Question 1.

(a) Show that the function $f(z) = \sin(1/z)$ has an isolated essential singularity at $z_0 = 0$.

(b) Explain the behaviour of $f(z)$ in the neighbourhood of z_0 using *Casorati–Weierstrass* theorem.

Answer

1(a) The Laurent series expansion of the function $\sin(1/z)$ is

$$\sin(1/z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots \qquad \text{for } 0 < |z| < \infty$$

having infinitely many non-zero terms in the principal part. Hence, $z_0 = 0$ is isolated essential singularity of f.

1(b) Using *Casoratti-Weirstrass* theorem, for any $\epsilon > 0$ and any complex number w_0, there exists $\delta > 0$ such that

$$|\sin(1/z) - w_0| < \epsilon \qquad \text{whenever } 0 < |z| < \delta.$$

In other words, the function $\sin(1/z)$ approaches arbitrarily close to any complex number w_0 in the deleted neighbourhood of 0.

Question 2.

Use the residue theorem to evaluate the following integral $\dfrac{1}{2\pi i}\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}dz$.

Answer

The function $f(z) = \dfrac{e^{iz}}{z^2(z-2)(z+5i)}$ has 3 isolated singularities ($2, -5i$ are simple poles, and 0 is pole of order 2). Further, the singularity $z = -5i$ does not lie interior to the positive oriented circle $|z| = 3$.

Hence, using residue theorem,
$\dfrac{1}{2\pi i}\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}dz = \text{Res}(f,0) + \text{Res}(f,2).$

At $z = 0$; $g_1(z) = z^2 f(z) = \dfrac{e^{iz}}{(z-2)(z+5i)}$ is analytic, $\text{Res}(f,0) = g_1'(0) = (-12+5i)/100$.

At $z = 2$; $g_2(z) = (z-2)f(z) = \dfrac{e^{iz}}{(z^2)(z+5i)}$ is analytic, $\text{Res}(f,2) = g_2(2) = e^{2i}/(8+20i)$.

Thus, $\dfrac{1}{2\pi i}\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}dz = \frac{-12+5i}{100} + \frac{e^{2i}}{8+20i}$

Rough work: Calculate the derivative of g_1 yourself to evaluate $g_1'(0)$.

Question 3.

(a) Show that $f(z) = \dfrac{\cos(z)}{z^2 - (\pi/2)^2}$ has isolated removable singularities at $z = \pm\pi/2$.

(b) Give an entire function g such that $g(z) = f(z)$ for $z \neq \pm\pi/2$.

Answer

3(a) The only singularities of the function $f(z) = \dfrac{\cos(z)}{z^2 - (\pi/2)^2}$ are the roots of $z^2 - (\pi/2)^2 = 0 \iff z = \pm\pi/2$.
Further, using L'Hôpital's rule,

$$\lim_{z\to\pi/2} f(z) = \lim_{z\to\pi/2} \frac{\cos(z)}{z^2 - (\pi/2)^2} = \lim_{z\to\pi/2} \frac{-\sin(z)}{2z} = \frac{-1}{\pi}$$
$$\lim_{z\to-\pi/2} f(z) = \lim_{z\to-\pi/2} \frac{\cos(z)}{z^2 - (\pi/2)^2} = \lim_{z\to-\pi/2} \frac{-\sin(z)}{2z} = \frac{-1}{\pi}.$$

Thus, $z = \pm\pi/2$ are the removable singularities of f as $\lim_{z\to\pm\pi/2} f(z)$ exists finitely.

3(b) As $z = \pm\pi/2$ are the removable singularities of f. Thus, f can be extended to an analytic function g by assigning the limit values at the removable singular points.

$$g(z) = \begin{cases} f(z) & \text{if } z \neq \pm\pi/2\ , \\ \lim\limits_{z\to\pm\pi/2} f(z) & \text{if } z = \pm\pi/2\,. \end{cases} = \begin{cases} \dfrac{\cos(z)}{z^2 - (\pi/2)^2} & \text{if } z \neq \pm\pi/2\ , \\ -1/\pi & \text{if } z = \pm\pi/2\,. \end{cases}$$

is the required entire function.

Question 4.

Suppose f is analytic and has a zero of order m at z_0. Show that $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\mathrm{Res}(g, z_0) = m$.

Answer

We are given f is analytic and z_0 is zero of order m, so f has the following representation:

$$f(z) = (z - z_0)^m \phi(z) \qquad \text{where } \phi \text{ is analytic and } \phi(z_0) \neq 0$$

Now, $g(z) = \dfrac{f'(z)}{f(z)} = \dfrac{m}{z - z_0} + \dfrac{\phi'(z)}{\phi(z)}$ has simple pole at z_0 as $\dfrac{1}{g(z)} = \dfrac{(z - z_0)\phi(z)}{(m\phi(z) + (z - z_0)\phi'(z))}$ has simple zero at z_0.

Thus, $\mathrm{Res}(g, z_0) = \lim\limits_{z\to z_0} (z - z_0) g(z) = m$
Rough work: Calculate the derivative of $f(z)$ yourself to evaluate $f'(z)/f(z)$.

Question 5.

Find the residue of the following functions at the mentioned point.

(a) $f(z) = \dfrac{1}{(1-z)}$ at $z = \infty$.

(b) $f(z) = \dfrac{\cos(z)}{\int_0^z g(w)dw}$ at $z = 0$, where g is an analytic function with $g(0) = 1$.

Answer

5(a) The function $f(z) = \dfrac{1}{1-z} = \dfrac{-1}{(z-1)}$ has only one singularity (simple pole at $z = 1$) with residue $\mathrm{Res}(f, 1)$.
where $\mathrm{Res}(f, 1) =$ Coefficient of $1/(z-1)$ in the principal part of Laurent series $= -1$.
Now, $\mathrm{Res}(f, \infty) = -\mathrm{Res}(f, 1) = 1$

5(b) The function $\phi(z) = \int_0^z g(w)dw$ has simple zero at $z = 0$ as $\phi(0) = 0$; $\phi'(0) = g(0) = 1$. Thus, $f(z)$ has simple pole at $z = 0$. Using residue for quotients with $p(z) = \cos(z)$ and $q(z) = \phi(z)$.

$$\mathrm{Res}(f, 0) = \frac{p(0)}{q'(0)} = \frac{\cos(0)}{g(0)} = 1$$

Question 6.

(a) Explain why Cauchy's integral formula can be viewed as a special case of Cauchy's residue theorem.

(b) Give a function f that is analytic in the punctured plane $\mathbb{C} - \{1\}$ with an isolated essential singularity at $z = 1$ and a simple zero at $z = 0$.

Answer

6(a) The Cauchy integral formula says: *Let C be simple closed positive oriented curve, f is analytic on and inside C then* $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-z_0}dz = f(z_0)$. The function $f(z)/(z-z_0)$ has simple pole at z_0 with residue $f(z_0)$.
Further, Cauchy's residue theorem says: $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-z_0}dz = \mathrm{Res}\left(\frac{f(z)}{z-z_0}, z_0\right) = f(z_0)$.

We get same result from both theorems.

The condition f is analytic on C can be relaxed, it is enough to consider f is analytic interior to C. Hence, Cauchy's integral formula can be viewed as special case of residue theorem.

6(b) $f(z) = ze^{1/(z-1)}$ has simple zero at $z = 0$ and isolated essential singularity at $z = 1$.

Question 7.

Show that $f(z) = (2\cos(z) - 2 + z^2)^2$ has a zero of order 8 at $z = 0$.

Answer

We use the Maclaurin series of $\cos(z)$ around 0 to show this.

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \sum_{n=3}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}; \qquad |z| < \infty$$

The function $f(z)$ can be rewritten as follows:

$$\begin{aligned} f(z) &= \left(2\cos(z) - 2 + z^2\right)^2 \\ &= \left(2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \sum_{n=3}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right) - 2 + z^2\right)^2 \\ &= \left(z^4\left(\frac{1}{12} + 2\sum_{n=3}^{\infty} \frac{(-1)^n z^{2n-4}}{(2n)!}\right)\right)^2 \\ &= z^8 g(z) \end{aligned}$$

where g is analytic with $g(0) = 1/144 \neq 0$. Hence, f has a zero of order 8 at $z = 0$.

Question 8.

Suppose z_0 is the isolated removable singularity of $f(z)$ then prove that f is bounded and analytic in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0.

Answer

See class notes or book for the proof.

Question 9.

If f is analytic everywhere in the complex plane except a finite number of singularities $z_1, z_2, z_3, \cdots, z_n$ interior to a simple closed positive oriented curve C, then show that

$$\frac{1}{2\pi i}\int_C f(z)dz = \text{Res}\left(\frac{1}{z^2} f(1/z), 0\right).$$

Note: You can use this result directly: $\text{Res}(f(z), \infty) = -\sum_{k=1}^{n} \text{Res}(f(z), z_k) = -\frac{1}{2\pi i}\int_C f(z)dz.$

Answer

See class notes or book for the proof.

Question 10.

Find all the singular points, their type (if pole, then mention the order as well), and the residue at each singular point of the function $f(z) = \pi\cot(\pi z)$.

Answer

The function $f(z) = \pi\cot(\pi z) = \dfrac{\pi\cos(\pi z)}{\sin(\pi z)}$ has singular points at the zeroes of function $\sin(\pi z)$. We know zeroes of $\sin(\pi z)$ are at each integer n, and are isolated and simple. Thus, $f(z)$ has simple poles at each integer n.

Using residue for quotients with $p(z) = \pi\cos(\pi z)$ and $q(z) = \sin(\pi z)$,

$$\text{Res}(f(z), n) = \frac{p(n)}{q'(n)} = \frac{\pi\cos(\pi z)}{\pi cos(\pi z)} = 1.$$