Unit 2: Differentiation: Definition and Fundamental Properties

2.5 - The Power Rule

- **power function:** $f(x) = x^n$ where n is a real number
  - examples include: $f(x) = x^5$ $v(t) = t^{\frac{7}{6}}$ $y = x^e$
  - functions can be rewritten in the form of power function
    - examples include: $f(x) = \frac{1}{x^2} = x^{-2}$ $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $y = \frac{1}{x} = x^{-1}$
  - if the exponent value isn't a constant, then the function isn't a power function
    - $p(x) = 2^x$ exponent is a variable
    - $y = e^x$ e is a constant, but it is not the exponent value
- **power rule:** $\frac{d}{dx}[x^n] = nx^{n-1}$ if $f(x)$ is a *power function*

$$f(x) = x^5 \\ f'(x) = 5x^{5-1} \\ f'(x) = 5x^4$$

$$v(t) = t^{\frac{7}{6}} \\ v'(t) = \frac{7}{6}t^{\frac{7}{6}-1} \\ v'(t) = \frac{7}{6}t^{\frac{1}{6}}$$

$$y = x^e \\ y' = ex^{e-1}$$

$$f(x) = x^{-2} \\ f'(x) = -2x^{-2-1} \\ f'(x) = -2x^{-3} = \frac{-2}{x^3}$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \\ f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} \\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$y = \frac{1}{x} = x^{-1} \\ \frac{dy}{dx} = -1x^{-1-1} \\ \frac{dy}{dx} = -1x^{-2} = -\frac{1}{x^2}$$

> **Power Function:** $f(x) = x^n$
> **Power Rule:** $\frac{d}{dx}[x^n] = nx^{n-1}$

2.6 - Derivative Rules - Constant, Sum, Difference, and Constant Multiple

- **constant rule:** the derivative of any constant term is 0
  - average rate of change of a horizontal line is 0
- **constant multiple rule:** $\frac{d}{dx}[kx^n] = k \cdot nx^{n-1}$

$$f(x) = -16x^2 \\ f'(x) = -16 \cdot 2x \\ f'(x) = -32x$$

$$f(x) = \frac{4}{3}\pi x^3 \\ f'(x) = \frac{4}{3}\pi \cdot 3x^2 \\ f'(x) = 4\pi x^2$$

$$f(x) = -0.75x \\ f'(x) = -0.75 \cdot 1 \\ f'(x) = -0.75$$

- **sum and difference rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$ where the derivative is equal to the respective sum or difference of the derivatives of each term

$$h(t) = \frac{1}{2}gt^2 + vt + k \\ h'(t) = (\frac{1}{2}g \cdot 2t) + v + 0 \\ h'(t) = gt + v$$

$$g(x) = 15x - 8\sqrt{x^3} \\ g'(x) = 15 - (8 \cdot \frac{3}{2}x^{\frac{1}{2}}) \\ g'(x) = 15 - 12x^{\frac{1}{2}} = 15 - 12\sqrt{x}$$

$$y = \frac{3x^5 + 2x - 7}{x^2} \\ y = \frac{3x^5}{x^2} + \frac{2x}{x^2} - \frac{7}{x^2} \\ y = 3x^3 + 2x^{-1} - 7x^{-2} \\ y' = (3 * 3x^2) + (2 * -1x^{-2}) - (7 * -2x^{-3}) \\ y' = 9x^2 - 2x^{-2} + 14x^{-3}$$

Constant Rule: $\frac{d}{dx}[k] = 0$
Constant Multiple Rule: $\frac{d}{dx}[kx^n] = k \cdot nx^{x-1}$
Sum and Difference Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

2.7 - Derivatives of $\cos x$, $\sin x$, e^x, and $\ln x$

- **sine:** $\frac{d}{dx}[\sin x] = \cos x$
- **cosine:** $\frac{d}{dx}[\cos x] = -\sin x$
- **e^x:** $\frac{d}{dx}[e^x] = e^x$
- **natural log:** $\frac{d}{dx}[\ln x] = \frac{1}{x}$
- limits can be changed to derivative form to which can be evaluated easier
  - if the function is defined as $\lim\limits_{h \to 0}[\frac{f(x+h)-f(x)}{h}]$
    - take the first term in the numerator and remove the h
    - if the term has a value plugged in, you are evaluating for the derivative at that value
    - if the term has a variable, evaluate the equation of the derivative
  - if the function is defined as $\lim\limits_{x \to a}[\frac{f(x)-f(a)}{x-a}]$
    - get the equation of the derivative from $f(x)$
    - evaluate for the derivative at the a value

$\lim\limits_{h \to 0}[\frac{(\sin\frac{\pi}{3}+h)-(sin\frac{\pi}{3})}{h}]$
$f(x) = \sin x$
$f'(x) = \cos x$
$f'(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\lim\limits_{h \to 0}[\frac{\frac{1}{h+2}-\frac{1}{2}}{h}]$
$f(x) = \frac{1}{x} = x^{-1}$
$f'(x) = -1x^{-2} = \frac{-1}{x^2}$
$f'(2) = \frac{-1}{2^2} = -\frac{1}{4}$

$\lim\limits_{h \to 0}[\frac{(x+h)^4-x^4}{h}]$
$f(x) = x^4$
$f'(x) = 4x^3$

$\lim\limits_{t \to 9}[\frac{\sqrt{t}-3}{t-9}]$
$f(x) = \sqrt{x} = x^{\frac{1}{2}}$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$\lim\limits_{\theta \to \pi}[\frac{\cos\theta - \cos\pi}{\theta - \pi}]$
$f(x) = \cos x$
$f'(x) = -\sin x$
$f'(\pi) = -\sin\pi = 0$

$\lim\limits_{x \to 3}[\frac{2x^2-5x-3}{x-3}]$
$\lim\limits_{x \to 3}[\frac{(2x+1)(x-3)}{(x-3)}]$
$\lim\limits_{x \to 3}[2x+1]$
$f'(x) = 2(3) + 1 = 7$

$\frac{d}{dx}[\sin x] = \cos x$
$\frac{d}{dx}[\cos x] = -\sin x$
$\frac{d}{dx}[e^x] = e^x$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$

2.8 - The Product Rule

- **product rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = g(x)f'(x) + f(x)g'(x)$ if f and g are differentiable at x
  - $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
  - can distribute first and then differentiate

$h(x) = (5x^2 - 2)(8x + 3)$
$h'(x) = (8x + 3) \cdot 10x + (5x^2 - 2) \cdot 8$
$h'(x) = 80x^2 + 30x + 40x^2 - 16$
$h'(x) = 120x^2 + 30x - 16$

$g(x) = (4x - 1)(2x^3 + 7)$
$g'(x) = (2x^3 + 7) \cdot 4 + (4x - 1) \cdot 6x^2$
$g'(x) = 8x^3 + 28 + 24x^3 - 6x^2$
$g'(x) = 32x^3 - 6x^2 + 28$

Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = g(x)f'(x) + f(x)g'(x)$

2.9 - The Quotient Rule

- **quotient rule:** $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ if f and g are differentiable at x and if $g(x) \neq 0$
  - $\frac{d}{dx}[\frac{u}{v}] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
  - quotient rule song: *lo dee hi mine hi dee lo o' lo lo*
  - sometimes you can simplify the equations so that you don't need the quotient rule

$$g(x) = \frac{5x^2+2}{8x-3}$$
$$g'(x) = \frac{(8x-3) \cdot 10x + (5x^2+2) \cdot 8}{(8x-3)^2}$$
$$g'(x) = \frac{80x^2 - 30x + 40x^2 + 16}{(8x-3)^2}$$
$$g'(x) = \frac{120x^2 - 30x + 16}{(8x-3)^2}$$

$$f(x) = \frac{5x + \frac{2}{x}}{8 - \frac{3}{x}} = \frac{5x^2+2}{8x-3}$$
$$f'(x) = \frac{(8x-3) \cdot 10x - (5x^2+2) \cdot 8}{(8x-3)^2}$$
$$f'(x) = \frac{80x^2 - 30x - 40x^2 - 16}{(8x-3)^2}$$
$$f'(x) = \frac{40x^2 - 30x - 16}{(8x-3)^2}$$

$$g(x) = \frac{5x^2+2}{3} = \frac{1}{3}(5x^2+2)$$
$$g'(x) = \frac{1}{3} \cdot 10x$$
$$g'(x) = \frac{10}{3}x$$

$$h(x) = \frac{3}{5x^2} = \frac{3}{5}x^{-2}$$
$$h'(x) = \frac{3}{5} \cdot -2x^{-3}$$
$$h'(x) = \frac{-6}{5x^3}$$

Quotient Rule: $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

2.10 - Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

- **tangent:** $\frac{d}{dx}[\tan x] = \sec^2 x$
  - use $\frac{\sin x}{\cos x}$ and the *quotient rule* to get the derivative of tangent
- **cosecant:** $\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x$
- **secant:** $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$
- **cotangent:** $\frac{d}{dx}[\cot x] = -\csc^2 x$
- point-slope form $y - f(a) = f'(a)(x-a)$ can be used to determine slope of the line tangent to a point

$$g(x) = \frac{\tan x + 4}{\sec x}$$
$$g'(x) = \frac{\tan x}{\sec x} + \frac{4}{\sec x}$$
$$g'(x) = \sin x + 4\cos x$$

$$h(x) = \frac{7}{5\sec x}$$
$$h'(x) = \frac{7}{5}\sec^{-1} x$$
$$h'(x) = -\frac{7}{5}(\sec x \tan x)$$

$$f(x) = 3\cot x + \ln x \csc x$$
$$f'(x) = 3(-\csc^2 x) + (\csc x)(\tfrac{1}{x}) + (\ln x)(-\csc x \cot x)$$
$$f'(x) = -3\csc^2 x + \frac{\csc x}{x} - \ln x \csc x \cot x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$
$$\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x$$
$$\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$$
$$\frac{d}{dx}[\cot x] = -\csc^2 x$$