Math 135 – Differential Equations
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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3rd by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

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§1 | Lec 1: Sep 27, 2021

§1.1   Laplace Transforms

Consider the following questions

1. What is a transform?

2. What is a Laplace transform?

3. What are some examples?

4. What are some general properties?

5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions } \to \text{ functions}$$

Example 1.1   • differentiation: $\frac{d}{dx} : \ f \mapsto f'$

- integration: $\int^x dx : \ f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L} : f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

where $f : [0, \infty) \to \mathbb{R}$ and $F : \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2   • $f(t): \ t \mapsto 0 \implies \mathscr{L}[0] = 0$

- $f(t) = 1$

$$\mathscr{L}[1] = \lim_{t \to \infty} \int_0^t e^{-st}\, dt$$

$$= \lim_{t \to \infty} \left[\frac{e^{-st}}{-s}\right]_0^t$$

$$= \lim_{t \to \infty} \left(\frac{e^{-st}}{-s} + \frac{1}{s}\right)$$

$$= \frac{1}{s} \quad \text{if } \ \text{Re}(s) > 0$$

Example 1.3   • Consider

$$\mathscr{L}[t] = \int_0^\infty t e^{-st}\, dt$$

$$= \left[\frac{t e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s^2} \text{ if } \mathrm{Re}(s) > 0$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \mathrm{Re}(s) > 0, \ \ n \in \mathbb{N}$$

In addition,

$$\mathscr{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{1}{s-a}, \qquad \mathrm{Re}(s) > a$$

$$\mathscr{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

4. Properties:

   a) Linear!

   $$\mathscr{L}[f+g] = \mathscr{L}[f] + \mathscr{L}[g]$$
   $$\mathscr{L}[af] = a\mathscr{L}[f]$$

   b) Consider:

   $$\mathscr{L}\left[e^{at}f(t)\right] = \int_0^\infty f(t)e^{-(s-a)t}\, dt$$

   $$= F(s-a) \quad \text{if } \mathrm{Re}(s-a) > 0$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

   $$\mathscr{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st}\, dt = \int_0^\infty f(t')e^{-st'}\, dt'e^{-sa}$$

   where $t' = t - a$. So

   $$\mathscr{L}[f(t-a)] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

   $$\mathscr{L}[f'] = \int_0^\infty f'(t)e^{-st}\, dt$$

   $$= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\, dt$$

   $$= sF(s) - f(0)$$

§2 | Lec 2: Sep 29, 2021

§2.1   Laplace Transform (Cont'd)

Recap: $\mathscr{L}: f \to F$

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1   • $\mathscr{L}[t^n] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$

• $\mathscr{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear

- shifting $\leftrightarrow$ multiplying by exponential

- $\mathscr{L}[f'] = s\mathscr{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0, \ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\mathscr{L}[f'' + af' + bf] = \mathscr{L}[g]$$
$$\mathscr{L}[f''] + a\mathscr{L}[f'] + b\mathscr{L}[f] = G(s)$$

Notice that

$$\mathscr{L}[f''] = s^2 F - sf(0) - f'(0)$$

So

$$\left(s^2 + as + b\right)F(s) = G(s) + (s+a)f_0 + f_0'$$
$$F(s) = \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$(s^2 + 4)F(s) = \frac{4}{s^2} + s + 5$$

$$F(s) = \frac{\frac{4}{s^2} + s + 5}{s^2 + 4}$$

$$= \frac{4}{s^2\,(s^2 + 4)} + \frac{s}{s^2 + 4} + \frac{5}{s^2 + 4}$$

Notice that we need to use partial fractions to decompose the first term.

$$\frac{4}{s^2(s^2 + 4)} = \frac{A}{s^2} + \frac{B}{s^2 + 4}$$

$$4 = A(s^2 + 4) + Bs^2$$

$$= (A + B)s^2 + 4A$$

So, $A = 1$, $B = -1$. Then,

$$F(s) = \frac{1}{s^2} - \frac{1}{s^2 + 4} + \frac{s}{s^2 + 4} + \frac{5}{s^2 + 4}$$

$$= \frac{1}{s^2} + \frac{4}{s^2 + 4} + \frac{s}{s^2 + 4}$$

$$\mathscr{L}[f] = \mathscr{L}[t + 2\sin 2t + \cos 2t]$$

$$\implies f = t + 2\sin 2t + \cos 2t$$

§3 | Lec 3: Oct 1, 2021

§3.1   Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

Note: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall\varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

<u>Comparison Test</u>: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)

2. fe^{-st} doesn't decay fast enough as $t\to\infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.

2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \ \& \ c$$

   Have

$$c^{-t} \le 1\cdot e^{-t} \quad \forall t > 0$$
$$1 \le 1\cdot e^{0t} \quad \forall t > 0$$
$$t \le 1\cdot e^{t} \quad \forall t > 0$$

> **Theorem 3.1**
> If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$.

Proof. Have

$$\mathscr{L}\left[f\right](s) = \int_0^\infty f(t)e^{-st}\, dt$$

$$\lim_{\tau \to \infty} \int_0^\tau f(t)e^{-st}\, dt \leq \lim_{\tau \to \infty} \int_0^\tau \left| f(t)e^{-st} \right| dt$$

$$= \lim_{\tau \to \infty} \int_0^\tau \left| f(t) \right| e^{-s_r t}\, dt$$

$$\leq \lim_{\tau \to \infty} \int_0^\tau M e^{ct} \cdot e^{-s_r t}\, dt$$

$$= \lim_{\tau \to \infty} M \left[\frac{e^{(c-s_r)t}}{-(c - s_r)} \right]_0^\tau$$

$$= \frac{1}{s_r - c} \quad \text{if } s_r > c$$

$$< \infty$$

Thus, $\mathscr{L}\left[f\right]$ exists (for $\mathrm{Re}(s) > c$) by comparison test.   □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}} \right] = \int_0^\infty t^{-\frac{1}{2}} e^{-st}\, dt$$

$$= s^{-\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$$

$$= s^{-\frac{1}{2}} 2 \int_0^\infty e^{-z^2}\, dz$$

$$= \sqrt{\frac{\pi}{s}}$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0, \infty)$.

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§4 | Lec 4: Oct 4, 2021

§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\
&= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt
\end{aligned}
$$

Let's define $\tau = t + t' \implies d\tau = dt'$
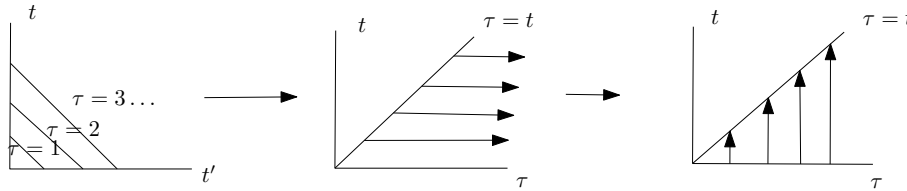
$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\
&= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau}\,d\tau\,dt \\
&= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)e^{-s\tau}\,dt \right) d\tau \\
&= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)\,dt \right) e^{-s\tau}\,d\tau \\
&= \mathscr{L}\left[\int_0^\tau f(t)g(\tau - t)\,dt \right]
\end{aligned}
$$

Theorem 4.1 (Convolution)

We have

$$
(f * g)(\tau) = \int_0^\tau f(t)g(\tau - t)\,dt
$$
$$
\mathscr{L}[f * g] = \mathscr{L}[f] \cdot \mathscr{L}[g]
$$

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§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$
f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t)\,dt
$$

Notice

$$\mathbf{f} = \mathbf{g} + K \cdot \mathbf{f}$$
$$f(\tau) \approx f_i$$
$$g(\tau) \approx g_i$$
$$k(\tau - t) \approx K_{ij}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\mathscr{L}[f] = \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f]$$
$$\mathscr{L}[f] = \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}$$

Example 4.2

Consider $f(t) = t^3 + \int_0^t \sin(t - \tau)f(\tau)d\tau$.

$$F(s) = \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s)$$

$$\vdots$$

$$F(s) = 3!(s^{-4} + s^{-6})$$

$$f(t) = t^3 + \frac{t^5}{20}$$

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§5 | Lec 5: Oct 6, 2021

§5.1   Dirac Delta "Function"

Visually:
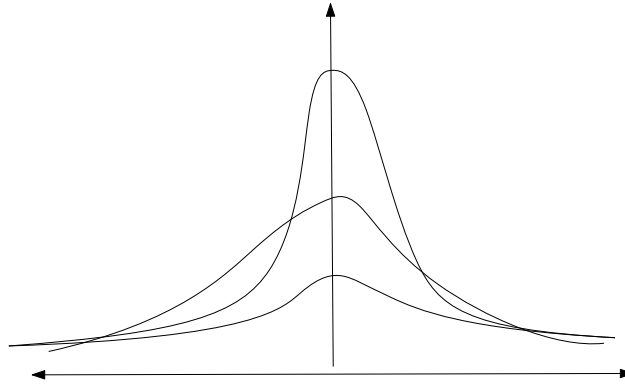
The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t) : \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\, d\tau$$

where $\delta(\tau - t) = \delta(t - \tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n \times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\mathbf{e}_i : \quad \text{unit vector}$$
$$\mathbf{f}_i : \quad \text{unit response vector}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$

2. Linear superposition means

$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$Lf = L\left(\sum_i h_i f_i\right)$$

$$= \sum_i h_i L f_i$$

$$= \sum_i h_i e_i$$

$$= h$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$

2. Then,

$$f = h * g$$

$$= \int_0^t g_t(\tau) h(\tau)\, d\tau$$

$$= \int_0^t g(t - \tau) h(\tau)\, d\tau$$

where g is known as the Green function.

$$e_i \approx \delta_t$$

$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

§6 | Lec 6: Oct 08, 2021

§6.1   Existence & Uniqueness of ODE Solutions

Intuitively, $f(t, x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.

$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$
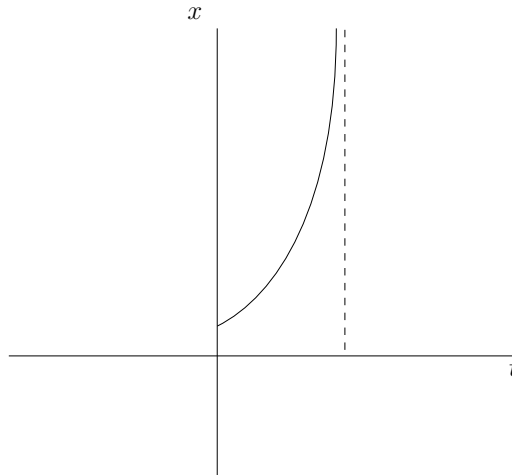
We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?

$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$
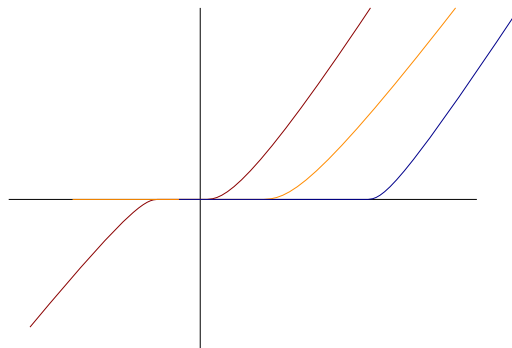
This has infinite number of solution through $(0, 0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation

2. Devise an iterative algorithm to approximate solutions (Picard iteration)

3. Prove the algorithm converges to a unique solution

§7 | Lec 7: Oct 11, 2021

§7.1   Picard Iteration

<u>Goal</u>: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

<u>Idea</u>:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

   Then, we can transform it into an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t')\right) \, dt'$$

   Notice that f is continuous and x is continuous imply x is differentiable.

2. Iteration: If we can't solve it at first, try again.

> **Example 7.1**
> Newton's root-finding algorithm
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

<u>Picard Iteration</u>: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t)\right) \, dt'$$

Start with a guess for the function $x_0(t) = x_0$ (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f\left(t', x_n(t')\right) \, dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \xrightarrow[\infty]{} \ldots$$

If $x_{n+1}(t) = x_n(t) = \overline{x}(t)$, then $\overline{x}(t)$ has to solve the IE. We want $\lim_{n \to \infty} x_n(t) \to x(t)$ solves IE.

Example 7.2

Consider $\dot{x}(t) = x(t)$, $x(0) = 1$. This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t') \, dt'$$

<u>Picard</u>:

$$x_0(t) = 1$$

$$x_1(t) = 1 + \int_0^t x_0(t') \, dt' = 1 + \int_0^t 1 \, dt'$$

$$= 1 + t$$

$$x_2(t) = 1 + \int_0^t 1 + t \, dt$$

$$= 1 + t + \frac{t^2}{2!}$$

$$\vdots$$

$$x_n(t) = \sum_{k=0}^n \frac{t^k}{k!}$$

Thus,

$$\lim_{n \to \infty} x_n(t) \to e^t$$

§8 | Lec 8: Oct 13, 2021

§8.1   Continuity

Limit of continuous function is not necessarily continuous.

Example 8.1

Consider $x_n(t) = t^n$ on $[0,1]$

$$x_0 = 1$$
$$x_1 = t$$
$$x_2 = t^2$$
$$\vdots$$
$$\overline{x} = \lim_{n \to \infty} x_n = \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases}$$

which is discontinuous.

Idea: We need "more" continuity. Given x, and given any $\varepsilon > 0$, if $|x - x'| < \delta(x, \varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Example 8.2

Consider $f(x) = x$ on $\mathbb{R}$. We can see that

$$|x - x'| < \varepsilon \quad \forall |x - x'| < \varepsilon$$

in which we pick $\delta(x, \varepsilon) = \varepsilon$.
How about $f(x) = x^2$ on $\mathbb{R}$?

$$|x^2 - y^2| < \varepsilon$$

If we pick $\delta(x, \varepsilon) = \varepsilon$, then $|x - y| < \delta = \varepsilon$ which does not necessarily imply $|x^2 - y^2| < \varepsilon$ because

$$|x^2 - y^2| = |(x+y)(x-y)|$$
$$= |x + y| \, |x - y|$$
$$\leq \varepsilon \, |x + y|$$

$|f(x) - f(y)| > \varepsilon$. So we need to pick smaller δ as x and y get larger. It would work for $\delta = \frac{\varepsilon}{2 \max(|x|, |y|)}$.

Question 8.1. Is $\frac{1}{x}$ continuous?

Ans: It depends on the domain. If we're talking about $\mathbb{R}$, it doesn't work at 0; on $(0, \infty)$, yes it's continuous.

> **Definition 8.3** (Uniform Continuity) — $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark 8.4. Notice that the definition is similar to continuity except that δ doesn't depend on x.

Example 8.5

x^2 on $\mathbb{R}$ is not uniformly continuous but x^2 on $(a, b) \subseteq \mathbb{R}$ is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|, |y|)} = \frac{\varepsilon}{\max(|a|, |b|)}$$

Remark 8.6. Uniform continuity also depends on the domain as continuity does.

Exercise 8.1. Is $x^{\frac{1}{2}}$ uniformly continuous on $[0, 1]$?

Lipschitz Continuity: "gradient is bounded"

$$\frac{|f(x) - f(y)|}{|x - y|} < L < \infty$$

We can pick $\delta = \frac{\varepsilon}{L}$ everywhere.

Example 8.7   • x^2 on $\mathbb{R}$ is not Lipschitz but it is on a finite interval.

   • $x^{\frac{1}{2}}$ is not Lipschitz continuous on $[0, 1]$. However, it's uniformly continuous.

§9 | Lec 9: Oct 15, 2021

§9.1   Picard's Theorem

Let's prove local existence of the theorem.

Theorem 9.1 (Picard)

If $f(t, x)$ and $\partial_x f(t, x)$ are continuous function on a bounded rectangle $R = [t_1, t_2] \times [x_1, x_2]$ and (t_0, x_0) is in interior of R $(t_1 < t_0 < t_2, \ x_1 < x_0 < x_2)$. Then $\exists$ a smaller rectangle $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ s.t. ODE

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution in R'.
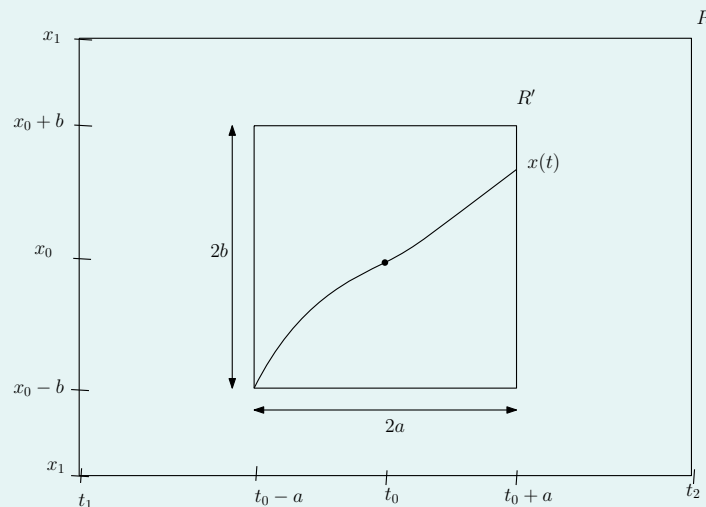
Note: Since R closed and bounded, then f, $\partial_x f$ are bounded, i.e.,

$$\max_{R} f(t, x) = M$$

$$\max_{R} \partial_x f(t, x) = L$$

Thus, f is Lipschitz.
Proof Outline:

1. Solving ODE $\iff$ Soling IE

2. Approximate solutions using Picard iteration

$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^{t} f(t', x_{n-1}(t')) \ dt'$$

3. Prove Picard iterates converges
$$\lim_{n \to \infty} x_n(t) \to \overline{x}(t)$$

4. Prove limit $\overline{x}(t)$ solves IE.

5. Prove limit $\overline{x}(t)$ is continuous.

6. Prove limit $\bar{x}(t)$ is unique.

7. How big is $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$?

$$\text{Pick } a \ni aL < 1 \ \& \ b = Ma \leq |x_0 - x_1| |x_0 - x_2|$$

Proof.   2. Prove Picard iterates converge

a) We have

$$\lim_{n \to \infty} x_n(t) \iff \lim_{n \to \infty} x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

telescoping sum!

b) Series $x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$ converges by <u>Weierstrass M-test</u> – If $|f_n(x)| < M_n$ $\forall n \in \mathbb{N}$, $x \in D$ and $\sum_{n=0}^{\infty} M_n$ converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

i) Show $x_i(t)$ are all in $R' \subseteq R$ so we can use bounds L, M.

$$|x_0(t) - x_0| = 0$$

$$|x_1(t) - x_0| = \left| \int_{t_0}^{t} f(t', x_0(t')) \, dt' \right|$$

$$\leq \int_{t_0}^{t} |f(t', x_0(t'))| \, dt$$

$$\leq \int_{t_0}^{t} M \, dt$$

$$\leq Ma = b$$

Thus, $x_1(t)$ is in the rectangle. By induction, every $x_n(t)$ in $R' \subseteq R$.

ii) Show $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$ is bounded.
Define $\Delta = \max_{R'} |x_1(t) - x_0|$. Then

$$|x_2(t) - x_1(t)| = \left| \int_{t_0}^{t} f(t', x_1(t')) - f(t', x_0(t')) \, dt' \right|$$

$$\leq \int_{t_0}^{t} |f(t', x_1(t')) - f(t', x_0(t'))| \, dt'$$

$$\leq \int_{t_0}^{t} L |x_1(t') - x_0(t')| \, dt'$$

$$\leq \Delta a L$$

and

$$|x_3(t) - x_2(t)| = \left| \int_{t_0}^{t} f(t, x_2(t)) - f(t, x_1(t)) \, dt \right|$$

$$\leq \int_{t_0}^{t} |f(t, x_2(t)) - f(t, x_1(t))| \, dt$$

$$\leq \int_{t_0}^{t} L |x_2(t') - x_1(t')| \, dt'$$

$$\leq L (\Delta a L)(t - t_0)$$

$$\leq \Delta (aL)^2$$

Every $|x_n(t) - x_{n-1}(t)|$ depends on $|x_{n-1}(t) - x_{n-2}(t)|$ recursively. The general pattern is

$$|x_n(t) - x_{n-1}(t)| \leq \Delta(aL)^{n-1}$$

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}| \leq \sum_{n=0}^{\infty} \Delta(aL)^n$$

$$= \frac{\Delta}{1 - aL}$$

$$< \infty$$

Thus, $\sum x_n - x_{n-1}$ converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n \to \infty} x_n(t) = \overline{x}(t) \text{ exists!}$$

3. $\overline{x}$ solves I.E.

   <u>Idea</u>: We know $|\overline{x} - x_n|$ gets small so break $\left| \overline{x} - x_0 - \int_{t_0}^{t} f(t', \overline{x}(t')) \, dt' \right|$ into pieces like $|\overline{x} - x_n(t)|$.

$$\text{subtract } x_n(t) - x_0 - \int_{t_0}^{t} f(t', x_{n-1}(t')) \, dt' = 0$$

   Let $\kappa = \left| \overline{x} - x_0 - \int_{t_0}^{t} f(t', \overline{x}(t')) \, dt' \right|$.

$$\kappa = \left| -(x_n - x_0 - \int_{t_0}^{t} f(t', x_{n-1}(t')) \, dt' \right|$$

$$\leq |\overline{x} - x_n| + \left| \int_{t_0}^{t} f(t, \overline{x}) - f(t, x_{n-1}) \, dt \right|$$

$$\leq |\overline{x} - x_n| + \int_{t_0}^{t} |f(t, \overline{x}) - f(t, x_{n-1})| \, dt$$

$$\leq |\overline{x} - x_n| + aL |\overline{x} - x_{n-1}|$$

   which approaches 0 as $n \to \infty$ because $\lim_{n \to \infty} x_n = \overline{x}$.

4. $\overline{x} = \lim_{n \to \infty} x_n$ is continuous, i.e., given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|t - t'| < \delta \implies |\overline{x}(t) - \overline{x}(t')| < \varepsilon$$

   Idea: Split into known things

$$|\overline{x}(t) - \overline{x}(t')| = |\overline{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \overline{x}(t)|$$
$$\leq |\overline{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \overline{x}(t)|$$

   We pick n s.t. $|\overline{x}(t) - x_n(t)| < \frac{\varepsilon}{3}$ $\forall t$ which is possible because Weierstrass implies uniform convergence. Then pick δ s.t.

$$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall \, |t - t'| < \delta$$

   which is possible because x_n is continuous.

5. $\overline{x}$ is unique.

   <u>Idea</u>: Prove $|\overline{x} - \tilde{x}| \leq |\overline{x} - \tilde{x}|$.

   - If $\tilde{u}$ is other solution, it also exists in R'.

Proof. (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some $|t_* - t| < a$. But

$$\begin{aligned}
|\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| \, dt' \\
&\leq M\left(t_* - t_0\right) \\
&< Ma = b
\end{aligned}$$

Contradiction! □

- Have

$$\begin{aligned}
|\overline{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^{t} f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t} |f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right)| \, dt' \\
&\leq \int_{t_0}^{t} L \max |\overline{x}(t') - \tilde{x}(t')| \, dt \\
&\leq La \max |\overline{x}(t') - \tilde{x}(t')| \\
\max |\overline{x}(t) - \tilde{x}(t)| &\leq \max |\overline{x}(t) - \tilde{x}(t)|
\end{aligned}$$

which is only possible if $\overline{x}(t) - \tilde{x}(t) = 0$, i.e., solution is unique. □

§10 | Lec 10: Oct 18, 2021

§10.1   Fourier Series

<u>Goal</u>: Solve linear PDE: 3 canonical examples

1. Heat/Diffusion equation

$$\partial_t u(t,x) - \partial_x^2 u(t,x) = 0$$
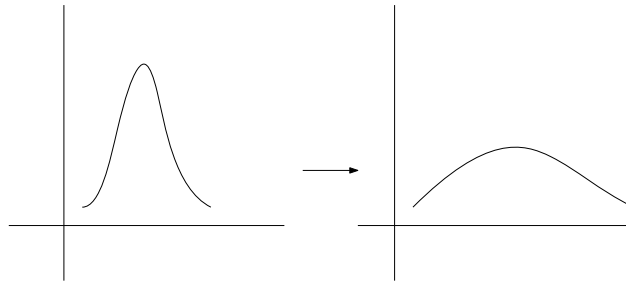
2. Wave equation

$$\partial_t^2 u = \partial_x^2 u$$
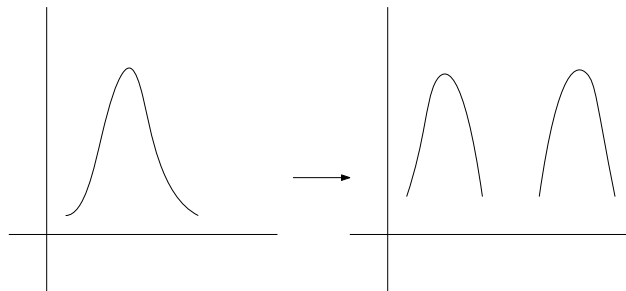
3. Laplace equation:

$$\partial_x^2 u + \partial_y^2 u = 0$$

Question 10.1. How do we solve linear PDEs?

Use linearity to split big problems into small ones that you can solve (find the eigenvectors). Then we split 1 PDE → ∞ ODEs. First, let's define Fourier series.

> **Definition 10.1** (Fourier Series) — Fourier Series is a function written as a sum of sines and cosines
>
> $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$
>
> $$= \sum_{-\infty}^{\infty} c_n e^{inx}$$
>
> where $c_n = c_r + i c_{in}$.

They have <u>amazing properties</u>:

1. They can approximate almost anything

   - analytic function
   - smooth function
   - periodic function
   - differentiable function
   - continuous/discontinuous function

2. They simplify differentiation!

$$\frac{d}{dx}e^{ikx} = ike^{ikx}$$

$$\frac{d^2}{dx^2}\sin kx = -k^2 \sin kx$$

$$\frac{d^2}{dx^2}\cos kx = -k^2 \cos kx$$

   Just like Laplace transform, Fourier series transform differentiation into multiplication problem (easier to deal with).

3. Fourier series are <u>orthogonal</u>

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

   or

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad \text{if } m \neq n$$

   or

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{if } m \neq n$$

   This gives easy formulas

From these facts follow from linear algebra, because Fourier series are eigenfuncitons of differentiation. They are the correct basis to solve linear PDEs.

§11 | Lec 11: Oct 20, 2021

§11.1 Coefficients of Fourier Series

Question 11.1. How do we calculate Fourier Series $a_n, b_n = ?$

Consider the domain: $[-\pi, \pi]$, finite dimensions N, vector

$$\mathbf{u} = \sum u_i \mathbf{e}_i$$

How do we calculate u_i?

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{e}_i &= \left(\sum_{i=1}^{N} u_i e_i \right) \cdot e_j \\
&= \sum_{i=1}^{N} u_i \left(e_i - e_j \right) \\
&= \sum_{i=1}^{N} \delta_{ij}
\end{aligned}
$$

where

$$
\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
$$

We want to do this in ∞ dimensions – inner product

$$N: \quad \langle u, v \rangle = u \cdot v = \sum_{i=1}^{N} u_i v_i$$

$$\infty: \quad \langle u, v \rangle \propto \int_a^b u(x) v(x) \, dx$$

<u>Inner Product</u>: $\langle u, v \rangle \to \mathbb{R}$ takes in two function & spits out a number. It has to satisfy the following properties

1. Bilinear

$$\langle au + bv, \rangle = a \langle u, vw \rangle + b \langle v, w \rangle$$

2. Symmetric $\langle u, v \rangle = \langle u, v \rangle$.

3. Positivity: $\langle u, u \rangle > 0$ unless $u = 0$.

Inner products are important

- They imply a norm $\|u\| = \sqrt{\langle u, u \rangle}$

- Cauchy-Schwarz Inequality
$$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$$

- Triangle inequality
$$\|u + v\| \leq \|u\| + \|v\|$$

Exercise 11.1. Prove these properties.

Now, we will use inner products to calculate Fourier. Define

$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x)\, dx$$

Under this inner product, $\sin kl$, $\cos kl$ are orthogonal functions, i.e.,

$$\langle \sin kx, \cos lx \rangle = 0 \quad \forall k, l$$
$$\langle \sin kx, \sin lx \rangle = 0 \quad \text{if } k \neq l$$
$$\langle \cos kx, \cos lx \rangle = 0 \quad \text{if } k \neq l$$

<u>Note</u>: $1 = \cos 0x$

Proof. Left as exercise, but use

$$\cos\left((k+l)x\right) = \cos kx \cos lx - \sin kx \sin lx$$
$$\sin\left((k+l)x\right) = \sin kx \cos lx + \sin lx \cos kx$$

Also,

$$\langle \sin kx, \sin kx \rangle = 1$$
$$\langle \cos kx, \cos kx \rangle = 1 \quad k \neq 0$$
$$\langle 1, 1 \rangle = 2 \hspace{6cm} \square$$

We have

$$f(x) = \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx$$
$$\langle f, \cos lx \rangle = \langle \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx, \cos lx \rangle$$
$$= \frac{a_0}{2} \langle 1, \cos lx \rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx, \cos lx \rangle + \sum_{k=1}^{\infty} b_n \langle \sin kx, \cos lx \rangle$$
$$\langle f, \cos lx \rangle = a_l$$
$$\langle f, \sin lx \rangle = b_l$$

So we can write any function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

$$a_k = \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx$$
$$b_k = \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx$$

Question 11.2. Are these orthogonal functions under $\langle u, v \rangle$?

Question 11.3. Are there any other kind of L^2 inner product?

Example 11.1

Consider $f(x) = x$
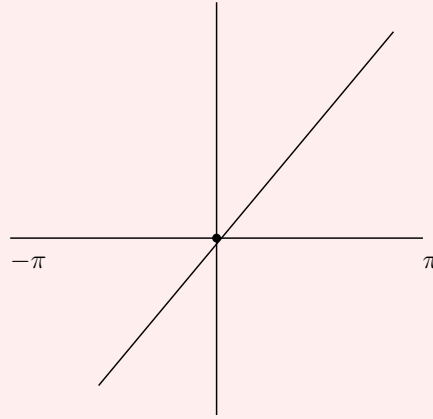
We have

$$x = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$a_k = \langle x, \cos kx \rangle$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx$$

$$= 0 - 0 - 0 = 0 \quad \text{(integration by parts)}$$

$$b_k = \langle x, \sin kx \rangle$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx$$

$$= \frac{1}{\pi} \left[-\pi \frac{\cos k\pi}{k} - (-(-\pi)) \frac{\cos(-k\pi)}{k} \right] \quad \text{(integration by parts)}$$

$$= \frac{2(-1)^{k+1}}{k}$$

Thus,

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

To show that infinite series converges

$$\sum_{k=1}^{\infty} \left| \frac{2(-1)^{k+1}}{k} \right| < 2 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is conclusive (by Weierstrass-M test).

§12 | **Lec 12: Oct 22, 2021**

§12.1   Convergence of Fourier Series

Consider the last example from last lecture

$$f(x) = x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

Question 12.1. In what sense does it converge? (What's happening at $\pm\pi$)

Fourier series must be 2π periodic (because $\cos kx$, $\sin kx$ are 2π-periodic) so the y must converge to a 2π-periodic extension of the function.

$$\tilde{f}(x + 2\pi) = \tilde{f}(x)$$

<u>*Note*</u>: x is C' (derivative continuous) but $\tilde{x}$ is not C'. It is piecewise C' (C': f continuous and $\frac{df}{dx}$ is continuous).
<u>Piecewise C'</u> on $[a, b]$
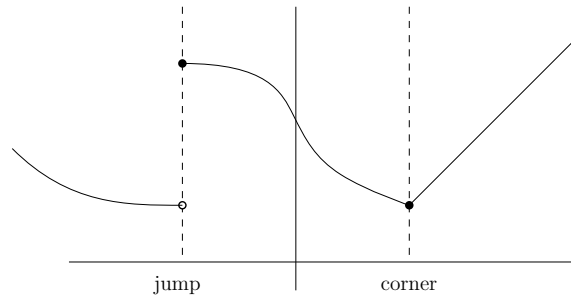
f is C' except at finitely many points. At any bad point we have

$$\begin{cases} f(x^-) = \lim_{h \to 0} f(x - h) & \text{if } f(x^+) \neq f(x^-) \text{ jump} \\ f(x^+) = \lim_{h \to 0} f(x + h) \\ f'(x^-) = \lim_{h \to 0} f'(x - h) & \text{if } f(x^+) = f(x^-) \\ f'(x^+) = \lim_{h \to 0} f'(x + h) & \text{but } f'(x^+) \neq f(x^-) \text{ corner} \end{cases}$$

> **Theorem 12.1** (Fourier Convergence)
> If $\tilde{f}(x)$ is 2π-periodic, piecewise C' function, then its Fourier series converges to $\tilde{f}$ everywhere except jump points x where the series converges to $\frac{f(x^+) + f(x^-)}{2}$

Question 12.2. Recall the example at the beginning, why is there no cosines for x?

Odd/even symmetries!

Fact 12.1. We have

$$\text{odd } + \text{odd} = \text{odd}$$
$$\text{even } + \text{even} = \text{even}$$

and

$$\text{odd} \times \text{odd} = \text{even}$$
$$\text{even} \times \text{even} = \text{even}$$
$$\text{odd} \times \text{even} = \text{odd}$$

and

$$\int_{-a}^{a} \text{odd } dx = 0$$
$$\int_{-a}^{a} \text{even } dx = 2 \int_{0}^{a} \text{even } dx$$

This implies odd functions f have sine series and even functions have cosine series.

§13 | Lec 13: Oct 27, 2021

Recap:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where the coefficients are calculated as follows

$$a_k = \langle f, \cos kx \rangle$$
$$b_k = \langle f, \sin kx \rangle$$
$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x)\, dx$$

Symmetry simplifies a_k, b_k. Fourier series converges for periodic and piecewise C^1 functions.

§13.1   Complex Fourier Series

Recall the Euler's formula
$$e^{ikx} = \cos kx + i \sin kx$$

Also,

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$$
$$\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

So,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

We want $c_k = \langle f, e^{ikx} \rangle$

$$\langle e^{ikx}, e^{ikx} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2ikx}\, dx$$

which is not necessarily positive and we want it to be strictly positive, i.e., norm.

$$\int_{-\pi}^{\pi} e^{2ikx}\, dx = \left[\frac{e^{2ikx}}{2ik} \right]_{-\pi}^{\pi}$$
$$= \frac{e^{2\pi ki} - e^{-2\pi ki}}{2ik}$$
$$= \frac{\sin 2\pi k}{k}$$
$$= 0$$

To fix this, let's define Hermitian inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

where $x \in (-\pi, \pi]$ and $f, g : (-\pi, \pi] \to \mathbb{C}$. So

$$c_k = \langle f, e^{ikx} \rangle$$
$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx$$

Question 13.1. How do Fourier series work with integration?

Integration makes things smoother. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$\int f(x)\, dx \sim \int \frac{a_0}{2}\, dx + \sum_{k=1}^{\infty} a_k \int \cos kx\, dx + b_k \int \sin kx\, dx$$

Question 13.2. Is this okay?

Notice that

$$\int \cos kx\, dx = \frac{\sin kx}{k} \qquad \int \sin kx\, dx = \frac{-\cos kx}{k}$$

<u>Problem</u>: If $f(x) = 1$, then

$$f \sim 1$$

$$\int_0^x f\, dx \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Constants terms in Fourier series are bad under integration.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Integration is fine if the function has mean 0

$$\int_{-\pi}^{\pi} f(x)\, dx = 0$$

Compare $f(x) = 1$ and $g(x) = x$.

> **Remark 13.1.** Fourier series need piecewise C^1. To have Fourier of f', it must be C^1 so f must be continuous (can have corners but not jumps).
>
> $$f = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
> $$f' = kb_k \cos kx - ka_k \sin kx$$
>
> if f is continuous.

Summary:

- Integrate: divide by k
- Differentiation: multiply by k

§14 | Lec 14: Oct 29, 2021

§14.1   Rescaling Intervals of Fourier Series

We know Fourier series on $[-\pi, \pi]$. What about $[-l, l]$? We use coordinate transformation

$$y = \frac{\pi}{l}x$$
$$F(y) = f\left(x(y)\right)$$
$$F\left(y(x)\right) = f(x)$$

We have

$$F(y) = f\left(x(y)\right) = f\left(\frac{l}{\pi}y\right)$$

So $F(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos ky + b_k \sin ky$

$$
\begin{aligned}
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos ky \, dy \\
&= \frac{1}{\pi} \int_{-l}^{l} F\left(y(x)\right) \cos ky(x) \frac{\pi}{l} \, dx \\
&= \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{k\pi}{l}x\right) dx
\end{aligned}
$$

So

$$f(x) = F\left(y(x)\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x$$

We can find b_k similarly.

Duc Vu (Fall 2021)   15   Lec 15: Nov 1, 2021

§15 | Lec 15: Nov 1, 2021

§15.1   The Relationship between Smoothness and Fourier Coefficients

Smoother functions (more differentiable) have faster decaying Fourier coefficients. (infinitely differentiable leads to exponential decay).

Example 15.1   • Discontinuous function $\rightarrow c_k \propto \frac{1}{k}$

- $C^0 \rightarrow c_k \propto \frac{1}{k^2}$

- $C^1 \rightarrow c_k \propto \frac{1}{k^3}$

- $C^2 \rightarrow c_k \propto \frac{1}{k^4}$
Why?

Recall these definitions

Definition 15.2 — $\forall \varepsilon, x \ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N(x)$$

Then, $f_n(x) \rightarrow f(x)$ (pointwise convergence).

Definition 15.3 — $\forall \varepsilon, \ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \quad \forall x$$

Then, $f_n(x) \rightarrow f(x)$ (uniform convergence).

Series converges $\sum_{k=1}^{\infty} f_k(x) \rightarrow g(x)$ if

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \rightarrow g(x) \text{ as } n \rightarrow \infty$$

<u>Weierstrass M-test</u>: If $|f_n(x)| < M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely/uniformly). So the limit is continuous if f_n are continuous.
Consider a complex Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_n e^{ikx}$$

Theorem 15.4

If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then the Fourier series is "good", i.e., the limit of the Fourier series is continuous.

Proof. Weierstrass!
$$\left| c_k e^{ikx} \right| \leq |c_k| \left| e^{ikx} \right| = |c_k| \qquad \qquad \square$$

> **Corollary 15.5**
> If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > 1$. Then Fourier series is continuous.

Proof. $\sum_{k=1}^{\infty} \frac{M}{k^\alpha} < \infty$ for $\alpha > 1$ by comparison test. $\qquad\square$

<u>Note</u>:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$f' \sim \sum_{k=-\infty}^{\infty} ik c_k e^{ikx}$$

Differentiation: $c_k \to ik c_k$ or $|c_k| \to k|c_k|$

> **Theorem 15.6**
> If $\sum_{k=1}^{\infty} |k|^n |c_k| < \infty$ where $f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, then $f^{(n)}$ is continuous (f is C^n)

Proof. Have

$$f^{(n)} \sim \sum_{k=-\infty}^{\infty} (ik)^n c_k e^{ikx}$$

then Weierstrass $\left|(ik)^n c_k e^{ikx}\right| \leq |k|^n c_k$ $\qquad\square$

> **Corollary 15.7**
> If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > n+1$ then f is n times differentiable.

Proof. Comparison test: $c_k = \frac{1}{k^2}$, then

$$|c_k| < \frac{1}{k^{1.5}} \propto \frac{1}{k}$$

So,

$$\frac{1}{k^2} \to C^0$$
$$\frac{1}{k^3} \to C^1$$
$$\frac{1}{k^4} \to C^2$$
$$\vdots \qquad\qquad\square$$

§16 | Lec 16: Nov 3, 2021

§16.1   Hilbert Spaces & Convergence in Norm

<u>Goal</u>: Prove Fourier series converge "in norm". First, we need some definitions.

> **Definition 16.1** (L^2 integrable) — f is L^2 integrable if $\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$.

> **Definition 16.2** (Hilbert Space) — Hilbert space is vector space of L^2 integrable function

Proof. Have

- Has a 0 (0 function)

- Closed under addition
$$\|f + g\| \le \|f\| + \|g\| < \infty$$

- Closed under scalar multiplication

$$\|cf\| = |c|\|f\| < \infty \qquad\qquad \square$$

- test other axioms . . .

<u>Note</u>: L^2 function have Fourier series.

Proof. $|c_k| = \left|\langle f, e^{ikx} \rangle\right| \le \|f\|\|e^{ikx}\| < \infty$ (Cauchy-Schwarz). $\square$

   <u>Note</u>: L^2 functions are "abnormal" TBA

Fact 16.1. Hilbert spaces are complete (every "convergent" sequence has a limit that is L^2)

> **Definition 16.3** — "Convergent" means <u>Cauchy sequence</u> for sequence $a_n \to a$. We need
>
> $$\text{Cauchy} : \forall \varepsilon, \ \exists N \ni \ |a_m - a_n| < \varepsilon \ \ \forall m, n > N$$

<u>Aside</u>: Completeness is the difference between rationals $\mathbb{Q}$, and reals $\mathbb{R}$ ($\mathbb{Q}$ isn't complete because π is limit of sequence in $\mathbb{Q}$ but $\pi \notin \mathbb{Q}$). Completeness matters for taking limits.

> **Definition 16.4** (Convergence in Norm) — $f_n(x) \to f(x)$ if $\|f_n(x) - f(x)\| \to 0$ as $n \to \infty$.

We'll prove Fourier series converge to their function in norm in a general way for a general ∞-dim vector space V with an inner product.

> **Definition 16.5** (Orthonormal System) — Orthonormal system $\phi_1, \phi_2, \ldots \in V$
>
> $$\langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$$

> **Definition 16.6** (General Fourier Series) — $f \sim \sum_{k=1}^{\infty} c_k \phi_k$ where $c_l = \langle f, \phi_l \rangle$ and c_k comes from $\langle f, \phi_l \rangle$ on both sides.

> **Theorem 16.7**
>
> The truncated Fourier series
>
> $$s_n = \sum_{k=1}^{\infty} c_k \phi_k$$
>
> is the best approximation to f in least squares sense, that is, consider $V_n = \text{span}\,\{\phi_1, \dots, \phi_n\}$ and take any $p_n = \sum_{k=1}^{n} d_k \phi_k \in V_n$ then
>
> $$\|s_n - f\| \le \|p_n - f\| \quad \forall p_n \in V_n$$

Proof. We have

$$p_n = \sum_{k=1}^{n} d_k \phi_k$$

$$s_n = \sum_{k=1}^{n} c_k \phi_k$$

$$c_k = \langle f, \phi_k \rangle$$

Then,

$$
\begin{aligned}
\|p_n\|^2 &= \langle p_n, p_n \rangle \\
&= \langle \sum_{k=1}^{n} d_k \phi_k, \sum_{l=1}^{n} d_l \phi_l \rangle \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} d_k d_l \langle \phi_k, \phi_l \rangle \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} d_k d_l \delta_{kl} \\
&= \sum_{k=1}^{n} |d_k^2|
\end{aligned}
$$

and

$$
\begin{aligned}
\|p_n - f\|^2 &= \langle p_n - f, p_n - f \rangle \\
&= \langle p_n, p_n \rangle - 2\langle p_n, f \rangle + \langle f, f \rangle \\
&= \sum_{k=1}^{n} |d_k|^2 - 2\left(\sum_{k=1}^{n} d_k \langle \phi_k, f \rangle \right) + \|f\|^2 \\
&= \sum_{k=1}^{n} |d_k - c_k|^2 - \sum_{k=1}^{n} |c_k|^2 + \|f\|^2
\end{aligned}
$$

Pick $d_k = c_k$ – norm minimized by s_n . $\square$

§17 | Lec 17: Nov 5, 2021

§17.1   Pointwise Convergence of Fourier Series

We know Fourier series converge in norms for continuous, piecewise C^1, periodic functions. But Fourier series seemed to work even for discontinuous functions too (Gibbs phenomenon!). Today we will prove it works pointwise for discontinuous function if $s_n = \sum_{k=-n}^{n} c_k e^{ikx}$. Prove

$$\lim_{n \to \infty} s_n(x) = \frac{1}{2} \left(f(x^+) + f(x^-) \right)$$

1. Use the formulas for c_k

$$s_n = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} \, dy \right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left(\sum_{k=-n}^{n} e^{ik(x-y)} \right)}_{} dy?$$

Notice that $\sum_{k=-n}^{n} e^{ikx}$ is a geometric series

$$\sum_{k=-n}^{n} e^{ikx} = e^{-inx} \left(\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} \right)$$

$$= \vdots$$

$$= \frac{\sin\left(\left(n + \frac{1}{2} \right) x \right)}{\sin \frac{1}{2} x}$$

So

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \frac{\sin\left(\left(n + \frac{1}{2} \right)(x - y) \right)}{\sin \frac{1}{2}(x - y)} \, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + y) \frac{\sin\left(n + \frac{1}{2} \right) y}{\sin \frac{1}{2} y} \, dy$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} f(x + y) \frac{\sin\left(n + \frac{1}{2} \right) y}{\sin \frac{1}{2} y} \, dy + \frac{1}{2\pi} \int_{-\pi}^{0} f(x + y) \frac{\sin\left(n + \frac{1}{2} \right) y}{\sin \frac{1}{2} y} \, dy$$

WTS:

$$\lim_{n \to \infty} \text{top} = f(x^+)$$

$$\lim_{n \to -\infty} \text{bottom} = f(x^-)$$

Note

$$\frac{1}{\pi} \int_{0}^{\pi} \frac{\sin\left(n + \frac{1}{2} \right) y}{\sin y} \, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^{n} e^{iky} \, dy$$

$$= 1$$

which is valid if only e^{i0} counts.

2. Prove a difference integral $\to 0$ by showing that it's a Fourier coefficient. Prove

$$\frac{1}{2\pi} \int_{0}^{\pi} \left(f(x + y) - f(x^+) \right) \frac{\sin\left(n + \frac{1}{2} \right) y}{\sin \frac{1}{2} y} \, dy = 0$$

Notice that

$$g(y) \equiv \frac{f(x+y) + f(x^+)}{\sin \frac{1}{2}y}$$

is piecewise continuous $\forall y \in [0, \pi]$. We need $\int_0^\pi g(y) \sin \left(n + \frac{1}{2}\right) y \, dy = 0$. Note that

$$\sin \left(n + \frac{1}{2}\right) y = \sin \frac{1}{2} y \cos ny + \cos \frac{1}{2} y \sin ny$$

Then,

$$\int_0^\pi \left(g(y) \sin \frac{y}{2}\right) (0) ny + \left(g(y) \cos \frac{y}{2} \sin ny\right)$$

But we know Fourier coefficients decay for all L^2 integrable functions. So these terms $\to 0$ and we prove pointwise convergence.

§18 | Lec 18: Nov 8, 2021

§18.1   Heat Equation

We've been learning about Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Question 18.1. Why is Fourier series useful?

It's used to solve PDEs; specifically, we want to investigate the heat equation in this lecture. For us,
$$\partial_t u(t, x) - \kappa \partial_x^2 u(t, x) = 0$$
where κ is constant (diffusing constant). Note that

- 1x time derivative $\implies$ 1x initial condition

- 2x space derivative $\implies$ 2x boundary condition

Types of boundary condition

$$\begin{aligned}
u(t, 0) &= \alpha(t) \quad \text{(Dirichlet boundaries)} \\
\partial_x u(t, 0) &= \mu(t) \quad \text{(Neumann boundaries)} \\
\partial_x u + \beta(t) u &= \tau(t) \quad \text{(Robin/Mixed boundaries)}
\end{aligned}$$

Homogeneous $\implies$ RHS = 0, i.e., $u = 0$, $\partial_x u = 0$, or $\partial_x u + \beta u = 0$.

Question 18.2. How do we solve this? (infinitely harder than an ODE!)

Assume $u(t, x) = T(t)X(x)$. Substitute into $\partial_t u - \kappa \partial_x^2 u = 0$

$$T'(t)X(x) - \kappa T(t)X''(x) = 0$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}$$

This can only be true if neither depends on t or x, i.e., constant. So

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \lambda$$

What sign is λ?

- If $\lambda > 0$, we get exponential growth which isn't physical.

$$\begin{aligned}
X'' &= \lambda X \\
X &= A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} \\
T' &= \kappa \lambda T \\
T &= T e^{\kappa \lambda t}
\end{aligned}$$

- $\lambda = 0$,

$$X'' = 0$$
$$X = Ax + B$$
$$T' = 0$$
$$T = T_0$$

- If $\lambda < 0$, redefine $\lambda \to -\lambda$

$$X'' = -\lambda X$$
$$X = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$$
$$T' = -\kappa\lambda T$$
$$T = T_0 e^{-\kappa\lambda t}$$

So either $\lambda = 0$, or $\lambda < 0$.

Example 18.1

Consider

$$\partial_t u - \partial_x^2 u = 0$$
$$u(0, x) = u_0(x)$$
$$u(t, 0) = 0$$
$$u(t, l) = 0$$
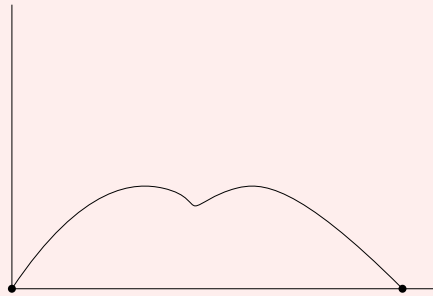
If we assume $u(t, x) = T(t)X(x)$ we find

$$\frac{T'}{\kappa T} = \frac{X'}{X} = \lambda$$

If $\lambda = 0$, $X = Ax + B$, but $u(t, 0) = u(t, l) = 0 \implies A = B = 0$. So $\lambda < 0$. Let's write $\lambda = -\omega^2$,

$$X'' = -\omega^2 X$$
$$X = A\cos\omega x + B\sin\omega x$$

Use BC's to get A, B.

$$u(t, 0) = T(t)X(0) = 0 \implies X(0) = A = 0$$
$$X = \sin\omega x$$

Example 18.2 (Cont'd)

And we have

$$u(t, l) = T(t)X(l) = B \sin \omega l = 0$$
$$\implies \omega l = k\pi, \quad k = 1, 2, \dots$$

Thus, $\omega = \frac{k\pi}{l}$, and it's an eigenvalue and $\sin \frac{n\pi}{l} x$ is an eigenfunction.

The final solution is

$$u(t, x) = \sum_{n=1}^{\infty} \left(\hat{u}_{0,k} e^{-\frac{n^2 \pi^2}{l^2} \kappa t} \sin \left(\frac{n\pi}{l} x \right) \right)$$

We get $\hat{u}_{0,k}$ from the Fourier series of the function $u_0(x)$.

<u>Note</u>: The Fourier coefficients decay more quickly as k gets larger so diffusion smooth things out. If we have source term

$$\partial_t u - \partial_x^2 u = f(t, x)$$

We can express f as a Fourier series and solve an ODE for each Fourier coefficient.

Summary:

1. We assume separable solution: $u(t, x) = T(t)X(x)$

2. Substituting gives an eigenvalue problem

$$X'' = \lambda X$$

3. The boundary conditions imply $\lambda = -\omega^2$ where $\omega = \frac{n\pi}{l}$, $x\alpha \sin \left(\frac{n\pi}{l} x \right)$.

4. Linearity mean we sum up all the eigenfunctions

$$u(t, x) = \sum_{n=1}^{\infty} \left(\hat{u}_{0,k} e^{-\kappa \frac{(k-\pi)^2}{2^2} t} \right) \sin k \frac{\pi}{l} x$$

5. We use the initial condition to determine the Fourier series coeff, $\hat{u}_{0,k}$

§19 | Lec 19: Nov 10, 2021

§19.1   Wave Equation

<u>Goal</u>: Solve the wave equation

1. Look for separable solutions
$$u(t,x) = T(t)X(x)$$

to
$$\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x)$$

c: wave speed $\left(\frac{\text{space}}{\text{time}}\right)$.

$$\partial_t^2(TX) = c^2 \partial_x^2(TX)$$
$$T''X = c^2 TX''$$
$$\frac{T''(t)}{T(t)} = \frac{c^2 X''(x)}{X(x)} = \lambda$$

a) $\lambda = \omega^2 > 0$, $T'' = \omega^2 T \implies T = e^{\omega t}$ or $T = e^{-\omega t}$ and $X'' = \frac{\omega^2}{c^2}X$, $X = e^{\frac{\omega x}{c}}$ or $X = e^{-\frac{\omega x}{c}}$

b) $\lambda = 0$:

$$T'' = 0 \implies T = A + Bt$$
$$X'' = 0 \implies X = C + Dx$$
$$TX = a + bt + cx + d + x$$

c) $\lambda = -\omega^2 < 0$

$$T'' = -\omega^2 T \implies T = \sin \omega t \text{ or } T = \cos \omega t$$
$$X'' = \frac{-\omega^2}{c^2}X \implies X = \frac{\sin \omega x}{c} \text{ or } X = \frac{\cos \omega x}{c}$$

Next, let's decide on the sign of λ using the boundary conditions, e.g., homogeneous, Dirichlet, boundary conditions
$$u(t,0) = u(t,l) = 0$$

If $\lambda > 0$
$$u = Ae^{\omega t}e^{\frac{\omega x}{c}} + Be^{-\omega t}e^{\omega \frac{x}{c}} + Ce^{\omega t}e^{-\omega \frac{x}{c}} + De^{-\omega t}e^{-\omega \frac{x}{c}}$$

but the boundary condition implies that $A = B = C = D = 0$.

If $\lambda = 0$, similarly $u = 0$ is only possibility.

If $\lambda < 0$, $u = T(t)X(x)$, $T = \sin \omega t, \cos \omega t$, $X = \sin \frac{\omega x}{c}, \cos \frac{\omega x}{c}$

$$u(t,0) = T(t)X(0) = X(0) = 0$$
$$\implies A\sin \frac{\omega 0}{c} + B\cos \frac{\omega 0}{c} = B = 0$$

So $X = A\sin \frac{\omega x}{c}$.

$$u(t,l) = T(t)X(l) = A\sin \frac{\omega l}{c}$$
$$\implies \frac{\omega l}{c} = n\pi, \quad n = 1, \dots$$
$$\omega = \frac{n\pi c}{l}$$

In general, the solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t \right) \sin \frac{n\pi x}{l}$$

TO find A_n, B_n, we use the initial conditions

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

Example 19.1

Consider $\partial_t^2 u = \partial_x^2 u$, $x \in [0,1]$

$$u(t,0) = u(t,1) = 0$$

$$u(0,x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\partial_t u(0,x) = 0$$

$$c = 1, \quad l = 1 \implies \omega_n = n\pi$$

Using Dirichlet, the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(a_n \cos n\pi t + b_n \sin n\pi t \right) \sin n\pi x$$

We want to get a_n, b_n with ICs. For $t = 0$,

$$u(0,x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

$$= \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$a_n = \frac{1}{2} \int_0^1 f(x) \sin n\pi x\, dx$$

In general,

$$\sum_{n=1}^{\infty} a_n \int_0^1 \sin n\pi x \sin m\pi x\, dx = \int_0^{\frac{1}{2}} x \sin n\pi x\, dx + \int_{\frac{1}{2}}^1 (1-x) \sin n\pi x\, dx$$

$$= \begin{cases} 0, & \text{n even} \\ 2\left(-\left[\frac{1}{2} \frac{\cos n\pi/2}{n\pi}\right] + \frac{1}{(n\pi)^2} [\sin n\pi]_0^{\frac{1}{2}} \right) = \ldots = \frac{4}{\pi^2} \frac{\sum (-1)^k \cos(2k+1)\pi k \sin(2k+1)\pi k}{2k+1} \end{cases}$$

---

§20 | Lec 20: Nov 12, 2021

§20.1   Midterm 2

Need to know:

- How to calculate Fourier series

- How to test convergence?

- How does convergence relate to continuity?

- How does symmetry affect Fourier series?

- How does rescaling affect Fourier series?

- How does differentiation/integration affect Fourier series?

- How does smoothness (C^n) relate to Fourier series?

- How to analyze generalized Fourier series in Hilbert spaces?

- How to use Fourier series to solve PDEs?

§20.2   Laplace Equations

Consider

$$\partial_x^2 u + \partial_y^2 u = 0$$
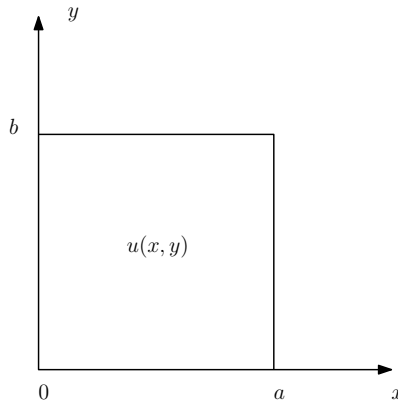
The Laplace equation satisfies the Maximum principle, i.e., the maximum (and minimum) value of a solution to the Laplace equation must occur on the boundary,

Proof. If a function u has a maximum at (x, y) then

1. $\partial_x^2 u < 0$

2. $\partial_y^2 u < 0$

But Laplace says $\partial_x^2 u + \partial_y^2 u = 0$. Thus, we can't have local maximum (or minimum) in the domain.   □

> **Theorem 20.1**
> Solutions to Laplace equation, with given boundary conditions, are unique.

Proof. Suppose there exist u_1 and u_2 where

$$\partial_x^2 u_1 + \partial_y^2 u_1 = 0, \quad u_1 = f \text{ on the boundary}$$
$$\partial_x^2 u_2 + \partial_y^2 u_2 = 0, \quad u_2 = f \text{ on the boundary}$$

Consider $u_1 - u_2 = \Delta u$. Since Laplace equation is linear, Δu solves Laplace. And we know that $\Delta u = 0$ on the boundary. Therefore, $\Delta u = 0$ everywhere (by maximum principle). □

Example 20.2

Consider

$$1 \qquad u(x,1) = 0 \qquad 1$$

$$u(0,y) = 0 \qquad \boxed{\partial_x^2 u + \partial_y^2 u = 0} \qquad u(1,y) = 0$$

$$0 \qquad\qquad\qquad 1$$

and

$$u(0,x) = \begin{cases} x \\ 1 - x \end{cases}$$

Have $u = X(x)Y(y)$

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

- $\lambda = -\omega^2 < 0$

$$X'' = -\omega^2 X \implies X = \cos \omega x, \ \sin \omega x$$
$$Y'' = \omega^2 Y \implies Y = e^{\omega y}, \ e^{-\omega y}, \ \cosh y, \ \sinh y$$

- $\lambda = 0,$

$$X = 1, \ x$$
$$Y = 1, \ y$$

- $\lambda = \omega^2 > 0$

$$X = e^{\omega x}, \ e^{-\omega x}$$
$$y = \cos \omega y, \ \sin \omega y$$

Example 20.3 (Cont'd)

Only $\lambda = -\omega^2 < 0$ works. For $X = \sin n\pi x$,

$$X(0) = X(1) = 0 \implies \omega = n\pi$$

Therefore,

$$Y_n = A e^{n\pi y} + B e^{-n\pi y}$$
$$Y(1) = 0 \implies A e^{n\pi} + B e^{-n\pi} = 0$$
$$\iff Y(y) \propto \sinh n\pi(1-y)$$

In general,

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi(1-y) \sin n\pi x$$

and

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi x = \begin{cases} x, & x < \frac{1}{2} \\ 1-x, & x > \frac{1}{2} \end{cases}$$
$$= \sum_{k=1}^{\infty} (-1)^k \frac{4}{\pi^2(2k+1)^2} \sin n\pi x$$

where

$$a_{2k+1} = \frac{(-1)^k 4}{\pi^2(2k+1)^2}$$

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§21 | Lec 21: Nov 17, 2021

§21.1   Adjoints

For $u \in U$, $v \in V$ and linear operator L

$$L : U \to V \text{ or } L[u] = v$$

Then,

$$\langle u_1, u_2 \rangle : U \times U \to \mathbb{R}/\mathbb{C}, \qquad \langle\langle v_1, v_2 \rangle\rangle : V \times V \to \mathbb{R}/\mathbb{C}$$
$$\langle u, L^*[v] \rangle = \langle\langle L[u], v \rangle\rangle$$

Example 21.1

$U : \mathbb{R}^m$, $V : \mathbb{R}^n$, $L : A$ matrix with n rows and m columns

$$\langle u_1, u_2 \rangle = u_1 \cdot u_2, \qquad \langle\langle v_1, v_2 \rangle\rangle = v_1 \cdot v_2$$

What is A^*?

$$\begin{aligned}
\langle\langle Au, v \rangle\rangle = (Au)^\top v = u^\top A^\top v &= u^\top (A^\top v) \\
&= u^\top (A^\top v) \\
&= \langle u, A^\top v \rangle
\end{aligned}$$

Example 21.2

$\langle u_1, u_2 \rangle = u_1^\top M u_2$, $\langle\langle v_1, v_2 \rangle\rangle = v_1^\top C v_2$

$$\begin{aligned}
\langle\langle Au, v \rangle\rangle = (Au)^\top C v = u^\top A^\top C v &= u^\top M M^{-1} A^\top C v \\
&= u^\top M \left(M^{-1} A^\top C v \right) \\
&= \langle u, M^{-1} A^\top C v \rangle
\end{aligned}$$

$A^* = M^{-1} A^\top C$ – adjoint depends on inner products.

Differential operators as linear operators

$$u \in U = C', \qquad D[u] = \frac{du}{dx}$$

D is linear! $D[u, ru_2] = \frac{du_1}{dx} \cdot r \frac{du_2}{dx} = D[u_1] + D[u_2]$ and $D[cu] = c\frac{du}{dx} = cD[u]$. Say

$$\langle u_1, u_2 \rangle = \int_a^b u_1(x) u_2(x) \, dx$$

$$\langle\langle v_1, v_2 \rangle\rangle = \int_a^b v_1(x) v_2(x) \, dx$$

Question 21.1. What is D^*?

$$\langle\langle D\left[u\right], v\rangle\rangle = \int_a^b \frac{du}{dx} v \, dx = \ldots? = \langle u, D^*\left[v\right]\rangle$$

$$= \left[uv\right]_a^b - \int_a^b u \frac{dv}{dx} dx$$

$$= -\int_a^b u \frac{dv}{dx} dx = \langle u, -D\left[v\right]\rangle$$

If $\left[uv\right]_a^b = u(b)v(b) - u(a)v(a) = 0$,

1. Dirichlet BC on u: $u(b) = u(a) = 0$

2. Dirichlet BC on v: $v(b) = v(a) = 0 \iff$ Neumann BC on u : $u'(b) = u'(a) = 0$

3. Periodic BC

Boundary conditions matter. If they're nice then

$$D^* = -D \text{ for standard inner products}$$

Question 21.2. What if

$$\langle u_1, u_2 \rangle = \int_a^b u_1(x)u_2(x)k(x)\,dx$$

$$\langle v_1, v_2 \rangle = \int_a^b v_1(x)v_2(x)p(x)\,dx$$

Show $D^*\left[v\right]$ is $-\frac{1}{p}\frac{d}{dx}(kv)$ if $\left[uvk\right]_a^b = 0$.

Fact 21.1. $(L^*)^* = L$

Fact 21.2. $U \underset{L^*}{\overset{L}{\rightleftarrows}} V \underset{M^*}{\overset{M}{\rightleftarrows}} W$

$$M \circ L = M\left[L\left[u\right]\right]$$
$$(M \circ L)^* = L^* \circ M^* = L^*\left[M^*\left[w\right]\right]$$

<u>Self-Adjoint:</u> $L^* = L$

$$D^2 = D \circ D, \qquad D^2\left[u\right] = \frac{d^2u}{dx^2}$$

Provided correct BCs and inner product

$$D^{2*} = D^* \circ D^* = (-D) \cdot (-D) = D^2$$

<u>Fredholm Alternative:</u>

Question 21.3. When can we solve a linear problem?

$$L\left[u\right] = f$$

If $f \perp \text{coker } L \iff \langle\langle f, v \rangle\rangle = 0$ for each $v \in \text{coker } L$.

$$\ker L : \{u | L\left[u\right] = 0\}$$
$$\text{coker } L : \ker L^* = \{v | L^*\left[v\right] = 0\}$$

Example 21.3

Consider

$$Au = f$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Way # 1: Direct Gaussian elimination

$$\begin{bmatrix} 1 & 0 & -1 & | & f_1 \\ 0 & 1 & -2 & | & f_2 \\ 1 & -2 & 3 & | & f_3 \end{bmatrix} \ldots \sim \begin{bmatrix} 1 & 0 & -1 & | & f_1 \\ 0 & 1 & -2 & | & f_2 \\ 0 & 0 & 0 & | & f_3 - f_1 + 2f_2 \end{bmatrix}$$

Thus, $f_3 - f_1 + 2f_2$ must be 0.

Way # 2: Find kernel of A^*.

$$A^*v = \begin{bmatrix} v_1 + v_3 \\ v_2 - 2v_3 \\ -v_1 - 2v_2 + 3v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Fredholm $\implies t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$ which is the same as above.

Example 21.4

What about differential operators?

$$u'' = f, \quad u'(0) = u'(l) = 0$$

Way # 1: Direct

$$u'' = f$$

$$u' = a + \int_0^x f(y) \, dy$$

$$u = ax + b + \int_0^x \int_0^y f(z) \, dz \, dy$$

BCs $\implies u'(0) = a = 0$ and $u'(l) = \int_0^l f(x)dx = 0$

Way # 2: Fredholm

   Adjoint of $\frac{d^2}{dx^2}$ is $\frac{d^2}{dx^2}$ for standard inner products

$$\text{cokernel} = \text{kernel} = \{g(x)|g'' = 0, \ g'(0) = g'(l) = 0\}$$
$$= \{c|c \in \mathbb{R}\}$$

Thus, $\langle c, f \rangle = c\langle 1, f \rangle = c \int_0^l f(x)dx = 0$.

Proof. If $v \in \text{coker } L$ and $L[u] = f$, then

$$\langle\langle f, v \rangle\rangle = \langle\langle L[u], v \rangle\rangle = \langle u, L^*[v] \rangle = \langle u, 0 \rangle = 0 \qquad \square$$

§22 | Lec 22: Nov 19, 2021

§22.1   Self-Adjoint & Positive Definite Linear Operators

Definition 22.1 (Self-Adjoint Opeator) — Self-adjoint linear operator $S : u \to u$ if

$$\langle S[u_1], u_2 \rangle = \langle u_1, S[u_2] \rangle \quad \forall u_1, u_2 \in u$$

Example 22.2

$\frac{d^2}{dx^2}$ with standard $L^2 \langle u_1, u_2 \rangle$ and appropriate boundary conditions

$$\begin{aligned}
\langle S[u_1], u_2 \rangle &= \int_a^b u_1'' u_2 \, dx \\
&= [u_1' u_2]_a^b - \int_a^b u_1' u_2' \, dx \\
&= [u_1' u_2 - u_1 u_2']_a^b + \int_a^b u_1 u_2'' \, dx \\
&= \langle u_1, S[u_2] \rangle
\end{aligned}$$

If $u_1'(b)u_2(b) - u_1(b)u_2'(b) - u_1'(a)u_2(a) + u_1(a)u_2'(a) = 0$ (homogeneous Dirichlet, mixed, and periodic BCs all work)

Definition 22.3 (Positive Definite Opeator) — Positive definite linear operator $S : u \to u$ on inner product space u,

$$\text{positive definite} : s > 0 \text{ if } \langle S[u], u \rangle > 0 \ \forall u \neq 0$$
$$\text{positive semi-definite} : s \geq 0 \text{ if } \langle S[u], u \rangle \geq 0 \ \forall u$$

Example 22.4

$S = -\frac{d^2}{dx^2}$ wit standard inner product and appropriate BCs,

$$\langle S[u], u \rangle = -\int_a^b u'' u \, dx$$

$$= -[u'u]_a^b + \int_a^b (u')^2 \, dx$$

$$= \int_a^b (u')^2 \, dx$$

for appropriate BCs.

Question 22.1. Is this positive?

1. If homogeneous Dirichlet/Mixed BCs, then no constants solutions with $u' = 0$ and so $S > 0$.

2. If homogeneous Neumann/Periodic BCs, constants allowed where $\langle S[u], u \rangle = 0$ for $u \neq 0$, so $S \geq 0$.

Boundary conditions matter.

Proposition 22.5

Positive definite operators have unique solutions.

Proof. $u \in \ker S \implies S[u] = 0 \implies \langle S[u], u \rangle = 0 \implies u = 0$. So $\ker S = \{0\}$. Then, we get unique solutions (If instead $v \neq 0$, $v \in \ker S$, then $S[u] = f \implies S[u+v] = S[u]+0 = f$ too!) $\square$

If $L : u \to v$, then $L^* \circ L$ is self-adjoint and positive semi-definite, where $\ker S = \ker L$ and positive definite $\iff \ker L = \ker S = \{0\}$

Proof. 1. $S = L^* \circ L$

$$S^* = (L^* \circ L)^* = L^* \circ L^{**} = L^* \circ L = S$$

2. Have

$$\langle u, S[u] \rangle = \langle u, L^*[L[u]] \rangle$$
$$= \langle \langle L[u], L[u] \rangle \rangle$$
$$= \|L[u]\|^2 > 0 \text{ unless } L[u] = 0, \text{ i.e., } u \in \ker L = \ker S$$

$\square$

Question 22.2. Why are $L^* \circ L$ special?

Simple solvability conditions!

Theorem 22.6

If $S = L^* \circ L$, and $S[u] = f$ has solution then $\langle z, f \rangle = 0$ for each $z \in \ker S = \ker S^*$. If multiple solutions $S[u_1] = f, S[u_2] = f$, then $u_2 - u_1 \in \ker S$

Proof. 1. Fredholm alternative for self-adjoint

2. $S[u_1 - u_2] = S[u_1] - S[u_2] = f - f = 0$ so $u_1 - u_2 \in \ker S$. □

Example 22.7

For standard inner products, $S = D^* \circ D = -D \circ D$ is self-adjoint and depending on BCs is positive definite or positive semi-definite.

What about the non-standard inner products?

$$\langle u_1, u_2 \rangle = \int_a^b k u_1 u_2 \, dx$$

$$\langle\langle v_1, v_2 \rangle\rangle = \int_a^b p u_1 u_2 \, dx$$

Let $L[u] = D[u] \implies L^*[v] = -\frac{1}{p}(kv)'$. So

$$S = L^* \circ L = -\frac{1}{p}\frac{d}{dx}\left(k\frac{du}{dx}\right)$$

is self-adjoint and positive semi-definite. And

$$-\frac{1}{p}\frac{d}{dx}\left(k\frac{du}{dx}\right) = f(x) \text{ has unique solutions}$$

§23 | Lec 23: Nov 22, 2021

§23.1   Minimization Problem

> **Theorem 23.1**
>
> Consider
> $$\mathbf{S}u = f \tag{1}$$
> where $\mathbf{S}$ is positive definite and self-adjoint operator. If u_* solves 1), then u_* minimizes the following optimization problem
> $$\frac{1}{2}\langle u, Su \rangle - \langle f, u \rangle = Q(u)$$

Proof. Have

$$\langle u, Su \rangle - \langle u, f \rangle = \langle u, Su \rangle - \langle u, Su_* \rangle$$
$$= \langle u_* + u - u_*, S\left(u_* + u - u_*\right)\rangle - \langle u, Su_* \rangle$$
$$= \frac{1}{2}\langle u_*, Su_* \rangle + \frac{1}{2}\langle u - u_*, S(u - u_*)\rangle + \frac{1}{2}\langle u_*, S(u - u_*)\rangle + \frac{1}{2}\langle (u - u_*), Su_* \rangle - \langle u, Su_* \rangle$$
$$= \frac{1}{2}\langle u - u_*, S(u - u_*)\rangle - \frac{1}{2}\langle u_*, Su_* \rangle$$

where the first term is non-negative and the second term is a constant. Take $u = u_*$ and that's the solution to minimization problem. $\qquad\square$

> **Example 23.2**
>
> $-u''(x) = f(x)$ on $u(0) = u(b) = 0$.
> $$S = -\frac{\partial^2}{\partial x^2}$$
> which is self-adjoint and positive definite operator (under certain BCs) and also
> $$S = -D \circ D$$
> If u_* solves this, then u_* solves
> $$Q(u) = \langle Su, u \rangle - \langle f, u \rangle$$
> $$\min_u(\|Du\|^2 - \langle f, u \rangle)$$

§23.2   Sturm-Liouville Problem

Special class of B.V.P. For $x \in [a, b]$, p, q, r real valued function defined on $[a, b]$

1. $(Ly)(x) = p(x)y''(x) + q(x)y'(x) + r(x)y(x)$

2. BC:
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

    where
$$(\alpha_1, \alpha_2) \neq (0,0), \quad (\beta_1, \beta_2) \neq (0,0)$$

3. BVP/Sturm Liouville problem
$$Ly + \lambda wy = f$$
where w is a known function that is positive and defined on $[a, b]$.

So the S.L. problem is finding solutions to 2.
We need $q = p'$ so that L is self-adjoint and positive definite. Then, 1. becomes

$$(Ly)(x) = (py')' + ry$$

Properties:

1. S.L. is self-adjoint and positive definite.

2. $\lambda > 0$ and eigenvectors are real-valued and orthogonal

3. Eigenvectors will form a complete basis

§24 | Lec 24: Nov 24, 2021

§24.1   Review

Motivation: $u(t,x) = v(t)w(x)$

$$\implies \frac{d^2w}{dx^2} = \lambda w$$

We have countably infinite number of solutions λ_i, w_i

$$u(t,x) = \sum_{i=1}^{\infty} c_i e^{\lambda_i t} w_i(x)$$

and

$$c_i = \frac{\langle u_0(x), w_i \rangle}{\langle w_i, w_i \rangle}$$

Consider the general PDE:

$$\frac{\partial u}{\partial t} = \frac{d}{dx} p(x) \frac{\partial u}{\partial x} + r(x)u(x)$$

where

$$\frac{d}{dx} p(x) \frac{d}{dx} w + r(x)w(x) = \lambda w(x)$$

§24.2   Eigenvalue and Eigenvectors of Self-Adjoint and PD Operator

Consider

$$\langle Su, v \rangle = \langle u, Sv \rangle$$

and $\langle Su, u \rangle > 0 \ \forall u \in \mathcal{H}$ (Hilbert space). Given the operator S, the eigenvalue problem can be described as

$$Su = \lambda u$$

where $\lambda \in \mathbb{C}$ is the eigenvalue and $u \in \mathcal{H}$ is known as the eigenvector.

1. Self-adjoint operators have only purely real set of eigenvalues.

2. If $\lambda_1, \lambda_2 \in \mathbb{C}$ are eigenvalues corresponding to two distinct eigenvectors $u_1, u_2 \in \mathcal{H}$. Then $\langle u_1, u_2 \rangle = 0$.

3. If S is positive (semi)-definite, then $\lambda \geq 0$ or $\lambda > 0$.

Theorem 24.1

If S is self-adjoint, then $\lambda \in \mathbb{R}$.

Proof. Have

$$\begin{aligned}
\lambda \langle u, u \rangle &= \langle \lambda u, u \rangle \\
&= \langle Su, u \rangle \\
&= \langle u, Su \rangle \\
&= \langle u, \lambda u \rangle \\
&= \overline{\lambda} \langle u, u \rangle
\end{aligned}$$

$\implies \lambda = \overline{\lambda} \implies \lambda \in \mathbb{R}.$ $\square$

Theorem 24.2

If S is self-adjoint, $\lambda_1, \lambda_2 \in \mathbb{C}$ are eigenvalues corresponding to two distinct eigenvectors $u_1, u_2 \in \mathcal{H}$. Then $\langle u_1, u_2 \rangle = 0$.

Proof. Have

$$
\begin{aligned}
\lambda_1 \langle u_1, u_2 \rangle &= \langle \lambda_1 u_1, u_2 \rangle \\
&= \langle S u_1, u_2 \rangle \\
&= \langle u_1, S u_2 \rangle \\
&= \langle u_1, \lambda_2 u_2 \rangle \\
&= \overline{\lambda_2} \langle u_1, u_2 \rangle \\
&= \lambda_2 \langle u_1, u_2 \rangle
\end{aligned}
$$

$\implies \langle u_1, u_2 \rangle = 0$. □

Example 24.3

Suppose $A = A^\top \in \mathbb{R}^{n \times n}$. Find an inner product on $\mathbb{R}^n$ s.t. AD is self-adjoint where $D \in \mathbb{R}^{n \times n}$ with positive diagonal elements.
Want

$$
\langle ADx, y \rangle_D = \langle x, ADy \rangle_D
$$

Have

$$
\begin{aligned}
\langle x, y \rangle_D &= \langle Dx, y \rangle = \langle x, Dy \rangle \\
\langle ADx, y \rangle_D &= \langle DADx, y \rangle \\
&= \langle ADx, Dy \rangle \\
&= \langle Dx, ADy \rangle \\
&= \langle x, ADy \rangle_D
\end{aligned}
$$

Theorem 24.4

If S is self-adjoint and positive definite, the smallest eigenvalue λ_1 of S is given by

$$
\lambda_1 = \min_{u \in \mathcal{H} \setminus \{0\}} R[u]
$$

where

$$
R[u] = \frac{\langle u, Su \rangle}{\langle u, u \rangle}
$$