Math 151A – Applied Numerical Methods I

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This is math 151A – Applied Numerical Methods taught by Professor Jiang. We meet weekly on MWF from 1:00 pm to 1:50 pm for lecture. The recommended textbook for the class is *Numerical Analysis* 10^{th} by *Burden*, *Faires* and *Burden*. Other course notes can be found at my blog site. Please let me know through my email if you spot any typos in the note.

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§1 Lec 1: Sep 24, 2021

§1.1 Calculus Review

- Intermediate Value Theorem (IVT): For continuous function $C([a,b])$, let $f \in C([a,b])$. Let $k \in \mathbb{R}$ s.t. k is strictly between $f(a)$ and $f(b)$. Then, $\exists$ some $c \in (a,b)$ s.t. $f(c) = k$.

  Question 1.1. Why is IVT useful?

  It guarantees the existence of solution to some nonlinear equations.

  Example 1.1

  Let $f(x) = 4x^2 - e^x$. IVT tells us $\exists x^*$ s.t. $f(x^*) = 0$.

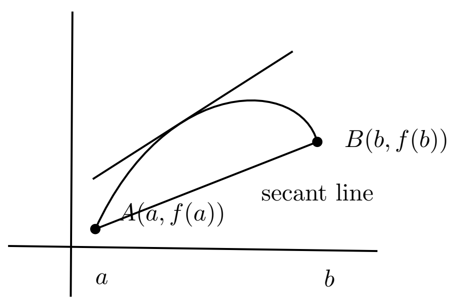
  $$f(0) = 0 - e^0 = -1 < 0$$
  $$f(1) = 4 - e > 0$$

  With $k = 0$, by IVT, $\exists c \in (0,1)$ s.t. $f(c) = 0$.

- Mean Value Theorem (MVT): If $f \in C([a,b])$ and f is differentiable in (a,b), then $\exists c \in (a,b)$ s.t.

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  in which $f'(c)$ is essentially the slope of the tangent line at $(c, f(c))$.



- Taylor's Theorem: Apply for a differentiable function, $f \in C^m([a,b])$ – f is m times continuously differentiable.

  Theorem 1.2 (Taylor)

  Let $f \in C^n([a,b])$. Let $x_0 \in [a,b]$. Assume $f^{(n+1)}$ exists on $[a,b]$. Then $\forall x \in [a,b]$, $\exists \xi(x) \in \mathbb{R}$ s.t. $x_0 < \xi < x$ or $x < \xi < x_0$. Then, we can express f as

  $$f(x) = P_n(x) + R_n(x)$$

  where

  $$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

  and

  $$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Example 1.3

$f(x) = \cos(x)$, $x_0 = 0$

$$\begin{aligned} f(x) = \cos(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 \\ &= 1 + 0 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin(\xi(x)) \end{aligned}$$

Note: Saying $f \in C^1$ is different from saying $f'(x)$ exists.

Example 1.4

Consider

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) \\ &= 0 \end{aligned}$$

But $f'(x)$ is not continuous. Specifically,

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Take sequence $\frac{1}{2k\pi}$, $f' \to -1$ and $\frac{1}{(2k+1)\pi}$, $f' \to 1$. Thus, the function is not continuous as it converges to two different values.

§2 Lec 2: Sep 27, 2021

§2.1 Errors and Convergence Rate

Fact 2.1.

1. Computers have finite memory
2. Only a subset of rational numbers $\mathbb{Q}$ can be exactly represented/stored.
3. Working with floating numbers instead of reals produces round-off error

Definition 2.1 (Error) — Let $p \in \mathbb{R}$, $\tilde{p}$ approximate to p. We define absolute error as

$$e_{\text{abs}} := |p - \tilde{p}|$$

We define relative error as

$$e_{\text{rel}} := \left|\frac{p - \tilde{p}}{p}\right|$$

Example 2.2

- $p = 1$, $\tilde{p} = 0.9$. In this case,

$$\begin{aligned} e_{\text{abs}} &= 0.1 \\ e_{\text{rel}} &= 0.1 \end{aligned}$$

- $p = 1000$, $\tilde{p} = 900$

$$\begin{aligned} e_{\text{abs}} &= 100 \\ e_{\text{rel}} &= 0.1 \end{aligned}$$

Finite Digit Arithmetic

Example 2.3

- π is rounded/chopped by computers
- $x = \frac{5}{7} = 0.\overline{714285}$, $y = \frac{1}{3} = 0.\overline{3}$
  Let $fl(x)$ is the floating point approx. to x. For example, we assume 5 digit rounding.

$$fl(x) = 0.71428, \qquad fl(y) = 0.33333$$

  Say if we want to add $x + y$ on computer

$$fl\left(fl(x) + fl(y)\right) = fl\left(1.04761\right) = 1.0476$$

Example 2.4

$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ where $x = 4.71$. The exact value of $f(x)$ at $x = 4.71$ is -14.263899. Let's assume 3 digit rounding

$$\begin{aligned} fl(x^2) &= fl(4.71 \cdot 4.71) = fl(22.1841) = 22.2 \\ fl(x^3) &= 105 \\ fl(3.2x) &= 15.1 \\ fl\left(f(4.71)\right) &= -13.4 \end{aligned}$$

The relative error here is approximately 6% which is huge. Our example has 7 floating point operations (FLOPs). In order to reduce the floating point error, we want to nest the function

$$f(x) = ((x - 6.1)x + 3.2)x + 1.5 \quad - 5 \text{ FLOPs}$$

So $fl\left(f(4.71)\right) = -14.3$ and the $e_{\text{rel}} = 0.25\%$.

Remark 2.5. Every operation introduces error.

Order of convergence for Sequences:

Definition 2.6 (Order of Convergence) — For a convergent sequence $(p_n) = (p_1, p_2, p_3, \ldots)$. Let $p_n \to p$ as $n \to \infty$. Assume $p_n \neq p$. Then, if $\exists \lambda,\ \alpha$ with $0 < \lambda < \infty$ and $\alpha > 0$ s.t.

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

Then we say p_n converges to p with order α.

Example 2.7

$p_1 = 1, p_2 = \frac{1}{5}, \ldots, p_n = \frac{1}{5}p_{n-1}$ or $p_n = \frac{1}{5^{n-1}}$ where $p_n \to 0$ as $n \to \infty$.

$$\frac{|p_{n+1} - 0|}{|p_n - 0|^1} = \frac{\left(\frac{1}{5}\right)^n}{\left(\frac{1}{5}\right)^{n-1}} = \frac{1}{5}$$

So p_n converges with $\alpha = 1$

Problem 2.1. Test with $\alpha = 2$.

Definition 2.8 (Big O Notation) — We have $a(t) = \mathcal{O}\left(b(t)\right)$ where a is on the order of b $\iff$

$$\exists C > 0 \quad \ni \quad |a(t)| \leq Cb(t) \quad \text{for } t \to 0 \text{ or } t \to \infty$$

In practice, the definition is equivalent to

$$\lim_{t \to 0} \frac{|a(t)|}{b(t)} \text{ is bounded by a positive number}$$

§3 Lec 3: Sep 29, 2021

§3.1 Lec 2 (Cont'd)

Example 3.1

The Taylor's theorem for $\cos(h)$ about 0 is

$$\cos(h) = 1 - \frac{1}{2}h^2 + \frac{1}{24}h^4 \cos(\xi(h)) \text{ with some } 0 < \xi(h) < h$$

Denote $f(h) = \cos(h) + \frac{1}{2}h^2 - 1 = \frac{1}{24}h^4 \cos(\xi(h))$

$$\lim_{h \to 0} \frac{|f(h)|}{h^4} = \lim_{h \to 0} \frac{1}{24} |\cos(\xi(h))| \leq \frac{1}{24}$$

Thus y definition of big $\mathcal{O}$ notation,

$$f(h) = \mathcal{O}(h^4)$$

§3.2 Root Finding with Bisection

The goal is to find a root, or a zero, of a function f, i.e., find p s.t. $f(p) = 0$. First, let's assume

1. $f \in C([a,b])$
2. $f(a)f(b) < 0$

Then, $\exists p$ s.t. $f(p) = 0$ (by IVT).

Example 3.2

Consider:

$$f(x) = \sqrt{x} - \cos x, \qquad [a,b] = [0,1]$$

Then,

$$f(0) = -1, \qquad f(1) = 1 - \cos 1 > 0$$

Therefore, by IVT, $\exists p \in (0,1)$ s.t. $\sqrt{p} - \cos p = 0$.

Bisection Method (B.M): is an algorithm to approximate TBA

Algorithm 1: Bisection method (given $f(x) \in C([a,b])$, with $f(a)f(b) < 0$)

1. Set $a_1 = a$, $b_1 = b$
2. Set $p_1 = \frac{a_1+b_1}{2}$
3. *if* $f(p_1) == 0$ *then* we are done!
4. *else if* $f(p_1)$ has same sign as $f(a_1)$ *then* $p \in (p_1, b_1)$
   - Set $a_2 = b_1$, $b_2 = p_1$
5. *else if* $f(p_1)$ has same sign as $f(b_1)$ *then* $p \in (a_1, p_1)$
   - Set $a_2 = a_1$, $b_2 = p_1$
6. *end*

7. Set $p_2 = \frac{a_2+b_2}{2}$

8. Reset the entire if/else process.

> **Remark 3.3.** B.M. is similar to binary search in computer algorithms. If there exists multiple roots, e.g., $\{p, q, r\} \in [a, b]$, then the B.M. is guaranteed to find exactly one root, not all of them (but no guarantee exists for which one the method will find).

Stopping Criteria: We need a sequence $(p_1, p_2, \dots)$ and need specified tolerance ε. Choices for when to stop an algorithm:

- $|p_n - p_{n-1}| < \varepsilon$ – absolute difference between successive elements of the sequence
- $\frac{|p_n - p_{n-1}|}{|p_n|} < \varepsilon$ (assume $p_n \neq 0$) – relative difference
- $|f(p_n)| < \varepsilon$ – sometimes called a residual (how close are we to the answer).

§4 Lec 4: Oct 1, 2021

§4.1 Bisection Method (Cont'd)

Remark 4.1. B.M. is a global method, $f \in C([a, b])$ as long as the assumptions are satisfied, $f(a)f(b) < 0$, the B.M. will converge. In particular, it will converge to some point p s.t. $f(p) = 0$. Here "global" means the algorithm doesn't need a good initial guess p_0 unlike some "local" methods that we will cover later.

Example 4.2

The B.M. won't work for functions like $f(x) = x^2$ even though it has a root at $p = 0$ because we can't find any a, b that satisfies $f(a)f(b) < 0$.

Theorem 4.3 (Convergence Order of B.M.)

The sequence provided by B.M. satisfies

$$|p_n - p| \leq \frac{b-a}{2^n}$$

which approaches to 0 as $n \to \infty$.

This further tells us that the error bound of B.M. converges linearly. Recall from previous lectures that linear convergence for a convergent sequence (p_n) means that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^1} = \lambda \quad \text{for some finite positive } \lambda$$

and

$$p_n = \frac{b-a}{2^n}, \quad p = 0$$

We can easily show that $\lambda = \frac{1}{2}$.

Remark 4.4. The B.M. converges slowly compared to other methods. We will soon see that Newton's method has quadratic order of convergence.

§4.2 Fixed Points

Definition 4.5 (Fixed Point) — Let function g be $g : [a, b] \to \mathbb{R}$ and $p \in [a, b]$ s.t. $g(p) = p$. Then p is a fixed point of g.

Theorem 4.6

Let p be a fixed point of g, then p is also a root of $G(x) := g(x) - x$.

Proof. Obvious by definition. □

Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways. For example, as $g(x) = x - f(x)$ or as $g(x) = x + 3f(x)$.
A fixed point for g just corresponds to the intersection between $y = g(x)$ and $y = x$.
Fixed Point Iteration (F.P.I): the F.P.I method is quite simple. For $g \in C([a, b])$ and $p_0 \in [a, b]$ we set $p_{n+1} = g(p_n)$. We also need $g(x) \in [a, b]$, otherwise at some point of the algorithm we won't be able to proceed to evaluate g. Also note that the initial guess p_0 is arbitrary.

$$p_1 = g(p_0), \quad p_2 = g(p_1), \ldots, p_{n+1} = g(p_n)$$

We use the same stopping criteria as in B.M.

Example 4.7 (F.P.I Failure Case)
To solve $x^2 - 7 = 0$, it is equivalent to $x = \frac{7}{x}$. We want to use F.P.I to find $p = \sqrt{7} \approx 2.64575\ldots$, so we can set

$$g_1(x) = \frac{7}{x}$$

then the goal is to find p s.t. $p = g_1(p)$. Another option is to use

$$g_2(x) := \frac{x + \frac{7}{x}}{2} = x$$

Let $p_0 = 3$ we can show that

- $g_1(x)$: $p_0 = 3$, $p_1 = \frac{7}{3}$, $p_2 = 3, \ldots$, oscillates between 2 numbers
- $g_2(x)$: $p_0 = 3$, $p_1 = 2.666\ldots$, $p_2 = 2.645833\ldots, \ldots$

Example 4.8
$x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$, i.e. $p = 1.365230013$.

a) $x = g_1(x) = x - x^3 - 4x^2 + 10$ – does not converge

b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{\frac{1}{2}}$ – does not converge

c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$ – converge

d) $x = g_4(x) = \left(\frac{10}{x+4}\right)^{\frac{1}{2}}$ – converge

e) $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$ – converge

We can see that the choice of $g(x)$ is critical to determine whether the algorithm converges. Before delving into that problem, let's first establish a theorem about the existence of a fixed point.

Theorem 4.9 (Existence of a Fixed Point)
Let $g \in \mathbb{C}([a, b])$ with $a \leq g(x) \leq b$. Then, $\forall x \in [a, b]$, $\exists$ at least one fixed point p s.t. $g(p) = p$.