

# Math 135 – Differential Equations

University of California, Los Angeles

Duc Vu

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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3<sup>rd</sup> by *Simmons*. Other course notes can be found at my [blog site](#). Please let me know through my [email](#) if you spot any concerning typos in the note.

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## List of Definitions

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# §1 | Lec 1: Sep 27, 2021

## §1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets  $\rightarrow$  sets. Transform is in higher hierarchy, i.e.,

Transform/Operator: functions  $\rightarrow$  functions

**Example 1.1** • differentiation:  $\frac{d}{dx} : f \mapsto f'$

- integration:  $\int^x dx : f \mapsto \int^x f'(x)dx$
- multiplication by  $g(x)$ :  $f(x) \rightarrow g(x)f(x)$
- shifting:  $f(x) \rightarrow f(x - a)$

2. Laplace transform  $\mathcal{L}$

$$\mathcal{L} : f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st} dt$$

where  $f : [0, \infty) \rightarrow \mathbb{R}$  and  $F : \mathbb{C} \rightarrow \mathbb{C}$

3. Examples:

**Example 1.2** •  $f(t) : t \mapsto 0 \implies \mathcal{L}[0] = 0$

- $f(t) = 1$

$$\begin{aligned} \mathcal{L}[1] &= \lim_{t \rightarrow \infty} \int_0^t e^{-st} dt \\ &= \lim_{t \rightarrow \infty} \left[ \frac{e^{-st}}{-s} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{e^{-st}}{-s} + \frac{1}{s} \right) \\ &= \frac{1}{s} \text{ if } \operatorname{Re}(s) > 0 \end{aligned}$$

**Example 1.3** • Consider

$$\begin{aligned}\mathcal{L}[t] &= \int_0^\infty t e^{-st} dt \\ &= \left[ \frac{t e^{-st}}{-s} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt \\ &= \frac{1}{s^2} \text{ if } \operatorname{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathcal{L}[t^n] = \frac{1}{s^{n+1}}, \quad \operatorname{Re}(s) > 0, \quad n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathcal{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t} dt \\ &= \frac{1}{s-a}, \quad \operatorname{Re}(s) > a \\ \mathcal{L}[\cos \omega t] &= \frac{s}{s^2 + \omega^2} \\ \mathcal{L}[\sin \omega t] &= \frac{\omega}{s^2 + \omega^2}\end{aligned}$$

4. Properties:

a) Linear!

$$\begin{aligned}\mathcal{L}[f+g] &= \mathcal{L}[f] + \mathcal{L}[g] \\ \mathcal{L}[af] &= a\mathcal{L}[f]\end{aligned}$$

b) Consider:

$$\begin{aligned}\mathcal{L}[e^{at}f(t)] &= \int_0^\infty f(t)e^{-(s-a)t} dt \\ &= F(s-a) \quad \text{if } \operatorname{Re}(s-a) > 0\end{aligned}$$

Multiply an exponential in  $t$ -space  $\xrightarrow{\mathcal{L}}$  shift in  $s$ -space.

5. In reverse,

$$\mathcal{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st} dt = \int_0^\infty f(t')e^{-st'} dt' e^{-sa}$$

where  $t' = t - a$ . So

$$\mathcal{L}[f(t-a)] = F(s)e^{-sa}$$

Thus, a shift in  $t$ -space  $\xrightarrow{\mathcal{L}}$  multiply an exponential in  $s$ -space.

6. Differentiation:

$$\begin{aligned}\mathcal{L}[f'] &= \int_0^\infty f'(t)e^{-st} dt \\ &= [f e^{-st}]_0^\infty + \int_0^\infty f(t) s e^{-st} dt \\ &= sF(s) - f(0)\end{aligned}$$

## § 2 | Lec 2: Sep 29, 2021

### § 2.1 Laplace Transform (Cont'd)

Recap:  $\mathcal{L} : f \rightarrow F$

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

where  $t > 0$  and  $s \in \mathbb{C}$ .

**Example 2.1** •  $\mathcal{L}[t^n] = \frac{1}{s^{n+1}}, n \in \mathbb{N}$

•  $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting  $\leftrightarrow$  multiplying by exponential
- $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \quad f(0) = f_0, \quad f'(0) = f'_0$$

Apply  $\mathcal{L}$ ,

$$\begin{aligned} \mathcal{L}[f'' + af' + bf] &= \mathcal{L}[g] \\ \mathcal{L}[f''] + a\mathcal{L}[f'] + b\mathcal{L}[f] &= G(s) \end{aligned}$$

Notice that

$$\mathcal{L}[f''] = s^2F - sf(0) - f'(0)$$

So

$$\begin{aligned} (s^2 + as + b)F(s) &= G(s) + (s + a)f_0 + f'_0 \\ F(s) &= \frac{G(s) + (s + a)f_0 + f'_0}{s^2 + as + b} \end{aligned}$$

To get  $f(t)$  we need to invert  $\mathcal{L}$ .

**Example 2.2**

Consider:

$$f'' + 4f = 4t, \quad f(0) = 1, \quad f'(0) = 5$$

Apply  $\mathcal{L}$ , we get

$$\begin{aligned} (s^2 + 4)F(s) &= \frac{4}{s^2} + s + 5 \\ F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2 + 4} \\ &= \frac{4}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4} + \frac{5}{s^2 + 4} \end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned} \frac{4}{s^2(s^2 + 4)} &= \frac{A}{s^2} + \frac{B}{s^2 + 4} \\ 4 &= A(s^2 + 4) + Bs^2 \\ &= (A + B)s^2 + 4A \end{aligned}$$

So,  $A = 1$ ,  $B = -1$ . Then,

$$\begin{aligned} F(s) &= \frac{1}{s^2} - \frac{1}{s^2 + 4} + \frac{s}{s^2 + 4} + \frac{5}{s^2 + 4} \\ &= \frac{1}{s^2} + \frac{4}{s^2 + 4} + \frac{s}{s^2 + 4} \\ \mathcal{L}[f] &= \mathcal{L}[t + 2\sin 2t + \cos 2t] \\ \implies f &= t + 2\sin 2t + \cos 2t \end{aligned}$$

## §3 | Lec 3: Oct 1, 2021

### §3.1 Existence of Laplace Transform

**Question 3.1.** When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathcal{L}[f] = \int_0^{\infty} f(t)e^{-st} dt$$

Note: Beware of  $\infty$  – only trust limits.

$$\mathcal{L}[f] = \lim_{\tau \rightarrow \infty} \int_0^{\tau} f(t)e^{-st} dt$$

Laplace transform exists when this limit exists?

$\lim_{\tau \rightarrow \infty} f^*(\tau)$  converges to  $f_{\infty} \in \mathbb{R}$  if  $\forall \varepsilon > 0, \exists M > 0$  s.t.

$$|f^*(\tau) - f_{\infty}| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau \rightarrow \infty} \int_0^{\tau} f(t) dt$$

Comparison Test: If  $|f(t)| < g(t)$  and  $\int_0^{\infty} g(t) < \infty$  (converges) then

$$\int_0^{\infty} f(t) dt \leq \int_0^{\infty} |f(t)| dt \leq \int_0^{\infty} g(t) dt < \infty$$

i.e.,  $\int_0^{\infty} f(t) dt$  converges. Now, back to the Laplace transform

$$\mathcal{L}[f] = \int_0^{\infty} f(t)e^{-st} dt$$

What could break this integral?

1.  $fe^{-st}$  diverges/unbounded ( $\lim_{t \rightarrow t^*} f(t) = \infty$ )
2.  $fe^{-st}$  doesn't decay fast enough as  $t \rightarrow \infty$ .

What could prevent these issues?

1. Piecewise continuous:  $\lim_{t \rightarrow t^-} f(t)$  and  $\lim_{t \rightarrow t^+} f(t)$  exist.
2. Exponential order

$$|f(t)| < Me^{ct} \text{ for some } M > 0 \text{ \& } c$$

Have

$$\begin{aligned} c^{-t} &\leq 1 \cdot e^{-t} & \forall t > 0 \\ 1 &\leq 1 \cdot e^{0t} & \forall t > 0 \\ t &\leq 1 \cdot e^t & \forall t > 0 \end{aligned}$$

#### Theorem 3.1

If  $f$  is piecewise continuous and of exponential order  $c$  then  $\mathcal{L}[f]$  exists for  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > c$ .

*Proof.* Have

$$\begin{aligned}
 \mathcal{L}[f](s) &= \int_0^\infty f(t)e^{-st} dt \\
 \lim_{\tau \rightarrow \infty} \int_0^\tau f(t)e^{-st} dt &\leq \lim_{\tau \rightarrow \infty} \int_0^\tau |f(t)e^{-st}| dt \\
 &= \lim_{\tau \rightarrow \infty} \int_0^\tau |f(t)| e^{-s_r t} dt \\
 &\leq \lim_{\tau \rightarrow \infty} \int_0^\tau M e^{ct} \cdot e^{-s_r t} dt \\
 &= \lim_{\tau \rightarrow \infty} M \left[ \frac{e^{(c-s_r)t}}{-(c-s_r)} \right]_0^\tau \\
 &= \frac{1}{s_r - c} \quad \text{if } s_r > c \\
 &< \infty
 \end{aligned}$$

Thus,  $\mathcal{L}[f]$  exists (for  $\text{Re}(s) > c$ ) by comparison test. □

This is a sufficient condition but not necessary.

### Example 3.2

Consider the function  $f(t) = \frac{1}{\sqrt{t}}$

$$\begin{aligned}
 \mathcal{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}} e^{-st} dt \\
 &= s^{-\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\
 &= s^{-\frac{1}{2}} 2 \int_0^\infty e^{-z^2} dz \\
 &= \sqrt{\frac{\pi}{s}}
 \end{aligned}$$

However, we can see that  $\frac{1}{t^{\frac{1}{2}}}$  isn't continuous on  $[0, \infty)$ .



## §4 | Lec 4: Oct 4, 2021

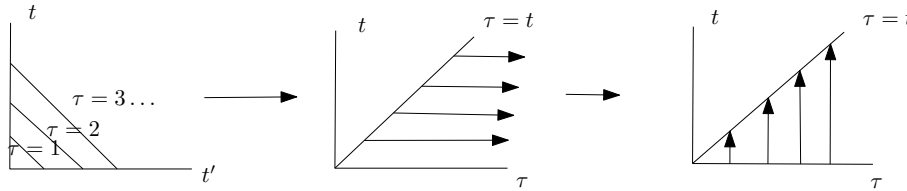
### §4.1 Convolution

**Question 4.1.** Can we invert  $\mathcal{L}[f] \cdot \mathcal{L}[g]$ ?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st} dt \int_0^\infty g(t')e^{-st'} dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')} dt' dt \end{aligned}$$

Let's define  $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')} dt' dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau} d\tau dt \\ &= \int_0^\infty \left( \int_0^\tau f(t)g(\tau - t)e^{-s\tau} dt \right) d\tau \\ &= \int_0^\infty \left( \int_0^\tau f(t)g(\tau - t) dt \right) e^{-s\tau} d\tau \\ &= \mathcal{L} \left[ \int_0^\tau f(t)g(\tau - t) dt \right] \end{aligned}$$

#### Theorem 4.1 (Convolution)

We have

$$\begin{aligned} (f * g)(\tau) &= \int_0^\tau f(t)g(\tau - t) dt \\ \mathcal{L}[f * g] &= \mathcal{L}[f] \cdot \mathcal{L}[g] \end{aligned}$$

### §4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t) dt$$

Notice

$$\begin{aligned}\mathbf{f} &= \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) &\approx f_i \\ g(\tau) &\approx g_i \\ k(\tau - t) &\approx K_{ij}\end{aligned}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathcal{L}[f] &= \mathcal{L}[g] + \mathcal{L}[k] \cdot \mathcal{L}[f] \\ \mathcal{L}[f] &= \frac{\mathcal{L}[g]}{1 - \mathcal{L}[k]}\end{aligned}$$

**Example 4.2**

Consider  $f(t) = t^3 + \int_0^t \sin(t - \tau)f(\tau)d\tau$ .

$$F(s) = \frac{3!}{s^4} + \mathcal{L}[\sin t] F(s)$$

$$\vdots$$

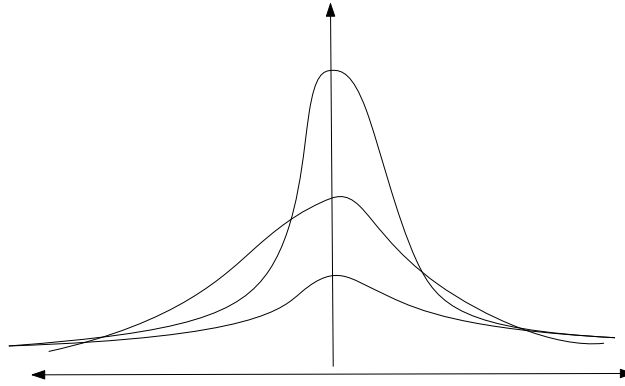
$$F(s) = 3!(s^{-4} + s^{-6})$$

$$f(t) = t^3 + \frac{t^5}{20}$$

## §5 | Lec 5: Oct 6, 2021

### §5.1 Dirac Delta “Function”

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Formally:

$$\delta : f(t) = \int_{-\infty}^{\infty} f(\tau) \delta(t - \tau) d\tau \implies f = f * \delta$$

$\delta$  “picks out” a pointwise value of any function we integrate against/convolve with. For finite dimension, let  $\mathbf{f} \in \mathbb{R}^n$  and  $\mathbf{e}_i = [0, \dots, 0, 1, 0, \dots]$ . So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension,  $f(t) : \mathbb{R} \rightarrow \mathbb{R}$  for  $t \in \mathbb{R}$ ,

$$f(t) = \int_{\mathbb{R}} f(\tau) \delta(t - \tau) d\tau$$

where  $\delta(\tau - t) = \delta(t - \tau) = \delta_t(\tau)$ . These two notions are analogous, in a sense. Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n \times n}$$

Solve  $L\mathbf{f} = \mathbf{h}$ . If we know  $L\mathbf{f}_i = \mathbf{e}_i$  where

$\mathbf{e}_i$  : unit vector

$\mathbf{f}_i$  : unit response vector

1.  $\mathbf{h} = \sum h_i \mathbf{e}_i$

2. Linear superposition means

$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned} L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\ &= \sum_i h_i L\mathbf{f}_i \\ &= \sum_i h_i \mathbf{e}_i \\ &= \mathbf{h} \end{aligned}$$

Solving  $\infty$ -dim ODE

$$f'' + af' + bf = h(t) (L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1.  $h = h * \delta$
2. Then,

$$\begin{aligned} f &= h * g \\ &= \int_0^t g_t(\tau) h(\tau) d\tau \\ &= \int_0^t g(t - \tau) h(\tau) d\tau \end{aligned}$$

where  $g$  is known as the Green function.

$$\begin{aligned} e_i &\approx \delta_t \\ \mathbf{f}_i &\approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g \end{aligned}$$

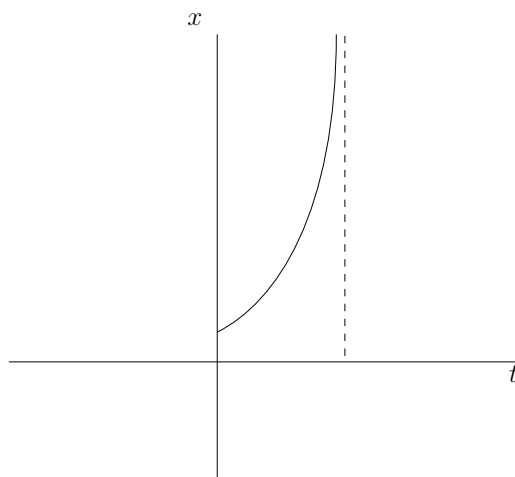
## §6 | Lec 6: Oct 08, 2021

### §6.1 Existence & Uniqueness of ODE Solutions

Intuitively,  $f(t, x)$  is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over  $\mathbb{R}$ .

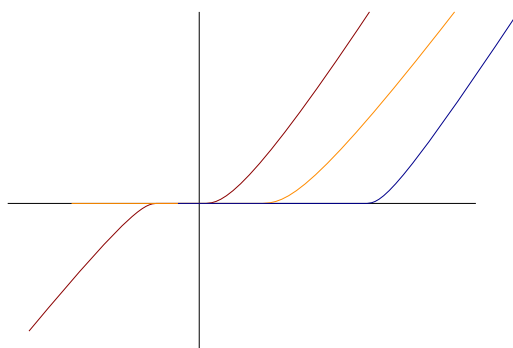
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$



We can easily solve this and obtain  $x(t) = \frac{1}{1-t}$  which blows up in finite time.

2. What about uniqueness?

$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$



This has infinite number of solution through  $(0, 0)$  – non-unique. Notice that  $x' = 3x^{\frac{2}{3}}$  is an autonomous ODE where the solution is  $x(t) = t^3$ . However,  $x(t) = 0$  is also a solution which shows that solutions are not unique.

**Question 6.1.** What can prove existence and uniqueness?

1. Converting to “nicer” problem,  $DE \iff$  integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution

## §7 | Lec 7: Oct 11, 2021

### §7.1 Picard Iteration

Goal: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f(t, x(t)), \quad x(t_0) = x_0$$

Idea:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

Then, we can transform it into an integral equation

$$x(t) = x_0 + \int_{t_0}^t f(t', x(t')) dt'$$

Notice that  $f$  is continuous and  $x$  is continuous imply  $x$  is differentiable.

2. Iteration: If we can't solve it at first, try again.

#### Example 7.1

Newton's root-finding algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Picard Iteration: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(t', x(t')) dt'$$

Start with a guess for the function  $x_0(t) = x_0$  (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^t f(t', x_n(t')) dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \xrightarrow{\infty} \dots$$

If  $x_{n+1}(t) = x_n(t) = \bar{x}(t)$ , then  $\bar{x}(t)$  has to solve the IE. We want  $\lim_{n \rightarrow \infty} x_n(t) \rightarrow x(t)$  solves IE.

**Example 7.2**

Consider  $\dot{x}(t) = x(t)$ ,  $x(0) = 1$ . This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t') dt'$$

Picard:

$$x_0(t) = 1$$

$$\begin{aligned} x_1(t) &= 1 + \int_0^t x_0(t') dt' = 1 + \int_0^t 1 dt' \\ &= 1 + t \end{aligned}$$

$$\begin{aligned} x_2(t) &= 1 + \int_0^t 1 + t dt \\ &= 1 + t + \frac{t^2}{2!} \end{aligned}$$

$\vdots$

$$x_n(t) = \sum_{k=0}^n \frac{t^k}{k!}$$

Thus,

$$\lim_{n \rightarrow \infty} x_n(t) \rightarrow e^t$$

## §8 | Lec 8: Oct 13, 2021

### §8.1 Continuity

Limit of continuous function is not necessarily continuous.

#### Example 8.1

Consider  $x_n(t) = t^n$  on  $[0, 1]$

$$x_0 = 1$$

$$x_1 = t$$

$$x_2 = t^2$$

$$\vdots$$

$$\bar{x} = \lim_{n \rightarrow \infty} x_n = \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases}$$

which is discontinuous.

Idea: We need “more” continuity. Given  $x$ , and given any  $\varepsilon > 0$ , if  $|x - x'| < \delta(x, \varepsilon)$  then  $|f(x) - f(x')| < \varepsilon$ .

#### Example 8.2

Consider  $f(x) = x$  on  $\mathbb{R}$ . We can see that

$$|x - x'| < \varepsilon \quad \forall |x - x'| < \varepsilon$$

in which we pick  $\delta(x, \varepsilon) = \varepsilon$ .

How about  $f(x) = x^2$  on  $\mathbb{R}$ ?

$$|x^2 - y^2| < \varepsilon$$

If we pick  $\delta(x, \varepsilon) = \varepsilon$ , then  $|x - y| < \delta = \varepsilon$  which does not necessarily imply  $|x^2 - y^2| < \varepsilon$  because

$$\begin{aligned} |x^2 - y^2| &= |(x + y)(x - y)| \\ &= |x + y| |x - y| \\ &\leq \varepsilon |x + y| \end{aligned}$$

$|f(x) - f(y)| > \varepsilon$ . So we need to pick smaller  $\delta$  as  $x$  and  $y$  get larger. It would work for  $\delta = \frac{\varepsilon}{2 \max(|x|, |y|)}$ .

**Question 8.1.** Is  $\frac{1}{x}$  continuous?

Ans: It depends on the domain. If we're talking about  $\mathbb{R}$ , it doesn't work at 0; on  $(0, \infty)$ , yes it's continuous.

**Definition 8.3** (Uniform Continuity) —  $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$  s.t.  $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ .



**Remark 8.4.** Notice that the definition is similar to continuity except that  $\delta$  doesn't depend on  $x$ .

**Example 8.5**

$x^2$  on  $\mathbb{R}$  is not uniformly continuous but  $x^2$  on  $(a, b) \subseteq \mathbb{R}$  is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|, |y|)} = \frac{\varepsilon}{\max(|a|, |b|)}$$

**Remark 8.6.** Uniform continuity also depends on the domain as continuity does.

**Exercise 8.1.** Is  $x^{\frac{1}{2}}$  uniformly continuous on  $[0, 1]$ ?

Lipschitz Continuity: “gradient is bounded”

$$\frac{|f(x) - f(y)|}{|x - y|} < L < \infty$$

We can pick  $\delta = \frac{\varepsilon}{L}$  everywhere.

**Example 8.7** •  $x^2$  on  $\mathbb{R}$  is not Lipschitz but it is on a finite interval.

•  $x^{\frac{1}{2}}$  is not Lipschitz continuous on  $[0, 1]$ . However, it's uniformly continuous.

## §9 | Lec 9: Oct 15, 2021

### §9.1 Picard's Theorem

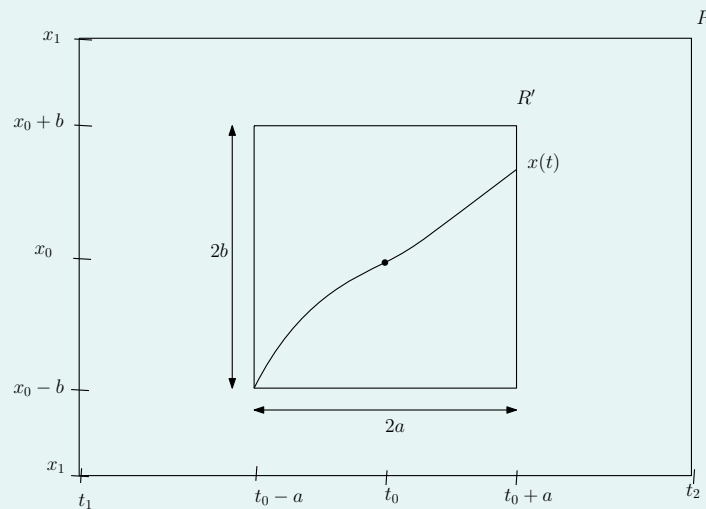
Let's prove local existence of the theorem.

#### Theorem 9.1 (Picard)

If  $f(t, x)$  and  $\partial_x f(t, x)$  are continuous function on a bounded rectangle  $R = [t_1, t_2] \times [x_1, x_2]$  and  $(t_0, x_0)$  is in interior of  $R$  ( $t_1 < t_0 < t_2$ ,  $x_1 < x_0 < x_2$ ). Then  $\exists$  a smaller rectangle  $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$  s.t. ODE

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution in  $R'$ .



Note: Since  $R$  closed and bounded, then  $f$ ,  $\partial_x f$  are bounded, i.e.,

$$\begin{aligned} \max_R f(t, x) &= M \\ \max_R \partial_x f(t, x) &= L \end{aligned}$$

Thus,  $f$  is Lipschitz.

Proof Outline:

1. Solving ODE  $\iff$  Soling IE
2. Approximate solutions using Picard iteration

$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^t f(t', x_{n-1}(t')) dt'$$

3. Prove Picard iterates converges

$$\lim_{n \rightarrow \infty} x_n(t) \rightarrow \bar{x}(t)$$

4. Prove limit  $\bar{x}(t)$  solves IE.
5. Prove limit  $\bar{x}(t)$  is continuous.

6. Prove limit  $\bar{x}(t)$  is unique.
7. How big is  $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ ?

$$\text{Pick } a \ni aL < 1 \text{ \& } b = Ma \leq |x_0 - x_1| |x_0 - x_2|$$

*Proof.* 2. Prove Picard iterates converge

a) We have

$$\lim_{n \rightarrow \infty} x_n(t) \iff \lim_{n \rightarrow \infty} x_0(t) + \sum_{k=1}^n x_k(t) - x_{k-1}(t)$$

telescoping sum!

- b) Series  $x_0(t) + \sum_{k=1}^n x_k(t) - x_{k-1}(t)$  converges by Weierstrass M-test - If  $|f_n(x)| < M_n$   
 $\forall n \in \mathbb{N}, x \in D$  and  $\sum_{n=0}^{\infty} M_n$  converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

- i) Show  $x_i(t)$  are all in  $R' \subseteq R$  so we can use bounds  $L, M$ .

$$\begin{aligned} |x_0(t) - x_0| &= 0 \\ |x_1(t) - x_0| &= \left| \int_{t_0}^t f(t', x_0(t')) dt' \right| \\ &\leq \int_{t_0}^t |f(t', x_0(t'))| dt \\ &\leq \int_{t_0}^t M dt \\ &\leq Ma = b \end{aligned}$$

Thus,  $x_1(t)$  is in the rectangle. By induction, every  $x_n(t)$  in  $R' \subseteq R$ .

- ii) Show  $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$  is bounded.

Define  $\Delta = \max_{R'} |x_1(t) - x_0|$ . Then

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \int_{t_0}^t f(t', x_1(t')) - f(t', x_0(t')) dt' \right| \\ &\leq \int_{t_0}^t |f(t', x_1(t')) - f(t', x_0(t'))| dt' \\ &\leq \int_{t_0}^t L |x_1(t') - x_0(t')| dt' \\ &\leq \Delta aL \end{aligned}$$

and

$$\begin{aligned} |x_3(t) - x_2(t)| &= \left| \int_{t_0}^t f(t, x_2(t)) - f(t, x_1(t)) dt \right| \\ &\leq \int_{t_0}^t |f(t, x_2(t)) - f(t, x_1(t))| dt \\ &\leq \int_{t_0}^t L |x_2(t') - x_1(t')| dt' \\ &\leq L(\Delta aL)(t - t_0) \\ &\leq \Delta(aL)^2 \end{aligned}$$

Every  $|x_n(t) - x_{n-1}(t)|$  depends on  $|x_{n-1}(t) - x_{n-2}(t)|$  recursively. The general pattern is

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| &\leq \Delta(aL)^{n-1} \\ \sum_{n=1}^{\infty} |x_n - x_{n-1}| &\leq \sum_{n=0}^{\infty} \Delta(aL)^n \\ &= \frac{\Delta}{1 - aL} \\ &< \infty \end{aligned}$$

Thus,  $\sum x_n - x_{n-1}$  converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n \rightarrow \infty} x_n(t) = \bar{x}(t) \text{ exists!}$$

3.  $\bar{x}$  solves I.E.

Idea: We know  $|\bar{x} - x_n|$  gets small so break  $\left| \bar{x} - x_0 - \int_{t_0}^t f(t', \bar{x}(t')) dt' \right|$  into pieces like  $|\bar{x} - x_n(t)|$ .

$$\text{subtract } x_n(t) - x_0 - \int_{t_0}^t f(t', x_{n-1}(t')) dt' = 0$$

$$\text{Let } \kappa = \left| \bar{x} - x_0 - \int_{t_0}^t f(t', \bar{x}(t')) dt' \right|.$$

$$\begin{aligned} \kappa &= \left| -(x_n - x_0 - \int_{t_0}^t f(t', x_{n-1}(t')) dt') \right| \\ &\leq |\bar{x} - x_n| + \left| \int_{t_0}^t f(t, \bar{x}) - f(t, x_{n-1}) dt \right| \\ &\leq |\bar{x} - x_n| + \int_{t_0}^t |f(t, \bar{x}) - f(t, x_{n-1})| dt \\ &\leq |\bar{x} - x_n| + aL |\bar{x} - x_{n-1}| \end{aligned}$$

which approaches 0 as  $n \rightarrow \infty$  because  $\lim_{n \rightarrow \infty} x_n = \bar{x}$ .

4.  $\bar{x} = \lim_{n \rightarrow \infty} x_n$  is continuous, i.e., given  $\varepsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$|t - t'| < \delta \implies |\bar{x}(t) - \bar{x}(t')| < \varepsilon$$

Idea: Split into known things

$$\begin{aligned} |\bar{x}(t) - \bar{x}(t')| &= |\bar{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \bar{x}(t)| \\ &\leq |\bar{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \bar{x}(t)| \end{aligned}$$

We pick  $n$  s.t.  $|\bar{x}(t) - x_n(t)| < \frac{\varepsilon}{3} \forall t$  which is possible because Weierstrass implies uniform convergence. Then pick  $\delta$  s.t.

$$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall |t - t'| < \delta$$

which is possible because  $x_n$  is continuous.

5.  $\bar{x}$  is unique.

Idea: Prove  $|\bar{x} - \tilde{x}| \leq |\bar{x} - \tilde{x}|$ .

- If  $\tilde{u}$  is other solution, it also exists in  $R'$ .

*Proof.* (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some  $|t_* - t| < a$ . But

$$\begin{aligned} |\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f(t', \tilde{x}(t')) dt' \right| \\ &\leq \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| dt' \\ &\leq M(t_* - t_0) \\ &< Ma = b \end{aligned}$$

Contradiction! □

- Have

$$\begin{aligned} |\bar{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^t f(t', \bar{x}(t')) - f(t', \tilde{x}(t')) dt' \right| \\ &\leq \int_{t_0}^t |f(t', \bar{x}(t')) - f(t', \tilde{x}(t'))| dt' \\ &\leq \int_{t_0}^t L \max |\bar{x}(t') - \tilde{x}(t')| dt' \\ &\leq La \max |\bar{x}(t') - \tilde{x}(t')| \\ \max |\bar{x}(t) - \tilde{x}(t)| &\leq \max |\bar{x}(t) - \tilde{x}(t)| \end{aligned}$$

which is only possible if  $\bar{x}(t) - \tilde{x}(t) = 0$ , i.e., solution is unique. □

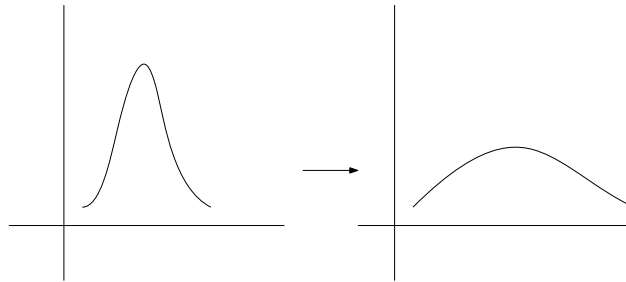
# §10 | Lec 10: Oct 18, 2021

## §10.1 Fourier Series

Goal: Solve linear PDE: 3 canonical examples

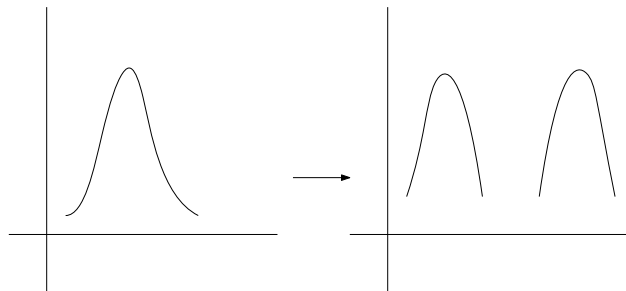
1. Heat/Diffusion equation

$$\partial_t u(t, x) - \partial_x^2 u(t, x) = 0$$



2. Wave equation

$$\partial_t^2 u = \partial_x^2 u$$



3. Laplace equation:

$$\partial_x^2 u + \partial_y^2 u = 0$$

**Question 10.1.** How do we solve linear PDEs?

Use linearity to split big problems into small ones that you can solve (find the eigenvectors). Then we split 1 PDE  $\rightarrow \infty$  ODEs. First, let's define Fourier series.

**Definition 10.1 (Fourier Series)** — Fourier Series is a function written as a sum of sines and cosines

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx) \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \end{aligned}$$

where  $c_n = c_r + ic_{in}$ .

They have amazing properties:

1. They can approximate almost anything

- analytic function
- smooth function
- periodic function
- differentiable function
- continuous/discontinuous function

2. They simplify differentiation!

$$\begin{aligned}\frac{d}{dx}e^{ikx} &= ik e^{ikx} \\ \frac{d^2}{dx^2} \sin kx &= -k^2 \sin kx \\ \frac{d^2}{dx^2} \cos kx &= -k^2 \cos kx\end{aligned}$$

Just like Laplace transform, Fourier series transform differentiation into multiplication problem (easier to deal with).

3. Fourier series are orthogonal

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

or

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad \text{if } m \neq n$$

or

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{if } m \neq n$$

This gives easy formulas

From these facts follow from linear algebra, because Fourier series are eigenfunctions of differentiation. They are the correct basis to solve linear PDEs.

# §11 | Lec 11: Oct 20, 2021

## §11.1 Coefficients of Fourier Series

**Question 11.1.** How do we calculate Fourier Series  $a_n, b_n = ?$

Consider the domain:  $[-\pi, \pi]$ , finite dimensions  $N$ , vector

$$\mathbf{u} = \sum u_i \mathbf{e}_i$$

How do we calculate  $u_i$ ?

$$\begin{aligned} \mathbf{u} \cdot \mathbf{e}_j &= \left( \sum_{i=1}^N u_i \mathbf{e}_i \right) \cdot \mathbf{e}_j \\ &= \sum_{i=1}^N u_i (\mathbf{e}_i \cdot \mathbf{e}_j) \\ &= \sum_{i=1}^N \delta_{ij} \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We want to do this in  $\infty$  dimensions – inner product

$$\begin{aligned} N : \langle u, v \rangle &= u \cdot v = \sum_{i=1}^N u_i v_i \\ \infty : \langle u, v \rangle &\propto \int_a^b u(x) v(x) dx \end{aligned}$$

Inner Product:  $\langle u, v \rangle \rightarrow \mathbb{R}$  takes in two function & spits out a number. It has to satisfy the following properties

1. Bilinear

$$\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$$

2. Symmetric  $\langle u, v \rangle = \langle v, u \rangle$ .

3. Positivity:  $\langle u, u \rangle > 0$  unless  $u = 0$ .

Inner products are important

- They imply a norm  $\|u\| = \sqrt{\langle u, u \rangle}$
- Cauchy-Schwarz Inequality

$$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$$

- Triangle inequality

$$\|u + v\| \leq \|u\| + \|v\|$$

**Exercise 11.1.** Prove these properties.



Now, we will use inner products to calculate Fourier. Define

$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x) dx$$

Under this inner product,  $\sin kl$ ,  $\cos kl$  are orthogonal functions, i.e.,

$$\begin{aligned}\langle \sin kx, \cos lx \rangle &= 0 \quad \forall k, l \\ \langle \sin kx, \sin lx \rangle &= 0 \quad \text{if } k \neq l \\ \langle \cos kx, \cos lx \rangle &= 0 \quad \text{if } k \neq l\end{aligned}$$

Note:  $1 = \cos 0x$

*Proof.* Left as exercise, but use

$$\begin{aligned}\cos((k+l)x) &= \cos kx \cos lx - \sin kx \sin lx \\ \sin((k+l)x) &= \sin kx \cos lx + \sin lx \cos kx\end{aligned}$$

Also,

$$\begin{aligned}\langle \sin kx, \sin kx \rangle &= 1 \\ \langle \cos kx, \cos kx \rangle &= 1 \quad k \neq 0 \\ \langle 1, 1 \rangle &= 2\end{aligned}$$

□

We have

$$\begin{aligned}f(x) &= \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx \\ \langle f, \cos lx \rangle &= \left\langle \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx, \cos lx \right\rangle \\ &= \frac{a_0}{2} \langle 1, \cos lx \rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx, \cos lx \rangle + \sum_{k=1}^{\infty} b_k \langle \sin kx, \cos lx \rangle \\ \langle f, \cos lx \rangle &= a_l \\ \langle f, \sin lx \rangle &= b_l\end{aligned}$$

So we can write any function  $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

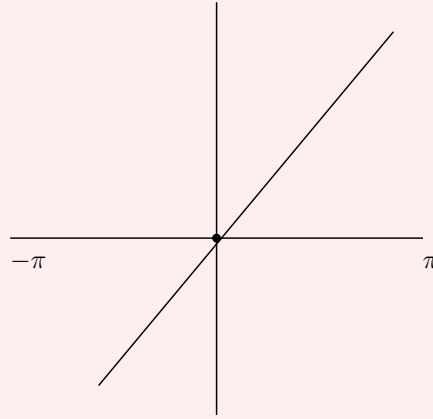
$$\begin{aligned}a_k &= \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\ b_k &= \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx\end{aligned}$$

**Question 11.2.** Are these orthogonal functions under  $\langle u, v \rangle$ ?

**Question 11.3.** Are there any other kind of  $L^2$  inner product?

**Example 11.1**

Consider  $f(x) = x$



We have

$$\begin{aligned}
 x &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\
 a_k &= \langle x, \cos kx \rangle \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx \\
 &= 0 - 0 - 0 = 0 \quad (\text{integration by parts}) \\
 b_k &= \langle x, \sin kx \rangle \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx \\
 &= \frac{1}{\pi} \left[ -\pi \frac{\cos k\pi}{k} - (-(-\pi)) \frac{\cos(-k\pi)}{k} \right] \quad (\text{integration by parts}) \\
 &= \frac{2(-1)^{k+1}}{k}
 \end{aligned}$$

Thus,

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

To show that infinite series converges

$$\sum_{k=1}^{\infty} \left| \frac{2(-1)^{k+1}}{k} \right| < 2 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is conclusive (by Weierstrass-M test).

## §12 | Lec 12: Oct 22, 2021

### §12.1 Convergence of Fourier Series

Consider the last example from last lecture

$$f(x) = x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

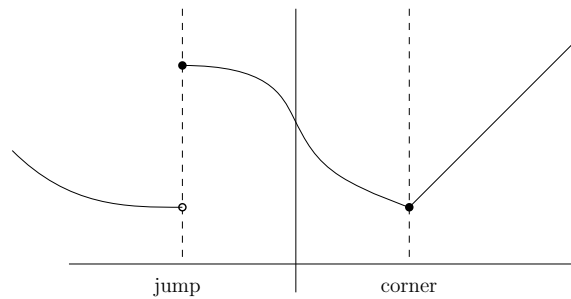
**Question 12.1.** In what sense does it converge? (What's happening at  $\pm\pi$ )

Fourier series must be  $2\pi$  periodic (because  $\cos kx$ ,  $\sin kx$  are  $2\pi$ -periodic) so the  $y$  must converge to a  $2\pi$ -periodic extension of the function.

$$\tilde{f}(x + 2\pi) = \tilde{f}(x)$$

Note:  $x$  is  $C'$  (derivative continuous) but  $\tilde{x}$  is not  $C'$ . It is piecewise  $C'$  ( $C'$ :  $f$  continuous and  $\frac{df}{dx}$  is continuous).

Piecewise  $C'$  on  $[a, b]$



$f$  is  $C'$  except at finitely many points. At any bad point we have

$$\begin{cases} f(x^-) = \lim_{h \rightarrow 0} f(x-h) & \text{if } f(x^+) \neq f(x^-) \text{ jump} \\ f(x^+) = \lim_{h \rightarrow 0} f(x+h) \\ f'(x^-) = \lim_{h \rightarrow 0} f'(x-h) & \text{if } f(x^+) = f(x^-) \\ f'(x^+) = \lim_{h \rightarrow 0} f'(x+h) & \text{but } f'(x^+) \neq f'(x^-) \text{ corner} \end{cases}$$

#### Theorem 12.1 (Fourier Convergence)

If  $\tilde{f}(x)$  is  $2\pi$ -periodic, piecewise  $C'$  function, then its Fourier series converges to  $\tilde{f}$  everywhere except jump points  $x$  where the series converges to  $\frac{f(x^+) + f(x^-)}{2}$

**Question 12.2.** Recall the example at the beginning, why is there no cosines for  $x$ ?

Odd/even symmetries!

**Fact 12.1.** We have

$$\begin{aligned} \text{odd} + \text{odd} &= \text{odd} \\ \text{even} + \text{even} &= \text{even} \end{aligned}$$

and

$$\begin{aligned}\text{odd} \times \text{odd} &= \text{even} \\ \text{even} \times \text{even} &= \text{even} \\ \text{odd} \times \text{even} &= \text{odd}\end{aligned}$$

and

$$\begin{aligned}\int_{-a}^a \text{odd } dx &= 0 \\ \int_{-a}^a \text{even } dx &= 2 \int_0^a \text{even } dx\end{aligned}$$

This implies odd functions  $f$  have sine series and even functions have cosine series.

## §13 | Lec 13: Oct 27, 2021

Recap:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where the coefficients are calculated as follows

$$\begin{aligned} a_k &= \langle f, \cos kx \rangle \\ b_k &= \langle f, \sin kx \rangle \\ \langle u, v \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x) dx \end{aligned}$$

Symmetry simplifies  $a_k, b_k$ . Fourier series converges for periodic and piecewise  $C^1$  functions.

### §13.1 Complex Fourier Series

Recall the Euler's formula

$$e^{ikx} = \cos kx + i \sin kx$$

Also,

$$\begin{aligned} \cos kx &= \frac{e^{ikx} + e^{-ikx}}{2} \\ \sin kx &= \frac{e^{ikx} - e^{-ikx}}{2i} \end{aligned}$$

So,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

We want  $c_k = \langle f, e^{ikx} \rangle$

$$\langle e^{ikx}, e^{ikx} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2ikx} dx$$

which is not necessarily positive and we want it to be strictly positive, i.e., norm.

$$\begin{aligned} \int_{-\pi}^{\pi} e^{2ikx} dx &= \left[ \frac{e^{2ikx}}{2ik} \right]_{-\pi}^{\pi} \\ &= \frac{e^{2\pi ki} - e^{-2\pi ki}}{2ik} \\ &= \frac{\sin 2\pi k}{k} \\ &= 0 \end{aligned}$$

To fix this, let's define Hermitian inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

where  $x \in (-\pi, \pi]$  and  $f, g : (-\pi, \pi] \rightarrow \mathbb{C}$ . So

$$\begin{aligned} c_k &= \langle f, e^{ikx} \rangle \\ c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \end{aligned}$$

**Question 13.1.** How do Fourier series work with integration?

Integration makes things smoother. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$\int f(x) dx \sim \int \frac{a_0}{2} dx + \sum_{k=1}^{\infty} a_k \int \cos kx dx + b_k \int \sin kx dx$$

**Question 13.2.** Is this okay?

Notice that

$$\int \cos kx dx = \frac{\sin kx}{k} \quad \int \sin kx dx = \frac{-\cos kx}{k}$$

Problem: If  $f(x) = 1$ , then

$$f \sim 1$$

$$\int_0^x f dx \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Constants terms in Fourier series are bad under integration.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Integration is fine if the function has mean 0

$$\int_{-\pi}^{\pi} f(x) dx = 0$$

Compare  $f(x) = 1$  and  $g(x) = x$ .

**Remark 13.1.** Fourier series need piecewise  $C^1$ . To have Fourier of  $f'$ , it must be  $C^1$  so  $f$  must be continuous (can have corners but not jumps).

$$f = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$f' = -ka_k \sin kx + kb_k \cos kx$$

if  $f$  is continuous.

Summary:

- Integrate: divide by  $k$
- Differentiation: multiply by  $k$

## §14 | Lec 14: Oct 29, 2021

### §14.1 Rescaling Intervals of Fourier Series

We know Fourier series on  $[-\pi, \pi]$ . What about  $[-l, l]$ ? We use coordinate transformation

$$\begin{aligned} y &= \frac{\pi}{l} x \\ F(y) &= f(x(y)) \\ F(y(x)) &= f(x) \end{aligned}$$

We have

$$F(y) = f(x(y)) = f\left(\frac{l}{\pi}y\right)$$

So  $F(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos ky + b_k \sin ky$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos ky \, dy \\ &= \frac{1}{\pi} \int_{-l}^l F(y(x)) \cos ky(x) \frac{\pi}{l} \, dx \\ &= \frac{1}{l} \int_{-l}^l f(x) \cos \left(\frac{k\pi}{l}x\right) \, dx \end{aligned}$$

So

$$f(x) = F(y(x)) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x$$

We can find  $b_k$  similarly.

## §15 | Lec 15: Nov 1, 2021

### §15.1 The Relationship between Smoothness and Fourier Coefficients

Smother functions (more differentiable) have faster decaying Fourier coefficients. (infinitely differentiable leads to exponential decay).

**Example 15.1** • Discontinuous function  $\rightarrow c_k \propto \frac{1}{k}$

- $C^0 \rightarrow c_k \propto \frac{1}{k^2}$
  - $C^1 \rightarrow c_k \propto \frac{1}{k^3}$
  - $C^2 \rightarrow c_k \propto \frac{1}{k^4}$
- Why?

Recall these definitions

**Definition 15.2** —  $\forall \varepsilon, x \exists N$  s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N(x)$$

Then,  $f_n(x) \rightarrow f(x)$  (pointwise convergence).

**Definition 15.3** —  $\forall \varepsilon, \exists N$  s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \quad \forall x$$

Then,  $f_n(x) \rightarrow f(x)$  (uniform convergence).

Series converges  $\sum_{k=1}^{\infty} f_k(x) \rightarrow g(x)$  if

$$s_n(x) = \sum_{k=1}^n f_k(x) \rightarrow g(x) \text{ as } n \rightarrow \infty$$

Weierstrass M-test: If  $|f_n(x)| < M_n$  and  $\sum_{n=1}^{\infty} M_n < \infty$ , then  $\sum_{n=1}^{\infty} f_n(x)$  converges (absolutely/uniformly). So the limit is continuous if  $f_n$  are continuous.

Consider a complex Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

#### Theorem 15.4

If  $\sum_{k=-\infty}^{\infty} |c_k| < \infty$ , then the Fourier series is “good”, i.e., the limit of the Fourier series is continuous.

*Proof.* Weierstrass!

$$|c_k e^{ikx}| \leq |c_k| |e^{ikx}| = |c_k|$$

□



**Corollary 15.5**

If  $|c_k| < \frac{M}{|k|^\alpha}$  where  $\alpha > 1$ . Then Fourier series is continuous.

*Proof.*  $\sum_{k=1}^{\infty} \frac{M}{k^\alpha} < \infty$  for  $\alpha > 1$  by comparison test. □

Note:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$f' \sim \sum_{k=-\infty}^{\infty} ik c_k e^{ikx}$$

Differentiation:  $c_k \rightarrow ikc_k$  or  $|c_k| \rightarrow k|c_k|$

**Theorem 15.6**

If  $\sum_{k=1}^{\infty} |k|^n |c_k| < \infty$  where  $f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ , then  $f^{(n)}$  is continuous ( $f$  is  $C^n$ )

*Proof.* Have

$$f^{(n)} \sim \sum_{k=-\infty}^{\infty} (ik)^n c_k e^{ikx}$$

then Weierstrass  $|(ik)^n c_k e^{ikx}| \leq |k|^n |c_k|$  □

**Corollary 15.7**

If  $|c_k| < \frac{M}{|k|^\alpha}$  where  $\alpha > n + 1$  then  $f$  is  $n$  times differentiable.

*Proof.* Comparison test:  $c_k = \frac{1}{k^2}$ , then

$$|c_k| < \frac{1}{k^{1.5}} \propto \frac{1}{k}$$

So,

$$\begin{aligned} \frac{1}{k^2} &\rightarrow C^0 \\ \frac{1}{k^3} &\rightarrow C^1 \\ \frac{1}{k^4} &\rightarrow C^2 \\ &\vdots \end{aligned}$$

□

## §16 | Lec 16: Nov 3, 2021

### §16.1 Hilbert Spaces & Convergence in Norm

Goal: Prove Fourier series converge “in norm”. First, we need some definitions.

**Definition 16.1** ( $L^2$  integrable) —  $f$  is  $L^2$  integrable if  $\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$ .

**Definition 16.2** (Hilbert Space) — Hilbert space is vector space of  $L^2$  integrable function

*Proof.* Have

- Has a 0 (0 function)
- Closed under addition

$$\|f + g\| \leq \|f\| + \|g\| < \infty$$

- Closed under scalar multiplication

$$\|cf\| = |c|\|f\| < \infty$$

□

- test other axioms ...

Note:  $L^2$  function have Fourier series.

*Proof.*  $|c_k| = |\langle f, e^{ikx} \rangle| \leq \|f\| \|e^{ikx}\| < \infty$  (Cauchy-Schwarz).

□

Note:  $L^2$  functions are “abnormal” TBA

**Fact 16.1.** Hilbert spaces are complete (every “convergent” sequence has a limit that is  $L^2$ )

**Definition 16.3** — “Convergent” means Cauchy sequence for sequence  $a_n \rightarrow a$ . We need

$$\text{Cauchy} : \forall \varepsilon, \exists N \ni |a_m - a_n| < \varepsilon \quad \forall m, n > N$$

Aside: Completeness is the difference between rationals  $\mathbb{Q}$ , and reals  $\mathbb{R}$  ( $\mathbb{Q}$  isn’t complete because  $\pi$  is limit of sequence in  $\mathbb{Q}$  but  $\pi \notin \mathbb{Q}$ ). Completeness matters for taking limits.

**Definition 16.4** (Convergence in Norm) —  $f_n(x) \rightarrow f(x)$  if  $\|f_n(x) - f(x)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

We’ll prove Fourier series converge to their function in norm in a general way for a general  $\infty$ -dim vector space  $V$  with an inner product.

**Definition 16.5** (Orthonormal System) — Orthonormal system  $\phi_1, \phi_2, \dots \in V$

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

**Definition 16.6** (General Fourier Series) —  $f \sim \sum_{k=1}^{\infty} c_k \phi_k$  where  $c_l = \langle f, \phi_l \rangle$  and  $c_k$  comes from  $\langle f, \phi_l \rangle$  on both sides.

**Theorem 16.7**

The truncated Fourier series

$$s_n = \sum_{k=1}^{\infty} c_k \phi_k$$

is the best approximation to  $f$  in least squares sense, that is, consider  $V_n = \text{span}\{\phi_1, \dots, \phi_n\}$  and take any  $p_n = \sum_{k=1}^n d_k \phi_k \in V_n$  then

$$\|s_n - f\| \leq \|p_n - f\| \quad \forall p_n \in V_n$$

*Proof.* We have

$$\begin{aligned} p_n &= \sum_{k=1}^n d_k \phi_k \\ s_n &= \sum_{k=1}^n c_k \phi_k \\ c_k &= \langle f, \phi_k \rangle \end{aligned}$$

Then,

$$\begin{aligned} \|p_n\|^2 &= \langle p_n, p_n \rangle \\ &= \left\langle \sum_{k=1}^n d_k \phi_k, \sum_{l=1}^n d_l \phi_l \right\rangle \\ &= \sum_{k=1}^n \sum_{l=1}^n d_k d_l \langle \phi_k, \phi_l \rangle \\ &= \sum_{k=1}^n \sum_{l=1}^n d_k d_l \delta_{kl} \\ &= \sum_{k=1}^n |d_k|^2 \end{aligned}$$

and

$$\begin{aligned} \|p_n - f\|^2 &= \langle p_n - f, p_n - f \rangle \\ &= \langle p_n, p_n \rangle - 2\langle p_n, f \rangle + \langle f, f \rangle \\ &= \sum_{k=1}^n |d_k|^2 - 2 \left( \sum_{k=1}^n d_k \langle \phi_k, f \rangle \right) + \|f\|^2 \\ &= \sum_{k=1}^n |d_k - c_k|^2 - \sum_{k=1}^n |c_k|^2 + \|f\|^2 \end{aligned}$$

Pick  $d_k = c_k$  – norm minimized by  $s_n$ . □

# §17 | Lec 17: Nov 5, 2021

## §17.1 Pointwise Convergence of Fourier Series

We know Fourier series converge in norms for continuous, piecewise  $C^1$ , periodic functions. But Fourier series seemed to work even for discontinuous functions too (Gibbs phenomenon!). Today we will prove it works pointwise for discontinuous function if  $s_n = \sum_{k=-n}^n c_k e^{ikx}$ . Prove

$$\lim_{n \rightarrow \infty} s_n(x) = \frac{1}{2} (f(x^+) + f(x^-))$$

1. Use the formulas for  $c_k$

$$\begin{aligned} s_n &= \sum_{k=-n}^n \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy \right) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left( \sum_{k=-n}^n e^{ik(x-y)} \right)}_{dy?} dy \end{aligned}$$

Notice that  $\sum_{k=-n}^n e^{ikx}$  is a geometric series

$$\begin{aligned} \sum_{k=-n}^n e^{ikx} &= e^{-inx} \left( \frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} \right) \\ &= \vdots \\ &= \frac{\sin\left((n + \frac{1}{2})x\right)}{\sin \frac{1}{2}x} \end{aligned}$$

So

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \frac{\sin\left((n + \frac{1}{2})(x - y)\right)}{\sin \frac{1}{2}(x - y)} dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + y) \frac{\sin\left((n + \frac{1}{2})y\right)}{\sin \frac{1}{2}y} dy \\ &= \frac{1}{2\pi} \int_0^{\pi} f(x + y) \frac{\sin\left((n + \frac{1}{2})y\right)}{\sin \frac{1}{2}y} dy + \frac{1}{2\pi} \int_{-\pi}^0 f(x + y) \frac{\sin\left((n + \frac{1}{2})y\right)}{\sin \frac{1}{2}y} dy \end{aligned}$$

WTS:

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{top} &= f(x^+) \\ \lim_{n \rightarrow -\infty} \text{bottom} &= f(x^-) \end{aligned}$$

Note

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} \frac{\sin\left((n + \frac{1}{2})y\right)}{\sin y} dy &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^n e^{iky} dy \\ &= 1 \end{aligned}$$

which is valid if only  $e^{i0}$  counts.

2. Prove a difference integral  $\rightarrow 0$  by showing that it's a Fourier coefficient. Prove

$$\frac{1}{2\pi} \int_0^{\pi} (f(x + y) - f(x^+)) \frac{\sin\left((n + \frac{1}{2})y\right)}{\sin \frac{1}{2}y} dy = 0$$

Notice that

$$g(y) \equiv \frac{f(x+y) + f(x^-)}{\sin \frac{1}{2}y}$$

is piecewise continuous  $\forall y \in [0, \pi]$ . We need  $\int_0^\pi g(y) \sin \left(n + \frac{1}{2}\right) y dy = 0$ . Note that

$$\sin \left(n + \frac{1}{2}\right) y = \sin \frac{1}{2}y \cos ny + \cos \frac{1}{2}y \sin ny$$

Then,

$$\int_0^\pi \left(g(y) \sin \frac{y}{2}\right) (0)ny + \left(g(y) \cos \frac{y}{2} \sin ny\right)$$

But we know Fourier coefficients decay for all  $L^2$  integrable functions. So these terms  $\rightarrow 0$  and we prove pointwise convergence.

## §18 | Lec 18: Nov 8, 2021

### §18.1 Heat Equation

We've been learning about Fourier series

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \end{aligned}$$

**Question 18.1.** Why is Fourier series useful?

It's used to solve PDEs; specifically, we want to investigate the heat equation in this lecture. For us,

$$\partial_t u(t, x) - \kappa \partial_x^2 u(t, x) = 0$$

where  $\kappa$  is constant (diffusing constant). Note that

- 1x time derivative  $\implies$  1x initial condition
- 2x space derivative  $\implies$  2x boundary condition

Types of boundary condition

$$\begin{aligned} u(t, 0) &= \alpha(t) && \text{(Dirichlet boundaries)} \\ \partial_x u(t, 0) &= \mu(t) && \text{(Neumann boundaries)} \\ \partial_x u + \beta(t)u &= \tau(t) && \text{(Robin/Mixed boundaries)} \end{aligned}$$

Homogeneous  $\implies$  RHS = 0, i.e.,  $u = 0$ ,  $\partial_x u = 0$ , or  $\partial_x u + \beta u = 0$ .

**Question 18.2.** How do we solve this? (infinitely harder than an ODE!)

Assume  $u(t, x) = T(t)X(x)$ . Substitute into  $\partial_t u - \kappa \partial_x^2 u = 0$

$$\begin{aligned} T'(t)X(x) - \kappa T(t)X''(x) &= 0 \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

This can only be true if neither depends on  $t$  or  $x$ , i.e., constant. So

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \lambda$$

What sign is  $\lambda$ ?

- If  $\lambda > 0$ , we get exponential growth which isn't physical.

$$\begin{aligned} X'' &= \lambda X \\ X &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ T' &= \kappa \lambda T \\ T &= Te^{\kappa \lambda t} \end{aligned}$$

- $\lambda = 0$ ,

$$\begin{aligned} X'' &= 0 \\ X &= Ax + B \\ T' &= 0 \\ T &= T_0 \end{aligned}$$

- If  $\lambda < 0$ , redefine  $\lambda \rightarrow -\lambda$

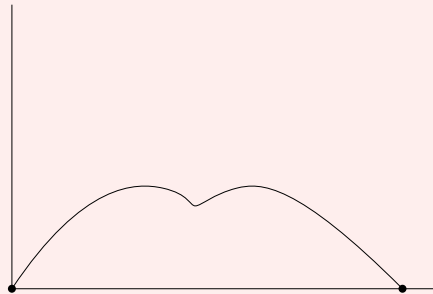
$$\begin{aligned} X'' &= -\lambda X \\ X &= A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x \\ T' &= -\kappa \lambda T \\ T &= T_0 e^{-\kappa \lambda t} \end{aligned}$$

So either  $\lambda = 0$ , or  $\lambda < 0$ .

### Example 18.1

Consider

$$\begin{aligned} \partial_t u - \partial_x^2 u &= 0 \\ u(0, x) &= u_0(x) \\ u(t, 0) &= 0 \\ u(t, l) &= 0 \end{aligned}$$



If we assume  $u(t, x) = T(t)X(x)$  we find

$$\frac{T'}{\kappa T} = \frac{X'}{X} = \lambda$$

If  $\lambda = 0$ ,  $X = Ax + B$ , but  $u(t, 0) = u(t, l) = 0 \implies A = B = 0$ . So  $\lambda < 0$ . Let's write  $\lambda = -\omega^2$ ,

$$\begin{aligned} X'' &= -\omega^2 X \\ X &= A \cos \omega x + B \sin \omega x \end{aligned}$$

Use BC's to get  $A, B$ .

$$\begin{aligned} u(t, 0) = T(t)X(0) = 0 &\implies X(0) = A = 0 \\ X &= \sin \omega x \end{aligned}$$

**Example 18.2 (Cont'd)**

And we have

$$\begin{aligned} u(t, l) = T(t)X(l) &= B \sin \omega l = 0 \\ \implies \omega l &= k\pi, \quad k = 1, 2, \dots \end{aligned}$$

Thus,  $\omega = \frac{k\pi}{l}$ , and it's an eigenvalue and  $\sin \frac{n\pi}{l}x$  is an eigenfunction.

The final solution is

$$u(t, x) = \sum_{n=1}^{\infty} \left( \hat{u}_{0,k} e^{-\frac{n^2 \pi^2}{l^2} \kappa t} \sin \left( \frac{n\pi}{l} x \right) \right)$$

We get  $\hat{u}_{0,k}$  from the Fourier series of the function  $u_0(x)$ .

Note: The Fourier coefficients decay more quickly as  $k$  gets larger so diffusion smooth things out. If we have source term

$$\partial_t u - \partial_x^2 u = f(t, x)$$

We can express  $f$  as a Fourier series and solve an ODE for each Fourier coefficient.

Summary:

1. We assume separable solution:  $u(t, x) = T(t)X(x)$
2. Substituting gives an eigenvalue problem

$$X'' = \lambda X$$

3. The boundary conditions imply  $\lambda = -\omega^2$  where  $\omega = \frac{n\pi}{l}$ ,  $x \in [0, l]$ .
4. Linearity mean we sum up all the eigenfunctions

$$u(t, x) = \sum_{n=1}^{\infty} \left( \hat{u}_{0,k} e^{-\kappa \frac{(k-\pi)^2}{2^2} t} \right) \sin k \frac{\pi}{l} x$$

5. We use the initial condition to determine the Fourier series coeff,  $\hat{u}_{0,k}$



# §19 | Lec 19: Nov 10, 2021

## §19.1 Wave Equation

Goal: Solve the wave equation

1. Look for separable solutions

$$u(t, x) = T(t)X(x)$$

to

$$\partial_t^2 u(t, x) = c^2 \partial_x^2 u(t, x)$$

$c$ : wave speed ( $\frac{\text{space}}{\text{time}}$ ).

$$\partial_t^2(TX) = c^2 \partial_x^2(TX)$$

$$T''X = c^2 TX''$$

$$\frac{T''(t)}{T(t)} = \frac{c^2 X''(x)}{X(x)} = \lambda$$

a)  $\lambda = \omega^2 > 0$ ,  $T'' = \omega^2 T \implies T = e^{\omega t}$  or  $T = e^{-\omega t}$  and  $X'' = \frac{\omega^2}{c^2} X$ ,  $X = e^{\frac{\omega x}{c}}$  or  $X = e^{-\frac{\omega x}{c}}$

b)  $\lambda = 0$ :

$$T'' = 0 \implies T = A + Bt$$

$$X'' = 0 \implies X = C + Dx$$

$$TX = a + bt + cx + d + x$$

c)  $\lambda = -\omega^2 < 0$

$$T'' = -\omega^2 T \implies T = \sin \omega t \text{ or } T = \cos \omega t$$

$$X'' = \frac{-\omega^2}{c^2} X \implies X = \frac{\sin \omega x}{c} \text{ or } X = \frac{\cos \omega x}{c}$$

Next, let's decide on the sign of  $\lambda$  using the boundary conditions, e.g., homogeneous, Dirichlet, boundary conditions

$$u(t, 0) = u(t, l) = 0$$

If  $\lambda > 0$

$$u = Ae^{\omega t} e^{\frac{\omega x}{c}} + Be^{-\omega t} e^{\frac{\omega x}{c}} + Ce^{\omega t} e^{-\frac{\omega x}{c}} + De^{-\omega t} e^{-\frac{\omega x}{c}}$$

but the boundary condition implies that  $A = B = C = D = 0$ .

If  $\lambda = 0$ , similarly  $u = 0$  is only possibility.

If  $\lambda < 0$ ,  $u = T(t)X(x)$ ,  $T = \sin \omega t, \cos \omega t$ ,  $X = \sin \frac{\omega x}{c}, \cos \frac{\omega x}{c}$

$$u(t, 0) = T(t)X(0) = X(0) = 0$$

$$\implies A \sin \frac{\omega 0}{c} + B \cos \frac{\omega 0}{c} = B = 0$$

So  $X = A \sin \frac{\omega x}{c}$ .

$$u(t, l) = T(t)X(l) = A \sin \frac{\omega l}{c}$$

$$\implies \frac{\omega l}{c} = n\pi, \quad n = 1, \dots$$

$$\omega = \frac{n\pi c}{l}$$

In general, the solution is

$$u(t, x) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t \right) \sin \frac{n\pi x}{l}$$

TO find  $A_n, B_n$ , we use the initial conditions

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x)$$

### Example 19.1

Consider  $\partial_t^2 u = \partial_x^2 u$ ,  $x \in [0, 1]$

$$\begin{aligned} u(t, 0) &= u(t, 1) = 0 \\ u(0, x) &= \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1 - x, & \frac{1}{2} \leq x \leq 1 \end{cases} \\ \partial_t u(0, x) &= 0 \\ c = 1, \quad l = 1 &\implies \omega_n = n\pi \end{aligned}$$

Using Dirichlet, the general solution is

$$u(t, x) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \sin n\pi x$$

We want to get  $a_n, b_n$  with ICs. For  $t = 0$ ,

$$\begin{aligned} u(0, x) &= \sum_{n=1}^{\infty} a_n \sin n\pi x \\ &= \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1 - x, & \frac{1}{2} \leq x \leq 1 \end{cases} \\ a_n &= \frac{1}{2} \int_0^1 f(x) \sin n\pi x dx \end{aligned}$$

In general,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_0^1 \sin n\pi x \sin m\pi x dx &= \int_0^{\frac{1}{2}} x \sin n\pi x dx + \int_{\frac{1}{2}}^1 (1 - x) \sin n\pi x dx \\ &= \begin{cases} 0, & n \text{ even} \\ 2 \left( -\left[ \frac{1}{2} \frac{\cos n\pi/2}{n\pi} \right] + \frac{1}{(n\pi)^2} [\sin n\pi]_0^{\frac{1}{2}} \right) & n \text{ odd} \end{cases} = \dots = \frac{4}{\pi^2} \frac{\sum (-1)^k \cos(2k+1)\pi/2 \sin(2k+1)\pi/2}{2k+1} \end{aligned}$$

## §20 | Lec 20: Nov 12, 2021

### §20.1 Midterm 2

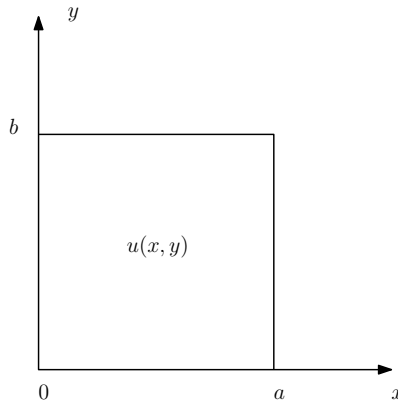
Need to know:

- How to calculate Fourier series
- How to test convergence?
- How does convergence relate to continuity?
- How does symmetry affect Fourier series?
- How does rescaling affect Fourier series?
- How does differentiation/integration affect Fourier series?
- How does smoothness ( $C^n$ ) relate to Fourier series?
- How to analyze generalized Fourier series in Hilbert spaces?
- How to use Fourier series to solve PDEs?

### §20.2 Laplace Equations

Consider

$$\partial_x^2 u + \partial_y^2 u = 0$$



The Laplace equation satisfies the Maximum principle, i.e., the maximum (and minimum) value of a solution to the Laplace equation must occur on the boundary,

*Proof.* If a function  $u$  has a maximum at  $(x, y)$  then

1.  $\partial_x^2 u < 0$
2.  $\partial_y^2 u < 0$

But Laplace says  $\partial_x^2 u + \partial_y^2 u = 0$ . Thus, we can't have local maximum (or minimum) in the domain.  $\square$

**Theorem 20.1**

Solutions to Laplace equation, with given boundary conditions, are unique.

*Proof.* Suppose there exist  $u_1$  and  $u_2$  where

$$\begin{aligned}\partial_x^2 u_1 + \partial_y^2 u_1 &= 0, & u_1 &= f \text{ on the boundary} \\ \partial_x^2 u_2 + \partial_y^2 u_2 &= 0, & u_2 &= f \text{ on the boundary}\end{aligned}$$

Consider  $u_1 - u_2 = \Delta u$ . Since Laplace equation is linear,  $\Delta u$  solves Laplace. And we know that  $\Delta u = 0$  on the boundary. Therefore,  $\Delta u = 0$  everywhere (by maximum principle).  $\square$

**Example 20.2**

Consider

$$\begin{array}{ccccc} & 1 & u(x, 1) = 0 & 1 & \\ & \downarrow & & \downarrow & \\ u(0, y) = 0 & \square & \partial_x^2 u + \partial_y^2 u = 0 & \square & u(1, y) = 0 \\ & \uparrow & & \uparrow & \\ & 0 & & 1 & \end{array}$$

and

$$u(0, x) = \begin{cases} x \\ 1 - x \end{cases}$$

Have  $u = X(x)Y(y)$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

- $\lambda = -\omega^2 < 0$

$$\begin{aligned}X'' &= -\omega^2 X \implies X = \cos \omega x, \sin \omega x \\ Y'' &= \omega^2 Y \implies Y = e^{\omega y}, e^{-\omega y}, \cosh y, \sinh y\end{aligned}$$

- $\lambda = 0,$

$$\begin{aligned}X &= 1, x \\ Y &= 1, y\end{aligned}$$

- $\lambda = \omega^2 > 0$

$$\begin{aligned}X &= e^{\omega x}, e^{-\omega x} \\ y &= \cos \omega y, \sin \omega y\end{aligned}$$

**Example 20.3 (Cont'd)**

Only  $\lambda = -\omega^2 < 0$  works. For  $X = \sin n\pi x$ ,

$$X(0) = X(1) = 0 \implies \omega = n\pi$$

Therefore,

$$\begin{aligned} Y_n &= Ae^{n\pi y} + Be^{-n\pi y} \\ Y(1) = 0 &\implies Ae^{n\pi} + Be^{-n\pi} = 0 \\ \iff Y(y) &\propto \sinh n\pi(1 - y) \end{aligned}$$

In general,

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi(1 - y) \sin n\pi x$$

and

$$\begin{aligned} u(x, 0) &= \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi x = \begin{cases} x, & x < \frac{1}{2} \\ 1 - x, & x > \frac{1}{2} \end{cases} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{4}{\pi^2(2k+1)^2} \sin n\pi x \end{aligned}$$

where

$$a_{2k+1} = \frac{(-1)^k 4}{\pi^2(2k+1)^2}$$

## §21 | Lec 21: Nov 17, 2021

### §21.1 Adjoints

For  $u \in U$ ,  $v \in V$  and linear operator  $L$

$$L : U \rightarrow V \text{ or } L[u] = v$$

Then,

$$\begin{aligned} \langle u_1, u_2 \rangle : U \times U &\rightarrow \mathbb{R}/\mathbb{C}, & \langle \langle v_1, v_2 \rangle \rangle : V \times V &\rightarrow \mathbb{R}/\mathbb{C} \\ \langle u, L^*[v] \rangle &= \langle \langle L[u], v \rangle \rangle \end{aligned}$$

#### Example 21.1

$U : \mathbb{R}^m$ ,  $V : \mathbb{R}^n$ ,  $L : A$  matrix with  $n$  rows and  $m$  columns

$$\langle u_1, u_2 \rangle = u_1 \cdot u_2, \quad \langle \langle v_1, v_2 \rangle \rangle = v_1 \cdot v_2$$

What is  $A^*$ ?

$$\begin{aligned} \langle \langle Au, v \rangle \rangle &= (Au)^\top v = u^\top A^\top v = u^\top (A^\top v) \\ &= u^\top (A^\top v) \\ &= \langle u, A^\top v \rangle \end{aligned}$$

#### Example 21.2

$$\langle u_1, u_2 \rangle = u_1^\top M u_2, \quad \langle \langle v_1, v_2 \rangle \rangle = v_1^\top C v_2$$

$$\begin{aligned} \langle \langle Au, v \rangle \rangle &= (Au)^\top C v = u^\top A^\top C v = u^\top M M^{-1} A^\top C v \\ &= u^\top M (M^{-1} A^\top C v) \\ &= \langle u, M^{-1} A^\top C v \rangle \end{aligned}$$

$A^* = M^{-1} A^\top C$  – adjoint depends on inner products.

Differential operators as linear operators

$$u \in U = C', \quad D[u] = \frac{du}{dx}$$

$D$  is linear!  $D[u, ru_2] = \frac{du_1}{dx} \cdot r \frac{du_2}{dx} = D[u_1] + D[u_2]$  and  $D[cu] = c \frac{du}{dx} = cD[u]$ . Say

$$\begin{aligned} \langle u_1, u_2 \rangle &= \int_a^b u_1(x) u_2(x) dx \\ \langle \langle v_1, v_2 \rangle \rangle &= \int_a^b v_1(x) v_2(x) dx \end{aligned}$$

**Question 21.1.** What is  $D^*$ ?

$$\begin{aligned}
\langle D[u], v \rangle &= \int_a^b \frac{du}{dx} v dx = \dots? = \langle u, D^*[v] \rangle \\
&= [uv]_a^b - \int_a^b u \frac{dv}{dx} dx \\
&= - \int_a^b u \frac{dv}{dx} dx = \langle u, -D[v] \rangle
\end{aligned}$$

If  $[uv]_a^b = u(b)v(b) - u(a)v(a) = 0$ ,

1. Dirichlet BC on  $u$ :  $u(b) = u(a) = 0$
2. Dirichlet BC on  $v$ :  $v(b) = v(a) = 0 \iff$  Neumann BC on  $u$ :  $u'(b) = u'(a) = 0$
3. Periodic BC

Boundary conditions matter. If they're nice then

$$D^* = -D \text{ for standard inner products}$$

**Question 21.2.** What if

$$\begin{aligned}
\langle u_1, u_2 \rangle &= \int_a^b u_1(x) u_2(x) k(x) dx \\
\langle v_1, v_2 \rangle &= \int_a^b v_1(x) v_2(x) p(x) dx
\end{aligned}$$

Show  $D^*[v]$  is  $-\frac{1}{p} \frac{d}{dx}(kv)$  if  $[uvk]_a^b = 0$ .

**Fact 21.1.**  $(L^*)^* = L$

**Fact 21.2.**  $U \xrightleftharpoons[L^*]{L} V \xrightleftharpoons[M^*]{M} W$

$$\begin{aligned}
M \circ L &= M[L[u]] \\
(M \circ L)^* &= L^* \circ M^* = L^*[M^*[w]]
\end{aligned}$$

Self-Adjoint:  $L^* = L$

$$D^2 = D \circ D, \quad D^2[u] = \frac{d^2 u}{dx^2}$$

Provided correct BCs and inner product

$$D^{2*} = D^* \circ D^* = (-D) \cdot (-D) = D^2$$

Fredholm Alternative:

**Question 21.3.** When can we solve a linear problem?

$$L[u] = f$$

If  $f \perp \text{coker } L \iff \langle f, v \rangle = 0$  for each  $v \in \text{coker } L$ .

$$\begin{aligned}
\ker L &: \{u | L[u] = 0\} \\
\text{coker } L &: \ker L^* = \{v | L^*[v] = 0\}
\end{aligned}$$

**Example 21.3**

Consider

$$Au = f$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Way # 1: Direct Gaussian elimination

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & f_1 \\ 0 & 1 & -2 & f_2 \\ 1 & -2 & 3 & f_3 \end{array} \right] \dots \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & f_1 \\ 0 & 1 & -2 & f_2 \\ 0 & 0 & 0 & f_3 - f_1 + 2f_2 \end{array} \right]$$

Thus,  $f_3 - f_1 + 2f_2$  must be 0.

Way # 2: Find kernel of  $A^*$ .

$$A^*v = \begin{bmatrix} v_1 + v_3 \\ v_2 - 2v_3 \\ -v_1 - 2v_2 + 3v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}$$

$$\text{Fredholm} \implies t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0 \text{ which is the same as above.}$$

**Example 21.4**

What about differential operators?

$$u'' = f, \quad u'(0) = u'(l) = 0$$

Way # 1: Direct

$$u'' = f$$

$$u' = a + \int_0^x f(y) dy$$

$$u = ax + b + \int_0^x \int_0^y f(z) dz dy$$

$$\text{BCs} \implies u'(0) = a = 0 \text{ and } u'(l) = \int_0^l f(x) dx = 0$$

Way # 2: Fredholm

Adjoint of  $\frac{d^2}{dx^2}$  is  $\frac{d^2}{dx^2}$  for standard inner products

$$\begin{aligned} \text{cokernel} &= \text{kernel} = \{g(x) | g'' = 0, g'(0) = g'(l) = 0\} \\ &= \{c | c \in \mathbb{R}\} \end{aligned}$$

$$\text{Thus, } \langle c, f \rangle = c \langle 1, f \rangle = c \int_0^l f(x) dx = 0.$$



*Proof.* If  $v \in \operatorname{coker} L$  and  $L[u] = f$ , then

$$\langle \langle f, v \rangle \rangle = \langle \langle L[u], v \rangle \rangle = \langle u, L^*[v] \rangle = \langle u, 0 \rangle = 0 \quad \square$$

## § 22 | Lec 22: Nov 19, 2021

### § 22.1 Self-Adjoint & Positive Definite Linear Operators

**Definition 22.1** (Self-Adjoint Operator) — Self-adjoint linear operator  $S : u \rightarrow u$  if

$$\langle S[u_1], u_2 \rangle = \langle u_1, S[u_2] \rangle \quad \forall u_1, u_2 \in u$$

#### Example 22.2

$\frac{d^2}{dx^2}$  with standard  $L^2$   $\langle u_1, u_2 \rangle$  and appropriate boundary conditions

$$\begin{aligned} \langle S[u_1], u_2 \rangle &= \int_a^b u_1'' u_2 \, dx \\ &= [u_1' u_2]_a^b - \int_a^b u_1' u_2' \, dx \\ &= [u_1' u_2 - u_1 u_2']_a^b + \int_a^b u_1 u_2'' \, dx \\ &= \langle u_1, S[u_2] \rangle \end{aligned}$$

If  $u_1'(b)u_2(b) - u_1(b)u_2'(b) - u_1'(a)u_2(a) + u_1(a)u_2'(a) = 0$  (homogeneous Dirichlet, mixed, and periodic BCs all work)

**Definition 22.3** (Positive Definite Operator) — Positive definite linear operator  $S : u \rightarrow u$  on inner product space  $u$ ,

positive definite :  $s > 0$  if  $\langle S[u], u \rangle > 0 \quad \forall u \neq 0$

positive semi-definite :  $s \geq 0$  if  $\langle S[u], u \rangle \geq 0 \quad \forall u$

**Example 22.4**

$S = -\frac{d^2}{dx^2}$  with standard inner product and appropriate BCs,

$$\begin{aligned}\langle S[u], u \rangle &= -\int_a^b u'' u \, dx \\ &= -[u' u]_a^b + \int_a^b (u')^2 \, dx \\ &= \int_a^b (u')^2 \, dx\end{aligned}$$

for appropriate BCs.

**Question 22.1.** Is this positive?

1. If homogeneous Dirichlet/Mixed BCs, then no constant solutions with  $u' = 0$  and so  $S > 0$ .
2. If homogeneous Neumann/Periodic BCs, constants allowed where  $\langle S[u], u \rangle = 0$  for  $u \neq 0$ , so  $S \geq 0$ .

Boundary conditions matter.

**Proposition 22.5**

Positive definite operators have unique solutions.

*Proof.*  $u \in \ker S \implies S[u] = 0 \implies \langle S[u], u \rangle = 0 \implies u = 0$ . So  $\ker S = \{0\}$ . Then, we get unique solutions (If instead  $v \neq 0$ ,  $v \in \ker S$ , then  $S[u] = f \implies S[u + v] = S[u] + 0 = f$  too!)  $\square$

If  $L : u \rightarrow v$ , then  $L^* \circ L$  is self-adjoint and positive semi-definite, where  $\ker S = \ker L$  and positive definite  $\iff \ker L = \ker S = \{0\}$

*Proof.* 1.  $S = L^* \circ L$

$$S^* = (L^* \circ L)^* = L^* \circ L^{**} = L^* \circ L = S$$

2. Have

$$\begin{aligned}\langle u, S[u] \rangle &= \langle u, L^*[L[u]] \rangle \\ &= \langle L[u], L[u] \rangle \\ &= \|L[u]\|^2 > 0 \text{ unless } L[u] = 0, \text{ i.e., } u \in \ker L = \ker S\end{aligned}$$

$\square$

**Question 22.2.** Why are  $L^* \circ L$  special?

Simple solvability conditions!

**Theorem 22.6**

If  $S = L^* \circ L$ , and  $S[u] = f$  has solution then  $\langle z, f \rangle = 0$  for each  $z \in \ker S = \ker S^*$ . If multiple solutions  $S[u_1] = f, S[u_2] = f$ , then  $u_2 - u_1 \in \ker S$

*Proof.* 1. Fredholm alternative for self-adjoint

2.  $S[u_1 - u_2] = S[u_1] - S[u_2] = f - f = 0$  so  $u_1 - u_2 \in \ker S$ . □

**Example 22.7**

For standard inner products,  $S = D^* \circ D = -D \circ D$  is self-adjoint and depending on BCs is positive definite or positive semi-definite.

What about the non-standard inner products?

$$\langle u_1, u_2 \rangle = \int_a^b k u_1 u_2 \, dx$$

$$\langle v_1, v_2 \rangle = \int_a^b p u_1 u_2 \, dx$$

Let  $L[u] = D[u] \implies L^*[v] = -\frac{1}{p}(kv)'$ . So

$$S = L^* \circ L = -\frac{1}{p} \frac{d}{dx} \left( k \frac{du}{dx} \right)$$

is self-adjoint and positive semi-definite. And

$$-\frac{1}{p} \frac{d}{dx} \left( k \frac{du}{dx} \right) = f(x) \text{ has unique solutions}$$