Fractional-order SVIR-model with two infection classes

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Abstract

This paper describes the fractional-order transmission dynamics of the two variant of SARS-CoV-2, for delta and omicron variants are chosen. An *SVIR* model is proposed with two infection classes corresponding to Delta and Omicron variant. The equilibrium points of the model are determined and next-generation matrix method is used to calculate the corresponding basic reproduction number. The stability conditions of the proposed model is investigated around the equilibrium points. Numerical L1 scheme is used to study the memory effect of the virus variants in the suggested model. Further, numerical simulations are provided for a better insight of the proposed model.

Keywords: Caputo fractional-order derivative, COVID-19, Omicron, Next-generation matrix method, L1 scheme, Memory effect

1. Introduction

Infectious illnesses with pandemic potential have emerged and spread on a regular basis throughout the history. In the past, the world faced serious infectious diseases, like Spanish Flu, Smallpox, Tuberculosis and many more. In recent times, the world is facing novel corona-virus disease known as COVID-19. The disease appeared first in Wuhan city of China and subsequently elsewhere. The causing virus SARS-CoV-2 like other viruses changes its behavior over time due to mutations and combination of mutations. World Health Organization named new variant B.1.1.529 of virus SARS-CoV-2 as Omicron and designated

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it as Variant of Concern (VoC)[19]. The first case of this variant was reported in last week of November 2021 from South Africa [19]. Dr Angelique Coetzee described the variant infection symptoms as very mild compared to Delta variants. Further, the first case of Omicron variant in India was reported to union health ministry of India from Karnataka state on December 2, 2021.

One of the most important things in epidemic research is predicting future patterns, such as how many people will be infected every day, when epidemics turn endemic, and so on. Further, in particular case of COVID-19, the additional questions about the omicron or any new variant are as follows [20]:

- To what extent does Omicron or any new variant bypass vaccination induced immunity and natural immunity induced from recovery / the either variant.
- Several questions related to which of the two variant are more infectious and severe.

Background. In the literature, various dynamical methods are used to forecast epidemic patterns and to make government policies to limit the spread. Basic compartmental SIR model[1] has been widely used to describe the pattern of disease. From then, there are numerous integer-order models obtained by varying basic SIR model [1]. However, corona-virus, like all viruses, evolves throughout time. The majority of alterations have little or no impact on the virus's properties [2]. This kind of real phenomena are better represented by system of fractional-order equations due to its hereditary properties. Further, the fractional-order improves the consistency of the model with real data and observations as it has a degree of freedom to fit the real data as compared to integer-order models[3, 4].

Literature Survey. Many scholars from various fields have contributed to the prediction, study, and creation of crucial epidemic-fighting strategies. Grassly

[1]created by W.O. Kermack and A.G. McKendrick in 1927.

discussed the importance of developing models that can capture key features of the spread of an infection [5]. Yang *et al.* [1] discussed many integer-order models obtained by varying basic SIR model.

It is found that the fractional-order models are very effective to study the dynamics of diseases like COVID-19 as fractional-order derivatives depend not only on local conditions but also on the past and the history of the phenomenon studied [6]. Koziol *et al.* [7] presented the fractional-order generalization of the susceptible-infected-recovered (SIR) epidemic model for predicting the spread of the COVID-19 disease. The time-domain model implementation was based on the fixed-step method using the nabla fractional-order difference defined by Grünwald-Letnikov formula. Fatma *et al.* [2] proposed a fractional-order model of COVID-19 Omicron variant containing heart attack effect with real data from the United Kingdom. Niak *et al.* [8] investigated a non-linear fractional-order SIR epidemic model and applied L1 scheme in fractional-order disease model. Teka *et al.* [9] suggested a fractional-order model based on spiking activities of transmitting information in brain through neurons and used L1 scheme for its memory trace effects emerged from the past activities of neuro. Otunuga [10] in his paper proposed an epidemic SEIRS model with vital dynamics for COVID-19 and estimated its epidemiological parameters. Shah *et al.* [11] applied the optimal control theory in the COVID-19 fractional-order model to pretend the impact of various intervention strategies.

This paper is arranged in the following manner, section 2 introduces preliminaries definitions and results to be used in the paper. In section 3, model is formulated and its equilibrium points are calculated. In section 4, next-generation matrix method is used to calculate the basic reproduction number. Local stability around the equilibrium points is discussed in section 5. Memory trace and heredity trait is discussed in section 6 and numerical simulation is done in section 7.

2. Preliminaries

Definition 1. *Caputo's definition of fractional-order ϕ-derivative of the function f on the interval $(0,t)$ is defined by*[2]

$$ {}_0^C D_t^{\phi} f(t) = \frac{1}{\Gamma(n-\phi)} \int_0^t (t-s)^{n-\phi-1} f^{(n)}(s) ds $$

where $n = \lceil \phi \rceil$ is the least integer greater than or equals to ϕ and $\Gamma(x)$ denotes the gamma function.

As $\phi \in (0,1]$, in Caputo's sense , ϕ-derivative of function f can be written as:

$$ {}_0^C D_t^{\phi} f(t) = \frac{1}{\Gamma(1-\phi)} \int_0^t \frac{f'(s)}{(t-s)^{\phi}} ds $$

Definition 2. *The Laplace transform of the function f is defined as*

$$ \mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{\infty} f(t) e^{-st} dt $$

The Laplace transform of the Caputo fractional ϕ-derivative of the function f is defined as:

$$ \mathcal{L}\{{}_0^C D_t^{\phi} f(t)\}(s) = s^{\phi} L(f(t))(s) - \sum_{k=0}^{n-1} s^{\phi-k-1} f^{(k)}(0) $$

Definition 3. *Mittag-Leffler function $E_{\phi,\eta}$ is a complex function depending upon two parameters ϕ and η is defined as,*

$$ E_{\phi,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\phi k + \eta)} $$

For particular case of $\eta = 1$, Mittag-Leffler function E_{ϕ} in one parameter ϕ can be described as,

$$ E_{\phi}(z) = E_{\phi,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\phi k + 1)} \tag{1} $$

[2]It was introduced in 1967 by Italian mathematician Michele Caputo.

Some well known results based on Mittag-Leffler function and their Laplace transform are given below that will be used further in this paper.

$$\mathcal{L}\{t^{\eta-1}E_{\phi,\eta}(-\lambda t^{\phi})\}(s) = \frac{s^{\phi-\eta}}{s^{\phi}+\lambda} \tag{2}$$

where $\mathrm{res}(s) > |\lambda|^{1/\phi}$ and

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^{\phi}+\lambda)}\right\}(t) &= t^{\phi}E_{\phi,\phi+1}(-\lambda t^{\phi}) \\ &= \frac{1}{\lambda}\left[1-E_{\phi}(-\lambda t^{\phi})\right]\end{aligned} \tag{3}$$

where $\mathrm{res}(s) > |\lambda|^{1/\phi}$.

3. Formulation of Model

In this epidemic model, the total population denoted by $n(t)$ is divided into five compartments. s, v, i_0, i_δ, r where each compartment denotes the specific stage of the disease transmission. To understand the system and to reduce the number of variables, the new variables S, V, I_0, I_δ, R are introduced by dividing each compartment by total population $n(t)$ (see Table 1). The flow diagram of the model is given in Figure 1.

Notation	Meaning
S	Fraction of individuals which are vulnerable to get infection and are not vaccinated.
V	Fraction of individuals which are vulnerable to get infection and are vaccinated.
I_0	Fraction of individuals that are responsible for spreading Omicron variant among population.
I_δ	Fraction of individuals that are responsible for spreading Delta variant among population.
R	Fraction of individuals which have been recovered.

Table 1: Compartments used in proposed model

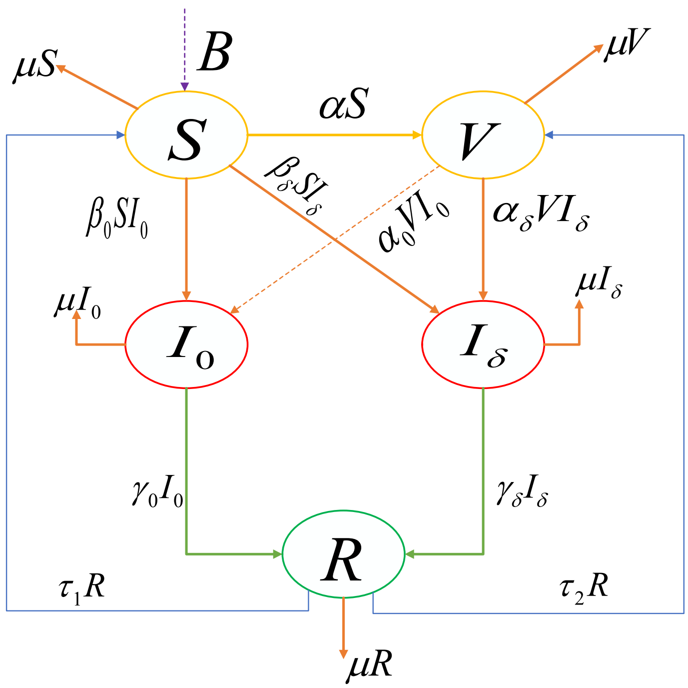


Figure 1: The flow diagram of the proposed model

Assumptions for proposed model :

(a) The infected population is homogeneously mixed with susceptible people.

(b) The recovered individuals may have temporary immunity but are vulnerable to catch the infection again.

(c) The vaccinated individuals are prone to both disease variants.

(d) The newly infected people are immediately contagious.

Parameter	Meaning	Value	Reference
B	Birth rate of population	0.000052705	Calculated
α	Vaccination rate	0.060925	Estimated
β_0	Transmission Rate of Non-Vaccinated getting Omicron	0.35	Estimated
β_δ	Transmission rate of Non-Vaccinated getting Delta Variant	0.35	Estimated
α_0	Transmission rate of Vaccinated getting Omicron Variant	0.25	Estimated
α_δ	Transmission rate of Vaccinated getting Delta Variant	0.25	Estimated
γ_0	Recovery Rate from Omicron Variant	0.14	Calculated
γ_δ	Recovery Rate from Delta Variant	0.14	Calculated
μ_0	Death rate due to omicron variant	0.005	Calculated
μ_δ	Death rate due to delta variant	0.0001	Calculated
μ	Death rate	0.000019135	Calculated
τ_1	Rate of recovered individuals moving to non-vaccinated susceptible	0.1	Estimated
τ_2	Rate of recovered individuals moving to vaccinated susceptible	0.9	Estimated

Table 2: Values and meaning of parameters used in the proposed model.

The governing system of the fractional-order non-linear differential equations which describes the proposed epidemic model is as follows:

$$\begin{aligned}
{}_0^C D_t^{\phi} S &= B - \beta_0 S I_0 - \beta_\delta S I_\delta - (\mu + \alpha) S + \tau_1 R \\
{}_0^C D_t^{\phi} V &= \alpha S - \mu V - \alpha_0 V I_0 - \alpha_\delta V I_\delta + \tau_2 R \\
{}_0^C D_t^{\phi} I_0 &= \beta_0 S I_0 + \alpha_0 V I_0 - (\mu + \mu_0 + \gamma_0) I_0 \\
{}_0^C D_t^{\phi} I_\delta &= \beta_\delta S I_\delta + \alpha_\delta V I_\delta - (\mu + \mu_\delta + \gamma_\delta) I_\delta \\
{}_0^C D_t^{\phi} R &= \gamma_0 I_0 + \gamma_\delta I_\delta - (\mu + \tau_1 + \tau_2) R
\end{aligned} \tag{4}$$

with non-negative initial conditions $S(0) = S_0$, $V(0) = V_0$, $I_0(0) = I_{00}$, $I_\delta(0) = I_{\delta 0}$, $R(0) = R_0$. **Note that:** The parameters used in the system (4) are non-negative.

Theorem 1. *The feasible region of the system (4) is given by*

$$\Omega = \left\{ (S, V, I_0, I_\delta, R) \in \mathbb{R}_+^5 \mid S + V + I_0 + I_\delta + R \leq N(0) + \frac{B}{\mu} \right\}$$

Proof. Let us assume that $N(t) = S(t) + V(t) + I_0(t) + I_\delta(t) + R(t)$. Adding all the equations in system (4),

$${}_0^C D_t^{\phi} N(t) = B - \mu N(t) - \mu_0 I_0(t) - \mu_\delta I_\delta(t) \leq B - \mu N(t) \tag{5}$$

Further, applying Laplace transform both sides into inequality (5),

$$s^{\phi} \mathcal{L}\{N(t)\} - s^{\phi-1} N(0) \leq \frac{B}{s} - \mu \mathcal{L}\{N(t)\}$$

On solving,

$$\mathcal{L}\{N(t)\} \leq \frac{B}{s(s^{\phi} + \mu)} + N(0) \frac{s^{\phi-1}}{s^{\phi} + \mu}$$

Taking inverse Laplace transform,

$$\begin{aligned}
N(t) &\leq B t^{\phi} E_{\phi,\phi+1}(-\mu t^{\phi}) + N(0) E_{\phi,1}(-\mu t^{\phi}) \\
&= \frac{B}{\mu}(1 - E_{\phi}(-\mu t^{\phi})) + N(0) E_{\phi}(-\mu t^{\phi})
\end{aligned}$$

Using equations [1] , [2] and [3]

Since $0 \leq E_{\phi}(-\mu t^{\phi}) \leq 1$, Thus,

$$S(t) + V(t) + I_0(t) + I_\delta(t) + R(t) = N(t) \leq \frac{B}{\mu} + N(0)$$

□

3.1. Existence and Uniqueness of the solution

Lemma 1. *[12] The fractional-order system ${}_0^C D_t^{\phi}(X(t)) = F(t, X(t))$ such that $X(0) = X_0$ has unique solution if following holds:*

(i) $F(t, X(t))$ and $(\partial F/\partial X)(X)$ are continuous functions.

(ii) $\|F(X)\| \leq K_1 + K_2\|X\|$ where K_1 and K_2 are positive constants.

Theorem 2. *The solution of fractional order system (4) exists and is unique.*

Proof. The system (4) of fractional order differential equations can be written as:

$$ {}_0^C D_t^{\phi}(X(t)) = F(X) = M_1 + M_2 X + S M_3 X + V M_4 X \tag{6} $$

where $X(t) = \begin{bmatrix} S(t) \\ V(t) \\ I_0(t) \\ I_\delta(t) \\ R(t) \end{bmatrix}$ and $M_1 = \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ are column vectors and rest M_2, M_3, M_4 are square matrices.

$$ M_2 = \begin{bmatrix} -(\mu+\alpha) & 0 & 0 & 0 & \tau_1 \\ \alpha & -\mu & 0 & 0 & \tau_2 \\ 0 & 0 & -(\mu+\mu_0+\gamma_0) & 0 & 0 \\ 0 & 0 & 0 & -(\mu+\mu_\delta+\gamma_\delta) & 0 \\ 0 & 0 & \gamma_0 & \gamma_\delta & -(\mu+\tau_1+\tau_2) \end{bmatrix}, $$

$$ M_3 = \begin{bmatrix} 0 & 0 & -\beta_0 & -\beta_\delta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & \beta_\delta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_0 & -\alpha_\delta & 0 \\ 0 & 0 & \alpha_0 & \alpha_\delta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} $$

Now, using norm both sides in equation (6)

$$
\begin{aligned}
\|F(X)\| &= \|M_1 + M_2X + SM_3X + VM_4X\| \\
&\leq \|M_1\| + \|M_2X\| + \|SM_3X\| + \|VM_4X\| \\
&\leq \|M_1\| + (\|M_2\| + \|SM_3\| + \|VM_4\|)\,\|X\| \\
&\leq K_1 + K_2\|X\|
\end{aligned}
$$

by properties of norm on $C^*[0,T]$

where $K_1 = \|M_1\|$ and $K_2 = (\|M_2\| + \|SM_3\| + \|VM_4\|)$ are positive constants. Hence, using lemma (1), system (4) has the unique solution. □

3.2. Equilibrium Points

In this section, the equilibrium points of the system (4) are evaluated. The points are the steady state solution of the system (4). There are two equilibrium points of the proposed system to be analyzed for the proposed model. The disease free equilibrium point E^0 is given by:

$$E^0 = \left(S = \frac{B}{\mu+\alpha}, V = \frac{B\alpha}{\mu(\mu+\alpha)}, I_\delta = 0, I_0 = 0, R = 0\right)$$

The endemic equilibrium point E^1 is given by

$$E^1 = (S^*, V^*, I_\delta^*, I_0^*, R^*)$$

where

$$
\begin{aligned}
S^* &= \frac{\alpha_\delta(\mu+\mu_0+\gamma_0) - \alpha_0(\mu+\mu_\delta+\gamma_\delta)}{\beta_0\alpha_\delta - \alpha_0\beta_\delta} \\
V^* &= \frac{\beta_0(\mu+\mu_\delta+\gamma_\delta) - \beta_\delta(\gamma_0+\mu+\mu_0)}{\beta_0\alpha_\delta - \alpha_0\beta_\delta}
\end{aligned}
$$

Further, $I_0^* = {}^{A_1}/_{A_4}, I_\delta^* = {}^{A_2}/_{A_4}, R^* = {}^{A_3}/_{A_4}$ where each A_1, A_2, A_3 and A_4 can be written in terms of S^*, V^* and the disease parameters as follows:

$$
\begin{aligned}
A_1 = &-\beta_\delta\alpha(\mu+\tau_1+\tau_2){S^*}^2 + ((\mu+\tau_1+\tau_2)((-\mu-\alpha)\alpha_\delta + \mu\beta_\delta)V^* \\
&+ \gamma_\delta(\mu\tau_2 + \alpha(\tau_1+\tau_2)))S^* + (B(\mu+\tau_1+\tau_2)\alpha_\delta - \gamma_\delta\mu\tau_1)V^* - B\tau_2\gamma_\delta
\end{aligned}
$$

$$
\begin{aligned}
A_2 = &\,\alpha\beta_0(\mu+\tau_1+\tau_2){S^*}^2 + (-(\mu+\tau_1+\tau_2)((-\mu-\alpha)\alpha_0 + \beta_0\mu)V^* \\
&- \gamma_0(\mu\tau_2 + \alpha(\tau_1+\tau_2)))S^* + (-B(\mu+\tau_1+\tau_2)\alpha_0 + \gamma_0\mu\tau_1)V^* + B\gamma_0\tau_2
\end{aligned}
$$

$$A_3 = -\alpha(-\beta_0\gamma_\delta + \gamma_0\beta_\delta){S^*}^2 + V^*(-\gamma_0(\mu+\alpha)\alpha_\delta + ((\mu+\alpha)\alpha_0 - \beta_0\mu)\gamma_\delta + \gamma_0\beta_\delta\mu)S^* + BV^*(-\alpha_0\gamma_\delta + \gamma_0\alpha_\delta)$$

$$A_4 = (-(\mu+\tau_1+\tau_2)(\alpha_0\beta_\delta - \beta_0\alpha_\delta)V^* + \tau_2(-\beta_0\gamma_\delta + \gamma_0\beta_\delta))S^* - V^*\tau_1(-\alpha_0\gamma_\delta + \gamma_0\alpha_\delta)$$

4. Calculation of basic reproduction number ($\mathsf{R_0}$)

The next-generation matrix method [10, 13] is used to calculate the basic reproduction number [14] of the system. Consider the vectors $\vec{x} = (I_0, I_\delta)^T$ and $\vec{y} = (S, V, R)^T$,

$$\frac{d\vec{x}}{dt} = f(\vec{x}, \vec{y}) + v(\vec{x}, \vec{y})$$

Let F and V be the corresponding Jacobian matrix at the disease free equilibrium point E^0.

$$F(E^0) = \frac{B}{\mu(\mu+\alpha)}\begin{bmatrix} \mu\beta_0 + \alpha\alpha_0 & 0 \\ 0 & \mu\beta_\delta + \alpha\alpha_\delta \end{bmatrix},\ V(E^0) = \begin{bmatrix} \gamma_0 + \mu + \mu_0 & 0 \\ 0 & \mu + \mu_\delta + \gamma_\delta \end{bmatrix}$$

The matrix V is non-singular. The basic reproduction number $\mathsf{R_0}$ is the spectral radius of the matrix FV^{-1}.

$$\mathsf{R_0} = \rho(FV^{-1}) = \sup\left[\frac{B(\alpha\alpha_0 + \beta_0\mu)}{\mu(\mu+\alpha)(\gamma_0+\mu+\mu_0)}, \frac{B(\alpha\alpha_\delta + \mu\beta_\delta)}{\mu(\mu+\alpha)(\mu+\mu_\delta+\gamma_\delta)}\right]$$

The basic reproduction number ($\mathsf{R_0}$) denotes the average number of secondary infections induced by an infected individual in a fully susceptible population. In the above context, first term in sup expression of $\mathsf{R_0}$ denotes the basic reproduction number corresponding to Omicron infection and the second term denotes reproduction-number for Delta variant.

5. Stability

In this section, the stability conditions of the model are investigated about equilibrium points.

5.1. Local Stability of the disease-free equilibrium E^0

Lemma 2. *[15, 16] Consider the fractional-order system*

$$ {}_0^C D_t^{\phi} x = Ax \, , x(0) = x_0 $$

where A is arbitirary matrix and $\phi \in (0, 1)$

- *The trivial solution is asymptotically stable if and only if all the eigenvalues λ_j of matrix A satisfy $|arg(\lambda_j)| > \phi\pi/2$ for $j = 1, 2, 3, \cdots, n$.*
- *The trivial solution is stable if and only if all the eigenvalues of matrix A satisfy $|arg(\lambda_j)| \geq \phi\pi/2$ and eigenvalues with $|arg(\lambda_j)| = \phi\pi/2$ have same algebraic and geometric multiplicity for $j = 1, 2, 3, \cdots, n$.*

Theorem 3. *The disease-free equilibrium point E^0 of the system (4) is locally asymptotically stable if $\mathsf{R}_0 < 1$ and unstable if $\mathsf{R}_0 > 1$.*

Proof. The Jacobian of the system (4) is given by

$$ J = \begin{pmatrix} -\beta_0 I_0 - \beta_\delta I_\delta - \alpha - \mu & 0 & -\beta_0 S & -\beta_\delta S & \tau_1 \\ \alpha & -\alpha_0 I_0 - \alpha_\delta I_\delta - \mu & -\alpha_0 V & -\alpha_\delta V & \tau_2 \\ \beta_0 I_0 & \alpha_0 I_0 & \beta_0 S + \alpha_0 V - \gamma_0 - \mu - \mu_0 & 0 & 0 \\ \beta_\delta I_\delta & \alpha_\delta I_\delta & 0 & \beta_\delta S + \alpha_\delta V - \mu - \gamma_\delta - \mu_\delta & 0 \\ 0 & 0 & \gamma_0 & \gamma_\delta & -\mu - \tau_1 - \tau_2 \end{pmatrix}. $$

At disease-free equilibrium point E^0,

$$ J(E^0) = \begin{pmatrix} -\alpha - \mu & 0 & -\frac{\beta_0 B}{\mu+\alpha} & -\frac{B\beta_\delta}{\mu+\alpha} & \tau_1 \\ \alpha & -\mu & -\frac{\alpha_0 B\alpha}{\mu(\mu+\alpha)} & -\frac{\alpha_\delta B\alpha}{\mu(\mu+\alpha)} & \tau_2 \\ 0 & 0 & \frac{\beta_0 B}{\mu+\alpha} + \frac{B\alpha\alpha_0}{\mu(\mu+\alpha)} - \gamma_0 - \mu - \mu_0 & 0 & 0 \\ 0 & 0 & 0 & \frac{B\beta_\delta}{\mu+\alpha} + \frac{B\alpha\alpha_\delta}{\mu(\mu+\alpha)} - \mu - \gamma_\delta - \mu_\delta & 0 \\ 0 & 0 & \gamma_0 & \gamma_\delta & -\mu - \tau_1 - \tau_2 \end{pmatrix} $$

The eigenvalues of the matrix $J(E^0)$ are as follows,

$$\begin{aligned}
\lambda_1 &= -\mu \\
\lambda_2 &= -\mu - \tau_1 - \tau_2 \\
\lambda_3 &= -\mu - \alpha \\
\lambda_4 &= \frac{B\alpha\alpha_\delta + B\mu\beta_\delta - \alpha\mu^2 - \alpha\mu\gamma_\delta - \alpha\mu\mu_\delta - \mu^3 - \mu^2\gamma_\delta - \mu^2\mu_\delta}{\mu(\mu+\alpha)} \\
\lambda_5 &= \frac{B\alpha\alpha_0 + B\mu\beta_0 - \alpha\mu^2 - \alpha\mu\gamma_0 - \alpha\mu\mu_0 - \mu^3 - \mu^2\gamma_0 - \mu^2\mu_0}{\mu(\mu+\alpha)}
\end{aligned}$$

It can be obsevrved that $\lambda_1\,, \lambda_2\,, \lambda_3$ are negative and

$$\lambda_4 < 0 \iff \frac{B(\alpha\alpha_\delta + \mu\beta_\delta)}{\mu(\mu+\alpha)(\mu+\mu_\delta+\gamma_\delta)} < 1$$

Similarly,

$$\lambda_5 < 0 \iff \frac{B(\alpha\alpha_0 + \beta_0\mu)}{\mu(\mu+\alpha)(\gamma_0+\mu+\mu_0)} < 1$$

Consequently,

$$\mathsf{R_0} = \sup\left[\frac{B(\alpha\alpha_0 + \beta_0\mu)}{\mu(\mu+\alpha)(\gamma_0+\mu+\mu_0)}, \frac{B(\alpha\alpha_\delta + \mu\beta_\delta)}{\mu(\mu+\alpha)(\mu+\mu_\delta+\gamma_\delta)}\right] < 1 \iff \lambda_4 \text{ and } \lambda_5 \text{ are negative.}$$

Therefore, all eigenvalues of matrix $J(E^0)$ have negative real part if and only if $\mathsf{R_0} < 1$. Thus, whenever $\mathsf{R_0} < 1$, argument of each eigenvalue has absolute value π.

$$|arg(\lambda_j)| = \pi > \frac{\phi\pi}{2} \qquad\qquad \text{for each } j = 1,2,3,4,5.$$

Hence, the disease-free equillibirium point E^0 of the system (4) is locally asymptotically stable if and only if $\mathsf{R_0} < 1$.
Whenever $\mathsf{R_0} > 1$, the eigenvalues λ_4 and λ_5 are positive. So, disease-free equilibrium point E^0 of the system (4) is unstable.

□

Next, the stability conditions of the endemic equilibrium point are observed.

5.2. Local stability of the endemic equilibrium point E^1

Theorem 4. *The endemic point E^1 of the system (4) is locally asymptotically stable if following condition holds,*

$$
\begin{aligned}
&(\beta_0+\beta_\delta)S^* + (\alpha_0+\alpha_\delta)V^* < (\alpha_0+\beta_0)I_0^* + (\alpha_\delta+\beta_\delta)I_\delta^* + \alpha + 4\mu + \mu_\delta \\
&+ \mu_0 + \gamma_\delta + \gamma_0
\end{aligned}
$$

Proof. The Jacobian of system (4) at endemic equilibrium point E^1 is given by

$$
J^* = \begin{pmatrix}
-\beta_0 I_0^* - \beta_\delta I_\delta^* - \alpha - \mu & 0 & -\beta_0 S^* & -\beta_\delta S^* & \tau_1 \\
\alpha & -\alpha_0 I_0^* - \alpha_\delta I_\delta^* - \mu & -\alpha_0 V^* & -\alpha_\delta V^* & \tau_2 \\
\beta_0 I_0^* & \alpha_0 I_0^* & \beta_0 S^* + \alpha_0 V^* - \gamma_0 - \mu - \mu_0 & 0 & 0 \\
\beta_\delta I_\delta^* & \alpha_\delta I_\delta^* & 0 & \beta_\delta S^* + \alpha_\delta V^* - \mu - \gamma_\delta - \mu_\delta & 0 \\
0 & 0 & \gamma_0 & \gamma_\delta & -\mu - \tau_1 - \tau_2
\end{pmatrix}.
$$

The characteristic polynomial of matrix J^* is as follows,

$$
P(\lambda) = (\lambda + \mu + \tau_1 + \tau_2)(\lambda^4 + k_3\lambda^3 + k_2\lambda^2 + k_1\lambda + k_0). \tag{7}
$$

where the coefficients k_0, k_1, k_2, k_3 are defined as,

$$
k_3 = (\alpha_0+\beta_0)I_0^* - (\beta_0+\beta_\delta)S^* - (\alpha_0+\alpha_\delta)V +^* (\alpha_\delta+\beta_\delta)I_\delta^* + \alpha + \gamma_0 + 4\mu + \mu_0 + \gamma_\delta + \mu_\delta
$$

$$
\begin{aligned}
k_2 = {} & 6\mu^2 + 3\mu((\beta_\delta+\alpha_\delta)I_\delta^* + (\beta_0+\alpha_0)I_0^* - (\beta_\delta+\beta_0)S^* - (\alpha_\delta V^* + \alpha_0)V^* + (\mu_\delta+\gamma_\delta) + (\mu_0+\alpha+\gamma_0)) + \\
& + {I_\delta^*}^2 \beta_\delta\alpha_\delta + ((\alpha_0\beta_\delta + \beta_0\alpha_\delta)I_0^* + (-\alpha_0 V^* + (-S^* - V^*)\beta_\delta - \beta_0 S^* + \mu_\delta + \gamma_\delta + \mu_0 + \alpha + \gamma_0)\alpha_\delta \\
& - \beta_\delta(\beta_0 S^* + \alpha_0 V^* - \gamma_0 - \mu_0 - \gamma_\delta - \mu_\delta))I_\delta^* + {I_0^*}^2 \alpha_0\beta_0 + (-V^*(\beta_0+\alpha_0)\alpha_\delta + \\
& (-\beta_\delta S^* + (-S^* - V^*)\beta_0 + \mu_\delta + \gamma_\delta + \mu_0 + \alpha + \gamma_0)\alpha_0 - \beta_0(\beta_\delta S^* - \gamma_0 - \mu_0 - \gamma_\delta - \mu_\delta))I_0^* \\
& + V^*(\beta_0 S^* + \alpha_0 V^* - \alpha - \gamma_0 - \mu_0)\alpha_\delta + V^*(\beta_\delta S^* - \alpha - \gamma_\delta - \mu_\delta)\alpha_0 + \\
& S^*(\beta_0 S^* - \alpha - \gamma_0 - \mu_0)\beta_\delta - S^*(\mu_\delta + \gamma_\delta + \alpha)\beta_0 + (\mu_\delta + \gamma_\delta + \mu_0 + \gamma_0)\alpha + (\mu_0 + \gamma_0)(\mu_\delta + \gamma_\delta)
\end{aligned}
$$

$$k_1 = 4\mu^3+((3\beta_\delta+3\alpha_\delta)I_\delta^*+(3\beta_0+3\alpha_0)I_0^*-3\beta_\delta S^*-3\beta_0 S^*-3\alpha_\delta V^*-3\alpha_0 V^*+3\mu_\delta+3\gamma_\delta+3\mu_0+3\alpha+3\gamma_0)\mu^2$$
$$+(2{I_\delta^*}^2\beta_\delta\alpha_\delta+((2\alpha_0\beta_\delta+2\beta_0\alpha_\delta)I_0^*+(-2\alpha_0 V^*+(-2S^*-2V^*)\beta_\delta-2\beta_0 S^*+2\mu_\delta+2\gamma_\delta+2\mu_0+2\alpha+2\gamma_0)\alpha_\delta$$
$$-2\beta_\delta(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_\delta^*+2{I_0^*}^2\alpha_0\beta_0+(-2V^*(\beta_0+\alpha_0)\alpha_\delta+(-2\beta_\delta S^*+(-2S^*-2V^*)\beta_0$$
$$+2\mu_\delta+2\gamma_\delta+2\mu_0+2\alpha+2\gamma_0)\alpha_0-2\beta_0(\beta_\delta S^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_0^*+2V^*(\beta_0 S^*+\alpha_0 V^*-\alpha-\gamma_0-\mu_0)\alpha_\delta+$$
$$+2V^*(\beta_\delta S^*-\alpha-\gamma_\delta-\mu_\delta)\alpha_0+2S^*(\beta_0 S^*-\alpha-\gamma_0-\mu_0)\beta_\delta-2S^*(\mu_\delta+\gamma_\delta+\alpha)\beta_0+(2\mu_\delta+2\gamma_\delta+2\mu_0+2\gamma_0)\alpha$$
$$+2(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\mu-\beta_\delta\alpha_\delta(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta){I_\delta^*}^2+(((-V^*(\beta_\delta+\beta_0)\alpha_0-\beta_0(\beta_\delta S^*$$
$$-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))\alpha_\delta-\beta_\delta\alpha_0(\beta_0 S^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_0^*+(((S^*+V^*)\beta_\delta-\mu_\delta-\gamma_\delta-\alpha)V^*\alpha_0+$$
$$(\beta_0 S^*-\gamma_0-\mu_0)(S^*+V^*)\beta_\delta-S^*(\mu_\delta+\gamma_\delta+\alpha)\beta_0+(\mu_\delta+\gamma_\delta+\mu_0+\gamma_0)\alpha+(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\alpha_\delta-\beta_\delta$$
$$(\mu_\delta+\gamma_\delta)(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0))I_\delta^*-\beta_0\alpha_0(\beta_\delta S^*+\alpha_\delta V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta){I_0^*}^2+((((S^*+V^*)$$
$$\beta_0-\mu_0-\alpha-\gamma_0)\alpha_0-\beta_0(\mu_0+\gamma_0))V^*\alpha_\delta+(((S^*+V^*)\beta_0-\mu_0-\alpha-\gamma_0)S^*\beta_\delta-(\mu_\delta+\gamma_\delta)(S^*+V^*)\beta_0$$
$$+(\mu_\delta+\gamma_\delta+\mu_0+\gamma_0)\alpha+(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\alpha_0-\beta_0(\mu_0+\gamma_0)(\beta_\delta S^*-\gamma_\delta-\mu_\delta))I_0^*+$$
$$+\alpha(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0)(\beta_\delta S^*+\alpha_\delta V^*-\gamma_\delta-\mu_\delta)$$

$$
\begin{aligned}
k_0 = {} & \mu^4+((\beta_\delta+\alpha_\delta)I_\delta^*+(\beta_0+\alpha_0)I_0^*-\beta_\delta S^*-\beta_0 S^*-\alpha_\delta V^*-\alpha_0 V^*+\mu_\delta+\gamma_\delta+\mu_0+\alpha+\gamma_0)\mu^3 \\
& +(I_\delta^{*2}\beta_\delta\alpha_\delta+((\alpha_0\beta_\delta+\beta_0\alpha_\delta)I_0^*+(-\alpha_0 V^*+(-S^*-V^*)\beta_\delta-\beta_0 S^*+\mu_\delta+\gamma_\delta+\mu_0+\alpha+\gamma_0)\alpha_\delta \\
& -\beta_\delta(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_\delta^*+I_0^{*2}\alpha_0\beta_0+(-V^*(\beta_0+\alpha_0)\alpha_\delta+(-\beta_\delta S^*+(-S^*-V^*)\beta_0 \\
& +\mu_\delta+\gamma_\delta+\mu_0+\alpha+\gamma_0)\alpha_0-\beta_0(\beta_\delta S^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_0^*+V^*(\beta_0 S^*+\alpha_0 V^*-\alpha-\gamma_0-\mu_0)\alpha_\delta \\
& +V^*(\beta_\delta S^*-\alpha-\gamma_\delta-\mu_\delta)\alpha_0+S^*(\beta_0 S^*-\alpha-\gamma_0-\mu_0)\beta_\delta-S^*(\mu_\delta+\gamma_\delta+\alpha)\beta_0+(\mu_\delta+\gamma_\delta+\mu_0+\gamma_0)\alpha \\
& +(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\mu^2+(-\beta_\delta\alpha_\delta(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta)I_\delta^{*2}+(((-V^*(\beta_\delta+\beta_0)\alpha_0 \\
& -\beta_0(\beta_\delta S^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))\alpha_\delta-\beta_\delta\alpha_0(\beta_0 S^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta))I_0^*+(((S^*+V^*)\beta_\delta-\mu_\delta-\gamma_\delta-\alpha)V^*\alpha_0 \\
& +(\beta_0 S^*-\gamma_0-\mu_0)(S^*+V^*)\beta_\delta-S^*(\mu_\delta+\gamma_\delta+\alpha)\beta_0+(\mu_\delta+\gamma_\delta+\mu_0+\gamma_0)\alpha+(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\alpha_\delta \\
& -\beta_\delta(\mu_\delta+\gamma_\delta)(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0))I_\delta^*-\beta_0\alpha_0(\beta_\delta S^*+\alpha_\delta V^*-\gamma_0-\mu_0-\gamma_\delta-\mu_\delta)I_0^{*2}+((((S^*+V^*)\beta_0 \\
& -\mu_0-\alpha-\gamma_0)\alpha_0-\beta_0(\mu_0+\gamma_0))V^*\alpha_\delta+(((S^*+V^*)\beta_0-\mu_0-\alpha-\gamma_0)S^*\beta_\delta-(\mu_\delta+\gamma_\delta)(S^*+V^*)\beta_0 \\
& +(\mu_\delta+\gamma_\delta+\mu_0+\gamma_0)\alpha+(\mu_0+\gamma_0)(\mu_\delta+\gamma_\delta))\alpha_0-\beta_0(\mu_0+\gamma_0)(\beta_\delta S^*-\gamma_\delta-\mu_\delta))I_0^*+\alpha(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0) \\
& (\beta_\delta S^*+\alpha_\delta V^*-\gamma_\delta-\mu_\delta))\mu-\beta_\delta\alpha_\delta(\mu_\delta+\gamma_\delta)(\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0)I_\delta^{*2}+(((-((\mu_0+\gamma_0)\beta_\delta+ \\
& \beta_0(\mu_\delta+\gamma_\delta))V^*\alpha_0-\beta_0(\mu_0+\gamma_0)(\beta_\delta S^*-\gamma_\delta-\mu_\delta))\alpha_\delta-\beta_\delta\alpha_0(\mu_\delta+\gamma_\delta)(\beta_0 S^*-\gamma_0-\mu_0))I_0^*-\alpha_\delta\alpha(\mu_\delta+\gamma_\delta) \\
& (\beta_0 S^*+\alpha_0 V^*-\gamma_0-\mu_0))I_\delta^*-I_0^*\alpha_0(\mu_0+\gamma_0)(I_0^*\beta_0+\alpha)(\beta_\delta S^*+\alpha_\delta V^*-\gamma_\delta-\mu_\delta)
\end{aligned}
$$

Using Routh-Hurwitz criteria [17, 18], characteristic polynomial (7) has eigenvalues with negative real part if $k_0, k_1, k_3 > 0$ and $k_1 k_2 k_3 > k_1^2 + k_3^2 k_0$ that gives the required stability conditions around E^1. □

6. Memory trace and hereditary trait

In this section, numerical solution is obtained using L1 scheme [8, 9]. The numerical approximation of the Caputo fractional-order derivative of the function $f(t)$ is described as

$$
{}_0^C D_t^\phi f(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} \left(f(t_{k+1}) - f(t_k)\right) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi}\right) \right]
$$

Now, the above approximation is done for each equation of the fractional-order system (4),

$$ {}_{0}^{C}D_t^{\phi} S(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} (S(t_{k+1}) - S(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

$$ {}_{0}^{C}D_t^{\phi} V(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} (V(t_{k+1}) - V(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

$$ {}_{0}^{C}D_t^{\phi} I_0(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} (I_0(t_{k+1}) - I_0(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

$$ {}_{0}^{C}D_t^{\phi} I_\delta(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} (I_\delta(t_{k+1}) - I_\delta(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

$$ {}_{0}^{C}D_t^{\phi} R(t) \approx \frac{(dt)^{-\phi}}{\Gamma(2-\phi)} \left[\sum_{k=0}^{T-1} (R(t_{k+1}) - R(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

Using the definition of Caputo derivative and above equations, the numerical solution of each equation is given by difference of Markov term and Memory trace.

- **For Omicron variant (I_0) :**

  $$ I_0(t_T) \approx dt^{\phi}\Gamma(2-\phi)F(X) + I_0(t_{T-1}) - \left[\sum_{k=0}^{T-2} (I_0(t_{k+1}) - I_0(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

  where Markov term and Memory trace for I_0 is given by:

  $$ \text{Markov Term} = dt^{\phi}\Gamma(2-\phi)F(X) + I_0(t_{T-1}) $$

  $$ \text{Memory Trace} = \sum_{k=0}^{T-2} (I_0(t_{k+1}) - I_0(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) $$

- **For Delta variant (I_δ) :**

  $$ I_\delta(t_T) \approx dt^{\phi}\Gamma(2-\phi)F(X) + I_\delta(t_{T-1}) - \left[\sum_{k=0}^{T-2} (I_\delta(t_{k+1}) - I_\delta(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) \right] $$

  where Markov term and Memory trace for I_δ is given by:

  $$ \text{Markov Term} = dt^{\phi}\Gamma(2-\phi)F(X) + I_\delta(t_{T-1}) $$

  $$ \text{Memory Trace} = \sum_{k=0}^{T-2} (I_\delta(t_{k+1}) - I_\delta(t_k)) \left((T-k)^{1-\phi} - (T-1-k)^{1-\phi} \right) $$

The term memory trace combines all previous activity and records of long term history of the system while Markov term refers to memory-less term where system's future state does not depend upon the past state of the system. Notice that whenever $\phi = 1$, the memory trace is zero and $0 < \phi < 1$ gives non-zero memory trace.

7. Numerical Simulation

In this section, the numerical simulation are performed using the MATLAB software for the proposed model. Parametric values for numerical simulation are taken from the daily case data being updated at [21]. Initial values $(S_0, V_0, I_{00}, I_{\delta 0}, R_0)$ are assumed to be $(0.78, 0.02, 0.05, 0.05, 0.1)$.

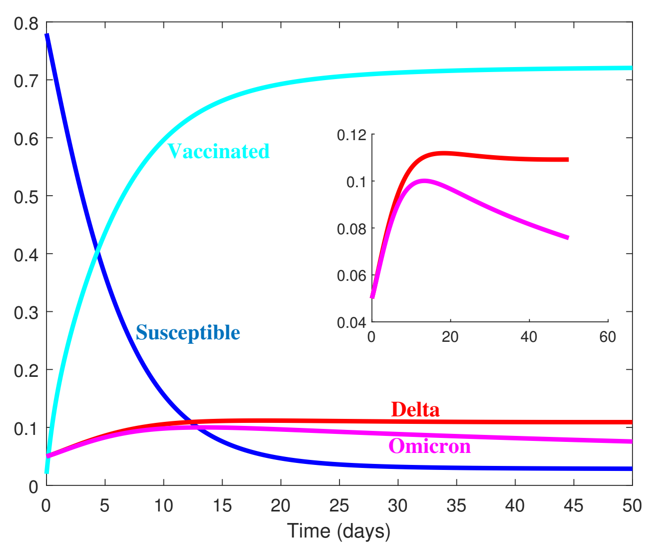
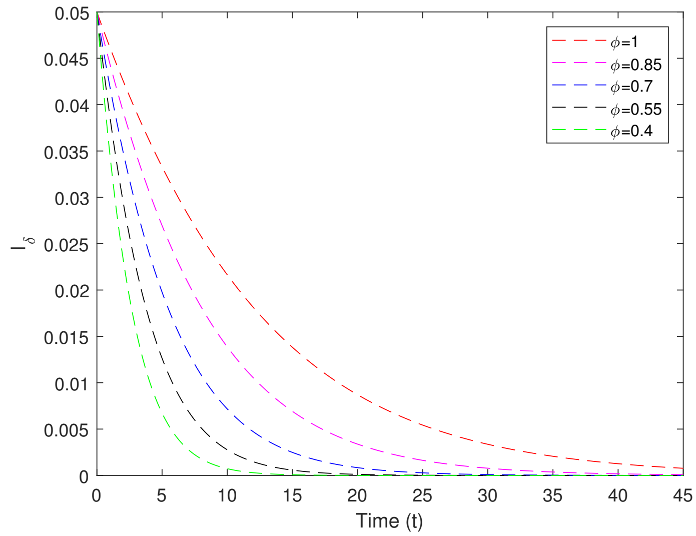


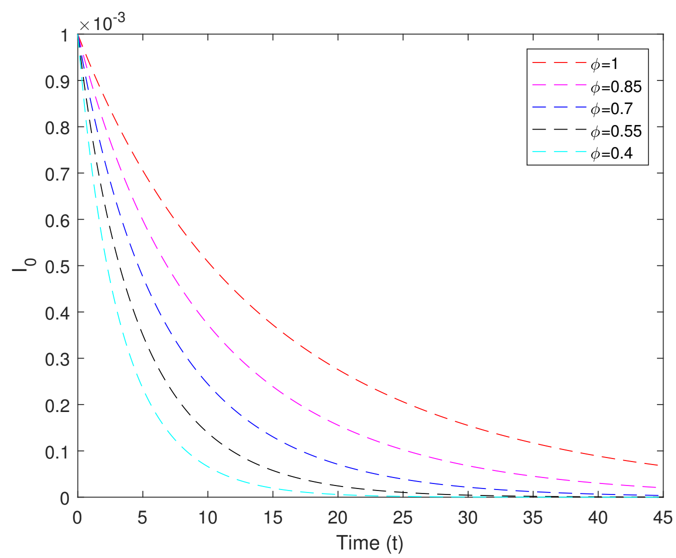
Figure 2: Variation in Susceptible, Vaccinated, Omicron and Delta variant over days.

Figure 2 represents transmission pattern of the susceptible, vaccinated, omicron, delta variant over a period of time (days). In initial days, omicron variant and delta variant have similar transmission pattern, but after a week omicron has slight decline in cases.



h

Figure 3: Changes in I_δ for different values of ϕ.



h

Figure 4: Changes in I_0 for different values of ϕ.

Figure 3 represents the variation in delta variant I_δ over different values of ϕ. Smaller values of fractional-order results to rapid decrement in values of I_δ over time. It can be observed that as memory effect increases , there is a sharp decrease in I_δ values. Figure 4 represents the variation in Omicron variant I_0 over different values of fractional-order ϕ. Similiar to I_δ, It can be observed that as memory effect increases, there is a sharp decrease in I_0 values as well. Figure 5 represents the variation in delta variant I_δ over vaccinated susceptibles S. Increment in value of vaccinated susceptible V results to decrement in values of I_δ. Figure 6 represents the variation in Omicron variant I_0 over vaccinated susceptibles V. Increment in values of vaccinated susceptible results to increment in values of I_0.

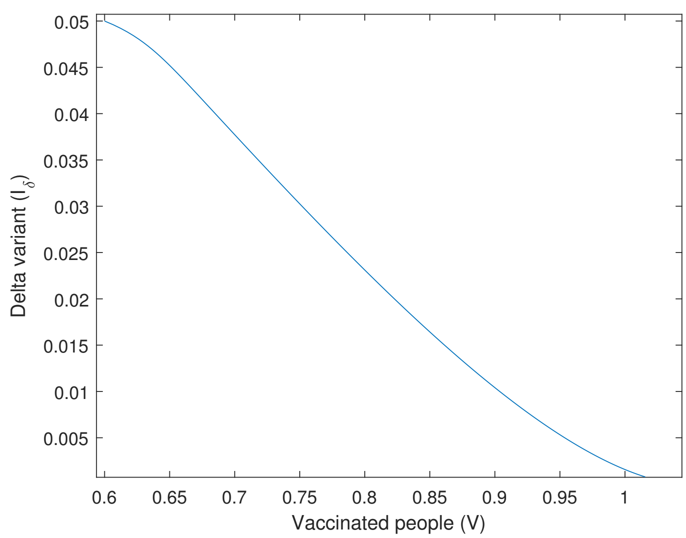


Figure 5: Variation in I_δ over V.

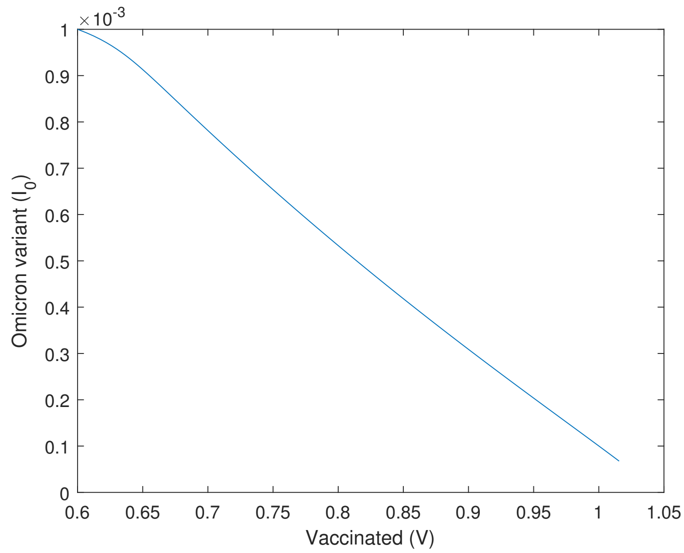


Figure 6: Variations in V w.r.t I_0.

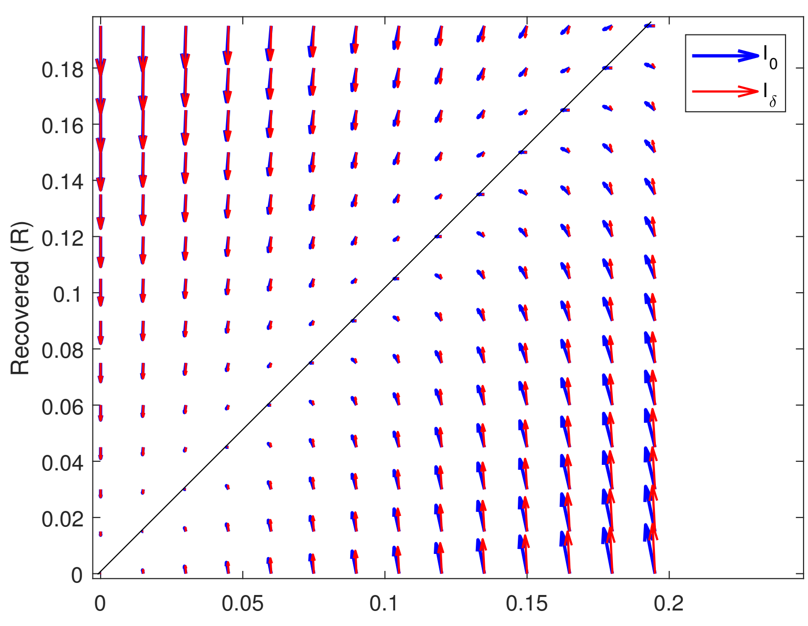


Figure 7: Quiver of I_δ, I_0 w.r.t R.

From figure 7 it is observed that individuals infectious from Delta and Omicron variant are getting recovered and are again vulnerable to catch the infection. Further, intensity of recovery also increase as number of infection (irrespective of variant) increases. As compared to red arrows representing I_δ), blue ones I_0 are slightly tilted towards recovered R represents early recovery of latter variants cases.

8. Conclusion

In this paper, a fractional-order $SVIR$ model is proposed for COVID-19 with two infection classes related to pre- and post-evolution of the SARS-CoV-2. Two important equilibrium points are obtained, disease-free E^0 and endemic E^1. The basic reproduction number R_0 is obtained using next-generation matrix method and is found to be maximum of the basic reproduction numbers corresponding to delta and omicron variant. The equilibrium point E^0 is found to be locally asymptotically stable only when $\mathsf{R}_0 < 1$ and E^1 is found to be asymptotically stable with certain conditions as mentioned in subsection (5.2). Numerical simulations are performed using MATLAB software. It is clear from numerical simulations that vaccination drive is very effective intervention measure and latter variant of virus is not much severe and infectious. Further, the memory effect graph shows as it increases, there is decline in infectious cases of each variant.

9. Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

10. Credit authorship contribution statement

Shah Nita: gave the concept. **Chaudhary Kapil:** analyzed model mathematically. **Jayswal Ekta:** carried out numerical simulation.

11. Acknowledgment

The authors acknowledge the valuable technical support given by DST-FIST to department. Chaudhary Kapil is supported by CSIR-JRF(Council of Scientific and Industrial Research) File No. 09/0070(13467)/2022-EMR-I. The authors are thankful to everyone who gave valuable suggestions to improve the quality of this manuscript.

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