Math 135 – Differential Equations

University of California, Los Angeles

Duc Vu

Fall 2021

This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3^{rd} by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

Contents

List of Theorems

List of Definitions

§1 Lec 1: Sep 27, 2021

§1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions} \to \text{functions}$$

Example 1.1
- differentiation: $\frac{d}{dx}: f \mapsto f'$
- integration: $\int^x dx: f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L}: f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f: [0,\infty) \to \mathbb{R}$ and $F: \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2
- $f(t): t \mapsto 0 \implies \mathscr{L}[0] = 0$
- $f(t) = 1$

$$\begin{aligned}\mathscr{L}[1] &= \lim_{t\to\infty}\int_0^t e^{-st}\,dt \\ &= \lim_{t\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^t \\ &= \lim_{t\to\infty}\left(\frac{e^{-st}}{-s} + \frac{1}{s}\right) \\ &= \frac{1}{s} \quad \text{if } \operatorname{Re}(s) > 0\end{aligned}$$

Example 1.3 • Consider

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty te^{-st}\,dt \\ &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2} \text{ if } \mathrm{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \mathrm{Re}(s) > 0,\;\; n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}, \qquad \mathrm{Re}(s) > a \\ \mathscr{L}\left[\cos\omega t\right] &= \frac{s}{s^2+\omega^2} \\ \mathscr{L}\left[\sin\omega t\right] &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

4. Properties:

   a) Linear!

$$\begin{aligned}\mathscr{L}\left[f+g\right] &= \mathscr{L}\left[f\right] + \mathscr{L}\left[g\right] \\ \mathscr{L}\left[af\right] &= a\mathscr{L}\left[f\right]\end{aligned}$$

   b) Consider:

$$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= F(s-a) \quad \text{if } \mathrm{Re}(s-a) > 0\end{aligned}$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

$$\mathscr{L}\left[f(t-a)\right] = \int_0^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(t')e^{-st'}\,dt' e^{-sa}$$

   where $t' = t-a$. So

$$\mathscr{L}\left[f(t-a)\right] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

$$\begin{aligned}\mathscr{L}\left[f'\right] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt \\ &= sF(s) - f(0)\end{aligned}$$

§2 Lec 2: Sep 29, 2021

§2.1 Laplace Transform (Cont'd)

Recap: $\mathscr{L}: f \to F$

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1
- $\mathscr{L}[t^n] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$
- $\mathscr{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting $\leftrightarrow$ multiplying by exponential
- $\mathscr{L}[f'] = s\mathscr{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0,\ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\begin{aligned}\mathscr{L}[f'' + af' + bf] &= \mathscr{L}[g]\\ \mathscr{L}[f''] + a\mathscr{L}[f'] + b\mathscr{L}[f] &= G(s)\end{aligned}$$

Notice that

$$\mathscr{L}[f''] = s^2F - sf(0) - f'(0)$$

So

$$\begin{aligned}\left(s^2 + as + b\right)F(s) &= G(s) + (s+a)f_0 + f_0'\\ F(s) &= \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b}\end{aligned}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$\begin{aligned}
(s^2+4)F(s) &= \frac{4}{s^2} + s + 5 \\
F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2+4} \\
&= \frac{4}{s^2\left(s^2+4\right)} + \frac{s}{s^2+4} + \frac{5}{s^2+4}
\end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned}
\frac{4}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\
4 &= A(s^2+4) + Bs^2 \\
&= (A+B)s^2 + 4A
\end{aligned}$$

So, $A = 1,\ B = -1$. Then,

$$\begin{aligned}
F(s) &= \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{5}{s^2+4} \\
&= \frac{1}{s^2} + \frac{4}{s^2+4} + \frac{s}{s^2+4} \\
\mathscr{L}[f] &= \mathscr{L}[t + 2\sin 2t + \cos 2t] \\
\implies f &= t + 2\sin 2t + \cos 2t
\end{aligned}$$

§3 Lec 3: Oct 1, 2021

§3.1 Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

<u>*Note*</u>: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall\varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

<u>Comparison Test</u>: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)
2. fe^{-st} doesn't decay fast enough as $t \to \infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.
2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \text{ \& } c$$
Have
$$\begin{aligned} c^{-t} &\le 1\cdot e^{-t} &\quad \forall t > 0\\ 1 &\le 1\cdot e^{0t} &\quad \forall t > 0\\ t &\le 1\cdot e^{t} &\quad \forall t > 0 \end{aligned}$$

Theorem 3.1

If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$.

Proof. Have

$$\begin{aligned}
\mathscr{L}\left[f\right](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
\lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt &\le \lim_{\tau\to\infty}\int_0^\tau \left|f(t)e^{-st}\right|\,dt \\
&= \lim_{\tau\to\infty}\int_0^\tau \left|f(t)\right| e^{-s_r t}\,dt \\
&\le \lim_{\tau\to\infty}\int_0^\tau Me^{ct}\cdot e^{-s_r t}\,dt \\
&= \lim_{\tau\to\infty} M\left[\frac{e^{(c-s_r)t}}{-(c-s_r)}\right]_0^\tau \\
&= \frac{1}{s_r-c} \quad \text{if } s_r > c \\
&< \infty
\end{aligned}$$

Thus, $\mathscr{L}\left[f\right]$ exists (for $\mathrm{Re}(s) > c$) by comparison test. □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$\begin{aligned}
\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt \\
&= s^{-\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx \\
&= s^{-\frac{1}{2}}2\int_0^\infty e^{-z^2}\,dz \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0,\infty)$.

§4 Lec 4: Oct 4, 2021

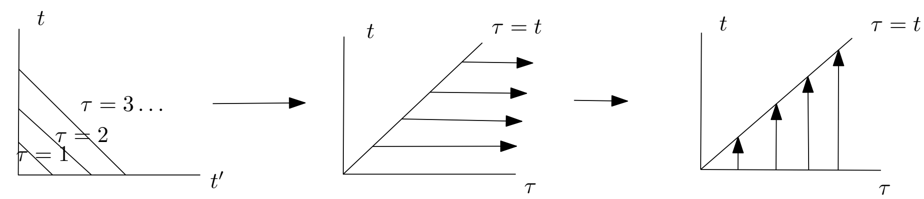
§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \end{aligned}$$

Let's define $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau}\,d\tau\,dt \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau-t)e^{-s\tau}\,dt\right)d\tau \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau-t)\,dt\right)e^{-s\tau}\,d\tau \\ &= \mathscr{L}\left[\int_0^\tau f(t)g(\tau-t)\,dt\right] \end{aligned}$$

Theorem 4.1 (Convolution)
We have

$$\begin{aligned} (f * g)(\tau) &= \int_0^\tau f(t)g(\tau - t)\,dt \\ \mathscr{L}[f * g] &= \mathscr{L}[f] \cdot \mathscr{L}[g] \end{aligned}$$

§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t)\,dt$$

Notice

$$\begin{gathered}\mathbf{f} = \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) \approx f_i \\ g(\tau) \approx g_i \\ k(\tau - t) \approx K_{ij}\end{gathered}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathscr{L}[f] &= \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f] \\ \mathscr{L}[f] &= \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}\end{aligned}$$

Example 4.2

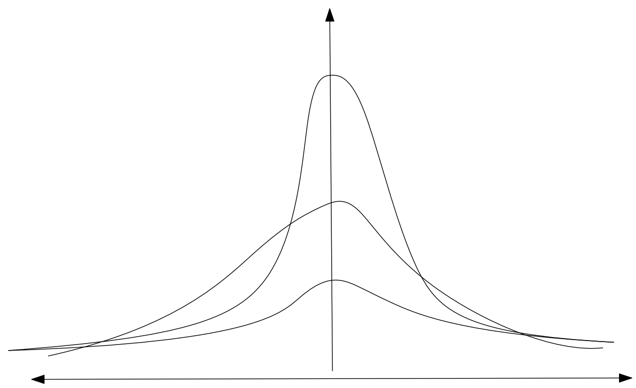
Consider $f(t) = t^3 + \int_0^t \sin(t-\tau) f(\tau) d\tau$.

$$\begin{aligned}F(s) &= \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s) \\ &\vdots \\ F(s) &= 3!(s^{-4} + s^{-6}) \\ f(t) &= t^3 + \frac{t^5}{20}\end{aligned}$$

§5 Lec 5: Oct 6, 2021

§5.1 Dirac Delta "Function"

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t) : \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\, d\tau$$

where $\delta(\tau - t) = \delta(t - \tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n \times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\begin{aligned} \mathbf{e}_i &: \text{ unit vector} \\ \mathbf{f}_i &: \text{ unit response vector} \end{aligned}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$
2. Linear superposition means
$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned}
L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\
&= \sum_i h_i L\mathbf{f}_i \\
&= \sum_i h_i \mathbf{e_i} \\
&= \mathbf{h}
\end{aligned}$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$
2. Then,

$$\begin{aligned}
f &= h * g \\
&= \int_0^t g_t(\tau)h(\tau)\,d\tau \\
&= \int_0^t g(t-\tau)h(\tau)\,d\tau
\end{aligned}$$

where g is known as the Green function.

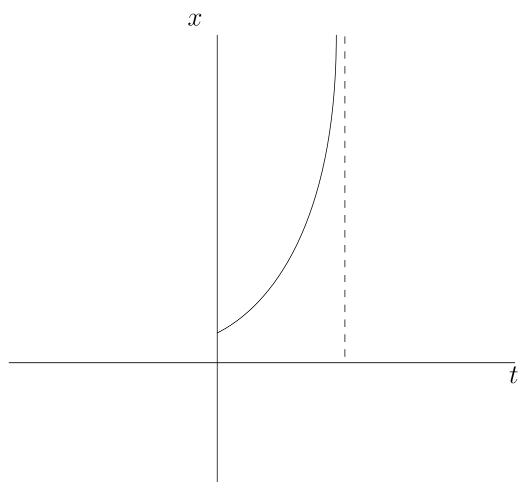
$$e_i \approx \delta_t$$
$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

§6 Lec 6: Oct 08, 2021

§6.1 Existence & Uniqueness of ODE Solutions

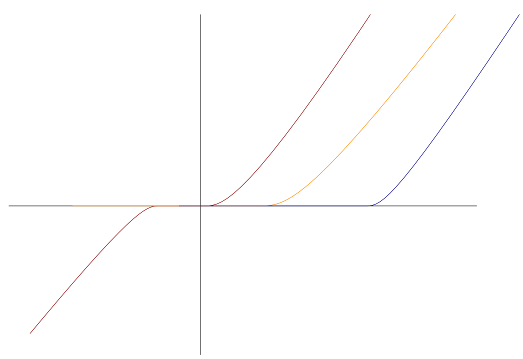
Intuitively, $f(t,x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$



   We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?
$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$



   This has infinite number of solution through $(0,0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution

§7 Lec 7: Oct 11, 2021

§7.1 Picard Iteration

Goal: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

Idea:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating
$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$
Then, we can transform it into an integral equation
$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t')\right) \, dt'$$
Notice that f is continuous and x is continuous imply x is differentiable.

2. Iteration: If we can't solve it at first, try again.

> **Example 7.1**
> Newton's root-finding algorithm
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Picard Iteration: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t)\right) \, dt'$$

Start with a guess for the function $x_0(t) = x_0$ (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f\left(t', x_n(t')\right) \, dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \xrightarrow[\infty]{} \dots$$

If $x_{n+1}(t) = x_n(t) = \overline{x}(t)$, then $\overline{x}(t)$ has to solve the IE. We want $\lim_{n\to\infty} x_n(t) \to x(t)$ solves IE.

Example 7.2

Consider $\dot{x}(t) = x(t)$, $x(0) = 1$. This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t')\,dt'$$

Picard:

$$\begin{aligned}
x_0(t) &= 1 \\
x_1(t) &= 1 + \int_0^t x_0(t')\,dt' = 1 + \int_0^t 1\,dt' \\
&= 1 + t \\
x_2(t) &= 1 + \int_0^t 1 + t\,dt \\
&= 1 + t + \frac{t^2}{2!} \\
&\vdots \\
x_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!}
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n(t) \to e^t$$

§8 Lec 8: Oct 13, 2021

§8.1 Continuity

Limit of continuous function is not necessarily continuous.

Example 8.1

Consider $x_n(t) = t^n$ on $[0,1]$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= t \\ x_2 &= t^2 \\ &\vdots \\ \overline{x} = \lim_{n\to\infty} x_n &= \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases} \end{aligned}$$

which is discontinuous.

<u>Idea</u>: We need "more" continuity. Given x, and given any $\varepsilon > 0$, if $|x - x'| < \delta(x, \varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Example 8.2

Consider $f(x) = x$ on $\mathbb{R}$. We can see that

$$|x - x'| < \varepsilon \quad \forall\, |x - x'| < \varepsilon$$

in which we pick $\delta(x, \varepsilon) = \varepsilon$.
How about $f(x) = x^2$ on $\mathbb{R}$?

$$\left|x^2 - y^2\right| < \varepsilon$$

If we pick $\delta(x, \varepsilon) = \varepsilon$, then $|x - y| < \delta = \varepsilon$ which does not necessarily imply $\left|x^2 - y^2\right| < \varepsilon$ because

$$\begin{aligned} \left|x^2 - y^2\right| &= |(x+y)(x-y)| \\ &= |x+y|\,|x-y| \\ &\le \varepsilon\,|x+y| \end{aligned}$$

$|f(x) - f(y)| > \varepsilon$. So we need to pick smaller δ as x and y get larger. It would work for $\delta = \frac{\varepsilon}{2\max(|x|,|y|)}$.

Question 8.1. Is $\frac{1}{x}$ continuous?

Ans: It depends on the domain. If we're talking about $\mathbb{R}$, it doesn't work at 0; on $(0, \infty)$, yes it's continuous.

Definition 8.3 (Uniform Continuity) — $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark 8.4. Notice that the definition is similar to continuity except that δ doesn't depend on x.

Example 8.5

x^2 on $\mathbb{R}$ is not uniformly continuous but x^2 on $(a,b) \subseteq \mathbb{R}$ is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|,|y|)} = \frac{\varepsilon}{\max(|a|,|b|)}$$

Remark 8.6. Uniform continuity also depends on the domain as continuity does.

Exercise 8.1. Is $x^{\frac{1}{2}}$ uniformly continuous on $[0,1]$?

Lipschitz Continuity: "gradient is bounded"

$$\frac{|f(x)-f(y)|}{|x-y|} < L < \infty$$

We can pick $\delta = \frac{\varepsilon}{L}$ everywhere.

Example 8.7

- x^2 on $\mathbb{R}$ is not Lipschitz but it is on a finite interval.
- $x^{\frac{1}{2}}$ is not Lipschitz continuous on $[0,1]$. However, it's uniformly continuous.

§9 Lec 9: Oct 15, 2021

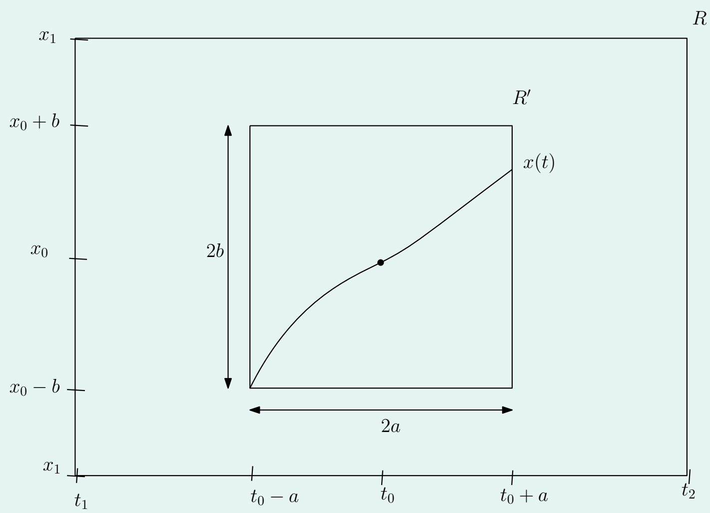
§9.1 Picard's Theorem

Let's prove local existence of the theorem.

Theorem 9.1 (Picard)
If $f(t,x)$ and $\partial_x f(t,x)$ are continuous function on a bounded rectangle $R = [t_1, t_2] \times [x_1, x_2]$ and (t_0, x_0) is in interior of R ($t_1 < t_0 < t_2$, $x_1 < x_0 < x_2$). Then $\exists$ a smaller rectangle $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ s.t. ODE

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

has a solution in R'.



<u>*Note*</u>: Since R closed and bounded, then f, $\partial_x f$ are bounded, i.e.,

$$\max_R f(t,x) = M$$
$$\max_R \partial_x f(t,x) = L$$

Thus, f is Lipschitz.
<u>Proof Outline</u>:

1. Solving ODE $\iff$ Soling IE
2. Approximate solutions using Picard iteration
$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^{t} f\left(t', x_{n-1}(t')\right)\, dt'$$
3. Prove Picard iterates converges
$$\lim_{n\to\infty} x_n(t) \to \overline{x}(t)$$
4. Prove limit $\overline{x}(t)$ solves IE.
5. Prove limit $\overline{x}(t)$ is continuous.

6. Prove limit $\overline{x}(t)$ is unique.
7. How big is $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$?

$$\text{Pick } a \ni aL < 1 \text{ \& } b = Ma \leq |x_0 - x_1|\,|x_0 - x_2|$$

Proof. 2. Prove Picard iterates converge

a) We have

$$\lim_{n\to\infty} x_n(t) \iff \lim_{n\to\infty} x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

telescoping sum!

b) Series $x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$ converges by Weierstrass M-test – If $|f_n(x)| < M_n$ $\forall n \in \mathbb{N}$, $x \in D$ and $\sum_{n=0}^{\infty} M_n$ converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

i) Show $x_i(t)$ are all in $R' \subseteq R$ so we can use bounds L, M.

$$\begin{aligned}
|x_0(t) - x_0| &= 0 \\
|x_1(t) - x_0| &= \left| \int_{t_0}^{t} f(t', x_0(t'))\, dt' \right| \\
&\leq \int_{t_0}^{t} |f(t', x_0(t'))|\, dt \\
&\leq \int_{t_0}^{t} M\, dt \\
&\leq Ma = b
\end{aligned}$$

Thus, $x_1(t)$ is in the rectangle. By induction, every $x_n(t)$ in $R' \subseteq R$.

ii) Show $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$ is bounded.
Define $\Delta = \max_{R'} |x_1(t) - x_0|$. Then

$$\begin{aligned}
|x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} f(t', x_1(t')) - f(t', x_0(t'))\, dt' \right| \\
&\leq \int_{t_0}^{t} |f(t', x_1(t')) - f(t', x_0(t'))|\, dt' \\
&\leq \int_{t_0}^{t} L\, |x_1(t') - x_0(t')|\, dt' \\
&\leq \Delta a L
\end{aligned}$$

and

$$\begin{aligned}
|x_3(t) - x_2(t)| &= \left| \int_{t_0}^{t} f(t, x_2(t)) - f(t, x_1(t))\, dt \right| \\
&\leq \int_{t_0}^{t} |f(t, x_2(t)) - f(t, x_1(t))|\, dt \\
&\leq \int_{t_0}^{t} L\, |x_2(t') - x_1(t')|\, dt' \\
&\leq L(\Delta a L)(t - t_0) \\
&\leq \Delta (aL)^2
\end{aligned}$$

Every $|x_n(t) - x_{n-1}(t)|$ depends on $|x_{n-1}(t) - x_{n-2}(t)|$ recursively. The general pattern is

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| &\le \Delta (aL)^{n-1} \\ \sum_{n=1}^{\infty} |x_n - x_{n-1}| &\le \sum_{n=0}^{\infty} \Delta (aL)^n \\ &= \frac{\Delta}{1-aL} \\ &< \infty \end{aligned}$$

Thus, $\sum x_n - x_{n-1}$ converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n\to\infty} x_n(t) = \overline{x}(t) \text{ exists!}$$

3. $\overline{x}$ solves I.E.

   Idea: We know $|\overline{x} - x_n|$ gets small so break $\left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$ into pieces like $|\overline{x} - x_n(t)|$.

   $$\text{subtract } x_n(t) - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\ dt' = 0$$

   Let $\kappa = \left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$.

   $$\begin{aligned} \kappa &= \left| -(x_n - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\, dt' \right| \\ &\le |\overline{x} - x_n| + \left| \int_{t_0}^t f(t, \overline{x}) - f(t, x_{n-1})\, dt \right| \\ &\le |\overline{x} - x_n| + \int_{t_0}^t |f(t, \overline{x}) - f(t, x_{n-1})|\, dt \\ &\le |\overline{x} - x_n| + aL\, |\overline{x} - x_{n-1}| \end{aligned}$$

   which approaches 0 as $n \to \infty$ because $\lim_{n\to\infty} x_n = \overline{x}$.

4. $\overline{x} = \lim_{n\to\infty} x_n$ is continuous, i.e., given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

   $$|t - t'| < \delta \implies |\overline{x}(t) - \overline{x}(t')| < \varepsilon$$

   Idea: Split into known things

   $$\begin{aligned} |\overline{x}(t) - \overline{x}(t')| &= |\overline{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \overline{x}(t)| \\ &\le |\overline{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \overline{x}(t)| \end{aligned}$$

   We pick n s.t. $|\overline{x}(t) - x_n(t)| < \frac{\varepsilon}{3}$ $\forall t$ which is possible because Weierstrass implies uniform convergence. Then pick δ s.t.

   $$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall\, |t - t'| < \delta$$

   which is possible because x_n is continuous.

5. $\overline{x}$ is unique.

   Idea: Prove $|\overline{x} - \tilde{x}| \le |\overline{x} - \tilde{x}|$.

   - If $\tilde{u}$ is other solution, it also exists in R'.

Proof. (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some $|t_* - t| < a$. But

$$\begin{aligned}
|\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\le \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| \, dt' \\
&\le M\left(t_* - t_0\right) \\
< Ma = b
\end{aligned}$$

Contradiction! □

- Have

$$\begin{aligned}
|\overline{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^{t} f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\le \int_{t_0}^{t} \left| f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) \right| dt' \\
&\le \int_{t_0}^{t} L \max |\overline{x}(t') - \tilde{x}(t')| \, dt \\
&\le La \max |\overline{x}(t') - \tilde{x}(t')| \\
\max |\overline{x}(t) - \tilde{x}(t)| &\le \max |\overline{x}(t) - \tilde{x}(t)|
\end{aligned}$$

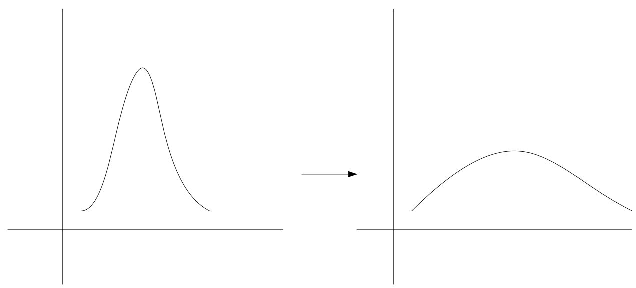
which is only possible if $\overline{x}(t) - \tilde{x}(t) = 0$, i.e., solution is unique. □

§10 Lec 10: Oct 18, 2021

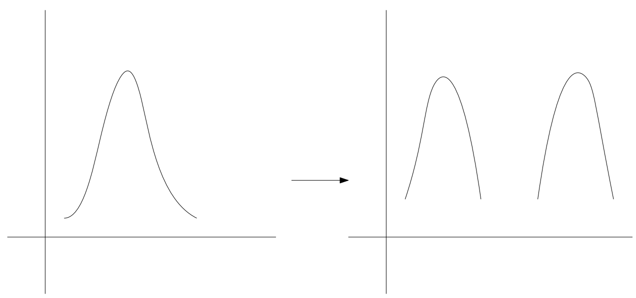
§10.1 Fourier Series

Goal: Solve linear PDE: 3 canonical examples

1. Heat/Diffusion equation
$$\partial_t u(t,x) - \partial_x^2 u(t,x) = 0$$



2. Wave equation
$$\partial_t^2 u = \partial_x^2 u$$



3. Laplace equation:
$$\partial_x^2 u + \partial_y^2 u = 0$$

Question 10.1. How do we solve linear PDEs?

Use linearity to split big problems into small ones that you can solve (find the eigenvectors). Then we split 1 PDE $\to \infty$ ODEs. First, let's define Fourier series.

Definition 10.1 (Fourier Series) — Fourier Series is a function written as a sum of sines and cosines

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx) \\ &= \sum_{-\infty}^{\infty} c_n e^{inx} \end{aligned}$$

where $c_n = c_r + ic_{in}$.

They have amazing properties:

1. They can approximate almost anything
   - analytic function
   - smooth function
   - periodic function
   - differentiable function
   - continuous/discontinuous function
2. They simplify differentiation!
   $$\begin{aligned}\frac{d}{dx}e^{ikx} &= ike^{ikx}\\ \frac{d^2}{dx^2}\sin kx &= -k^2\sin kx\\ \frac{d^2}{dx^2}\cos kx &= -k^2\cos kx\end{aligned}$$
   Just like Laplace transform, Fourier series transform differentiation into multiplication problem (easier to deal with).
3. Fourier series are <u>orthogonal</u>
   $$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = 0$$
   or
   $$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = 0 \quad \text{if } m \neq n$$
   or
   $$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = 0 \quad \text{if } m \neq n$$
   This gives easy formulas

From these facts follow from linear algebra, because Fourier series are eigenfuncitons of differentiation. They are the correct basis to solve linear PDEs.

§11 Lec 11: Oct 20, 2021

§11.1 Coefficients of Fourier Series

Question 11.1. How do we calculate Fourier Series $a_n, b_n =$?

Consider the domain: $[-\pi, \pi]$, finite dimensions N, vector

$$\mathbf{u} = \sum u_i \mathbf{e}_i$$

How do we calculate u_i?

$$\begin{aligned}\mathbf{u} \cdot \mathbf{e}_i &= \left(\sum_{i=1}^{N} u_i e_i\right) \cdot e_j \\ &= \sum_{i=1}^{N} u_i \left(e_i - e_j\right) \\ &= \sum_{i=1}^{N} \delta_{ij}\end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We want to do this in ∞ dimensions – inner product

$$N: \ \langle u, v\rangle = u \cdot v = \sum_{i=1}^{N} u_i v_i$$

$$\infty: \ \langle u, v\rangle \propto \int_a^b u(x)v(x)\,dx$$

Inner Product: $\langle u, v\rangle \to \mathbb{R}$ takes in two function & spits out a number. It has to satisfy the following properties

1. Bilinear

$$\langle au + bv, \rangle = a\langle u, vw\rangle + b\langle v, w\rangle$$

2. Symmetric $\langle u, v\rangle = \langle u, v\rangle$.
3. Positivity: $\langle u, u\rangle > 0$ unless $u = 0$.

Inner products are important

- They imply a norm $\|u\| = \sqrt{\langle u, u\rangle}$
- Cauchy-Schwarz Inequality

$$\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle$$

- Triangle inequality

$$\|u + v\| \leq \|u\| + \|v\|$$

Exercise 11.1. Prove these properties.

Now, we will use inner products to calculate Fourier. Define

$$\langle u, v\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)v(x)\,dx$$

Under this inner product, $\sin kl$, $\cos kl$ are orthogonal functions, i.e.,

$$\begin{aligned}\langle \sin kx, \cos lx\rangle &= 0 \quad \forall k, l\\ \langle \sin kx, \sin lx\rangle &= 0 \quad \text{if } k \neq l\\ \langle \cos kx, \cos lx\rangle &= 0 \quad \text{if } k \neq l\end{aligned}$$

<u>*Note*</u>: $1 = \cos 0x$

Proof. Left as exercise, but use

$$\begin{aligned}\cos\left((k+l)x\right) &= \cos kx \cos lx - \sin kx \sin lx\\ \sin\left((k+l)x\right) &= \sin kx \cos lx + \sin lx \cos kx\end{aligned}$$

Also,

$$\begin{aligned}\langle \sin kx, \sin kx\rangle &= 1\\ \langle \cos kx, \cos kx\rangle &= 1 \quad k \neq 0\\ \langle 1, 1\rangle &= 2\end{aligned}$$

□

We have

$$\begin{aligned}f(x) &= \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx\\ \langle f, \cos lx\rangle &= \langle \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx, \cos lx\rangle\\ &= \frac{a_0}{2}\langle 1, \cos lx\rangle + \sum_{k=1}^{\infty} a_k\langle \cos kx, \cos lx\rangle + \sum_{k=1}^{\infty} b_n\langle \sin kx, \cos lx\rangle\\ \langle f, \cos lx\rangle &= a_l\\ \langle f, \sin lx\rangle &= b_l\end{aligned}$$

So we can write any function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

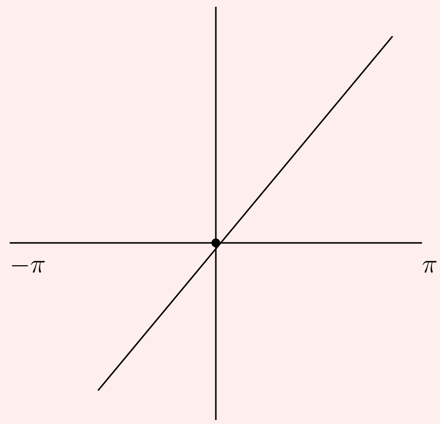
$$\begin{aligned}a_k &= \langle f, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\\ b_k &= \langle f, \sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\end{aligned}$$

Question 11.2. Are these orthogonal functions under $\langle u, v\rangle$?

Question 11.3. Are there any other kind of L^2 inner product?

Example 11.1

Consider $f(x) = x$



We have

$$\begin{aligned}
x &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\
a_k &= \langle x, \cos kx \rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx \\
&= 0 - 0 - 0 = 0 \quad \text{(integration by parts)} \\
b_k &= \langle x, \sin kx \rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx \\
&= \frac{1}{\pi} \left[-\pi \frac{\cos k\pi}{k} - (-(-\pi)) \frac{\cos(-k\pi)}{k} \right] \quad \text{(integration by parts)} \\
&= \frac{2(-1)^{k+1}}{k}
\end{aligned}$$

Thus,

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

To show that infinite series converges

$$\sum_{k=1}^{\infty} \left| \frac{2(-1)^{k+1}}{k} \right| < 2 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is conclusive (by Weierstrass-M test).

§12 Lec 12: Oct 22, 2021

§12.1 Convergence of Fourier Series

Consider the last example from last lecture

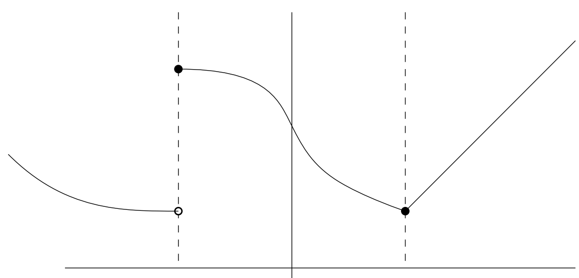
$$f(x) = x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

Question 12.1. In what sense does it converge? (What's happening at $\pm\pi$)

Fourier series must be 2π periodic (because $\cos kx$, $\sin kx$ are 2π-periodic) so the y must converge to a 2π-periodic extension of the function.

$$\tilde{f}(x + 2\pi) = \tilde{f}(x)$$

<u>*Note*</u>: x is C' (derivative continuous) but $\tilde{x}$ is not C'. It is piecewise C' (C': f continuous and $\frac{df}{dx}$ is continuous).
<u>Piecewise C'</u> on $[a, b]$



f is C' except at finitely many points. At any bad point we have

$$\begin{cases} f(x^-) = \lim_{h\to 0} f(x-h) & \text{if } f(x^+) \neq f(x^-) \text{ jump} \\ f(x^+) = \lim_{h\to 0} f(x+h) & \\ f'(x^-) = \lim_{h\to 0} f'(x-h) & \text{if } f(x^+) = f(x^-) \\ f'(x^+) = \lim_{h\to 0} f'(x+h) & \text{but } f'(x^+) \neq f(x^-) \text{ corner} \end{cases}$$

Theorem 12.1 (Fourier Convergence)
If $\tilde{f}(x)$ is 2π-periodic, piecewise C' function, then its Fourier series converges to $\tilde{f}$ everywhere except jump points x where the series converges to $\frac{f(x^+)+f(x^-)}{2}$

Question 12.2. Recall the example at the beginning, why is there no cosines for x?

Odd/even symmetries!

Fact 12.1. We have

$$\begin{aligned} \text{odd } &+ \text{odd} = \text{odd} \\ \text{even } &+ \text{even} = \text{even} \end{aligned}$$

and

$$\begin{aligned} \text{odd } &\times \text{odd} = \text{even} \\ \text{even } &\times \text{even} = \text{even} \\ \text{odd } &\times \text{even} = \text{odd} \end{aligned}$$

and

$$\begin{aligned} \int_{-a}^{a} \text{odd } dx &= 0 \\ \int_{-a}^{a} \text{even } dx &= 2\int_{0}^{a} \text{even } dx \end{aligned}$$

This implies odd functions f have sine series and even functions have cosine series.