Math 135 – Differential Equations

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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3^{rd} by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

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List of Theorems

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§1 Lec 1: Sep 27, 2021

§1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions } \to \text{ functions}$$

Example 1.1
- differentiation: $\frac{d}{dx}: \ f \mapsto f'$
- integration: $\int^x dx: \ f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L}: f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f: [0,\infty) \to \mathbb{R}$ and $F: \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2
- $f(t): \ t \mapsto 0 \implies \mathscr{L}[0] = 0$
- $f(t) = 1$

$$\begin{aligned}
\mathscr{L}[1] &= \lim_{t\to\infty} \int_0^t e^{-st}\,dt \\
&= \lim_{t\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^t \\
&= \lim_{t\to\infty} \left(\frac{e^{-st}}{-s} + \frac{1}{s}\right) \\
&= \frac{1}{s} \ \text{ if } \ \mathrm{Re}(s) > 0
\end{aligned}$$

Example 1.3 • Consider

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty te^{-st}\,dt\\ &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\\ &= \frac{1}{s^2} \text{ if } \operatorname{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \operatorname{Re}(s) > 0,\ \ n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t}\,dt\\ &= \frac{1}{s-a}, \qquad \operatorname{Re}(s) > a\\ \mathscr{L}[\cos\omega t] &= \frac{s}{s^2+\omega^2}\\ \mathscr{L}[\sin\omega t] &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

4. Properties:

   a) Linear!

   $$\begin{aligned}\mathscr{L}[f+g] &= \mathscr{L}[f] + \mathscr{L}[g]\\ \mathscr{L}[af] &= a\mathscr{L}[f]\end{aligned}$$

   b) Consider:

   $$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{-(s-a)t}\,dt\\ &= F(s-a) \quad \text{if } \operatorname{Re}(s-a) > 0\end{aligned}$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

   $$\mathscr{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(t')e^{-st'}\,dt' e^{-sa}$$

   where $t' = t - a$. So

   $$\mathscr{L}[f(t-a)] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

   $$\begin{aligned}\mathscr{L}[f'] &= \int_0^\infty f'(t)e^{-st}\,dt\\ &= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt\\ &= sF(s) - f(0)\end{aligned}$$

§2 Lec 2: Sep 29, 2021

§2.1 Laplace Transform (Cont'd)

Recap: $\mathscr{L}: f \to F$

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1
- $\mathscr{L}[t^n] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$
- $\mathscr{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting $\leftrightarrow$ multiplying by exponential
- $\mathscr{L}[f'] = s\mathscr{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0,\ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\begin{aligned} \mathscr{L}[f'' + af' + bf] &= \mathscr{L}[g] \\ \mathscr{L}[f''] + a\mathscr{L}[f'] + b\mathscr{L}[f] &= G(s) \end{aligned}$$

Notice that

$$\mathscr{L}[f''] = s^2F - sf(0) - f'(0)$$

So

$$\begin{aligned} \left(s^2 + as + b\right)F(s) &= G(s) + (s+a)f_0 + f_0' \\ F(s) &= \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b} \end{aligned}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$\begin{aligned}
(s^2+4)F(s) &= \frac{4}{s^2} + s + 5 \\
F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2+4} \\
&= \frac{4}{s^2\left(s^2+4\right)} + \frac{s}{s^2+4} + \frac{5}{s^2+4}
\end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned}
\frac{4}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\
4 &= A(s^2+4) + Bs^2 \\
&= (A+B)s^2 + 4A
\end{aligned}$$

So, $A = 1,\ B = -1$. Then,

$$\begin{aligned}
F(s) &= \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{5}{s^2+4} \\
&= \frac{1}{s^2} + \frac{4}{s^2+4} + \frac{s}{s^2+4} \\
\mathscr{L}\left[f\right] &= \mathscr{L}\left[t + 2\sin 2t + \cos 2t\right] \\
\implies f &= t + 2\sin 2t + \cos 2t
\end{aligned}$$

§3 Lec 3: Oct 1, 2021

§3.1 Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

Note: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall\varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

Comparison Test: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)
2. fe^{-st} doesn't decay fast enough as $t\to\infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.
2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \ \&\ c$$
Have
$$\begin{aligned} c^{-t} &\le 1\cdot e^{-t} && \forall t > 0 \\ 1 &\le 1\cdot e^{0t} && \forall t > 0 \\ t &\le 1\cdot e^{t} && \forall t > 0 \end{aligned}$$

Theorem 3.1

If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$.

Proof. Have

$$
\begin{aligned}
\mathscr{L}\left[f\right](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
\lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt &\le \lim_{\tau\to\infty}\int_0^\tau \left|f(t)e^{-st}\right|\,dt \\
&= \lim_{\tau\to\infty}\int_0^\tau |f(t)|\,e^{-s_r t}\,dt \\
&\le \lim_{\tau\to\infty}\int_0^\tau Me^{ct}\cdot e^{-s_r t}\,dt \\
&= \lim_{\tau\to\infty} M\left[\frac{e^{(c-s_r)t}}{-(c-s_r)}\right]_0^\tau \\
&= \frac{1}{s_r - c} \quad \text{if } s_r > c \\
&< \infty
\end{aligned}
$$

Thus, $\mathscr{L}\left[f\right]$ exists (for $\mathrm{Re}(s) > c$) by comparison test. □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$
\begin{aligned}
\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt \\
&= s^{-\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx \\
&= s^{-\frac{1}{2}}2\int_0^\infty e^{-z^2}\,dz \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}
$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0,\infty)$.

§4 Lec 4: Oct 4, 2021

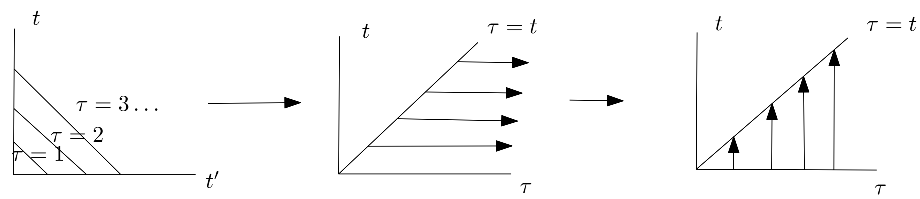
§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \end{aligned}$$

Let's define $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau}\,d\tau\,dt \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)e^{-s\tau}\,dt \right) d\tau \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)\,dt \right) e^{-s\tau}\,d\tau \\ &= \mathscr{L}\left[\int_0^\tau f(t)g(\tau - t)\,dt \right] \end{aligned}$$

Theorem 4.1 (Convolution)
We have

$$\begin{aligned} (f * g)(\tau) &= \int_0^\tau f(t)g(\tau - t)\,dt \\ \mathscr{L}[f * g] &= \mathscr{L}[f] \cdot \mathscr{L}[g] \end{aligned}$$

§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t)\,dt$$

Notice

$$\begin{gathered}\mathbf{f} = \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) \approx f_i \\ g(\tau) \approx g_i \\ k(\tau - t) \approx K_{ij}\end{gathered}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathscr{L}[f] &= \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f] \\ \mathscr{L}[f] &= \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}\end{aligned}$$

Example 4.2

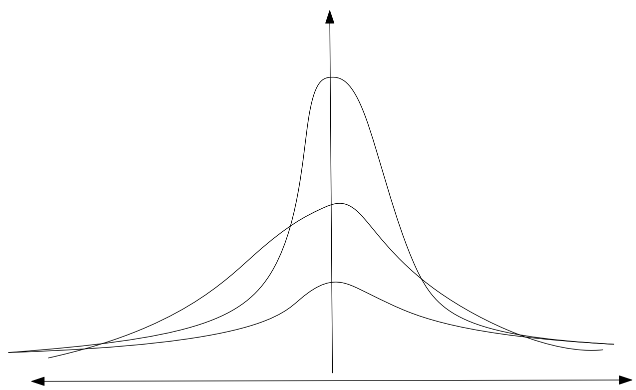
Consider $f(t) = t^3 + \int_0^t \sin(t-\tau) f(\tau) d\tau$.

$$\begin{aligned}F(s) &= \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s) \\ &\vdots \\ F(s) &= 3!(s^{-4} + s^{-6}) \\ f(t) &= t^3 + \frac{t^5}{20}\end{aligned}$$

§5 Lec 5: Oct 6, 2021

§5.1 Dirac Delta "Function"

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t) : \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\,d\tau$$

where $\delta(\tau - t) = \delta(t - \tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n\times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\begin{aligned} \mathbf{e}_i &: \text{ unit vector} \\ \mathbf{f}_i &: \text{ unit response vector} \end{aligned}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$
2. Linear superposition means
$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned} L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\ &= \sum_i h_i L\mathbf{f}_i \\ &= \sum_i h_i \mathbf{e_i} \\ &= \mathbf{h} \end{aligned}$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L\left[f\right] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$
2. Then,

$$\begin{aligned} f &= h * g \\ &= \int_0^t g_t(\tau) h(\tau)\, d\tau \\ &= \int_0^t g(t-\tau) h(\tau)\, d\tau \end{aligned}$$

where g is known as the Green function.

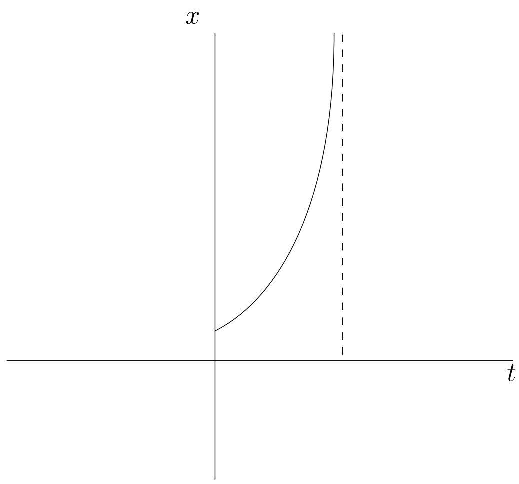
$$e_i \approx \delta_t$$
$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

§6 Lec 6: Oct 08, 2021

§6.1 Existence & Uniqueness of ODE Solutions

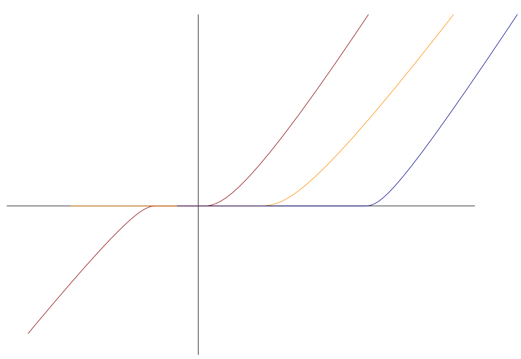
Intuitively, $f(t,x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$



We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?
$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$



This has infinite number of solution through $(0,0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution

§7 Lec 7: Oct 11, 2021

§7.1 Picard Iteration

Goal: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f(t, x(t)), \quad x(t_0) = x_0$$

Idea:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating
$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$
Then, we can transform it into an integral equation
$$x(t) = x_0 + \int_{t_0}^{t} f(t', x(t'))\, dt'$$
Notice that f is continuous and x is continuous imply x is differentiable.

2. Iteration: If we can't solve it at first, try again.

> **Example 7.1**
> Newton's root-finding algorithm
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Picard Iteration: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(t', x(t))\, dt'$$

Start with a guess for the function $x_0(t) = x_0$ (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(t', x_n(t'))\, dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \underset{\infty}{\longrightarrow} \dots$$

If $x_{n+1}(t) = x_n(t) = \overline{x}(t)$, then $\overline{x}(t)$ has to solve the IE. We want $\lim_{n\to\infty} x_n(t) \to x(t)$ solves IE.

Example 7.2

Consider $\dot{x}(t) = x(t)$, $x(0) = 1$. This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t')\,dt'$$

Picard:

$$\begin{aligned}
x_0(t) &= 1 \\
x_1(t) &= 1 + \int_0^t x_0(t')\,dt' = 1 + \int_0^t 1\,dt' \\
&= 1 + t \\
x_2(t) &= 1 + \int_0^t 1 + t\,dt \\
&= 1 + t + \frac{t^2}{2!} \\
&\vdots \\
x_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!}
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n(t) \to e^t$$

§8 Lec 8: Oct 13, 2021

§8.1 Continuity

Limit of continuous function is not necessarily continuous.

Example 8.1

Consider $x_n(t) = t^n$ on $[0,1]$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= t \\ x_2 &= t^2 \\ &\vdots \\ \overline{x} = \lim_{n\to\infty} x_n &= \begin{cases} 0, & t<1 \\ 1, & t=1 \end{cases} \end{aligned}$$

which is discontinuous.

<u>Idea</u>: We need "more" continuity. Given x, and given any $\varepsilon > 0$, if $|x - x'| < \delta(x,\varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Example 8.2

Consider $f(x) = x$ on $\mathbb{R}$. We can see that

$$|x - x'| < \varepsilon \quad \forall\, |x - x'| < \varepsilon$$

in which we pick $\delta(x,\varepsilon) = \varepsilon$.
How about $f(x) = x^2$ on $\mathbb{R}$?

$$\left|x^2 - y^2\right| < \varepsilon$$

If we pick $\delta(x,\varepsilon) = \varepsilon$, then $|x - y| < \delta = \varepsilon$ which does not necessarily imply $\left|x^2 - y^2\right| < \varepsilon$ because

$$\begin{aligned} \left|x^2 - y^2\right| &= |(x+y)(x-y)| \\ &= |x+y|\,|x-y| \\ &\le \varepsilon\,|x+y| \end{aligned}$$

$|f(x) - f(y)| > \varepsilon$. So we need to pick smaller δ as x and y get larger. It would work for $\delta = \frac{\varepsilon}{2\max(|x|,|y|)}$.

Question 8.1. Is $\frac{1}{x}$ continuous?

Ans: It depends on the domain. If we're talking about $\mathbb{R}$, it doesn't work at 0; on $(0,\infty)$, yes it's continuous.

Definition 8.3 (Uniform Continuity) — $\forall\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark 8.4. Notice that the definition is similar to continuity except that δ doesn't depend on x.

Example 8.5

x^2 on $\mathbb{R}$ is not uniformly continuous but x^2 on $(a,b) \subseteq \mathbb{R}$ is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|,|y|)} = \frac{\varepsilon}{\max\left(|a|,|b|\right)}$$

Remark 8.6. Uniform continuity also depends on the domain as continuity does.

Exercise 8.1. Is $x^{\frac{1}{2}}$ uniformly continuous on $[0,1]$?

Lipschitz Continuity: "gradient is bounded"

$$\frac{|f(x)-f(y)|}{|x-y|} < L < \infty$$

We can pick $\delta = \frac{\varepsilon}{L}$ everywhere.

Example 8.7

- x^2 on $\mathbb{R}$ is not Lipschitz but it is on a finite interval.
- $x^{\frac{1}{2}}$ is not Lipschitz continuous on $[0,1]$. However, it's uniformly continuous.

§9 Lec 9: Oct 15, 2021

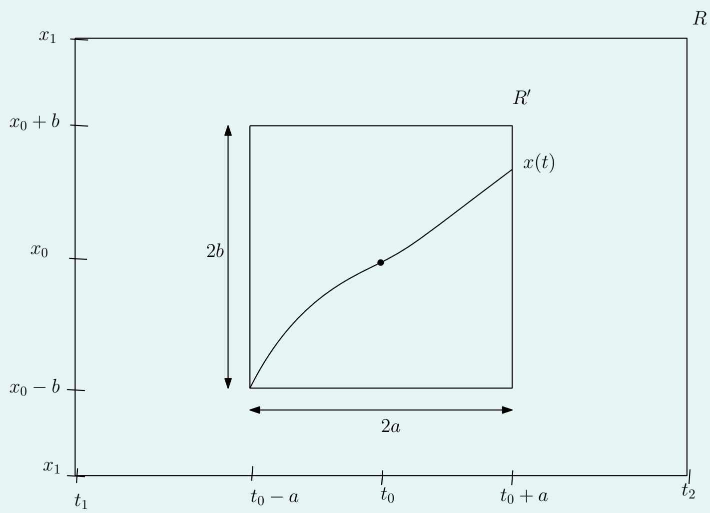
§9.1 Picard's Theorem

Let's prove local existence of the theorem.

Theorem 9.1 (Picard)
If $f(t,x)$ and $\partial_x f(t,x)$ are continuous function on a bounded rectangle $R = [t_1, t_2] \times [x_1, x_2]$ and (t_0, x_0) is in interior of R ($t_1 < t_0 < t_2$, $x_1 < x_0 < x_2$). Then $\exists$ a smaller rectangle $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ s.t. ODE

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

has a solution in R'.



Note: Since R closed and bounded, then f, $\partial_x f$ are bounded, i.e.,

$$\max_R f(t,x) = M$$
$$\max_R \partial_x f(t,x) = L$$

Thus, f is Lipschitz.
Proof Outline:

1. Solving ODE $\iff$ Soling IE
2. Approximate solutions using Picard iteration
$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^t f\left(t', x_{n-1}(t')\right)\, dt'$$
3. Prove Picard iterates converges
$$\lim_{n\to\infty} x_n(t) \to \overline{x}(t)$$
4. Prove limit $\overline{x}(t)$ solves IE.
5. Prove limit $\overline{x}(t)$ is continuous.

6. Prove limit $\overline{x}(t)$ is unique.
7. How big is $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$?

$$\text{Pick } a \ \ni aL < 1 \ \& \ b = Ma \le |x_0 - x_1| \, |x_0 - x_2|$$

Proof. 2. Prove Picard iterates converge

a) We have

$$\lim_{n\to\infty} x_n(t) \iff \lim_{n\to\infty} x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

telescoping sum!

b) Series $x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$ converges by Weierstrass M-test – If $|f_n(x)| < M_n$ $\forall n \in \mathbb{N}$, $x \in D$ and $\sum_{n=0}^{\infty} M_n$ converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

i) Show $x_i(t)$ are all in $R' \subseteq R$ so we can use bounds L, M.

$$\begin{aligned} |x_0(t) - x_0| &= 0 \\ |x_1(t) - x_0| &= \left| \int_{t_0}^{t} f\left(t', x_0(t')\right) \, dt' \right| \\ &\le \int_{t_0}^{t} \left| f\left(t', x_0(t')\right)\right| dt \\ &\le \int_{t_0}^{t} M \, dt \\ &\le Ma = b \end{aligned}$$

Thus, $x_1(t)$ is in the rectangle. By induction, every $x_n(t)$ in $R' \subseteq R$.

ii) Show $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$ is bounded.
Define $\Delta = \max_{R'} |x_1(t) - x_0|$. Then

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right) \, dt' \right| \\ &\le \int_{t_0}^{t} \left| f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right)\right| dt' \\ &\le \int_{t_0}^{t} L \left| x_1(t') - x_0(t')\right| dt' \\ &\le \Delta a L \end{aligned}$$

and

$$\begin{aligned} |x_3(t) - x_2(t)| &= \left| \int_{t_0}^{t} f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right) \, dt \right| \\ &\le \int_{t_0}^{t} \left| f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right)\right| dt \\ &\le \int_{t_0}^{t} L \left| x_2(t') - x_1(t')\right| dt' \\ &\le L \left(\Delta a L\right)\left(t - t_0\right) \\ &\le \Delta (aL)^2 \end{aligned}$$

Every $|x_n(t) - x_{n-1}(t)|$ depends on $|x_{n-1}(t) - x_{n-2}(t)|$ recursively. The general pattern is

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| &\leq \Delta(aL)^{n-1} \\ \sum_{n=1}^{\infty} |x_n - x_{n-1}| &\leq \sum_{n=0}^{\infty} \Delta(aL)^n \\ &= \frac{\Delta}{1-aL} \\ &< \infty \end{aligned}$$

Thus, $\sum x_n - x_{n-1}$ converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n\to\infty} x_n(t) = \overline{x}(t) \text{ exists!}$$

3. $\overline{x}$ solves I.E.

   Idea: We know $|\overline{x} - x_n|$ gets small so break $\left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$ into pieces like $|\overline{x} - x_n(t)|$.

   $$\text{subtract } x_n(t) - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\ dt' = 0$$

   Let $\kappa = \left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$.

   $$\begin{aligned} \kappa &= \left| -(x_n - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\, dt' \right| \\ &\leq |\overline{x} - x_n| + \left|\int_{t_0}^t f(t,\overline{x}) - f(t, x_{n-1})\, dt\right| \\ &\leq |\overline{x} - x_n| + \int_{t_0}^t |f(t,\overline{x}) - f(t, x_{n-1})|\, dt \\ &\leq |\overline{x} - x_n| + aL\, |\overline{x} - x_{n-1}| \end{aligned}$$

   which approaches 0 as $n \to \infty$ because $\lim_{n\to\infty} x_n = \overline{x}$.

4. $\overline{x} = \lim_{n\to\infty} x_n$ is continuous, i.e., given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

   $$|t - t'| < \delta \implies |\overline{x}(t) - \overline{x}(t')| < \varepsilon$$

   Idea: Split into known things

   $$\begin{aligned} |\overline{x}(t) - \overline{x}(t')| &= |\overline{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \overline{x}(t)| \\ &\leq |\overline{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \overline{x}(t)| \end{aligned}$$

   We pick n s.t. $|\overline{x}(t) - x_n(t)| < \frac{\varepsilon}{3}$ $\forall t$ which is possible because Weierstrass implies uniform convergence. Then pick δ s.t.

   $$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall\, |t - t'| < \delta$$

   which is possible because x_n is continuous.

5. $\overline{x}$ is unique.

   Idea: Prove $|\overline{x} - \tilde{x}| \leq |\overline{x} - \tilde{x}|$.

   - If $\tilde{u}$ is other solution, it also exists in R'.

Proof. (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some $|t_* - t| < a$. But

$$\begin{aligned}
|\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| \, dt' \\
&\leq M\left(t_* - t_0\right) \\
< Ma = b
\end{aligned}$$

Contradiction! ☐

- Have

$$\begin{aligned}
|\overline{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^{t} f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) dt' \right| \\
&\leq \int_{t_0}^{t} \left| f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) \right| dt' \\
&\leq \int_{t_0}^{t} L \max |\overline{x}(t') - \tilde{x}(t')| \, dt \\
&\leq La \max |\overline{x}(t') - \tilde{x}(t')| \\
\max |\overline{x}(t) - \tilde{x}(t)| &\leq \max |\overline{x}(t) - \tilde{x}(t)|
\end{aligned}$$

which is only possible if $\overline{x}(t) - \tilde{x}(t) = 0$, i.e., solution is unique. ☐