Unit 6: Integration and Accumulation of Change

6.4 - The Fundamental Theorem of Calculus and Accumulation Functions

- **integral-defined function:** if f is continuous on $[a, b]$ and $g(x) = \int_a^x f(t)dt$ for all x in $[a, b]$, then g is a function defined by an integral, and $g'(x) = f(x)$

$h(x) = \int_0^x (64 - 12x)dx$
$h'(x) = 64 - 12x$

$f(x) = \int_0^x (x^2 + 3)dx$
$f'(x) = x^2 + 3$

$g(x) = \int_0^x (3x^{\frac{1}{2}} + \sin x)$
$g'(x) = 3x^{\frac{1}{2}} + \sin x$

- if the lower limit of integration is the independent variable, just swap the limits and put a negative sign in front of the integral to make the lower limit the constant again
    - the lower limit a is supposed to be the constant, but sometimes it will be a variable

$\frac{d}{dx}[\int_x^3 (2^{-t^2} + t^4)dt] = \frac{d}{dx}[-\int_3^x (2^{-t^2} + t^4)dt]$

- when taking the derivative of an integral function, plug in the upper limit to the variable and take the derivative as normal like $\frac{d}{dx}[\int_1^{h(x)} f(t)dt] = f(h(x)) \cdot h'(x)$

$\frac{d}{dx}[\int_0^{2x} \ln(t^3 + 1)dt] = \ln((2x)^3 + 1)dt \cdot 2$

$\frac{d}{dx}[\int_3^{x^2} \cos(e^t)dt] = \cos(e^{x^2}) \cdot 2x$

- if there are functions of x in both limits of integration, rewrite the integral as a difference of two integrals with constant lower limits

$\frac{d}{dx}[\int_{3x^2}^{x^3} \sqrt{16 - t^2}dt]$
$\frac{d}{dx}(\int_0^{x^3} \sqrt{16 - t^2}dt - \int_0^{3x^2} \sqrt{16 - t^2})$
$3x^2\sqrt{16 - (x^3)^2} - 6x\sqrt{16 - (3x^2)^2}$

$\frac{d}{dx}[\int_{2x}^{\sin x} f(t)dt]$
$\frac{d}{dx}(\int_0^{\sin x} f(t)dt - \int_0^{2x} f(t)dt)$
$f(\sin x) \cdot \cos x - f(2x) \cdot 2$

- If the lower limit of integration is an independent variable, reverse the upper and lower limits and put a negative sign in front of the integral
- If the upper limit of integration has an independent variable in it, plug the variable into the equation and derive using the chain rule
- If both the upper and lower limits of integrations have variables, rewrite the integral into the difference of two integrals with constant and solve

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6.5 - Interpreting the Behavior of Accumulation Functions Involving Area

- increasing derivative → concave up
- decreasing derivative → concave down
- derivative maximum or minimum → point of inflection
- break down integrals into sections on the FRQs and keep track of first/second derivatives ?
- to find a horizontal tangent line of $f(x)$, look for when $f'(x)$ is equal to 0
    - to find its local maxima or minima, check if the sign changes from positive to negative or vice versa as the graph of $f'(x)$ crosses 0
    - also found when the second derivative changes signs, either from positive to negative, or from increasing to decreasing

📢 - Note which functions or graphs are first and second derivatives

6.6 - Applying Properties of Definite Integrals

- integration properties
  - $\int_a^a f(x)dx = 0$
  - $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function (symmetrical)
  - $\int_{-a}^a f(x)dx = 2$ if $f(x)$ is an even function (non-symmetrical)
  - given $a < b < c$, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
  - $\int_a^b f(x)dx \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
  - $\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$
  - $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- integrate using geometry with semicircle and absolute value equations

📢 - Use geometric shapes to calculate area under the curve

6.7 - The Fundamental Theorem of Calculus and Definite Integrals

- $\int_a^b f(x)dx = g(b) - g(a)$ if $g(x)$ is the antiderivative of $f(x)$

$\int_0^{\frac{\pi}{2}} (\cos t)dt = \sin(t)\Big|_0^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(0) = 1$

$\int_1^4 3x^2dx = x^3\Big|_1^4 = (4)^3 - (1)^3 = 63$

6.8 - Finding Antiderivatives and Indefinite Integrals - Basic Rules and Notations

- **antiderivative:** function g is the antiderivative of f if f is the derivative of g
  - x^2 is the antiderivative of $2x$
  - $\sin x$ is the antiderivative of $\cos x$
  - write antiderivatives by adding a constant C onto the end of the equation
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
- $\int kdx = kx + C$
- $\int (f'(x) + (g'(x))dx = f(x) + g(x) + C$
- $\int a \cdot f(x)dx = a\int f(x)dx$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln(a)} + C$

- Every derivative rule has an opposite integral rule

6.9 - Integrate Using Substitution

- when working with complicated equations to differentiate, assign the variable u to a section of the equation and substitute it in, then find the derivative with respect to x and plug the section of the equation back in

$\int 2x^2(3x^3-5)^{20}dx$
$u=(3x^3-5)$
$\frac{du}{dx}=9x^2$
$dx=\frac{du}{9x^2}$
$\int 2x^2(u)^{20}\cdot\frac{du}{9x^2}=2\int(u)^{20}\frac{du}{9}$
$\frac{2}{9}\int u^{20}du=\frac{2}{9}\cdot\frac{1}{21}u^{21}+C$
$\frac{2}{189}(3x^3-5)^{21}+C$

$\int 5x(x^2-5)^2dx$
$u=(x^2-5)$
$\frac{du}{dx}=2x$
$dx=\frac{du}{2x}$
$\int 5x(u)^2\cdot\frac{du}{2x}=5\int(u)^2\frac{du}{2}$
$\frac{5}{2}\int(u)^2du=\frac{5}{2}\cdot\frac{1}{3}u^3+C$
$\frac{5}{6}(x^2-5)+C$

$\int\tan(3x)dx$
$u=\cos(3x)$
$\frac{du}{dx}=-\sin(3x)\cdot 3$
$dx=\frac{du}{-3\sin(3x)}$
$\int\frac{\sin(3x)}{u}\cdot\frac{du}{-3\sin(3x)}=\int\frac{1}{u}\cdot\frac{du}{-3}$
$-\frac{1}{3}\int\frac{1}{u}\cdot du=-\ln|u|+C$
$-\frac{1}{3}\ln|\cos(3x)|+C$

$\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$
$u=\sqrt{x}$
$\frac{du}{dx}=\frac{1}{2\sqrt{x}}$
$dx=2du\sqrt{x}$
$\int\frac{e^u}{u}\cdot 2du\cdot u=2\int e^u du$
$2\int e^{\sqrt{x}}+C$

- certain functions call for a specific section to be the u in u-substitution
  - given an exponential function, let u be the power
  - given a trigonometric function, let u be the angle
  - given multiple trigonometric functions, let u be one of the functions, specifically the one whose derivative is the other function
- when looking at definite integrals, given $\int_a^b f'(g(x))g'(x)dx$, rearrange the integral to be $\int_{g(a)}^{g(b)} f'(u)du=f(u)\Big|_{g(a)}^{g(b)}$

$\int_0^5 3x(e^{4x^2-1})dx$
$u=4x^2-1$
$dx=\frac{du}{8x}$
$\int_{u(0)}^{u(5)} 3x(e^u)\cdot\frac{du}{8x}=\frac{3}{8}\int_{-1}^{99} e^u du=\frac{3}{8}e^u\Big|_{-1}^{99}$
$\frac{3}{8}(e^{99}-e^{-1})$

$\int_0^{\frac{\pi}{4}}(\tan x)(\sec^2 x)dx$
$u=\tan x$
$dx=\frac{du}{\sec^2 x}$
$\int_{u(0)}^{u(\frac{\pi}{4})} u\cdot\sec^2 x\cdot\frac{du}{\sec^2 x}=\int_0^1 u\cdot du=\frac{u^2}{2}\Big|_0^1$
$\frac{\tan^2(1)}{2}-\frac{tan^2(0)}{2}$

- U-substitution is where you replace an inner function with a letter to more easily differentiate the entire function using the chain rule, and then plugging the inner function back in at the end
- Substitution works best when you can get rid of a variable within the equation by cancelling it out
- For definite integrals, the upper and lower bounds are replaced by their respective values plugged into the inner function that's being substituted

6.10 - Integrating Functions Using Long Division and Completing the Square

- if the degree of the numerator is larger than the degree of the denominator, use long division to rewrite the integrand

$\int\frac{4x+4}{2x-1}dx$

$\int\frac{3x^{+}2+5x+3}{3x-1}dx$

$\frac{4x+4}{2x-1} = 2 + \frac{6}{2x-1}$
$\int 2 + \frac{6}{2x-1} dx = 2x + 3\ln|2x-1| + C$

$\frac{3x^2+5x+3}{3x-1} = x + 2 + \frac{5}{3x-1}$
$\int x + 2 + \frac{5}{3x-1} dx = \frac{1}{2}x^2 + 2x + \frac{5}{3}\ln|3x-1| + C$

- if after long division, the remainder has two terms in its numerator, split them into two separate fractions and integrate using u-substitution

$\int \frac{x^3+3x^2-1}{x^2+1} dx$
$\frac{x^3+3x^2-1}{x^2+1} = x + 3 - \frac{x+4}{x^2-1}$
$\int x + 3 - \frac{x+4}{x^2-1} = \frac{x^2}{2} + 3x - \int \frac{x}{x^2+1} dx - \int \frac{4}{x^2+1} dx$
$\frac{x^2}{2} + 3x - \frac{1}{2}\ln|x^2+1| - 4\tan^{-1}x + C$

- inverse trig rules can be used to help integrate rational functions
  - if the radicand isn't factorable, we can factor out values to complete the square

$\int \frac{8}{\sqrt{1-5x^2}} dx$
$a^2 = 1$
$a = 1$
$u^2 = 5x^2$
$u = \sqrt{5}x$
$dx = \frac{du}{\sqrt{5}}$
$\int \frac{8}{\sqrt{1-u^2}} \cdot \frac{du}{\sqrt{5}} = \frac{8}{\sqrt{5}} \int \frac{du}{\sqrt{1-u^2}}$
$\frac{8}{\sqrt{5}} \sin^{-1}(\frac{\sqrt{5}}{1}) + C$

$\int \frac{8}{\sqrt{1-50x-5x^2}} dx = 8\int \frac{dx}{\sqrt{1-5(x^2+10x)}}$
$1 - 5(x^2+10x) = 126 - 5(x^2+10x+25) = 126 - 5(x+5)^2$
$8\int \frac{dx}{\sqrt{126-5(x+5)^2}}$
$a = \sqrt{126}$
$u = \sqrt{5}(x+5)$
$\frac{8}{\sqrt{5}} \int \frac{du}{126-u^2} = \frac{8}{\sqrt{5}} \sin^{-1}(\frac{\sqrt{5}(x+5)}{\sqrt{126}}) + C$

> 📢 - Use long division to rewrite the integrand when the degree of the denominator is smaller than the degree of the numerator
> - Use complete the square when the denominator of the radicand is a square root that isn't able to be separated into a and u

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6.14 - Selecting Techniques for Antidifferentiation

- "short" division: when the numerator of the integrand is made up of two terms that can be split apart ($\int \frac{1+x^2}{2x} = \int \frac{1}{2x} + \int \frac{x^2}{2x}$)
- long division: when the degree of the numerator is equal to or greater than the degree of the denominator
- completing the square: when there is a quadratic polynomial in the denominator that can't be easily substituted ($\int \frac{12}{\sqrt{9-x^2+6x}}$)
- u-substitution: when the derivative of the inner function is equal to the outer function so that it can be cancelled out
  - if the inside function is linear, you don't need u-substitution because you can just multiply by the derivative of the inside

> 📢 - There are a lot of different ways to solve antiderivatives, but each method is for a specific type of function