Stats 231A – Machine Learning

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This is stats 231A – an intro graduate level course on **Pattern Recognition and Machine Learning** taught by Professor Wu. We meet weekly on TR from 3:30 pm to 4:45 pm for lecture. Other course notes can be found at my blog site. Please let me know through my email if you spot any typos in the note.

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§1 Lec 1: Sep 28, 2021

§1.1 Modes of Learning

Table 1: Supervised Learning

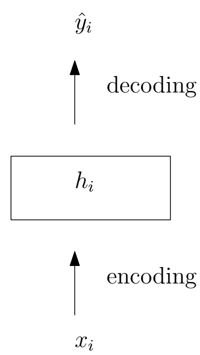
1		
⋮		
i	x_i	y_i
⋮		
n		

Table 2: Unsupervised Learning

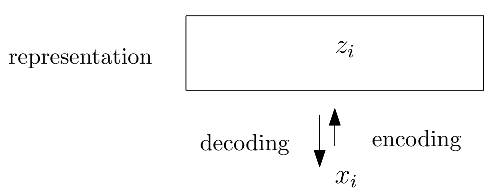
i	x_i	?

Representation Learning:

supervised



This is known as thought vector, features, base learners, or hidden variables.



Each argument in unsupervised learning is known as latents, code, embedding. The decoding part also requires generative model (auto-encoder).
Reinforcement Learning

$$s_0 \to \dots \to \underbrace{s_t \xrightarrow{a_t}}_{r_t(\text{reward})} s_{t+1} \underbrace{\xrightarrow{a_{t+1}} \dots}_{r_{t+1}+\dots}$$

where s represents state and a stands for action and

$$\begin{aligned} \text{policy}: &\ \pi(a|s) \\ \text{value}: &\ v(s) = E\left(r_t + r_{t+1} + \ldots | s_t = s\right) \end{aligned}$$

Supervised Learning: Consider regression problems where y_i is continuous or in classification where y_i is categorical binary $(+/-, 1/0)$

- Regression:

$$\begin{aligned} y_i &\sim N(s_i, \sigma^2) \\ s_i &= f(x_i) \end{aligned}$$

- Classification:

$$\begin{aligned} p_i = p_r\left(y_r = 1 | x_i\right) &= \frac{e^{s_i}}{1 + e^{s_i}} \\ &= \frac{1}{1 + e^{s_i}} = \text{sigmoid}(s_i) \end{aligned}$$

Before logistic regression, we also have perceptron (non-probabilistic)

$$\hat{y}_i = \text{sign}(s_i) = \begin{cases} 1 \text{ if } s_i \geq 0 \\ 0 \text{ if } s_i < 0 \end{cases}$$

Consider the linear model

$$s_i = \beta_0 + \sum_{j=1}^{p} \beta_j X_{ij}$$

where β_0 is the bias and $\beta_1, \ldots, \beta_p$ are the connection weights. We can use this for linear regression.
One hidden layer:

$$\begin{aligned} s_i &= f(x_i) \\ &= h_i^\top \beta \\ &= h(x_i)^\top \beta \end{aligned}$$

We can divide R into different partitions where $h_{ik} = 1(x_i \in R_k)$ and $h_{ik} = \text{tree}$.

$$\begin{gathered} \text{classification tree} \longrightarrow \text{adaboost} \\ \text{regression tree} \longrightarrow XGB \end{gathered}$$

Kernel:

$$k(x, x') = \langle h(x), h(x') \rangle$$

where k is explicit.
Two Layer Neural Network:

$$\begin{aligned} s_i &= \sum_{k=1}^{d} \beta_k h_{ik} \\ h_{ik} &= r\left(\sum_{j=1}^{p} d_{kj} x_{ij}\right) \\ &= \text{Relu}\left(\sum_{j=1}^{p} d_{kj} x_{ij}\right) = \max\left(0, \sum_{j=1}^{p} d_{kj} x_{ij}\right) \end{aligned}$$

1D:

$$s = \beta_0 + \sum_{k=1}^{d} \beta_k \max(0, x - a_k)$$

This can be very flexible and it can be used to approximate any nonlinear function (by piecewise linear function/line). For β_k (curvature),

$$\underbrace{|\beta|_{l_2}^2}_{\text{smoothness}} = \sum_{k=1}^{d} \beta_k^2$$

2D:

$$h_k = \max(0, \alpha_{k_1} x_1 + \alpha_{k_2} x_2 - b_k)$$

§2 Lec 2: Sep 30, 2021

§2.1 Gradient Descent

Two layer NN:

$$\hat{y} = h^\top \beta = \sum_{k=1}^{d} \beta_k h_k$$

$$h_k = r\left(\sum_{j=1}^{p} \alpha_{kj} \cdot x_j\right)$$

Overfitting: over interpret noise signal – model absorbs all noise $\implies$ training error

- Avoid overfitting: split data to training & testing (cross validation)
- Modern situation: p big, $p \gg n$ and $d \gg n$.

Gradient descent:

$$\text{initialize } \beta^{(0)} = 0$$
$$\beta^{t+1} = \beta^t - J_t \cdot \nabla_\beta \operatorname{Loss}(\beta_t)$$

$$\begin{aligned}
\operatorname{Loss}(\beta) &= \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \sum_{k=1}^{d} \beta_k \cdot h_k(x_i)\right)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \langle h(x_i), \beta\rangle\right)^2 \\
\nabla_\beta \operatorname{Loss}(\beta) &= -\sum_{i=1}^{n} \underbrace{\left(y_i - h(x_i)^\top \beta\right)}_{\text{error}} \cdot \underbrace{h(x_i)}_{\text{latent vec. of } x_i}
\end{aligned}$$

Grad. Descent:

$$\begin{aligned}
\beta^{t+1} &= \beta^t + J_t \cdot \sum_{i=1}^{n}\left(y_i - h(x_i)^\top \beta\right) - h(x_i) \\
\hat{\beta} &= \sum_{i=1}^{n} c_i h(x_i) \\
y_i &= h(x_i)^\top \hat{\beta}, \quad i = 1, \ldots, n
\end{aligned}$$

Another $\tilde{\beta} : y_i = h(x_i)^\top \tilde{\beta} \; \forall i = 1, \ldots, n$

$$\langle h(x_i), \hat{\beta} - \tilde{\beta}\rangle = 0$$

Representer theorem

$$\hat{\beta} = \sum_{i=1}^{n} c_i h(x_i)$$

Testing x :

$$\begin{aligned}
h(x)^\top \hat{\beta} &= \langle \sum_{i=1}^{n} c_i h(x_i) \cdot h(x)\rangle \\
&= \sum_{i=1}^{n} c_i \langle h(x_i) \cdot h(x)\rangle
\end{aligned}$$

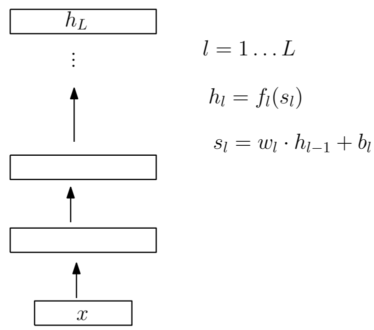
Double descent:

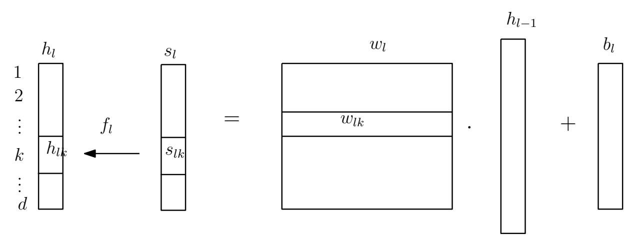
$$\text{testing error} = \text{training error} + \text{overfitting}$$

$\implies$ back propagation: based on chain rule.

§2.2 Multi-Layer Perceptron

Consider:





$$x(h_0) \to h_1(w_1 \cdot b_1) \to h_2(w_2 \cdot b_2) \to \dots \to h_{l-1} \to h_l \to \dots \to h_L \to \hat{y}$$

Then,

$$\begin{aligned}
\frac{\partial \text{loss}}{\partial h_{l-1}^\top} &= \sum_{k=1}^{d} \frac{\partial \text{loss}}{\partial h_{lk}} \cdot \frac{\partial h_{lk}}{\partial s_{lk}} \cdot \frac{\partial s_{lk}}{\partial h_{l-1}^\top} \\
&= \sum_{k=1}^{d} \frac{\partial \text{loss}}{\partial h_{lk}} \cdot f_i(s_{lk}) \cdot w_{lk} \\
&= \left(\frac{\partial \text{loss}}{\partial h_l} \odot f_l' \right)^\top \cdot w_l
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \text{loss}}{\partial w_{lk}} &= \frac{\partial \text{loss}}{\partial h_{lk}} \cdot \frac{\partial h_{lk}}{\partial s_{lk}} \cdot \frac{\partial s_{lk}}{\partial w_{lk}} \\
&= \frac{\partial \text{loss}}{\partial h_{lk}} \cdot f_l'(s_{lk}) \cdot h_{l-1}^\top \\
\frac{\partial \text{loss}}{\partial w_l} &= \left(\frac{\partial \text{loss}}{\partial h_l} \odot f_i \right) \cdot h_{l-1}^\top \\
\text{loss} &= \frac{1}{2} |y - h_l|^2 \\
\frac{\partial \text{loss}}{\partial h_l} &= -(y - h_l) = -\text{error}
\end{aligned}$$

Classification: $\frac{\partial \text{loss}}{\partial h_l} = -(y - p) = -$ error.