Unit 4: Contextual Applications of Differentiation

4.1 - Interpreting the Meaning of the Derivative in Context

- units of the derivative is also the units of $f(x)$ over units of x
- interpret the derivative by using the sentence starter "at $x =$ [x-value], the [context of $f(x)$] is increasing/decreasing at a rate of [units]"
    - be sure to include the context, rate of change, units, and instant at which the derivative is taken

📢 - Interpret rate of change in context and watch out for units

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4.2 - Straight-Line Motion - Connecting Position, Velocity, and Acceleration

- the derivative of a position function is velocity, and the derivative of a velocity function is acceleration (second derivative of position)
- for a velocity function:
    - an object is at rest when $v(t) = 0$
    - an object is moving in the negative direction when $v(t) < 0$
    - an object is moving in the positive direction when $v(t) > 0$
- an object is speeding up when the direction of acceleration and direction of velocity are the same
    - when the velocity changes signs, that means the object switched the direction it was moving
- when given a table, estimate rate of change with $\frac{v(t_2)-v(t_1)}{t_2-t_1}$

📢 - The first derivative of position is velocity and the second derivative of position is acceleration

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4.3 - Rates of Change in Other Applied Contexts (Nonmotion Problems)

- rate of change functions can help predict the rate at which something is changing
- the derivative of a rate of change function is already the second derivative of the original unit
- given the original function $Q(t)$:
    - if $Q(t)$ is not changing, then $Q'(t) = 0$
    - if $Q(t)$ is increasing, then $Q'(t) > 0$
    - if $Q(t)$ is decreasing, then $Q'(t) < 0$
- some problems are determined by more than one rate, where one is a rate-in and one is a rate-out
    - the overall function is equal to the rate-in function minus the rate-out function

📢 - The overall rate of a situation is the difference between the rate-in and the rate-out

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4.4 - Introduction to Related Rates

- before starting a related rates problem, first find the derivative of the function using implicit differentiation with respect to t
  - make sure to apply any derivative rules in the original function to its derivative

$A = lw$
$\frac{dA}{dt} = \frac{dl}{dt} \cdot w + \frac{dw}{dt} \cdot l$

$A = \pi r^2$
$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}$

$x^2 + 3y = 4$
$2x\frac{dx}{dt} + 3\frac{dy}{dt} = 0$

> 📢 - Chain rule and implicit differentiation are used to find derivatives with respect to t to set up related rates problems

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4.5 - Solving Related Rates Problems

- be careful when setting up the problem, because there will be rate-in and rate-out

$P = 2l + 2w$
$\frac{dl}{dt} = 3$
$\frac{dw}{dt} = -5$
$\frac{dP}{dt} = 2(3) + 2(-5)$
$\frac{dP}{dt} = -4$

$A = lw$
$l = 10$
$w = 5$
$\frac{dl}{dt} = 3$
$\frac{dw}{dt} = 0$
$\frac{dA}{dt} = 3 \cdot 5 + 0 \cdot 10 = 15$

$x^2 + y^2 = z^2$
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$
$x = 4,\ y = 3,\ z = 5$
$\frac{dx}{dt} = -15$
$\frac{dy}{dt} = 10$
$2(4)(-15) + 2(3)(10) = 2(5)(\frac{dz}{dt})$
$\frac{dz}{dt} = -6$

- a common type of problem is the water-in-a-cube type of problem where the question asks for how fast the depth of the water is changing
  - the only thing that changes is the height of the water, the other two variables (length and width) stay constant

$l = 5,\ w = 5$
$\frac{dV}{dt} = 20$
$V = lwh$
$V = 25h$
$\frac{dV}{dt} = 25\frac{dh}{dt}$
$25\frac{dh}{dt} = 20$
$\frac{dh}{dt} = \frac{4}{5}$

$h = r,\ h = 5$
$\frac{dV}{dt} = 8$
$V = \frac{1}{3}\pi r^2 h$
$V = \frac{1}{3}\pi h^3$
$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$
$8 = \pi \cdot 5^2 \frac{dh}{dt}$
$\frac{dh}{dt} = \frac{8}{25\pi}$

> 📢 - Constants can be substituted into variables for related rate derivatives
> - Related rates questions can include area, volume, and angles
> - Find the speed of the rate of change and then consider units in context

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4.6 - Approximating Values of a Function Using Local Linearity and Linearization

- tangent lines can be used to roughly approximate a value of a function
  - given a point on a curve $a, f(a)$ and the derivative at that point $f'(a)$, we can write the equation of the tangent line
  - use point slope for $y - f(a) = f'(a)(x - a)$ because there's no reason to find the y-intercept
- the value of the function close to a tangent line is approximately the value at that point on the tangent line
  - if I'm trying to find the y value at $x = 4$ on the function $f(a)$ but I only have the tangent line equation of $x = 3$, then the value of y on the tangent at $x = 4$ should be pretty close to the real answer
- if the slope of the tangent line is decreasing, the tangent line will be an overestimate (concave)

- if the slope of the tangent line is increasing, the tangent line will be an underestimate (convex)

📢 - Tangent lines can be used to estimate a value from the original function
- The value will be an overestimate or an underestimate depending on the direction of the slope at that time

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4.7 - Using L'Hospital's Rule for Finding Limits of Indeterminate Forms

- **indeterminate form:** when $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$
- if a limit produces an indeterminate form, the new limit will be equal to the quotient of the derivatives of both the numerator and denominator

$\lim_{x \to 0} \frac{1-\cos(2x)}{5x}$
$\lim_{x \to 0}(1-\cos(2x)) = 0$
$\lim_{x \to 0}(5x) = 0$
$\lim_{x \to 0} \frac{1-\cos(2x)}{5x} = \lim_{x \to 0} \frac{2\sin(2x)}{5} = \frac{0}{5} = 0$

$\lim_{x \to 1} \frac{\ln x}{x^2-1}$
$\lim_{x \to 1}(\ln x) = 0$
$\lim_{x \to 1}(x^2-1) = 0$
$\lim_{x \to 1} \frac{\ln x}{x^2-1} = \lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \lim_{x \to 1} \frac{1}{2x^2} = \frac{1}{2}$

$\lim_{x \to 2} \frac{\sin(x-2)}{x^2-4}$
$\lim_{x \to 2}(\sin(x-2)) = 0$
$\lim_{x \to 2}(x^2-4) = 0$
$\lim_{x \to 2} \frac{\sin(x-2)}{x^2-4} = \lim_{x \to 2} \frac{\cos(x-2)}{2x} = \frac{1}{4}$

$\lim_{x \to 0} \frac{e^x-\cos x-2x}{x^2-2x}$
$\lim_{x \to 0}(e^x-\cos x-2x) = 0$
$\lim_{x \to 0}(x^2-2x) = 0$
$\lim_{x \to 0} \frac{e^x-\cos x-2x}{x^2-2x} = \lim_{x \to 0} \frac{e^x+\sin x-2}{2x-2} = \frac{1}{2}$

📢 - L'Hospital's Rule is only used when the original limit produces an indeterminate form
- If the first derivative produces another indeterminate form, keep taking derivatives of the numerator and denominator until it doesn't