

Complex analysis and power series *

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1. Analytic functions

1.1 Notation In this text, z refers to a complex number ($z \in \mathbb{C}$), while f and g represent functions $f: E \rightarrow \mathbb{C}$ and $g: F \rightarrow \mathbb{C}$, where $B(a; \bar{\delta}) \subseteq E \subseteq \mathbb{C}$, $B(a; \bar{\delta}) \subseteq F \subseteq \mathbb{C}$, $B(a; \bar{\delta}) = \{z \in \mathbb{C} : |z - a| < \bar{\delta}\}$, $0 < \bar{\delta} \leq \infty$ and $a \in \mathbb{C}$. In other words, f and g are functions with complex values whose domains are subsets E and F of the complex numbers containing an open ball centered at the point a .

1.2 Definition LIMIT OF A COMPLEX FUNCTION. Let $b \in \mathbb{C}$. We say that $f(z)$ converges to b when z tends to a , denoted

$$\lim_{z \rightarrow a} f(z) = b ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - a| < \delta \text{ and } z \neq a \Rightarrow |f(z) - b| < \varepsilon .$$

1.3 Definition RIGHT AND LEFT LIMITS. Let $b \in \mathbb{C}$, $x \in \mathbb{R}$ and $f: E \rightarrow \mathbb{C}$, where $B(a; \bar{\delta}) \subseteq E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say that $f(x)$ converges to b when x tends to a from the left, denoted

$$\lim_{x \rightarrow a^-} f(x) = b \text{ or } f(a-) = b ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x < a \Rightarrow |f(x) - b| < \varepsilon .$$

Similarly, we say that $f(x)$ converges to b when x tends to a from the right, denoted

$$\lim_{x \rightarrow a^+} f(x) = b \text{ or } f(a+) = b ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x > a \Rightarrow |f(x) - b| < \varepsilon .$$

1.4 Definition CONTINUOUS FUNCTION. We say that the function f is continuous at point a iff

$$\lim_{z \rightarrow a} f(z) = f(a) .$$

1.5 Definition DERIVATIVE OF A COMPLEX FUNCTION. We say that the function f is differentiable at point a iff there exists a number $f'(a) \in \mathbb{C}$ such that

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = f'(a) .$$

We call $f'(a)$ the derivative of $f(z)$ at a .

1.6 Remark We also denote $f'(z)$ by $\frac{d}{dz}f(z)$.

1.7 Proposition CONTINUITY OF DIFFERENTIABLE FUNCTIONS. *If the function f is differentiable at point a , then it is continuous at point a .*

1.8 Theorem PROPERTIES OF DIFFERENTIATION. *Let $z \in B(a; \bar{\delta}) \subseteq E \cap F$. If the functions f and g are differentiable at point z , then*

- (1) $\frac{d}{dz} [c f(z)] = c f'(z),$
- (2) $\frac{d}{dz} [f(z) + g(z)] = f'(z) + g'(z),$
- (3) $\frac{d}{dz} [f(z) g(z)] = f'(z) g(z) + f(z) g'(z),$
- (4) $\frac{d}{dz} \left[\frac{f(z)}{g(z)} \right] = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2},$ provided $g(z) \neq 0.$

1.9 Theorem CHAIN RULE. *Let $h: G \rightarrow \mathbb{C}$ where $B(f(a); \delta_0) \subseteq f(E) \subseteq G \subseteq \mathbb{C}$, $B(f(a); \delta_0) = \{z \in \mathbb{C} : |z - f(a)| < \delta_0\}$ and $0 < \delta_0 \leq \infty$. If the function f is differentiable at point a and if h is differentiable at point $f(a)$, then the composite function $H(z) = h[f(z)]$ is differentiable at point a and*

$$H'(a) = h'[f(a)] f'(a).$$

1.10 Theorem DERIVATIVES OF IMPORTANT FUNCTIONS.

- (1) *If c is a complex constant, then*

$$\frac{d}{dz}(c) = 0.$$

- (2) *If n is a real constant,*

$$\frac{d}{dz}(z^n) = n z^{n-1}, \text{ provided } z \neq 0 \text{ when } n < 1.$$

- (3) $\frac{d}{dz}(e^z) = e^z.$

1.11 Theorem DERIVATIVE OF A REAL FUNCTION OF A COMPLEX VARIABLE. *Suppose the function f only takes real values at all points of the open ball $B(a; \bar{\delta})$, i.e. $f(z) \in \mathbb{R}$ for any $z \in B(a; \bar{\delta})$. If f is differentiable at point a , then $f'(a) = 0$.*

1.12 Definition ANALYTIC FUNCTION. *If there exists a positive constant $\varepsilon > 0$ such that the function f is differentiable at all points z such that $|z - a| < \varepsilon$, we say that the function is analytic at point a . If the function f is analytic at all points of a domain $D \subseteq \mathbb{C}$, we say that f is analytic on the domain D .*

1.13 Remark An analytic function is also called a holomorphic function.

1.14 Definition SINGULAR POINT. *If a function f is not analytic at point z_0 , but for any $\varepsilon > 0$ there exists a point z_1 such that $|z_1 - z_0| < \varepsilon$ and f is analytic at z_1 , we say that z_0 is a singular point (or a singularity) of the function f . If, furthermore, there exists a radius $R > 0$ such that f is analytic on the disk $0 < |z - z_0| < R$, we say that z_0 is an isolated singular point of the function f .*

1.15 Theorem OPERATIONS ON ANALYTIC FUNCTIONS. *If the functions f and g are analytic at point a , then*

- (1) *the functions $f(z) + g(z)$ and $f(z)g(z)$ are analytic at point a ;*
- (2) *the function $f(z)/g(z)$ is analytic at point a provided $g(a) \neq 0$.*

1.16 Theorem COMPOSITION OF ANALYTIC FUNCTIONS. *Let $h : G \rightarrow \mathbb{C}$ where $B(f(a); \delta_0) \subseteq f(E) \subseteq G \subseteq \mathbb{C}$, $B(f(a); \delta_0) = \{z \in \mathbb{C} : |z - f(a)| < \delta_0\}$ and $0 < \delta_0 \leq \infty$. If the function f is analytic at point a and if h is analytic at point $f(a)$, then the composed function $(h \circ f)(z) = h[f(z)]$ is analytic at point a .*

1.17 Theorem INFINITE DIFFERENTIABILITY OF ANALYTIC FUNCTIONS. *If the function f is analytic at point $a \in \mathbb{C}$, then f has derivatives of all orders at a , and the derivative functions are also analytic at point a .*

1.18 Theorem IMPORTANT ANALYTIC FUNCTIONS.

- (1) *Any polynomial of degree n ,*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (1.1)$$

where $a_0, a_1, \dots, a_n \in \mathbb{C}$, is analytic at all points $z \in \mathbb{C}$.

- (2) *A rational function*

$$f(z) = P(z)/Q(z) \quad (1.2)$$

where $P(z)$ and $Q(z)$ are polynomials of degrees p and q , is analytic everywhere, except when $Q(z) = 0$.

- (3) *The functions e^z , $\cos(z)$ and $\sin(z)$ are analytic everywhere.*
- (4) *The function $\log(z)$ is analytic everywhere except at $z = 0$.*

2. Power series

2.1 Definition POWER SERIES. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}$. We call the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ a power series centered at z_0 . The numbers a_n are the coefficients of the series.*

2.2 Remark In this definition and the sequel, we will use the convention $0^0 = 1$.

2.3 Theorem CONVERGENCE RADIUS OF A POWER SERIES (ABEL-HADAMARD). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ a power series and

$$\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}, \quad R = 1/\alpha,$$

where $R = \infty$ when $\alpha = 0$, and $R = 0$ when $\alpha = \infty$. Then the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0| > R$. Further, if $0 \leq \rho < R$, the convergence is uniform for $|z - z_0| \leq \rho$.

2.4 Remark We call R the *convergence radius* of the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The expression $1/R = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ is the *Hadamard* formula for the convergence radius.

2.5 Corollary ABSOLUTE CONVERGENCE OF POWER SERIES. If the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $z = z_1$, where $z_1 \neq z_0$, then it converges absolutely for any z such that $|z - z_0| < |z_1 - z_0|$.

2.6 Corollary BOUNDS ON COEFFICIENTS OF POWER SERIES. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ a power series whose convergence radius is R , and let $\varepsilon > 0$.

- (1) If $0 < R \leq \infty$, there exists an integer N , such that $|a_n| < \left(\frac{1}{R} + \varepsilon\right)^n$ for $n > N$.
- (2) If $0 < R < \infty$, there is an infinity of values of n for which $|a_n| > \left(\frac{1}{R} - \varepsilon\right)^n$.
- (3) If $R = 0$, there is an infinity of values of n for which $|a_n| > \varepsilon^n$.

2.7 Theorem UNIFORM ABSOLUTE CONVERGENCE OF POWER SERIES. If the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely for $z = z_1$, where $z_1 \neq z_0$, then it converges absolutely and uniformly on the closed disk $D = \{z \in \mathbb{C} : |z - z_0| \leq |z_1 - z_0|\}$.

2.8 Proposition CONVERGENCE RADIUS AND RATIO CRITERION. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series whose convergence radius is R . Then

$$\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq R \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Further, if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n|$ exists or $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = \infty$, then $R = \lim_{n \rightarrow \infty} |a_{n+1}/a_n|$.

2.9 Theorem CONVERGENCE CONDITION ON THE UNIT CIRCLE. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series whose convergence radius is 1. If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that

- (a) $a_{n+1} \leq a_n, \forall n$, and
- (b) $\lim_{n \rightarrow \infty} a_n = 0$,

then the series $\sum_{n=0}^{\infty} a_n z^n$ converges at any point of the circle $|z| = 1$, except possibly at $z = 1$.

2.10 Theorem CONTINUITY OF POWER SERIES ON THE UNIT CIRCLE (ABEL). *If the series $\sum_{n=0}^{\infty} a_n$ converges, then the function $\sum_{n=0}^{\infty} a_n z^n$, where $|z| < 1$, tends to $\sum_{n=0}^{\infty} a_n$ when $z \rightarrow 1$ so that $|1 - z| / (1 - |z|)$ remains bounded.*

2.11 Corollary CONTINUITY OF REAL POWER SERIES ON THE UNIT CIRCLE. *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that $\sum_{n=0}^{\infty} a_n$ converges, and if the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$, where $x \in \mathbb{R}$, then $\lim_{x \rightarrow 1^-} f(x)$ exists and*

$$\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} a_n .$$

2.12 Remark If the series $\sum_{n=0}^{\infty} a_n$ does not converge, the limit $\lim_{x \rightarrow 1^-} f(x)$ may or may not exist. In general, the existence of the limit $\lim_{x \rightarrow 1^-} f(x)$ does not guarantee the convergence of the series $\sum_{n=0}^{\infty} a_n$. There are however cases where the existence of the limit $\lim_{x \rightarrow 1^-} f(x)$ implies the convergence of $\sum_{n=0}^{\infty} a_n$ (Tauberian theorems). The following theorem provides an example.

2.13 Theorem CRITERION FOR CONVERGENCE AND CONTINUITY OF REAL POWER SERIES ON THE UNIT CIRCLE (TAUBER). *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$, where $x \in \mathbb{R}$, if $\lim_{n \rightarrow \infty} (n a_n) = 0$ and if $\lim_{x \rightarrow 1^-} f(x)$ exists, then the series $\sum_{n=0}^{\infty} a_n$ converges and $\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} a_n$.*

2.14 Theorem UNICITY OF POWER SERIES COEFFICIENTS. *If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ are two power series which converge for $|z - z_0| < R$, where $R > 0$, and if the limits of these series coincide on a sequence of points $\{z_k\}_{k=1}^{\infty}$ such that $0 < |z_k| < R$, $\forall k$, and $\lim_{k \rightarrow \infty} z_k = z_0$, then*

$$a_n = b_n , \quad \forall n .$$

2.15 Corollary UNICITY OF POWER SERIES COEFFICIENTS IN A CIRCLE. *If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ are two power series which converge for $|z - z_0| < R$, where $R > 0$, and if the limits of these series coincide for any z in the circle $|z - z_0| < R$, then*

$$a_n = b_n , \quad \forall n .$$

2.16 Theorem DIFFERENTIABILITY OF POWER SERIES. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$, where $R > 0$ and $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series whose convergence radius is R . Then the function $f(z)$ is analytic (and thus differentiable) on the disk $|z - z_0| < R$, and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

where the power series $\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ has convergence radius R . If, furthermore, $0 < R < \infty$ and $f(z)$ is a function such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ at every point where the series

$\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges, then there is at least one point on the circle $|z - z_0| = R$ where the function $f(z)$ is not analytic.

2.17 Remark In other words, we can obtain the derivative of the function $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$ by differentiating the series term by term, and the derivative series has the same convergence radius as the original series.

2.18 Corollary DIFFERENTIABILITY AT ALL ORDERS OF POWER SERIES. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$, where $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series whose convergence radius is R . Then the function $f(z)$ has derivatives of all orders, and these derivatives can be obtained by differentiating the series term by term. The derivative series all have the same convergence radius R , and

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \dots$$

where $f^{(n)}(z)$ is the derivative of order n of $f(z)$.

2.19 Theorem INTEGRABILITY OF POWER SERIES. Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series whose convergence radius is R , let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$, C a contour (continuous curve) in the interior of the convergence circle $|z - z_0| < R$, and $g(z)$ a continuous function on C . Then

$$\int_C f(z) g(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n dz.$$

2.20 Definition TWO-SIDED POWER SERIES. Let $\{a_n\}_{n=-\infty}^{\infty}$, $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}$. We call two-sided power series a series of the form $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. This series converges when the two series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ converge. Otherwise, we say it diverges.

2.21 Proposition CONVERGENCE ANNULUS OF TWO-SIDED POWER SERIES. Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^n$ be power series whose convergence radii are R_1 and R_2 respectively, where $R_1 > 0$ and $R_2 > 0$.

- (1) If $1/R_2 < R_1$, the series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ converges for $1/R_2 < |z - z_0| < R_1$ and diverges when $|z - z_0| > R_1$ or $|z - z_0| < 1/R_2$.
- (2) If $R_1 < 1/R_2$, the series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ diverges everywhere.
- (3) If $R_1 = 1/R_2$, the series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ diverges everywhere except possibly on the circle $|z - z_0| = R_1$.

3. Taylor and Laurent series

3.1 Theorem TAYLOR SERIES. Let f be an analytic function at any point of the open disk

$$D = \{y \in \mathbb{C} : |z - z_0| < R\}, \text{ where } z_0 \in \mathbb{C} \text{ and } 0 < R \leq \infty.$$

Then there exists a unique sequence $\{a_n\}_{n=0}^{\infty}$ in \mathbb{C} such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D.$$

Further,

$$a_n = f^{(n)}(z_0)/n! = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}, n = 0, 1, 2, \dots$$

where $C = \{z \in \mathbb{C} : |z - z_0| = \rho\}$ and ρ is any radius such that $0 < \rho < R$.

3.2 Remark In other words, an analytic function on the interior of a circle centered at z_0 can be written in the interior of this circle as a power series of $(z - z_0)$. Further, this series is unique. The integral \int_C is evaluated counterclockwise.

3.3 Corollary CAUCHY INEQUALITIES. Under the conditions of Theorem 3.1, suppose that $|f(z)| \leq M$ for $z \in C(\rho)$, where $C(\rho) = \{z \in \mathbb{C} : |z - z_0| = \rho\}$ and $0 < \rho < R$. Then

$$|a_n| = |f^{(n)}(z_0)|/n! \leq M/\rho^n, n = 0, 1, 2, \dots$$

3.4 Remark The Cauchy inequalities entail: for $\rho < R$, the coefficients of the Taylor series must decline at an exponential rate which depends on the convergence radius.

3.5 Corollary EQUIVALENCE BETWEEN ANALYTICITY AND THE EXISTENCE OF A TAYLOR SERIES. Let $D = \{z \in \mathbb{C} : |z - z_0| < R\}$ where $z_0 \in \mathbb{C}$ and $0 < R \leq \infty$. Then a function f is analytic on the domain D iff there exists a unique sequence $\{a_n\}_{n=0}^{\infty}$ in \mathbb{C} such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D.$$

3.6 Theorem ZEROS OF ANALYTIC FUNCTIONS. Let f be an analytic function at point z_0 , such that $f(z_0) = 0$. If $f^{(n)}(z_0) = 0$, $n = 1, 2, \dots, m-1$, but $f^{(m)}(z_0) \neq 0$, where $m \geq 1$, then there exists a radius $R > 0$ such that the function f can be written

$$f(z) = (z - z_0)^m g(z)$$

for $|z - z_0| < R$, where the function $g(z)$ is analytic at z_0 , and $g(z) \neq 0$ for $|z - z_0| < R$. If $f^{(n)}(z_0) = 0$, $n = 1, 2, \dots$, then there exists a radius $R > 0$ such that $f(z) = 0$ for $|z - z_0| < R$.

3.7 Remark The latter theorem implies that the zeros of a non-zero analytic function are *isolated*: unless all the derivatives of f are zero, we can find a radius $R > 0$ such that z_0 is the only point where the function cancels in the disk $|z - z_0| < R$. We call z_0 a *root* of the function f , and m its *multiplicity*.

3.8 Theorem FACTORIZATION OF AN ANALYTIC FUNCTION. *Let f be an analytic function on an open convex domain $U \subseteq \mathbb{C}$. If the function f has only a finite number p of distinct roots z_1, \dots, z_p , then the function f can be written*

$$f(z) = (z - z_1)^{m_1} \dots (z - z_p)^{m_p} g(z), z \in U$$

where m_1, \dots, m_p are the multiplicities of the roots z_1, \dots, z_p and $g(z)$ is an analytic function on U such that $g(z) \neq 0$ for any $z \in U$.

3.9 Remark In other words, an analytic function with a finite number of roots is finite can be written as the product of a polynomial with the same roots and an analytic function which is different from zero everywhere. An open disk $C = \{z \in \mathbb{C} : 0 \leq (z - z_0) < R\}$ where $R > 0$ is a convex set. The latter theorem remains valid when U is a convex and connected set.

3.10 Theorem SIMPLIFICATION RULE. *Let $U \subseteq \mathbb{C}$ an open and connected set. If f and g are two analytic functions on U such that*

$$f(z)g(z) = 0, \quad \forall z \in U,$$

then $f(z) = 0, \forall z \in U$, or $g(z) = 0, \forall z \in U$.

3.11 Remark If f, g and h are three analytic functions on U such that $f(z)h(z) = g(z)h(z), \forall z \in U$, and if $h(z) \neq 0$ for at least one value of $z \in U$, then

$$[f(z) - g(z)]h(z) = 0$$

and we can conclude that $f(z) = g(z), \forall z \in U$.

3.12 Theorem LOCAL SEPARABILITY OF ANALYTIC FUNCTIONS. *Let f be an analytic function which is not constant on an open connected set U . Then, for $w \in \mathbb{C}$ and $z_0 \in U$, there exists a radius $R > 0$ such that $f(z) \neq w$ for $0 < (z - z_0) < R$.*

3.13 Remark In other words, if the function is not constant, we can find a radius $R > 0$ such that $f(z)$ takes the value w at least one time in the disk $0 \leq |z - z_0| < R$.

3.14 Theorem LAURENT SERIES. *Let C_0 and C_1 be two circles centered at z_0 such that C_0 is contained in C_1 , i.e.*

$$C_0 = \{z \in \mathbb{C} : |z - z_0| = R_0\}, C_1 = \{z \in \mathbb{C} : |z - z_0| = R_1\} \text{ where } 0 \leq R_0 < R_1 \leq \infty.$$

Let f be an analytic function on C_0 and C_1 as well as on the domain between these two circles. Then there exists a unique two-sided sequence $\{a_n\}_{n=-\infty}^{\infty}$ in \mathbb{C} such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for any z such that $R_0 < |z - z_0| < R_1$, where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{C_1} \frac{f(z) dz}{(z - z_0)^{n+1}}, \text{ for } n = 0, 1, 2, \dots \\ &= \frac{1}{2\pi i} \int_{C_0} \frac{f(z) dz}{(z - z_0)^{n+1}}, \text{ for } n = -1, -2, \dots \end{aligned}$$

Further, for any circle $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ where $R_0 < R < R_1$,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \dots$$

3.15 Remark The line integrals \int_{C_0} , \int_{C_1} and \int_C are evaluated counterclockwise.

3.16 Corollary LAURENT SERIES NEAR AN ISOLATED SINGULARITY. *If f is an analytic function at any point of the disk $|z - z_0| < R$, where $R > 0$, except possibly at z_0 , then there exists a unique two-sided sequence $\{a_n\}_{n=-\infty}^{\infty}$ in \mathbb{C} such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for any z such that $0 < |z - z_0| < R$.

3.17 Remark In other words, if z_0 is a singular point of the function f , the function f can be represented by a Laurent series on the disk $0 < |z - z_0| < R$. If, furthermore, $a_n = 0$ for $n < 0$, the Laurent series reduces to a Taylor series, and we can redefine the function f at z_0 so that the latter is analytic at z_0 and thus everywhere on the disk $0 \leq |z - z_0| < R$. In such a case, we say that the singular point z_0 is removable. When a function f is analytic at any point of the disk $|z - z_0| < R$, it is clear we must have $a_n = 0$ for $n < 0$.

3.18 Corollary GENERALIZED CAUCHY INEQUALITIES. *Under the conditions of Theorem 3.14, suppose that $|f(z)| \leq M$ for $z \in C(R)$, where $C(R) = \{z \in \mathbb{C} : |z - z_0| = R\}$ and $R_0 < R < R_1$. Then*

$$|a_n| \leq M/R^n, \quad n = 0, \pm 1, \pm 2, \dots$$

3.19 Definition PRINCIPAL AND REGULAR PARTS OF A LAURENT SERIES. *In a Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, we call the series $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ the principal part of the series, while the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the regular part of the series.*

4. Sums, products and ratios of power series

4.1 Theorem POINTWISE CONVERGENCE. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ be two convergent power series whose limits are $f(z)$ and $g(z)$ respectively at a given point z . Then the*

following properties hold:

- (1) $c f(z) = \sum_{n=0}^{\infty} c a_n (z - z_0)^n, \forall c \in \mathbb{C};$
- (2) $f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) (z - z_0)^n;$
- (3) if $f(z)$ or $g(z)$ converges absolutely, then

$$f(z) g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (4.1)$$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$; furthermore, if the two series $f(z)$ and $g(z)$ converge absolutely, the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges absolutely;

(4) if

- (a) $b_0 \neq 0,$
- (b) the series $h(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n$ where the coefficients d_n are obtained by solving the equations $\sum_{k=0}^n a_k b_{n-k} = a_n, n = 0, 1, \dots,$ converges,
- (c) $g(z)$ or $h(z)$ converges absolutely,
- (d) $g(z) \neq 0,$

then

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z - z_0)^n. \quad (4.2)$$

4.2 Theorem CONVERGENCE IN A CIRCLE. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ be two power series whose convergence radii are R_1 and R_2 respectively, where $R_1 > 0$ and $R_2 > 0$. Then

- (1) for any $c \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c a_n (z - z_0)^n$ converges absolutely for $|z - z_0| < R_1$ and

$$\sum_{n=0}^{\infty} c a_n (z - z_0)^n = c f(z) \text{ for } |z - z_0| < R_1; \quad (4.3)$$

- (2) the series $\sum_{n=0}^{\infty} (a_n + b_n) (z - z_0)^n$ converges absolutely for $|z - z_0| < \min\{R_1, R_2\}$ and

$$\sum_{n=0}^{\infty} (a_n + b_n) (z - z_0)^n = f(z) + g(z) \text{ for } |z - z_0| < \min\{R_1, R_2\}; \quad (4.4)$$

- (3) the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$, converges absolutely for $|z - z_0| < \min\{R_1, R_2\}$, and

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n = f(z) g(z), \text{ for } |z - z_0| < \min\{R_1, R_2\}; \quad (4.5)$$

- (4) if $g(z) \neq 0$ for $|z - z_0| < R$, where $0 < R \leq \min\{R_1, R_2\}$, and $\{d_n\}_{n=0}^\infty$ is the sequence of coefficients obtained by solving the equations

$$\sum_{k=0}^n d_k b_{n-k} = a_n, n = 0, 1, \dots, \quad (4.6)$$

then the series $\sum_{n=0}^\infty d_n (z - z_0)^n$ converges absolutely for $|z - z_0| < R$, and

$$\sum_{n=0}^\infty d_n (z - z_0)^n = f(z)/g(z), \text{ for } |z - z_0| < R; \quad (4.7)$$

when $g(z_0) = b_0 \neq 0$, the coefficients d_n are unique and there exists a radius $R > 0$ such that $g(z) \neq 0$ for $|z - z_0| < R$.

4.3 Theorem MACLAURIN SERIES FOR A RATIONAL FUNCTION. *If*

$$f(z) = P(z)/Q(z) \quad (4.8)$$

where $P(z) = \sum_{n=0}^p a_n z^n$ and $Q(z) = \sum_{n=0}^q a_n z^n$ are polynomials of degree p and q respectively, and $Q(0) \neq 0$, then

$$f(z) = \sum_{n=0}^\infty d_n z^n, \text{ for } |z| < R,$$

where $R = \min\{|z_1^*|, \dots, |z_q^*|\} > 0$, z_1^*, \dots, z_q^* are the roots (possibly non distinct) of polynomial $Q(z)$ and the coefficients d_n are obtained by solving the equations

$$\sum_{k=0}^n d_k b_{n-k} = a_n, n = 0, 1, \dots, \quad (4.9)$$

with $a_n \equiv 0$ for $n > p$ and $b_n \equiv 0$ for $n > q$. Further, the series $\sum_{n=0}^\infty d_n z^n$ converges absolutely for $|z| < R$.

5. Singularities

5.1 Definition POLE AND ESSENTIAL SINGULARITY. *Let f be an analytic function on the disk $0 < |z - z_0| < R$. We say that f has a pole at point z_0 if $\lim_{z \rightarrow z_0} |f(z)| = \infty$. If the point z_0 is a singular point which is neither removable nor a pole, we say that it is an essential singular point.*

5.2 Theorem CHARACTERIZATION OF ISOLATED SINGULARITIES. *Let f be an analytic function with an isolated singular point at z_0 . Then*

- (1) z_0 is a removable singular point

$$\Leftrightarrow \lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$$

$$\Leftrightarrow \lim_{z \rightarrow z_0} f(z) = c, \text{ for some } c \in \mathbb{C}.$$

(2) z_0 is a pole

- \Leftrightarrow the function $g(z) = 1/f(z)$ has a removable singular point at z_0
- \Leftrightarrow there is a positive integer m ($m > 0$) and an analytic function $h(z)$ on a disk $|z - z_0| < R$, where $R > 0$, such that $h(z_0) \neq 0$ and $f(z) = h(z)/(z - z_0)^m$
- \Leftrightarrow there is a positive integer m such that $\lim_{z \rightarrow z_0} (z - z_0)^m f(z) = c$, where $c \in \mathbb{C}$
- \Leftrightarrow there is a positive integer m such that the function $g(z) = (z - z_0)^m f(z)$ has a removable singular point at z_0 .

5.3 Definition ORDER OF A POLE. If z_0 is a pole of the function f such that

$$\lim_{z \rightarrow z_0} (z - z_0)^m f(z) = c \neq 0, \text{ for some } c \in \mathbb{C},$$

we say that z_0 is a pole of order m .

5.4 Theorem SINGULARITIES AND LAURENT SERIES. Let f be an analytic function with an isolated singular point at z_0 with Laurent series is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ for } 0 < |z - z_0| < R. \quad (5.1)$$

Then

- (1) z_0 is a removable singular point $\Leftrightarrow a_n = 0, \forall n < 0$;
- (2) z_0 is a pole of order $m \Leftrightarrow a_{-m} \neq 0$ and $a_n = 0$ for $n < -m$;
- (3) z_0 is an essential singular point $\Leftrightarrow a_n \neq 0$ for an infinite number of negative values of n .

5.5 Theorem BEHAVIOR OF AN ANALYTIC FUNCTION NEAR AN ESSENTIAL SINGULARITY (PICARD). Let f be an analytic function on the disk $0 < |z - z_0| < R$. If z_0 is an essential singular point, then for any complex number $c \in \mathbb{C}$, except possibly one, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ converging to z_0 such that $f(z_n) = c, \forall n$.

5.6 Remark Picard's theorem means that in any neighborhood of z_0 and for any complex number c (except possibly one), the function f takes the value c an infinite number of times.

6. Partial fractions

6.1 Theorem PARTIAL FRACTION EXPANSION OF RATIONAL FUNCTIONS. Consider the rational function $f(z) = P(z)/Q(z)$ where $P(z) = \sum_{n=0}^p a_n z^n$ is a polynomial of degree p ($a_p \neq 0$) and $Q(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_q)^{m_q}$ is a polynomial of degree $q_* = \sum_{j=1}^q m_j$ with q distinct roots z_1, \dots, z_q of multiplicities m_1, \dots, m_q respectively ($q \geq 1, m_j \geq 1$ for $j = 1, \dots, q$). Then the function $f(z)$ can be uniquely written in the form

$$f(z) = G(z) + \sum_{j=1}^q G_j[1/(z - z_j)]$$

for any $z \in \mathbb{C}$ such that $z \neq z_j, j = 1, \dots, q$, where

$$G_j[1/(z - z_j)] = \sum_{k=1}^{m_j} A_{jk} / (z - z_j)^k,$$

$A_{jk} \in \mathbb{C}$, and $G(z)$ is a polynomial. Further,

- (1) if $p < q_*$, $G(z) \equiv 0$,
- (2) if $p \geq q_*$ and the polynomials $P(z)$ and $Q(z)$ have no common root, the degree of $G(z)$ is $p - q_*$.

6.2 Theorem FACTORIZATION OF AN ANALYTIC FUNCTION WITH FINITE NUMBER OF POLES.

Let f be an analytic function everywhere on an open domain $U \subseteq \mathbb{C}$ except at a finite number of singular points z_1, \dots, z_q which are poles of orders m_1, \dots, m_q respectively ($q \geq 1, m_j \geq 1$ for $j = 1, \dots, q$). Then there exists a function $g(z)$ analytic everywhere on U such that $g(z_j) \neq 0, j = 1, \dots, q$, and

$$f(z) = g(z) / [(z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_q)^{m_q}]$$

for $z \in U$ and $z \neq z_j, j = 1, \dots, q$. If, furthermore, the function f has a finite number of zeros, the function f can be written

$$f(z) = \frac{P(z)}{Q(z)} h(z)$$

for $z \in U$ and $z \neq z_j, j = 1, \dots, q$, where $P(z)$ and $Q(z)$ are polynomials with no common root, $Q(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_q)^{m_q}$ and $h(z) \neq 0$ for $z \in U$.

6.3 Theorem PARTIAL FRACTION EXPANSION OF AN ANALYTIC FUNCTION WITH FINITE NUMBER OF POLES. Let f be an analytic function everywhere on an open domain $U \subseteq \mathbb{C}$ except at a finite number of singular points z_1, \dots, z_q which are poles of orders m_1, \dots, m_q ($q \geq 1, m_j \geq 1$ for $j = 1, \dots, q$). Then the function f can be written in a unique way in the form

$$f(z) = g(z) + \sum_{j=1}^q G_j[1/(z - z_j)]$$

for any $z \in U$ such that $z \neq z_j$, $j = 1, \dots, q$, where

$$G_j[1/(z - z_j)] = \sum_{k=1}^{m_j} A_{jk}/(z - z_j)^k,$$

$A_{jk} \in \mathbb{C}$, and $G(z)$ is analytic everywhere on U .

7. Proofs and references

1. Churchill and Brown (1984, chapters 2 and 3) and Ahlfors (1979, chapter 2).
- 1.2. Ahlfors (1979, section 1.1, p. 22).
- 1.4. Ahlfors (1979, section 1.1, p. 23).
- 1.5. Ahlfors (1979, section 1.1, p. 23).
- 1.7. Ahlfors (1979, section 1.2, p. 24).
- 1.8 - 1.9. Churchill and Brown (1984, section 15, p. 41, and section 21, pp. 57-58).
- 1.10. Churchill and Brown (1984, section 15, p. 41 et section 21, pp. 57-58).
- 1.11. Ahlfors (1979, section 1.1, p. 23).
- 1.12. Churchill and Brown (1984, section 19, p. 50).
- 1.14. Churchill and Brown (1984, section 54, p. 156).
- 1.15 - 1.16. Churchill and Brown (1984, section 19, p. 51).
- 1.17. Churchill and Brown (1984, section 39, pp. 111-114).
- 1.18. Churchill and Brown (1984, section 9, p. 27, and Chapter 3).
2. Ahlfors (1979, Chapter 2), Churchill and Brown (1984, Chapter 5), and Rudin (1976, Chapter 3).
- 2.3. Ahlfors (1979, section 2.4, p. 38) and Rudin (1976, section 3.39, p. 69).
- 2.6. Wilf (1990, section 2.4, Theorem 2.4.2, p. 44).
- 2.7 - 2.8. Deshpande (1986, section 6.1, pp. 62-64).
- 2.9. Rudin (1976, section 3.44, p. 71).
- 2.10. Ahlfors (1979, section 2.5, pp. 41-42).
- 2.11 - 2.13. Devinatz (1968, section 4.5, pp. 170-171).
- 2.14. Gillert, Küstner, Kellwich and Kästner (1986, section 21.2, p. 527).
- 2.16. Ahlfors (1979, section 2.4, p. 38), Churchill and Brown (1984, section 49, p. 144) and Wilf (1990, section 2.4, Theorem 2.4.2, pp. 44-45).
- 2.18. Churchill and Brown (1984, section 50, pp. 146-147).
- 2.19. Churchill and Brown (1984, section 44, pp. 126-128, and section 39,).
- 3.1. Churchill and Brown (1984, section 44, pp. 126-128, and section 39, pp. 111-114.).
- 3.3. Silverman (1974, section 10.1, p. 139).
- 3.5. This is a direct consequence of Theorem 3.1.
- 3.6. Churchill and Brown (1984, section 53, pp. 152-153).
- 3.8. Deshpande (1986, section 10.2, p. 139).
- 3.10 - 3.12. Deshpande (1986, section 10.1, Propositions 10.3 and 10.4, p. 135).
- 3.14. Churchill and Brown (1984, sections 46 and 50, pp. 132-136 and 146-148).

- 3.16.** Silverman (1974, section 11.1, p. 158).
- 4.** Churchill and Brown (1984, Section 51, pp. 148-153) and Deshpande (1986, Section 6.1).
- 4.1.** Statement (3) is a consequence of the Cauchy-Mertens theorem; see Devinatz (1968, Section 4.5, pp. 168-169). Statement (4) is a consequence of (3).
- 5.** Deshpande (1986, Chapter 12).
- 5.1.** Deshpande (1986, Section 12.2, Propositions 12.2 - 12.3, pp. 154-155).
- 5.2 - 5.3.** Deshpande (1986, section 10.2, p. 139).
- 5.4.** Deshpande (1986, Section 12.3, Proposition 12.9, p. 163).
- 5.5.** Silverman (1974, Section 57, p. 241) and Churchill and Brown (1984, Section 56, p. 161).
- 6.1.** Ahlfors (1979, Section 1.4, pp. 31-32) and Lentin and Rivaud (1964, chapitre II, section 19, pp. 234-238).
- 6.2.** Deshpande (1986, Section 13.1, Proposition 13.1, pp. 169-170).
- 6.3.** Deshpande (1986, Section 13.1, p. 171).

Other useful references include: Cartan (1961), Gillert et al. (1986), Gradshteyn and Ryzhik (1980), Knopp (1956), Knopp (1990), Rudin (1987), Silverman (1972), Spiegel (1964).

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