Math 135 – Differential Equations

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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3^{rd} by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

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§1 Lec 1: Sep 27, 2021

§1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions } \to \text{ functions}$$

Example 1.1
- differentiation: $\frac{d}{dx}: \; f \mapsto f'$
- integration: $\int^x dx: \; f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L}: f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f: [0,\infty) \to \mathbb{R}$ and $F: \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2
- $f(t): \; t \mapsto 0 \implies \mathscr{L}[0] = 0$
- $f(t) = 1$

$$\begin{aligned}
\mathscr{L}[1] &= \lim_{t\to\infty} \int_0^t e^{-st}\,dt \\
&= \lim_{t\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^t \\
&= \lim_{t\to\infty} \left(\frac{e^{-st}}{-s} + \frac{1}{s}\right) \\
&= \frac{1}{s} \;\text{ if } \; \mathrm{Re}(s) > 0
\end{aligned}$$

Example 1.3 • Consider

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty te^{-st}\,dt \\ &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2} \text{ if } \operatorname{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \operatorname{Re}(s) > 0, \;\; n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}, \qquad \operatorname{Re}(s) > a \\ \mathscr{L}\left[\cos\omega t\right] &= \frac{s}{s^2+\omega^2} \\ \mathscr{L}\left[\sin\omega t\right] &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

4. Properties:

   a) Linear!

$$\begin{aligned}\mathscr{L}\left[f+g\right] &= \mathscr{L}\left[f\right] + \mathscr{L}\left[g\right] \\ \mathscr{L}\left[af\right] &= a\mathscr{L}\left[f\right]\end{aligned}$$

   b) Consider:

$$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= F(s-a) \quad \text{if } \operatorname{Re}(s-a) > 0\end{aligned}$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

$$\mathscr{L}\left[f(t-a)\right] = \int_0^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(t')e^{-st'}\,dt'e^{-sa}$$

   where $t' = t-a$. So

$$\mathscr{L}\left[f(t-a)\right] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

$$\begin{aligned}\mathscr{L}\left[f'\right] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt \\ &= sF(s) - f(0)\end{aligned}$$

§2 Lec 2: Sep 29, 2021

§2.1 Laplace Transform (Cont'd)

Recap: $\mathscr{L}: f \to F$

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1
- $\mathscr{L}[t^n] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$
- $\mathscr{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting $\leftrightarrow$ multiplying by exponential
- $\mathscr{L}[f'] = s\mathscr{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0,\ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\mathscr{L}[f'' + af' + bf] = \mathscr{L}[g]$$
$$\mathscr{L}[f''] + a\mathscr{L}[f'] + b\mathscr{L}[f] = G(s)$$

Notice that

$$\mathscr{L}[f''] = s^2F - sf(0) - f'(0)$$

So

$$\left(s^2 + as + b\right) F(s) = G(s) + (s+a)f_0 + f_0'$$
$$F(s) = \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$\begin{aligned}(s^2+4)F(s) &= \frac{4}{s^2} + s + 5 \\ F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2+4} \\ &= \frac{4}{s^2\left(s^2+4\right)} + \frac{s}{s^2+4} + \frac{5}{s^2+4}\end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned}\frac{4}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\ 4 &= A(s^2+4) + Bs^2 \\ &= (A+B)s^2 + 4A\end{aligned}$$

So, $A = 1,\ B = -1$. Then,

$$\begin{aligned}F(s) &= \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{5}{s^2+4} \\ &= \frac{1}{s^2} + \frac{4}{s^2+4} + \frac{s}{s^2+4} \\ \mathscr{L}\left[f\right] &= \mathscr{L}\left[t + 2\sin 2t + \cos 2t\right] \\ \implies f &= t + 2\sin 2t + \cos 2t\end{aligned}$$

§3 Lec 3: Oct 1, 2021

§3.1 Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

<u>*Note*</u>: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall\varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

<u>Comparison Test</u>: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)
2. fe^{-st} doesn't decay fast enough as $t \to \infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.
2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \;\&\; c$$
Have
$$\begin{aligned} c^{-t} &\le 1\cdot e^{-t} \qquad \forall t > 0\\ 1 &\le 1\cdot e^{0t} \qquad \forall t > 0\\ t &\le 1\cdot e^{t} \qquad \forall t > 0 \end{aligned}$$

Theorem 3.1

If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$.

Proof. Have

$$
\begin{aligned}
\mathscr{L}\left[f\right](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
\lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt &\le \lim_{\tau\to\infty}\int_0^\tau \left|f(t)e^{-st}\right|\,dt \\
&= \lim_{\tau\to\infty}\int_0^\tau |f(t)|\,e^{-s_r t}\,dt \\
&\le \lim_{\tau\to\infty}\int_0^\tau Me^{ct}\cdot e^{-s_r t}\,dt \\
&= \lim_{\tau\to\infty} M\left[\frac{e^{(c-s_r)t}}{-(c-s_r)}\right]_0^\tau \\
&= \frac{1}{s_r-c} \quad \text{if } s_r > c \\
&< \infty
\end{aligned}
$$

Thus, $\mathscr{L}\left[f\right]$ exists (for $\mathrm{Re}(s) > c$) by comparison test. □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$
\begin{aligned}
\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt \\
&= s^{-\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx \\
&= s^{-\frac{1}{2}}2\int_0^\infty e^{-z^2}\,dz \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}
$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0,\infty)$.

§4 Lec 4: Oct 4, 2021

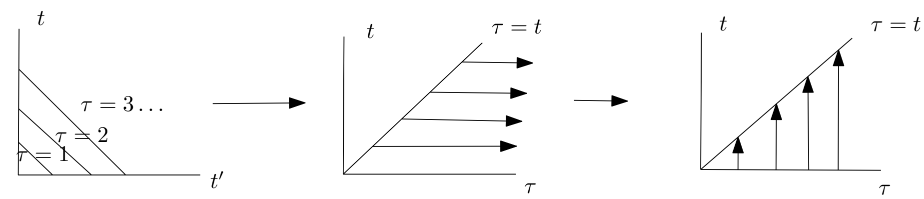
§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \end{aligned}$$

Let's define $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau}\,d\tau\,dt \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)e^{-s\tau}\,dt \right) d\tau \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)\,dt \right) e^{-s\tau}\,d\tau \\ &= \mathscr{L}\left[\int_0^\tau f(t)g(\tau - t)\,dt \right] \end{aligned}$$

Theorem 4.1 (Convolution)
We have

$$\begin{aligned} (f * g)(\tau) &= \int_0^\tau f(t)g(\tau - t)\,dt \\ \mathscr{L}[f * g] &= \mathscr{L}[f] \cdot \mathscr{L}[g] \end{aligned}$$

§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t)\,dt$$

Notice

$$\begin{gathered}\mathbf{f} = \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) \approx f_i \\ g(\tau) \approx g_i \\ k(\tau - t) \approx K_{ij}\end{gathered}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathscr{L}[f] &= \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f] \\ \mathscr{L}[f] &= \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}\end{aligned}$$

Example 4.2

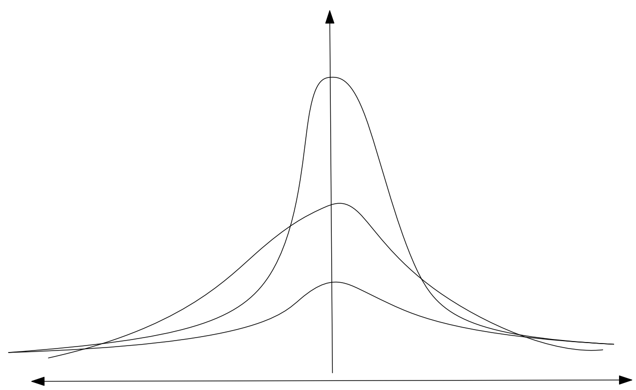
Consider $f(t) = t^3 + \int_0^t \sin(t - \tau) f(\tau) d\tau$.

$$\begin{aligned}F(s) &= \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s) \\ &\vdots \\ F(s) &= 3!(s^{-4} + s^{-6}) \\ f(t) &= t^3 + \frac{t^5}{20}\end{aligned}$$

§5 Lec 5: Oct 6, 2021

§5.1 Dirac Delta "Function"

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \dots, 0, 1, 0, \dots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t): \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\, d\tau$$

where $\delta(\tau - t) = \delta(t-\tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n\times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\begin{aligned} \mathbf{e}_i &: \text{ unit vector} \\ \mathbf{f}_i &: \text{ unit response vector} \end{aligned}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$
2. Linear superposition means
$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned} L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\ &= \sum_i h_i L\mathbf{f}_i \\ &= \sum_i h_i \mathbf{e_i} \\ &= \mathbf{h} \end{aligned}$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$
2. Then,

$$\begin{aligned} f &= h * g \\ &= \int_0^t g_t(\tau)h(\tau)\,d\tau \\ &= \int_0^t g(t-\tau)h(\tau)\,d\tau \end{aligned}$$

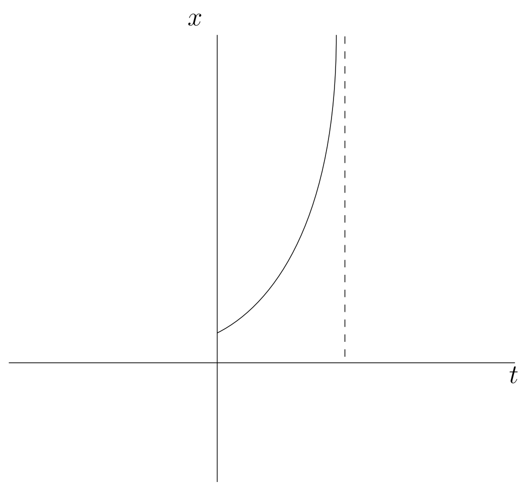
where g is known as the Green function.

$$e_i \approx \delta_t$$
$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

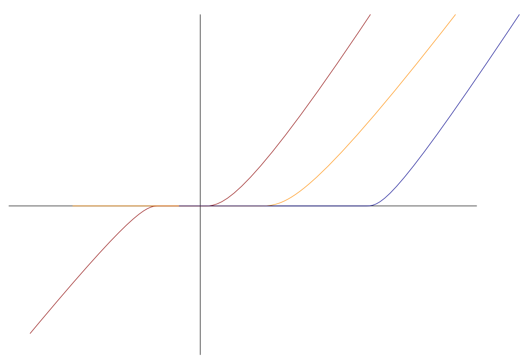
§6 Lec 6: Oct 08, 2021

§6.1 Existence & Uniqueness of ODE Solutions

Intuitively, $f(t,x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$


We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?
$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$

This has infinite number of solution through $(0,0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution