Question 1.

(a) Show that the function $f(z) = \sin(1/z)$ has an isolated essential singularity at $z_0 = 0$.

(b) Explain the behaviour of $f(z)$ in the neighbourhood of z_0 using *Casorati–Weierstrass* theorem.

Question 2.

Use the residue theorem to evaluate the following integral $\dfrac{1}{2\pi i}\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}$.

Question 3.

(a) Show that $f(z) = \dfrac{\cos(z)}{z^2 - (\pi/2)^2}$ has isolated removable singularities at $z = \pm\pi/2$.

(b) Give an entire function g such that $g(z) = f(z)$ for $z \neq \pm\pi/2$.

Question 4.

Suppose f is analytic and has a zero of order m at z_0. Show that $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\mathrm{Res}(g, z_0) = m$.

Question 5.

Find the residue of the following functions at the mentioned point.

(a) $f(z) = \dfrac{1}{(1-z)}$ at $z = \infty$.

(b) $f(z) = \dfrac{\cos(z)}{\int_0^z g(w)dw}$ at $z = 0$, where g is an analytic function with $g(0) = 1$.

Question 6.

(a) Explain why Cauchy's integral formula can be viewed as a special case of Cauchy's residue theorem.

(b) Give a function f that is analytic in the punctured plane $\mathbb{C} - \{1\}$ with an isolated essential singularity at $z = 1$ and a simple zero at $z = 0$.

Question 7.

Show that $f(z) = (2\cos(z) - 2 + z^2)^2$ has a zero of order 8 at $z = 0$.

Question 8.

Suppose z_0 is the isolated removable singularity of $f(z)$ then prove that f is bounded and analytic in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0.

Question 9.

If f is analytic everywhere in the complex plane except a finite number of singularities $z_1, z_2, z_3, \cdots, z_n$ interior to a simple closed positive oriented curve C, then show that

$$\frac{1}{2\pi i}\int_C f(z)dz = \mathrm{Res}\left(\frac{1}{z^2}f(1/z), 0\right).$$

Note: You can use this result directly: $\mathrm{Res}(f(z), \infty) = -\sum_{k=1}^{n} \mathrm{Res}(f(z), z_k) = -\frac{1}{2\pi i}\int_C f(z)dz.$

Question 10.

Find all the singular points, their type (if pole, then mention the order as well), and the residue at each singular point of the function $f(z) = \pi\cot(\pi z)$.

ENDS