Unit 3: Differentiation: Composite, Implicit, and Inverse Functions

3.1 - The Chain Rule

- **composite function:** a function that's written inside another function
- **chain rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g(x)$
  - $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
  - find the inner and outer function
  - if there are more than two terms, take the derivative of the two outermost functions and then work inwards

$y = (x^3 + 5x)^4$
$y' = 4(x^3+5x)^3 \cdot (3x^2+5)$

$y = \sqrt[3]{h(x)} = (h(x))^{\frac{1}{3}}$
$y' = \frac{1}{3}(h(x))^{-\frac{2}{3}} \cdot h'(x)$

$y = \ln(5x+1)$
$y' = \frac{1}{5x+1} \cdot 5$

$y = e^{3x}$
$y' = 3e^{3x}$

$y = e^{(\sin x - 3x)}$
$y' = e^{\sin x - 3x} \cdot (\cos x - 3)$

$y = \ln(\sin x)$
$y' = \frac{1}{\sin x} \cdot \cos x$

$y = \sin^3(7x) = (\sin(7x))^3$
$y' = 3(\sin(7x))^2 \cdot \cos(7x) \cdot 7$

$y = \sec(2x)$
$y' = \sec(2x)\tan(2x) \cdot 2$

📢 **Chain Rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g(x)$

3.2 - Implicit Differentiation

- implicit differentiation
  - used when we can't solve for y or when it would be too difficult to
    - for example, the equation of a circle $x^2 + y^2 = r^2$ where y can be a positive or a negative value
    - $\cos y = 3x + 3y$ where y is in a term that we can't easily solve for
    - $3x^4 - xy - y^3 = 12$ where there are too many y terms
- steps to implicitly differentiate
  1. take the derivative of all terms with respect to x
  2. simplify the $\frac{dx}{dx}$ terms because it's equal to just multiplying by 1
  3. solve for $\frac{dy}{dx}$

$x^2 + y^2 = 25$
$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$
$2x + 2y\frac{dy}{dx} = 0$
$2y\frac{dy}{dx} = -2x$
$\frac{dy}{dx} = \frac{2x}{2y} = -\frac{x}{y}$

$\cos y = 3x + 3y$
$-\sin y\frac{dy}{dx} = 3\frac{dx}{dx} + 3\frac{dy}{dx}$
$-\sin y\frac{dy}{dx} = 3 + 3\frac{dy}{dx}$
$-3 = \sin y\frac{dy}{dx} + 3\frac{dy}{dx}$
$\frac{dy}{dx} = -\frac{3}{\sin y + 3}$

$\tan y + x^3 + 9 = 0$
$\sec^2 y\frac{dy}{dx} + 3x^2\frac{dx}{dx} + 0 = 0$
$\sec^2 y\frac{dy}{dx} + 3x^2 = 0$
$\sec^2 y\frac{dy}{dx} = -3x^2$
$\frac{dy}{dx} = \frac{-3x^2}{\sec^2 y}$

$3x^4 - xy - y^3 = 12$
$12x^3\frac{dx}{dx} - (x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx}) + 3y^2\frac{dy}{dx} = 0$
$12x^3 - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$
$-x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -12x^3 + y$
$\frac{dy}{dx}(-x + 3y^2) = -12x^3 + y$
$\frac{dy}{dx} = \frac{-12x^3+y}{-x+3y^2}$

$y = \ln(3x + 4y)$
$\frac{dy}{dx} = \frac{1}{3x+4y}\frac{dx}{dx} \cdot (3\frac{dx}{dx} + 4\frac{dy}{dx})$
$\frac{dy}{dx} = \frac{1}{3x+4y} \cdot (3 + 4\frac{dy}{dx})$
$(3x+4y)\frac{dy}{dx} = 3 + 4\frac{dy}{dx}$
$(3x+4y)\frac{dy}{dx} - 4\frac{dy}{dx} = 3$
$\frac{dy}{dx} = \frac{3}{(3x+4y)-4}$

1. Take the derivative of all terms with respect to x
2. Simplify the $\frac{dx}{dx}$ terms
3. solve for $\frac{dy}{dx}$

3.3 - Differentiating Inverse Functions

- **definition of inverse function:** $f(g(x)) = g(f(x)) = x$, given that $f(x)$ and $g(x)$ are inverses, if the point (a, b) is on the graph of $f(x)$, then the point (b, a) is on the graph of $g(x)$
  - the functions will reflect over the $y = x$ line
  - use the chain rule to differentiate
- **derivatives of inverses:** if $f(x)$ and $g(x)$ are inverses, and if $f(a) = b$, and further if $f'(a) = m$, then $g'(b) = \frac{1}{m}$

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3.4 - Differentiating Inverse Trigonometric Functions

- **sine:** $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$
- **cosine:** $\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$
- **tangent:** $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$
- **cosecant:** $\frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$
- **secant:** $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$
- **cotangent:** $\frac{d}{dx}[\tan^{-1} x] = \frac{-1}{1+x^2}$

$$\frac{dy}{dx}[e^{\cos^{-1} 3x}] = e^{\cos^{-1} 3x} \cdot \frac{-1}{\sqrt{1-(3x)^2}} \cdot 3$$
$$\frac{dy}{dx}[e^{\cos^{-1} 3x}] = \frac{-3e^{\cos^{-1} 3x}}{\sqrt{1-9x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x^3] = \frac{1}{1+(x^3)^2} \cdot 3x^2$$
$$\frac{d}{dx}[\tan^{-1} x^3] = \frac{3x^2}{1+x^6}$$

$$\frac{d}{dt}[\cot^{-1}(\cos t)] = \frac{-1}{1+(\cos t)^2} \cdot -\sin t$$
$$\frac{d}{dt}[\cot^{-1}(\cos t)] = \frac{\sin t}{1+\cos^2 t}$$

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$
$$\frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\tan^{-1} x] = \frac{-1}{1+x^2}$$

3.5 - Selecting Procedures for Calculating Derivatives

- first step is to figure out what derivative rule to start with
- take whichever derivative is on the outside, or comes first in the equation

Think before you calculate?

3.6 - Calculating Higher-Order Derivatives

- taking the derivative of an equation once is the first derivative
- taking the derivative of a derivative is the second derivative, denoted by y'', $f''(x)$, $\frac{d^2y}{dx}$, or $\frac{d^2}{dx^2}[f(x)]$
  - the number of primes can be replaced with n number of derivatives taken

$$f(x) = 2x^4 - 5x + \sin x$$
$$f'(x) = 8x^3 - 5 + \cos x$$
$$f''(x) = 24x^2 - \sin x$$
$$f'''(x) = 48x - \cos x$$

$$y = 2\sec x + e^x$$
$$\frac{dy}{dx} = 2\sec x \tan x + e^x$$
$$\frac{d^2y}{dx} = 2(\sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x) + e^x$$

$$f(x) = 8\sin(\tfrac{x}{2})$$
$$f'(x) = 8\cos(\tfrac{x}{2}) \cdot \tfrac{1}{2} = 4\cos(\tfrac{x}{2})$$
$$f''(x) = -4\sin(\tfrac{x}{2}) \cdot \tfrac{1}{2} = -2\sin(\tfrac{x}{2})$$

$$y^2 = 7y + 4x$$
$$2y\frac{dy}{dx} = 7\frac{dy}{dx} + 4$$
$$\frac{dy}{dx} = \frac{4}{2y-7} = 4(2y-7)^{-1}$$
$$\frac{d^2y}{2x^2} = -4(2y-7)^{-2} \cdot (2 \cdot 4(2y-7)^{-1})$$
$$\frac{d^2y}{2x^2} = -32(2y-7)^{-3}$$

📢 Finding a higher order derivative is taking the derivative of a derivative n amount of times