Infinity-harmonic functions on SG

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1 Background

1.1 The infinity Laplacian Δ_∞

In this paper, we seek to explore how the **infinity Laplacian** operator Δ_∞ could be defined on the fractal SG. The infinity Laplacian is defined on $\mathbb{R}^n$ as

$$\Delta_\infty u = |\nabla u|^{-2} \sum_{i,j} u_{x_i} u_{x_i,x_j} u_{x_j},$$

which can be understood informally as "the second derivative of u in the direction of the gradient." It has been studied in connection with absolutely minimizing Lipschitz extensions and game theory [3]. As with the standard Laplacian, the infinity Laplacian has a discrete analog. Given a graph G with node set X, the **graph infinity Laplacian** is defined as

$$\Delta_\infty u(x) = \frac{1}{2}\left(\sup_{y\in N(x)} u(y) + \inf_{y\in N(x)} u(y)\right) - u(x), \tag{1}$$

where $N(x)$ is the neighbor set of x in G. Note how this compares with the definition of the discrete Laplacian,

$$\Delta u(x) = \frac{1}{|N(x)|} \sum_{y\in N(x)} u(y) - u(x).$$

1.2 The Lazarus algorithm

If u is ∞-harmonic (satisfies $\Delta_\infty u = 0$), then we have that

$$u(x) = \frac{1}{2}\left(\sup_{y\in N(x)} u(y) + \inf_{y\in N(x)} u(y)\right). \tag{2}$$

Given a certain subset $B \subseteq X$, defined to be the boundary set ∂G of G, a natural question to consider is the existence of a function u which takes prescribed values on ∂G and satisfies $\Delta_\infty u = f$ on $X - \partial G$. Such a function can be shown to exist in great generality, but for our purposes it suffices to know that such a u exists as long as $f > 0$ or $f = 0$ everywhere and $u|_B$ is finite [3]. An iterative scheme for computing such a function on a given graph with prescribed boundary values was developed by Adam Oberman in [2]. This is useful for numerical work, but a second interesting algorithm for computing u exists, due to Lazarus [1]. Despite computing the exact values that u takes on X in finite time, for numerical computations Oberman's scheme is more efficient. However, for our theoretical work, understanding the Lazarus algorithm is useful.

Let G be a finite graph with boundary set B. In a slight abuse of notation, we will consider paths ρ through G both as a subset of $\rho \subseteq G$ with vertex set

$\{x_1, \ldots, x_n\}$, and as the sequence $(x_1, \ldots, x_n)$ itself. The *endpoints* of ρ then are $\partial\rho = \{x_1, x_2\}$. Which perspective is being used should be clear from context.

For $x, y \in V$, define $d_B(x, y)$ to be the number of edges in the shortest path connecting x and y through $G - B$ ($d(x, y)$ when clear from context), and let $\rho_B(x, y) \subseteq X$ be this shortest path. For each pair $b, c \in B$, compute the **slope** of $\rho_B(b, c)$ as

$$S(\rho_B(b, c)) = \frac{|u(b) - u(c)|}{d_B(b, c)}. \tag{3}$$

Moreover, as it will sometimes be useful, for $x, y \in V$ define

$$S(x, y) = S(\rho_B(x, y)).$$

For the pair $b^*, c^* \in B$ which maximizes (3), Lazarus shows that along $\rho_B(b^*, c^*)$, u is linear [1]. So if $d(b^*, c^*) = \ell$, for each $x \in \rho_B(b^*, c^*)$ we find that

$$u(x) = \frac{d(x, b^*)}{\ell} u(c^*) + \frac{d(x, c^*)}{\ell} u(b^*).$$

Then, since this is computed correctly, the next step of the Lazarus algorithm is the same as the first, with an expanded boundary set $B_1 = B \cup \rho(x, y)$. Along the path between elements of B_1 through $G - B_1$ with greatest slope, u is again linear. We interpolate again to find the values of u, and repeat this procedure until all nodes are assigned values. The resulting u satisfies the equation $\Delta_\infty u = 0$, where Δ_∞ is the graph infinity Laplacian.

Let us call the path $\ell_i(G)$ of maximum slope during the i^{th} iteration of the Lazarus algorithm the $i^{\textbf{th}}$ **Lazarus path** through G. One property of these Lazarus paths which will be useful to us is that slopes are nonincreasing. If $i > j$, $S(\ell_i) \leq S(\ell_j)$. The proof can be found in [1].

Note that there is an ambiguity in this definition of Lazarus paths. Namely, it may occur that for two pairs $x, y, x', y' \in B$, $S(x, y) = S(x', y')$ is the greatest slope. There is then a question of which path to choose. It is easy to see that the choice of path is irrelevant, the u values we get via the Lazarus algorithm will be the same. Where this issue arises in the work here, we resolve the problem on an ad-hoc basis - since it does not matter which path is chosen first, we choose as is convenient.

Actually, we will use a slight variant on the Lazarus algorithm as described here, with a simple class of weighted graph metrics. Instead of the weight of each edge being 1 as above, each edge will have weight c for some $c > 0$, but for any particular graph, each edge will be of the same weight. The results Lazarus proved using the link-counting metric still hold here, since these new metrics differ from the link-counting metric by a constant multiple.

1.3 The construction of SG

The fractal SG and its analytic properties have been studied extensively over the past 30 years, with Strichartz's book [4] a good overview of basic results. One method for constructing SG, described in [4], is as a limit of graph approximations. This series of approximating graphs can be constructed entirely without reference to an ambient space, but it is cleaner and intuitively easier to understand if we consider the graphs as embedded in $\mathbb{R}^2$. Start with a graph Γ_0 with node set $V_0 = \{x_0, x_1, x_2\}$ as vertices of a nondegenerate triangle.

The boundary of SG (and of each Γ_n) is defined to be

$$\partial SG \ (\partial\, \Gamma_n) := V_0.$$

For each $x_i \in \partial SG$, we define a contraction mapping $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F_i(x) = \frac{1}{2}(x - x_i) + x_i.$$

So, we can define vertex sets inductively, starting with V_0 and then

$$V_n = \bigcup_i F_i(V_{n-1}).$$

Note that the contracting mappings can send distinct elements of V_k to the same element of V_{k+1}. For instance, $F_0(x_2) = F_2(x_0)$. Then starting with Γ_0, we inductively define the $n^{\,th}$*-level approximation graph* Γ_n as

$$\Gamma_n = \bigcup_i F_i(\Gamma_{n-1}),$$

identifying together nodes of the graphs which map to the same node in V_n. $\{\Gamma_n\}$ approximates SG in that SG is the inverse limit of the sequence $\Gamma_1, \Gamma_2, \ldots$, ordered by graph inclusion. The traditional analytic machinery, starting with the Laplacian, is constructed on SG making use of a relationship between the object on the fractal and discrete objects on $\{\Gamma_n\}$. To understand the infinity Laplacian, we will not require the Laplacian construction. However, we will still proceed by establishing a relationship between the discrete infinity Laplacian and any plausible definition on SG. The first thing we set out to do is defining what could be called Δ_∞*-harmonic* functions on SG: functions $u : SG \to \mathbb{R}$ that are a limit in some sense of functions $u_n : \Gamma_n \to \mathbb{R}$ which satisfy ${\Delta_\infty}^{(n)}\, u_n = 0$.

2 Δ_∞-harmonic functions on SG

The harmonic functions of the standard Laplacian on SG can be constructed either via a weak formulation, or pointwise as a limit of values on the approximation graphs $\{\Gamma_n\}$. The two constructions are known to be equivalent [4].

To define ∞-harmonic functions, we pursue a pointwise formulation as a limit of graph approximations. As we prove, this converges and therefore defines the values of an ∞-harmonic function on the elements of SG that are also elements of Γ_n for some n, $\bigcup_n \Gamma_n$. Continuity of u on any Cauchy sequence in $\bigcup_n \Gamma_n$ can be proven, and so we can define the value of $u(x)$ on $SG - \bigcup \Gamma_n$ as

$$u(x) = \lim_{n \to \infty} u(x_n),$$

where $\{x_n\} \subset \bigcup \Gamma_n$ converges to x. [**Note**]: **the continuity proof is almost done being written up formally. Essentially, if two Cauchy sequences converging to some $x \in SG$ approach different u values as they converge to x, paths through SG connecting elements of the two Cauchy sequences have unboundedly large slopes, and from this we can derive a contradiction.**

Definition 1. *A function $u : SG \to \mathbb{R}$ is **infinity-harmonic** if*

$$u(x) := \lim_{n \to \infty} u_n(x) : \Gamma_n \to \mathbb{R} \mid \Delta_\infty u_n = 0.$$

we demonstrate that this limit exists by showing that for any fixed level graph Γ_k,

$$|u_n|_{\Gamma_k} - u_{n-1}(\Gamma_k)| = 0$$

for all n greater than some finite N, and that therefore a limit value exists for u. To do this requires examining the relationship between sequences of harmonic functions with the same boundary conditions on successive graphs $\Gamma_1, \Gamma_2, \ldots$. In order to simplify this, a trick can be used to reduce the dimensionality of the space of ∞-harmonic functions that we need to consider.

2.1 Eccentricity

Since an Δ_∞-harmonic function on Γ_n is uniquely determined by its boundary values, there is a three-dimensional space of such functions on any Γ_n. However, taking into account two properties of infinity-harmonic functions, we can essentially reduce this to a one-dimensional space. We can verify directly from the definition of the graph infinity Laplacian that, if u defined on G satisfies $\Delta_\infty u = 0$ on $G \setminus B$, then v defined by the affine transformation

$$v(x) = u(x) - K$$

for some constant K satisfies $\Delta_\infty v = 0$ on $G \setminus B$ as well. If we define w by the scaling

$$w = u(x)/K,$$

this also satisfies $\Delta_\infty w = 0$ on $G \setminus B$. So, consider an infinity-harmonic function u on Γ_n with $u(x_0) = a$, $u(x_1) = b$, and $u(x_2) = c$. WLOG we can assume

$a \le b \le c$. If $a = b = c$, u is trivially just a constant function and equivalent to the 0 function by an affine transformation. Otherwise, we can construct

$$v(x) = \frac{1}{c-a}\,(u(x) - a)$$

to obtain an infinity-harmonic function v with boundary data $v(x_0) = 0$, $v(x_1) = e$, and $v(x_2) = 1$ for some $e \in [0, 1]$. Thus, if we can determine the behavior of v on the interior of Γ_n, this entirely determines the behavior of u as well, since

$$u(x) = (c - a)v(x) + a.$$

So, we can associate each ∞-harmonic function, up to a scaled affine transformation, with an e value taking values in $[0, 1]$. This can be reduced further to taking values in $[0, 1/2]$. If u has boundary data with eccentricity $e > 1/2$, We can construct $v(x) = 1-u(x)$, also ∞-harmonic, with eccentricity $1-e \in [0, 1/2]$. Thus, every ∞-harmonic function has associated with it a parameter $e \in [0, 1/2]$ known as the **eccentricity** of the function.

2.2 Sequences of ∞-harmonic functions

As the preceding subsection makes clear, to determine the behavior of all ∞-harmonic functions on any Γ_n, it is enough to consider functions with boundary data $(0, 1, e)$, $e \in [0, 1/2]$. We started by examining the behavior computationally. The number of levels of refinement (i.e. passing from Γ_n to Γ_{n+1}) needed before the values of u_n on Γ_1 ceases to change upon refinement seems to increase as $e \to 0$. This then is what we would like to prove. Note that this immediately would imply that $u_n|_{\Gamma_k}$ is fixed on Γ_n for all n greater than some N depending on e and k. This gives us a stronger-than-pointwise convergence of the sequence $\{u_k\}$ to a limiting object on u, though uniform convergence is not always possible, as a known counterexample shows.

Our first theorem for infinity-harmonic functions then is

Theorem 1. *For all $k \in \mathbb{N}$ and any boundary data, $u_n|_{\Gamma_k} = u_{n+1}|_{\Gamma_k}$ for all n greater than some $N \in \mathbb{N}$, depending on e and k.*

Proving this requires some build-up of graph material.

3 Useful definitions and results

3.1 Graph refinements

Consider graphs G_1, G_2 with vertex sets V_1, V_2 respectively such that $V_1 \subset V_2$ and $\partial G_1 = \partial G_2 := B$. Moreover, let $x \in V_1$, and $y \in N_1(x)$, the (G_1) neighbor set of x. For such points, there exists a path connecting x and y in G_2 which does not contain any other element of V_1. For pairs of graphs satisfying these properties, let d_1, d_2 be metrics on G_1 and G_2 respectively. If the choice of metric d_j is clear from context, we will relegate this to an understood distinction and refer to d. Then, a path $\rho_2 \subseteq G_2$ is a **refinement** of $\rho_1 \subseteq G_1$ if $\partial\rho_1 = \partial\rho_2$, each $x \in \rho_1$ is also in ρ_2, and if for every $x_i, x_{i+1} \in \rho_1$,

$$\rho_{G_2}(x_i, x_{i+1}) \cap V_1 = \{x_i, x_{i+1}\}.$$

If each path $\rho_i \subset G_1$ has such a refinement in G_2, let us say that

$$G_1 < G_2.$$

Moreover, let us also assume that for each $x, y \in V_1$, $d_1(x, y) = d_2(x, y)$. Let us denote two graphs related in this way as

$$G_1 \prec G_2.$$

It is easy to see that this relation forms a partial order on graphs.

Using this language, we can state the first proposition we seek to prove.

Proposition 1. *Let $\{\Gamma_n\}, n = 1, \ldots$ be the standard sequence of graph approximations to SG. Let $u_n : \Gamma_n \to \mathbb{R}$ be Δ_∞-harmonic such that $u_n|_{\partial\Gamma_n} = u_m|_{\partial\Gamma_m}$ for all m, n. Then, $\ell_k(\Gamma_m)$ is a refinement of $\ell_k(\Gamma_k)$ for all $m \geq k$.*

Demonstrating that this holds will require looking at relationships between Δ_∞-harmonic functions on graphs likes G_1 and G_2 above.

3.2 Δ_∞-harmonic functions on pairs of graphs

Consider the relation between the values of infinity-harmonic functions defined on these graphs, $u_1 : G_1 \to \mathbb{R}$ and $u_2 : G_2 \to \mathbb{R}$, with $G_1 \prec G_2$ and $u_1|_B = u_2|_B$. We want to consider which path is first accepted by the Lazarus algorithm, both through G_1 and through G_2.

A selection of path depends on calculated slopes, so let us adopt the following notation in addition to the previous: define the slope $S(x)$ to be $S(x_1, x_2)$ for the x_1, x_2 forming the terminal nodes of the Lazarus path from which $u(x)$ is computed.

Lazarus says that for $x^*, y^* \in B$ such that $S(x^*, y^*) \geq S(x, y)$ for all $x, y \in B$, $u|_{\rho(x^*, y^*)}$ is linear. Specifically, for the generalized Lazarus algorithm which takes into account constant ($\neq 1$) link weights, at each point x_i on the path $(x^* = x_1, x_2, \ldots, x_n = y^*)$,

$$u(x_i) = \frac{d(x_i, x_n)}{d(x_1, x_n)} u(x_1) + \frac{d(x_1, x_i)}{d(x_1, x_n)} u(x_n).$$

So, we see that the first path accepted by Lazarus in G_1 is the same as the first path accepted by Lazarus in G_2: $u_1 = u_2$ on B, and $d(x, y)$ is the same on both G_1 and G_2 for, so $S(x, y)$ is equal for $x, y \in B$ in both G_1 and G_2. So, for each $x \in V_1$ along these paths,

$$u_1(x) = u_2(x).$$

The Lazarus algorithm is of course a multi-step process: We now repeat the procedure on G_1 and G_2 with expanded boundary sets, but

$$B \cup \ell_1(G_1) \neq B \cup \ell_1(G_2).$$

So, we cannot apply the same reasoning as before to conclude that $\ell_2(G_1) = \ell_2(G_2)$.

A reinterpretation of the Lazarus algorithm

We can reinterpret this iterative procedure though: For a graph G with boundary set B, some path $(x_1, \ldots, x_n)$ is first accepted by the Lazarus algorithm We compute u along $(x_1, \ldots, x_n)$. With this information, we can create a new graph G^1 with the same vertices as G and the same link set with the following exception: for every link in the path $(x_1, \ldots, x_n) \subset G$, we delete the corresponding link in G^1, and construct our boundary set B^1 in G^1 as $B \cup \{x_1, x_2, \ldots, x_n\}$. The second path accepted under Lazarus in G then is the same as the first path accepted in G^1, and so we can create a graph G^2 analogously from G^1 and repeat this procedure to find the third accepted path in G, and so on. In this interpretation, we need not talk about multiple paths generated by the Lazarus algorithm, as at each G^i, a single path is computed, which induces a new graph G^{i+1}. Therefore, let us talk about "*the* Lazarus path" $\ell(G^i)$ in reference to the first path computed by the Lazarus algorithm on some G^i.

The preceding paragraphs highlighted interplays between different graphs with respect to the Lazarus algorithm in two different ways, individually. Now, we want to consider a scenario in which both are occurring simultaneously, and try to examine how they relate. Let $\{G_i\}$ be a (possibly infinite) sequence of graphs such that $G_i \prec G_{i+1}$ for all i. For each G_j, let G_j^k be the kth subgraph induced by the Lazarus algorithm, with $G_j = G_j^0$, the entire graph before any links have been removed by Lazarus. The standard sequence of graph approximations $\{\Gamma_n\}$ to SG is a sequence of this type. It would be nice if some similar relation existed among $\{\Gamma_j^i\}$ for some j and i. Specifically, to prove Proposition 1, we would like to show that if $i \leq j$,

$$\Gamma_j^i \prec \Gamma_{j+1}^i.$$

We can show that this does not hold, though that is omitted here. However, we can try to construct a new relation $\lhd$ such that $\Gamma_j^i \lhd \Gamma_{j+1}^i$, and $\ell(\Gamma_{j+1}^i)$ is a refinement of $\ell(\Gamma_j^i)$, capturing the crucial properties of the $\prec$ relation that allowed the same to hold there. For a sequence of graphs $\{G_n\}$ and its Lazarus-induced subgraphs $\{G_n^i\}, i = 0, 1, \ldots$, assume $B = B^0$ is the boundary whose data is prescribed prior to the calculation of any Lazarus paths. For SG, this would be V_0, the vertex set of Γ_0. Which Lazarus path is calculated on each G_n^i is completely determined by the prescribed values on B and our procedure for breaking ties, should there be more than one potential Lazarus path on a G_n^i, since all other elements of B_n^i are given values by linear interpolation. So, for a given (fixed) tie-breaking procedure (such as the ad-hoc one we will use), let $\mathcal{B}_n^i \subset B_n^i$ be the set of all nodes in B^i ever needed to compute a Lazarus path on G^i for all choices of u values on B. $G_n^i \prec G_{n+1}^i$ implies graph inclusion and that the set of all shortest paths starting and ending in the vertex set of G_i did not shorten upon inclusion into G_{n+1}^i. However, in proving that this implied that the Lazarus path on G_n^i is the same as the Lazarus path on G_{n+1}^i, we only really need the fact that $\mathcal{B}_n^i = \mathcal{B}_{n+1}^i$*, and* $d_n(x,y) = d_{n+1}(x,y)$ *for all* $x, y \in \mathcal{B}_n^i$. Let us say that

$$G_n^i \triangleleft G_{n+1}^i$$

if $G_n^i < G_{n+1}^i$, $\mathcal{B}_n^i = \mathcal{B}_{n+1}^i$, and $d_n(x,y) = d_{n+1}(x,y)$ for all $x, y \in \mathcal{B}_n^i$. It is then straightforward to modify the proof that $G \prec H$ implies G and H have the same Lazarus path (i.e. $\ell(H)$ is a refinement of $\ell(G)$) to show that $G \triangleleft H$ implies the same thing. It should also be clear that

$$G_n^i \prec G_{n+1}^i \Rightarrow G_n^i \triangleleft G_{n+1}^i.$$

Then, we can rephrase Proposition 1 as follows:

Proposition 2. *Let* $\{\Gamma_n\}, n = 1, \ldots$ *be the standard sequence of graph approximations to SG. For all* $j \geq 0$ *and all* $m \geq j$, $\Gamma_m^j \triangleleft \Gamma_{m+1}^j$.

In a slightly more graphical form, we are saying the following: We know the following relations hold true *a priori*:

$$\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \cdots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & & \Gamma_2^1 & & \Gamma_3^1 & & \cdots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & & \Gamma_3^2 & & \cdots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & & \cdots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

What we would like to prove, via Proposition 2, is that the diagram in the following figure is also correct.

$$
\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & \triangleleft & \Gamma_2^1 & \triangleleft & \Gamma_3^1 & \triangleleft & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & \triangleleft & \Gamma_3^2 & \triangleleft & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & \triangleleft & \dots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & \triangleleft & \ddots
\end{array}
$$

4 A proof of Proposition 2

For the rest of the paper, let V_m be the set of vertices in Γ_m, which is also the set of vertices in Γ_m^j for all j. Let B_m^j be the set of boundary nodes in Γ_m along with all nodes in the first j accepted paths on Γ_m under the Lazarus algorithm, and $\mathcal{B}_m^j \subset B_m^j$ as before. Moreover, for convenience, let us identify SG with its standard embedding into $\mathbb{R}^2$, so that we can use the restriction of the Euclidean metric on $\mathbb{R}^2$ onto SG as a metric d on this space, and likewise create a family of metrics d_m^j for the graph approximations in the same way. This can be turned into an embedding-independent family of metrics on $\{\Gamma_n\}$, but the details are unnecessary for us here.

A proof of Proposition 2 depends on each graph approximation Γ_m^j having the following two properties for $m \geq j$:

Lemma 1. *Let $\{\Gamma_n\}$ be the graph approximations to SG. Then:*

P1. $\mathcal{B}_m^j = B_{j-1}^j$

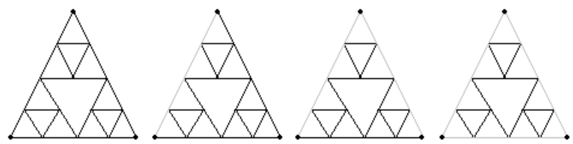
P2. If the shortest path between $x, y \in B_m^j$ in Γ_{m+1}^i is not a refinement of the shortest path between $x, y \in B_m^j$ in Γ_m, then x or y is in $B_m^j \setminus B_{m-1}^j$.

It is easy to see that $\Gamma_0^0 \prec \Gamma_1^0 \prec \dots$, that this implies $\Gamma_0^0 \triangleleft \Gamma_1^0 \triangleleft \dots$, and that P1 and P2 hold for these graphs. We want to use this inductively to prove that, if all of these properties (besides the $\prec$ relation) hold for all Γ_m^i, $m \geq i$ for all i less than some j, then P1 and P2 hold for Γ_j^j as well. Then, with this established, we prove inductively that $\Gamma_n^j \triangleleft \Gamma_{n+1}^j$ for all $n \geq j$.

4.1 k-cells

Before doing so however, let us introduce a bit more terminology. Define a **k-cell** $C_k \subseteq \Gamma_m$, $m > k$ to be a cell in Γ_m with at least one boundary point in $V_k - V_{k-1}$, with boundary set $\{c_1, c_2, c_3\}$. It is easy to show that for such a

k-cell, in fact two elements of the boundary set are in $V_k - V_{k-1}$. Then, for any i, consider the subgraph of Γ_m^i with the same vertex set as C_k, call this C_k^i. Let us say that C_k^i is **type 0** if C_k had no links removed by the first i iterations of the Lazarus algorithm on Γ_m, so that each link in C_k is contained in C_k^i. C_k^i is **type 1** if exactly one of the straight paths connecting $c_1 - c_2$, $c_1 - c_3$, or $c_2 - c_3$ has been entirely removed under the first i iterations of the Lazarus algorithm on Γ_m, and no other links have been removed. C_k^i is **type 2** if two such paths have been entirely removed, and no other links have been removed, and **type 3** if all three paths have been removed.



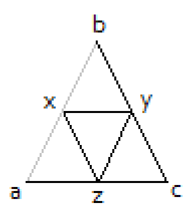
With that in mind, we can prove the following:

Lemma 2. *For all $j > 0$, if $\Gamma_m^i \lhd \Gamma_{m+1}^i$ for all $i < j$ and for all $m \geq i$, and P1 and P2 hold for all such graphs, then P1 and P2 hold for all Γ_m^j, $m \geq j$.*

Proof. For $j = 0$, this same result is clear, so consider some $j > 0$. First, we will show that P1 holds on Γ_j^j, and extend this result to all Γ_m^j, $m \geq j$. Suppose for contradiction that there were a path ρ computed on Γ_j^j whose terminal or initial node lies in $B_j^j \setminus V_{j-1}$ and whose slope is strictly greater than that of any path with terminal and initial nodes in B_{j-1}^j. Since by hypothesis, the Lazarus path on Γ_j^{j-1} is the same as the Lazarus path Γ_{j-1}^{j-1}, it cannot change direction at any node in $V_j \setminus V_{j-1}$. So, any node $x \in B_j^j \setminus V_{j-1}$ used to compute ρ lies in the interior of a $(j-1)$-cell of type 1 or 2, since only elements of V_0 can be contained in type 0 cells. Let us knock out the type 1 case first:

Case: Type 1

Suppose x is contained in a type 1 $(j-1)$-cell, as below:



There are three paths that ρ could take from x inside this cell *a priori*: (x, z, a), (x, y, b), (x, y, c) or (x, z, c). Neither (x, z, a) nor (x, y, b) can be part of ρ though, since

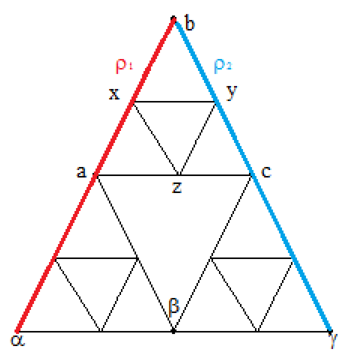
$$S((a, z, y, b)) \geq S((x, z, a)) = S((x, z, b)).$$

(x, y, c) and (x, z, c) cannot be a part of ρ either: Since $x = (a+b)/2$, either the path (a, z, c) or (b, y, z) will have a greater slope than (x, y, c) or (x, y, z). So, ρ cannot contain a node x in $B_j^j \setminus V_{j-1}$ lying in a type 1 $(j-1)$-cell. We will show that same is true for x in a type 2 $(j-1)$-cell:

Case: Type 2

For x in this situation, we must pull back one layer and consider the $(j-2)$-cell C_{j-2}^j in which this $(j-1)$-cell lies. There will be a few subcases here: First we want to note that in the preceding picture, the accepted paths (b, x, a) and (b, y, c) cannot be part of the same Lazarus path, since (a, z, c) has strictly greater slope for nonconstant boundary data (in which case the entire function is trivial). So, WLOG we will assume that (b, x, a) is part of a Lazarus path ρ_1 accepted prior to the path ρ_2 containing (b, y, c) and then prove that no ρ with terminal node x or y exists. To do so, we note that ρ_1 was at latest the $(j-1)$st Lazarus path, and so cannot change direction on any nodes in $V_j - V_{j-1}$. Likewise, ρ_2 was then at latest the $(j-2)$nd Lazarus path, and so cannot change direction on any nodes in $V_j - V_{j-2}$. So, ρ_2 does not change direction on C_{j-2}^j, and there are only three possible paths ρ_1 can take through C_{j-2}^j.

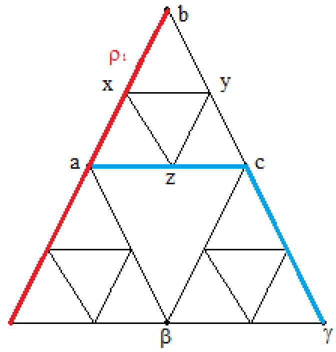
Subcase 1:



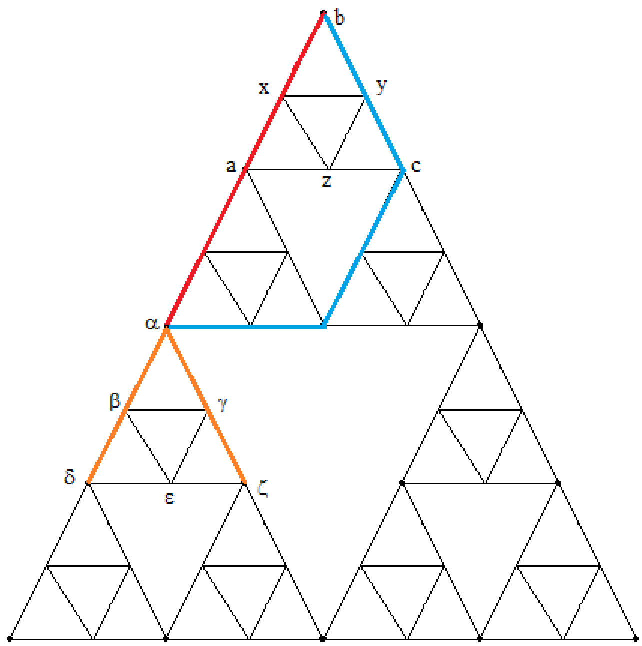
Here, note that $S((x, z, a)) = S((a, \beta, \alpha))$, $S((x, z, c)) = S((a, \beta, \gamma))$, $S((y, z, a)) = S((c, \beta, \alpha))$, and $S((y, z, c)) = S((c, \beta, \gamma))$, so no strictly dominating path exists starting from x or y.

Subcase 2: (Figure on following page)

This case is knocked out easily: $S((a, \beta, \gamma)) = S(\rho_1)$, so no path in C_{j-2}^j dominates.



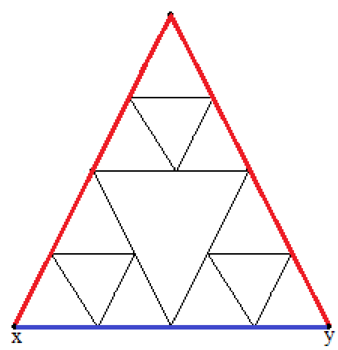
Subcase 3:



If ρ_2 stays in the $(j-2)$-cell, then we must consider how ρ_1 behaves in the $(j-3)$-cell C^j_{j-3} containing C^j_{j-2}. Note that since 2 paths have been accepted already, $j \geq 3$, so C^j_{j-3} exists. The orange lines in the picture above indicate which paths ρ_1 could take through this $(j-1)$ cell. ρ_1 of course must continue beyond this $(j-1)$-cell, but this is the only part we're interested in. If the path in this cell is (α, β, δ), then $S((\alpha, \gamma, \epsilon, \delta)) = S(\rho_2)$, and since Lazarus produces monotonically decreasing slopes, no path in C^j_{j-1} can have a larger slope than this. Similarly if the path through this new $(j-1)$-cell is (α, γ, ζ), then $S((\alpha, \beta, \epsilon, \zeta)) = S(\rho_2)$, so we get the same conclusion. *This proves that P1 holds for* Γ_j^j.

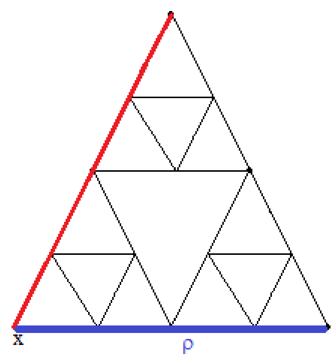
Before proving that P1 holds for all Γ_m^j, $m \geq j$, let's show that P2 holds for all Γ_m^j, $m \geq j$. Let x, y be a pair of points in B_m^j such that $d_{m+1}^j(x,y) < d_m^j(x,y)$. Call the shortest path connecting them in Γ_{m+1}^j as $\rho(x,y)$. Suppose that both x and y are in V_{m-1}. We can decompose $\rho(x,y)$ into a union of subpaths $\rho_1, \rho_2, \ldots, \rho_k$ through the k $(m-1)$-cells $C_{m-1,1}^j, C_{m-1,2}^j, \ldots, C_{m-1,k}^j$ containing ρ. In order for ρ to be strictly shorter than the shortest path connecting x and y in Γ_m^j, it follows that in at least one of these $(m-1)$-cells, the shortest path through the cell considered as a part of Γ_m^j must be longer than ρ restricted to the cell. Let us show that this does not happen.

So, we want to consider what sort of $(m-1)$ cells could contain x and y. First, suppose that one (and therefore the other) is contained in a type 2 $(m-1)$-cell C_{m-1}^j on both levels m and $m+1$. Since by assumption $x, y \in V_{m-1}$, this implies that they are boundary points of C_{m-1}^j, and we are in the situation in this picture:



where the red paths are accepted, and the blue path is the shortest between x and y. This is present in both Γ_m^j and Γ_{m+1}^j and has the same length in both graphs, contradicting the lengthening assumption.

We can therefore assume that x and y lie in type 1 $(m-1)$-cells. They cannot lie in the same $(m-1)$-cell for the same reason that they can they could not lie in a type 2 $(m-1)$-cell: they will be on the boundary of the cell, and the straight line path connecting them is the shortest on both levels m and $m+1$. So, they lie in different $(m-1)$-cells, and we can look at how ρ behaves in each such cell. Let us start with the interior cells, the $(m-1)$-cells through which ρ passes, but which does not contain x or y. Any such cell must be type 0, since any Lazarus path entering a type 1 $(m-1)$-cell must terminate in that cell, and no Lazarus path can even enter a type 2 or 3 cell. But a type 0 $(m-1)$-cell in Γ_m^j is isomorphic as a graph to Γ_1^0, and in Γ_{m+1}^j it is isomorphic to Γ_2^0. We already know that P2 holds for these graphs, so ρ cannot lengthen on interior cells. Therefore, this lengthening must occur in one of the cells containing x or y. Again, however, we see that by assumption x and y lie on the boundary of their respective $(m-1)$-cells as in the following picture, and the shortest path exiting the cell is the same on both levels.



Therefore, there is no $(m-1)$ cell which ρ passes through, in which the shortest path on Γ^j_m is longer than ρ. So, *P2 holds for all* Γ^j_m, $m \geq j$.

Using this fact, we can finish off the proof, by showing that P2 holding for all such Γ^j_m and P1 holding for Γ^j_j implies that P1 holds for all such Γ^j_m. We'll induct on m: given that P1 holds for all Γ^j_n with $j \leq n < m$, consider a path ρ as before with at least one boundary node in $B^j_m - V_{j-1}$ whose slope is strictly greater than that of any path with both boundary nodes in V_{j-1}. Since no such path exists lying in $B^j_n - V_{j-1}$ for any $n < m$, and by P2 no shortest paths shorten for elements of B^j_n with $n < m-1$, we conclude that our new pathological points must lie in $B^j_{m-1} - V_{j-1}$.

Given this, the remainder of the proof that P1 holds proceeds exactly as it did for the base case, replacing the $(j-1)$-cells in the arguments with $(m-1)$ cells, along which previous Lazarus paths have still not changed direction, and likewise for $(m-2)$ and $(m-3)$ cells. Given this, Lemma 1 is proven. $\square$

The last step in the proof is to show that if P1 and P2 hold for Γ^j_m and Γ^j_{m+1}, then $\Gamma^j_m \lhd \Gamma^j_{m+1}$.

Lemma 3. *Suppose P1 and P2 hold for* Γ^j_m *and* Γ^j_{m+1}. *Then,* $\Gamma^j_m \lhd \Gamma^j_{m+1}$.

Proof. By P1, $\mathcal{B}^j_m = \mathcal{B}^j_{m+1} = B^j_{j-1}$, and by P2 $d^j_m(x,y) = d^j_{m+1}(x,y)$ for all $x, y \in B^j_{j-1}$, and so it follows immediately from the definition that

$$\Gamma^j_m \lhd \Gamma^j_{m+1}.$$

$\square$

The proof of Proposition 2 then follows immediately from Lemmas 2 and 3, and therefore Proposition 1 is correct.

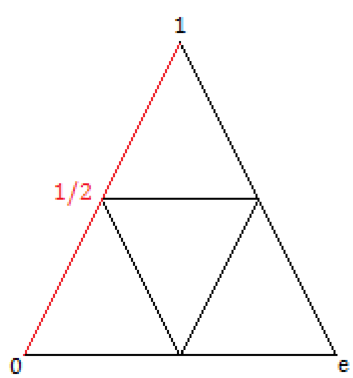
From Proposition 1, we can get some results immediately. If $e \in [1/3, 1/2]$, the second Lazarus path on Γ_2 passes through all of the V_1 vertices, so Theorem 1 is true for e in this range. However, Proposition 1 does not immediately help us for $e \in [0, 1/3)$. In fact, just calculating Lazarus paths for this e range suggests that none of these always-accepted Lazarus paths pass through $z \in V_1$ for any Γ_n.

What is needed is a generalization of Proposition 2.

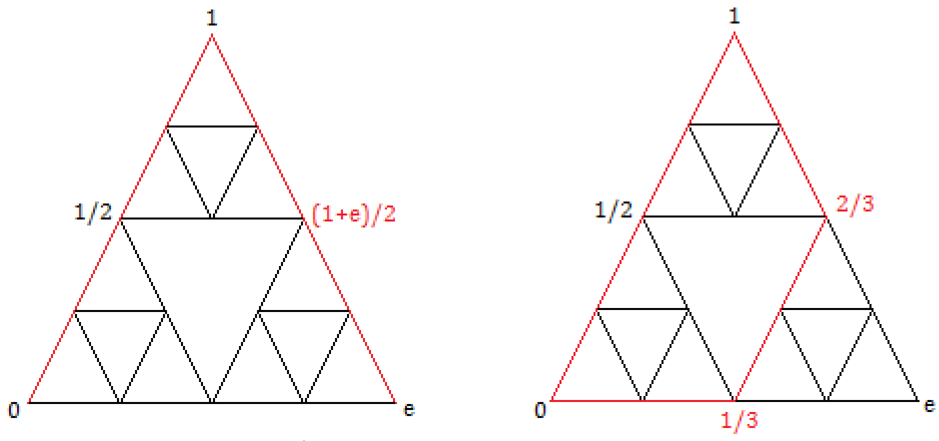
5 A generalization of Proposition 2

The overall goal is to show that each Lazarus path, under certain conditions at least, partitions each Γ_n into sequences of unions of subgraphs, each of which is amenable to the same techniques used on the original graph. In doing so, we are then able to calculate Lazarus paths on each subgraph at each step, enlarging the set of points which we can show to be computed correctly at each level of approximation to SG.

Recall Theorem 1 on page six. Proposition 1 gives us some results in proving this. Consider as before the following level 1 graph:



Since the V_0 nodes are always correct trivially we're only concerned with the nodes $a, b, c \in V_1 - V_0$. We know the first Lazarus path is the left path from 0 to 1 for any choice of e, so $a = 1/2$ is given. The second Lazarus path on Γ_2 does depend on e, but in a simple way. We have two choices:



Regardless, for $e \in [1/3, 1/2]$, the first and second Lazarus paths on Γ_2 run through all nodes in V_1, and therefore will do so for all Γ_k, $k \geq 2$. So, Theorem 1 is true for $e \in [1/3, 1/2]$. For $e \in (0, 1/3)$ however, we do not get the same behavior. But then, consider Γ_k for $k \geq 2$. For each such Γ_k, we know that the first two Lazarus paths are a refinement of the first two Lazarus paths on Γ_2. Moreover, we can calculate that for any e in this range, the third Lazarus path on Γ_k is fixed, as drawn below. So, the problem of running the Lazarus algorithm on Γ_k^j for $j, k > 3$ splits into running it on three separated domains,

$$\Gamma_{k,1}, \Gamma_{k,2}, \Gamma_{k,3},$$

pictured on the next page in Figure 1. As an aside: the doubly-indexed subscripts will come up a few times here, so a bit more notation might be handy here. Just as $SG = \lim\limits_{k \to \infty} \Gamma_k$, let

$$\Gamma_{\cdot,i} := \lim_{k \to \infty} \Gamma_{k,i}\,.$$

It would be nice if we could say that the set of graphs $\{\Gamma_{k,3}\}$ in the above picture, with appropriate choice of initial boundary data, had the same properties possessed by $\{\Gamma_k\}$ which allowed for Proposition 2 to hold true. Then, we could restrict our focus to $\Gamma_{k,3}$ and apply the procedure to see what information is gained about the nodes of Γ_k therein. It would be nicer to generalize this result to a wider class of graphs than just $\{\Gamma_{k,3}\}$.

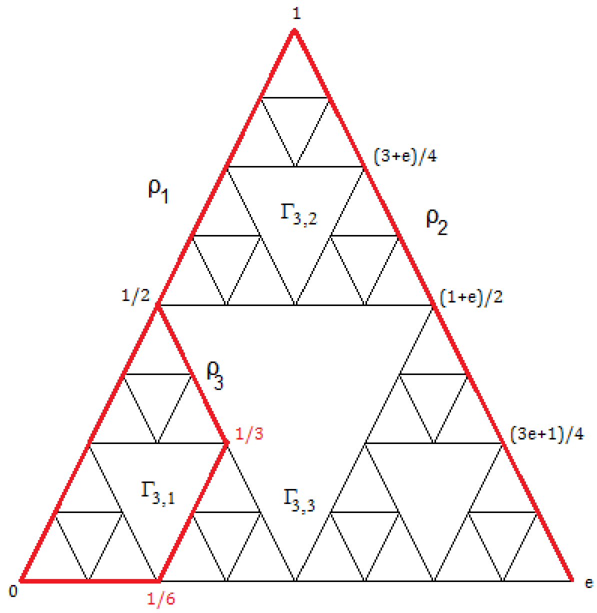


Figure 1: The $e \in (0, 1/3)$ case

5.1 Regular finite fractafolds

Definition 2: A *regular finite fractafold* is the quotient space

$$\left(\bigsqcup_{i=1}^{n} SG_i \right) / \varphi,$$

where φ is a quotient map such that in the ith copy of SG (SG_i), φ identifies each boundary point of SG_i with at most one boundary point of at most one SG_j, $i \neq j$, and φ never identifies two different boundary points in SG_i with boundary points in one other SG_j.

For instance, each of the $\Gamma_{\cdot,i}$, $i \in [1,3]$ constructed above is a regular finite fractafold. In the course of this paper, we also demonstrate constructively that each partition created by the Lazarus paths in $\{\Gamma_{k,3}\}$ consists of rff's.

Let $F = \bigsqcup_{i=1}^{n} SG_i / \varphi$ be a rff. On analogy with Γ_k, let

$$F_k := \bigsqcup_{i=1}^{n} \Gamma_{k,i} \, / \varphi,$$

where $\Gamma_{k,i}$ is the level-k approximation graph of SG_i.

We define ∂F to be the subset of $\bigcup_i \partial SG_i$ of nodes which have only two neighbors in F. Therefore, there may be elements of F_0 which are not elements of ∂F, unlike the situation with SG. Again though, let us say that $\mathcal{B}_0 = B_0 := \partial F$.

With our boundary defined in this way, let us say the following. It is easy to show that $F_k \prec F_{k+1}$, with $\prec$ defined as in the previous writeup. Also analogously to that writeup, let us define F_k^j to be the jth Lazarus subgraph of F_k. Then, is is also easy to show that $F_k^j < F_k^{j-1}$, too. This gives us the following picture:

$$\begin{array}{ccccccccc}
F_0^0 & \prec & F_1^0 & \prec & F_2^0 & \prec & F_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^1 & & F_1^1 & & F_2^1 & & F_3^1 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^2 & & F_1^2 & & F_2^2 & & F_3^2 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^3 & & F_1^3 & & F_2^3 & & F_3^3 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

We want to show that the following is also true:

$$\begin{array}{ccccccccc}
F_0^0 & \prec & F_1^0 & \prec & F_2^0 & \prec & F_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^1 & & F_1^1 & \lhd & F_2^1 & \lhd & F_3^1 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^2 & & F_1^2 & & F_2^2 & \lhd & F_3^2 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^3 & & F_1^3 & & F_2^3 & & F_3^3 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Lemma 1: Let F be a regular finite fractafold. Then, for all j and $m \geq j$,

$$F_m^j \lhd F_{m+1}^j.$$

Proof. Again, this will depend solely on the sequences $\{F_m^j\}, m \geq j > 0$ for all $j \in N$ satisfying P1 and P2. The proof is much simpler here though, since we know the same is true for $F = SG$.

P1: For all $m \geq j > 0$, $\mathcal{B}_m^j = \mathcal{B}_{j-1}^j$.

Proof. Suppose that for some $\Gamma_{m,i}^j$ comprising F_m^j, there exists some $b \in B_m^j - \mathcal{B}_{j-1}^j$ necessary for computing the next Lazarus path on F_m^j. Just as in the proof of P1 for SG, we can look at the $(j-1)$ and $(j-2)$ cells in which b lies. These are still inside $\Gamma^j_{m,i}$, and so the proof that no such b exists follows directly from the proof for the graph approximations to SG, since these cells are identical.

$\square$

P2: If the shortest path between $x, y \in B^j_m$ in F^j_{m+1} is not a refinement of the shortest path between $x, y \in B^j_m$ in Γ^j_m, then x or y is in $B^j_m - B^j_{m-1}$

Proof. Suppose this were not true. Let x, y be such elements of B^j_m and let ρ_m be the path connecting them in F^j_m, while ρ_{m+1} is the path connecting them in F^j_{m+1}. If ρ_{m+1} is not a refinement of ρ_m, then there is some $(m-1)$-cell contained in exactly one SG_i in which the path first changes. So, we can restrict ourself to looking at how the path behaves in SG_i, in which case we are back to the conditions of the original proof, and know that no such shortening occurs.

$\square$

So, the sequences $\{F^j_m\}$, $j \geq m$, satisfy P1 and P2. The results of Lemma 1 and Lemma 2 from the previous writeup transfer immediately over to the regular finite fractafold setting, and so we conclude that

$$F^j_m \triangleleft F^j_{m+1}$$

for all j and all $m \geq j$.

$\square$

This is good news. Going back to the situation illustrated in Figure 1, Lemma 1 tells us that we can run this compute-Lazarus-path-then-refine procedure on each subgraph individually. For now, let us focus on $F = \Gamma_{3,3}$, as that is where all the interesting things are happening. If we let it inherit the boundary data from the computed values on Γ^3_3, the graph of F^0_0 as depicted in Figure 2 on the next page.

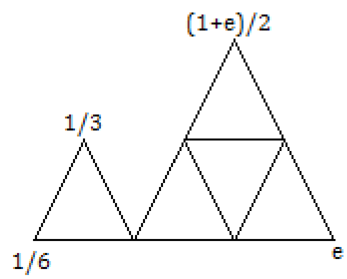
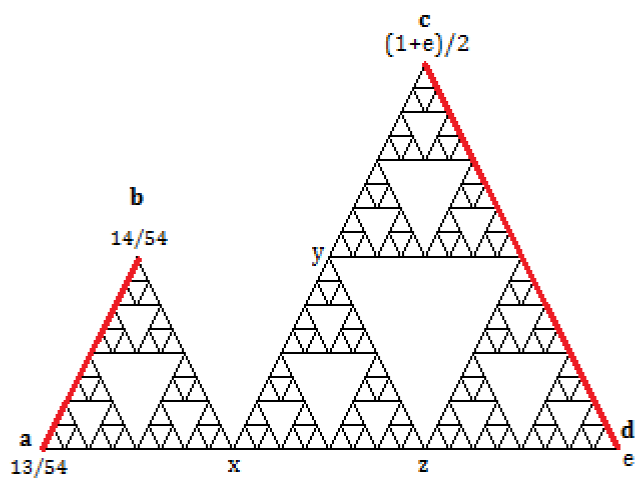


Figure 2: F^0_0

Note that, in calling this F_0, we are going to introduce some confusion that should be clarified. This is level 0 for this particular fractafold, but included into Γ_2 it is a level 2 portion of the copy of Γ_2 from which it was taken. A result about F_k will translate into a result about Γ_{k+2}. Let us leave this at the level of tacit understanding where there is no chance for confusion, and otherwise, call the level of refinement on the graph F the *local level*, and the level of refinement if F were to be considered as included in the original copy of SG the *inherited level*. This causes some problems for our slope renormalization scheme, as well. Which level do we use when calculating path slopes? For consistency, let us always use the inherited level, not the local level. For this particular F we can directly verify that the first Lazarus path on F_1 connects e and $(1+e)/2$, and the second Lazarus path on F_2 connects $1/6$ and $1/3$. So, this choice of boundary data on F gives us Lazarus paths which agree with the paths in the inherited approximation graphs. We get the following graph for F_3^2:



The next Lazarus path on this domain depends on e. Again, we can verify that it will be one of two paths,

$$\rho_1 = (a, x, y, c)$$

or

$$\rho_2 = (c, y, z, e).$$

We are on local level 3, so our inherited level is 5. Therefore, our slope renormalization factor is 2^5. The shortest path connecting a and b has renormalized slope of $\frac{1/12}{12} \cdot 2^5 = 4/9$ and the shortest path b and d has slope $\frac{4-12e}{9}$, while

$$S(\rho_1) = \frac{4+6e}{9}$$

and

$$S(\rho_2) = \frac{2-2e}{3}.$$

For any $e \in (0, 1/3)$, either slope strictly dominates the slopes of the $a, \ldots, b$ and $b, \ldots, d$ paths (as well as the rest of the possible paths if you care to check).

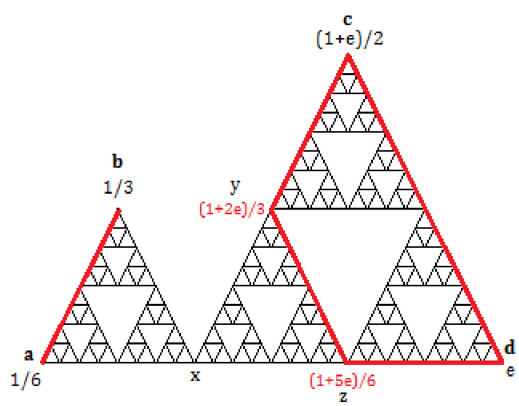
However, $S(\rho_1) > S(\rho_2)$ for $e \in (1/6, 1/3)$, while $S(\rho_2) > S(\rho_1)$ for $e \in (0, 1/6)$. This means that x is computed correctly with value

$$x = \frac{5 + 3e}{18}$$

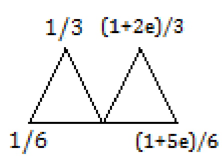
at local level 3 (and each thereafter) if $e \in (1/6, 1/3)$. This corresponds to x being computed correctly on Γ_5. But note: $x \in V_1$ (here, V_1 understood as the vertex set of the original copy of Γ_1), and the other elements of V_1 were computed correctly by Γ_2.

For $e \in (1/6, 1/3)$, $u_j^e(x)$ *is computed correctly for all* $x \in V_1$ *for all* $j \geq 5$.

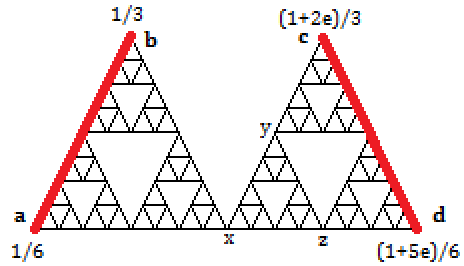
Before this, we had that Theorem 1 was true for $e \in [1/3, 1/2]$ easily. This last bit extends that, we now have that Theorem 1 is true for $e \in [1/6, 1/2]$. Moreover, the same techniques used to extend the result here can be used to greater advantage. Suppose $e \in (0, 1/3)$. Then, on F_3^2 our Lazarus path is ρ_2, and we have this situation:



Our node x is now contained in a different regular finite fractafold (rff), call it G, with boundary nodes a, b, y, z. Let it inherit boundary data on these nodes as before, so that G_0 is the following:



Keeping in mind that we compute no paths on this level, we can refine one level to see that the first Lazarus path on G_1 is the one connecting 1/6 and 1/3, with renormalized slope 2/3 again (our renormalization constant on local G level 1 is 2^3), and the second Lazarus path on G_2 is the line connecting $(1 + 5e)/6$ and $(1 + 2e)/3$, with slope $(2 - 2e)/3$, putting us here at local G level 3:



As before, the renormalized slope of the shortest path connecting a and b (call it ρ_0) is 4/9. The slope of the path γ connecting b to d is $\frac{1-5e}{3}$ The other contender paths are $\rho_1 = (a, x, y, c)$ and $\rho_2 = (d, z, y, c)$. Here,

$$S(\rho_1) = \frac{1+4e}{3}$$

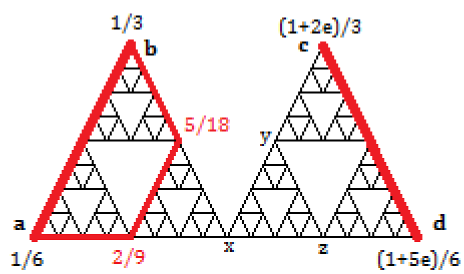
and

$$S(\rho_2) = \frac{4-4e}{9}.$$

Note that γ is still out of the running but, $S(\rho_2) < S(\rho_0)$, so our competition is between ρ_0 and ρ_1. A little algebra shows us that ρ_0 wins out when $e \in (0, 1/12)$, and ρ_1 wins out when $e \in (1/12, 1/6)$. So, for $e \in (1/12, 1/6)$, once we get to this stage (inherited level 5 again), x is still computed correctly as

$$x = \frac{3+4e}{12},$$

just via a different route. For the $e \in (0, 1/12)$ range, however, we find ourself with the following picture:



Now, x is contained in yet another subgraph/rff, let us call this one H. Something interesting has happened here though: H is identical to F, scaled down by a factor of 2. The (renormalized) slopes along the boundaries of H are 2/3 of the corresponding slopes for F. On H, the first Lazarus path connects (1+23)/3 and $(1+5e)/6$, and the second connects 2/9 and 5/18, so that at local H level 3 we find ourselves in a familiar situation:

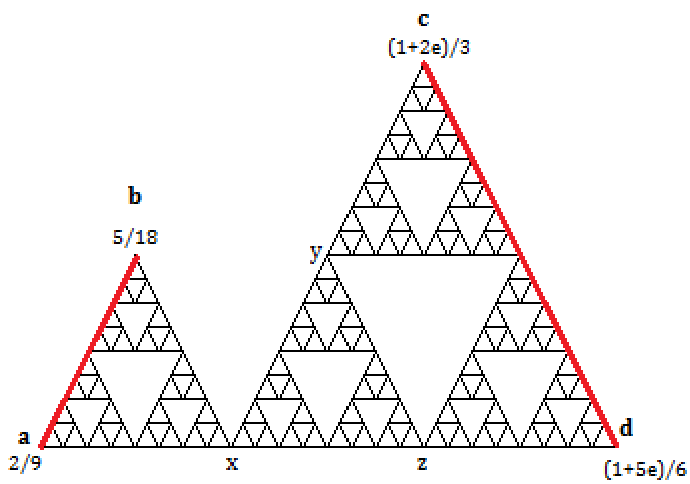


Figure 3: H_3^2 this time, not F_3^2

Let's see what happens this time with the third Lazarus path. Here, our inherited level is 6, so our scaling is 2^6. The shortest path ρ_0 connecting a and b has slope

$$S(\rho_0) = 8/27.$$

The path $\rho_1 = (a, x, y, c)$ has slope

$$S(\rho_1) = \frac{8 + 48e}{27}$$

and the path $\rho_2 = (d, z, y, c)$ has slope

$$S(\rho_2) = \frac{4 - 4e}{9}.$$

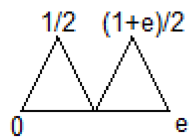
Again, we can just use a little algebra to find that ρ_1 dominates for $e \in (1/15, 1/12)$. So, for e in this range, u^e is computed correctly on the V_1 vertices by level 6, with

$$x = \frac{7 + 6e}{27}.$$

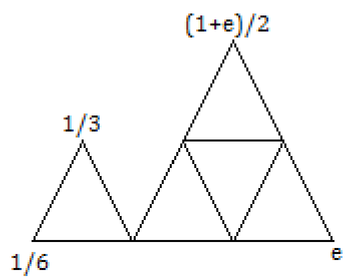
[I had a third example worked out, omitted to help shorten this thing. Should this extra info go on a website?]

5.2 A general formula

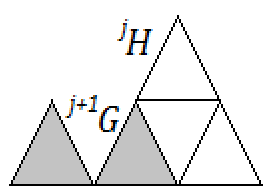
However, by this point we notice a pattern is forming. We seem to be alternating between scaled copies of two different graphs/rff's, one consisting of two cells of the same size joined together at a point, and the other consisting of two cells of different sizes joined together at a point. If we can show that this is indeed the case in general, and get closed-form expressions for the boundary data on each copy of these graphs, we can hope to get the results we want. We will want some terminology to talk about these things. Despite the confusion with previous names and general ugliness, let us call the former graph, two same-sized cells glued together, ${}^{j}G$, and the latter as ${}^{k}H$, where the j and k work as follows: ${}^{j}G$ is the rff produced by the gluing together of two $(j+1)$-cells, and ${}^{k}H$ is the gluing together of a $(k+1)$-cell and a $(k+2)$-cell. So for instance, ${}^{0}G_0$ is



and ${}^{1}H_0$ is

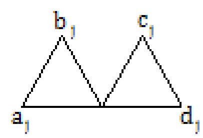


The remaining graphs are defined inductively as: ${}^{j+1}G$ is ${}^{j}H$ minus the upper and right $(j+2)$-cells contained within the $(j+1)$-cell that is a part of ${}^{j}H$.



${}^{j+1}H$ is formed from ${}^{j+1}G$ by removing the upper and left $(j+3)$-cells from the left $(j+2)$-cell of ${}^{j+1}G$.

With this in mind, we want to show the following: Given that ${}^{1}G$ is as above, we want to show that the either the Lazarus path on ${}^{j}G_3^2$ partitions the graph into ${}^{j}H$ and its complement or passes through the remaining V_1 node for all j, and likewise that the Lazarus path on ${}^{j}H_3^2$ partitions the graph into ${}^{j+1}G$ and its complement or passes through the V_1 node. In the process, we will also derive an expression for the boundary data inherited from Lazarus paths on each level j for both of these graphs, and therefore the range of e values for which this process hits the V_1 node for each j, ultimately giving an upper bound for the level of the correct computation of the V_1 set as a function of e. First, let us name the boundary data values as a function of the level. The boundary data for ${}^jG$ is



and similarly for ${}^jH$:

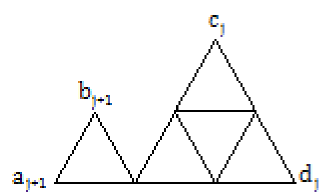


Figure 4: Boundary data

By looking at the numbers we have gotten for the first few j values by hand, we can conjecture that

$$\begin{aligned}
a_j &= \frac{1-(1/3)^j}{4} \\
b_j &= \frac{1+(1/3)^j}{4} \\
c_j &= \frac{1+(1/3)^j}{4} + \frac{3-(1/3)^j}{4}e \\
d_j &= \frac{1-(1/3)^j}{4} + \frac{3+(1/3)^j}{4}e
\end{aligned}$$

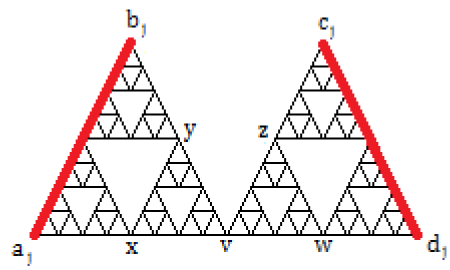
Moreover, it is easy to prove this by induction, assuming that the procession of graphs we conjectured holds true, since

$$a_{j+1} = \frac{2}{3}a_j + \frac{1}{3}b_j,$$

and similarly for the others.

Given this, let us look at ${}^jG_1$. To avoid needlessly complicating arguments with inconsequential renormalization factors, let us ignore them for now. To make the distinction clear though, unrenormalized slopes of paths ρ will be denoted as

$S'(\rho)$, instead of $S(\rho)$. We compute that the first two Lazarus paths on ${}^0G$ are the left side of the left 1-cell and the right side of the right 1-cell. For $j > 0$, we use the fact that the boundary data has (by inductive hypothesis) been inherited from previous Lazarus paths on ${}^{j-1}H$. Since Lazarus paths are decreasing in slope, their restriction to this graph (i.e. the left side of the left $(j+1)$-cell and right side of the right $(j+1)$-cell again) must again be the first two Lazarus paths here, as anything with steeper slope through the interior would have been part of a Lazarus path previously. So, let us jump right to ${}^jG_3^2$.



There are essentially four possible Lazarus paths (since the rest are just contractions of these with the same slope or clearly worse), $\rho_0 = (a_j, x, v, z, c_j)$, $\rho_1 = (b_j, y, v, w, d_j)$, $\rho_2 = (a_j, x, y, b_j)$, and $\rho_3 = (c_j, z, w, d_j)$. These slopes can be calculated as

$$\begin{aligned}
S'(\rho_0) &= \frac{3(1/3)^j + e[9/2 - 1/2(1/3)^{j-1}]}{96} \\
S'(\rho_1) &= \frac{3(1/3)^j + e[-9/2 - 1/2(1/3)^{j-1}]}{96} \\
S'(\rho_2) &= \frac{4(1/3)^j}{96} \\
S'(\rho_3) &= \frac{4(1/3)^j - e[4(1/3)^j]}{96}.
\end{aligned}$$

For $e > 0$, $S'(\rho_0) > S'(\rho_1)$ and $S'(\rho_2) > S'(\rho_3)$, so we compare ρ_0 and ρ_2. The point of equality is

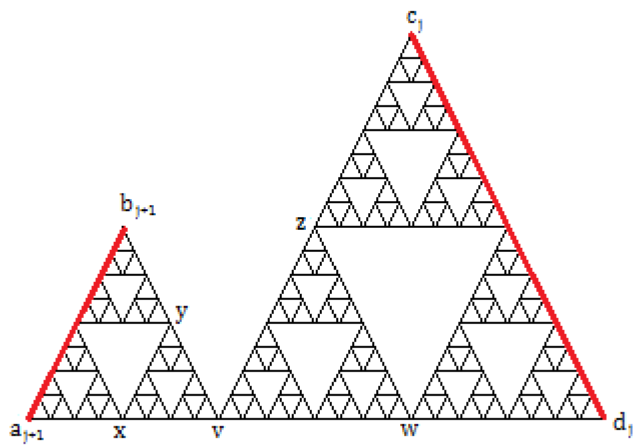
$$e = \frac{2}{3^{j+2} - 3}.$$

For e larger than this, the Lazarus path ρ_0 passes through v and so we take care of V_1, with

$$v = \frac{2 + (3 - (1/3)^j)e}{8}.$$

Below this value of e, ρ_2 is the Lazarus path and we go must go into the ${}^jH$ graph. This shows that at least the first of our putative transitions happen the way we would like them to. For ${}^jH_0$ we are back in the situation of Figure 4 on page 26.

Note that, since Lazarus paths have decreasing slope, the first two Lazarus paths on ${}^{j}H_0$ must be the same as the restriction of the previous Lazarus paths on ${}^{j}G$ to this subgraph, since any path through the interior must have smaller slope. So, we again have



in the general case, and essentially four paths to choose from again: $\rho_0 = (a_{j+1}, x, v, z, c_j)$, $\rho_1 = (b_{j+1}, y, v, w, d_j)$, $\rho_2 = (a_{j+1}, x, y, b_{j+1})$, and $\rho_3 = (c_j, z, w, d_j)$. We can calculate these slopes as

$$
\begin{aligned}
S'(\rho_0) &= \frac{(1/3)^j + (1/3)^{j+1} - e[(1/3)^j - 3]}{96} \\
S'(\rho_1) &= \frac{(1/3)^j + (1/3)^{j+1} - e[(1/3)^j + 3]}{96} \\
S'(\rho_2) &= \frac{4(1/3)^{j+1}}{96} \\
S'(\rho_3) &= \frac{6(1/3)^{j+1} - 6(1/3)^{j+1}}{96}
\end{aligned}
$$

Again we see that $S'(\rho_0) > S'(\rho_1)$, and now $S'(\rho_3) > S'(\rho_2)$ for $e < 1/3$, with $S'(\rho_3) > S'(\rho_0)$ for e small. Again we can calculate the point of equality as

$$e = \frac{2}{3^{j+2} + 3}$$

with

$$v = \frac{3 + (1/3)^j + (3 - (1/3)^j)e}{12}.$$

Finally, we notice that if we are below this e cutoff, our next subgraph containing v is indeed ${}^{j+1}G$, so the induction is basically complete. All that remains now is to relate these j values back to levels on the original graph. First, v is computed correctly by level 2 for $e > 1/3$, as we know. On ${}^{0}G$ we get nothing, since our formula would only give us a cutoff of $e = 1/3$ and we are already in this case. On ${}^{0}H$, we get v in the third Lazarus path for $e \in (1/6, 1/3)$, on local level 3, which corresponds with an inherited level of 5, so we compute v correctly for

$$e \in (1/6, 1/3).$$

Then, on local level 3 of ${}^{1}G$ we compute v correctly for $e \in (1/12, 1/6)$, which corresponds to inherited level 5 as well. Since we are always computing the correct v (in the appropriate range) on local level 3 for all ${}^{j}G$ and ${}^{j}H$, an easy induction argument shows that this corresponds to inherited level $j+4$ for ${}^{j}G$, $j > 0$, and inherited level $j+5$ for ${}^{j}H$. Thus, on any (global) level k, we compute v correctly from ${}^{k-5}H$ for

$$e \in \left(\frac{2}{3^{k-3}+3}, \frac{2}{3^{k-3}-3} \right)$$

and we compute v correctly from ${}^{k-4}G$ for

$$e \in \left(\frac{2}{3^{k-2}-3}, \frac{2}{3^{k-3}+3} \right).$$

Since the latter subsumes the former, if all we care about is computing an upper bound for the level at which v is correctly computed as a function of e, we can lump these results together, and say that by level k,

$$v \text{ is computed correctly for all } e > \frac{2}{3^{k-2}-3}, \quad k > 4.$$

So, the lower bound for the e values for which V_1 has been computed correctly by level k is $O(1/3^k)$, giving us definite convergence to 0 as $k \to \infty$. Since we know how to deal with the $e = 0$ case already, this is everything we need to show that Theorem 1 is true, completing our proof.

6 Conclusion

Theorem 1 proves the existence of a function u on SG which is a limit of functions u_k on each Γ_k satisfying $\Delta_\infty u_k = 0$. Naturally, we would like to get a much more general result, proving the existence of a function $u : SG \to \mathbb{R}$ as a limit of functions $u_k : \Gamma_k \to \mathbb{R}$ such that $-c^k \Delta_\infty u_k = f_k$, where c is a possible renormalization constant and $f_k = f|_{V_k}$ for some suitably wide class of functions f, so that we could define

$$-\Delta_\infty u = f$$

on SG. The proof of existence of infinity-harmonic functions depended on the use of the Lazarus algorithm, which has no immediate analogue for non-harmonic functions. One could try to construct some sort of similar procedure for non-zero f on the right-hand side of the previous equation, and that is one direction for possible future exploration.

The use here of "regular finite fractafolds" to understand the behavior of Lazarus paths suggests a much more general theory for the behavior of Lazarus paths on this space of objects, and is another direction in which this work could be extended.

This aside, we can take stock of what has been accomplished. We have a plausible definition for infinity-harmonic functions on SG, and a proof of their existence. Perhaps equally important is the new understanding of how the discrete infinity-harmonic functions behave on the graph approximations $\{\Gamma_k\}$. It suggests that extensions of these results could be found for other, similarly defined fractal sets.

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