Question 1.

Let f be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad \text{whenever } |z| < \infty$$

- Show that
$$f[f(z)] = z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \cdots \quad \text{whenever } |z| < \infty$$
- Use this to find a series representation of the function $\sin(\sin(z))$.

Question 2.

Use multiplication or division of the series to find the series representation of the following functions and their respective region of convergence.
(i) $f(z) = 1/(e^z - 1)$ (ii) $f(z) = \tan(z)$ (iii) $f(z) = z/\sinh(2z)$

Question 3.

Let $\displaystyle\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ converges to a function $f(z)$ in some annular region $R_1 < |z - z_0| < R_2$, then show that for each $n \in \mathbb{Z}$ and for any positive oriented simple closed contour C in the annular region,

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

In other words, Prove the uniqueness of the Laurent series of $f(z)$ in the annular region.

Question 4.

Consider the function $f(z) = \dfrac{1}{z^2 \sin(z)}$ and the positive oriented contour C_N denoting the square whose edges lie along the lines

$$x = \pm(N + \frac{1}{2})\pi \quad y = \pm(N + \frac{1}{2})\pi$$

- Show that f has a pole of order 3 at $z = 0$ with residue $1/6$.
- Show that $z = \pm n\pi$ are the simple poles of f with residue $\dfrac{(-1)^n}{n^2\pi^2}$.

  Hint: Use theorem of residue calculation for quotients with $p(z) = 1$ and $q(z) = z^2 \sin(z)$
- Use the Cauchy-Residue theorem to show that

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i}\int_{C_N} f(z)dz$$

- Letting $N \to \infty$, show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Question 5.

Use the Cauchy-residue theorem to evaluate the following integrals:

(i) $\displaystyle \frac{1}{2\pi i}\int_{|z|=2} \frac{\cosh \pi z}{z(z^2+1)}dz$ (ii) $\displaystyle \frac{1}{2\pi i}\int_{|z|=3} \frac{e^{-z}}{(z-1)^2}dz$ (iii) $\displaystyle \frac{1}{2\pi i}\int_{|z|=3} \frac{1-e^{2z}}{z^4}dz$

Question 6.

Use the term-by term integration approach of series to show that
$$\frac{1}{2\pi i}\int_{|z|=1} e^z \left(\sum_{n=0}^{\infty} \frac{1}{n! z^n}\right) dz = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$$

Question 7.

Show that function is analytic in the mentioned domain. $f(z) = \begin{cases} \frac{\mathrm{Log}(z)}{z-1}, & \text{if } z \neq 1 \\ 1, & \text{otherwise} \end{cases}$

$|z| < \infty$, $-\pi < \mathrm{Arg}(z) < \pi$

Question 8.

Find the residue of the given functions at $z = 0$. Given that g is analytic function s.t. $g(0) \neq 0$.
(i) $f(z) = \dfrac{1}{[zg(z)]^2}$ (ii) $f(z) = \csc^2(z)$ (iii) $f(z) = \dfrac{1}{(z+z^2)^2}$ **Hint:** use (i) with suitable g.

Question 9.

Consider $p(z)$ and $q(z)$ are two functions satisfying $p(z)q(z) = 1$. Show that if z_0 is the zero of order m of $p(z)$ then it is pole of order m of $q(z)$ and if z_0 is the pole of order m of $p(z)$ then it is removable singularity of $q(z)$.

Question 10.

Assume an entire function $P(z)$ with n zeroes (z_k of order m_k), where $k = 1, 2, 3, \cdots, n$. What can be concluded about the singularities of $P'(z)/P(z)$, the residue at each z_k and the following integral?
$$\frac{1}{2\pi i}\int_{|z|=R} \frac{P'(z)}{P(z)}dz \quad \text{where } R = 1 + \max_k\{|z_k| : k = 1, 2, 3, \cdots, n\}.$$
Hint: Think first about the case $k = 1$ with representation $P(z) = (z-z_1)^{m_1} g(z)$ where $g(z_1) \neq 0$.