Math 142 – Mathematical Modeling

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This is math 142 – Mathematical Modeling taught by Professor Huang. We meet weekly on MWF from 9:00am – 9:50am for lecture. There is one textbook used for the class, which is *Mathematical Models* by *Haberman*. You can find other lecture notes at my blog site. Please let me know through my email if you spot any mathematical errors/typos.

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§1 Lec 1: Sep 24, 2021

§1.1 Intro to Mathematical Modeling

First, let's examine the following question

Question 1.1. Why do we learn mathematical modeling?

There are lots of question that math may provide some explanation so that we could understand the question deeply.

Example 1.1
1. How is Covid-19 spread? How can we control the spread of Covid-19?
2. How to control the spreading of the forest fire and how to reduce the loss?
3. How does the population of human evolve over time?

So,

Question 1.2. What is a mathematical model and how can we create the model?

Definition 1.2 (Mathematical Model) — A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is called mathematical modeling.

To create a mathematical model, we

1. formulate the problem: approximations and assumptions based on experiments and observations
2. solve the problem that is formulated above
3. interpret the mathematical results in the context of the problem

Let's now explain the three steps above in more details.

1. Formulation
   a) State the question: If the question is vague, then make it to be precise. If the question is too "big", then subdivide it into several simple and manageable parts.
   b) Identify factors: Decide important quantities and assign some notation to the corresponding quantity. Then, we need to determine the relationship between the quantities and represent each relationship with an equation.
2. Solve the problem above: This may entail calculations that involve algebraic equations, some ODE, PDE, etc; provide some theorems or doing some simulations, etc.
3. Interpretation/Evaluation: We need to translate the mathematical result in step 2 back to the real world situations and evaluate whether the model is good or not by asking the following questions:
   a) Has the model explained the real-world observations?
   b) Are the answers we found accurate enough?
   c) Were our assumptions good?

d) What are the strengths and weaknesses of our model?

e) Did we make any mistake in step 2?

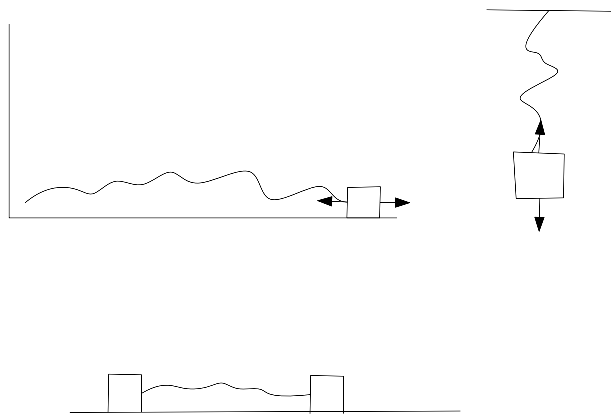
If the answer to any of the above question is not favorable, we need to go back to step 1 and go through all the steps again until we get some satisfying results.

§2 Lec 2: Sep 27, 2021

§2.1 An Example of Modeling a Mass-Spring System

Consider the following question

Question 2.1. How does the spring-mass system move/work?

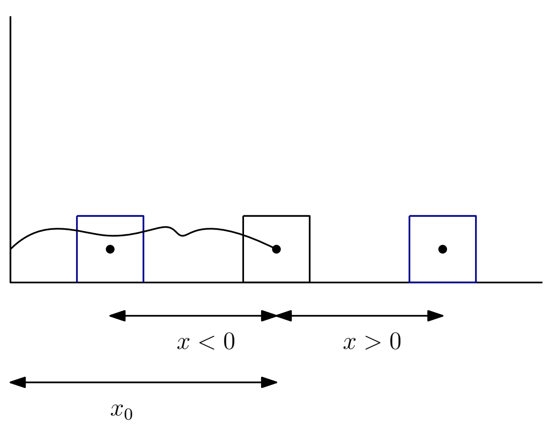


Formulation:

a) State the question: What formula can describe how the spring-mass system work?

b) Identify factors:

   (a) initial position x_0 (called natural length)
   (b) the spring constant k
   (c) friction f_c
   (d) mass of the object m
   (e) position x
   (f) velocity v
   (g) acceleration a
   (h) force F



Now, we try to find some relations between factors we listed above. First, let's describe our observations. If we contract the spring ($x < 0$), there is some force to push the spring outward ($F > 0$). If we stretch the spring ($x > 0$), there is some force that restores the initial shape of the spring ($F < 0$). So, we can observe that

$$F \cdot x < 0$$

The relation between F and x can be summarized by the Hooke's Law

$$F = -kx \tag{*}$$

Next, let's find the relation between the force and the movement of the object (F, m, v, a) by assuming that the movement of the object only depends on the force of the spring (not on other factors). This can be summarized by Newton's second law of motion.

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d}{dt}\left(\frac{d\vec{x}}{dt}\right) = m\frac{d^2\vec{x}}{dt^2} \tag{**}$$

By (*) and (**), we deduce

$$F = -kx = m\frac{d^2x}{dt^2}$$

Mathematical analysis: we need to find the solution of the ODE:

$$mx'' + kx = 0$$

To solve the ODE, we want to find the solution to the characteristic equation

$$m\lambda^2 + k = 0 \implies x = \pm\sqrt{\frac{k}{m}}i$$

Thus,

$$\begin{aligned} x(t) &= c_1 e^{t\sqrt{\frac{k}{m}}i} + c_2 e^{-t\sqrt{\frac{k}{m}}i} \\ &= (c_1 + c_2)\cos\left(\sqrt{\frac{k}{m}}t\right) + (c_1 - c_2)i\sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= c_3\cos\left(\sqrt{\frac{k}{m}}t\right) + c_4\sin\left(\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

§3 Lec 3: Sep 29, 2021

§3.1 An Example (Cont'd)

Recall that we have

$$x(t) = c_3 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_4 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

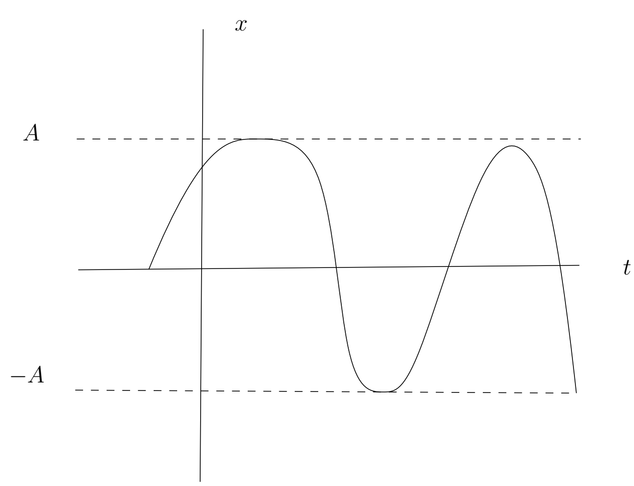
Let $\theta_2 = \sqrt{\frac{k}{m}}t$. Then,

$$x(t) = \sqrt{c_3^2 + c_4^2}\left(\frac{c_3}{\sqrt{c_3^2+c_4^2}}\cos(\theta_2) + \frac{c_4}{\sqrt{c_3^2+c_4^2}}\sin(\theta_2)\right)$$

Let $\sin\theta_1 = \frac{c_3}{\sqrt{c_3^2+c_4^2}}$ and $\cos\theta_1 = \frac{c_4}{\sqrt{c_3^2+c_4^2}}$ with $\tan\theta_1 = \frac{c_3}{c_4}$ or $\theta_1 = \arctan\left(\frac{c_3}{c_4}\right)$. So,

$$\begin{aligned} x(t) &= \sqrt{c_3^2+c_4^2}\sin(\theta_1+\theta_2) \\ &= \sqrt{c_3^2+c_4^2}\left(\sqrt{\frac{k}{m}} + \theta_1\right) \end{aligned}$$

Evaluation of $x(t) = A\sin(\omega t + \theta)$



From the figure above, we know $x(t)$ is periodic with period $T = \frac{2\lambda}{\omega} = 2\lambda\sqrt{\frac{m}{k}}$

$$\max_t x(t) = A, \qquad \min_t x(t) = -A$$

where A is the amplitude and $\omega t + TBA$

Since $x(t)$ is a periodic function, this means the spring will oscillate forever. However, in practice, it is impossible. Thus, we need to modify our model by removing or adding some assumption.

Now, we may consider the case that there is friction when spring oscillates.

$$F_f = -c\frac{dx}{dt}$$

Then,

$$m\frac{d^2x}{dt^2} = -kx - c\cdot\frac{dx}{dt}$$

§3.2 Population Dynamics

Consider the following question

Question 3.1. Can we predict whether a species or its population will thrive or go extinct?

In order to answer it, let's first investigate an example.

Example 3.1

How many people will there be in the U.S. in the next 4 years?

First let's reformulate the question in the example to be more specific:

Question 3.2. Can we build a math model to predict the number of people in the U.S. in 1, 2, 3, 4 year?

Assumption	Factor
the death and birth rate are constant	birth rate: b
the counting period (of the population) is fixed	death rate: d
the growth of the population only depends on	the period
the death and birth rate	initial population: N_0
	the distribution of the population: $N^{(a)}$
	migration rate
	the # of years from the current time: t
	the # of population at time t: $N(t)$
	the growth rate: R

To study $N(t)$ we need to consider the relation between $N(t)$ and $N(t+\Delta t)$

$$\begin{aligned} N(t+\Delta t) &= N(t) + \#\text{ of new birth at } [t, t+\Delta t] - \#\text{ of death at } [t, t+\Delta t] \\ &= N(t) + (b-d)\Delta t \cdot N(t) \\ &= (1 + (b-d)\Delta t) \cdot N(t) \end{aligned}$$

Thus,

$$\begin{aligned} N(t+\Delta t) &= (1 + R\Delta t)\, N(t) \\ N(1) &= (1+R)N_0 \\ N(2) &= (1+R)N(1) = (1+R)^2 N_0 \\ N(3) &= (1+R)N(2) = (1+R)^3 N_0 \\ N(4) &= (1+R)N(3) = (1+R)^4 N_0 \end{aligned}$$

§4 Lec 4: Oct 1, 2021

§4.1 Population Dynamics (Cont'd)

Example 4.1

$N_0 = 300$ millions, $R = 0.6\%$, $\Delta t = 1$

$$\begin{aligned} N(1) &= (1+r)N_0 = (1+0.6\%)\cdot 300 \\ &= 300 + 1.8 = 301.8 \text{ millions} \\ N(2) &= (1+r)^2 N_0 = (1+0.6\%)\cdot 300 \\ &= 301.8 \cdot 100.6\% \\ N(3) &= (1+R)^3 N_0 = (1+0.6\%)^3 \cdot 300 \\ N(4) &= (1+R)^4 \cdot N_0 = (1+0.6\%)^4 \cdot 300 \end{aligned}$$

Consider:

$$N(t+\Delta t) = (1 + R\cdot \Delta t)\cdot N(t)$$

where $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t, \ldots$, $t_n = n\Delta t$

$$\implies N(n\cdot \Delta t) = (1 + R\cdot \Delta t) N\left((nt)\Delta t\right) = \ldots = \left(1 + R\Delta t\right)^n N_0$$

We have

$$\left(1 + R\Delta t\right)^{\frac{1}{\Delta t R}\cdot Rn\Delta t}\cdot N_0 = \left(1+R\Delta t\right)^{\frac{1}{R\Delta t}Rt} N_0$$

Set $\Delta t \to 0$, we obtain $\left(1+R\Delta t\right)^{\frac{1}{R\Delta t}} \to e$. Then,

$$N(t) = e^{Rt} N_0 \text{ as } \Delta t \to 0$$

Next, let's analyze the property of the model above:

$$N\left(n\Delta t\right) = \left(1 + R\Delta t\right)^n N_0$$

1. $1 + R\Delta t > 1$, then $N(n\Delta t) \to +\infty$, as $n \to +\infty$
2. $0 < 1 + R\Delta t < 1$, then $N(n\Delta t) \to 0$ as $n \to +\infty$

Conclusion: When $0 < 1 + R\Delta t < 1$, the model is acceptable; however, when $1 + R\Delta t > 1 (R > 0)$, the model should be modified. Thus, we may change our assumption: the growth rate is constant (e.g., the growth rate depends on the population itself)

§4.2 Continuous Population Model

Have:

$$N(t) = e^{Rt} N_0$$

Let's start from the previous lecture

$$N\left(t + \Delta t\right) = N(t) + R\Delta t \cdot N(t)$$

So

$$\frac{N(t+\Delta t)-N(t)}{\Delta t}=R\cdot N(t)$$
$$\lim_{\Delta t\to 0}\frac{N(t+\Delta t)-N(t)}{\Delta t}=\lim_{\Delta t\to 0}R\cdot N(t)=R\cdot N(t)$$
$$\frac{dN(t)}{dt}=R\cdot N(t)$$
$$\int\frac{dN(t)}{N(t)}=\int Rdt$$
$$\ln(N(t))=Rt+C$$
$$N(t)=e^Ce^{Rt}=N_0e^{Rt}$$

Evaluate the continuous model $N(t)=e^{Rt}N_0$

1. $0<R<1$: $N(t)\to\infty$ as $t\to\infty$ and $N(t)\uparrow$ as $t\uparrow$
2. $-1<R<0$: $N(t)\to 0$ as $t\to\infty$ and $N(t)\downarrow$ as $t\uparrow$

Conclusion: When $R<0$, the model is acceptable; however, when the growth rate $R>0$,the individuals (of a species) will compete each other as the resource is limited, $N(t)\to\infty$ as $t\to\infty$. Now, let's consider the density-dependent growth. Assumption:

- The growth rate is density dependent, i.e., $R(t)=R\left(N(t)\right)$
- If the population is small, then the influence of the environment is small, then we hope that the population has exponential growth.
- As $N(t)$ gets large enough, we don't expect the growth of $N(t)$. In other word, the growth rate $R\left(N(t)\right)\leq 0$ when $N(t)$ is large enough (since $R(t)$ is usually assume to be smooth, $R\left(N(t)\right)=0$ when $N(t)$ is large enough)

$$\frac{dN}{dt}=R\left(N(t)\right)\cdot N(t)$$

From our assumption, $R\left(N(t)\right)$ should be a constant when $N(t)$ is small and $R\left(N(t)\right)=0$ as $N(t)$ is large enough. So we can consider $R\left(N(t)\right)$ of the form

$$R\left(N(t)\right)=a-bN(t)$$

Thus, the model becomes

$$\frac{dN}{dt}=(a-bN)N$$

This is known as the logistic model.

Remark 4.2. The discrete-time population model is called Beverton-Holt model.

$$\begin{cases}N(t\cdot\Delta t)=\frac{R_0(N(t-1)\cdot\Delta t)}{1+N((t-1)\Delta t)/M}\\ R(N)=\frac{R_0}{1+N((t-1)\cdot\Delta t)/M}\end{cases}$$

§5 Lec 5: Oct 4, 2021

§5.1 Continuous and Discrete Population Models

Recall the continuous logistic population model

$$\frac{dN}{dt} = N(a - bN)$$

Let's manipulate this

$$\begin{aligned}
\frac{dN}{N(a-bN)} &= dt \\
\int \frac{1}{aN} + \frac{b}{a(a-bN)} dN &= \int dt \\
\frac{1}{a}\ln N - \frac{1}{a}\ln|a-bN| &= t + c \\
\ln\left|\frac{N}{a-bN}\right| &= at + \tilde{c} \\
\frac{N}{a-bN} &= e^{at+\tilde{c}} = Ce^{at} \\
N &= \frac{a}{b+Ce^{-at}}
\end{aligned}$$

Since $N(0) = N_0 \implies N_0 = \frac{a}{b+C}$, we have

$$N(t) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}}$$

Let's now consider the relation between continuous logistic population and discrete-time logistic model for $\Delta t = 1$. For the discrete case,

$$\begin{cases} N(t) = \frac{R_0 N(t-1)}{1+N(t-1)/M} \\ R(N(t)) = \frac{R_0}{1+N(t-1)/M} \end{cases}$$

For the continuous case,

$$N(t) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}}$$

Then,

$$N(t-1) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}e^{a}}$$

Notice that

$$\begin{aligned}
\frac{1}{N(t)} &= \frac{b}{a} + \left(\frac{a}{N_0} - b\right)e^{at}/a \\
e^a \cdot \frac{1}{N(t-1)} &= \left(\frac{b}{a} + \left(\frac{a}{N_0} - b\right)e^{at}e^{-a}/a\right)\cdot e^a \\
\frac{1}{N(t)} - \frac{e^a}{N(t-1)} &= \frac{b}{a} - \frac{b}{a}e^a
\end{aligned}$$

For the continuous model, as $t \to \infty$, we can see that $N(t) \to \frac{a}{b}$ which is a good model.

§5.2 Discrete One-Species Model with an Age Distribution

Motivation: The birth and death rates will vary a lot if state A has more young citizens than state B.
Let's consider the period $\Delta t = 1$ year, define variables for a population at each age

$$\begin{aligned} N_0(t) &= \# \text{ individuals whose age } < 1 \\ N_1(t) &= \# \text{ of individuals one year old} \\ N_2(t) &= \# \text{ of individuals two years old} \\ &\vdots \\ N_M(t) &= \# \text{ of individuals M years old} \end{aligned}$$

where M is the oldest age with proper population. Suppose

$$\begin{aligned} b_m &= \text{ birth rate for a population that is m years old} \\ d_m &= \text{ death rate for a population that is m years old} \end{aligned}$$

Let's consider the population $N_m(t+1)$

$$\begin{aligned} N_0(t+1) &= b_0 N_0(t) + b_1 N_1(t) + \ldots + b_M N_M(t) \\ N_1(t+1) &= N_0(t) - d_0 N_0(t) = (1-d_0) N_0(t) \\ N_2(t+1) &= N_1(t) - d_1 N_1(t) = (1-d_1) N_1(t) \\ &\vdots \\ N_M(t+1) &= N_{M-1}(t) - d_{M-1} N_{M-1}(t) = (1-d_{M-1}) N_{M-1}(t) \end{aligned}$$

In matrix notation,

$$\vec{N}(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ \vdots \\ N_M(t) \end{bmatrix}$$

Then,

$$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \\ \vdots \\ N_M(t+1) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \ldots & b_M \\ 1-d_0 & 0 & \ldots & 0 \\ 0 & 1-d_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1-d_{M-1} & 0 \end{bmatrix} \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_M(t) \end{bmatrix}$$

$\implies \vec{N}(t+1) = L\vec{N}(t)$ – the matrix is called Leslie matrix.

§6 Lec 6: Oct 6, 2021

§6.1 Stable Age Distribution

Definition 6.1 (Stable Age Distribution) — A stable age distribution exists if the populations approach an age distribution that is independent of time as time increases, i.e., $\frac{1}{\|\vec{N}(t)\|_1}\vec{N}(t) \to \vec{v}$ as $t \to \infty$ where

$$\|\vec{N}(t)\|_1 = \sum_{i=0}^{M} |N_i(t)|$$

Assume that the Leslie matrix

$$L = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}$$

and

$$\vec{N}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

Let's track the evolution of the population age groups. We have

$$\vec{N}(t+1) = L \cdot \vec{N}(t)$$
$$\vec{N}(1) = L\vec{N}(0) = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 300 \\ 44 \end{bmatrix}$$
$$\vec{N}(2) = L\vec{N}(1) = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}\begin{bmatrix} 300 \\ 44 \end{bmatrix} = \begin{bmatrix} 644 \\ 132 \end{bmatrix}$$

Continue this process we obtain

$$\vec{N}(3) = \begin{bmatrix} 1420 \\ 2834 \end{bmatrix}, \quad \begin{bmatrix} 3123.4 \\ 624.8 \end{bmatrix}, \dots$$

Observation: The population appears to grow over time without bound.a The ratio $\frac{N_0(t+1)}{N_0(t)}$ and $\frac{N_1(t+1)}{N_1(t)}$

$$\frac{N_0(1)}{N_0(0)} = \frac{300}{100} = 3 \qquad \frac{N_0(2)}{N_0(1)} = \frac{644}{300} = 2.1467$$
$$\frac{N_0(3)}{N_0(2)} = \frac{1420}{300} = 2.2050 \qquad \frac{N_0(4)}{N_0(3)} = 2.1996$$

Apply the same process to N_1 and we can notice that they both approach 2.2, i.e.,

$$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix} \approx 2.2 \begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix}$$

The fraction of the population in age 0 and fraction of the population in age 0 is 1.

$$\frac{N_0(0)}{N_0(0)+N_1(0)} = \frac{100}{100+100} = \frac{1}{2} \qquad \frac{N_0(1)}{N_0(1)+N_1(1)} = \frac{300}{344} \approx 0.872$$
$$\frac{N_0(2)}{N_0(2)+N_1(2)} \approx 0.8407 \qquad \frac{N_0(3)}{N_0(3)+N_1(3)} \approx 0.8336 \qquad \dots$$

With these calculations, we can see that

$$\frac{N_0(t)}{N_0(t)+N_1(t)} \to 0.833 \implies \frac{N_1(t)}{N_0(t)+N_1(t)} \to 0.167$$

So

$$\frac{1}{\|\vec{N}(t)\|}\begin{bmatrix}N_0(t)\\N_1(t)\end{bmatrix} \to \begin{bmatrix}0.833\\0.167\end{bmatrix}$$

Recall that

$$\begin{bmatrix}N_0(t+1)\\N_1(t+1)\end{bmatrix} = L\begin{bmatrix}N_0(t)\\N_1(t)\end{bmatrix} \approx 2.2\begin{bmatrix}N_0(t)\\N_1(t)\end{bmatrix}$$

Claim 6.1. 2.2 is one eigenvalue of the Leslie matrix L.

Guess: $\begin{bmatrix}0.833\\0.167\end{bmatrix}$ is an eigenvector of the Leslie matrix L. Let's check.

$$\begin{aligned}\det(L-\lambda I) &= \det\left(\begin{bmatrix}2 & 1\\0.44 & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right)\\ &= (2-\lambda)(-\lambda) - 0.44\\ &= (\lambda - 2.2)(\lambda + 0.2)\end{aligned}$$

Thus, $\lambda = 2.2$, $\lambda = -0.2$ which verifies our claim. When $\lambda = 2.2$, we can find the corresponding eigenvector as follows

$$\begin{aligned}L - 2.2I &= \begin{bmatrix}2 & 1\\0.44 & 0\end{bmatrix} - \begin{bmatrix}2.2 & 0\\0 & 2.2\end{bmatrix}\\ &= \begin{bmatrix}-0.2 & 1\\0.44 & -2.2\end{bmatrix}\end{aligned}$$

We need to find the null space of $L - 2.2I$, i.e.

$$\begin{bmatrix}-0.2 & 1\\0.44 & -2.2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

which is

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}5x_2\\x_2\end{bmatrix} = 6x_2\begin{bmatrix}\frac{5}{6}\\\frac{1}{6}\end{bmatrix}$$

Thus, $\begin{bmatrix}\frac{5}{6}\\\frac{1}{6}\end{bmatrix} \approx \begin{bmatrix}0.833\\0.167\end{bmatrix}$ is the corresponding eigenvector (of 2.2).
From this example, we may guess in order to find the stable age distribution, we need to find the maximum eigenvalue of the Leslie matrix and then find the corresponding normalized eigenvector. Now, we will try to check our guess for the general Leslie model.

$$\vec{N}(t+\Delta t) = L\vec{N}(t)$$

with

$$\vec{N}(t) = \begin{bmatrix}N_0(t)\\N_1(t)\\\vdots\\\vec{N}_M(t)\end{bmatrix} \quad \text{and} \quad L \in \mathbb{R}^{(M+1)\times(M+1)}$$

being a non-negative. Let's assume that $\vec{N}(0) = \vec{N}_0$, then we have $\vec{N}(n\cdot\Delta t) = L\vec{N}\left((n-1)\cdot\Delta t\right) = \ldots = L^n \cdot \vec{N}_0$. Suppose that the Leslie matrix L is diagonalizable, i.e., there are $M+1$ eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{M+1}$ and $M+1$ linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_{M+1}$.

§7 Lec 7: Oct 7, 2021

§7.1 Stable Age Distribution (Cont'd)

Assume that $\vec{N}(0) = \vec{N}_0$, then we have $\vec{N}(n \cdot \Delta t) = L\vec{N}\left((n-1)\cdot\Delta t\right) = \ldots = L^n\vec{N}_0$. Suppose that the Leslie matrix L is diagonalizable, i.e., there are $M+1$ eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{M+1}$ and $M+1$ linearly indep. eigenvectors $\vec{v}_1, \ldots, \vec{v}_{M+1}$.

$$L = VDV^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{M+1} \end{bmatrix}, \quad V = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_{M+1} \end{bmatrix}$$

Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{M+1}$ are linearly independent, $\{\vec{v}_1, \ldots, \vec{v}_{M+1}\}$ is a basis for $\mathbb{R}^{M+1}$. Then, there exists $c_1, c_2, \ldots, c_{M+1}$ s.t.

$$\vec{N}_0 = \sum_{i=1}^{M+1} c_i\vec{v}_i$$

Thus,

$$\begin{aligned} \vec{N}(n\cdot\Delta t) &= L^n\vec{N}_0 \\ &= L^n\left(\sum_{i=1}^{M+1} c_i\vec{v}_i\right) \\ &= \sum_{i=1}^{M+1} c_i(L^n\vec{v}_i) \\ &= \sum_{i=1}^{M+1} c_i\lambda_i^n\vec{v}_i \\ &= c_1\vec{v}_1 + \sum_{i=2}^{M+1} c_i\left(\frac{\lambda_i}{\lambda_1}\right)^n\vec{v}_i \end{aligned}$$

If $|\lambda_1| > |\lambda_i|$ for $i \geq 2$, then $\frac{|\lambda_i|}{|\lambda_1|} < 1$ which means

$$\left|\frac{\lambda_i}{\lambda_1}\right|^n \to 0 \text{ as } n \to \infty \text{ for } i \geq 2$$

Therefore, we have

$$\frac{1}{\lambda_1^n}\vec{N}(n\cdot\Delta t) = c_1\vec{v}_1 + \sum_{i=2}^{M+1} c_i\left(\frac{\lambda_i}{\lambda_1}\right)^n\vec{v}_i \to c_1\vec{v}_1$$

as $n \to \infty$. Thus, for large value of n, we can approximate $\vec{N}(n\cdot\Delta t)$ by $c_1\lambda_1^n\vec{v}_1$.
The process to find "stable age distribution":

1. Find the maximum eigenvalue of the Leslie matrix L

$$\det(L - \lambda I) = 0$$

2. $|\lambda_1| > |\lambda_i|$
3. Find one corresponding eigenvector $\vec{v}_i$ associated to λ_1
4. Normalize $\vec{v}_1$: $\frac{\vec{v}_1}{\|\vec{v}_1\|}$

§7.2 Logistic Equations with Phase Plane Solution

Definition 7.1 (Phase Plane) — A phase plane is a visual display of certain characteristics of certain kinds of differential equations. A coordinate plane with axes being the values of two variables.

Logistic Equation:

$$\frac{dN}{dt} = N \cdot (a - bN)$$

Notice that this is an autonomous differential equation. One important thing for autonomous DE is the stability of the equilibrium points.

$$N(a - bN) = 0 \implies N = 0, \quad N = \frac{a}{b}$$

We can observe that the equilibrium point $N(t) = \frac{a}{b}$ is stable and $N(t) = 0$ is unstable. Now, let's show the stability of equilibrium points from an analytical aspect. We will first analyze the solution in the neighborhood of $N = \frac{a}{b}$. Let's consider the Taylor's expansion of $f(N) = N(a - bN)$ at $N = \frac{a}{b}$.

$$\begin{aligned}
f(N) &= N \cdot (a - bN) \\
&= f\left(\frac{a}{b}\right) + \frac{d}{dN} f(N)\Big|_{N=\frac{a}{b}(N-\frac{a}{b})} + \frac{d^2 f(N)}{dN^2}\Big|_{N=\frac{a}{b}\cdot\frac{1}{2}(N-\frac{a}{b})^2} \\
&= 0 + (-a)\left(N - \frac{a}{b}\right) + (-b)\left(N - \frac{a}{b}\right)^2 \\
&\approx -a \cdot \left(N - \frac{a}{b}\right)
\end{aligned}$$

Therefore,

$$\frac{dN}{dt} = N \cdot (a - bN) \approx (-a)\left(N - \frac{a}{b}\right)$$

near the neighborhood of $N = \frac{a}{b}$.

$$\frac{dN}{dt} = -a\left(N - \frac{a}{b}\right)$$

Let $y = N - \frac{a}{b} \implies \frac{dy}{dt} = \frac{dN}{dt}$

$$\begin{aligned}
\frac{dy}{dt} &= -ay \implies y = Ce^{-at} \\
N - \frac{a}{b} &= Ce^{-at} \\
N(t) &= \frac{a}{b} + Ce^{-at}
\end{aligned}$$

as $t \to \infty$, we have $N(t) \to \frac{a}{b}$. Thus, $N(t) = \frac{a}{b}$ is stable.

§8 Lec 8: Oct 11, 2021

§8.1 Logistic Equation with Phase Plane Solution (Cont'd)

We'd like to illustrate $N(t) = \frac{a}{b}$ is stable from perturbation analysis point of view. Let $N(t) = \frac{a}{b} + \varepsilon \cdot N_1(t)$ by assuming that

$$|\varepsilon N_1(t)| \ll \frac{a}{b}$$

Let's substitute $N(t) = \frac{a}{b} + \varepsilon N_1(t)$ into the original DE:

$$\begin{aligned}
\frac{dN}{dt} &= N\left(a - bN\right) \\
\frac{d}{dt}\left(\frac{a}{b} + \varepsilon N_1(t)\right) &= \varepsilon \frac{d}{dt} N_1(t) \\
&= \left(\frac{a}{b} + \varepsilon N_1(t)\left(a - \left(a + \varepsilon b N_1(t)\right)\right)\right) \\
&= -\frac{a}{b}\varepsilon b N_1(t) - \varepsilon^2 b N_1^2(t) \\
&= -a\varepsilon N_1(t) - \varepsilon^2 b N_1^2(t) \\
\frac{d}{dt} N_1(t) &= -aN_1(t) - \varepsilon b N_1^2(t) \\
&\approx -aN_1(t)
\end{aligned}$$

Thus, $N_1(t) = Ce^{-at} \to 0$ as $t \to \infty$ and $N(t) \to \frac{a}{b}$ as $t \to \infty$. So, $N(t) = \frac{a}{b}$ is stable.

§8.2 SIR Model

The SIR model was first used by Kermack and McKendrick in 1947. Now this model is popularly used to study the spread of infectious disease such as measles, Covid 19, etc. It consists of three parts:

- S: the number of susceptible individuals
- I: the number of infected individuals
- R: the number of recovered individuals

The process of the spread of the infectious disease is at the beginning where all the individuals are susceptible. The some of them become infectious and then become recovered individuals.

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

We assume that the total population

$$N = S + I + R$$

is fixed. Let β be the contact rate (individuals who come into contact with each other). Let γ be the recovery rate for the infected individuals.

§9 Lec 9: Oct 13, 2021

§9.1 SIR Model (Cont'd)

SIR model without vital dynamics

- We assume that the course of the infection is short.
- The birth and death can be ignored.
- The total number N can be treated as a constant.

Observation: The more interactions between the people in S and I the more individuals in S will "transfer" to I.

$$\frac{dS}{dt} = -\beta \cdot S \cdot I/N \tag{1}$$

The change of I will involve two parts: $S \to I$ which will increase I, and $I \to R$ which will decrees I

$$\frac{dI}{dt} = \beta \cdot S \cdot I/n - \gamma \cdot I \tag{2}$$

$$\frac{dR}{dt} = \gamma I \tag{3}$$

Let's combine the three equations.

$$\begin{cases} \frac{dS}{dt} = \frac{-\beta SI}{N} \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

with $S + I + R = N$ being a constant. Thus, to understand the model, we only need to understand

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \end{cases}$$

Let's normalize S, I, R first by setting

$$s = \frac{S}{N}, \quad i = \frac{I}{n}, \quad r = \frac{R}{N}$$

$$\frac{ds}{dt} = \frac{1}{N}\frac{dS}{dt} = \frac{1}{N}\left(\frac{-\beta SI}{N}\right) = -\beta si$$

$$\frac{di}{dt} = \frac{1}{N}\frac{dI}{dt} = \frac{1}{N}\left(\frac{\beta SI}{N} - \gamma I\right) = \beta si - \gamma i$$

and we know $r = 1 - i - s$.

> **Remark 9.1.** $s \in [0,1]$, $i \in [0,1]$, $r \in [0,1]$.

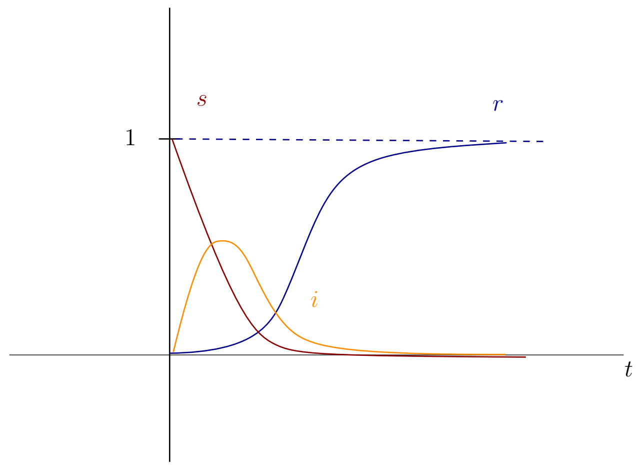
Next, let's analyze the new model

$$\begin{cases} \frac{ds}{dt} = -\beta si \\ \frac{di}{dt} = \beta si - \gamma i = (\beta s - \gamma) i \end{cases}$$

Observe that

1. $\frac{ds}{dt} = -\beta si \le 0 \implies s \downarrow$
2. $\frac{di}{dt} = (\beta s - \gamma)i = 0 \implies i = 0,\ s = \frac{\gamma}{\beta}$. When $\frac{di}{dt} > 0$, we know that $s > \frac{\gamma}{\beta}$. Similarly, when $\frac{di}{dt} < 0$, $s < \frac{\gamma}{\beta}$.

Let's draw the graph for s, i, r together.

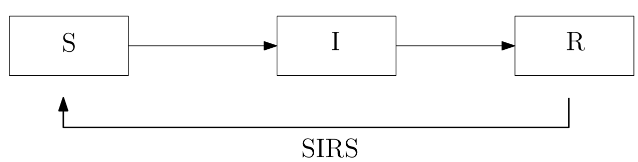


SIR Model with Vital Dynamics:
For this model, the disease will last for a long period. It is not reasonable to ignore the birth and death rate. It is not a reasonable assumption that $S + I + R = N$ where N is a constant. For this case, let's introduce new parameters birth rate b and death rate d.

$$\frac{dS}{dt} = \frac{-\beta SI}{N} + bN - dS$$
$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - dI$$
$$\frac{dR}{dt} = \gamma I - dR$$

§9.2 SIRS Model

SIRS Model without Vital Dynamics:



$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \\ \frac{dR}{dt} = \gamma I - \alpha R \end{cases} \qquad \text{and } S + I + R = N \text{ fixed}$$

SIRS with Vital Dynamics: Similar to SIR with vital dynamics, we need to take the birth and death rate into account.

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R + bN - dS \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - dI \\ \frac{dR}{dt} = \gamma I - \alpha R - dR \end{cases} \qquad \text{and } N(t) = S + I + R \text{ not fixed}$$

Intro to Two-Species Models: There are several different relations: competition, predator and prey, symbiosis, mutualism.

§10 Lec 10: Oct 15, 2021

§10.1 Solutions to System of Differential Equations

Theorem 10.1

If $(\lambda, \vec{v})$ is an eigen pair of M, then $e^{\lambda t}\vec{v}$ is a solution of $\frac{d\vec{y}(t)}{dt} = M\vec{y}(t)$.

Proof. Set $\vec{y}(t) = e^{\lambda t}\vec{v}$. Then we have

$$\frac{d}{dt}\vec{y}(t) = \frac{d}{dt}\left(e^{\lambda t}\vec{v}\right) = \left(\frac{d}{dt}e^{\lambda t}\right)\vec{v} = \lambda e^{\lambda t}\vec{v} \tag{1}$$

and

$$\begin{aligned} M\vec{y}(t) &= M\left(e^{\lambda t}\vec{v}\right) \\ &= e^{\lambda t}M\vec{v} \\ &= e^{\lambda t}(\lambda\vec{v}) \\ &= \lambda e^{\lambda t}\vec{v} \end{aligned} \tag{2}$$

Combining (1) and (2) we have $\vec{y}(t) = e^{\lambda t}\vec{v}$ is a solution of $\frac{d}{dt}\vec{y}(t) = M\vec{y}(t)$. $\square$

From the above theorem, we could find n solutions $e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n$.

Question 10.1. Are these n solutions linearly independent?

If $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ where $c_i = 0$ in which $i = 1, \ldots, n$, then $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent. Know: $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ and $M\vec{v}_i = \lambda_i\vec{v}_i$. We want to show $c_i = 0$ for all i. Let's use mathematical induction to show this.

- When $n = 1$, $c_1\vec{v}_1 = \vec{0} \implies c_1 = 0$ because $\vec{v}_1 \neq 0$
- Assume that the statement is correct when $n = k$.
- We want to show now that the statement also applies for the case $n = k + 1$. Have

$$\sum_{i=1}^{k+1} c_i M\vec{v}_i = \sum_{i=1}^{k+1} c_i\lambda_i\vec{v}_i = \vec{0} \tag{3}$$

Idea: get rid of one term so that we could use the induction assumption.

$$\sum_{i=1}^{k+1} c_i\vec{v}_i = \vec{0} \implies \sum_{i=1}^{k+1} c_i\lambda_{k+1}\vec{v}_i = \vec{0} \tag{4}$$

So (3) – (4),

$$\begin{gathered} \sum_{i=1}^{k} c_i\left(\lambda_i - \lambda_{k+1}\right)\vec{v}_i = \vec{0} \\ c_i\left(\lambda_i - \lambda_{k+1}\right) = 0 \end{gathered}$$

Thus, $c_i = 0$ since λ_i are distinct.

$$\sum_{i=1}^{k+1} c_i\vec{v}_i = c_{k+1}\vec{v}_{k+1} = \vec{0} \implies c_{k+1} = 0$$

Thus, the statement is true for $n = k + 1$.

Theorem 10.2

If M has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with the corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ then $\left\{e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n\right\}$ are linearly independent.

Proof. Left as exercise. □

Example 10.3

Solve the following ODE:

$$\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = x - 2y \end{cases}$$

Let's rewrite the ODE into the matrix vector form.

$$\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

Now, let's find the eigenvalues and the corresponding eigenvectors of M.

$$\begin{aligned} \det\left(M - \lambda I\right) &= \det \begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

So, $\lambda_{1,2} = \pm 1$.

- For $\lambda_1 = -1$,

$$\begin{aligned} (M+I)\vec{v}_1 &= \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \\ &= -\begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \vec{v}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

- For $\lambda = 1$, using the same process we obtain $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Therefore,

$$\vec{Y}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is the general solution for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$.

§11 Lec 11: Oct 18, 2021

§11.1 Solutions to System of Differential Equations (Cont'd)

Example 11.1 (Cont'd of the last example from last lecture)
Suppose that the initial conditions are $x(0) = 8$ and $y(0) = 4$. Find the explicit solution for the DE. Recall

$$\vec{Y}(t) = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{t}\begin{bmatrix}3\\1\end{bmatrix}$$

is the general solution. So,

$$\begin{aligned} c_1 e^{-0}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{0}\begin{bmatrix}3\\1\end{bmatrix} &= \begin{bmatrix}8\\4\end{bmatrix} \\ \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} &= \begin{bmatrix}8\\4\end{bmatrix} \\ \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}8\\4\end{bmatrix} &= \begin{bmatrix}2\\2\end{bmatrix} \end{aligned}$$

Question 11.1. If there are some complex eigenvalues for the real matrix M, how can we find the general real solutions for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$?

Example 11.2
Find the real solution for the ODE

$$\begin{cases} \frac{dx}{dt} = x(t) - y(t) \\ \frac{dy}{dt} = x(t) + y(t) \end{cases}$$

Notice that

$$\vec{Y}(t) = \begin{bmatrix}x(t)\\y(t)\end{bmatrix}, \quad M = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

First, let's find the eigenvalues and their corresponding eigenvectors of M.

$$\begin{aligned} \det(M - \lambda I) &= \det\left(\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) \\ &= \lambda^2 - 2\lambda + 2 = 0 \end{aligned}$$

So, $\lambda = 1 \pm i$.

- For $\lambda = 1 + i$, we have $\begin{bmatrix}i\\1\end{bmatrix}$ is a corresponding eigenvector.
- For $\lambda = 1 - i$, we have $\begin{bmatrix}-i\\1\end{bmatrix}$ is a corresponding eigenvector.

Thus,

$$\vec{Y}(t) = c_1 e^{(1+i)t}\begin{bmatrix}i\\1\end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix}-i\\1\end{bmatrix}$$

is the general solution for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$.

Question 11.2. How do we transform the general solution to general real solution?

Recall that

$$e^{ai} = \cos(a) + i\sin(a), \quad a \in \mathbb{R}$$

So,

$$\begin{aligned}
\vec{Y}(t) &= c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t e^{ti} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^t e^{-ti} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t \left(\cos(t) + i\sin(t)\right) \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^t \left(\cos(-t) + i\sin(-t)\right) \begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t \begin{bmatrix} \left(\cos(t) + i\sin(t)\right) i \\ \cos(t) + i\sin(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \left(\cos(-t) + i\sin(-t)\right)(-i) \\ \cos(-t) + i\sin(-t) \end{bmatrix} \\
&= c_1 e^t \begin{bmatrix} -\sin(t) + \cos(t)i \\ \cos(t) + \sin(t)i \end{bmatrix} + c_2 e^t \begin{bmatrix} -\sin(t) - \cos(t)i \\ \cos(t) - \sin(t)i \end{bmatrix} \\
&= (c_1 + c_2) e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + (c_1 - c_2) i e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}
\end{aligned}$$

Because c_1 and c_2 are arbitrary numbers we could choose $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$ or $c_1 + c_2 = 0$ and $(c_1 - c_2)i = 1$.

$$e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \quad e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

are two linearly independent real solutions of $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$. The general real solutions can be represented by

$$\vec{Y}(t) = \tilde{c_1} e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + \tilde{c_2} e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

where $\tilde{c_1}, \tilde{c_2} \in \mathbb{R}$.

Method II: Exponential Method

When $n = 1$, we have ODE

$$\frac{dx}{dt} = mx \implies x(t) = e^{mt} x_0$$

is the solution of $\frac{dx}{dt} = mx$. Recall that

$$\begin{aligned}
e^{mt} &= \sum_{j=0}^{\infty} \frac{(mt)^j}{j!} \\
e^{Mt} &= \sum_{j=0}^{\infty} \frac{(Mt)^j}{j!} = \sum_{j=1}^{\infty} \frac{t^j M^j}{j!}
\end{aligned}$$

To get a clearer look at e^{Mt}, let's consider the case that M is diagonal, e.g., $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\begin{aligned} e^{Mt} &= \sum_{j=0}^{\infty} \frac{t^j M^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j \begin{bmatrix} 2^j & 0 \\ 0 & 3^j \end{bmatrix}}{j!} \\ &= \begin{bmatrix} \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{(3t)^j}{j!} \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \end{aligned}$$

If M is diagonalizable, how can we compute e^{Mt}?

$$\begin{aligned} M &= JDJ^{-1} \\ e^{Mt} &= \sum_{j=0}^{\infty} \frac{t^j M_j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j (JDJ^{-1})^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j JD^j J^{-1}}{j!} \\ &= J \left(\sum_{j=0}^{\infty} \frac{t^j D^j}{j!} \right) J^{-1} \\ &= Je^{Dt} J^{-1} \end{aligned}$$

§12 Lec 12: Oct 22, 2021

§12.1 Asymptotic Properties of Solutions to Linear ODE System

Consider:

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

Then,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

So,

$$\begin{aligned} \det(M - \lambda I) &= \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \end{aligned}$$

Set $p = a + d$, $q = ad - bc$. Then,

$$\begin{aligned} \det(M - \lambda I) = \lambda^2 - p\lambda + q = 0 \\ \Delta = p^2 - 4q \end{aligned}$$

Thus the eigenvalues distribution of the matrix M are as follows

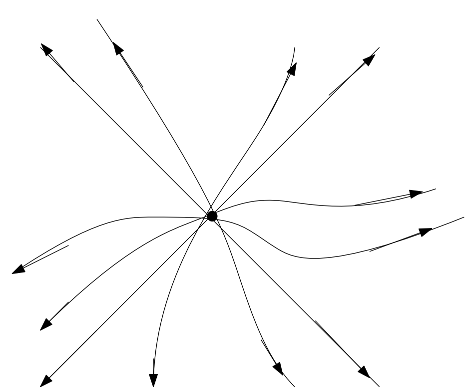
1. $\Delta > 0$, the eigenvalues are real and distinct (node or saddle)
2. $\Delta = 0$, repeated real eigenvalues (improper node)
3. $\Delta < 0$, the eigenvalues are complex (spiral)

First, let's consider the case where we have two real roots: $\Delta > 0$.

a) positive real roots $p > 0$, $q > 0$

$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Since $\lambda_1, \lambda_2 > 0 \implies e^{\lambda_1 t} -> \infty$



Example 12.1

Consider

$$M = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

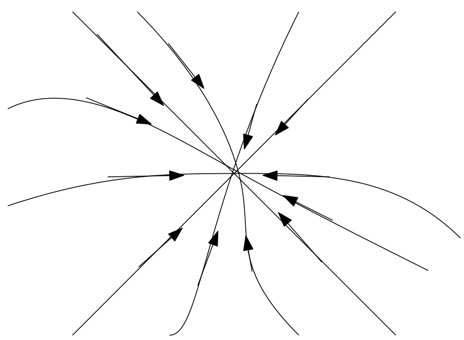
then

$$\det(M - \lambda I) = \lambda^2 - 6\lambda + 7 = 0$$
$$\lambda = 3 \pm \sqrt{2} > 0$$
$$\vec{Y}(t) = c_1 e^{(3+\sqrt{2})t} \begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix} + c_2 e^{(3-\sqrt{2})t} \begin{bmatrix} 1 \\ -\sqrt{2} - 1 \end{bmatrix}$$

b) Two negative real solutions: $p < 0$, $q > 0$.

$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

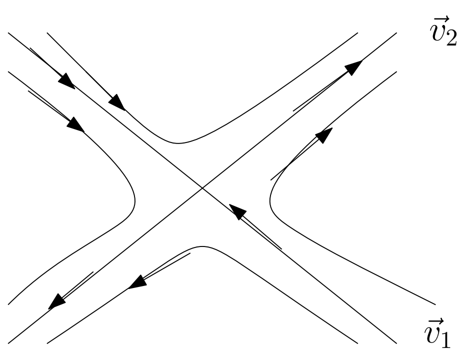
Since $\lambda_1, \lambda_2 < 0 \implies e^{\lambda_1 t} \to 0$, $e^{\lambda_2 t} \to 0$ as $t \to \infty$. So the equilibrium solution is stable.



c) $\lambda_1 < 0$ and $\lambda_2 > 0$ and so $q < 0$

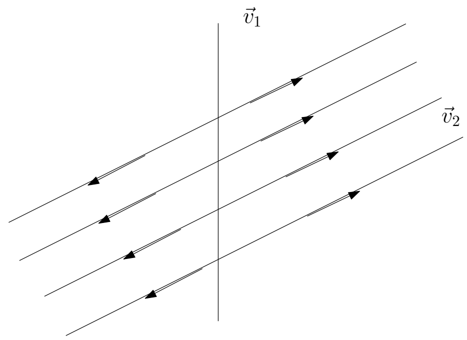
$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Since $\lambda_1 < 0 \implies e^{\lambda_1 t} \to 0$ as $t \to \infty$ and $\lambda_2 > 0 \implies e^{\lambda_2 t} \to \infty$ as $t \to \infty$.



d) One root is 0: $q = 0$ and another root is positive: $p > 0$. Let's assume that $\lambda_1 = 0$, $\lambda_2 > 0$

$$\vec{Y}(t) = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$



e) One root is 0: $q = 0$, and another root is negative: $p < 0$

$$\vec{Y}(t) = c_1\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$