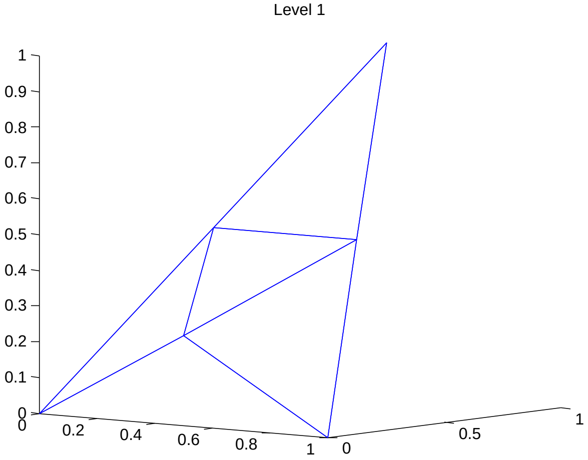

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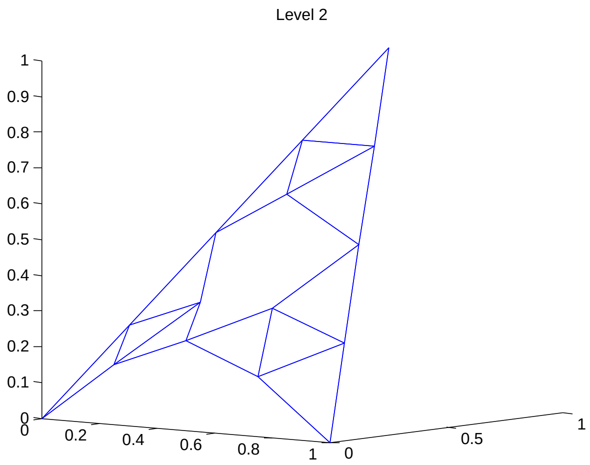

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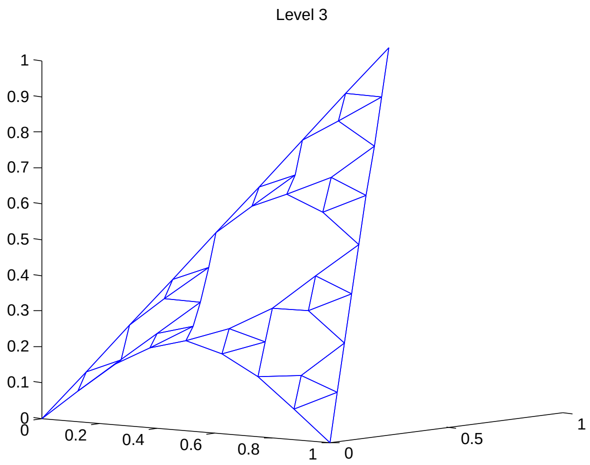

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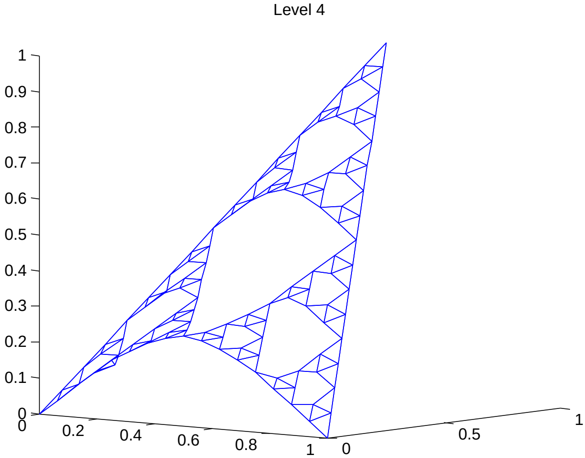

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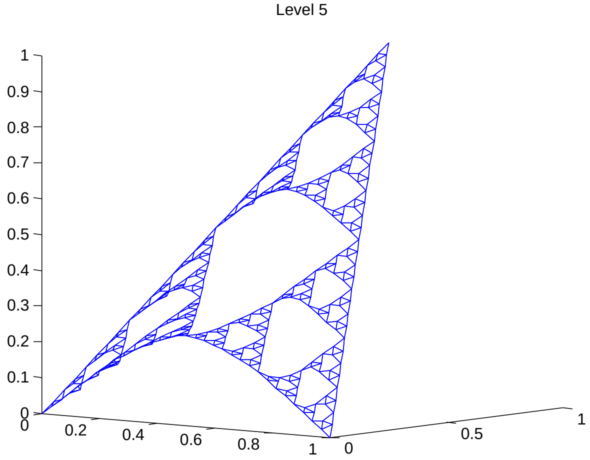

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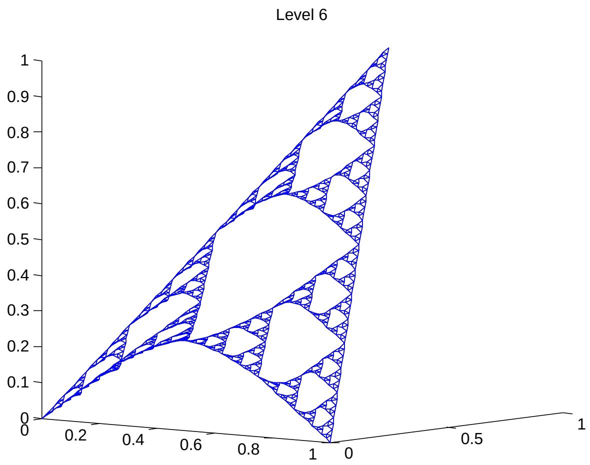
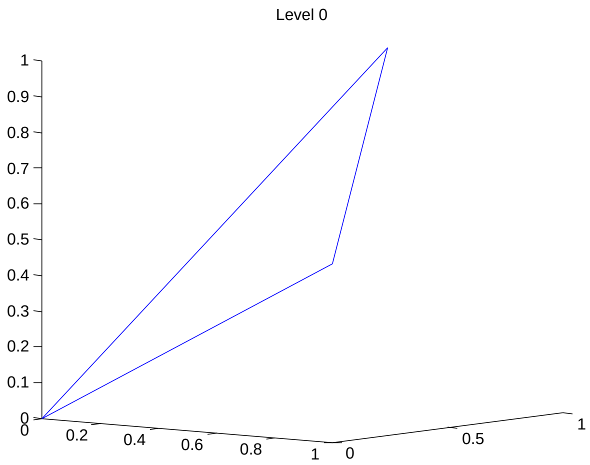


Examples: The Skew-Symmetric Case



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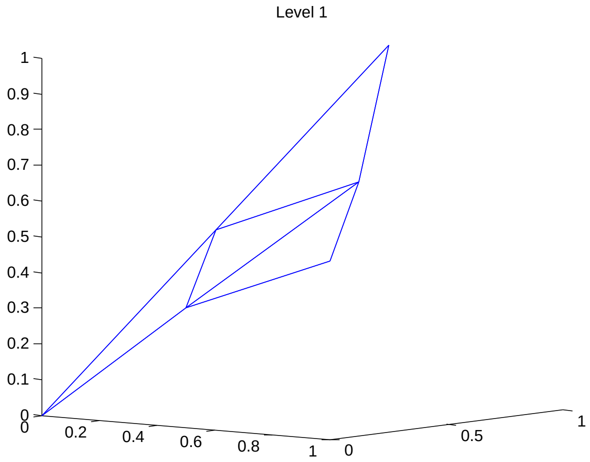

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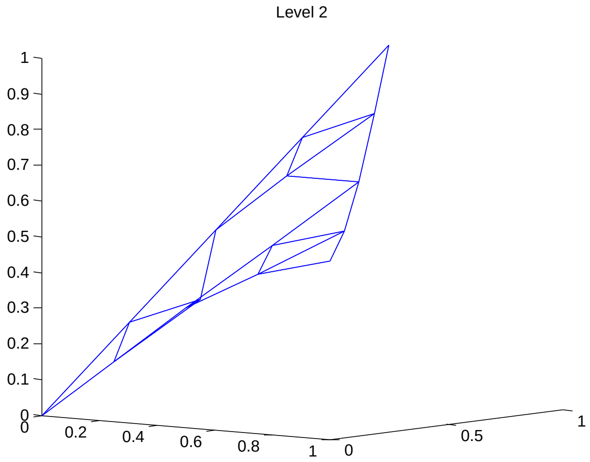
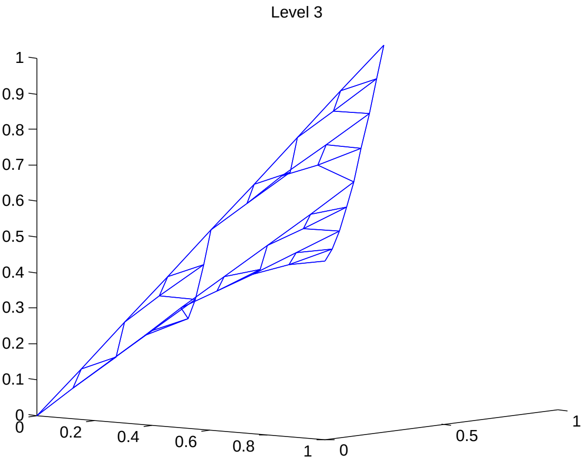


Examples: The Skew-Symmetric Case



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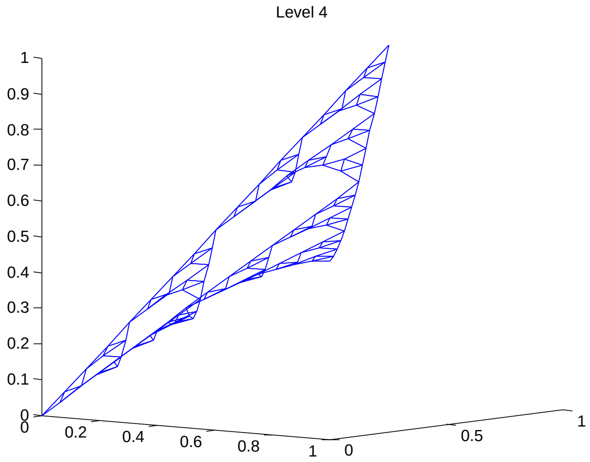

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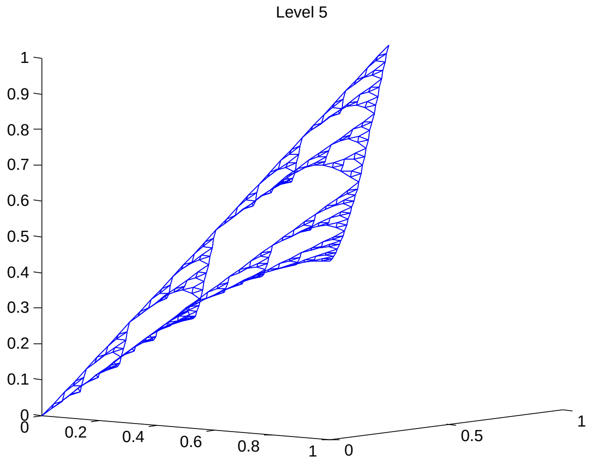

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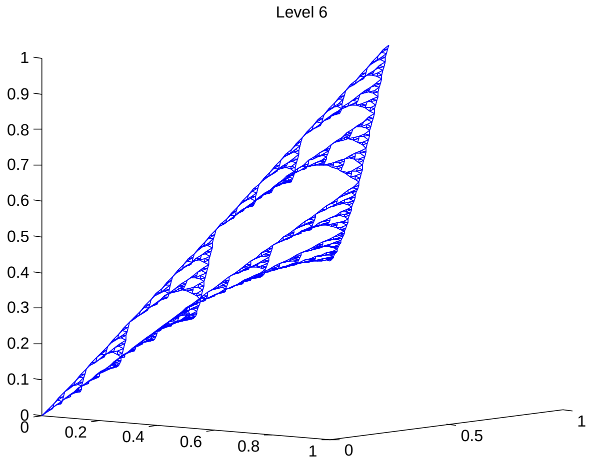


The Importance Of Eccentricity

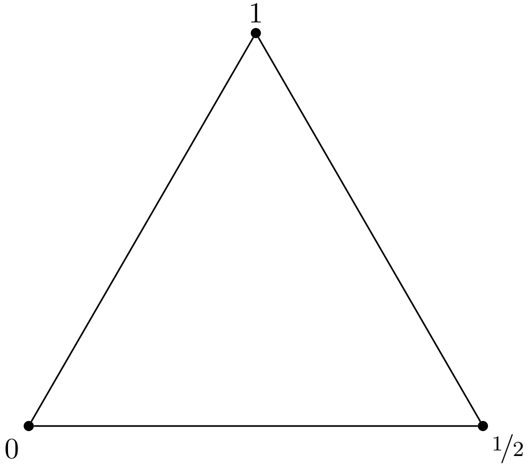
- ∞-harmonic functions (on Γ_n or SG) are not additive, but they do have the following properties:

$$\text{If } \Delta_\infty u = 0, \text{ then } \Delta_\infty (cu) = 0 \ \forall c \in \mathbb{R}\,.$$

$$\text{If } \Delta_\infty u = 0, \text{ then } \Delta_\infty (u - c) = 0 \ \forall c \in \mathbb{R}\,.$$

- These transformations let us reduce the 3-dimensional space of ∞-harmonic functions on SG to a 1-dimensional space with boundary data $(0, e, 1)$ for $e \in [0, 1/2]$.
- The eccentricity of Γ_0 determines how the V_1 values change on refinement. Numerical evidents suggests the following:
- If $e \geq 1/3$, $u_n|_{V_1} = u_1|_{V_1}$ for all $n \geq 1$.
- As $e \to 0$, the level of refinement at which $u_n|_{V_1}$ stops changing increases.

An Example Which Does Not Change



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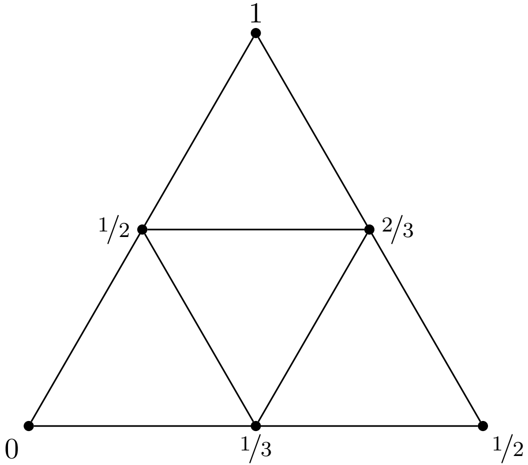

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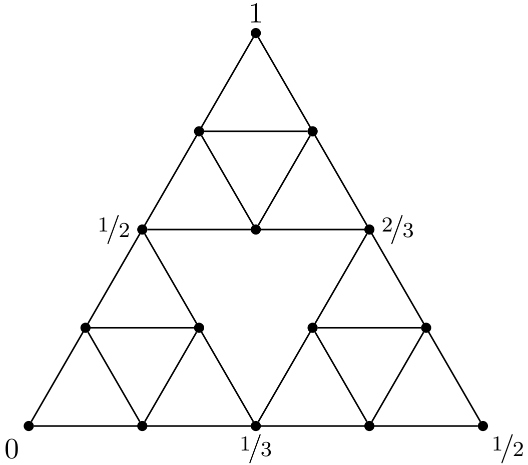
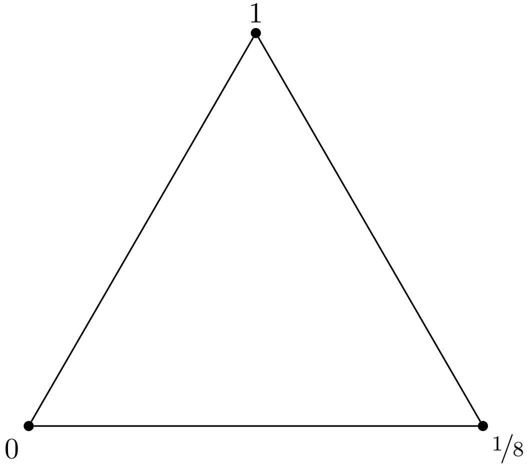


An Example Which *Does* Change



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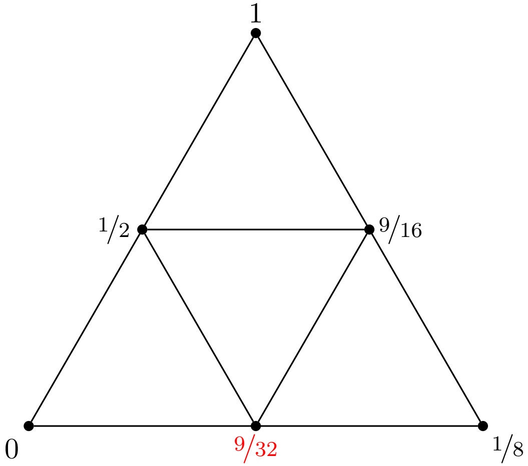

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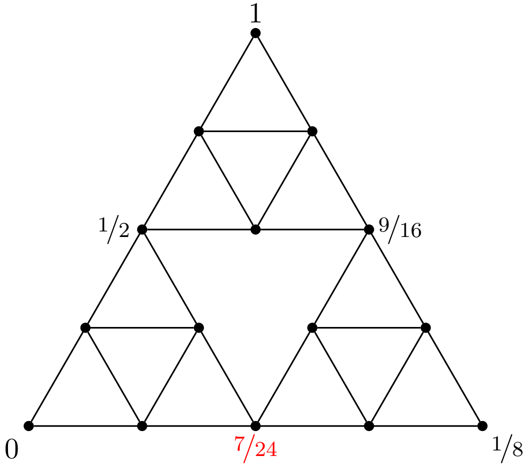


The Point of this Talk

- We use the concept of eccentricity to state the main theorem to be proved:

  Theorem 1: For any $e \in [0, 1/2]$, there exists a constant N depending on e such that $u_n|_{V_1} = u_m|_{V_1}$ for all $n, m \geq N$.

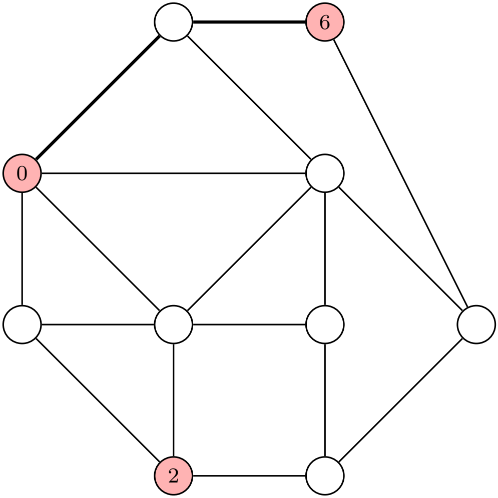
- Proving this requires a better understanding of the relations between successive graph approximations to SG.
- The behavior of the **Lazarus algorithm** with regards to this graph refinement is of primary importance.

The Lazarus Algorithm

- The Lazarus algorithm provides a slow but non-iterative procedure for computing ∞-harmonic functions on graphs.
- For a graph G as before and a set of accepted nodes A, define a path $x_1 - x_2 - \ldots - x_n$ to be **admissible** if $x_1, x_n \in A$ and $x_i \notin A$ for all $i \in [2, n-1]$.
- For an admissible path with n nodes and given values $f(\mathbf{x})$ for $\mathbf{x} \in A$, define the **slope** of the path to be $(f(\mathbf{x}_n) - f(\mathbf{x}_1))/n$

Given this, the Lazarus algorithm proceeds as follows.

1. In our graph G, with nodes X and boundary nodes B, assume there is some boundary data $q(\mathbf{x})$ for $\mathbf{x} \in B$. Then, initialize $A := B$.
2. For each pair of nodes $\mathbf{x}_i, \mathbf{x}_j \in A$, compute the slope of the shortest admissible path from $\mathbf{x}_i$ to $\mathbf{x}_j$, if one exists.
3. Along the path with highest slope computed in 2, our ∞-harmonic function u is linear. Fill in the new values along this path, and add the unaccepted nodes of the path to A.
4. If any nodes remain, return to 2.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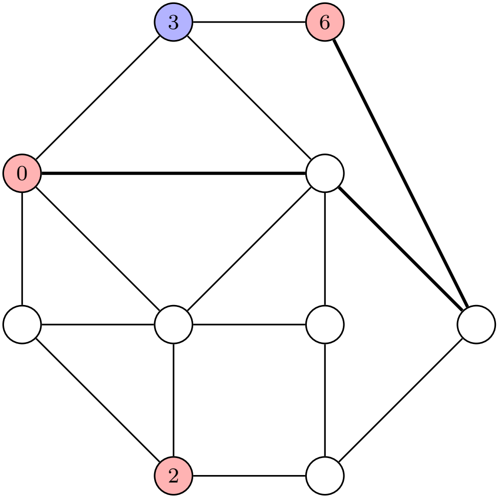
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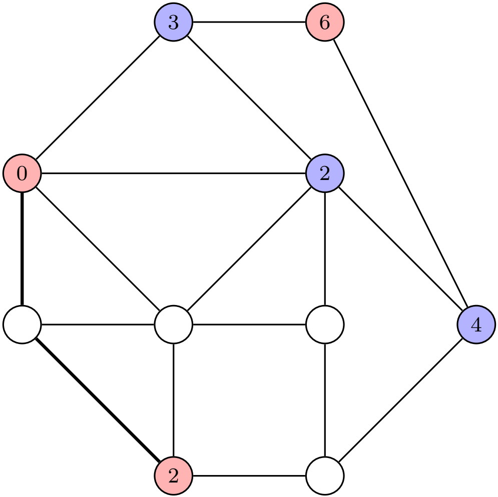
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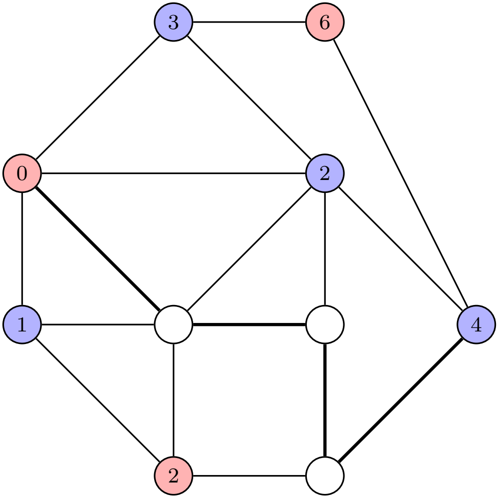
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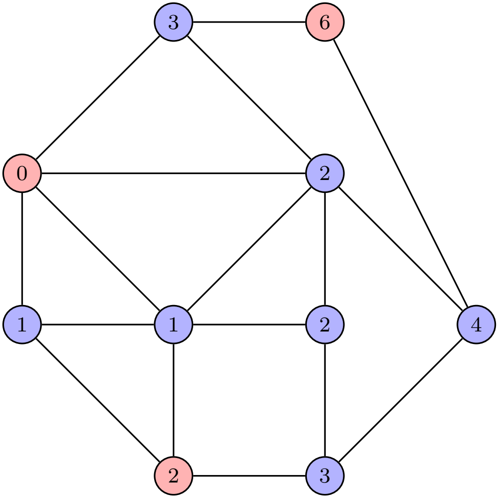
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Lazarus Paths and Graph Refinements

- Understanding how Lazarus paths change between consecutive graph approximations to SG is key to proving convergence of ∞-harmonic functions.
- G_1, G_2 graphs with vertex sets V_1, V_2 such that $V_1 \subset V_2$. Assume metrics d_1, d_2.
- For every $x, y \in V_1$ such that $y \in N_1(x)$, the shortest path connecting x and y in G_2 contains no other elements of V_1.
- A path $\rho_2 \subseteq G_2$ is a **refinement** of $\rho_1 \subseteq G_1$ if $x \in \rho_1 \Rightarrow x \in \rho_2$, the terminal nodes of the paths are the same, and the ρ_2 subpath connecting each successive x_i, x_{i+1} is a (d_2) geodesic.

Lazarus Path Refinements on $\{\Gamma_n\}$

- A crucial proposition, due to A. Vladimirsky, is the following:

  Proposition 1: $\ell^k(\Gamma_{m+1})$ is a refinement of $\ell^k(\Gamma_m)$ for all $k > 0$ and all $m \geq k$.

- With this Proposition, it is easy to show that Theorem 1 is true for $e \in [1/3, 1/2]$ with $N = 2$.

The $<$ and $\prec$ Graph Relations

- If each path ρ_1 in G_1 has a refinement in G_2, let us say that

$$G_1 < G_2.$$

- B_1^0 is the boundary set for the Lazarus algorithm on G_1, B_2^0 for G_2.
- If $B_1^0 = B_2^0$ and $d_1(x,y) = d_2(x,y)$ for all $x, y \in V_1$, let us say that

$$(G_1, B_1^0) \prec (G_2, B_2^0)$$

 ($G_1 \prec G_2$ when there is no danger of ambiguity).
- Then, *the first Lazarus path $\ell^1(G_2)$ on G_2 is a refinement of the first Lazarus path $\ell^1(G_1)$ on G_1.*
- This implies that the ∞-harmonic function values are equal on the V_1 vertices.

Problems with $\prec$!

- The boundary sets for the second Lazarus path are $B_1^1 = B_1^0 \cup \ell^1(G_1)$ and $B_2^1 = B_2^0 \cup \ell^1(G_2)$.
- $B_1^1 \neq B_2^1$, so
$$(G_1, B_1^1) \not\prec (G_2, B_2^1).$$

- This machinery cannot tell us anything about the equality of any Lazarus paths beyond the first.
- A new graph relation is needed.

The $\lhd$ Graph Relation

- The conditions prescribed by the $\prec$ relation are stronger than necessary for showing that Lazarus paths are equal.
- For a fixed graph G and boundary set B, define

$$\mathcal{B} = \{x \in B \mid \text{for some bdry data, } x \text{ is a terminal node of } \ell^1(G)\}.$$

- In order for $\mathcal{B}$ to be well-defined, assume a fixed tie-breaking procedure.
- For $G_1 < G_2$ as before with boundary sets B_1, B_2 and $\mathcal{B}_1 \subseteq B_1$, $\mathcal{B}_2 \subseteq B_2$, say

$$(G_1, B_1) \lhd (G_2, B_2)$$

  if $\mathcal{B}_1 = \mathcal{B}_2$ and $d_1(x, y) = d_2(x, y)$ for all $x, y \in \mathcal{B}_1$.
- With this relation, it can be shown that $\ell^1(G_2)$ is a refinement of $\ell^1(G_1)$.

A Reinterpretation of the Lazarus Algorithm

- on a graph G^0 with vertices V^0, edges E^0, and boundary set B^0, the first Lazarus path is some $\ell^1(G^0) = (x_1, \ldots, x_n), x_i \in V^0$.
- Create a new graph G^1 (the **Lazarus subgraph** of G^0) with vertex set $V^1 = V^0$, boundary set $B^1 = B^0 \cup \ell^1(G^0)$, and edge set $E^1 = E^0 - \ell^1(G^0)$.
- Then,
$$\ell^1(G^1) = \ell^2(G^0).$$
- We can create a G^2 analogously from G^1 such that
$$\ell^1(G^2) = \ell^3(G^0),$$
and so on.
- From this perspective, it is enough to talk about the Lazarus path $\ell(G^i)$ on a graph, since each has only one.

$\lhd$ and $\{\Gamma_n\}$

- Define $\Gamma_j^0 := \Gamma_j$ for $j \in \mathbb{N}_0$, and Γ_j^i to be the Lazarus subgraph of Γ_j^{i-1} for $i \geq 1$.
- *A priori* we know the following to be true:

$$\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & & \Gamma_2^1 & & \Gamma_3^1 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & & \Gamma_3^2 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & & \dots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

$\lhd$ and $\{\Gamma_n\}$

- To prove Proposition 1, it suffices to show that the following relations hold true:

$$
\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & \lhd & \Gamma_2^1 & \lhd & \Gamma_3^1 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & \lhd & \Gamma_3^2 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & \lhd & \dots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & \lhd & \ddots
\end{array}
$$

- This is the motivation for

  Proposition 2: Let $\{\Gamma_n\}$ be the standard sequence of graph approximations to SG. Then, $\Gamma_m^j \lhd \Gamma_{m+1}^j$ for all $j \geq 0$ and all $m \geq j$.

A Sketch of a Proof for Proposition 2

- The proof relies on each Γ_m^j having the following two properties for $m \geq j$:
  - P1. $\mathcal{B}_m^j = \mathcal{B}_{j-1}^j$
  - P2. If the shortest path between $x, y \in B_m^j$ in Γ_{m+1}^j is not a refinement of the shortest path in Γ_m^j, then x or y is in $B_m^j - B_{m-1}^j$.
- These properties can be directly verified for Γ_m^0, and then the proof proceeds inductively.
- The proof that Proposition 2 $\Rightarrow$ Proposition 1 is straightforward.

A generalization of Proposition 2

- We were seeking a sequence of Lazarus paths on members of $\{\Gamma_n\}$ which partitioned the graphs into subgraphs isomorphic to members of $\{\Gamma_n\}$.
- It is useful to consider Lazarus paths partitioning graphs into other subgraphs, as well.
- A **regular finite fractafold** (rff) is the quotient space

$$\left(\bigsqcup_{i=1}^{n} SG\right)/\varphi,$$

where φ is a quotient map such that in the ith copy of SG (call it SG_i), φ identifies each boundary point of SG_i with at most one boundary point of at most one SG_j, $i \neq j$, and φ never identifies two different boundary points in SG_i with boundary points in a single SG_j.

Regular Finite Fractafolds

- If $F = \left(\bigsqcup_{i=1}^{n} SG\right)/\varphi$, define

$$F_k = \left(\bigsqcup_{i=1}^{n} \Gamma_k\right)/\varphi.$$

- Using Proposition 2, it can be shown that the analogue of Proposition 2 holds: $F_m^j \triangleleft F_{m+1}^j$ for $j > 0$, $m \geq j$.
- If we start with $\{\Gamma_n\}$ approximations to SG, we can directly show that each Lazarus path partitions the approximation graphs into rff's.

Convergence results on SG

- Using the generalization of Proposition 2, we finally get convergence bounds which allow us to prove Theorem 1.

  Result: $u_n|_{V_1}$ does not change upon refinement for all eccentricities $e > \frac{2}{3^{n-2}-3}$, $n > 4$.

- This proves Theorem 1 for all $e \in (0, 1/2]$. The $e = 0$ case can be taken care of by an argument due to R. Strichartz, completing the proof of Theorem 1!

Conclusion, Future Work

- This gives us a plausible definition for $\Delta_\infty u = 0$, but nothing with nonzero RHS.
- We would like to prove something analogous for functions $u_n : \Gamma_n \to \mathbb{R}$ that are solutions to

$$-c^n \Delta_\infty^{(n)} u_n = f_n,$$

where c is some renormalization constant.