Math 135 – Differential Equations

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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3^{rd} by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

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List of Theorems

List of Definitions

§1 Lec 1: Sep 27, 2021

§1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions } \to \text{ functions}$$

Example 1.1
- differentiation: $\frac{d}{dx}: \ f \mapsto f'$
- integration: $\int^x dx: \ f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L} : f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f : [0, \infty) \to \mathbb{R}$ and $F : \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2
- $f(t): \ t \mapsto 0 \implies \mathscr{L}[0] = 0$
- $f(t) = 1$

$$\begin{aligned}
\mathscr{L}[1] &= \lim_{t\to\infty} \int_0^t e^{-st}\,dt \\
&= \lim_{t\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^t \\
&= \lim_{t\to\infty} \left(\frac{e^{-st}}{-s} + \frac{1}{s}\right) \\
&= \frac{1}{s} \ \text{ if } \ \mathrm{Re}(s) > 0
\end{aligned}$$

Example 1.3 • Consider

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty te^{-st}\,dt \\ &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2} \text{ if } \operatorname{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \operatorname{Re}(s) > 0,\ \ n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}, \qquad \operatorname{Re}(s) > a \\ \mathscr{L}\left[\cos\omega t\right] &= \frac{s}{s^2+\omega^2} \\ \mathscr{L}\left[\sin\omega t\right] &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

4. Properties:

   a) Linear!

$$\begin{aligned}\mathscr{L}\left[f+g\right] &= \mathscr{L}\left[f\right] + \mathscr{L}\left[g\right] \\ \mathscr{L}\left[af\right] &= a\mathscr{L}\left[f\right]\end{aligned}$$

   b) Consider:

$$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= F(s-a) \quad \text{if } \operatorname{Re}(s-a) > 0\end{aligned}$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

$$\mathscr{L}\left[f(t-a)\right] = \int_0^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(t')e^{-st'}\,dt' e^{-sa}$$

   where $t' = t-a$. So

$$\mathscr{L}\left[f(t-a)\right] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

$$\begin{aligned}\mathscr{L}\left[f'\right] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt \\ &= sF(s) - f(0)\end{aligned}$$

§2 Lec 2: Sep 29, 2021

§2.1 Laplace Transform (Cont'd)

Recap: $\mathscr{L}: \ f \to F$

$$\mathscr{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1
- $\mathscr{L}\left[t^n\right] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$
- $\mathscr{L}\left[e^{at}\right] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting $\leftrightarrow$ multiplying by exponential
- $\mathscr{L}\left[f'\right] = s\mathscr{L}\left[f\right] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0,\ \ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\begin{aligned} \mathscr{L}\left[f'' + af' + bf\right] &= \mathscr{L}\left[g\right] \\ \mathscr{L}\left[f''\right] + a\mathscr{L}\left[f'\right] + b\mathscr{L}\left[f\right] &= G(s) \end{aligned}$$

Notice that

$$\mathscr{L}\left[f''\right] = s^2F - sf(0) - f'(0)$$

So

$$\begin{aligned} \left(s^2 + as + b\right)F(s) &= G(s) + (s+a)f_0 + f_0' \\ F(s) &= \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b} \end{aligned}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$\begin{aligned}(s^2+4)F(s) &= \frac{4}{s^2} + s + 5 \\ F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2+4} \\ &= \frac{4}{s^2\left(s^2+4\right)} + \frac{s}{s^2+4} + \frac{5}{s^2+4}\end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned}\frac{4}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\ 4 &= A(s^2+4) + Bs^2 \\ &= (A+B)s^2 + 4A\end{aligned}$$

So, $A = 1,\ B = -1$. Then,

$$\begin{aligned}F(s) &= \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{5}{s^2+4} \\ &= \frac{1}{s^2} + \frac{4}{s^2+4} + \frac{s}{s^2+4} \\ \mathscr{L}\left[f\right] &= \mathscr{L}\left[t + 2\sin 2t + \cos 2t\right] \\ \implies f &= t + 2\sin 2t + \cos 2t\end{aligned}$$

§3 Lec 3: Oct 1, 2021

§3.1 Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

Note: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall\varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

Comparison Test: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)
2. fe^{-st} doesn't decay fast enough as $t\to\infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.
2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \;\&\; c$$
Have
$$\begin{aligned} c^{-t} &\le 1\cdot e^{-t} \qquad \forall t > 0\\ 1 &\le 1\cdot e^{0t} \qquad \forall t > 0\\ t &\le 1\cdot e^{t} \qquad \forall t > 0 \end{aligned}$$

Theorem 3.1

If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$.

Proof. Have

$$\begin{aligned}
\mathscr{L}\left[f\right](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
\lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt &\le \lim_{\tau\to\infty}\int_0^\tau \left|f(t)e^{-st}\right|\,dt \\
&= \lim_{\tau\to\infty}\int_0^\tau |f(t)|\,e^{-s_r t}\,dt \\
&\le \lim_{\tau\to\infty}\int_0^\tau Me^{ct}\cdot e^{-s_r t}\,dt \\
&= \lim_{\tau\to\infty} M\left[\frac{e^{(c-s_r)t}}{-(c-s_r)}\right]_0^\tau \\
&= \frac{1}{s_r-c} \quad \text{if } s_r > c \\
&< \infty
\end{aligned}$$

Thus, $\mathscr{L}\left[f\right]$ exists (for $\mathrm{Re}(s) > c$) by comparison test. □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$\begin{aligned}
\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt \\
&= s^{-\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx \\
&= s^{-\frac{1}{2}}2\int_0^\infty e^{-z^2}\,dz \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0, \infty)$.

§4 Lec 4: Oct 4, 2021

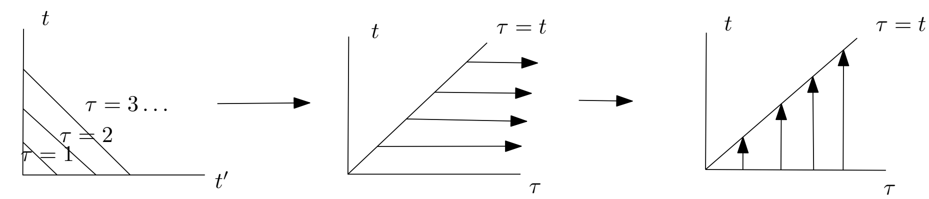
§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \end{aligned}$$

Let's define $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau-t)e^{-s\tau}\,d\tau\,dt \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau-t)e^{-s\tau}\,dt\right)d\tau \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau-t)\,dt\right)e^{-s\tau}\,d\tau \\ &= \mathscr{L}\left[\int_0^\tau f(t)g(\tau-t)\,dt\right] \end{aligned}$$

Theorem 4.1 (Convolution)
We have

$$\begin{aligned} (f*g)(\tau) &= \int_0^\tau f(t)g(\tau-t)\,dt \\ \mathscr{L}[f*g] &= \mathscr{L}[f]\cdot\mathscr{L}[g] \end{aligned}$$

§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau-t)f(t)\,dt$$

Notice

$$\begin{aligned}\mathbf{f} &= \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) &\approx f_i \\ g(\tau) &\approx g_i \\ k(\tau - t) &\approx K_{ij}\end{aligned}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathscr{L}[f] &= \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f] \\ \mathscr{L}[f] &= \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}\end{aligned}$$

Example 4.2

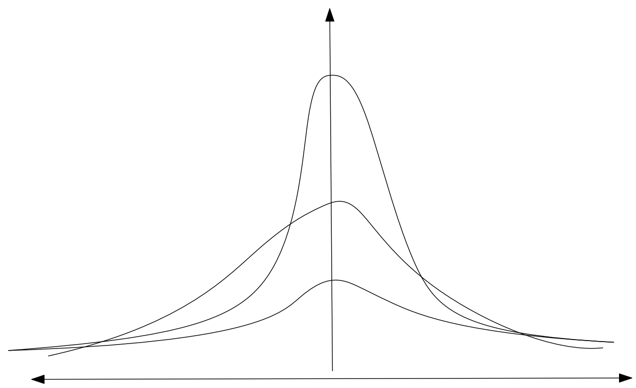
Consider $f(t) = t^3 + \int_0^t \sin(t-\tau) f(\tau) d\tau$.

$$\begin{aligned}F(s) &= \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s) \\ &\vdots \\ F(s) &= 3!(s^{-4} + s^{-6}) \\ f(t) &= t^3 + \frac{t^5}{20}\end{aligned}$$

§5 Lec 5: Oct 6, 2021

§5.1 Dirac Delta "Function"

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t) : \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\,d\tau$$

where $\delta(\tau - t) = \delta(t - \tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n \times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\begin{aligned} \mathbf{e}_i &: \text{ unit vector} \\ \mathbf{f}_i &: \text{ unit response vector} \end{aligned}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$
2. Linear superposition means
$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned} L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\ &= \sum_i h_i L\mathbf{f}_i \\ &= \sum_i h_i \mathbf{e_i} \\ &= \mathbf{h} \end{aligned}$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$
2. Then,

$$\begin{aligned} f &= h * g \\ &= \int_0^t g_t(\tau)h(\tau)\,d\tau \\ &= \int_0^t g(t-\tau)h(\tau)\,d\tau \end{aligned}$$

where g is known as the Green function.

$$e_i \approx \delta_t$$
$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

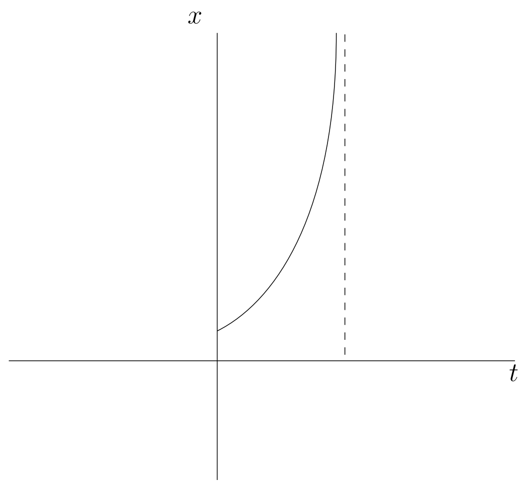
§6 Lec 6: Oct 08, 2021

§6.1 Existence & Uniqueness of ODE Solutions

Intuitively, $f(t,x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.

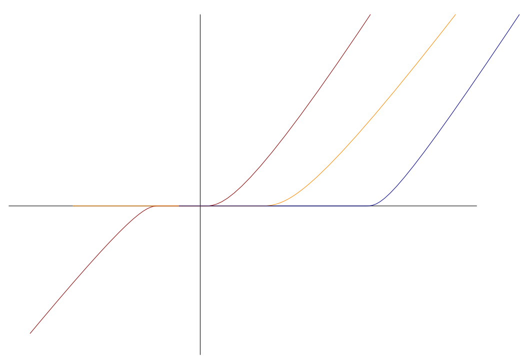
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$



   We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?

$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$



   This has infinite number of solution through $(0,0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution

§7 Lec 7: Oct 11, 2021

§7.1 Picard Iteration

Goal: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

Idea:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating
$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$
Then, we can transform it into an integral equation
$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t')\right)\, dt'$$
Notice that f is continuous and x is continuous imply x is differentiable.
2. Iteration: If we can't solve it at first, try again.

> **Example 7.1**
> Newton's root-finding algorithm
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Picard Iteration: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t)\right)\, dt'$$

Start with a guess for the function $x_0(t) = x_0$ (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f\left(t', x_n(t')\right)\, dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \underset{\infty}{\longrightarrow} \dots$$

If $x_{n+1}(t) = x_n(t) = \overline{x}(t)$, then $\overline{x}(t)$ has to solve the IE. We want $\lim_{n\to\infty} x_n(t) \to x(t)$ solves IE.

Example 7.2

Consider $\dot{x}(t) = x(t)$, $x(0) = 1$. This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t')\,dt'$$

Picard:

$$\begin{aligned}
x_0(t) &= 1 \\
x_1(t) &= 1 + \int_0^t x_0(t')\,dt' = 1 + \int_0^t 1\,dt' \\
&= 1 + t \\
x_2(t) &= 1 + \int_0^t 1 + t\,dt \\
&= 1 + t + \frac{t^2}{2!} \\
&\vdots \\
x_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!}
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n(t) \to e^t$$

§8 Lec 8: Oct 13, 2021

§8.1 Continuity

Limit of continuous function is not necessarily continuous.

Example 8.1

Consider $x_n(t) = t^n$ on $[0,1]$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= t \\ x_2 &= t^2 \\ &\vdots \\ \overline{x} = \lim_{n\to\infty} x_n &= \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases} \end{aligned}$$

which is discontinuous.

<u>Idea</u>: We need "more" continuity. Given x, and given any $\varepsilon > 0$, if $|x - x'| < \delta(x, \varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Example 8.2

Consider $f(x) = x$ on $\mathbb{R}$. We can see that

$$|x - x'| < \varepsilon \quad \forall\, |x - x'| < \varepsilon$$

in which we pick $\delta(x, \varepsilon) = \varepsilon$.
How about $f(x) = x^2$ on $\mathbb{R}$?

$$\left|x^2 - y^2\right| < \varepsilon$$

If we pick $\delta(x, \varepsilon) = \varepsilon$, then $|x - y| < \delta = \varepsilon$ which does not necessarily imply $\left|x^2 - y^2\right| < \varepsilon$ because

$$\begin{aligned} \left|x^2 - y^2\right| &= |(x+y)(x-y)| \\ &= |x+y|\,|x-y| \\ &\le \varepsilon\,|x+y| \end{aligned}$$

$|f(x) - f(y)| > \varepsilon$. So we need to pick smaller δ as x and y get larger. It would work for $\delta = \frac{\varepsilon}{2\max(|x|,|y|)}$.

Question 8.1. Is $\frac{1}{x}$ continuous?

Ans: It depends on the domain. If we're talking about $\mathbb{R}$, it doesn't work at 0; on $(0, \infty)$, yes it's continuous.

Definition 8.3 (Uniform Continuity) — $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark 8.4. Notice that the definition is similar to continuity except that δ doesn't depend on x.

Example 8.5

x^2 on $\mathbb{R}$ is not uniformly continuous but x^2 on $(a, b) \subseteq \mathbb{R}$ is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|, |y|)} = \frac{\varepsilon}{\max(|a|, |b|)}$$

Remark 8.6. Uniform continuity also depends on the domain as continuity does.

Exercise 8.1. Is $x^{\frac{1}{2}}$ uniformly continuous on $[0, 1]$?

Lipschitz Continuity: "gradient is bounded"

$$\frac{|f(x) - f(y)|}{|x - y|} < L < \infty$$

We can pick $\delta = \frac{\varepsilon}{L}$ everywhere.

Example 8.7

- x^2 on $\mathbb{R}$ is not Lipschitz but it is on a finite interval.
- $x^{\frac{1}{2}}$ is not Lipschitz continuous on $[0, 1]$. However, it's uniformly continuous.

§9 Lec 9: Oct 15, 2021

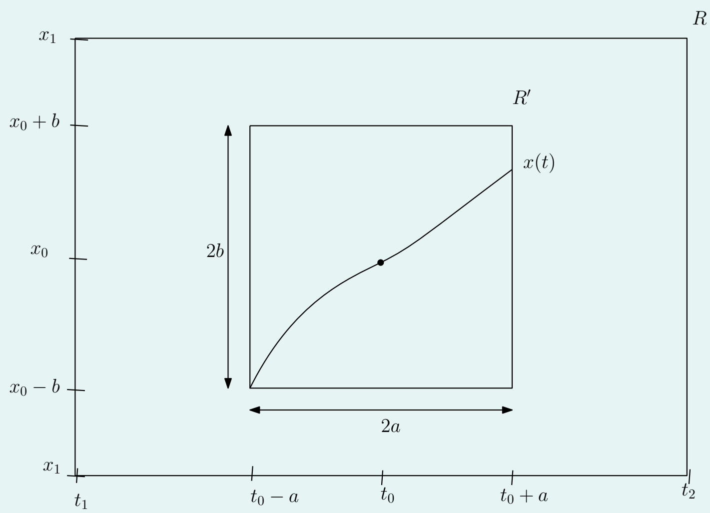
§9.1 Picard's Theorem

Let's prove local existence of the theorem.

Theorem 9.1 (Picard)
If $f(t,x)$ and $\partial_x f(t,x)$ are continuous function on a bounded rectangle $R = [t_1, t_2] \times [x_1, x_2]$ and (t_0, x_0) is in interior of R ($t_1 < t_0 < t_2$, $x_1 < x_0 < x_2$). Then $\exists$ a smaller rectangle $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ s.t. ODE

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

has a solution in R'.



Note: Since R closed and bounded, then f, $\partial_x f$ are bounded, i.e.,

$$\max_R f(t,x) = M$$
$$\max_R \partial_x f(t,x) = L$$

Thus, f is Lipschitz.
Proof Outline:

1. Solving ODE $\iff$ Soling IE
2. Approximate solutions using Picard iteration
$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^{t} f\left(t', x_{n-1}(t')\right)\, dt'$$
3. Prove Picard iterates converges
$$\lim_{n\to\infty} x_n(t) \to \overline{x}(t)$$
4. Prove limit $\overline{x}(t)$ solves IE.
5. Prove limit $\overline{x}(t)$ is continuous.

6. Prove limit $\overline{x}(t)$ is unique.
7. How big is $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$?

$$\text{Pick } a \ \ni aL < 1 \ \& \ b = Ma \le |x_0 - x_1| \, |x_0 - x_2|$$

Proof. 2. Prove Picard iterates converge

a) We have

$$\lim_{n\to\infty} x_n(t) \iff \lim_{n\to\infty} x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

telescoping sum!

b) Series $x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$ converges by Weierstrass M-test – If $|f_n(x)| < M_n$ $\forall n \in \mathbb{N}$, $x \in D$ and $\sum_{n=0}^{\infty} M_n$ converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

i) Show $x_i(t)$ are all in $R' \subseteq R$ so we can use bounds L, M.

$$\begin{aligned} |x_0(t) - x_0| &= 0 \\ |x_1(t) - x_0| &= \left| \int_{t_0}^{t} f\left(t', x_0(t')\right) \, dt' \right| \\ &\le \int_{t_0}^{t} \left| f\left(t', x_0(t')\right) \right| dt \\ &\le \int_{t_0}^{t} M \, dt \\ &\le Ma = b \end{aligned}$$

Thus, $x_1(t)$ is in the rectangle. By induction, every $x_n(t)$ in $R' \subseteq R$.

ii) Show $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$ is bounded.
Define $\Delta = \max_{R'} |x_1(t) - x_0|$. Then

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right) \, dt' \right| \\ &\le \int_{t_0}^{t} \left| f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right) \right| dt' \\ &\le \int_{t_0}^{t} L \left| x_1(t') - x_0(t') \right| dt' \\ &\le \Delta a L \end{aligned}$$

and

$$\begin{aligned} |x_3(t) - x_2(t)| &= \left| \int_{t_0}^{t} f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right) \, dt \right| \\ &\le \int_{t_0}^{t} \left| f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right) \right| dt \\ &\le \int_{t_0}^{t} L \left| x_2(t') - x_1(t') \right| dt' \\ &\le L \left(\Delta a L\right) \left(t - t_0\right) \\ &\le \Delta (aL)^2 \end{aligned}$$

Every $|x_n(t) - x_{n-1}(t)|$ depends on $|x_{n-1}(t) - x_{n-2}(t)|$ recursively. The general pattern is

$$\begin{aligned}|x_n(t) - x_{n-1}(t)| &\leq \Delta(aL)^{n-1} \\ \sum_{n=1}^{\infty} |x_n - x_{n-1}| &\leq \sum_{n=0}^{\infty} \Delta(aL)^n \\ &= \frac{\Delta}{1-aL} \\ &< \infty\end{aligned}$$

Thus, $\sum x_n - x_{n-1}$ converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n\to\infty} x_n(t) = \overline{x}(t) \text{ exists!}$$

3. $\overline{x}$ solves I.E.

   Idea: We know $|\overline{x} - x_n|$ gets small so break $\left|\overline{x} - x_0 - \int_{t_0}^t f\left(t', \overline{x}(t')\right) dt'\right|$ into pieces like $|\overline{x} - x_n(t)|$.

   $$\text{subtract } x_n(t) - x_0 - \int_{t_0}^t f\left(t', x_{n-1}(t')\right)\, dt' = 0$$

   Let $\kappa = \left|\overline{x} - x_0 - \int_{t_0}^t f\left(t', \overline{x}(t')\right) dt'\right|$.

   $$\begin{aligned}\kappa &= \left|-(x_n - x_0 - \int_{t_0}^t f\left(t', x_{n-1}(t')\right) dt'\right| \\ &\leq |\overline{x} - x_n| + \left|\int_{t_0}^t f(t, \overline{x}) - f\left(t, x_{n-1}\right) dt\right| \\ &\leq |\overline{x} - x_n| + \int_{t_0}^t \left|f(t, \overline{x}) - f\left(t, x_{n-1}\right)\right| dt \\ &\leq |\overline{x} - x_n| + aL\,|\overline{x} - x_{n-1}|\end{aligned}$$

   which approaches 0 as $n \to \infty$ because $\lim_{n\to\infty} x_n = \overline{x}$.

4. $\overline{x} = \lim_{n\to\infty} x_n$ is continuous, i.e., given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

   $$|t - t'| < \delta \implies |\overline{x}(t) - \overline{x}(t')| < \varepsilon$$

   Idea: Split into known things

   $$\begin{aligned}|\overline{x}(t) - \overline{x}(t')| &= |\overline{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \overline{x}(t)| \\ &\leq |\overline{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \overline{x}(t)|\end{aligned}$$

   We pick n s.t. $|\overline{x}(t) - x_n(t)| < \frac{\varepsilon}{3}$ $\forall t$ which is possible because Weierstrass implies uniform convergence. Then pick δ s.t.

   $$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall\, |t - t'| < \delta$$

   which is possible because x_n is continuous.

5. $\overline{x}$ is unique.

   Idea: Prove $|\overline{x} - \tilde{x}| \leq |\overline{x} - \tilde{x}|$.

   - If $\tilde{u}$ is other solution, it also exists in R'.

Proof. (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some $|t_* - t| < a$. But

$$\begin{aligned}
|\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| \, dt' \\
&\leq M\left(t_* - t_0\right) \\
< Ma = b
\end{aligned}$$

Contradiction! ☐

- Have

$$\begin{aligned}
|\overline{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^{t} f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t} \left| f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) \right| dt' \\
&\leq \int_{t_0}^{t} L \max |\overline{x}(t') - \tilde{x}(t')| \, dt \\
&\leq La \max |\overline{x}(t') - \tilde{x}(t')| \\
\max |\overline{x}(t) - \tilde{x}(t)| &\leq \max |\overline{x}(t) - \tilde{x}(t)|
\end{aligned}$$

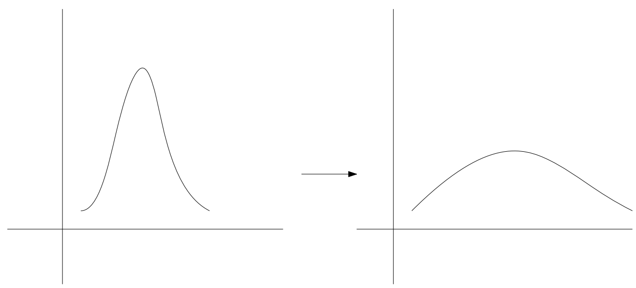
which is only possible if $\overline{x}(t) - \tilde{x}(t) = 0$, i.e., solution is unique. ☐

§10 Lec 10: Oct 18, 2021

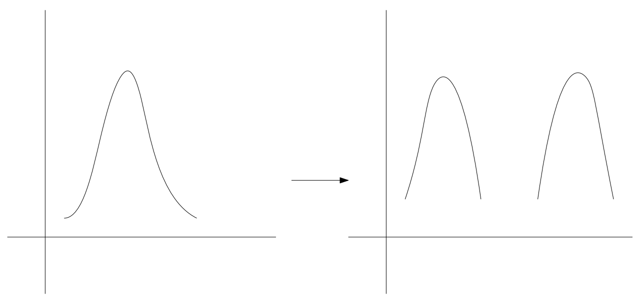
§10.1 Fourier Series

Goal: Solve linear PDE: 3 canonical examples

1. Heat/Diffusion equation
$$\partial_t u(t,x) - \partial_x^2 u(t,x) = 0$$



2. Wave equation
$$\partial_t^2 u = \partial_x^2 u$$



3. Laplace equation:
$$\partial_x^2 u + \partial_y^2 u = 0$$

Question 10.1. How do we solve linear PDEs?

Use linearity to split big problems into small ones that you can solve (find the eigenvectors). Then we split 1 PDE $\to \infty$ ODEs. First, let's define Fourier series.

Definition 10.1 (Fourier Series) — Fourier Series is a function written as a sum of sines and cosines

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx) \\ &= \sum_{-\infty}^{\infty} c_n e^{inx} \end{aligned}$$

where $c_n = c_r + ic_{in}$.

They have amazing properties:

1. They can approximate almost anything
   - analytic function
   - smooth function
   - periodic function
   - differentiable function
   - continuous/discontinuous function
2. They simplify differentiation!

$$\frac{d}{dx}e^{ikx} = ike^{ikx}$$
$$\frac{d^2}{dx^2}\sin kx = -k^2\sin kx$$
$$\frac{d^2}{dx^2}\cos kx = -k^2\cos kx$$

   Just like Laplace transform, Fourier series transform differentiation into multiplication problem (easier to deal with).

3. Fourier series are orthogonal

$$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = 0$$

   or

$$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = 0 \quad \text{if } m \neq n$$

   or

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = 0 \quad \text{if } m \neq n$$

   This gives easy formulas

From these facts follow from linear algebra, because Fourier series are eigenfuncitons of differentiation. They are the correct basis to solve linear PDEs.

§11 Lec 11: Oct 20, 2021

§11.1 Coefficients of Fourier Series

Question 11.1. How do we calculate Fourier Series $a_n, b_n =$?

Consider the domain: $[-\pi, \pi]$, finite dimensions N, vector

$$\mathbf{u} = \sum u_i \mathbf{e}_i$$

How do we calculate u_i?

$$\begin{aligned}\mathbf{u} \cdot \mathbf{e}_i &= \left(\sum_{i=1}^{N} u_i e_i\right) \cdot e_j \\ &= \sum_{i=1}^{N} u_i \left(e_i - e_j\right) \\ &= \sum_{i=1}^{N} \delta_{ij}\end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We want to do this in ∞ dimensions – inner product

$$N: \ \langle u, v\rangle = u \cdot v = \sum_{i=1}^{N} u_i v_i$$

$$\infty: \ \langle u, v\rangle \propto \int_a^b u(x)v(x)\, dx$$

Inner Product: $\langle u, v\rangle \to \mathbb{R}$ takes in two function & spits out a number. It has to satisfy the following properties

1. Bilinear

$$\langle au + bv, \rangle = a\langle u, vw\rangle + b\langle v, w\rangle$$

2. Symmetric $\langle u, v\rangle = \langle u, v\rangle$.
3. Positivity: $\langle u, u\rangle > 0$ unless $u = 0$.

Inner products are important

- They imply a norm $\|u\| = \sqrt{\langle u, u\rangle}$
- Cauchy-Schwarz Inequality

$$\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle$$

- Triangle inequality

$$\|u + v\| \leq \|u\| + \|v\|$$

Exercise 11.1. Prove these properties.

Now, we will use inner products to calculate Fourier. Define

$$\langle u, v\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)v(x)\,dx$$

Under this inner product, $\sin kl$, $\cos kl$ are orthogonal functions, i.e.,

$$\begin{aligned}
\langle \sin kx, \cos lx\rangle &= 0 \quad \forall k, l\\
\langle \sin kx, \sin lx\rangle &= 0 \quad \text{if } k \neq l\\
\langle \cos kx, \cos lx\rangle &= 0 \quad \text{if } k \neq l
\end{aligned}$$

<u>*Note*</u>: $1 = \cos 0x$

Proof. Left as exercise, but use

$$\begin{aligned}
\cos((k+l)x) &= \cos kx \cos lx - \sin kx \sin lx\\
\sin((k+l)x) &= \sin kx \cos lx + \sin lx \cos kx
\end{aligned}$$

Also,

$$\begin{aligned}
\langle \sin kx, \sin kx\rangle &= 1\\
\langle \cos kx, \cos kx\rangle &= 1 \quad k \neq 0\\
\langle 1, 1\rangle &= 2
\end{aligned}$$

□

We have

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx\\
\langle f, \cos lx\rangle &= \langle \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx, \cos lx\rangle\\
&= \frac{a_0}{2}\langle 1, \cos lx\rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx, \cos lx\rangle + \sum_{k=1}^{\infty} b_n \langle \sin kx, \cos lx\rangle\\
\langle f, \cos lx\rangle &= a_l\\
\langle f, \sin lx\rangle &= b_l
\end{aligned}$$

So we can write any function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

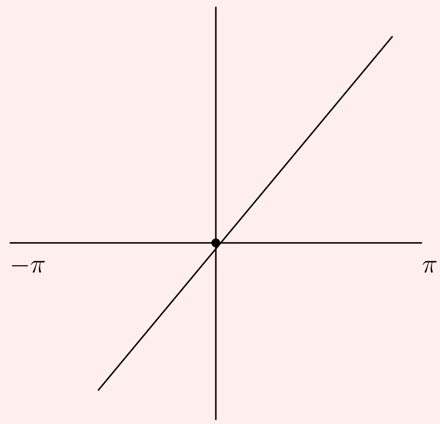
$$\begin{aligned}
a_k &= \langle f, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\\
b_k &= \langle f, \sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx
\end{aligned}$$

Question 11.2. Are these orthogonal functions under $\langle u, v\rangle$?

Question 11.3. Are there any other kind of L^2 inner product?

Example 11.1

Consider $f(x) = x$



We have

$$\begin{aligned}
x &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\
a_k &= \langle x, \cos kx \rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx \\
&= 0 - 0 - 0 = 0 \quad \text{(integration by parts)} \\
b_k &= \langle x, \sin kx \rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx \\
&= \frac{1}{\pi} \left[-\pi \frac{\cos k\pi}{k} - (-(-\pi)) \frac{\cos(-k\pi)}{k} \right] \quad \text{(integration by parts)} \\
&= \frac{2(-1)^{k+1}}{k}
\end{aligned}$$

Thus,

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

To show that infinite series converges

$$\sum_{k=1}^{\infty} \left| \frac{2(-1)^{k+1}}{k} \right| < 2 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is conclusive (by Weierstrass-M test).

§12 Lec 12: Oct 22, 2021

§12.1 Convergence of Fourier Series

Consider the last example from last lecture

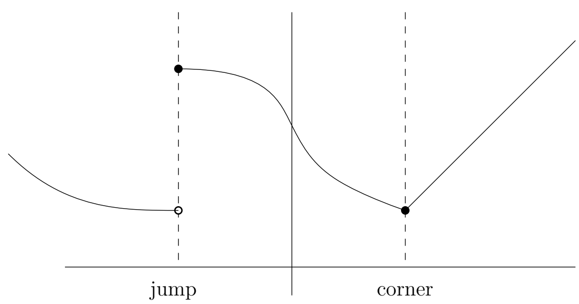
$$f(x) = x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

Question 12.1. In what sense does it converge? (What's happening at $\pm\pi$)

Fourier series must be 2π periodic (because $\cos kx$, $\sin kx$ are 2π-periodic) so the y must converge to a 2π-periodic extension of the function.

$$\tilde{f}(x + 2\pi) = \tilde{f}(x)$$

Note: x is C' (derivative continuous) but $\tilde{x}$ is not C'. It is piecewise C' (C': f continuous and $\frac{df}{dx}$ is continuous).
Piecewise C' on $[a, b]$



f is C' except at finitely many points. At any bad point we have

$$\begin{cases} f(x^-) = \lim_{h\to 0} f(x-h) & \text{if } f(x^+) \neq f(x^-) \text{ jump} \\ f(x^+) = \lim_{h\to 0} f(x+h) & \\ f'(x^-) = \lim_{h\to 0} f'(x-h) & \text{if } f(x^+) = f(x^-) \\ f'(x^+) = \lim_{h\to 0} f'(x+h) & \text{but } f'(x^+) \neq f(x^-) \text{ corner} \end{cases}$$

Theorem 12.1 (Fourier Convergence)
If $\tilde{f}(x)$ is 2π-periodic, piecewise C' function, then its Fourier series converges to $\tilde{f}$ everywhere except jump points x where the series converges to $\frac{f(x^+)+f(x^-)}{2}$

Question 12.2. Recall the example at the beginning, why is there no cosines for x?

Odd/even symmetries!

Fact 12.1. We have

$$\begin{aligned} \text{odd } &+ \text{odd} = \text{odd} \\ \text{even } &+ \text{even} = \text{even} \end{aligned}$$

and

$$\begin{aligned}\text{odd } &\times \text{odd} = \text{even}\\ \text{even } &\times \text{even} = \text{even}\\ \text{odd } &\times \text{even} = \text{odd}\end{aligned}$$

and

$$\begin{aligned}\int_{-a}^{a} \text{odd } dx &= 0\\ \int_{-a}^{a} \text{even } dx &= 2\int_{0}^{a} \text{even } dx\end{aligned}$$

This implies odd functions f have sine series and even functions have cosine series.

§13 Lec 13: Oct 27, 2021

Recap:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where the coefficients are calculated as follows

$$\begin{aligned} a_k &= \langle f, \cos kx \rangle \\ b_k &= \langle f, \sin kx \rangle \\ \langle u, v \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x)\, dx \end{aligned}$$

Symmetry simplifies a_k, b_k. Fourier series converges for periodic and piecewise C^1 functions.

§13.1 Complex Fourier Series

Recall the Euler's formula

$$e^{ikx} = \cos kx + i \sin kx$$

Also,

$$\begin{aligned} \cos kx &= \frac{e^{ikx} + e^{-ikx}}{2} \\ \sin kx &= \frac{e^{ikx} - e^{-ikx}}{2i} \end{aligned}$$

So,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

We want $c_k = \langle f, e^{ikx} \rangle$

$$\langle e^{ikx}, e^{ikx} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2ikx}\, dx$$

which is not necessarily positive and we want it to be strictly positive, i.e., norm.

$$\begin{aligned} \int_{-\pi}^{\pi} e^{2ikx}\, dx &= \left[\frac{e^{2ikx}}{2ik} \right]_{-\pi}^{\pi} \\ &= \frac{e^{2\pi ki} - e^{-2\pi ki}}{2ik} \\ &= \frac{\sin 2\pi k}{k} \\ &= 0 \end{aligned}$$

To fix this, let's define Hermitian inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

where $x \in (-\pi, \pi]$ and $f, g : (-\pi, \pi] \to \mathbb{C}$. So

$$\begin{aligned} c_k &= \langle f, e^{ikx} \rangle \\ c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \end{aligned}$$

Question 13.1. How do Fourier series work with integration?

Integration makes things smoother. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$\int f(x)\,dx \sim \int \frac{a_0}{2}\,dx + \sum_{k=1}^{\infty} a_k \int \cos kx\,dx + b_k \int \sin kx\,dx$$

Question 13.2. Is this okay?

Notice that

$$\int \cos kx\,dx = \frac{\sin kx}{k} \qquad \int \sin kx\,dx = \frac{-\cos kx}{k}$$

Problem: If $f(x) = 1$, then

$$f \sim 1$$
$$\int_0^x f\,dx \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Constants terms in Fourier series are bad under integration.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Integration is fine if the function has mean 0

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Compare $f(x) = 1$ and $g(x) = x$.

Remark 13.1. Fourier series need piecewise C^1. To have Fourier of f', it must be C^1 so f must be continuous (can have corners but not jumps).

$$f = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$f' = kb_k \cos kx - ka_k \sin kx$$

if f is continuous.

Summary:

- Integrate: divide by k
- Differentiation: multiply by k

§14 Lec 14: Oct 29, 2021

§14.1 Rescaling Intervals of Fourier Series

We know Fourier series on $[-\pi, \pi]$. What about $[-l, l]$? We use coordinate transformation

$$\begin{aligned} y &= \frac{\pi}{l}x \\ F(y) &= f\left(x(y)\right) \\ F\left(y(x)\right) &= f(x) \end{aligned}$$

We have

$$F(y) = f\left(x(y)\right) = f\left(\frac{l}{\pi}y\right)$$

So $F(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos ky + b_k \sin ky$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ky\, dy \\ &= \frac{1}{\pi}\int_{-l}^{l} F\left(y(x)\right)\cos ky(x)\frac{\pi}{l}\, dx \\ &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi}{l}x\right)\, dx \end{aligned}$$

So

$$f(x) = F\left(y(x)\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x$$

We can find b_k similarly.

§15 Lec 15: Nov 1, 2021

§15.1 The Relationship between Smoothness and Fourier Coefficients

Smoother functions (more differentiable) have faster decaying Fourier coefficients. (infinitely differentiable leads to exponential decay).

Example 15.1
- Discontinuous function $\to c_k \propto \frac{1}{k}$
- $C^0 \to c_k \propto \frac{1}{k^2}$
- $C^1 \to c_k \propto \frac{1}{k^3}$
- $C^2 \to c_k \propto \frac{1}{k^4}$

Why?

Recall these definitions

Definition 15.2 — $\forall \varepsilon, x\ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N(x)$$

Then, $f_n(x) \to f(x)$ (pointwise convergence).

Definition 15.3 — $\forall \varepsilon$, $\exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \quad \forall x$$

Then, $f_n(x) \to f(x)$ (uniform convergence).

Series converges $\sum_{k=1}^{\infty} f_k(x) \to g(x)$ if

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \to g(x) \text{ as } n \to \infty$$

Weierstrass M-test: If $|f_n(x)| < M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely/uniformly). So the limit is continuous if f_n are continuous.
Consider a complex Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_n e^{ikx}$$

Theorem 15.4
If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then the Fourier series is "good", i.e., the limit of the Fourier series is continuous.

Proof. Weierstrass!

$$\left|c_k e^{ikx}\right| \le |c_k| \left|e^{ikx}\right| = |c_k|$$

□

Corollary 15.5

If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > 1$. Then Fourier series is continuous.

Proof. $\sum_{k=1}^{\infty} \frac{M}{k^\alpha} < \infty$ for $\alpha > 1$ by comparison test. □

<u>*Note*</u>:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$
$$f' \sim \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$$

Differentiation: $c_k \to ikc_k$ or $|c_k| \to k|c_k|$

Theorem 15.6

If $\sum_{k=1}^{\infty} |k|^n |c_k| < \infty$ where $f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, then $f^{(n)}$ is continuous (f is C^n)

Proof. Have

$$f^{(n)} \sim \sum_{k=-\infty}^{\infty} (ik)^n c_k e^{ikx}$$

then Weierstrass $\left|(ik)^n c_k e^{ikx}\right| \le |k|^n c_k$ □

Corollary 15.7

If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > n+1$ then f is n times differentiable.

Proof. Comparison test: $c_k = \frac{1}{k^2}$, then

$$|c_k| < \frac{1}{k^{1.5}} \propto \frac{1}{k}$$

So,

$$\begin{gathered} \frac{1}{k^2} \to C^0 \\ \frac{1}{k^3} \to C^1 \\ \frac{1}{k^4} \to C^2 \\ \vdots \end{gathered}$$

□

§16 Lec 16: Nov 3, 2021

§16.1 Hilbert Spaces & Convergence in Norm

Goal: Prove Fourier series converge "in norm". First, we need some definitions.

Definition 16.1 (L^2 integrable) — f is L^2 integrable if $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$.

Definition 16.2 (Hilbert Space) — Hilbert space is vector space of L^2 integrable function

Proof. Have

- Has a 0 (0 function)
- Closed under addition
$$\|f+g\| \le \|f\| + \|g\| < \infty$$
- Closed under scalar multiplication
$$\|cf\| = |c|\|f\| < \infty \qquad \square$$
- test other axioms ...

Note: L^2 function have Fourier series.

Proof. $|c_k| = \left|\langle f, e^{ikx}\rangle\right| \le \|f\|\|e^{ikx}\| < \infty$ (Cauchy-Schwarz). $\square$

Note: L^2 functions are "abnormal" TBA

Fact 16.1. Hilbert spaces are complete (every "convergent" sequence has a limit that is L^2)

Definition 16.3 — "Convergent" means Cauchy sequence for sequence $a_n \to a$. We need

$$\text{Cauchy}: \forall \varepsilon, \ \exists N \ni \ \ |a_m - a_n| < \varepsilon \quad \forall m, n > N$$

Aside: Completeness is the difference between rationals $\mathbb{Q}$, and reals $\mathbb{R}$ ($\mathbb{Q}$ isn't complete because π is limit of sequence in $\mathbb{Q}$ but $\pi \notin \mathbb{Q}$). Completeness matters for taking limits.

Definition 16.4 (Convergence in Norm) — $f_n(x) \to f(x)$ if $\|f_n(x) - f(x)\| \to 0$ as $n \to \infty$.

We'll prove Fourier series converge to their function in norm in a general way for a general ∞-dim vector space V with an inner product.

Definition 16.5 (Orthonormal System) — Orthonormal system $\phi_1, \phi_2, \ldots \in V$

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$$

Definition 16.6 (General Fourier Series) — $f \sim \sum_{k=1}^{\infty} c_k\phi_k$ where $c_l = \langle f, \phi_l\rangle$ and c_k comes from $\langle f, \phi_l\rangle$ on both sides.

Theorem 16.7
The truncated Fourier series

$$s_n = \sum_{k=1}^{\infty} c_k\phi_k$$

is the best approximation to f in least squares sense, that is, consider $V_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ and take any $p_n = \sum_{k=1}^{n} d_k\phi_k \in V_n$ then

$$\|s_n - f\| \le \|p_n - f\| \quad \forall p_n \in V_n$$

Proof. We have

$$\begin{aligned} p_n &= \sum_{k=1}^{n} d_k\phi_k \\ s_n &= \sum_{k=1}^{n} c_k\phi_k \\ c_k &= \langle f, \phi_k\rangle \end{aligned}$$

Then,

$$\begin{aligned} \|p_n\|^2 &= \langle p_n, p_n\rangle \\ &= \langle \sum_{k=1}^{n} d_k\phi_k, \sum_{l=1}^{n} d_l\phi_l\rangle \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} d_k d_l\langle\phi_k, \phi_l\rangle \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} d_k d_l\delta_{kl} \\ &= \sum_{k=1}^{n} |d_k^2| \end{aligned}$$

and

$$\begin{aligned} \|p_n - f\|^2 &= \langle p_n - f, p_n - f\rangle \\ &= \langle p_n, p_n\rangle - 2\langle p_n, f\rangle + \langle f, f\rangle \\ &= \sum_{k=1}^{n} |d_k|^2 - 2\left(\sum_{k=1}^{n} d_k\langle\phi_k, f\rangle\right) + \|f\|^2 \\ &= \sum_{k=1}^{n} |d_k - c_k|^2 - \sum_{k=1}^{n} |c_k|^2 + \|f\|^2 \end{aligned}$$

Pick $d_k = c_k$ – norm minimized by s_n. $\square$

§17 Lec 17: Nov 5, 2021

§17.1 Pointwise Convergence of Fourier Series

We know Fourier series converge in norms for continuous, piecewise C^1, periodic functions. But Fourier series seemed to work even for discontinuous functions too (Gibbs phenomenon!). Today we will prove it works pointwise for discontinuous function if $s_n = \sum_{k=-n}^{n} c_k e^{ikx}$. Prove

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

1. Use the formulas for c_k

$$\begin{aligned} s_n &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy\right) e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \underbrace{\left(\sum_{k=-n}^{n} e^{ik(x-y)}\right)} dy? \end{aligned}$$

Notice that $\sum_{k=-n}^{n} e^{ikx}$ is a geometric series

$$\begin{aligned} \sum_{k=-n}^{n} e^{ikx} &= e^{-inx}\left(\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1}\right) \\ &= \vdots \\ &= \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin \frac{1}{2}x} \end{aligned}$$

So

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-y)\right)}{\sin\frac{1}{2}(x-y)} dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y} dy \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy + \frac{1}{2\pi}\int_{-\pi}^{0} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy \end{aligned}$$

WTS:

$$\lim_{n\to\infty} \text{top} = f(x^+)$$
$$\lim_{n\to-\infty} \text{bottom} = f(x^-)$$

Note

$$\begin{aligned} \frac{1}{\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)y}{\sin y}\,dy &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{iky}\,dy \\ &= 1 \end{aligned}$$

which is valid if only e^{i0} counts.

2. Prove a difference integral $\to 0$ by showing that it's a Fourier coefficient. Prove

$$\frac{1}{2\pi}\int_0^{\pi}\left(f(x+y) - f(x^+)\right)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy = 0$$

Notice that

$$g(y) \equiv \frac{f(x+y)+f(x^+)}{\sin\frac{1}{2}y}$$

is piecewise continuous $\forall y \in [0,\pi]$. We need $\int_0^\pi g(y)\sin\left(n+\frac{1}{2}\right)ydy = 0$. Note that

$$\sin\left(n+\frac{1}{2}\right)y = \sin\frac{1}{2}y\cos ny + \cos\frac{1}{2}y\sin ny$$

Then,

$$\int_0^\pi \left(g(y)\sin\frac{y}{2}\right)(0)ny + \left(g(y)\cos\frac{y}{2}\sin ny\right)$$

But we know Fourier coefficients decay for all L^2 integrable functions. So these terms $\to 0$ and we prove pointwise convergence.

§18 Lec 18: Nov 8, 2021

§18.1 Heat Equation

We've been learning about Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Question 18.1. Why is Fourier series useful?

It's used to solve PDEs; specifically, we want to investigate the heat equation in this lecture. For us,

$$\partial_t u(t,x) - \kappa \partial_x^2 u(t,x) = 0$$

where κ is constant (diffusing constant). Note that

- 1x time derivative $\implies$ 1x initial condition
- 2x space derivative $\implies$ 2x boundary condition

Types of boundary condition

$$\begin{aligned} u(t,0) &= \alpha(t) \quad \text{(Dirichlet boundaries)} \\ \partial_x u(t,0) &= \mu(t) \quad \text{(Neumann boundaries)} \\ \partial_x u + \beta(t) u &= \tau(t) \quad \text{(Robin/Mixed boundaries)} \end{aligned}$$

Homogeneous $\implies$ RHS $= 0$, i.e., $u = 0$, $\partial_x u = 0$, or $\partial_x u + \beta u = 0$.

Question 18.2. How do we solve this? (infinitely harder than an ODE!)

Assume $u(t,x) = T(t)X(x)$. Substitute into $\partial_t u - \kappa \partial_x^2 u = 0$

$$\begin{aligned} T'(t)X(x) - \kappa T(t) X''(x) &= 0 \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

This can only be true if neither depends on t or x, i.e., constant. So

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \lambda$$

What sign is λ?

- If $\lambda > 0$, we get exponential growth which isn't physical.

$$\begin{aligned} X'' &= \lambda X \\ X &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ T' &= \kappa\lambda T \\ T &= Te^{\kappa\lambda t} \end{aligned}$$

- $\lambda = 0$,

$$\begin{aligned} X'' &= 0 \\ X &= Ax + B \\ T' &= 0 \\ T &= T_0 \end{aligned}$$

- If $\lambda < 0$, redefine $\lambda \to -\lambda$

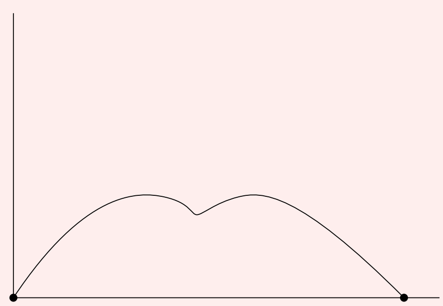
$$\begin{aligned} X'' &= -\lambda X \\ X &= A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x \\ T' &= -\kappa \lambda T \\ T &= T_0 e^{-\kappa \lambda t} \end{aligned}$$

  So either $\lambda = 0$, or $\lambda < 0$.

Example 18.1

Consider

$$\begin{aligned} \partial_t u - \partial_x^2 u = 0 \\ u(0, x) = u_0(x) \\ u(t, 0) = 0 \\ u(t, l) = 0 \end{aligned}$$



If we assume $u(t, x) = T(t)X(x)$ we find

$$\frac{T'}{\kappa T} = \frac{X'}{X} = \lambda$$

If $\lambda = 0$, $X = Ax + B$, but $u(t, 0) = u(t, l) = 0 \implies A = B = 0$. So $\lambda < 0$. Let's write $\lambda = -\omega^2$,

$$\begin{aligned} X'' &= -\omega^2 X \\ X &= A \cos \omega x + B \sin \omega x \end{aligned}$$

Use BC's to get A, B.

$$\begin{aligned} u(t, 0) = T(t)X(0) = 0 &\implies X(0) = A = 0 \\ X &= \sin \omega x \end{aligned}$$

Example 18.2 (Cont'd)

And we have

$$u(t, l) = T(t)X(l) = B \sin \omega l = 0$$
$$\implies \omega l = k\pi, \quad k = 1, 2, \ldots$$

Thus, $\omega = \frac{k\pi}{l}$, and it's an eigenvalue and $\sin \frac{n\pi}{l} x$ is an eigenfunction.

The final solution is

$$u(t, x) = \sum_{n=1}^{\infty} \left(\hat{u}_{0,k} e^{-\frac{n^2\pi^2}{l^2}\kappa t} \sin\left(\frac{n\pi}{l} x\right)\right)$$

We get $\hat{u}_{0,k}$ from the Fourier series of the function $u_0(x)$.
Note: The Fourier coefficients decay more quickly as k gets larger so diffusion smooth things out. If we have source term

$$\partial_t u - \partial_x^2 u = f(t, x)$$

We can express f as a Fourier series and solve an ODE for each Fourier coefficient.
Summary:

1. We assume separable solution: $u(t, x) = T(t)X(x)$
2. Substituting gives an eigenvalue problem

$$X'' = \lambda X$$

3. The boundary conditions imply $\lambda = -\omega^2$ where $\omega = \frac{n\pi}{l}$, $x\alpha \sin\left(\frac{n\pi}{l} x\right)$.
4. Linearity mean we sum up all the eigenfunctions

$$u(t, x) = \sum_{n=1}^{\infty} \left(\hat{u}_{0,k} e^{-\kappa \frac{(k-\pi)^2}{2^2} t} \right) \sin k \frac{\pi}{l} x$$

5. We use the initial condition to determine the Fourier series coeff, $\hat{u}_{0,k}$

§19 Lec 19: Nov 10, 2021

§19.1 Wave Equation

Goal: Solve the wave equation

1. Look for separable solutions
$$u(t,x) = T(t)X(x)$$
to
$$\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x)$$
c: wave speed $\left(\frac{\text{space}}{\text{time}}\right)$.
$$\begin{aligned}\partial_t^2(TX) &= c^2\partial_x^2(TX)\\ T''X &= c^2TX''\\ \frac{T''(t)}{T(t)} &= \frac{c^2X''(x)}{X(x)} = \lambda\end{aligned}$$

   a) $\lambda = \omega^2 > 0$, $T'' = \omega^2 T \implies T = e^{\omega t}$ or $T = e^{-\omega t}$ and $X'' = \frac{\omega^2}{c^2}X$, $X = e^{\frac{\omega x}{c}}$ or $X = e^{-\frac{\omega x}{c}}$

   b) $\lambda = 0$:
$$\begin{aligned}T'' = 0 &\implies T = A + Bt\\ X'' = 0 &\implies X = C + Dx\\ TX &= a + bt + cx + d + x\end{aligned}$$

   c) $\lambda = -\omega^2 < 0$
$$\begin{aligned}T'' = -\omega^2 T &\implies T = \sin\omega t \text{ or } T = \cos\omega t\\ X'' = \frac{-\omega^2}{c^2}X &\implies X = \frac{\sin\omega x}{c} \text{ or } X = \frac{\cos\omega x}{c}\end{aligned}$$

Next, let's decide on the sign of λ using the boundary conditions, e.g., homogeneous, Dirichlet, boundary conditions
$$u(t,0) = u(t,l) = 0$$
If $\lambda > 0$
$$u = Ae^{\omega t}e^{\frac{\omega x}{c}} + Be^{-\omega t}e^{\omega\frac{x}{c}} + Ce^{\omega t}e^{-\omega\frac{x}{c}} + De^{-\omega t}e^{-\omega\frac{x}{c}}$$
but the boundary condition implies that $A = B = C = D = 0$.

If $\lambda = 0$, similarly $u = 0$ is only possibility.

If $\lambda < 0$, $u = T(t)X(x)$, $T = \sin\omega t, \cos\omega t$, $X = \sin\frac{\omega x}{c}, \cos\frac{\omega x}{c}$
$$\begin{aligned}u(t,0) = T(t)X(0) = X(0) = 0\\ \implies A\sin\frac{\omega 0}{c} + B\cos\frac{\omega 0}{c} = B = 0\end{aligned}$$
So $X = A\sin\frac{\omega x}{c}$.
$$\begin{aligned}u(t,l) = T(t)X(l) &= A\sin\frac{\omega l}{c}\\ \implies \frac{\omega l}{c} &= n\pi, \quad n = 1,\dots\\ \omega &= \frac{n\pi c}{l}\end{aligned}$$

In general, the solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t\right) \sin \frac{n\pi x}{l}$$

TO find A_n, B_n, we use the initial conditions

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

Example 19.1

Consider $\partial_t^2 u = \partial_x^2 u$, $x \in [0,1]$

$$\begin{aligned} u(t,0) = u(t,1) &= 0 \\ u(0,x) &= \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases} \\ \partial_t u(0,x) &= 0 \\ c = 1, \quad l = 1 &\implies \omega_n = n\pi \end{aligned}$$

Using Dirichlet, the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \sin n\pi x$$

We want to get a_n, b_n with ICs. For $t = 0$,

$$\begin{aligned} u(0,x) &= \sum_{n=1}^{\infty} a_n \sin n\pi x \\ &= \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases} \\ a_n &= \frac{1}{2} \int_0^1 f(x) \sin n\pi x dx \end{aligned}$$

In general,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_0^1 \sin n\pi x \sin m\pi x dx &= \int_0^{\frac{1}{2}} x \sin n\pi x dx + \int_{\frac{1}{2}}^1 (1-x) \sin n\pi x dx \\ &= \begin{cases} 0, & \text{n even} \\ 2\left(-\left[\frac{1}{2}\frac{\cos n\pi/2}{n\pi}\right] + \frac{1}{(n\pi)^2}\left[\sin n\pi\right]_0^{\frac{1}{2}}\right) = \ldots = \frac{4}{\pi^2} \frac{\sum (-1)^k \cos(2k+1)\pi k \sin(2k+1)\pi k}{2k+1} \end{cases} \end{aligned}$$

§20 Lec 20: Nov 12, 2021

§20.1 Midterm 2

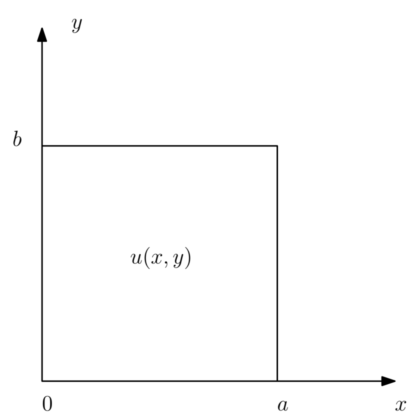
Need to know:

- How to calculate Fourier series
- How to test convergence?
- How does convergence relate to continuity?
- How does symmetry affect Fourier series?
- How does rescaling affect Fourier series?
- How does differentiation/integration affect Fourier series?
- How does smoothness (C^n) relate to Fourier series?
- How to analyze generalized Fourier series in Hilbert spaces?
- How to use Fourier series to solve PDEs?

§20.2 Laplace Equations

Consider

$$\partial_x^2 u + \partial_y^2 u = 0$$



The Laplace equation satisfies the Maximum principle, i.e., the maximum (and minimum) value of a solution to the Laplace equation must occur on the boundary,

Proof. If a function u has a maximum at (x, y) then

1. $\partial_x^2 u < 0$
2. $\partial_y^2 u < 0$

But Laplace says $\partial_x^2 u + \partial_y^2 u = 0$. Thus, we can't have local maximum (or minimum) in the domain. $\square$

Theorem 20.1

Solutions to Laplace equation, with given boundary conditions, are unique.

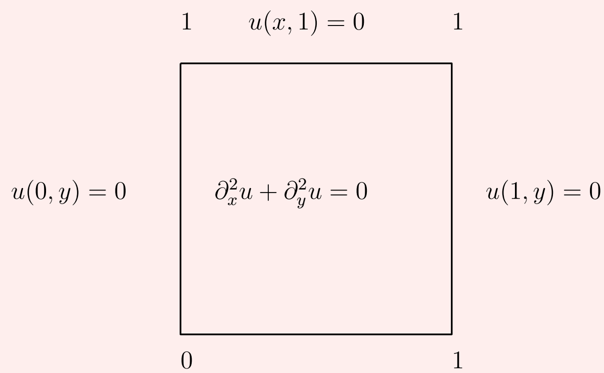
Proof. Suppose there exist u_1 and u_2 where

$$\begin{aligned} \partial_x^2 u_1 + \partial_y^2 u_1 = 0, &\quad u_1 = f \text{ on the boundary} \\ \partial_x^2 u_2 + \partial_y^2 u_2 = 0, &\quad u_2 = f \text{ on the boundary} \end{aligned}$$

Consider $u_1 - u_2 = \Delta u$. Since Laplace equation is linear, Δu solves Laplace. And we know that $\Delta u = 0$ on the boundary. Therefore, $\Delta u = 0$ everywhere (by maximum principle). ☐

Example 20.2

Consider



and

$$u(0,x) = \begin{cases} x \\ 1 - x \end{cases}$$

Have $u = X(x)Y(y)$

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

- $\lambda = -\omega^2 < 0$

$$X'' = -\omega^2 X \implies X = \cos \omega x,\ \sin \omega x$$
$$Y'' = \omega^2 Y \implies Y = e^{\omega y},\ e^{-\omega y},\ \cosh y,\ \sinh y$$

- $\lambda = 0$,

$$X = 1,\ x$$
$$Y = 1,\ y$$

- $\lambda = \omega^2 > 0$

$$X = e^{\omega x},\ e^{-\omega x}$$
$$y = \cos \omega y,\ \sin \omega y$$

Example 20.3 (Cont'd)

Only $\lambda = -\omega^2 < 0$ works. For $X = \sin n\pi x$,

$$X(0) = X(1) = 0 \implies \omega = n\pi$$

Therefore,

$$\begin{gathered} Y = Ae^{n\pi} Be^{-n\pi y} \\ Y(1) = 0 \implies Ae^{n\pi} + Be^{-n\pi} = 0 \\ \iff Y(y) \propto \sinh n\pi(y-1) \end{gathered}$$

In general,

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi(1-y) \sin n\pi x$$

and

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi x &= \begin{cases} x, & x < \frac{1}{2} \\ 1-x, & x > \frac{1}{2} \end{cases} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{4}{\pi^2 (2k+1)^2} \sin n\pi x \end{aligned}$$

where

$$a_{2k+1} = \frac{(-1)^k 4}{\pi^2 (2k+1)^2}$$