Math 151A – Applied Numerical Methods I

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This is math 151A – Applied Numerical Methods taught by Professor Jiang. We meet weekly on MWF from 1:00 pm to 1:50 pm for lecture. The recommended textbook for the class is *Numerical Analysis* 10^{th} by *Burden*, *Faires* and *Burden*. Other course notes can be found at my blog site. Please let me know through my email if you spot any typos in the note.

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§1 Lec 1: Sep 24, 2021

§1.1 Calculus Review

- Intermediate Value Theorem (IVT): For continuous function $C([a,b])$, let $f \in C([a,b])$. Let $k \in \mathbb{R}$ s.t. k is strictly between $f(a)$ and $f(b)$. Then, $\exists$ some $c \in (a,b)$ s.t. $f(c) = k$.

  Question 1.1. Why is IVT useful?

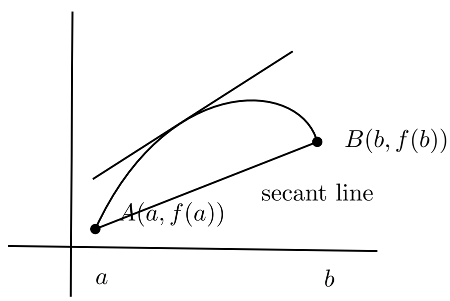
  It guarantees the existence of solution to some nonlinear equations.

  Example 1.1
  Let $f(x) = 4x^2 - e^x$. IVT tells us $\exists x^*$ s.t. $f(x^*) = 0$.

  $$f(0) = 0 - e^0 = -1 < 0$$
  $$f(1) = 4 - e > 0$$

  With $k = 0$, by IVT, $\exists c \in (0,1)$ s.t. $f(c) = 0$.

- Mean Value Theorem (MVT): If $f \in C([a,b])$ and f is differentiable in (a,b), then $\exists c \in (a,b)$ s.t.
  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
  in which $f'(c)$ is essentially the slope of the tangent line at $(c, f(c))$.



- Taylor's Theorem: Apply for a differentiable function, $f \in C^m([a,b])$ – f is m times continuously differentiable.

Theorem 1.2 (Taylor)
Let $f \in C^n([a,b])$. Let $x_0 \in [a,b]$. Assume $f^{(n+1)}$ exists on $[a,b]$. Then $\forall x \in [a,b]$, $\exists \xi(x) \in \mathbb{R}$ s.t. $x_0 < \xi < x$ or $x < \xi < x_0$. Then, we can express f as

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Example 1.3

$f(x) = \cos(x)$, $x_0 = 0$

$$\begin{aligned} f(x) = \cos(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 \\ &= 1 + 0 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin(\xi(x)) \end{aligned}$$

Note: Saying $f \in C^1$ is different from saying $f'(x)$ exists.

Example 1.4

Consider

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) \\ &= 0 \end{aligned}$$

But $f'(x)$ is not continuous. Specifically,

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Take sequence $\frac{1}{2k\pi}$, $f' \to -1$ and $\frac{1}{(2k+1)\pi}$, $f' \to 1$. Thus, the function is not continuous as it converges to two different values.